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Bose gases, Bose–Einstein condensation, and the Bogoliubov approximation

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We review recent progress towards a rigorous understanding of the Bogoliubov approximation for bosonic quantum many-body systems. We focus, in particular, on the excitation spectrum of a Bose gas in the mean-field (Hartree) limit. A list of open problems will be discussed at the end. © *2014 AIP Publishing LLC*. [http://dx.doi.org/10.1063/1.4881536]

I. INTRODUCTION

Bose–Einstein condensation (BEC) in cold atomic gases was first experimentally realized in 1995.[1,8] In these experiments, a large number of (bosonic) atoms are confined to a trap and cooled to very low temperatures. Below a critical temperature condensation of a large fraction of particles into the same one-particle state occurs.

Various interesting quantum phenomena have been explored in these and subsequent experiments, like the appearance of quantized vortices in rotating systems and the related property of superfluidity. The latter is related to the low-energy excitation spectrum of the system. We refer to Refs. 2, 5, 7, and 13 for reviews of the recent developments in this field.

BEC was predicted by Einstein in 1924,[12] building upon a previous derivation of Planck's formula for black-body radiation by Bose.[4] Einstein's considerations were based on an ideal (i.e., non-interacting) Bose gas. The presence of particle interactions represents a major difficulty for a rigorous derivation of this phenomenon, however. One of the key contributions to the theory of weakly interacting Bose gases is Bogoliubov's 1947 paper,[3] where he introduces an approximate model (now referred to as the Bogoliubov approximation) to explain its superfluid behavior. In this paper, we will summarize recent progress made towards a rigorous justification of this approximation.

II. THE BOSE GAS: A QUANTUM MANY-BODY PROBLEM

The quantum-mechanical description of a system of N bosons is given in terms of the Hamiltonian, acting as a linear operator in a suitable Hilbert space. For bosons interacting via a pair-interaction potential denoted by $v(x)$, it is given, in appropriate units, by

$$H_N = -\sum_{i=1}^{N} \Delta_i + \sum_{1\leq i<j\leq N} v(x_i - x_j). \tag{1}$$

The kinetic energy is described by Δ, the Laplacian on a suitable domain in $\mathbb{R}^3$, which we will typically take to be a cube of side length L, i.e., $[0, L]^3$. Suitable boundary conditions have to be imposed, with periodic boundary conditions being a typical example. The subscript i indicates, as usual, that the second derivative is with respect to $x_i \in \mathbb{R}^3$.

As appropriate for bosons, the Hamiltonian H_N acts on the Hilbert space of *permutation-symmetric* wave functions $\Psi(x_1, \ldots, x_N)$ in $\bigotimes^N L^2([0, L]^3)$. The interaction v is assumed to be short-range, i.e., it decays fast enough to be integrable at infinity, and mostly repulsive to ensure that

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the system behaves like a gas at low density and low temperature. A particularly simple example is the special case of hard spheres of diameter $a > 0$, where, formally, $v(x) = \infty$ for $|x| \leq a$, and $v(x) = 0$ for $|x| > a$.

The following quantities, derived from the Hamiltonian H_N, will interest us here.

- **Ground state energy**, defined as the lowest value of the spectrum of the Hamiltonian,

$$E_0(N, L) = \inf \operatorname{spec} H_N. \tag{2}$$

For large systems, one can take a thermodynamic limit $N \to \infty$, $L \to \infty$ with $N/L^3 = \varrho$ fixed, and consequently define the ground state energy density as

$$e(\varrho) = \lim_{L\to\infty} \frac{E_0(\varrho L^3, L)}{L^3}. \tag{3}$$

- At positive temperature $T = \beta^{-1} > 0$, one considers instead the **free energy**

$$F(N, L, T) = -\frac{1}{\beta} \ln \operatorname{Tr} \exp(-\beta H_N), \tag{4}$$

and the corresponding free energy density in the thermodynamic limit

$$f(\varrho, T) = \lim_{L\to\infty} \frac{F(\varrho L^3, L, T)}{L^3}. \tag{5}$$

- The ground state wave function Ψ_0, being a function of $N \gg 1$ variables, is for all practical purposes too complicated to compute. Instead one considers the corresponding reduced density matrices of Ψ_0, the simplest of which is the **one-particle density matrix**, given by the integral kernel

$$\gamma_0(x, x') = N \int_{\mathbb{R}^{3(N-1)}} \Psi_0(x, x_2, \ldots, x_N)\overline{\Psi_0(x', x_2, \ldots, x_N)}\, dx_2 \cdots dx_N. \tag{6}$$

It satisfies $0 \leq \gamma_0 \leq N$ as an operator, and $\operatorname{Tr} \gamma_0 = N$. With the aid of creation and annihilation operators (to be reviewed in Sec. IV below) one can also write

$$\gamma_0(x, x') = \left\langle a^\dagger(x')a(x)\right\rangle, \tag{7}$$

and this definition generalizes to arbitrary mixed states as well.
- The diagonal of the one-particle density matrix is the **particle density**

$$\varrho_0(x) = \gamma_0(x, x) = N \int_{\mathbb{R}^{3(N-1)}} |\Psi_0(x, x_2, \ldots, x_N)|^2 dx_2 \cdots dx_N, \tag{8}$$

with $\int \varrho_0(x)dx = N$. For translation invariant systems in the thermodynamic limit, ϱ_0 is a constant and does not depend on x, but for inhomogeneous systems the spatial variation of ϱ_0 represents a non-trivial question.
- By definition, **Bose–Einstein condensation** in a state Ψ_0 means that the one-particle density matrix γ_0 has an eigenvalue of order N, i.e., that $\|\gamma_0\|_\infty \geq cN$ for some $c > 0$ and all (large) N. The corresponding eigenfunction is called the *condensate wave function*. BEC is expected to occur below a critical temperature.
For translation invariant systems with Hamiltonian of the form (1) one always has

$$\|\gamma_0\|_\infty = \frac{1}{L^3} \int_{[0,L]^6} \gamma_0(x, x')dx\, dx' \tag{9}$$

in the ground state, or any Gibbs state at positive temperature. This being of the order $N = \varrho L^3$ means that $\gamma_0(x, x')$ does not decay as $|x - x'| \to \infty$, a property which is also termed *long range order*.
- Of particular interest to us will be the structure of the **excitation spectrum**, i.e., the spectrum of H_N above the ground state energy $E_0(N)$, and the relation of the corresponding eigenstates

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to the ground state. For translation invariant systems, H_N commutes with the total momentum operator

$$P = -i \sum_{j=1}^{N} \nabla_j, \tag{10}$$

and hence one can look at their joint spectrum. Of particular relevance is the infimum

$$E_q(N, L) = \inf \operatorname{spec} H_N \restriction_{P=q}, \tag{11}$$

and one can investigate the limit

$$e_q(\varrho) = \lim_{L\to\infty} \left(E_q(\varrho L^3, L) - E_0(\varrho L^3, L) \right) \tag{12}$$

for fixed ϱ and q. In contrast to the non-interacting case, where $e_q(\varrho) = 0$ for all q and ϱ, one expects a linear behavior of $e_q(\varrho)$ for small q for interacting particles. For a review of various questions related to the excitation spectrum of Bose gases we refer to Ref. 6.

