RESEARCH ARTICLE

Resurgence analysis of quantum invariants of Seifert fibered homology spheres

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Abstract

For a Seifert fibered homology sphere X, we show that the q-series invariant $\hat{Z}_0(X; q)$, introduced by Gukov–Pei–Putrov–Vafa, is a resummation of the Ohtsuki series $Z_0(X)$. We show that for every even $k \in \mathbb{N}$ there exists a full asymptotic expansion of $\hat{Z}_0(X; q)$ for q tending to $e^{2\pi i/k}$, and in particular that the limit $\hat{Z}_0(X; e^{2\pi i/k})$ exists and is equal to the Witten–Reshetikhin–Turaev quantum invariant $\tau_k(X)$. We show that the poles of the Borel transform of $Z_0(X)$ coincide with the classical complex Chern–Simons values, which we further show classifies the corresponding components of the moduli space of flat $SL(2, \mathbb{C})$-connections.

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1 | INTRODUCTION

Let X be a closed and oriented 3-manifold and consider the positive integer level k Witten–Reshetikhin–Turaev (WRT) quantum invariant [88, 89, 92]

$$\tau_k(X) \in \mathbb{C}. \tag{1.1}$$

This extends to a full topological quantum field theory (TQFT) [15] and was motivated from physics by Witten's work [94] on quantum Chern–Simons and the Jones polynomial [68, 69]. Classical Chern–Simons theory is a gauge theory with a Lagrangian formulation [45], and for

a SU(2)-connection $\alpha \in \Omega^1(X, \mathfrak{su}(2))$ we have the classical Chern–Simons action

$$S_{CS}(\alpha) = \frac{1}{8\pi^2} \int_X \text{tr}\left(\alpha \, d\alpha + \frac{2}{3}\alpha^3\right) \quad \text{mod } \mathbb{Z}.$$

The classical solutions in the theory are given by the moduli space of flat connections, which we denote by $\mathcal{M}_{\text{Flat}}(X, SU(2)))$. The asymptotic expansion conjecture [1–3, 7, 9, 94] postulates that to each classical Chern–Simons invariant

$$\theta \in CS(X) = S_{CS}(\mathcal{M}_{\text{Flat}}(X, SU(2))),$$

there exists a natural number $\mu_\theta, \in \mathbb{N}$ and constants $d_{\theta,j} \in \mathbb{Q}, j = 1, \dots, \mu_\theta$ and (possible divergent) formal power series with non-zero constant term $Z_{\theta,j}(k) \in \mathbb{C}[[k^{-1/2}]]^\times, j = 1, \dots, \mu_\theta$ giving a large level k Poincaré asymptotic expansion

$$\tau_k(X) \underset{k\to\infty}{\sim} \sum_{\theta \in CS(X)} e^{2\pi i k\theta} \sum_{j=1}^{\mu_\theta} k^{d_{\theta,j}} Z_{\theta,j}(k). \tag{1.2}$$

The Chern–Simons action have a holomorphic extension to the space of SL(2, $\mathbb{C}$)-connections, and this fact was used by Witten in his work on the analytic continuation of Chern–Simons theory [95]. This is in accordance with recent developments in quantum field theory [18, 39, 54, 76] which makes use of resurgence [38, 40, 41, 80] and Pham–Picard–Lefshetz theory [19, 20, 35, 66, 75, 85–87]. In particular, this motivates that the power series invariants (pertubative series) in the expansion (1.2) should be resurgent, with poles in the Borel-plane equal to (up to a shift) the set of classical complex Chern–Simons values

$$CS_{\mathbb{C}}(X) = S_{CS}(\mathcal{M}_{\text{Flat}}(X, SL(2, \mathbb{C}))).$$

The work [48] by Garoufalidis used these ideas to pose a rich series of conjectures concerning the resurgence properties of quantum invariants, and their connection to complex Chern–Simons theory. More recently, the use of resurgence in connection to the Teichmüller topological quantum field theory (TQFT) of the first author and Kashaev [8], which is the mathematical model of the partition function of complex Chern–Simons theory [52], has been initiated by Garoufalidis, Gu and Marino in [49].

In [55], Gukov, Pei, Putrov and Vafa used arguments from string theory and $3d - 3d$ correspondence to propose the existence of an invariant of (X, a), $a \in \text{Spin}^c(X)$, which is an integer power series convergent inside the unit disc

$$\hat{Z}_a(X; q) \in 2^{-c} q^\Delta \mathbb{Z}[[q]]. \tag{1.3}$$

The Gukov–Pei–Putrov–Vafa (GPPV) invariant was argued to be connected via resurgence to Chern–Simons in the work of Gukov, Marino and Putrov [54]. In the case of certain Brieskorn spheres $X = \Sigma(p_1, p_2, p_3)$ the invariant $\hat{Z}_a(X; q)$ was conceived as a Borel–Laplace resummation of the large level k asymptotic expansion of the Witten–Reshetikhin–Turaev (WRT) invariant (1.1). Subsequently, the contour integral formula for (1.3) for (negatively definite) plumbed 3-manifolds from [55] was proven mathematically to be a topological invariant in [53]. Moreover; the radial limit conjecture [28, 53, 55] postulates that if $b_1(X) = 0$, then the following holds:

$$\frac{\tau_k(X)}{\tau_k(S^2 \times S^1)} = (i\sqrt{2k})^{-1} \sum_{a,b \in \text{Spin}^c(X)/\mathbb{Z}_2} e^{2\pi i k l k(a,a)} |\mathcal{W}_b|^{-1} S_{a,b} \lim_{q \uparrow \exp(2\pi i/k)} \hat{Z}_b(X; q)$$

(see Conjecture 2, where the notation is introduced). The proof of this remarkable conjecture would give an analytic extension of $\tau_k(X)$ to the interior of the unit disc.

This paper concerns with WRT quantum invariants and their connection to Gukov–Pei–Putrov–Vafa (GPPV) invariants via resurgence and the radial limit conjecture. We now summarize our main results. Let $p_1, \ldots, p_n \in \mathbb{N}$ be pairwise coprime integers, and let for the rest of this paper X denote the oriented Seifert fibered integral homology 3-sphere with $n \geqslant 3$ exceptional fibers:

$$X = \Sigma(p_1, \ldots, p_n).$$

Let $Z_0(X)$ denote the Ohtsuki series of X. This is known by the work [73] to give the series in (1.2) attached to $0 \in CS(X)$. Let $\mathcal{B}$ denote the Borel transform (see the Appendix).

(1) Theorem 1 computes $CS_{\mathbb{C}}(X)$ and establishes that the Chern–Simons action S_{CS} induces a bijection between $\pi_0(\mathcal{M}(X, SL(2, \mathbb{C})))$ and $CS_{\mathbb{C}}(X)$.
(2) Theorem 2 establishes that $CS_{\mathbb{C}}(X)$ is equal to the set of poles of $\mathcal{B}(Z_0(X))$.
(3) Theorem 3 establishes that $\hat{Z}_0(X; q)$ is a Borel–Laplace resummation of $Z_0(X)$.
(4) Theorem 4 establishes that $\hat{Z}_0(X; q)$ admits an asymptotic expansion for $q \to e^{2\pi i/k}$, and that the radial limit conjecture is true for X, that is, there are explicit constants μ, δ such that

$$\frac{\tau_k(X)}{\tau_k(S^2 \times S^1)} = \lim_{q \uparrow \exp(2\pi i/k)} \frac{\mu}{\sqrt{k}} q^\delta \hat{Z}_0(X; q).$$

We stress that in this paper we work with the mathematical definition of $\hat{Z}_0(X; q)$ given in [53]. We now present our results in full detail.

1.1 | Complex Chern–Simons theory

For a Lie group G, let $\mathcal{M}(G) = \mathcal{M}(X, G)$ be the moduli space of flat G-connections on X. Set $P = \prod_{j=1}^n p_j$. For a rational number $x \in \mathbb{Q}$, let $[x] = x \mod \mathbb{Z}$. We prove

Theorem 1. *The Chern–Simons action S_{CS} induces a bijection between $\pi_0(\mathcal{M}(SL(2, \mathbb{C})))$ and the range of the Chern–Simons action, which is equal to*

$$CS_{\mathbb{C}}(X) = \{[0]\} \sqcup \left\{ \left[\frac{-m^2}{4P} \right] : m \in \mathbb{Z} \text{ is divisible by at most } n - 3 \text{ of the integers } p_j \right\}.$$

The natural inclusion

$$\mathcal{M}(SL(2, \mathbb{R})) \sqcup_{\mathcal{M}(U(1))} \mathcal{M}(SU(2)) \to \mathcal{M}(SL(2, \mathbb{C}))$$

induces an isomorphism on the level of π_0

$$\pi_0(\mathcal{M}(SL(2, \mathbb{R})) \sqcup_{\mathcal{M}(U(1))} \mathcal{M}(SU(2))) \cong \pi_0(\mathcal{M}(SL(2, \mathbb{C}))).$$

1.2 | The Borel transform

For $k \in \mathbb{N}^*$ set $q_k = \exp(2\pi i/k)$ and let $\varsigma \in \mathbb{C}^*$ and $\phi \in \mathbb{Q}$ be the constants introduced below in (2.2). We consider the normalized quantum invariant

$$\widetilde{Z}_k(X) = \varsigma q_k^{\frac{\phi}{4}} \frac{\tau_k(X)}{\tau_k(S^2 \times S^1)}. \tag{1.4}$$

Let $\mathrm{CS}_{\mathbb{C}}^*(X) = \mathrm{CS}_{\mathrm{CS}}(X) \setminus \{[0]\}$. Introduce the rational function

$$G(z) = \prod_{j=1}^{n} \left(z^{\frac{P}{p_j}} - z^{-\frac{P}{p_j}} \right) \left(z^P - z^{-P} \right)^{2-n}. \tag{1.5}$$

In Theorem 2 we use the notion of a resurgent function and the Borel transform, which are recalled in Definitions A1 and A2, respectively. Let $\kappa = \sqrt{2\pi i P}$. Building on the work of Lawrence and Rozansky[†] [73] and on our Theorem 1 we prove the following:

Theorem 2. *There are uniquely determined polynomials $Z_\theta = Z_\theta(X)$ for $\theta \in \mathrm{CS}_{\mathbb{C}}^*(X)$ of degree at most $n-3$ and a formal power series $Z_0 = Z_0(X) \in x^{-\frac{1}{2}}\mathbb{C}[[x^{-1}]]$ giving the full asymptotic expansion in the Poincaré sense*

$$\widetilde{Z}_k(X) \underset{k \to \infty}{\sim} \sum_{\theta \in \mathrm{CS}_{\mathbb{C}}(X)} e^{2\pi i k \theta} Z_\theta(k). \tag{1.6}$$

The series Z_0 is a normalization of the Ohtsuki series of X (see Equation 2.6) whose Borel transform $\mathcal{B}(Z_0)$ is the resurgent function:

$$\mathcal{B}(Z_0)(\zeta) = \frac{\kappa}{\pi i \sqrt{\zeta}} G\left(\exp\left(\frac{\kappa\sqrt{\zeta}}{P} \right) \right) = \frac{\kappa i}{4\pi} \frac{\prod_{j=1}^{n} \sinh\left(\frac{\kappa\sqrt{\zeta}}{p_j} \right)}{\sqrt{\zeta}\left(\sinh\left(\kappa\sqrt{\zeta} \right) \right)^{n-2}}. \tag{1.7}$$

Furthermore; if Ω is the set of poles of $\mathcal{B}(Z_0)$, then

$$\Omega = -2\pi i \, \mathrm{CS}_{\mathbb{C}}^*(X) + 2\pi i \mathbb{Z}. \tag{1.8}$$

Remark 1. In accordance with the asymptotic expansion conjecture we expect that the sum in (1.6) should only range over the Chern–Simons values of flat SU(2)-connections. This is known to be true for $n = 3$ [59] and in some cases for $n = 4$ [58]. However, we see from (1.8) that the quantum invariants via resurgence determine all the Chern–Simons values of flat SL$(2, \mathbb{C})$-connections.

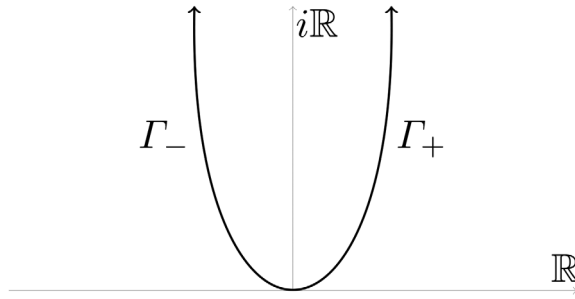[†] A comparison is given at the end of the introduction.

FIGURE 1 The integration contour $\Gamma = \Gamma_+ + \Gamma_-$

1.3 | A resurgence formula for the GPPV invariant $\hat{Z}$

Let $\Delta \in \mathbb{Q}$ be given by Equation (4.2). Let q denote a complex variable. Consider the GPPV invariant [55] which (up to the pre-factor $q^{-\Delta}$) is given by a power series invariant (1.9) with integer coefficients and radius of convergence equal to unity

$$\hat{Z}_0(X; q) \in q^{-\Delta}\mathbb{Z}[[q]]. \tag{1.9}$$

In this paper, we work with the mathematical definition of (1.9) given in [53] for a large class of plumbed 3-manifolds which includes X. This definition is recalled below in Definition 1. Set $m_0 = P(n - 2 - \sum_{j=1}^{n} p_j^{-1}) \in \mathbb{Z}$. There exists a sequence of integers $\{\chi_m\}_{m=m_0}^{\infty}$ such that for all $z \in \mathbb{C}$ with $|z| < 1$

$$G(z) = (-1)^n \sum_{m=m_0}^{\infty} \chi_m z^m \in \mathbb{Z}[[z]].$$

Let $\mathfrak{h}$ denote the upper half-plane. Let $\tau \in \mathfrak{h}$ and set

$$q = \exp(2\pi i \tau).$$

Let $\Gamma = \Gamma_+ + \Gamma_-$ be the oriented unbounded contour depicted in Figure 1 and let $\lambda = (-1)^n \frac{i}{2}(2P)^{-1/2}$. We show that the GPPV invariant $\hat{Z}_0(X; q)$ is a Borel–Laplace resummation (see the Appendix) of the Ohtsuki series $Z_0(X)$.

Theorem 3.

$$q^{\Delta}\hat{Z}_0(X; q) = \frac{\lambda}{\sqrt{\tau}} \int_{\Gamma} \exp(-\zeta/\tau) \mathcal{B}(Z_0(X))(\zeta) \, d\zeta = \sum_{m=m_0}^{\infty} \chi_m q^{\frac{m^2}{4P}}. \tag{1.10}$$

In this paper, we use the resurgence formula (1.10) to prove that the GPPV invariant $\hat{Z}_0(X; q)$ admits a full Poincaré asymptotic expansion when q tends to the kth root of unity equal to $e^{2\pi i/k}$. We further show that the constant term of this expansion is equal (up to a scaling factor) to the Witten–Reshetikhin–Turaev quantum invariant, and thus we use resurgence to prove the radial limit conjecture for X.

1.4 | The asymptotic expansion of the GPPV invariant $\hat{Z}$

We now present the asymptotic expansion theorem for $\hat{Z}_0(X)$, which in particular implies the radial limit conjecture for X [28, 53, 55]. This conjecture is recalled in Conjecture 2. Assume that P is even. Set $\delta = \Delta - \phi/4$ and $\mu = (2\lambda\varsigma)^{-1}$. For a positive parameter t set

$$q_{k,t} = \exp\left(\frac{2\pi i}{k - i\frac{2Pt}{\pi}}\right).$$

Theorem 4. *For each $\theta \in \mathrm{CS}^*_{\mathbb{C}}(X)$, there exists a unique polynomial (defined in (5.21)) in k of degree at most $n - 3$ with coefficients in formal power series without constant terms*

$$\check{Z}_\theta(k, t) \in t \cdot \mathbb{Q}[\pi i, k][[t]] \tag{1.11}$$

giving a full Poincaré asymptotic expansion for small t and fixed even k

$$(\sqrt{k}\lambda)^{-1} q_{k,t}^\Delta \hat{Z}_0(X; q_{k,t}) \underset{t \to 0}{\sim} \tilde{Z}_k(X) + \sum_{\theta \in \mathrm{CS}^*_{\mathbb{C}}(X)} e^{2\pi i k \theta} \check{Z}_\theta(k, t). \tag{1.12}$$

In particular, for every even k we have

$$\frac{\mu}{\sqrt{k}} \lim_{t \to 0} q^\delta \hat{Z}_0(X; q_{k,t}) = \frac{\tau_k(X)}{\tau_k(S^2 \times S^1)}. \tag{1.13}$$

Thus, the radial limit conjecture (Conjecture 2) holds for X.