III. THE IDEAL BOSE GAS

For *non-interacting bosons*, i.e., in the case $v \equiv 0$, the free energy can be calculated explicitly in terms of its Legendre transform. It is given by

$$f_0(\varrho, T) = \sup_{\mu\le 0} \left[\mu\varrho + \frac{1}{(2\pi)^3\beta} \int_{\mathbb{R}^3} \ln\left(1 - \exp(-\beta(p^2 - \mu))\right) dp \right], \tag{13}$$

and satisfies the simple scaling relation

$$f_0(\varrho, T) = \varrho^{5/3} f_0(\beta\varrho^{2/3}, 1). \tag{14}$$

From (13) one can immediately infer the following property: If ϱ exceeds a certain *critical density*, namely, if

$$\varrho \ge \varrho_c(\beta) \equiv \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} \frac{1}{e^{\beta p^2} - 1} dp = \left(\frac{T}{4\pi}\right)^{3/2} \zeta(3/2), \tag{15}$$

(where ζ denotes the Riemann zeta function) the supremum in (13) is achieved at $\mu = 0$, and hence $\partial f_0/\partial\varrho = 0$ for $\varrho \ge \varrho_c$. That is, the free energy becomes independent of the density above the critical density. Because of the scaling relation (14) one can equivalently talk about a *critical temperature*, which equals

$$T_c(\varrho) = \frac{4\pi}{\zeta(3/2)^{2/3}} \varrho^{2/3}. \tag{16}$$

The one-particle density matrix for the ideal Bose gas can also be calculated explicitly, and is given by

$$\gamma_0(x, y) = [\varrho - \varrho_c(\beta)]_+ + \sum_{n\ge 0} \frac{e^{\beta\mu_\varrho n}}{(4\pi\beta n)^{3/2}} e^{-|x-y|^2/(4\beta n)}. \tag{17}$$

Here, $[\,\cdot\,]_+ = \max\{0, \cdot\}$ denotes the positive part of a real number, and $\mu_\varrho \le 0$ is the μ where the supremum is achieved in (13). In particular, $\mu_\varrho = 0$ for $T \le T_c$. The last sum in (17) can easily be seen to decay exponentially in $|x - y|$ if $\mu_\varrho < 0$, while it decays algebraically (like $|x - y|^{-1}$, in fact) for $\mu_\varrho = 0$. We thus conclude the following asymptotic behavior of γ_0 depending on the temperature:

- For $T < T_c$ or, equivalently, $\varrho > \varrho_c$, γ_0 does not decay at infinity, but converges to the positive number $\varrho - \varrho_c$, which equals the condensate density.
- For $T > T_c$, γ_0 decays exponentially in $|x - y|$.
- At the critical value $T = T_c$, γ_0 decays algebraically.

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These features of the ideal Bose gas are expected to be correct even in the presence of interparticle interactions, with a different value of the critical temperature T_c, but so far no general result of this kind is known. In the case of purely repulsive interactions, one can show that at large enough temperature there is always exponential decay.[40] That is, one can derive an upper bound on the critical temperature in this case, but no (non-zero) lower bounds are available to this date.

The ground state energy of the ideal Bose gas is of course identically zero, and also the excitation spectrum can easily be computed explicitly. It is simply given by all finite sums of the form

$$\sum_{p\neq 0} p^2 n_p, \tag{18}$$

where the sum is over $p \in (\frac{2\pi}{L}\mathbb{Z})^3$ and $n_p \in \{0, 1, 2, \ldots\}$ for each p. In particular, one easily checks that $e_q(\varrho)$ defined in (12) is identically zero for all $q \in \mathbb{R}^3$.

IV. SECOND QUANTIZATION ON FOCK SPACE

In the following, it will be convenient to regard $\bigotimes_{\rm sym}^N L^2([0, L]^3)$ as a subspace of the bosonic *Fock space*

$$\mathcal{F} = \bigoplus_{n=0}^{\infty} \bigotimes_{\rm sym}^{n} L^2([0, L]^3). \tag{19}$$

On this space, the particle number N is now an operator, which acts simply as multiplication by n on the subspace $\bigotimes_{\rm sym}^n L^2([0, L]^3)$.

A basis of $L^2([0, L]^3)$ is given by the plane waves $L^{-3/2}e^{ip\cdot x}$ for $p \in (\frac{2\pi}{L}\mathbb{Z})^3$, and we introduce the corresponding *creation and annihilation operators*, which satisfy the canonical commutation relations (CCR)

$$\left[a_p, a_q\right] = \left[a_p^\dagger, a_q^\dagger\right] = 0\,, \quad \left[a_p, a_q^\dagger\right] = \delta_{p,q}. \tag{20}$$

The Hamiltonian H_N is equal to the restriction of

$$\mathbb{H} = \sum_p |p|^2 a_p^\dagger a_p + \frac{1}{2L^3}\sum_p \widehat{v}(p) \sum_{q,k} a_{q+p}^\dagger a_{k-p}^\dagger a_k a_q \tag{21}$$

to the subspace $\bigotimes_{\rm sym}^N L^2([0, L]^3) \subset \mathcal{F}$, where

$$\widehat{v}(p) = \int_{[0,L]^3} v(x) e^{-ipx}\, dx \tag{22}$$

denotes the Fourier transform of v.

V. THE BOGOLIUBOV APPROXIMATION

At low energy, and for sufficiently weak interactions, one expects the occurrence of Bose–Einstein condensation. That is, the zero momentum mode is expected to be macroscopically occupied, meaning that $a_0^\dagger a_0 \sim N$. In particular, the $p = 0$ mode plays a special role.

The *Bogoliubov approximation* consists of

- dropping all terms in $\mathbb{H}$ higher than quadratic in $a_p^\dagger$ and a_p for $p \neq 0$,
- replacing $a_0^\dagger$ and a_0 in $\mathbb{H}$ by $\sqrt{N}$.

The resulting Hamiltonian is quadratic in the $a_p^\dagger$ and a_p, and equals (note that the contribution of $p = 0$ to the second sum in (21) is exactly equal to $N(N-1)\widehat{v}(0)/(2L^3)$, hence the substitution of

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$a_0^\dagger$ and a_0 by $\sqrt{N}$ was not applied to this term)

$$\mathbb{H}^{\rm Bog} = \frac{N(N-1)}{2L^3}\widehat{v}(0) + \sum_{p\neq 0}\left(\left(|p|^2+\varrho\widehat{v}(p)\right)a_p^\dagger a_p + \tfrac{1}{2}\varrho\widehat{v}(p)\left(a_p^\dagger a_{-p}^\dagger + a_p a_{-p}\right)\right), \tag{23}$$

with $\varrho = N/L^3$ the particle density. It can be explicitly diagonalized via a *Bogoliubov transformation*:

Let $b_p = \cosh(\alpha_p)a_p + \sinh(\alpha_p)a_{-p}^\dagger$, with

$$\tanh(\alpha_p) = \frac{|p|^2+\varrho\widehat{v}(p) - \sqrt{|p|^4+2|p|^2\varrho\widehat{v}(p)}}{\varrho\widehat{v}(p)}. \tag{24}$$

Here, we have to *assume* that $|p|^2 + 2\varrho\widehat{v}(p) \geq 0$ for all p in order for the square root to be well-defined. The b_p and $b_p^\dagger$ again satisfy CCR (for any choice of real numbers α_p, in fact). A simple calculation shows that

$$\mathbb{H}^{\rm Bog} = E_0^{\rm Bog} + \sum_{p\neq 0} e_p b_p^\dagger b_p, \tag{25}$$

where

$$E_0^{\rm Bog} = \frac{N(N-1)}{2L^3}\widehat{v}(0) - \frac{1}{2}\sum_{p\neq 0}\left(|p|^2+\varrho\widehat{v}(p) - \sqrt{|p|^4+2|p|^2\varrho\widehat{v}(p)}\right), \tag{26}$$

and

$$e_p = \sqrt{|p|^4+2|p|^2\varrho\widehat{v}(p)}. \tag{27}$$

Note that in contrast to the non-interacting case, where $e_p = p^2$, the function e_p in (27) behaves linearly in p for small p (assuming that $\widehat{v}$ does not vanish near zero, i.e., that $\widehat{v}(0) > 0$).

The Bogoliubov approximation thus predicts that the ground state energy density in the thermodynamic limit equals

$$e^{\rm Bog}(\varrho) = \frac{1}{2}\varrho^2\widehat{v}(0) - \frac{1}{2(2\pi)^3}\int_{\mathbb{R}^3}\left(|p|^2+\varrho\widehat{v}(p) - \sqrt{|p|^4+2|p|^2\varrho\widehat{v}(p)}\right)dp. \tag{28}$$

For small ϱ, it turns out that

$$e^{\rm Bog}(\varrho) = \frac{1}{2}\varrho^2\left(\widehat{v}(0) - \frac{1}{2(2\pi)^3}\int_{\mathbb{R}^3}\frac{|\widehat{v}(p)|^2}{|p|^2}dp\right) + 4\pi\frac{128}{15\sqrt{\pi}}\left(\frac{\varrho\widehat{v}(0)}{8\pi}\right)^{5/2} + o(\varrho^{5/2}), \tag{29}$$

where the numeric factor in the last term arises from the integral

$$\frac{128}{15\sqrt{\pi}} = -\sqrt{\frac{8}{\pi^3}}\int_{\mathbb{R}^3}\left(|p|^2+1-\sqrt{|p|^4+2|p|^2} - \frac{1}{2|p|^2}\right)dp. \tag{30}$$

The expression $\widehat{v}(0) - \frac{1}{2(2\pi)^3}\int_{\mathbb{R}^3}\frac{|\widehat{v}(p)|^2}{|p|^2}dp$ can be identified with the first two terms in the Born series for $8\pi a$, where a denotes the *scattering length* of v. The latter can, e.g., be defined as[18]

$$a = \frac{1}{8\pi}\left\langle |v|^{1/2}\left|\frac{1}{1+\frac{1}{2}v^{1/2}\frac{1}{p^2}|v|^{1/2}}\right| v^{1/2}\right\rangle, \tag{31}$$

whenever the operator in question is invertible, i.e., whenever $p^2 + \frac{1}{2}v$ does not have a zero-energy resonance. Here, $v^{1/2}$ is defined as $v|v|^{-1/2}$ if $v \neq 0$, and as zero otherwise.

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Since the scattering length is the relevant physical parameter at low energy, this suggests that the true ground state energy density for small ϱ should be

$$e(\varrho) = 4\pi a \varrho^2 \left(1 + \frac{128}{15\sqrt{\pi}}\sqrt{\varrho a^3} + o(\varrho^{1/2})\right). \tag{32}$$

This expression for $e(\varrho)$ is known as the Lee–Huang–Yang formula.[21] Its rigorous justification is one of the open problems discussed below in Sec. VIII.

Not only does the Bogoliubov approximation make a prediction about the ground state energy of the system, it also allows to compute the complete excitation spectrum. In fact, from (25) we see that the spectrum of $\mathbb{H}^{\rm Bog} - E_0^{\rm Bog}$ is given by

$$\sum_p e_p n_p \quad \text{with } n_p \in \{0, 1, 2, \dots\}, \tag{33}$$

with e_p defined in (27). Moreover, the corresponding eigenstates can be constructed out of the ground state Ψ_0 by *elementary excitations* of the form

$$b^\dagger_{p_n} \cdots b^\dagger_{p_1} \Psi_0, \tag{34}$$

with $b^\dagger_p = \cosh(\alpha_p) a^\dagger_p + \sinh(\alpha_p) a_{-p}$, as before.

One can also calculate the ground state energy $E_q^{\rm Bog}$ in a sector of total momentum q, and arrives at

$$\begin{aligned} e_q(\varrho) = \lim_{L\to\infty} \left(E_q^{\rm Bog} - E_0^{\rm Bog}\right) &= \text{subadditive hull of } e_p \\ &= \inf_{\sum_p p n_p = q} \sum_p e_p n_p. \end{aligned} \tag{35}$$

In particular, also $e_q(\varrho)$ behaves linearly in q for small q.

For a detailed discussion of variants of the Bogoliubov approximation we refer the interested reader to Ref. 46.

VI. VALIDITY OF THE BOGOLIUBOV APPROXIMATION

There are only very few rigorous results concerning the validity of the Bogoliubov approximation:

- Quite generally, one can show that the pressure in the thermodynamic limit is unaffected by the substitution of $a_0^\dagger$ and a_0 (or any other mode) by complex numbers (called c-numbers in the physics literature), see Refs. 15, 29, and 42. This is true independent of whether or not BEC occurs in the system. Moreover, the value of the occupation number $a_0^\dagger a_0$ is correctly predicted, to leading order in the system size, by the approximate model with $a_0^\dagger$ and a_0 replaced by c-numbers.
- The *Lieb–Liniger model* of one-dimensional bosons with δ-function interaction is defined by the Hamiltonian

$$H_N = \sum_{j=1}^N -\frac{\partial^2}{\partial z_j^2} + g \sum_{1\le i<j\le N} \delta(z_i - z_j) \tag{36}$$

on $\bigotimes_{\rm sym}^N L^2([0, L])$. As shown in Ref. 25, this model is exactly solvable, and various quantities, like the ground state energy density and the excitation spectrum, can be computed. As far as the ground state energy is concerned, the Bogoliubov approximation turns out to become exact in the weak coupling/high density limit $g/\varrho \to 0$. The same is true for parts of the excitation spectrum, but the exact excitation spectrum has an additional branch (called the Lieb-mode) which is absent in the Bogoliubov approximation.

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- For *charged bosons* in a uniform background (also known as the "jellium" model) Foldy's law[14]

$$e(\varrho) \approx C\varrho^{5/4} \tag{37}$$

for the ground state energy density has been verified in Refs. 30 and 41. Also in this case, the Bogoliubov approximation becomes exact in the high density limit, i.e., as $\varrho \to \infty$. In this model, the interaction among the particles is given by the Coulomb potential $v(x) = |x|^{-1}$. For charge neutrality, an additional one-particle potential $\sum_{i=1}^N V(x_i)$ of the form

$$V(x) = -\int_{[0,L]^3} \frac{\varrho}{|x-y|} dy \tag{38}$$

is added, as well as a constant to take into account the self-interaction of the background, $\frac{1}{2}\varrho^2 \iint_{[0,L]^6} |x-y|^{-1} dx dy$. The total energy per unit volume then satisfies (37) with an explicit constant C given by the integral

$$\begin{aligned} C &= -\frac{1}{2(2\pi)^3} \int_{\mathbb{R}^3} \left(|p|^2 + \frac{4\pi}{|p|^2} - \sqrt{|p|^4 + 8\pi} \right) dp \\ &= -\frac{2}{5} \frac{\Gamma(3/4)}{\Gamma(5/4)} \left(\frac{2}{\pi}\right)^{1/4}, \end{aligned} \tag{39}$$