We remark that the existence of a full asymptotic expansion in terms of complex Chern–Simons invariants and polynomials in the level as in (1.12) is a new phenomenon not observed in the literature prior to this work (neither as a conjecture nor as a result). Thus the series (1.11) are new topological invariants of X.

In combination with Lemma 14, our Theorem 4 yields a remarkable resummation formula for the (normalized) Witten–Reshetikhin–Turaev quantum invariant in terms of the (normalized) Ohtsuki series Z_0. Recall that Ω denotes the set of poles of the Borel transform $\mathcal{B}(Z_0)$ of Z_0. Let k be even, let t be a small positive parameter and set

$$\tau_{k,t} = \left(k - i\frac{2Pt}{\pi}\right)^{-1} = \frac{\log(q_{k,t})}{2\pi i}.$$

Corollary 5.

$$\tilde{Z}_k(X) = \lim_{t \to 0} \left[\int_0^\infty \exp\left(-\frac{\zeta}{\tau_{k,t}}\right) \mathcal{B}(Z_0)(\zeta) \, d\zeta - \sum_{\omega \in \Omega} \mathrm{Res}_{\zeta = \omega} \left(\exp\left(-\frac{\zeta}{\tau_{k,t}}\right) \mathcal{B}(Z_0)(\zeta)\right)\right]. \tag{1.14}$$

Informally, the identity (1.14) can be rewritten as

$$\tilde{Z}_k(X) = \int_0^\infty e^{-k\zeta} \mathcal{B}(Z_0)(\zeta) \, d\zeta - \sum_{\omega \in \Omega} \mathrm{Res}_{\zeta = \omega} \left(e^{-k\zeta} \mathcal{B}(Z_0)(\zeta)\right),$$

provided the right-hand side is interpreted as the limit on the right-hand side of (1.14).

1.5 | Comparisons with the literature

We now give a brief comparison with the relevant works from the literature. The existence of an asymptotic expansion

$$\widetilde{Z}_k(X) \underset{k \to \infty}{\sim} \sum_{\theta \in R(Y)} e^{2\pi i k\theta} Z_\theta(k), \tag{1.15}$$

where $R(Y) \subset \mathbb{Q}/\mathbb{Z}$ is a finite set which was proven in [73]. In this work, it was also shown that Z_0 is a normalization of the Ohtsuki series. Our contribution in regard to (1.15) is to compute $\mathrm{CS}_\mathbb{C}(X)$ and to show $R(X) \subset \mathrm{CS}_\mathbb{C}(X)$. In [73], the authors do not address the Borel transform.

The q-series from Theorem 3

$$\Psi(q) = \sum_{m=m_0}^{\infty} \chi_m q^{\frac{m^2}{4P}} \tag{1.16}$$

was considered in the study of $\tau_k(\Sigma(p_1, p_2, p_3))$ by Lawrence and Zagier [74], and further explored by Hikami [58]. It is easy to show that for $n = 3$, the series (1.16) have periodic coefficients of mean value equal to zero. These facts, which are not true for $n \geqslant 3$, were used by Lawrence and Zagier to prove that when $q \to q_k$, the series $\Psi(q)$ tends to the Witten–Reshetikhin–Turaev invariant. For $n \geqslant 4$ Hikami in [59] considers a differently defined q-series. Our Theorem 4 generalizes the result from [74] from $n = 3$ to any number of exceptional fibers $n \geqslant 3$. We remark that to go beyond the case of $n = 3$, we use the resurgence formula (1.10).

The work [54] of Gukov, Marino and Putrov is one of the main inspirations for this paper. In [54], the authors analyze $\tau_k(X)$ for some examples with $n = 3$. The identity (1.8) was verified for these examples. For $\tau \in \mathfrak{h}$ set $h = 2\pi i\tau$ so that $q = \exp(h)$. Consider again the integral

$$\mathrm{I}(h) = \frac{\lambda}{\sqrt{\tau}} \int_\Gamma \exp\left(-\frac{2\pi i\xi}{h}\right) \mathcal{B}(Z_0)(\xi) \, d\xi. \tag{1.17}$$

In [54], identities of the form

$$\mathrm{I}(h) = \Psi(q) \tag{1.18}$$

were discovered. In a sense, the series $\Psi(q)$ was taken as a definition for $\hat{Z}_0(q)$ for $\Sigma(p_1, p_2, p_3)$, and the GPPV formula (Definition 1) was only later introduced in [55]. Prior to this work, it was not proven that the GPPV formula and the Borel–Laplace resummation (1.17) give the same result.

In the work [47] of Fuji, Iwaki, Murakami and Terashima, the q-series $\Psi(q)$ is also considered for general $n \geqslant 3$, and they prove a radial limit theorem, which is analogous to (1.13). They also prove an identity of the form (1.18). In [47], they do not however work with the definition of the GPPV invariant $\hat{Z}_0(q)$, although they conjecture that this is equal to $\Psi(q)$. They also consider the case of the WRT invariant of a knot inside X and prove a difference equation for $\Psi(q)$.

Our Theorem 3 shows

$$q^\Delta \hat{Z}_0(q) = \mathrm{I}(h) = \Psi(q)$$

for all Seifert fibered integral homology 3-spheres X with $n \geqslant 3$ singular fibers, where $\hat{Z}_0(q)$ is independently defined via the GPPV formula. We remark that those of our results that overlap

with [47] had been presented prior to their submission by the first author in the online seminar [5] and by the second author at a seminar [79] at IST, Austria. The q-series $\Psi(q)$ was also conjectured to be a normalization of $\hat{Z}_0(q)$ in the second author's thesis [78]. We also remark that our proof of the radial limit formula (1.13) differs from theirs; our stronger Theorem 4 is derived using the resurgence formula for $\hat{Z}_0(q)$ from Theorem 3, while their proof of their radial limit theorem uses Gaussian reciprocity directly on $\Psi(q)$. We warmly thank them for cordial coordination.

1.6 | Further perspectives

In a planned sequel to this paper, we give a resurgence analysis of Witten–Reshetikhin–Turaev quantum invariants of hyperbolic surgeries on the figure-eight knot. These manifolds are more complicated than Seifert fibered manifolds, and the resurgence analysis will fully use the Pham–Picard–Lefshetz theory [19, 20, 35, 66, 75, 85–87] developed for Laplace integrals with holomorphic phase (see the introduction to the second author's thesis [78] for a brief summary of the relevant results), as well as a detailed study of Faddeev's quantum dilogarithm [42, 43]. In connection hereto, we mention also the paper [50] on resurgence of Faddeev's quantum dilogarithm and the following work of the first author [4], which concerns resurgence of meromorphic transforms, which is a class of functions that includes Faddeev's quantum dilogarithm. To obtain our results, we prove a conjecture due to the first author and Hansen [6]. A preliminary version of our results in this direction, which assumed the conjecture of [6], featured in the PhD thesis of the second author [78].

The resurgence analysis in this paper was derived from concrete formulae for quantum invariants, obtained combinatorially from a surgery diagram of X. It is an important open problem to generalize this work by deriving a similar resurgence analysis for more general and complicated 3-manifolds using either the conformal field theory approach [11, 91], or the quantization approach [17, 65] to the Witten–Reshetikhin–Turaev TQFT. By a large body of work culminating in the works of the first author and Ueno [10–13], these approaches are equivalent to the combinatorial construction of τ_k.

Let us make a few more remarks in the quantization direction. For a closed oriented surface Σ (possible with labeled points), let $V_k(\Sigma)$ be the TQFT Hilbert space. By the works mentioned above, this Hilbert space can be obtained by quantization of $\mathcal{M}_{\text{Flat}}(\Sigma, SU(n))$ [17, 65] equipped with the Atiyah–Bott–Goldman symplectic form [16, 51]. Consider a closed oriented 3-manifold M_3, which contains Σ as an embedded surface. By cutting along Σ, we obtain a 3-manifold M_3^{cut} with boundary $\partial M_3^{\text{cut}} \simeq \Sigma \sqcup -\Sigma$ and there is an associated boundary state vector $\tau_k(M_3^{\text{cut}}) \in V_k(\Sigma \sqcup -\Sigma)$. The TQFT provides isomorphisms $V_k(\Sigma \sqcup -\Sigma) \simeq V_k(\Sigma) \otimes V_k(-\Sigma)$ and $V_k(\Sigma) \simeq V_k(\Sigma)^*$, and using these one can compute the Witten–Reshetikhin–Turaev invariant of M_3 as the trace

$$\tau_k(M_3) = \text{tr} \, \tau_k(M_3^{\text{cut}}). \tag{1.19}$$

In our joint work [9], we prove the asymptotic expansion conjecture for the mapping tori M_3 (with a special colored link) of a generic surface self-diffeomorphism $\varphi : \Sigma \to \Sigma$ (preserving a point $P \in \Sigma$ which traces out the link in M_3) using formula (1.19) and the quantization of moduli spaces approach to $V_k(\Sigma)$. This quantization approach to quantum invariants is also considered by other authors in the works of [25–27, 67]. By building on the work of the first author [2] and Toeplitz operator theory [70], the quantization approach allowed us in [9] to reduce the proof of the asymptotic conjecture to an application of stationary phase approximation applied to oscillatory integrals

(1.20) over the moduli space $\mathcal{M}_\Sigma = \mathcal{M}_{\text{Flat}}(\Sigma \setminus \{P\}, \text{SU}(n), e^{2\pi i/n} \, \text{Id})$ of flat connections on $\Sigma \setminus \{P\}$ with holonomomy around P conjugate to $e^{2\pi i/n} \, \text{Id}$

$$I_\varphi(k) = \int_{\mathcal{M}_\Sigma} \exp(k\hat{\varphi}) \cdot \Omega_\varphi. \tag{1.20}$$

Such integrals are amenable to resurgence analysis by means of Pham–Picard–Lefshetz theory [19, 20, 35, 66, 75, 85–87]. Moreover, the phase function $\hat{\varphi}$ appearing in (1.20) admits a holomorphic extensions to a suitable complexification of an open neighborhood of the fixed locus $\mathcal{M}_\Sigma^\varphi$. This is possible connected to Gukov and Witten's theory of brane quantization [56], in which complexification plays a central role, and where Hitchin's moduli space of Higgs bundles [64] plays the role of a complexification of $\mathcal{M}_{\text{Flat}}(\Sigma, \text{SU}(n))$. The moduli space of Higgs bundles is by non-abelian Hodge theory [31, 37, 64] isomorphic to $\mathcal{M}_{\text{Flat}}(\Sigma, \text{SL}(n, \mathbb{C}))$, and thus there is an immediate connection to complex Chern–Simons theory. In the future, we hope to use this approach to perform a general resurgence analysis of quantum invariants.

1.7 | Organization

In Section 2, we prove Theorem 1 in several steps. Theorem 7 gives a decomposition of the moduli space, Corollary 9 computes the Chern–Simons invariants and Theorem 10 proves that components of this moduli space are classified by their Chern–Simons value. In Section 3, we prove Theorem 2. Proposition 12 gives an exact formula for the generating function of $\tilde{Z}_k(X)$, verifying a special case of a conjecture of Garoufalidis [48]. In Section 4 we prove Theorem 3 and in Section 5 we prove Theorem 4. In the Appendix, we present generalities on resurgence.

2 | COMPLEX CHERN–SIMONS THEORY ON X

Let X be the oriented Seifert fibered homology 3-sphere from the introduction. Choose $q_1, \ldots, q_n \in \mathbb{Z}$ such that $(p_j, q_j) = 1$ and

$$\sum_{j=1}^n \frac{q_j}{p_j} = \frac{1}{P}.$$

Then X has a surgery diagram as depicted in Figure 2. Without loss of generality, we can assume that $p_2, \ldots, p_n$ are odd. The homeomorphism type of X is unaltered under a transformation
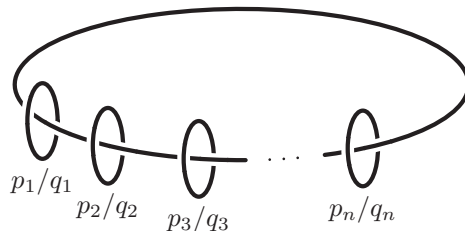

FIGURE 2 Surgery link for X

$q_j \mapsto q_j + y_j$ for any choice of integers $y_1, \ldots, y_n$ such that $(p_j, q_j + y_j) = 1$ and

$$\sum_{j=1}^{n} \frac{q_j}{p_j} = \sum_{j=1}^{n} \frac{q_j + y_j}{p_j}. \tag{2.1}$$

If q_j is odd for $j > 1$, we perform the transformation $q_j \mapsto q_j + p_j$ and $q_1 \mapsto q_1 - (n-1)p_1$ which does not change sum (2.1). Hence, we can assume without loss of generality that $q_2, \ldots, q_n$ are all even. Recall that under our assumptions $1 = P \sum_{j=1}^{n} \frac{q_j}{pj}$. Note that this implies that q_1 is odd.

We recall the computation of $\tau_k(X)$ from [73]. Let $S(\cdot, \cdot)$ be the Dedekind sum. Introduce the constants

$$\varsigma = -\frac{\sqrt{P}}{4} \exp\left(-\frac{\pi i 3}{4}\right), \qquad \phi = 3 - \frac{1}{P} + 12 \sum_{j=1}^{n} S\left(\frac{P}{p_j}, p_j\right). \tag{2.2}$$

The quantity ϕ is related to the Casson–Walker invariant $\lambda(X)$ [93] (in Casson's normalization) as follows:

$$-24\lambda(X) = \phi + P\left(n - 2 - \sum_{j=1}^{n} p_j^{-2}\right).$$

Define the meromorphic function $F \in \mathcal{M}(\mathbb{C})$ and $g \in \mathbb{C}[y]$ explicitly as follows:

$$F(y) = \frac{1}{4}\left(\sinh\left(\frac{y}{2}\right)\right)^{2-n} \prod_{j=1}^{n} \sinh\left(\frac{y}{2p_j}\right),$$

$$g(y) = \frac{iy^2}{8\pi P}. \tag{2.3}$$

In [73], Lawrence and Rozansky show the following results. There exists a finite subset $R^*(X) \subset \mathbb{Q}^*/\mathbb{Z}$ and non-vanishing polynomials $Z_\theta(z) \in \mathbb{C}[z]$, $\theta \in R^*(X)$ of degree at most $n - 3$ such that

$$\sum_{\theta \in R^*(X)} e^{2\pi i k \theta} Z_\theta(k) = -\sum_{m=1}^{2P-1} \mathrm{Res}\left(\frac{F(y)e^{kg(y)}}{1 - e^{-ky}}, y = 2\pi i m\right)$$

for all non-negative integers k. Let γ to be the contour from $(-1 - i)\infty$ to $(1 + i)\infty$. Observe that $\gamma(H)$ is a steepest descent path for g. Introduce the following notation:

$$Z^R(k) = \sum_{\theta \in R^*(X)} e^{2\pi i k \theta} Z_\theta(k)$$

$$Z^I(k) = \frac{1}{2\pi i} \int_\gamma F(y)e^{kg(y)} \, dy.$$

Recalling the definition of the normalized quantum invariant $\widetilde{Z}_k$ given in (1.4), Lawrence and Rozansky proved that it can be decomposed into a sum of an integral part Z^I and a residue part Z^R

$$\widetilde{Z}_k(X) = Z^I(k) + Z^R(k). \tag{2.4}$$

This is [73, Section 4.5, eq. 4.8]. We have used the same notation for F, g and ϕ, whereas constant B in their notation is equal to ς^{-1}. Thus, if we define

$$Z_0(x) = \frac{1}{2\pi i} \sqrt{P\pi i 8} \sum_{n=0}^{\infty} \frac{F^{(2n)}(0)(i8P\pi)^n}{(2n)!} \frac{\Gamma\left(n + \frac{1}{2}\right)}{x^{n+\frac{1}{2}}} \in x^{-1/2} \mathbb{C}[[x^{-1}]]$$

and set $R(X) = R^*(X) \sqcup \{0\}$ then we have an asymptotic expansion

$$\widetilde{Z}_k(X) \sim_{k\to\infty} \sum_{\theta \in R(X)} e^{2\pi i k \theta} Z_\theta(k). \tag{2.5}$$

In the work [73], it was observed that Z_0 is in fact a normalization of the Ohtsuki-series [81–83]. Let Z_∞ denote the Ohtsuki series (with the normalization used in [73]). Introduce the variable $h = q_k - 1$, where, as above, $q_k = \exp(2\pi i/k)$. In [74, Section 4.6], they show the following identity:

$$Z_0(k) = Z_\infty(h). \tag{2.6}$$

2.1 | The moduli space and complex Chern–Simons values

We now begin our investigation of $\mathcal{M}(X, \mathrm{SL}(2, \mathbb{C}))$, which closely follows [44]. We have the following presentation of the fundamental group of X:

$$\pi_1(X) \simeq \left\langle h, x_1, \dots, x_n \mid x_1 x_2 \cdots x_n, x_j^{p_j} h^{-q_j}, [x_j, h], j = 1, \dots, n \right\rangle.$$

Let us first recall a few of Fintushel and Stern's results concerning the moduli space $\mathcal{M}(X, SU(2))$ established in [44]. As X is an integral homology sphere, the only reducible representation into $SU(2)$ is the trivial one. For an irreducible representation $\rho : \pi_1(X) \to \mathrm{SU}(2)$ at most $n - 3$ of the $\rho(x_j)$ are $\pm I$, and if exactly $n - m$ of the $\rho(x_j)$ are equal to $\pm I$, then the component of ρ in $\mathcal{M}(X, SU(2))$ is of dimension $2(n - m) - 6$.

Let $L(p_1, \dots, p_n) \subset \mathbb{N}^n$ be the set of n-tuples $l = (l_1, \dots, l_n)$ which satisfies the following condition. We have $0 \leqslant l_1 \leqslant p_1$ and $0 \leqslant l_j \leqslant (p_j - 1)/2$, for $j = 2, \dots, n$ and there exist at least three distinct $j_1 < j_2 < j_3$ with $l_{j_t} \neq 0$ for $t = 1, 2, 3$. The following proposition is an adaptation and generalization of [21, Lemma 2; 44, Lemmas 2.1 and 2.2].

Proposition 6. *Let $l = (l_1, l_2, \dots, l_n) \in L(p_1, \dots, p_n)$. Then there exist matrices $Q_j \in \mathrm{SL}(2, \mathbb{C})$ and a representation $\rho_l : \pi_1(X) \to \mathrm{SL}(2, \mathbb{C})$ with*

$$\rho_l(x_1) = Q_1 \begin{pmatrix} e^{\frac{\pi i l_1}{p_1}} & 0 \\ 0 & e^{-\frac{\pi i l_1}{p_1}} \end{pmatrix} Q_1^{-1}, \quad \rho_l(x_j) = Q_j \begin{pmatrix} e^{\frac{2\pi i l_j}{p_j}} & 0 \\ 0 & e^{-\frac{2\pi i l_j}{p_j}} \end{pmatrix} Q_j^{-1}$$

for $j = 2, \dots n$. In fact we can choose $Q_j = I$ for $j \neq j_2, j_3$.
Furthermore we can choose functions Q_j such that

$$\rho_l : \pi_1(X) \to \mathrm{SL}(2, \mathbb{R})$$

or

$$\rho_l : \pi_1(X) \to \mathrm{SU}(2)$$

depending on properties of l. For any non-trivial representation $\rho : \pi_1(X) \to \mathrm{SL}(2, \mathbb{C})$, there exists $l' \in \mathbb{N}^n$ such that p_j divides l'_j for at most $n - 3$ of the indices $j = 1, 2, 3, \dots, n$ and such that ρ is of the form

$$\rho(x_1) = S_1 \begin{pmatrix} e^{\frac{\pi i l'_1}{p_1}} & 0 \\ 0 & e^{-\frac{\pi i l'_1}{p_1}} \end{pmatrix} S_1^{-1}, \quad \rho(x_j) = S_j \begin{pmatrix} e^{\frac{2\pi i l'_j}{p_j}} & 0 \\ 0 & e^{-\frac{2\pi i l'_j}{p_j}} \end{pmatrix} S_j^{-1} \tag{2.7}$$

for some $S_1, \dots, S_n \in \mathrm{SL}(2, \mathbb{C})$.

Finally, we have that the map which associates $l \in L(p_1, \dots, p_n)$ to a non-trivial representation $\rho : \pi_1(X) \to \mathrm{SL}(2, \mathbb{C})$ via (2.7) induces an isomorphism

$$\pi_0(\mathcal{M}^*(X, \mathrm{SL}(2, \mathbb{C}))) \cong L(p_1, \dots, p_n).$$

The family of Brieskorn integral homology spheres ($n = 3$) is very special because the moduli space $\mathcal{M}(\Sigma(p_1, p_2, p_3), \mathrm{SL}(2, \mathbb{C}))$ is finite with cardinality given by the $\mathrm{SL}(2, \mathbb{C})$ Casson invariant introduced by Curtis [33, 34]:

$$\lambda_{\mathrm{SL}(2,\mathbb{C})}(\Sigma(p_1, p_2, p_3)) = (p_1 - 1)(p_2 - 1)(p_3 - 1)/4.$$

This is shown by Boden and Curtis [21]. Prior to this and in relation to Floer homology, Fintuschel and Stern [44] analyzed the $\mathrm{SU}(2)$ moduli space $\mathcal{M}(X, \mathrm{SU}(2))$ of the Seifert fibered 3-manifold X considered in this paper and their work shows that the components are even-dimensional manifolds with the top dimension $2n - 6$. This is in stark contrast to the finiteness of the moduli space $\mathcal{M}(\Sigma(p_1, p_2, p_3), \mathrm{SL}(2, \mathbb{C}))$. In the three fibered case, Kitano and Yamaguchi [72] has given a decomposition

$$\mathcal{M}(\Sigma, \mathrm{SL}(2, \mathbb{C})) = \mathcal{M}(\Sigma, \mathrm{SL}(2, \mathbb{R})) \bigcup_{\mathcal{M}(\Sigma, \mathrm{U}(1))} \mathcal{M}(\Sigma, \mathrm{SU}(2)),$$

where $\Sigma = \Sigma(p_1, p_2, p_3)$. Here we can observe the following generalization of this work as an immediate corollary of Proposition 6.

Theorem 7. *The natural inclusion*

$$\mathcal{M}(X, \mathrm{SL}(2, \mathbb{R})) \sqcup_{\mathcal{M}(X, U(1))} \mathcal{M}(X, \mathrm{SU}(2)) \to \mathcal{M}(X, \mathrm{SL}(2, \mathbb{C}))$$

induces an isomorphism on the level of π_0

$$\pi_0(\mathcal{M}(X, \mathrm{SL}(2, \mathbb{R})) \sqcup_{\mathcal{M}(X, U(1))} \mathcal{M}(X, \mathrm{SU}(2))) \cong \pi_0(\mathcal{M}(X, \mathrm{SL}(2, \mathbb{C}))).$$

By this corollary we can, in particular, conclude that all Chern–Simons values are real and they only depend on $l \in L(p_1, \dots, p_n)$. In Proposition 8, we actually provide an explicit formula.

Before commencing the proof let us introduce the following notation:

$$\exp(x) = \begin{pmatrix} e^x & 0 \\ 0 & e^{-x} \end{pmatrix},$$

which should not cause any ambiguities as long as the context shows that we are dealing with a matrix.

Proof. We start with the construction of ρ_l. Introduce matrices

$$X_1 = \exp(\pi i l_1/p_1), \quad X_j = \exp(2\pi i l_j/p_j)$$

for $j \in \{2, \ldots, n\} \setminus \{j_2, j_3\}$. Rewrite the relation $\prod_{j=1}^n x_j = 1$ as the equivalent relation

$$x_{j_3+1} \cdots x_n x_1 \cdots x_{j_1} \cdots x_{j_2} \cdots x_{j_3-1} = x_{j_3}^{-1}.$$

Assume that we can chose $Q_{j_2}, Q_{j_3} \in \mathrm{SL}(2, \mathbb{C})$ such that

$$X_{j_3+1} \cdots X_n X_1 \cdots X_{j_1} \cdots Q_{j_2} X_{j_2} Q_{j_2}^{-1} \cdots X_{j_3-1} = Q_{j_3}^{-1} X_{j_3}^{-1} Q_{j_3}. \tag{2.8}$$

Taking $Q_j = I$ for $j \notin \{j_2, j_3\}$, we can define $\rho : \pi_1(X) \to \mathrm{SL}(2, \mathbb{C})$ by the assignment

$$\rho(x_j) = Q_j X_j Q_j^{-1}, \quad \rho(h) = X_1^{p_1}.$$

To see this, observe that $B := X_1^{p_1} = (-I)^{l_1}$ is central and as q_1 is odd whereas q_j is even for $j \geqslant 2$, we also have $X_j^{p_j} = B^{q_j}, \forall j$. The last relation in $\pi_1(X)$ is ensured by (2.8). Observe that it will suffice to choose $Q_{j_2} \in \mathrm{SL}(2, \mathbb{C})$ such that

$$\mathrm{tr}\left(X_{j_3+1} \cdots X_n X_1 \cdots X_{j_1} \cdots Q_{j_2} X_{j_2} Q_{j_2}^{-1} \cdots X_{j_3-1}\right) = 2\cos\left(\frac{2\pi l_{j_3}}{p_{j_3}}\right) \tag{2.9}$$

because this will ensure that there exists some $Q_{j_3} \in \mathrm{SL}(2, \mathbb{C})$ with the property that

$$Q_{j_3} X_{j_3+1} \cdots X_n X_1 \cdots X_{j_1} \cdots Q_{j_2} X_{j_2} Q_{j_2}^{-1} \cdots X_{j_3-1} Q_{j_3}^{-1} = X_{j_3},$$

since non-diagonalizable elements of $\mathrm{SL}(2, \mathbb{C})$ have trace ± 2, given that the unit determinant condition implies that the unique eigenvalue with multiplicity two for such elements must be either 1 or -1 and we have that

$$2\left|\cos\left(\frac{2\pi l_{j_3}}{p_{j_3}}\right)\right| < 2.$$

For (2.9) we used our assumption on j_3. Write

$$X_{j_3+1} \cdots X_n X_1 \cdots X_{j_1} \cdots X_{j_2-1} = \exp(ia),$$

$$X_{j_2} = \exp(ib),$$

$$X_{j_2+1} \cdots X_{j_3-1} = \exp(ic),$$

$$X_{j_3} = \exp(id).$$

We observe that by the conditions on (l_{j_2}, p_{j_2}) and (l_{j_3}, p_{j_3}) we have that

$$b, d \notin \pi\mathbb{Z}.$$

Let $Q_{j_2} = \tilde{Q}$ where

$$\tilde{Q} = \tilde{Q}(u, v, w, z) = \begin{pmatrix} u & v \\ w & z \end{pmatrix}$$

for u, v, w, z to be chosen below. Assume $uz - vw = 1$ so that $\tilde{Q} \in \mathrm{SL}(2, \mathbb{C})$. We compute

$$X_{j_3+1} \cdots X_n X_1 \cdots X_{j_1} \cdots X_{j_2-1} \tilde{Q} X_{j_2} \tilde{Q}^{-1} X_{j_2+1} \cdots X_{j_3-1}$$

$$= \begin{pmatrix} zue^{i(a+b+c)} - wve^{i(a-b+c)} & -uve^{i(a+b-c)} + uve^{i(a-b-c)} \\ zve^{i(b+c-a)} - zwe^{i(c-a-b)} & -vwe^{-i(a+b+c)} + zue^{i(b-a-c)} \end{pmatrix}.$$

Thus we have that

$$\mathrm{tr} \begin{pmatrix} zue^{i(a+b+c)} - wve^{i(a-b+c)} & -uve^{i(a+b-c)} + uve^{i(a-b-c)} \\ zwe^{i(b+c-a)} - zwe^{i(c-a-b)} & -vwe^{-i(a+b+c)} + zue^{i(b-a-c)} \end{pmatrix}$$

$$= 2zu \cos(a + b + c) - 2wv \cos(a + c - b).$$

It follows that we must solve

$$\begin{pmatrix} \cos(a+b+c) & \cos(a+c-b) \\ 1 & 1 \end{pmatrix} \begin{pmatrix} zu \\ -wv \end{pmatrix} = \begin{pmatrix} 2\cos(d) \\ 1 \end{pmatrix}. \tag{2.10}$$

Using the trigonometric identity $\cos(x + y) - \cos(x - y) = -2\sin(x)\sin(y)$ we get

$$\det \begin{pmatrix} \cos(a+b+c) & \cos(a+c-b) \\ 1 & 1 \end{pmatrix} = \cos((a+c)+b) - \cos((a+c)-b)$$

$$= -2\sin(a+c)\sin(b).$$

Thus it remains to argue $a + c \notin \pi\mathbb{Z}$ and $b \notin \pi\mathbb{Z}$. Assume toward a contradiction that $a + c = \pi m$ for some $m \in \mathbb{Z}$. Hence, we would have $P(a + c) = Pm\pi$ for some integer m, which would imply

$$l_{j_1} 2^\epsilon \prod_{t \neq j_1} p_t = 0 \mod p_{j_1}$$

for $\epsilon \in \{0, 1\}$, with $\epsilon = 0$ for $j_1 = 1$ and $\epsilon = 0$ otherwise. This is a contradiction, as $2^\epsilon \prod_{t \neq j_1} p_t$ is invertible in $\mathbb{Z}/p_{j_1}\mathbb{Z}$ and $1 \leqslant l_{j_1} \leqslant (p_{j_1} - 1)/2^\epsilon$. We see that $b \notin \pi\mathbb{Z}$ directly from the conditions on l_{j_2}. Thus we can solve (2.10), and hence find the needed u, v, w, z, which concludes the proof of the first part of the proposition.