which arises from the Bogoliubov approximation in a similar fashion as discussed in Sec. V. There is also a corresponding result for a two-component charged Bose gas, where the validity of the Bogoliubov approximation for the ground state energy was rigorously verified.[31,41]
- The leading term in the ground state energy of the *low density Bose gas*,

$$e(\varrho) \approx 4\pi a \varrho^2, \tag{40}$$

is rigorously known to be correct for purely repulsive interaction potentials, i.e., for $v \geq 0$, with finite scattering length. An upper bound of the correct form was provided by Dyson already in 1957,[10] while a proof of the lower bound was given only in 1998 by Lieb and Yngvason.[33] For generalizations to partially attractive interactions see Refs. 20 and 44, and for a corresponding result for the free energy at positive temperature see Refs. 37 and 45. We emphasize that these results do not, strictly speaking, concern a rigorous justification of the Bogoliubov approximation, since the latter does not actually predict the correct appearance of the scattering length, but only yields its first two orders in Born approximation, as discussed in Sec. V.
An *upper bound* on the ground state energy density of the conjectured form of the Lee–Huang–Yang formula

$$4\pi a \varrho^2 \left(1 + \frac{128}{15\sqrt{\pi}} \sqrt{\varrho a^3} + o(\varrho^{1/2}) \right) \tag{41}$$

was proved in Ref. 43 for positive, smooth, and sufficiently weak interaction potentials. A proof of a lower bound of this form is still an outstanding open problem.
- For low density ϱ, the Bogoliubov approximation can only be strictly valid if
  - the third term in the Born series for the scattering length is negligible,
  - the second term in the Born series for the scattering length is large compared with $a(a^3\varrho)^{1/2}$.

  Consider an interaction potential of the form

$$\frac{a_0}{R^3} v(x/R) \tag{42}$$

for "nice" v with $\int v = 8\pi$, and $R > 0$ an adjustable parameter, which is allowed to depend on the density of the system, and will be chosen large compared to a_0. Then $a \approx a_0$ to leading order. In terms of R, the two conditions above are equivalent to

$$\frac{a^3}{R^2} \ll a(a^3\varrho)^{1/2} \ll \frac{a^2}{R}. \tag{43}$$

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If we write $a/R \sim (a^3\varrho)^{1/2-\delta}$ for small $a^3\varrho$, this requires δ to satisfy $0 < \delta < 1/4$. Note that for this choice of R one has $R^3\varrho \sim (a^3\varrho)^{3\delta-1/2}$. In particular, the range R of the interaction potential is much smaller than the mean particle spacing $\varrho^{-1/3}$ only if $\delta > 1/6$.
In Ref. 16, the validity of the Lee–Huang–Yang formula (41) is proved for interaction potentials of the form (42) and small enough δ. An extension of this result to a larger range of δ, including δ slightly larger than 1/6, was announced in Ref. 32.

Most of these results concern the validity of the Bogoliubov approximation for the *ground state energy* of the system. Much less is known as far as the *excitation spectrum* is concerned. The only case where the Bogoliubov approximation can be rigorously justified even for the excitation spectrum concerns a system in the mean-field (or Hartree) limit, where the interactions among the particles are very weak and of long range. These results are fairly recent and we review them in Sec. VII.

VII. THE MEAN-FIELD (HARTREE) LIMIT

A simpler case where the analysis of the validity of the Bogoliubov approximation can be extended beyond the ground state energy is the Hartree limit. This is an extreme form of a mean-field limit where the interaction potential extends over the whole size of the system, but the interaction is sufficiently weak (of order $1/N$) in order for the interaction energy to be of the same order as the kinetic energy.

We consider again a system of N bosons in a cubic box, with periodic boundary conditions. For simplicity, let us choose units such that the length of the box L equals 1. The Hamiltonian of the systems is thus given by

$$H_N = -\sum_{i=1}^{N} \Delta_i + \frac{1}{N-1} \sum_{1\le i<j\le N} v(x_i - x_j). \tag{44}$$

Here we wrote the interaction potential as $(N-1)^{-1}v(x)$, reflecting the weakness of the potential as mentioned above. The case of fixed, N-independent v corresponds to the *mean-field* or *Hartree* limit.

It is not difficult to see that the ground state energy is determined, to leading order in N for large N, by minimizing the energy $\langle\Psi|H_N|\Psi\rangle$ over product states of the form

$$\Psi(x_1, \dots, x_N) = \phi(x_1)\cdots\phi(x_N). \tag{45}$$

This has been shown, in a much more general setting than what is discussed here, in Ref. 22. For a constant ϕ, corresponding to a homogeneous system, the resulting Hartree energy is then simply equal to $\frac{1}{2}N\int v$.

It is also known that starting from a product state of the form (45), a solution to the time-dependent Schrödinger equation $i\partial_t\Psi = H_N\Psi$ stays roughly a product at later times, with the factors in the limit $N\to\infty$ determined by the time-dependent Hartree equation

$$i\partial_t\phi = -\Delta\phi + \left(|\phi|^2 * v\right)\phi, \tag{46}$$

where $*$ denotes convolution. For a history of this problem and a review of recent results, we refer to Ref. 34.

Going beyond the leading order, where the Hartree equation applies, we can ask the following questions.

- Given that the ground state energy $E_0(N) = \inf \operatorname{spec} H_N$ satisfies $E_0(N) = \frac{1}{2}N\widehat{v}(0) + o(N)$ for fixed (i.e., N-independent) v, what is the next order correction? It turns out that it is actually $O(1)$, and the $O(1)$-term can be explicitly computed and agrees with the prediction from the Bogoliubov approximation.
- What is the spectrum of $H_N - E_0(N)$, i.e., the excitation spectrum of the system? Does it converge as $N\to\infty$? Is the Bogoliubov approximation valid? The latter predicts a dispersion law for elementary excitations that is *linear* for small momentum, as discussed in Sec. V.

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- What fraction of particles is in a Bose–Einstein condensate? Recall that Bose–Einstein condensation concerns the largest eigenvalue of the one-particle density matrix γ of a many-body wave function Ψ, defined via the matrix elements

$$\langle f|\gamma|g\rangle = N\int \overline{f(x)}\Psi(x, x_2, \ldots, x_N)g(y)\overline{\Psi(y, x_2, \ldots, x_N)}\,dx\,dy\,dx_2\cdots dx_N. \tag{47}$$

  For fixed v, the Bogoliubov approximation predicts that $\|\gamma\| \geq N - O(1)$ in the ground state, and this can actually be proved to be correct.

A. Main results

For our analysis of the excitation spectrum, we assume that $v(x)$ is bounded and of positive type, i.e.,

$$v(x) = \sum_{p\in(2\pi\mathbb{Z})^3} \widehat{v}(p)e^{ip\cdot x} \quad \text{with } \widehat{v}(p) \geq 0\ \forall p \in (2\pi\mathbb{Z})^3. \tag{48}$$

Under these assumptions, the following theorem holds.