Let us now prove that we can actually choose the functions Q_j such that we obtain an SL(2, $\mathbb{R}$)-representation. We will denote this new choice of the Q_j by $Q_j^{\mathbb{R}}$. We set $Q_j^{\mathbb{R}} = Q$ for $j \in \{1, ... j_2 - 1, j_3 + 1, ... , n\}$, where

$$Q = \frac{\sqrt{2}}{2} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix},$$

which has the following property:

$$Q \exp(a) Q^{-1} = \begin{pmatrix} \cos(a) & -\sin(a) \\ \sin(a) & \cos(a) \end{pmatrix}.$$

Introduce the notation $\tilde{X}_j = QX_jQ^{-1}$ and observe by the above computations that

$$\text{tr}(\tilde{X}_{j_3+1} \cdots \tilde{X}_n \tilde{X}_1 \cdots \tilde{X}_{j_1} \cdots \tilde{X}_{j_2-1} Q\tilde{Q}X_{j_2}(Q\tilde{Q})^{-1}) = \text{tr}(X_{j_3+1} \cdots X_n X_1 \cdots X_{j_1} \cdots X_{j_2-1} \tilde{Q}X_{j_2}\tilde{Q}^{-1})$$

$$= 2zu \cos(a + b) - 2wv \cos(a - b).$$

To understand which values, say t, this trace can take, we consider in analogy with (2.10) the equation

$$\begin{pmatrix} \cos(a + b) & \cos(a - b) \\ 1 & 1 \end{pmatrix} \begin{pmatrix} zu \\ -wv \end{pmatrix} = \begin{pmatrix} t \\ 1 \end{pmatrix}. \tag{2.11}$$

The determinant is $D = -2\sin(a)\sin(b)$, which is non-vanishing since $a, b \notin \pi\mathbb{Z}$. Then we have that

$$\begin{pmatrix} zu \\ -wv \end{pmatrix} = \frac{1}{D} \begin{pmatrix} t - \cos(a - b) \\ -t + \cos(a + b) \end{pmatrix}.$$

For $t \in \mathbb{R}$ we observe that $zu \in \mathbb{R}$ and $wv \in \mathbb{R}$. Now we compute

$$Q\tilde{Q}X_{j_2}(Q\tilde{Q})^{-1}$$

$$= \begin{pmatrix} (uz - wv)\cos(b) + (uv + wz)\sin(b) & (-(uz + wv) - i(uv - wz))\sin(b) \\ ((uz + wv) - i(uv - wz))\sin(b) & (uz - wv)\cos(b) - (uv + wz)\sin(b) \end{pmatrix}.$$

From which we see that

$$Q\tilde{Q}X_{j_2}(Q\tilde{Q})^{-1} \in SL(2, \mathbb{R})$$

if and only if

$$\text{Im}(uv + wz) = 0, \quad \text{Re}(uv - wz) = 0$$

or equivalently

$$uv = \overline{wz}$$

whenever $t \in \mathbb{R}$. But then this implies that

$$|u|^2|v|^2 = \frac{(t - \cos(a + b))(t - \cos(a - b))}{D^2}.$$

Which we can solve when $t > \cos(a \pm b)$ by letting

$$u = \frac{\sqrt{t - \cos(a + b)}}{D}, \quad v = \frac{\sqrt{t - \cos(a - b)}}{D}$$

and then

$$z = \sqrt{t - \cos(a + b)}, \quad w = \sqrt{t - \cos(a - b)}$$

and when $t < \cos(a \pm b)$ then we can take

$$u = \frac{\sqrt{\cos(a + b) - t}}{D}, \quad v = \frac{\sqrt{\cos(a - b) - t}}{D}$$

and then

$$z = \sqrt{\cos(a + b) - t}, \quad w = \sqrt{\cos(a - b) - t}.$$

This allows us to complete the construction as follows. First we assume that $a, c \notin \pi\mathbb{Z}$. For

$$\tilde{Q}_i = \tilde{Q}(u_i, v_i, w_i, z_i)$$

for $i = 1, 2$ we consider the equation

$$\left[\tilde{X}_{j_3+1} \cdots \tilde{X}_n \tilde{X}_1 \cdots \tilde{X}_{j_1} \cdots Q\tilde{Q}_1 X_{j_2}(Q\tilde{Q}_1)^{-1} \right] = \left[\left(\tilde{X}_{j_2+1} \ldots \tilde{X}_{j_3-1} Q\tilde{Q}_2 X_{j_3}(Q\tilde{Q}_2)^{-1} \right)^{-1} \right]$$

which is equivalent to

$$\mathrm{tr}\left(\tilde{X}_{j_3+1} \cdots \tilde{X}_n \tilde{X}_1 \cdots \tilde{X}_{j_1} \cdots Q\tilde{Q}_1 X_{j_2}(Q\tilde{Q}_1)^{-1} \right) = \mathrm{tr}\left(\tilde{X}_{j_2+1} \ldots \tilde{X}_{j_3-1} Q\tilde{Q}_2 X_{j_3}(Q\tilde{Q}_2)^{-1} \right),$$

since these are certainly all SL$(2, \mathbb{C})$ matrices. But now, using that we also have that $c, d \notin \pi\mathbb{Z}$, we can choose t bigger than $\cos(a \pm b)$ and $\cos(c \pm d)$ and fix $\tilde{Q}_i$ as above such that

$$\mathrm{tr}\left(\tilde{X}_{j_3+1} \cdots \tilde{X}_n \tilde{X}_1 \cdots \tilde{X}_{j_1} \cdots Q\tilde{Q}_1 X_{j_2}(Q\tilde{Q}_1)^{-1} \right) = t \tag{2.12}$$

and

$$\mathrm{tr}\left(\tilde{X}_{j_2+1} \ldots \tilde{X}_{j_3-1} Q\tilde{Q}_2 X_{j_3}(Q\tilde{Q}_2)^{-1} \right) = t. \tag{2.13}$$

Thus, we can now conclude that there exists $Q^{\mathbb{R}} \in \mathrm{SL}(2, \mathbb{R})$ such that

$$\tilde{X}_{j_3+1} \cdots \tilde{X}_n \tilde{X}_1 \cdots \tilde{X}_{j_1} \cdots Q\tilde{Q}_1 X_{j_2}(Q\tilde{Q}_1)^{-1} = Q^{\mathbb{R}} \left(\tilde{X}_{j_2+1} \ldots \tilde{X}_{j_3-1} Q\tilde{Q}_2 X_{j_3}(Q\tilde{Q}_2)^{-1} \right)^{-1} (Q^{\mathbb{R}})^{-1}$$

$$= Q^{\mathbb{R}} Q\tilde{Q}_2 X_{j_3}^{-1}(Q^{\mathbb{R}} Q\tilde{Q}_2)^{-1} Q^{\mathbb{R}} \tilde{X}_{j_3-1}^{-1}(Q^{\mathbb{R}})^{-1} \ldots Q^{\mathbb{R}} \tilde{X}_{j_2+1}^{-1}(Q^{\mathbb{R}})^{-1}.$$

Thus if we further set

$$Q_{j_2}^{\mathbb{R}} = Q\tilde{Q}_1, Q_{j_3}^{\mathbb{R}} = Q^{\mathbb{R}}Q\tilde{Q}_2$$

and

$$Q_j^{\mathbb{R}} = Q^{\mathbb{R}}Q$$

for $j \in \{j_2 + 1, \ldots, j_3 - 1\}$, then we find the needed conjugation to obtain an SL(2, $\mathbb{R}$)-representation. Let us now consider the remaining cases. Suppose that $a \in \pi Z$ but $c \notin \pi\mathbb{Z}$. Then the common trace t is by (2.12) forced to be $e^{ia}\cos b$, so we can solve (2.13) if and only if $e^{ia}\cos b$ is not contained in the interval spanned by the two values $\cos(c \pm d)$. If this is the case, we proceed with the argument as above. If on the other hand $e^{ia}\cos b$ is contained in the interval spanned by $\cos(c \pm d)$, then it is well known that we can choose $Q_j \in$ SU(2) so as to obtain an SU(2)-representation. A similar argument of course works in the case where $c \in \pi Z$ but $a \notin \pi\mathbb{Z}$. If we have $a, c \in \pi\mathbb{Z}$, then $a + c \in \pi\mathbb{Z}$, but this we have already argued is impossible.

Now let $\rho : \pi_1(X) \to$ SL(2, $\mathbb{C}$) be an arbitrary non-trivial representation. As remarked before any non-trivial representation is irreducible since X is an integral homology 3-sphere. Since $\rho(h)$ commutes with the image of ρ, we see that $\rho(h) = \pm I$. Hence, the relation $x_j^{p_j} = h^{q_j}$ implies that $\rho(x_j)^{p_j} = \pm I$, and for $j = 2, \ldots, n$ we must have $\rho(x_j)^{p_j} = I$, since q_j is even. Hence, ρ must be of the form (2.7) for some $l' \in \mathbb{N}^n$. It only remains to argue that at most $n - 3$ of the $\rho(x_j)$ are $\pm I$. If not, the relation $x_1 x_2 \cdots x_n = 1$ implies that there is $j_1 < j_2$ with $\rho(x_{j_1})\rho(x_{j_2}) = \pm I$. As p_{j_1} and p_{j_2} are relatively coprime, this is only possible if $\rho(x_{j_1}) = \pm I$ and $\rho(x_{j_2}) = \pm I$. This would imply that $\rho(\pi_1(X)) \subset \{\pm 1\} = Z(\text{SU}(2))$ which contradicts the fact that ρ is irreducible since it was assumed non-trivial.

We describe the connected components of $\mathcal{M}^*(X, \text{SL}(2, \mathbb{C}))$. First we assume that $l_1 \in \{1, \ldots, p_1 - 1\}$ and $l_2 > 0$ for an $l \in L(p_1, \ldots, p_n)$. We will now prove that the subset $\mathcal{M}_l^*(X, \text{SL}(2, \mathbb{C}))$ of $\mathcal{M}^*(X, \text{SL}(2, \mathbb{C}))$ consisting of conjugacy classes of representations ρ for which

$$\text{tr}(\rho(x_1)) = 2\cos(\pi l_1/p_1), \quad \text{tr}(\rho(x_j)) = 2\cos(2\pi l_j/p_j), \quad j = 2, \ldots n$$

is connected. Let

$$T = \bigtimes_{j=3}^{n} \left[\exp\left(\frac{2\pi i l_j}{p_j} \right) \right].$$

It is obvious that T is connected. Let now $m : T \to$ SL(2, $\mathbb{C}$) be the algebraic product map. Let $P \subset$ SL(2, $\mathbb{C}$) be the set of non-diagonalizable elements in SL(2, $\mathbb{C}$) and observe that P has complex co-dimension one, thus so does $m^{-1}(P) \subset T$, but then it follows that $T' = T - m^{-1}(P)$ is also connected. For any

$$(M_3, \ldots, M_n) \in T',$$

we observe that the set of $(Q_1, Q) \in$ SL(2, $\mathbb{C}$) which solves

$$\exp(\pi l_1/p_1)Q_1 \exp(2\pi l_2/p_2)Q_1^{-1} = QM_n^{-1}\ldots M_3^{-1}Q^{-1} \tag{2.14}$$

is non-empty and connected since it is acted transitively on by $(\mathbb{C}^*)^2 \times \mathbb{C}^*$, where the first factor comes from the ambiguity from solving (2.11) and the second comes from the stabiliser of

$M_n^{-1} \cdots M_3^{-1}$ under conjugation. But then we see that an open dense subset of $\mathcal{M}_l^*(X, \mathrm{SL}(2, \mathbb{C}))$ is connected, thus $\mathcal{M}_l^*(X, \mathrm{SL}(2, \mathbb{C}))$ itself must be connected. If $l_1 \in \{0, p_1\}$ or $l_2 = 0$, we proceed as follows. Choose $j_1, j_2 \in \{1, \dots, n-2\}$ such that

$$a = \pi \frac{l_1}{p_1} + 2\pi \sum_{j=2}^{j_1} \frac{l_j}{p_j}, \quad b = 2\pi \sum_{j=j_1+1}^{j_2} \frac{l_j}{p_j}$$

has the property that $a, b \notin \pi \mathbb{Z}$. Now consider the equation

$$\exp(a) Q_1 \exp(b) Q_1^{-1} = Q M_n^{-1} \dots M_{j_2+1}^{-1} Q^{-1}. \tag{2.15}$$

The connectedness is now argued in exactly the same way, with (2.15) in place of (2.14).   □

For an $\mathrm{SL}(2, \mathbb{C})$-connection a in the trivial $\mathrm{SL}(2, \mathbb{C})$-bundle on X we recall that the Chern–Simons action is given by

$$\mathrm{S}_{\mathrm{CS}}(a) = \frac{1}{8\pi^2} \int_X \mathrm{tr}\left(a\,\mathrm{d}a + \frac{2}{3}a^3\right) \mod \mathbb{Z}.$$

We now compute the Chern–Simons values of the representations constructed above.

Proposition 8. *For any representation $\rho : \pi_1(X) \to SL(2, \mathbb{C})$, define $l = (l_1, \dots, l_n) \in L(p_1, \dots, p_n)$, so that*

$$\mathrm{tr}(\rho(x_1)) = 2\cos(\pi l_1/p_1), \quad \mathrm{tr}(\rho(x_j)) = 2\cos(2\pi l_j/p_j), \quad j = 2, \dots, n.$$

Then we have that

$$\mathrm{S}_{\mathrm{CS}}(\rho) = -\frac{P}{4}\left(\frac{l_1}{p_1} + \sum_{j=2}^{n} \frac{2l_j}{p_j}\right)^2 \mod \mathbb{Z}. \tag{2.16}$$

Formula (2.16) was proven for SU(2) connections by Kirk and Klassen and it is stated in [71, Theorem 5.2]. It is proven using the following general result. Let M be a closed oriented 3-manifold containing a knot K. Let Y be the complement of a tubular neighborhood of K in M. With respect to an identification $M \setminus Y \simeq D^2 \times S^1$, choose simple closed curves μ, λ on ∂Y intersecting in a single point such that μ bounds a disc of the form $D^2 \times \{1\}$. Let $\rho_t : \pi_1(Y) \to \mathrm{SU}(2)$ be a path of representations such that $\rho_0(\mu) = \rho_1(\mu) = 1$, and for which there exist continuous piecewise differentiable functions

$$\alpha, \beta : I \to \mathbb{R}$$

with

$$\rho_t(\mu) = \begin{pmatrix} e^{2\pi i \alpha(t)} & 0 \\ 0 & e^{-2\pi i \alpha(t)} \end{pmatrix}, \quad \rho_t(\lambda) = \begin{pmatrix} e^{2\pi i \beta(t)} & 0 \\ 0 & e^{-2\pi i \beta(t)} \end{pmatrix}.$$

Thinking of ρ_1, ρ_0 as flat connections on M we have

$$S_{CS}(\rho_0) - S_{CS}(\rho_1) = -2 \int_0^1 \beta(t)\alpha'(t)\, dt \mod \mathbb{Z}. \tag{2.17}$$

Note that formula (2.17) differs from the corresponding formula in [71] by a sign. This discrepancy was already discussed by Freed and Gompf in [46] and is due to a sign convention; see the footnote in [46, p. 98]. The formula (2.17) was also used in the work [6] by the first author and Hansen.

Proof of Proposition 8. Let $K \subset X$ be the nth exceptional fiber. Let Y be the complement of a tubular neighborhood of K in X. Removing K has the effect on π_1 of removing the relation $x_n^{p_n} = h^{-q_n}$, that is, we have a presentation

$$\pi_1(Y) \simeq \left\langle h, x_1, \dots, x_n \mid x_1 x_2 \cdots x_n, x_1^{p_1} h^{-q_1}, \dots, x_{n-1}^{p_{n-1}} h^{-q_{n-1}}, \forall j [x_j, h] \right\rangle. \tag{2.18}$$

As the meridian and longitude of ∂Y we can take $\mu = x_n^{p_n} h^{q_n}$ and $\lambda = x_n^{-p_1 \cdots p_{n-1}} h^c$, respectively, where $c = \sum_{j=1}^{n-1} (p_1 \cdots p_{n-1} q_j)/p_j$. These choices of meridian and longitude coincide with the choices made in [44].

Let $\rho : \pi_1(X) \to SL(2, \mathbb{C})$ be any irreducible representation with its corresponding $l = (l_1, \dots, l_n) \in L(p_1, \dots, p_n)$. Now $\rho(h) = \exp(\pi i \upsilon)$ for some integer $\upsilon \in \mathbb{Z}$. Introduce the two quantities,

$$\xi = P\left(\frac{l_1}{p_1} + \sum_{j=2}^n \frac{2l_j}{p_j}\right)$$

and

$$\eta = \frac{\xi}{P}.$$

The proof of (2.16) presented here consists analogously with [71, Proof of Theorem 5.2] of two parts. In the first part, we find a path of $SL(2, \mathbb{C})$ connections on X connecting ρ to an abelian representation ρ_0. In fact ρ_0 will be an $SU(2)$ connection on X. In the second part, we then find a path from ρ_0 to the trivial representation ρ^{triv} and we then apply Kirk and Klassens formula (2.17). The only difference from the proof in [71] is that we need to explicitly ensure that our paths stay away from parabolic representations. The relevant paths are chosen such that λ, μ are mapped to the maximal $\mathbb{C}^*$ torus of diagonal matrices.

After conjugating by S_n^{-1} we have $\rho(x_n) = \exp(\frac{2\pi i l_n}{p_n})$. Consider the subset

$$S \subset \text{Hom}(\pi_1(Y), SL(2, \mathbb{C}))$$

of representations $\tilde{\rho}$ satisfying

$$\tilde{\rho}(h) = \rho(h), \quad [\tilde{\rho}(x_1)] = \left[\exp\left(\frac{\pi i l_1}{p_1}\right)\right]$$

and

$$[\tilde{\rho}(x_j)] = \left[\exp\left(\frac{2\pi i l_j}{p_j}\right)\right], \quad \text{for } 2 \leqslant j \leqslant n-1,$$

where $[Q]$ denotes the $SL(2, \mathbb{C})$ conjugacy class of $Q \in SL(2, \mathbb{C})$. By considering the presentation (2.18), we see that S is naturally homeomorphic to the product of the $n-1$ conjugacy classes

$$S \simeq \left[\exp\left(\frac{\pi i l_1}{p_1}\right)\right] \times \bigtimes_{j=2}^{n-1} \left[\exp\left(\frac{2\pi i l_j}{p_j}\right)\right].$$

Therefore, the connectedness of $SL(2, \mathbb{C})$ implies that S is connected. Following a similar argument use in the proof of the previous proposition, we let

$$m : S \to SL(2, \mathbb{C})$$

be the algebraic product map. Let $P \subset SL(2, \mathbb{C})$ be the set of non-diagonalizable elements in $SL(2, \mathbb{C})$ and observe that P has complex co-dimension one, thus so does $m^{-1}(P) \subset S$, but then it follows that $S' = S - m^{-1}(P)$ is also connected.

Write $\rho = \rho_1$ and observe that $\rho_1 \in S'$. Choose a smooth path ρ_t in S' connecting ρ_1 to $\rho_0 \in S'$ given by

$$\rho_0(x_1) = \exp\left(-\frac{\pi i l_1}{p_1}\right), \; \rho_0(x_j) = \exp\left(-\frac{2\pi i l_j}{p_j}\right), \; j = 2, \dots, n-1$$

and $\rho_0(x_n) = \exp(\pi i l_1/p_1 + \sum_{j=2}^{n-1} \frac{2\pi i l_j}{p_j})$. By an overall conjugation, we can choose the arc ρ_t such that $\rho_t(x_n) = \exp(2\pi i f(t))$ for a smooth function $f(t)$. We have $f(0) = \frac{l_1}{2p_1} + \sum_{j=2}^{n-1} \frac{l_j}{p_j}$ and $f(1) = \frac{l_n}{p_n}$. Note that $f(0) = (\eta/2) - f(1)$. As q_n is even, we have the following two equalities:

$$\rho_t(\mu) = \rho_t(x_n)^{p_n} \rho_t(h)^{q_n} = \exp(2\pi i p_n f(t)),$$

$$\rho_t(\lambda) = \rho_t(x_n)^{-p_1 \cdots p_{n-1}} \rho_t(h)^c = \exp\left(-2\pi i p_1 \cdots p_{n-1} f(t) + vc\pi i\right).$$

Write $y = vc \in \mathbb{Z}$. Define $\alpha_1(t) = p_n f(t)$ and $\beta_1(t) = -\frac{P}{p_n} f(t) + \frac{y}{2}$. We have that

$$
\begin{aligned}
-2\int_0^1 \alpha_1'(t)\beta_1(t)\,dt &= -2\int_0^1 p_n f'(t)\left(-\frac{P}{p_n}f(t) + \frac{y}{2}\right)dt \\
&= -2\int_{f(0}^{f(1)}\left(-Pu + \frac{p_n y}{2}\right)du \\
&= -2\left[-\frac{Pu^2}{2} + \frac{p_n yu}{2}\right]_{u=f(0)}^{u=f(1)} \\
&= Pf(1)^2 - yp_n f(1) - Pf(0)^2 + yp_n f(0) \\
&= Pf(1)^2 - Pf(0)^2 + yp_n f(0) \bmod \mathbb{Z}.
\end{aligned}
\tag{2.19}
$$

For the last identity we used that $yp_n f(1) = yl_n \in \mathbb{Z}$.

For the second part, we use the fact that $H_1(Y) \simeq \mathbb{Z}$ with generator μ to conclude that the abelian SU(2) connection ρ_0 can be connected to the trivial representation ρ^{triv} by a path of SU(2) representations σ_t with $\sigma_t(\mu) = \exp(2\pi i t \alpha_1(0))$ and $\sigma_t(\lambda) = \exp(2\pi i \beta_1(0))$. Let $\alpha_0(t) = t\alpha_1(0)$ and $\beta_0(t) = \beta_1(0)$. As $S_{CS}(\rho^{\text{triv}}) = 0$, we can apply Kirk and Klassen's formula (2.17) to obtain

$$- S_{CS}(\rho) = S_{CS}(\rho^{\text{triv}}) - S_{CS}(\rho) = -2 \int_0^1 \alpha_0'(t)\beta_0(t)\, dt - 2 \int_0^1 \alpha_1'(t)\beta_1(t)\, dt.$$

We have

$$-2 \int_0^1 \alpha_0'(t)\beta_0(t)\, dt = -2f(0)\left(-Pf(0) + \frac{y p_n}{2}\right) = 2Pf(0)^2 - y p_n f(0).$$

Comparing this with (2.19) we get that

$$- S_{CS}(\rho) = P(f(1)^2 + f(0)^2) \mod \mathbb{Z}$$

$$= P((f(1) + f(0))^2 - 2f(0)f(1)) \mod \mathbb{Z}$$

$$= \frac{\xi^2}{4P} \mod \mathbb{Z}.$$

For the last equality, we used that $2Pf(0)f(1) \in \mathbb{Z}$ and that

$$f(0) + f(1) = \frac{\eta}{2} = \frac{\xi}{2P}.$$

This is what we wanted. $\qquad\square$

For $x \in \mathbb{Q}$ let $[x] = x \mod \mathbb{Z}$. Introduce the set

$$\mathcal{W}(p_1, \dots, p_n) = \left\{ \left[\frac{-m^2}{4P}\right] \ : \ m \in \mathbb{Z} \text{ is divisible by at most } n - 3 \text{ of the functions } p_j \right\}.$$

Recall that the classical complex Chern–Simons values $CS_{\mathbb{C}}^*(X)$ is the range of the restriction of S_{CS} to $\mathcal{M}^*(X, SL(2, \mathbb{C}))$. Thus we can compute $CS_{\mathbb{C}}^*(X)$ as a corollary of Proposition 8.

Corollary 9. *We have that*

$$CS_{\mathbb{C}}^*(X) = \mathcal{W}(p_1, \dots, p_n).$$

Proof. It is clear that $CS_{\mathbb{C}}^*(X) \subset \mathcal{W}(p_1, \dots, p_n)$. We must show that for any $y \in \mathbb{Z}$ which is not divisible by more than three of the p_j we can find $l = (l_1, \dots, l_n) \in L(p_1, \dots, p_n)$ which solves the congruence equation

$$y^2 = \left(P\left(\frac{l_1}{p_1} + \sum_{j=2}^n \frac{2l_j}{p_j}\right)\right)^2 \mod 4P\mathbb{Z}. \tag{2.20}$$

For $x \in \mathbb{Z}$ and $d \in \mathbb{N}$ let $[x]_d$ denote the congruence class of x in the quotient ring $\mathbb{Z}/d\mathbb{Z}$. Since p_j is odd for $j \geqslant 2$, it follows that $4p_1, p_2, \ldots, p_n$ are also pairwise co-prime. Hence, the Chinese remainder theorem applies and the natural ring homomorphism $q : \mathbb{Z} \to \mathbb{Z}/4p_1\mathbb{Z} \oplus_{j=2}^{n} \mathbb{Z}/p_j\mathbb{Z}$, given by $x \mapsto ([x]_{4p_1}, \ldots, [x]_{p_n})$, descends to an isomorphism of rings

$$\bar{q} : \mathbb{Z}/4P\mathbb{Z} \xrightarrow{\sim} \mathbb{Z}/4p_1\mathbb{Z} \oplus \bigoplus_{j=2}^{n} \mathbb{Z}/p_j\mathbb{Z}.$$

It follows that (2.20) is in fact equivalent to the following n congruence equations

$$[y]_{4p_1}^2 = \left[l_1 \prod_{j=2}^{n} p_j + 2\left(\sum_{j=2}^{n} l_j \prod_{t \neq j} p_t \right) \right]_{4p_1}^2, \tag{2.21}$$

$$[y]_{p_j}^2 = \left[2l_j \prod_{t \neq j} p_t \right]_{p_j}^2, \ \forall j \geqslant 2.$$

The coprimality conditions ensures that $2 \prod_{t \neq j} p_t$ is an invertible element in $\mathbb{Z}/p_j\mathbb{Z}$ and therefore solving the last $n - 1$ of the equations in (2.21) can indeed be done with $0 \leqslant l_j \leqslant (p_j - 1)/2$. It remains only to consider the first of the equations in (2.21). To this end we first observe that

$$\left(l_1 \prod_{j=2}^{n} p_j + 2 \sum_{j=2}^{n} l_j \prod_{t \neq j} p_j \right)^2 = \left(l_1 \prod_{j=2}^{n} p_j \right)^2 \mod 4p_1.$$

But then we can solve

$$y^2 = l_1^2 \prod_{j=2}^{n} p_j^2 \mod 4p_1$$

for $0 \leqslant l_1 \leqslant 2p_1$. But we also have that

$$y^2 = (-l_1 \pm 2p_1)^2 \prod_{j=2}^{n} p_j^2 \mod 4p_1.$$

Thus it follows that we can in fact solve (2.21) with $l_1 \in \{0, \ldots, p_1\}$. Thus we have shown that $\mathrm{CS}_{\mathbb{C}}^*(X) = \mathcal{W}(p_1, \ldots, p_n)$. $\qquad \square$

Our analysis of the components of the $\mathrm{SL}(2, \mathbb{C})$ moduli space and the Chern–Simons values now allow us to prove the following.

Theorem 10. *The Chern–Simons action*

$$S_{\mathrm{CS}} : \pi_0(\mathcal{M}(X, \mathrm{SL}(2, \mathbb{C}))) \to \mathbb{R}/\mathbb{Z}$$

is injective and induces an isomorphism

$$\pi_0(\mathcal{M}(X, \mathrm{SL}(2, \mathbb{C}))) \cong \mathcal{W}(p_1, \ldots, p_n) \sqcup \{0\}.$$

Proof. We use the inverse of the isomorphism

$$\bar{q} : \mathbb{Z}/4P\mathbb{Z} \overset{\sim}{\to} \mathbb{Z}/4p_1\mathbb{Z} \bigoplus_{j=2}^{n} \mathbb{Z}/p_j\mathbb{Z}$$

to conclude that for each Chern–Simons value in $\mathcal{W}(p_1, \dots, p_n)$, there is a unique $l \in L(p_1, \dots, p_n)$ with the given Chern–Simons value, concluding the proof by the last statement of Proposition 6. □

Theorem 1 is a summary of the main results obtained in this section.

Proof of Theorem 1. This follows from Theorem 7, Corollary 9 and Theorem 10. □

3 | THE BOREL TRANSFORM AND COMPLEX CHERN–SIMONS

We now provide the proof of Theorem 2. The reader not familiar with the Borel transform $\mathcal{B}$ and its relation to the Laplace transform is encouraged to read the Appendix, before reading the proof of Theorem 2. For a measurable function $g : \mathbb{R}_+ \to \mathbb{C}$ of sufficient decay, we use the notation $\mathcal{L}_{\mathbb{R}_+}(g)$ for the Laplace transform — see Equation (A1).

Proof of Theorem 2. We start by giving a characterization of which of the phases in (2.4) give a non-zero contribution. Introduce for $\mu \in \mathbb{Q}/\mathbb{Z}$ the set

$$\mathcal{T}(\mu) = \{m = 1, \dots, 2P - 1 : -m^2/4P = \mu \text{ mod } \mathbb{Z}\}$$

$$= \{m = 1, \dots, 2P - 1 : g(2\pi im) = 2\pi i\mu \text{ mod } 2\pi i\mathbb{Z}\}.$$

The set of phases $2\pi iR^*(X)$ in (2.5) consists of the values $g(2\pi im) = \frac{-m^2 2\pi i}{4P}$ for which

$$\sum_{x \in \mathcal{T}(-m^2/4P)} \text{Res}\left(\frac{F(y)e^{kg(y)}}{1 - e^{-ky}}, y = 2\pi ix\right) \neq 0, \tag{3.1}$$

for $m = 1, \dots, 2P - 1$. Thus, by Corollary 9, we must prove that if (3.1) holds, then there exists $\tilde{m} \in \mathcal{T}(-m^2/4P)$ such that at most $n - 3$ of the functions p_j which divide $\tilde{m}$.

We start by noting that the set of poles of F is given by

$$\mathcal{P}_F = \{2\pi im \mid m \in \mathbb{Z} \text{ and } m \text{ is divisible by at most } n - 3 \text{ of the functions } p_j\}. \tag{3.2}$$

It follows that if $\tilde{m}$ is divisible by at least $n - 2$ of the p_j, then $F(y)$ does not have a pole at $y = 2\pi i\tilde{m}$ and we get for integral k

$$\text{Res}\left(\frac{F(y)e^{kg(y)}}{1 - e^{-ky}}, y = 2\pi i\tilde{m}\right) = F(2\pi i\tilde{m})e^{kg(2\pi i\tilde{m})}\text{Res}\left(\frac{1}{1 - e^{-ky}}, y = 2\pi i\tilde{m}\right)$$

$$= F(2\pi i\tilde{m})e^{kg(2\pi i\tilde{m})}\frac{1}{k}.$$

As we already noted above, Lawrence and Rozansky checked that all the k^{-1} terms cancels, so it follows that

$$\sum_{\substack{\tilde{m}\in\mathcal{T}(-m^2/4P),\\ \text{and } \tilde{m} \text{ is divisible by at least } n-2 \text{ of the } p_j}} \text{Res}\left(\frac{F(y)e^{kg(y)}}{1-e^{-ky}}, y=2\pi i\tilde{m}\right) = 0.$$

Therefore we see that if (3.1) holds, then there is some $\tilde{m} \in \mathcal{T}(-m^2/4P)$ which is divisible by at most $n-3$ of the p_j. This establishes $R^*(X) \subset \text{CS}^*_{\mathbb{C}}$ and we get (1.6). Observe that as a corollary we obtain for each $\theta \in \text{CS}^*_{\mathbb{C}}$ the formula

$$e^{2\pi ik\theta}Z_\theta(k) = -\sum_{\tilde{m}\in\mathcal{T}(\theta)} \text{Res}\left(\frac{F(y)e^{kg(y)}}{1-e^{-ky}}, y=2\pi i\tilde{m}\right). \tag{3.3}$$

We now turn to $\mathcal{B}(Z_0)$. The formal series Z_0 is the asymptotic expansion of the Laplace integral

$$Z^{\text{I}}(k) = \frac{1}{2\pi i}\int_\gamma F(y)e^{kg(y)}\,dy.$$

Let G be the rational function introduced in (1.5) and introduce the multivalued function $\mathcal{B}_0(\zeta)$ given by

$$\mathcal{B}_0(\zeta) = \frac{\kappa}{\pi i\sqrt{\zeta}}G\left(\exp\left(\frac{\kappa\sqrt{\zeta}}{P}\right)\right) = \frac{\kappa i}{4\pi}\frac{\prod_{j=1}^n \sinh\left(\frac{\kappa\sqrt{\zeta}}{p_j}\right)}{\sqrt{\zeta}\left(\sinh\left(\kappa\sqrt{\zeta}\right)\right)^{n-2}}.$$

With this notation, the equation for the Borel transform (1.7) which we want to prove, reads as follows:

$$\mathcal{B}(Z_0) = \mathcal{B}_0.$$

The function $\mathcal{B}_0$ is related to F as follows:

$$\mathcal{B}_0(\zeta) = \frac{\sqrt{2P}}{\sqrt{\pi i\zeta}}F\left(\sqrt{8\pi iP\zeta}\right). \tag{3.4}$$

Now as, $F(-y) = F(y)$ we have a convergent power series expansion valid for y close to 0

$$F(y) = \sum_{m=1}^\infty \frac{F^{(2m)}(0)}{(2m)!}y^{2m}.$$

Therefore Equation (3.4) implies that if we set

$$B_m = \frac{1}{2\pi i}\sqrt{8\pi iP}\frac{F^{(2m)}(0)}{(2m)!}(8\pi iP)^m,$$

then we have a convergent expansion valid for ζ close to 0 of the form

$$B_0(\zeta) = \sum_{m=1}^{\infty} B_m \zeta^{m-1/2}. \tag{3.5}$$

Introduce variable t defined by

$$-t = g(y) = \frac{iy^2}{8\pi P}.$$

Thus

$$dy = \frac{1}{2}\sqrt{\frac{8\pi iP}{t}}\, dt.$$

We now rewrite $Z^{\mathrm{I}}(k)$ as the Laplace transform of B_0

$$
\begin{aligned}
Z^{\mathrm{I}}(k) &= \frac{1}{2\pi i}\int_{\gamma} F(y) e^{kg(y)}\, dy = \frac{1}{2\pi i}\int_0^{\infty} e^{-kt}\int_{g=t}\frac{F}{dg}\, dt \\
&= \frac{1}{2\pi i}\int_0^{\infty} e^{-kt}\sqrt{\frac{8\pi iP}{t}} F\left(\sqrt{8\pi iPt}\right) dt \\
&= \int_0^{\infty} e^{-kt} B_0(t)\, dt = \mathcal{L}_{\mathbb{R}_+}(B_0)(k).
\end{aligned}
\tag{3.6}
$$

The existence of the asymptotic expansion (2.5)

$$Z^{\mathrm{I}}(k) = \mathcal{L}_{\mathbb{R}_+}(B_0)(k) \sim_{k\to\infty} Z_0(k)$$

can now be obtained by appealing to the first part of Lemma A1 where we set $B = B_0$. Here we use the existence of the expansion (3.5). Therefore the desired identity (1.7) $\mathcal{B}(Z_0) = B_0$ follows from the second part of Lemma A1 and the convergence of the expansion (3.5).