Theorem 1. *The ground state energy $E_0(N)$ of H_N equals*

$$E_0(N) = \frac{N}{2}\widehat{v}(0) + E_0^{\text{Bog}} + O(N^{-1/2}) \tag{49}$$

with

$$E_0^{\text{Bog}} = -\frac{1}{2}\sum_{p\neq 0}\left(|p|^2 + \widehat{v}(p) - \sqrt{|p|^4 + 2|p|^2\widehat{v}(p)}\right). \tag{50}$$

Moreover, the excitation spectrum of $H_N - E_0(N)$ below an energy ξ is equal to finite sums of the form

$$\sum_{p\in(2\pi\mathbb{Z})^3\setminus\{0\}} e_p\, n_p + O\left(\xi^{3/2}N^{-1/2}\right), \tag{51}$$

where

$$e_p = \sqrt{|p|^4 + 2|p|^2\widehat{v}(p)}, \tag{52}$$

and $n_p \in \{0, 1, 2, \ldots\}$ for all $p \neq 0$.

Theorem 1 is proved in Ref. 38. The proof consists of constructing a unitary operator U that makes $UH_NU^\dagger$ close to the operator:

$$\frac{N}{2}\widehat{v}(0) + E_0^{\text{Bog}} + \sum_{p\in(2\pi\mathbb{Z})^3\setminus\{0\}} e_p\, a_p^\dagger a_p. \tag{53}$$

In particular, the proof implies that the excited eigenfunctions can be (approximately) obtained by acting with products of $U^\dagger a_p^\dagger a_0 U$ on the ground state.

Let us comment on the error terms in (49) and (51). Both the ground state energy and all excited energy levels a distance $O(1)$ from the ground state agree with the prediction obtained via Bogoliubov's approximation up to errors of order $N^{-1/2}$ for large N. Moreover, an excitation energy a distance ξ from the ground state energy is necessarily of the form $\sum_p e_p n_p(1 + o(1))$ as long as $\xi^{3/2}N^{-1/2} \ll \xi$, i.e., for $\xi \ll N$. That is, the Bogoliubov approximation gives the correct excitation energies to leading order in a very large window above the ground state energy, whose size has to be small compared with N. This restriction is presumably optimal. The existence of Bose–Einstein condensation is only guaranteed for excitation energies small compared to N, and the existence of BEC is one of the key assumptions entering the Bogoliubov approximation.

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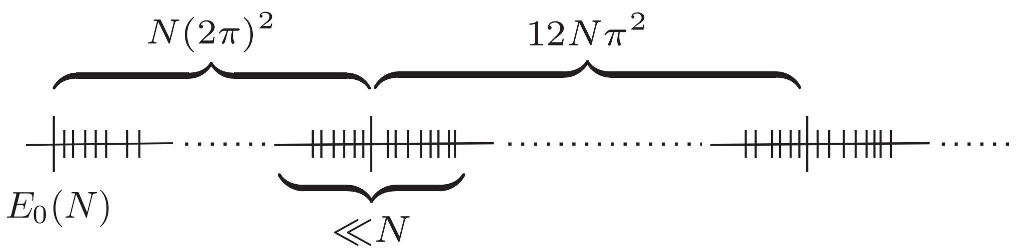

FIG. 1. Sketch of the parts of the spectrum that are correctly determined by the Bogoliubov approximation in the Hartree limit.

Theorem 1 implies the following corollary concerning the momentum dependence of the spectrum of H_N.

Corollary 1. Let $E_P(N)$ denote the ground state energy of H_N in the sector of total momentum P. We have

$$E_P(N) - E_0(N) = \min_{\{n_p\},\, \sum_p p\, n_p = P} \sum_{p\neq 0} e_p\, n_p \;+\; O\left(|P|^{3/2} N^{-1/2}\right). \tag{54}$$

In particular,

$$E_P(N) - E_0(N) \geq |P| \min_p \sqrt{2\widehat{v}(p) + |p|^2} + O(|P|^{3/2} N^{-1/2}). \tag{55}$$

The bound (55) implies that $E_P(N) - E_0(N)$ behaves linearly in P for not too large P (assuming that $\widehat{v}(p)$ does not vanish for small p). Note that this fact is caused by the interactions among the particles, non-interacting systems do not show this behavior. The linear behavior is very important physically and is responsible for the superfluid behavior of the system. According to Landau, the coefficient in front of $|P|$ is, in fact, the critical velocity for frictionless flow. We refer to Ref. 6 for further details on this correspondence.

Note that under the unitary transformation

$$\widetilde{U} = \exp\left(-iq \cdot \sum_{j=1}^{N} x_j\right), \quad q \in (2\pi\mathbb{Z})^3, \tag{56}$$

the Hamiltonian H_N transforms as

$$\widetilde{U}^\dagger H_N \widetilde{U} = H_N + N|q|^2 - 2q \cdot P, \tag{57}$$

where $P = -i\sum_{j=1}^{N} \nabla_j$ denotes again the total momentum operator. Hence our results apply equally also to the parts of the spectrum of H_N with excitation energies close to $N|q|^2$, corresponding to *collective excitations* where the particles move uniformly with momentum q; cf. Fig. 1.

B. Ideas in the proof

In the language of second quantization, the Hamiltonian H_N is the restriction of the operator

$$\mathbb{H} = \sum_{p\in(2\pi\mathbb{Z})^3} |p|^2 a_p^\dagger a_p + \frac{1}{2(N-1)} \sum_p \widehat{v}(p) \sum_{q,k} a_{q+p}^\dagger a_{k-p}^\dagger a_k a_q \tag{58}$$

to the N-particle subspace of the Fock space $\mathcal{F}$. Note that N has two different roles here. It determines the particle number, but also appears as a parameter in the Hamiltonian $\mathbb{H}$.

As discussed in Sec. V, the Bogoliubov approximation consists of

- dropping all terms higher than quadratic in $a_p^\dagger$ and a_p, $p \neq 0$;
- replacing $a_0^\dagger$ and a_0 by $\sqrt{N}$.

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The resulting quadratic Hamiltonian is $\frac{N}{2}\widehat{v}(0) + \mathbb{H}^{\rm Bog}$, where

$$\mathbb{H}^{\rm Bog} = \sum_{p\neq 0} \left(\left(|p|^2 + \widehat{v}(p)\right) a_p^\dagger a_p + \tfrac{1}{2}\widehat{v}(p) \left(a_p^\dagger a_{-p}^\dagger + a_p a_{-p} \right) \right). \tag{59}$$

It is diagonalized via a Bogoliubov transformation of the form

$$b_p = \cosh(\alpha_p) a_p + \sinh(\alpha_p) a_{-p}^\dagger, \tag{60}$$

leading to

$$\mathbb{H}^{\rm Bog} = E_0^{\rm Bog} + \sum_{p\neq 0} e_p b_p^\dagger b_p, \tag{61}$$

with $E_0^{\rm Bog}$ and e_p defined in (50) and (52), respectively.

The proof of Theorem 1 consists of *two main steps*:

1. As a first step, one shows that H_N is well approximated by an operator similar to the Bogoliubov Hamiltonian $\mathbb{H}^{\rm Bog}$, but with a_p and $a_p^\dagger$ replaced by

$$a_p^\dagger \to c_p^\dagger := \frac{a_p^\dagger a_0}{\sqrt{N}}, \quad a_p \to c_p := \frac{a_p a_0^\dagger}{\sqrt{N}}. \tag{62}$$

   Note that the operators c_p and $c_p^\dagger$ conserve the particle number. The resulting Hamiltonian is quadratic in $c_p^\dagger$ and c_p and is, in particular, also particle number conserving. Hence, it has a chance of being close to H_N on the subspace of particle number N. The original Bogoliubov Hamiltonian (59) does not leave this subspace invariant, and hence can not be directly compared with H_N.
2. Mimicking the Bogoliubov transformation (60), we introduce the operators $d_p = \cosh(\alpha_p)c_p + \sinh(\alpha_p)c_{-p}^\dagger$. It turns out that the modified Hamiltonian from Step 1 is close to

$$E_0^{\rm Bog} + \sum_{p\neq 0} e_p d_p^\dagger d_p, \tag{63}$$

   whose spectrum now has to be analyzed. This analysis is complicated by the fact that the operators d_p and $d_p^\dagger$ do not satisfy CCR. It turns out that they do, however, approximately on the subspace where $a_0^\dagger a_0$ is close to N, which is sufficient for our purpose.