As $F(-y) = F(y)$ we note that the factor

$$\zeta \mapsto F\left(\sqrt{8\pi iP\zeta}\right)$$

gives a well-defined meromorphic function. Thus $\mathcal{B}(Z_0)(\zeta)$ is a multivalued meromorphic function with a square root singularity at 0 and with singularities for $\sqrt{8\pi iP\zeta} \in \mathcal{P}_F$ where $\mathcal{P}_F$ is the set of poles of $F(y)$. This set was computed above (see Equation 3.2) and we conclude that the poles of $\mathcal{B}(Z_0)(\zeta)$ occur at

$$\zeta_m = \frac{-\pi m^2}{2iP} = \frac{-m^2}{4P}\frac{2\pi}{i}$$

with $m \in \mathbb{Z}$ being divisible by less than or equal to $n - 3$ of the functions p_j. This concludes the proof of (1.8). □

It is of course expected that only a Chern–Simons invariant θ of a flat SU(2) connection have a non-vanishing polynomial $Z_\theta \neq 0$, that is,

$$R^*(X) = S_{CS}(\mathcal{M}^*(X, SU(2))).$$

3.1 | Resummation of the WRT invariant

We now turn to the resummation of the normalized WRT invariant $\tilde{Z}_k(X)$. Recall that for $\mu \in \mathbb{Q}/\mathbb{Z}$ we introduced the set

$$\mathcal{T}(\mu) = \{m = 1, \dots, 2P - 1 : -m^2/4P = \mu \mod \mathbb{Z}\}.$$

We also introduce the residue operator $\mathcal{I}_\mu$ which for a meromorphic function $\hat{\varphi}$ is given by

$$\mathcal{I}_\mu(\hat{\varphi})(\xi) = - \sum_{x \in \mathcal{T}(\mu)} \mathrm{Res}\left(\frac{\exp\left(\xi \frac{iy^2}{8\pi P}\right)}{(1 - e^{-\xi y})} \frac{y}{4P} \hat{\varphi}\left(\frac{y^2}{i8\pi P}\right), y = 2\pi i x \right).$$

Observe that by definition $\mathcal{T}(\mu)$ is empty for all but finitely many $\mu \in \mathbb{Q}/\mathbb{Z}$ and therefore $\mathcal{I}_\mu$ is 0 for all but these finitely many μ.

Corollary 11. *The polynomials Z_θ and the quantum invariant $\tilde{Z}_k(X)$ are determined by $\mathcal{B}(Z_0)$ as follows:*

$$Z_\theta(k) = e^{-2\pi i k\theta} \mathcal{I}_\theta(\mathcal{B}(Z_0))(k). \tag{3.7}$$

$$\tilde{Z}_k(X) = \mathcal{L}_{\mathbb{R}_+}(\mathcal{B}(Z_0))(k) + \sum_{\theta \in \frac{i}{2\pi}\Omega \mod \mathbb{Z}} \mathcal{I}_\theta(\mathcal{B}(Z_0))(k). \tag{3.8}$$

The identity (3.8) of Corollary 11 is reminiscent of the typical resummation process from resurgence [14, 38]. The Ohtsuki series is known to determine $\tau_k(X)$. The new insight provided by resurgence is that it does so via resummation as stated in Corollary 11.

We now prove Corollary 11.

Proof. It easily follows from (1.7) that

$$F(\zeta) = \mathcal{B}(Z_0)\left(\frac{\zeta^2}{i8P\pi}\right)\frac{\zeta}{4P}. \tag{3.9}$$

Recall from (3.3) that

$$e^{2\pi i k\theta} Z_\theta(k) = - \sum_{\tilde{m} \in \mathcal{T}(\theta)} \mathrm{Res}\left(\frac{F(y)e^{kg(y)}}{1 - e^{-ky}}, y = 2\pi i \tilde{m}\right).$$

From this and Equation (3.9), one easily obtains (3.7).

In the proof of Theorem 2 we obtained the following exact formula:

$$\tilde{Z}_k(X) = \mathcal{L}_{\mathbb{R}_+}(\mathcal{B})(k) + \sum_{\theta \in \mathrm{CS}^*_{\mathbb{C}}(X)} e^{2\pi i k\theta} Z_\theta(k). \tag{3.10}$$

Thus, we see that (3.8) follows from this formula and (3.7). $\qquad\square$

3.2 | Resurgence of the generating function

Let G be a simple, simply connected compact Lie group, and let $\tau_{G,k}$ be the level k Reshetikhin–Turaev TQFT constructed from the quantum group $U_q(\mathfrak{g})$, where $\mathfrak{g}$ is the complexification of the Lie algebra $\mathfrak{h}$ of G. Let $\check{h}$ be the dual Coxeter number of $\mathfrak{h}$, and set $\check{k} = k + h$. For a closed oriented 3-manifold M (possibly containing a colored framed link) we consider the normalized invariant

$$Z_{G,k}(M) = \frac{\tau_{G,\check{k}}(M)}{\tau_{G,\check{k}}(S^2 \times S^1)}.$$

Let z be a formal variable and consider the generating function

$$Z_G(M; z) \in \mathbb{C}[[z]]$$

given by

$$Z_G(M; z) = \sum_{k=0}^{\infty} Z_{G,k}(M) z^k.$$

By the work of Garoufalidis, $Z_G(M; z)$ is known to be convergent on the unit disc. Motivated by the paradigm of analytic continuation and resurgence, Garoufalidis posed the following conjecture.

Conjecture 1 [48]. *The generating function $Z_G(M; z)$ has an analytic continuation to $\mathbb{C} \setminus e\Lambda$ where $e\Lambda$ is a finite set containing zero and the exponentials of the negatives of the complex classical Chern–Simons values.*

In other words, the conjecture is that the generating function $Z_G(M; z)$ determines the germ at zero of a resurgent function. This conjecture is formally motivated from resurgence of Laplace integrals and the (non-rigorous) path integral formula for the WRT invariant, as explained in [48].

We now specialize to the case of the Seifert fibered homology sphere X and $G = \mathrm{SU}(2)$. Set $K = k + 2$ and consider the generating function for the normalized quantum invariant $\tilde{Z}_k(X)$ given by

$$\tilde{Z}(X; z) = \sum_{k=0}^{\infty} \tilde{Z}_K(X) z^k \in \mathbb{C}[[z]].$$

For $s \in \mathbb{C}$ consider the polylogarithm

$$\mathrm{Li}_s(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^s}. \tag{3.11}$$

For $s = -m$, $m \in \mathbb{N}$ the polylogarithm is exact and in fact a rational function

$$\text{Li}_{-m}(z) = \left(z \frac{\partial}{\partial z} \right)^m \left(\frac{z}{1-z} \right).$$

We introduce the following notation for the exponentials of the negatives of the classical complex Chern–Simons values:

$$e\Lambda = \exp\left(-2\pi i\, \text{CS}^*_{\mathbb{C}}(X) \right).$$

We prove the following proposition.

Proposition 12. *The generating function $\tilde{Z}(X; z)$ is the germ at zero of a holomorphic function $\tilde{Z} \in \mathcal{O}(\mathbb{C} \setminus e\Lambda)$ given by the following formula:*

$$\tilde{Z}(X; z) = \int_0^\infty \frac{e^{-2y} \mathcal{B}(Z_0)(y)}{1 - ze^{-y}} \, dy + \sum_{\theta \in \text{CS}^*_{\mathbb{C}}(X)} e^{4\pi i\theta} \sum_{j=0}^{n-3} \frac{Z_\theta^{(j)}(0)}{j!} \left(2^j + \sum_{l=0}^{j} 2^{j-l} \binom{j}{l} \text{Li}_{-l}\left(ze^{2\pi i\theta} \right) \right). \tag{3.12}$$

Proof. From Equation (3.10), it follows that

$$\tilde{Z}(X; z) = \sum_{k=0}^\infty z^k \int_0^\infty e^{-y(k+2)} \mathcal{B}(Z_0)(y) \, dy \tag{3.13}$$

$$+ \sum_{\theta \in \text{CS}_{\mathbb{C}}(X)} \sum_{k=0}^\infty z^k Z_\theta(k+2) e^{2\pi i(k+2)\theta}.$$

The first term can be simplified by interchanging summation and integration and then using the geometric series expansion

$$\sum_{k=0}^\infty z^k \int_0^\infty e^{-y(k+2)} \mathcal{B}(Z_0)(y) \, dy = \int_0^\infty \sum_{k=0}^\infty (ze^{-y})^k \mathcal{B}(Z_0)(y) e^{-2y} \, dy \tag{3.14}$$

$$= \int_0^\infty \frac{e^{-2y} \mathcal{B}(Z_0)(y)}{1 - ze^{-y}} \, dy.$$

This can be justified by standard complex analysis arguments. To complete the proof, we can consider separately each term in (3.13) corresponding to a complex Chern–Simons value $\theta \in \text{CS}_{\mathbb{C}}(X)$. We get that

$$\sum_{k=0}^\infty z^k Z_\theta(k+2) e^{2\pi i(k+2)\theta}$$

$$= e^{4\pi i\theta} \sum_{j=0}^{n-3} \frac{Z_\theta^{(j)}(0)}{j!} \sum_{k=0}^\infty (k+2)^j (e^{2\pi i\theta} z)^k$$

$$= e^{4\pi i\theta} \sum_{j=0}^{n-3} \frac{Z_\theta^{(j)}(0)}{j!} \left(2^j + \sum_{l=0}^{j} 2^{j-l} \binom{j}{l} \sum_{k=1}^{\infty} k^l (e^{2\pi i\theta} z)^k \right)$$

$$= e^{4\pi i\theta} \sum_{j=0}^{n-3} \frac{Z_\theta^{(j)}(0)}{j!} \left(2^j + \sum_{l=0}^{j} 2^{j-l} \binom{j}{l} \mathrm{Li}_{-l}(e^{2\pi i\theta} z) \right). \tag{3.15}$$

In the last equality, we used the series expansion (3.11) of the polylogarithm. By substituting the identities (3.14) and (3.15) into (3.13), we obtain the desired identity (3.12). $\qquad\square$

4 | A RESURGENCE FORMULA FOR THE GPPV INVARIANT

We now turn to the q-series invariant $\hat{Z}_0(X, q)$. We follow [53]. Let (Γ, m) be an ordered weighted tree, that is, Γ is a tree together with an ordering of its set of vertices V and m is a map $m : V \to \mathbb{Z}$. Set $s = |V|$ and let $M = M(\Gamma, m)$ be the $s \times s$ matrix with entries given by

$$M_{i,j} = \begin{cases} m_v & \text{if } v_i = v_j = v, \\ 1 & \text{if } v_i \text{ and } v_j \text{ are joined by an edge,} \\ 0 & \text{otherwise.} \end{cases}$$

We say M is weakly negative definite if M is invertible and M^{-1} is negative definite on the subspace of $\mathbb{Z}^s$ spanned by vertices of degree at most 3. Let $Y = Y(\Gamma, m)$ be the oriented 3-manifold with surgery data $L = L(\Gamma, m)$ constructed as follows. For each vertex v the link L has an unknotted component L_v with framing m_v, and L_v is chained together with L_w if and only if v and w are joined by an edge. We call Y a plumbed manifold with plumbing graph Γ.

We recall that two plumbed 3-manifolds Y and Y' are diffeomorphic if and only their plumbing graphs are related by Neumann moves.

When Y is a plumbed manifold with weakly negative definite plumbing graph and Y is not necessarily a homology 3-sphere, the q-series invariant $\hat{Z}_a(Y, q)$ depend on a label $a \in \mathrm{Spin}^c(Y)$. Originally, these labels were thought to be abelian or 'almost abelian' flat connections (see [29]). For a 3-manifold with $b_1(Y) = 0$, the set of abelian flat connections and Spin^c can be identified. As $X = \Sigma(p_1, \ldots, p_n)$ is an integral homology 3-sphere, we have $a = 0$, and need not go deeper into this discussion. For the sake of completeness however, we recall the GPPV-formula definition as it is stated in terms of Spin^c structures. First, we recall how Spin^c structures can be described in terms of the adjacency matrix M. This is thoroughly explained in [53]. Let Y be a plumbed 3-manifold with plumbing graph Γ. Let $s = |V|$. Let $\vec{m} \in \mathbb{Z}^s$ be the weight vector, that is, $m_j = m(v_j)$. Let $\vec{\delta} \in \mathbb{Z}^s$, be the degree vector, that is, $\delta_j = \deg(v_j)$. We have isomorphisms

$$\mathrm{Spin}^c(Y) \simeq (\mathbb{Z}^s + \vec{m})/2M\mathbb{Z}^s \simeq (\mathbb{Z}^s + \vec{\delta})/2M\mathbb{Z}^s.$$

These isomorphisms are compatible with Neumann moves as explained in [53]. We now recall the GPPV formula (4.1).

Definition 1 [55]. Let Y be a plumbed 3-manifold with weakly definite plumbing graph Γ. Let ϕ denote the number of positive eigenvalues of M and let σ denote the signature of M.

Let $a = [\vec{a}] \in (\mathbb{Z}^s + \vec{\delta})/2M\mathbb{Z}^s \simeq \mathrm{Spin}^c(Y)$. The $\hat{Z}$-invariant of (Y, a) is given by

$$\hat{Z}_a(Y; q) = (-1)^{\phi} q^{\frac{3\sigma - \sum_v m_v}{4}} \cdot v.p. \oint_{|z_v|=1} \prod_{v \in V} \frac{dz_v}{2\pi i z_v} \left(z_v - \frac{1}{z_v} \right)^{2 - \deg(v)} \Theta_a^{-M}(\vec{z}), \qquad (4.1)$$

where $v \cdot p\cdot$ denotes the principal value and

$$\Theta_a^{-M}(\vec{z}) = \sum_{\vec{l} \in 2M\mathbb{Z}^s + \vec{a}} q^{-\frac{(\vec{l}, M^{-1}\vec{l})}{4}} \prod_{v \in V} z_v^{l_v}.$$

Remark 2. The invariance of (4.1) under Neumann moves is proved by Gukov and Manulescu in [53].