In the following, we shall explain these two steps in greater detail. For further details, we refer to Ref. 38.

1. Step 1: Approximation by a quadratic Hamiltonian

Under our assumptions on the interaction potential v, it is not difficult to see that

$$N - a_0^\dagger a_0 \leq \text{const.}\, [1 + H_N - E_0(N)]\,. \tag{64}$$

This proves that the excitation energy dominates the condensate depletion. In particular, if the excitation energy is small compared with N, most particles occupy the zero momentum mode, i.e., Bose–Einstein condensation occurs.

To show that cubic and quartic terms in $a_p^\dagger$ and a_p, $p \neq 0$, in the Hamiltonian are negligible, one needs to prove a stronger bound of the form

$$\left(N - a_0^\dagger a_0\right)^2 \leq \text{const.}\, \left[1 + (H_N - E_0(N))^2\right], \tag{65}$$

however. It implies that also the fluctuations in the number of particles outside the condensate are suitably small.

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The first statement (64) follows easily from positivity of $\widehat{v}(p)$. Positivity implies that

$$\sum_{p\in(2\pi\mathbb{Z})^3\setminus\{0\}} \widehat{v}(p) \left| \sum_{j=1}^{N} e^{ip\cdot x_j} \right|^2 \geq 0, \tag{66}$$

which can be rewritten as

$$\sum_{1\leq i<j\leq N} v(x_i - x_j) \geq \frac{N^2}{2}\widehat{v}(0) - \frac{N}{2}v(0). \tag{67}$$

Thus, H_N is bounded from below as

$$H_N \geq \frac{N}{2}\widehat{v}(0) + T - \frac{N}{2(N-1)}\left(v(0) - \widehat{v}(0)\right), \tag{68}$$

where T denotes the kinetic energy

$$T = -\sum_{i=1}^{N} \Delta_i. \tag{69}$$

The statement (64) follows, since $T \geq (2\pi)^2(N - a_0^\dagger a_0)$.

For the second statement (65) one has to work a bit more. It turns out to be useful to actually prove a slightly stronger bound, namely, the inequality

$$\left(N - a_0^\dagger a_0\right) T \leq \text{const.}\left[1 + (H_N - E_0(N))^2\right]. \tag{70}$$

Since $T \geq (2\pi)^2(N - a_0^\dagger a_0)$ (and the two operators commute), this indeed implies the bound (65).

For the proof of (70), let us introduce the notation

$$N^{>} = N - a_0^\dagger a_0 = \sum_{i=1}^{N} Q_i \tag{71}$$

for the number of particles outside the condensate, where Q denotes the projection onto the subspace of $L^2([0,\,1]^3)$ of co-dimension one orthogonal to the constant function. For any bosonic (i.e., permutation-symmetric) wave function Ψ, we can write

$$\begin{aligned}\left\langle\Psi\left|N^{>}T\right|\Psi\right\rangle &= N\left\langle\Psi\left|Q_1T\right|\Psi\right\rangle \\ &= N\left\langle\Psi\left|Q_1S\right|\Psi\right\rangle + \left\langle\Psi\left|N^{>}\left(H_N - E_0(N)\right)\right|\Psi\right\rangle,\end{aligned} \tag{72}$$

where

$$\begin{aligned}S &= T - H_N + E_0(N) \\ &= E_0(N) - (N-1)^{-1}\sum_{i<j} v(x_i - x_j).\end{aligned} \tag{73}$$

Using Schwarz's inequality, the last term in (72) can be bounded as

$$\left\langle\Psi\left|N^{>}\left(H_N - E_0(N)\right)\right|\Psi\right\rangle \leq \left\langle\Psi\left|(N^{>})^2\right|\Psi\right\rangle^{1/2}\left\langle\Psi\left|(H_N - E_0(N))^2\right|\Psi\right\rangle^{1/2}. \tag{74}$$

We split S into two parts, $S = S_a + S_b$, with

$$S_a = E_0(N) - \frac{1}{N-1}\sum_{2\leq i<j\leq N} v(x_i - x_j) \tag{75}$$

and

$$S_b = -\frac{1}{N-1}\sum_{j=2}^{N} v(x_1 - x_j). \tag{76}$$

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Note that S_a does not depend on x_1. Using the positivity of $\widehat{v}(p)$ as in (66), but with the sum over j running from 2 to N only, as well as the simple upper bound $E_0(N) \leq \frac{N}{2}\widehat{v}(0)$ on the ground state energy, we see that

$$S_a \leq \frac{1}{2}\left(\widehat{v}(0) + v(0)\right). \tag{77}$$

In particular, this implies that

$$N\,\langle \Psi\,| Q_1 S_a |\,\Psi\rangle \leq \frac{1}{2}\left(\widehat{v}(0) + v(0)\right)\left\langle \Psi\,\middle| N^> \middle|\,\Psi\right\rangle. \tag{78}$$

To bound the contribution of S_b, we use

$$\begin{aligned} -\langle \Psi | Q_1 S_b | \Psi\rangle = \langle \Psi\,| Q_1 v(x_1 - x_2) |\,\Psi\rangle = \langle \Psi\,| Q_1 Q_2 v(x_1 - x_2) |\,\Psi\rangle \\ + \langle \Psi\,| Q_1 P_2 v(x_1 - x_2) P_2 |\,\Psi\rangle \\ + \langle \Psi\,| Q_1 P_2 v(x_1 - x_2) Q_2 |\,\Psi\rangle\,, \end{aligned} \tag{79}$$

where $P = 1 - Q$ denotes the rank-one projection onto the constant function in $L^2([0, 1]^3)$. The second term on the right side of (79) is positive. For the first and the third, we use Schwarz's inequality and $\|v\|_\infty = v(0)$ to conclude that

$$\langle \Psi | Q_1 S_b | \Psi\rangle \leq v(0)\langle \Psi | Q_1 Q_2 | \Psi\rangle^{1/2} + v(0)\langle \Psi | Q_1 | \Psi\rangle. \tag{80}$$

Since

$$\langle \Psi | Q_1 Q_2 | \Psi\rangle = \frac{\langle \Psi | N^>(N^> - 1) | \Psi\rangle}{N(N-1)} \leq \frac{\langle \Psi | (N^>)^2 | \Psi\rangle}{N^2}, \tag{81}$$

we have thus shown that

$$\begin{aligned} \left\langle \Psi\,\middle| N^> T \middle|\,\Psi\right\rangle \leq \frac{1}{2}\left(\widehat{v}(0) + 3v(0)\right)\left\langle \Psi\,\middle| N^> \middle|\,\Psi\right\rangle \\ + \left(v(0) + \left\langle \Psi\,\middle| (H_N - E_0(N))^2 \middle|\,\Psi\right\rangle^{1/2}\right)\left\langle \Psi\,\middle| (N^>)^2 \middle|\,\Psi\right\rangle^{1/2}. \end{aligned} \tag{82}$$

Using that $N^> \leq (2\pi)^{-2} T$ in the last factor, this further implies that

$$\begin{aligned} \left\langle \Psi\,\middle| N^> T \middle|\,\Psi\right\rangle \leq \left(\frac{v(0) + \left\langle \Psi\,\middle| (H_N - E_0(N))^2 \middle|\,\Psi\right\rangle^{1/2}}{2\pi}\right)^2 \\ + \left(3v(0) + \widehat{v}(0)\right)\left\langle \Psi\,\middle| N^> \middle|\,\Psi\right\rangle. \end{aligned} \tag{83}$$

The desired result (70) then follows from (64).