We recall that the principal value $v.p.$ is defined such that for every sufficiently small $\epsilon > 0$ we have

$$v \cdot p \cdot \oint_{|z|=1} = \frac{1}{2} \left(\oint_{|z|=1+\epsilon} + \oint_{|z|=1-\epsilon} \right).$$

4.1 | Proof of Theorem 3

We now consider $X = \Sigma(p_1, \ldots, p_n)$ in more detail. Choose $q_1, \ldots, q_n \in \mathbb{N}$ such that for each $j = 1, \ldots, n$ we have $(p_j, q_j) = 1$, $p_j \leqslant q_j$ and

$$p_0 := \frac{-1}{P} - \sum_{j=1}^n \frac{q_j}{p_j} \in \mathbb{Z}_{<0}.$$

Then $e = -1/P < 0$ is the Seifert Euler number. Choose a continued fraction expansion of q_j/p_j for each $j = 1, \ldots, n$

$$\frac{q_j}{p_j} = k_{j,1} - \cfrac{1}{k_{j,2} - \cfrac{1}{\ddots - k_{j,s_j}}}.$$

As explained in [53], X has a negative definite plumbing graph Γ defined as follows. The graph Γ is star-shaped with n arms and central vertex v_0 with weight p_0. For each $j = 1, \ldots, n$ the jth arm has s_j vertices. If these are ordered $(v_{j,1}, \ldots, v_{j,s_j})$ with $v_{j,1}$ being closest to the central vertex v_0, then $v_{i,j}$ have weight $-k_{j,i}$. This graph is illustrated for $n = 3$ in Figure 3.

Before proving Theorem 3, we first give a formula for the rational exponent $\Delta \in \mathbb{Q}$. For each $j = 1, \ldots, n$ let X_j be the plumbed manifold whose graph Γ_j is identical to Γ except that on the jth arm, we delete the terminal vertex v_{j,s_j}. Define $h_j \in \mathbb{N}$ as

$$h_j = |H_1(X_j, \mathbb{Z})|.$$

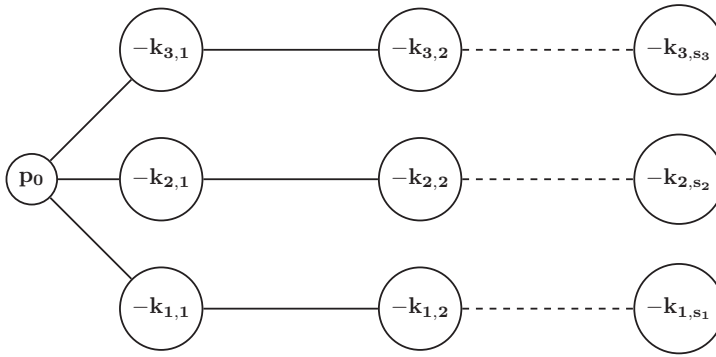**FIGURE 3** Plumbing graph for X in the case $n = 3$

Observe that the total number of vertices of Γ is given by $s = 1 + \sum_{j=1}^{n} s_j$. Define $\Delta \in \mathbb{Q}$ by

$$\Delta = -\frac{1}{4}\left(\sum_{j=1}^{n} h_j - 3s - p_0 + \sum_{j=1}^{n} \left(-\frac{P}{p_j^2} + \sum_{i=1}^{s_j} k_{j,i} \right) \right). \tag{4.2}$$

We now prove Theorem 3.

Proof. Recall that $q = \exp(2\pi i \tau)$ where $\tau \in \mathfrak{h}$. For the sake of notational simplicity, we also introduce the paramter $h = 2\pi i \tau$ so that $q = \exp(h)$. We start by proving that

$$I(h) = \Psi(q), \tag{4.3}$$

where $\Psi(q)$ is the series introduced in (1.16) and $I(h)$ is the contour integral introduced in (1.17) (with $h = 2\pi i \tau$). Observe that for the purpose of proving (4.3) we can and will assume that

$$\tau \in i\mathbb{R}_{>0},$$

because if the identity (4.3) holds true on this half-line, it has to hold on the entire upper half-plane $\mathfrak{h}$, since both functions are holomorphic in $\mathfrak{h}$.

Set

$$\mathcal{B}(\zeta) = \frac{1}{2\sqrt{P}} B(Z_0)\left(\frac{\zeta}{2\pi i} \right). \tag{4.4}$$

For all $t \in \mathbb{C}$ with $\sqrt{t} \in \{z \in \mathbb{C} \mid \mathrm{Re}(z) < 0\}$ the normalized Borel transform $\mathcal{B}$ satisfies by Theorem 2

$$\mathcal{B}(t) = \frac{1}{\sqrt{t}} \sum_{m=m_0}^{\infty} c_m \exp\left(m\sqrt{\frac{t}{P}} \right) = \frac{1}{\sqrt{t}} G\left(\exp\left(\sqrt{\frac{t}{P}} \right) \right), \tag{4.5}$$

where for all $m \in \mathbb{N}$ we have that
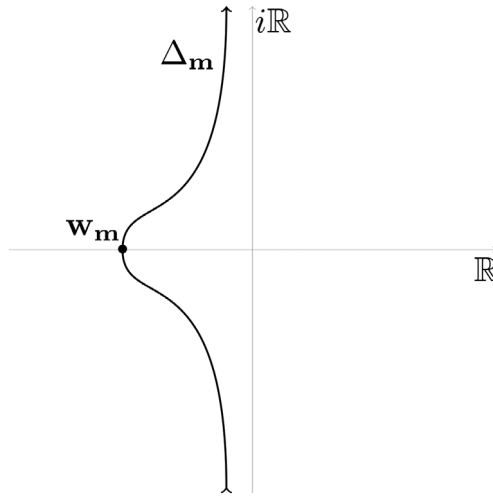
$$c_m = (-1)^n \chi_m.$$

FIGURE 4   The contour Δ_m

For each $m \in \mathbb{Z}_{\geqslant m_0}$ introduce the polynomial

$$p_m(w) = -\frac{w^2}{h} + \frac{m}{\sqrt{P}} w.$$

This is a Morse function with a unique saddle point at $w_m = \frac{hm}{2\sqrt{P}}$ and we have that

$$p_m(w_m) = h\frac{m^2}{4P}.$$

Let $D(R)$ be the closed ball centered at the origin with radius $R > |m_0|$. We can deform $i\mathbb{R}$ slightly to a contour $\Delta_m \subset \{z \in \mathbb{C} : \mathrm{Re}(z) < 0\} \cup D(R)$, which passes through the saddle point w_m and such that the function given by

$$w \mapsto \exp(p_m(w))$$

has exponential decay along Δ_m. The orientation of Δ_m is as depicted in Figure 4. We remark that Figure 4 depicts the situation where $m_0 \geqslant 0$. Recall that if Γ is a steepest descent contour through the unique saddle point of a degree two polynomial $p(z) = -\alpha z^2 + \beta z$, then we have the following exact formula known as Gaussian integration

$$\int_\Gamma \exp(p(z)) \, dz = \sqrt{\frac{\pi}{\alpha}} \exp\left(\frac{\beta^2}{4\alpha}\right).$$

Applying Gaussian integration to the polymonials p_m gives us the following identity:

$$(-1)^n \Psi(q) = \sum_{m=m_0}^{\infty} c_m q^{\frac{m^2}{4P}} = \sum_{m=m_0}^{\infty} c_m \frac{1}{\sqrt{\pi h}} \int_{i\mathbb{R}} \exp\left(p_m(w)\right) dw. \tag{4.6}$$
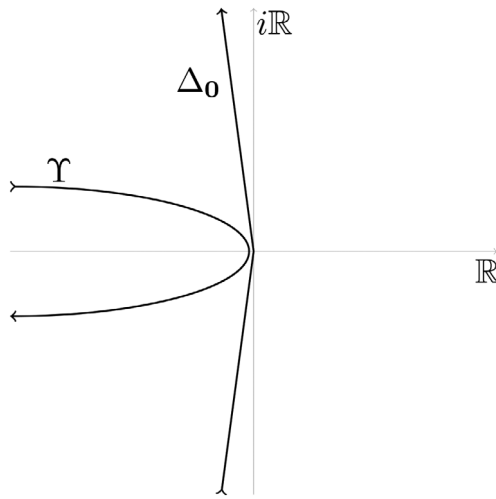
FIGURE 5   The contours Y and Δ_0

Choose a small positive parameter $\delta > 0$ and introduce the contour

$$\Delta_0 = e^{i\delta} i\mathbb{R}_+ \cup e^{-i\delta} i\mathbb{R}_- \subset \{z \in \mathbb{C} : \mathrm{Re}(z) < 0\}.$$

Let $\Upsilon \subset \{z \in \mathbb{C} : \mathfrak{R}(z) < 0\}$ be the Hankel contour which encloses $\mathbb{R}_-$ and satisfies $\sqrt{\Upsilon} = \Delta_0$. The orientation of these contours are given in Figure 5. We let

$$\Upsilon_\pm = \Upsilon \cap \{z \in \mathbb{C} : \mathrm{Im}(z) \in \mathbb{R}_\mp\},$$

so that $\sqrt{\Upsilon_+} = e^{i\delta} i\mathbb{R}_+$ and $-\sqrt{\Upsilon_-} = e^{-i\delta} i\mathbb{R}_-$ where $\sqrt{\cdot}$ denotes the principal branch of the square root. Introduce the variable

$$w^2 = v.$$

As Δ_0 is a small deformation of Δ_m for each $m \in \mathbb{Z}_{\geqslant m_0}$, we obtain

$$\sum_{m=m_0}^{\infty} \frac{c_m}{\sqrt{\pi h}} \int_{i\mathbb{R}} \exp(p_m(w)) \, dw = \sum_{m=m_0}^{\infty} \sum_{\epsilon=\pm 1} \frac{c_m}{\sqrt{\pi h}} \int_{\Upsilon(\epsilon)} \frac{\exp\left(-\frac{v}{h} + \epsilon \frac{m}{\sqrt{P}} \sqrt{v}\right)}{\epsilon 2 \sqrt{v}} \, dv$$

$$= \sum_{\epsilon=\pm 1} \frac{1}{2\sqrt{\pi h}} \int_{\epsilon \Upsilon(\epsilon)} \frac{\exp(-\frac{v}{h})}{\sqrt{v}} \sum_{m=m_0}^{\infty} c_m e^{\epsilon \frac{m\sqrt{v}}{\sqrt{P}}} \, dv \qquad (4.7)$$

$$= \sum_{\epsilon=\pm 1} \frac{1}{2\sqrt{\pi h}} \int_{\epsilon \Upsilon(\epsilon)} \exp\left(-\frac{v}{h}\right) \mathcal{B}(v) \, dv.$$

In the second equality of (4.7) we used that $\epsilon \sqrt{v} \in \{z \in \mathbb{C} : \mathrm{Re}\, z < 0\}$ for all $v \in \Upsilon(\epsilon)$, and the contour $-\Upsilon(-1)$ denotes $\Upsilon(-1)$ but oriented in the direction from the origin and toward infinity.

In the third equality of (4.7) we used Equation (4.5) and the identity

$$G(z) = G\left(\frac{1}{z}\right),$$

which follows directly from the definition of G. Now introduce the variable

$$\xi = \frac{v}{2\pi i}.$$

This identifies (up to a small deformation) the v contour $Y_+ - Y_-$ with the ξ contour Γ introduced in Figure 1.

We have

$$\sum_{\epsilon=\pm 1} \frac{1}{2\sqrt{\pi h}} \int_{\epsilon Y(\epsilon)} \exp\left(-\frac{v}{h}\right) \mathcal{B}(v)\, dv$$

$$= \sum_{\epsilon=\pm 1} \frac{2\pi i}{2\sqrt{\pi h}} \int_{\exp(\epsilon \frac{\pi i}{4})i\mathbb{R}_+} \exp\left(-\frac{2\pi i\xi}{h}\right) \mathcal{B}(2\pi i\xi)\, d\xi$$

$$= \sum_{\epsilon=\pm 1} \frac{2\pi i}{2\sqrt{\pi h}} \frac{\sqrt{2\pi i}}{8\kappa} \int_{\exp(\epsilon \frac{\pi i}{4})i\mathbb{R}_+} \exp\left(-\frac{2\pi i\xi}{h}\right) \mathcal{B}(Z_0)(\xi)\, d\xi. \tag{4.8}$$

In the last equality of (4.8) we used Equation (4.4), which relates $\mathcal{B}$ and $\mathcal{B}(Z_0)$. Now recall that $\kappa = \sqrt{2\pi i P}$, since $H = 1$ and recognize the pre-factor in the last line of (4.8) as

$$\frac{2\pi i}{2\sqrt{\pi h}} \frac{1}{\sqrt{P2}} = (-1)^n \frac{\lambda}{\sqrt{\tau}},$$

where λ is the scalar introduced in the statement of Theorem 3. By combining Equations (4.6), (4.7) and (4.8), we see that Equation (4.3) holds.

Write $\hat{Z}_0(X; q) = \hat{Z}_0(q)$. We now show that

$$\Psi(q) = q^\Delta \hat{Z}_0(q), \tag{4.9}$$

where $\Delta \in \mathbb{Q}$ is the scalar introduced in (4.2). This will establish (1.10) and thereby finish the proof. We start with $\hat{Z}_0(q)$. By Definition 1 and since in this case $\phi = 0$, we have that

$$\hat{Z}_0(q) = q^{\frac{3\sigma - \sum_v m_v}{4}} \sum_{\vec{l} \in 2M\mathbb{Z}^s} q^{-\frac{(\vec{l}, M^{-1}(\vec{l}))}{4}} \oint_{|z_v|=1} \prod_{v \in V} \frac{dz_v}{2\pi i z_v} \left(z_v - \frac{1}{z_v}\right)^{2-\deg(v)} z_v^{l_v}. \tag{4.10}$$

Here it is understood that we have taken the principal value of the integral as explained above. Recall that for a Laurent series $a(z) = \sum_{j \in \mathbb{Z}} a_j z^j$ we have that

$$v.p. \oint_{|z|=1} \frac{dz}{2\pi i z} a(z) = a_0.$$

For our star-shaped plumbing graph Γ, the non-zero contributions to (4.10) comes from $\vec{l} \in 2M\mathbb{Z}^s$ with $l_w = 0$ for all of the entries corresponding to an internal vertex w of an arm, and $l_v = \pm 1$ if v is a terminal vertex of an arm and then from the central vertex v_0, which we will now consider.

In comparing $\hat{Z}_0(q)$ with $\Psi(q)$ it is useful to introduce the integer sequence $\{a_j\}_{j=0}^\infty$ determined by

$$\left(t - t^{-1}\right)^{2-n} = \begin{cases} \sum_{j=0}^\infty a_j t^{2j+n-2}, & \text{if } |t| < 1 \\ \sum_{j=0}^\infty (-1)^n a_j t^{-2j-n+2}, & \text{if } |t| > 1. \end{cases} \tag{4.11}$$

The functions a_j can be explicitly evaluated: By the formula for the geometric series $(1-t)^{-1} = \sum_{j=0}^\infty t^j$ and Cauchy multiplication of power series, we see that

$$\frac{1}{(1-t)^m} = \sum_{j=0}^\infty \left(\sum_{j_1 + \cdots + j_m = j} 1 \right) t^j = \sum_{j=0}^\infty \binom{j+m-1}{j} t^j,$$

and therefore one sees that

$$a_j = (-1)^n \binom{j+n-3}{j}.$$

However for the comparison of $\hat{Z}_0(q)$ and $\Psi(q)$ given below, we do not need the closed form for a_j, but rather Equation (4.11).