2. An algebraic identity

The inequalities (64) and (70) allow us to conclude that $\mathbb{H}$ is, at low energy, well approximated by

$$\frac{N}{2}\widehat{v}(0) + \frac{1}{2}\sum_{p \neq 0}\left[A_p\left(c_p^\dagger c_p + c_{-p}^\dagger c_{-p}\right) + B_p\left(c_p^\dagger c_{-p}^\dagger + c_p c_{-p}\right)\right], \tag{84}$$

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where $A_p = |p|^2 + \widehat{v}(p)$ and $B_p = \widehat{v}(p)$, and the operators c_p are defined in (62). A simple identity, which does *not* use the CCR, is

$$A_p\left(c_p^\dagger c_p + c_{-p}^\dagger c_{-p}\right) + B_p\left(c_p^\dagger c_{-p}^\dagger + c_p c_{-p}\right)$$
$$= \sqrt{A_p^2 - B_p^2}\left(\frac{\left(c_p^\dagger + \beta_p c_{-p}\right)\left(c_p + \beta_p c_{-p}^\dagger\right)}{1-\beta_p^2} + \frac{\left(c_{-p}^\dagger + \beta_p c_p\right)\left(c_{-p} + \beta_p c_p^\dagger\right)}{1-\beta_p^2}\right)$$
$$- \frac{1}{2}\left(A_p - \sqrt{A_p^2 - B_p^2}\right)\left([c_p, c_p^\dagger] + [c_{-p}, c_{-p}^\dagger]\right), \tag{85}$$

where

$$\beta_p = \begin{cases} \frac{1}{B_p}\left(A_p - \sqrt{A_p^2 - B_p^2}\right) & \text{if } B_p > 0 \\ 0 & \text{if } B_p = 0. \end{cases} \tag{86}$$

Note that if the operators c_p and $c_p^\dagger$ satisfied CCR, the term in the last line of (85) would be a constant. Its deviation from a constant can be controlled in terms of the condensate depletion, and the inequality (70) can be used to control the error made by simply replacing it be the value it would take in the case of CCR.

Introducing the operators

$$d_p = \frac{c_p + \beta_p c_{-p}^\dagger}{\sqrt{1-\beta_p^2}} \tag{87}$$

and their adjoints, we conclude that $\mathbb{H}$ is, in fact, close to the operator

$$\frac{N}{2}\widehat{v}(0) + E_0^{\rm Bog} + \sum_{p\neq 0} e_p\, d_p^\dagger d_p, \tag{88}$$

where we used that

$$E_0^{\rm Bog} = -\frac{1}{2}\sum_{p\neq 0}\left(A_p - \sqrt{A_p^2 - B_p^2}\right) \tag{89}$$

and

$$e_p = \sqrt{A_p^2 - B_p^2}. \tag{90}$$

3. Step 2: The spectrum of $d_p^\dagger d_p$

If the operators d_p and $d_p^\dagger$ satisfied CCR, we could immediately read off the spectrum of the operator in (88), and we would be done. However, without CCR we do not know the spectrum of $d_p^\dagger d_p$ and, moreover, the various summands in (88) do not actually commute in our case.

The usual Bogoliubov transformation (60) is of the form

$$b_p = \cosh(\alpha_p) a_p + \sinh(\alpha_p) a_{-p}^\dagger = e^{-X} a_p e^X, \tag{91}$$

where X is the anti-hermitian operator

$$X = \frac{1}{2}\sum_{p\neq 0}\alpha_p\left(a_p^\dagger a_{-p}^\dagger - a_p a_{-p}\right). \tag{92}$$

This identity can easily be verified using the CCR $[a_p, a_q^\dagger] = \delta_{p,q}$. Our operators $c_p = a_p a_0^\dagger/\sqrt{N}$, on the other hand, satisfy

$$\left[c_p, c_q^\dagger\right] = \delta_{p,q}\frac{a_0 a_0^\dagger}{N} - \frac{a_p a_q^\dagger}{N}. \tag{93}$$

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We now define, in analogy to (92), the particle-number conserving anti-hermitian operator

$$\widetilde{X} = \frac{1}{2}\sum_{p\neq 0}\alpha_p\left(c_p^\dagger c_{-p}^\dagger - c_p c_{-p}\right). \tag{94}$$

In order to compute the spectrum of $d_p^\dagger d_p$, we apply the unitary $e^{\widetilde{X}}$, and argue that the resulting operator is close $a_p^\dagger a_p$, at least in the subspace of low energy. More precisely, we show that

$$e^{-\widetilde{X}} a_p e^{\widetilde{X}} = \overbrace{\cosh(\alpha_p)c_p + \sinh(\alpha_p)c_{-p}^\dagger}^{d_p} + \text{Error}_p \tag{95}$$

for suitable small error term. Here it is important that actually the sum over all error terms (depending on p) is still (relatively) small as long as $(N - a_0^\dagger a_0)^2 \ll N^2$. The proof of (95) is somewhat lengthy and will be skipped here. It proceeds by studying $e^{-t\widetilde{X}} a_p e^{t\widetilde{X}}$ as a function of $t \in [0, 1]$, using a Grönwall type estimate. The details are presented in Ref. 38.

C. Conclusions and generalizations

The mean-field or Hartree limit may be somewhat unphysical when it comes to the description of cold atomic gases. It can be used as a toy model, however, which is analytically much easier to handle than the Gross-Pitaevskii limit of dilute gases,[26–28,39] for instance. The results reviewed in this section are the first rigorous results concerning the excitation spectrum of an interacting Bose gas, in a suitable limit of weak, long-range interactions. With the notable exception of exactly solvable models in one dimension, this is the only model where rigorous results on the excitation spectrum are available. The results verify Bogoliubov's prediction that the spectrum consists of sums of elementary excitations. In the translation invariant case, the excitation energy turns out to be linear in the momentum for small momentum. In particular, Landau's criterion for superfluidity is verified.

The methods presented in this section can be generalized to inhomogeneous systems without translation invariance. This was shown in Ref. 17, where the excitation spectrum of the Hamiltonian

$$H_N = \sum_{i=1}^N(-\Delta_i + V(x_i)) + \frac{1}{N-1}\sum_{1\le i<j\le N} v(x_i - x_j) \tag{96}$$

on the Hilbert space $\otimes_{\text{sym}}^N L^2(\mathbb{R}^3)$ was studied, with a trap potential V that is locally bounded and tends to infinity at infinity, in order to ensure that all the particles are confined and cannot escape to infinity. To leading order in N, the ground state energy of (96) is determined by minimizing the *Hartree functional*

$$\begin{aligned}\mathcal{E}^{\text{H}}(\phi) = &\int_{\mathbb{R}^3}\left(|\nabla\phi(x)|^2 + V(x)|\phi(x)|^2\right)dx\\ &+ \frac{1}{2}\int_{\mathbb{R}^6}|\phi(x)|^2 v(x-y)|\phi(y)|^2 dx dy,\end{aligned} \tag{97}$$

with minimal energy $E^{\text{H}} = \inf\{\mathcal{E}^{\text{H}}(\phi) : \int|\phi|^2 = 1\}$. Under the stated conditions on v and V, it is not difficult to see that there exists a unique minimizer ϕ_0 (up to a constant phase, of course, which we can choose such that ϕ_0 is positive) with $E^{\text{H}} = \mathcal{E}^{\text{H}}(\phi_0)$. The corresponding Euler-Lagrange equation for the minimizer ϕ_0 can be written as $K^{\text{H}}\phi_0 = 0$, where K^{H} is the Hartree operator

$$K^{\text{H}} = -\Delta + V(x) + v * |\phi_0|^2(x) - \varepsilon_0, \tag{98}$$

with $\varepsilon_0 = E^{\text{H}} + \frac{1}{2}\int_{\mathbb{R}^6}|\phi_0(x)|^2 v(x-y)|\phi_0(y)|^2 dx dy$ and $*$ denoting convolution.

The excitation spectrum of (96) turns out to have a similar structure as in (51), i.e., it consists of sums of elementary excitations. These are described by an effective one-body operator given by

$$E = \left(\sqrt{K^{\text{H}}}\left(K^{\text{H}} + 2W\right)\sqrt{K^{\text{H}}}\right)^{1/2}, \tag{99}$$

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where W denotes the operator with integral kernel $\phi_0(x)v(x-y)\phi_0(y)$. More precisely, to leading order in N the spectrum of $H_N - E_0(N)$ is of the form $\sum_i e_i n_i$, with $n_i \in \{0, 1, 2, \dots\}$ and e_i the (non-zero) eigenvalues of E. We refer to Ref. 17 for details.

By using different techniques, this result was further generalized in Ref. 24, where the validity of the Bogoliubov approximation in the Hartree limit was shown for a much larger class of Hamiltonians and interaction potentials. The method of Ref. 24 does not require that v has positive Fourier transform, for instance, one merely needs to assume that the corresponding Hartree functional has a unique minimizer and that the Hessian is strictly positive at the minimum. While the result of Ref. 24 applies to a much larger class of models, it does not yield so precise estimates on the error terms as the ones obtained in Theorem 1, and is restricted to studying the excitation spectrum in a window of order one above the ground state energy.

It remains to be seen to what extent the methods in Refs. 17 and 38 or the method in Ref. 24 can be generalized to the study of less restrictive parameter regimes, away from the Hartree limit. A first step in this direction was recently taken in Ref. 9, where bounds were given on the maximally allowed rate at which the system size is allowed to grow with N in order for the Bogoliubov approximation to remain valid. Equivalently, one can let the interaction potential v depend on N and ask at what rate it is allowed to tend to a δ-function as $N \to \infty$. Since all error terms in Theorem 1 are explicit, an estimate of this kind is actually contained in Theorem 1, but the dependence of these error terms on v was greatly improved in Ref. 9.

Finally, we mention that the validity of the Bogoliubov approximation in the Hartree limit can also be investigated concerning the dynamics generated by the Hamiltonian H_N. We refer to Ref. 23 and the references there for recent results in this direction.

VIII. OPEN PROBLEMS

In this final section, we collect a list of open problems related to the Bogoliubov approximation for many-boson systems. Some of these problems have already been mentioned in Secs. VI and VII.

- One of the key assumptions motivating the Bogoliubov approximation is the existence of *Bose–Einstein condensation*. While this property is easy to demonstrate in the Hartree limit discussed in Sec. VII, it is not known how to prove it in more general cases. In particular, the existence of BEC in the usual thermodynamic limit remains an open problem. The only model where the occurrence of BEC has been proved in the thermodynamic limit is the hard-core lattice gas at exactly half-filling, which is equivalent to the quantum XY spin model.[11,19] BEC is also known to occur in the Gross-Pitaevskii limit of dilute trapped gases.[26–28,39]
- As discussed in Sec. VI, the ground state energy density of a dilute Bose gas with repulsive interactions is known to equal $4\pi a\varrho^2$ to leading order in ϱ, with a denoting the scattering length of the interaction potential. The first correction term to this expression is expected to be of the form of the *Lee–Huang–Yang formula* displayed in Eq. (41) above. In particular, also this correction term is expected to depend on the interaction v only via its scattering length. It is an interesting open problem to establish the formula (41) rigorously, i.e., to give a suitable lower bound on the ground state energy that agrees with (41) up to terms of lower order. As discussed in Sec. VI, an upper bound of the correct form was recently derived in Ref. 43, at least for smooth and suitably small interaction potentials such that the Born series for the scattering length converges. In the more general case, even an upper bound of the right form is unknown.
- The results in Sec. VII on the excitation spectrum concern the mean-field or Hartree limit, where the interaction among the particles is very weak and of long range. In fact, the range is of the same order as the system size. In view of applications to cold atomic gases, a physically more relevant limit would be the *Gross-Pitaevskii limit*,[26–28,39] where the interaction potential takes the form

$$v(x) = N^2 w(Nx) \tag{100}$$

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for some fixed, N-independent function w. As discussed in more detail in Ref. 17, one expects that in this limit the excitation spectrum is still of the form (51), but with $\widehat{v}(p)$ replaced by $8\pi a$, where a denotes the scattering length of w.

- An even more challenging problem concerns the low energy excitation spectrum in the *thermodynamic limit*, and to study its relation to the property of *superfluidity*. There are no rigorous results available up to now, not even rough bounds are known. In fact, not even the absence of a *spectral gap* in the thermodynamic limit of an interacting Bose gas is rigorously known. We refer to Ref. 6 for further discussion of this topic.
- Also in the Hartree limit discussed in Sec. VII there are interesting open problems. One of them concerns the existence of *collective excitations* which should be described by solutions of the Hartree equation

$$-\Delta\phi(x) + V(x)|\phi(x)|^2 + v * |\phi|^2(x)\phi(x) = \mu\phi(x) \tag{101}$$

for some $\mu \in \mathbb{R}$ which are different from ϕ_0 and hence correspond to (non-linear) excited states of the Hartree functional. In the translation invariant case, collective excitations are related to the ground state via a Galileo transformation, as explained in Sec. VII A. In the absence of translation invariance, there is no such symmetry, and the existence of such states is therefore an open problem in general.

Moreover, the results in Refs. 9, 17, 24, and 38 are all limited to the case where the Hartree functional (97) has a unique minimizer (up to a constant phase). However, at least in the case of attractive interactions, uniqueness will not hold, in general. Even with repulsive interactions, uniqueness can fail in the presence of magnetic fields or, equivalently, the case of rotating Bose gases.[35,36] In this case, there can even be uncountably many minimizers. This happens, for instance, in rotating systems if the system is rotation invariant with respect to the axis of rotation, and the rotation speed is large enough for quantized vortices to form. If there is more than one such vortex, the rotation symmetry is necessarily broken in the minimizer, and hence there are infinitely many minimizers, which are all related via rotation. It would be nice to extend the results about the excitation spectrum in the Hartree limit to the case of multiple Hartree minimizers.

Note added in proof: For recent progress in this direction, see Ref. 47.

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