Write $z_{v_0} = z$ and $l = l_{v_0}$. We obtain

$$v.p. \oint_{|z|=1} \frac{dz}{2\pi i z} \left(z - z^{-1}\right)^{2-n} z^l = \begin{cases} \frac{1}{2} a_{\frac{l-n+2}{2}} & \text{if } 2-n-l \in 2\mathbb{Z}_+, \\ \frac{1}{2}(-1)^n a_{\frac{2-n-l}{2}} & \text{if } l-n+2 \in 2\mathbb{Z}_+, \\ a_0 & \text{if } l = 0, n = 2. \end{cases}$$

We know that the adjacency matrix M is unimodular, and so $M\mathbb{Z}^s = \mathbb{Z}^s$. Define a map $\vec{l} : \{\pm 1\} \times \mathbb{N} \times \{\pm 1\}^n \to \mathbb{Z}^s$ as follows: For the central vertex v_0 we have

$$\vec{l}(\varepsilon, j, \epsilon)_{v_0} = \varepsilon(-2j+2-n).$$

For $m = 1, \ldots, n$ and the terminal vertex v of the mth arm, we have

$$\vec{l}(\varepsilon, j, \epsilon)_v = \epsilon_m,$$

and for every internal vertex w of the arms, we have

$$\vec{l}(\varepsilon, j, \epsilon)_w = 0.$$

With this notation, the above considerations show that

$$\hat{Z}_0(q) = q^{-\frac{3s+\sum_v m_v}{4}} \sum_{\varepsilon=\pm 1} \sum_{r=0}^{\infty} \sum_{\epsilon \in \{\pm 1\}^n} (-1)^{\frac{(1-\varepsilon)(n-2)}{2}} \frac{a_r}{2} \left(\prod_{j=1}^{n} \epsilon_j \right) q^{-\frac{\langle \vec{l}(\varepsilon,r,\epsilon), M^{-1} \vec{l}(\varepsilon,r,\epsilon), \rangle}{4}}.$$

If we apply the symmetry that simultaneously changes the sign of all ϵ_j and ε, then we obtain

$$\hat{Z}_0(q) = (-1)^n q^{-\frac{3s+\sum_v m_v}{4}} \sum_{r=0}^{\infty} \sum_{\epsilon \in \{\pm 1\}^n} a_r \left(\prod_{j=1}^{n} \epsilon_j \right) q^{-\frac{\langle \vec{l}(-1,r,\epsilon), M^{-1} \vec{l}(-1,r,\epsilon), \rangle}{4}}.$$

The quadratic form

$$\vec{l} \mapsto \langle \vec{l}, M^{-1} \vec{l} \rangle / 4$$

was computed for $n = 3$ in [53] in their proof of Proposition 4.8. The size of the matrix M^{-1} is irrelevant to their computation, and their formula can be generalized to our case to give the formula

$$\hat{Z}_0(q) = (-1)^n q^{-\Delta} \sum_{r=0}^{\infty} \sum_{\epsilon \in \{\pm 1\}^n} a_r \left(\prod_{j=1}^{n} \epsilon_j \right) q^{\frac{P}{4} \left(2r+(n-2)+\sum_{j=1}^{n} \epsilon_j \frac{1}{P_j} \right)^2}.$$

We now compute $\Psi(q)$. For $|z| < 1$ we have

$$G(z) = \prod_{j=1}^{n-2} \left(z^{\frac{P}{P_j}} - z^{-\frac{P}{P_j}} \right) \left(z^P - z^{-P} \right)^{2-n}$$

$$= \sum_{r=0}^{\infty} \sum_{\epsilon \in \{\pm 1\}^n} a_r \left(\prod_{j=1}^{n} \epsilon_j \right) z^{2Pr+P(n-2)+\sum_{j=1}^{n} \epsilon_j \frac{P}{P_j}}.$$

It follows that

$$\Psi(q) = (-1)^n \sum_{r=0}^{\infty} \sum_{\epsilon \in \{\pm 1\}^n} a_r \left(\prod_{j=1}^{n} \epsilon_j \right) q^{\frac{\left(2Pr+P(n-2)+\sum_{j=1}^{n} \epsilon_j \frac{P}{P_j} \right)^2}{4P}}.$$

This shows (4.9). □

We obtain the following corollary.

Corollary 13. *Let $Z_0 \in x^{-1/2} \mathbb{C}[[x^{-1}]]$ be the normalization of the Ohtsuki series from Theorem 2. We have an asymptotic expansion*

$$q^{\Delta} \hat{Z}_0(X; q) \sim_{q \to 1} \frac{2\lambda}{\sqrt{\tau}} Z_0(1/\tau).$$

Proof. This is a consequence of the integral formula (1.10) from Theorem 3

$$q^{\Delta}\hat{Z}_0(X;q) = \frac{\lambda}{\sqrt{\tau}} \int_{\Gamma} \exp(-\zeta/\tau)\mathcal{B}(Z_0)(\zeta)\,d\zeta$$

$$= \frac{\lambda}{\sqrt{\tau}} \sum_{\epsilon \in \{\pm 1\}} \mathcal{L}_{\Gamma_{\epsilon}}(\mathcal{B}(Z_0))(1/\tau)$$

and Borel–Laplace resummation, which is stated as Theorem A2. $\qquad\square$

Let us now recall previous work on the q-series Ψ. We start with the case $n = 3$, for which more is known. As already mentioned in the introduction, Lawrence and Zagier have shown in [74] that the quantum invariant $\tau_k(X)$ can be recovered as the radial limit of $\Psi(q)$, as q tends to $\exp(2\pi i/k)$. This was generalized to $n = 4$ by Hikami in [59] but with corrections terms appearing. The series Ψ have interesting arithmetic properties; the coefficients $\chi(m)$ are periodic functions of period $2P$ and Ψ is the so-called Eichler integral of a mock modular form with weight $3/2$. As mentioned in the introduction the connection between quantum invariants and number theory was further pursued by Hikami in a number of articles [57–61, 63]. For general $n \geqslant 3$ we mention again the work [47] of Fuji, Iwaki, Murakami and Terashima, which was discussed in the introduction.

Let us now discuss what was previously known about the q-series invariant $\hat{Z}_0(X)$. In [53] it was shown that when X is a Brieskorn sphere $\Sigma(p_1, p_2, p_3)$, (that is, $n = 3$) then $\hat{Z}_0(X)$ is a linear combination of so-called false theta functions. The q-series invariant $\hat{Z}_0$ was also considered for certain Seifert fibered manifolds (with up to $n = 4$ singular fibers) in the work [30], as well as a proposed analog of $\hat{Z}_0$ for higher rank gauge group — see also [84] for further developments in this direction. In this paper, we work exclusively with $G = \mathrm{SU}(2)$).

In connection with the work [74], Zagier invented the notion of a quantum modular form. This notion was generalized by Bringmann et al. in [22], where they introduce the notion of a higher depth quantum modular form. For any $n \geqslant 3$, it is known, that Ψ is a linear combination of derivatives of quantum modular forms [23, 24]. It is interesting to observe that Ψ is obtained from the Borel transform through a resummation process reminiscent of the median resummation of [32]. Moreover, as explained in [28] it is expected that for a general 3-manifold M, mock/false modular form duality is related to $\hat{Z}_a(M;q)$, that is, there exists an associated pair of a so-called Mock modular form and a so-called false modular form, and these are related by a $q \mapsto q^{-1}$ transformation and have the same transseries expression near $q \to 1$. This is quite possibly connected to [48, Conjecture 2] (called the symmetry conjecture). Let us also mention the work [36] by Dimofte–Garoufalidis, which connects modularity in quantum topology with complex Chern–Simons theory.

5 | THE ASYMPTOTIC EXPANSION OF THE GPPV INVARIANT

The invention of $\hat{Z}$ was party motivated by an attempt to generalize the following discovery of Lawrence and Zagier. Set $q_k = \exp(2\pi i/k)$. For $n = 3$ they proved in [74] the identity (for some $\sigma \in \mathbb{Q}$)

$$\tau_k(\Sigma(p_1, p_2, p_3))(q_k - 1)q_k^{\sigma} = -\frac{1}{2} \lim_{q \to q_k} \sum_{m=m_0}^{\infty} \chi_m q^{\frac{m^2}{4P}}.$$

For a closed oriented 3-manifold Y consider the normalized WRT invariant

$$Z_{CS}(Y;k) = \frac{\tau_k(Y)}{\tau_k(S^2 \times S^1)}.$$

Conjecture 2 (The radial limit conjecture [53]). *Let Y be a closed oriented 3-manifold with $b_1(Y) = 0$. Set $T = \mathrm{Spin}^c(Y)/\mathbb{Z}_2$. For every $a \in T$, there exists invariants*

$$\Delta_a \in \mathbb{Q}, \ c \in \mathbb{Z}_+, \ \hat{Z}_a(q) \in 2^{-c}q^{\Delta_a}\mathbb{Z}[[q]],$$

with the following properties. The series $\hat{Z}_a(q)$ is convergent inside the unit disc $\{q : |q| < 1\}$, and for infinitely many $k \in \mathbb{N}$ the radial limits $\lim_{q\uparrow\exp(2\pi i/k)} \hat{Z}_a(q)$ exists and we have that

$$Z_{CS}(Y;k) = (i\sqrt{2k})^{-1} \sum_{a,b\in T} e^{2\pi iklk(a,a)}|\mathcal{W}_b|^{-1}S_{a,b} \lim_{q\uparrow\exp(2\pi i/k)} \hat{Z}_b(q).$$

Here $\mathcal{W}_x$ is the $\mathbb{Z}_2$-stabilizer of x and

$$S_{a,b} = \frac{e^{2\pi ikl(a,b)} + e^{-2\pi ikl(a,b)}}{|\mathcal{W}_a|\sqrt{|H_1(Y;\mathbb{Z})|}}.$$

Remark 3. Conjecture 2 appeared in slightly different form in [28, 54, 55].

The level k WRT invariant $\tau_k(M)$ of a closed oriented 3-manifold M can be seen as a function of the k-root of unity $q_k = \exp(2\pi i/k)$, and as such it is a function of a certain subset of the boundary of the unit disc $D = \{q : |q| < 1\}$. Assume $b_1(M) = 0$ and define the k-dependent q-series

$$\hat{Z}_k(M;q) = (i\sqrt{2k})^{-1} \sum_{a,b\in T} e^{2\pi ik \, lk(a,a)}|\mathcal{W}_b|^{-1}S_{a,b}\hat{Z}_b(q).$$

Then $\hat{Z}_k(M;q)$ is convergent for $q \in D$ and the radial limit conjecture states

$$\lim_{q\uparrow q_k} \hat{Z}_k(M;q) = \tau_k(M).$$

Thus $\hat{Z}_k(M;q)$ can be seen as an analytic extension of $\tau_k(M)$ to the interior of the unit disc as illustrated in Figure 6.
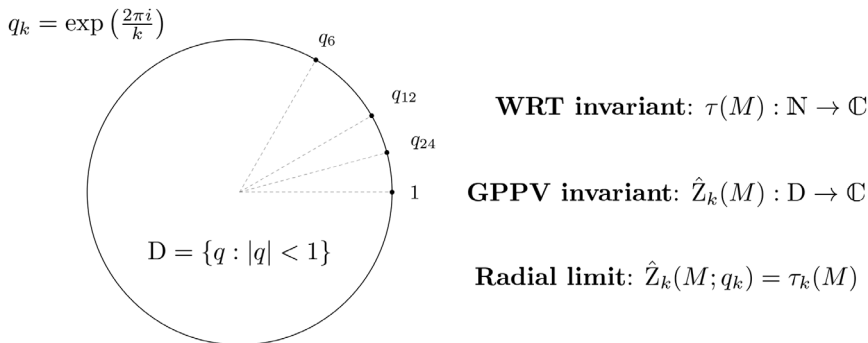

FIGURE 6　Analytic extension of $\tau_k(M)$

5.1 | Proof of Theorem 4

To simplify notation, we write

$$\hat{Z}_0(X; q) = \hat{Z}(q).$$

Recall the decomposition (2.4) of the normalized quantum invariant $\tilde{Z}_k(X)$ into an integral part Z^I and a residue part Z^R. In Lemma 14, we prove the existence of an analogous decomposition for $\hat{Z}_0(q)$ into a Laplace integral part $\mathcal{L}$ and a residue part R

$$q^\Delta \hat{Z}_0(q) = \mathcal{L}(\tau) + R(\tau), \qquad (5.1)$$

where we recall that $q = e^{2\pi i \tau}$. We present in Proposition 15 a standard result in complex analysis [74], which asserts that a q-series with periodic coefficients of mean value zero has an asymptotic expansion, as q tends to a root of unity. We then show in Proposition 16 that R satisfy this hypothesis. Finally, we apply Proposition 15 to prove Theorem 4.

5.1.1 | The decomposition of the GPPV invariant

Recall that $q = \exp(2\pi i \tau)$ with $\tau \in \mathfrak{h}$ where $\mathfrak{h}$ denotes the upper half-plane, and recall the definitions of $F \in \mathcal{M}(\mathbb{C})$ and $g \in \mathbb{C}[x]$ given in (2.3). Let $\tilde{\Gamma}_+ = e^{\frac{\pi i}{4}}$.

Lemma 14. *Introduce the holomorphic functions* $\mathcal{L}, R \in \mathcal{O}(\mathfrak{h})$ *given by*

$$\mathcal{L}(\tau) = \frac{2\lambda}{\sqrt{\tau}} \int_{\Gamma_+} e^{-x/\tau} \mathcal{B}(Z_0)(x) \, dx,$$

$$R(\tau) = -\frac{2\lambda}{\sqrt{\tau}} \sum_{m=0}^{\infty} \mathrm{Res}\left(\exp\left(\frac{g(\xi)}{\tau} \right) F(\xi), \xi = 2\pi i m \right).$$

Then we have that

$$q^\Delta \hat{Z}_0(q) = \mathcal{L}(\tau) + R(\tau). \qquad (5.2)$$

For τ in the first quadrant $\mathfrak{h}_+ = \{z \in \mathfrak{h} \mid \mathfrak{R}(z) > 0\}$ we have that

$$\mathcal{L}(\tau) = \frac{2\lambda}{\sqrt{\tau}} \int_0^\infty e^{-x/\tau} \mathcal{B}(Z_0)(x) \, dx. \qquad (5.3)$$

Proof of Lemma 14. Recall the contour formula from Theorem 3

$$q^\Delta \hat{Z}_0(q) = I(\tau) = \frac{\lambda}{\sqrt{\tau}} \int_\Gamma \exp(-x/\tau) \mathcal{B}(Z_0)(x) \, dx.$$

Under the coordinate change

$$x = y^2$$

the contour Γ corresponds to the contour

$$\Psi = e^{i\pi\frac{3}{8}}\mathbb{R}_+ + e^{i\pi\frac{1}{8}}\mathbb{R}_+$$

Therefore we have that

$$q^{\Delta}\hat{Z}_0(q) = \frac{\lambda}{\sqrt{\tau}}\int_{\Psi}\exp(-y^2/\tau)B(Z_0)(y^2)2y\,\mathrm{d}y.$$

Introduce the meromorphic function $B \in \mathcal{M}(\mathbb{C})$ given for all $y \in \mathbb{C}$ by

$$B(y) = 2yB(Z_0)(y^2). \tag{5.4}$$

By Theorem 2 we have that

$$B(y) = \frac{2\kappa}{\pi i}G\left(\exp\left(\frac{\kappa y}{P}\right)\right) = \frac{2\kappa}{\pi i}F\left(\frac{\kappa y}{2}\right). \tag{5.5}$$

From (5.5) we see that B is periodic with period κ, that is, for all $m \in \mathbb{Z}$ we have that

$$B(y + \kappa m) = B(y). \tag{5.6}$$

Let $\mathcal{P}$ be the set of poles of B. It follows from Theorem 2 that $\mathcal{P}$ is a subset of the axis $e^{\pi i/4}\mathbb{R}$ and that

$$\{-\omega^2 \mid \omega \in \mathcal{P}\} = 2\pi i\,\mathrm{CS}_{\mathbb{C}}(X) + 2\pi i\mathbb{Z}.$$

Write

$$\Psi_{\pm} = e^{i\pi(\frac{1}{4}\mp\frac{1}{8})}\mathbb{R}_+.$$

We will now apply Cauchy's residue theorem to move Ψ_- across $e^{i\pi/4}\mathbb{R}_+$ to Ψ_+ in order to obtain the formula (5.2). Deform $\Psi_{\pm}$ on the complement of a neighborhood around the origin to two curves $L_{\pm}$, which are parallel to $e^{i\pi/4}\mathbb{R}_+$ outside this neighborhood of the origin, as indicated in Figure 7. Set

$$L = L_+ \cup L_-.$$

We first show that

$$\int_{\Psi}\exp(-y^2/\tau)B(y)\,\mathrm{d}y = \int_{L}\exp(-y^2/\tau)B(y)\,\mathrm{d}y, \tag{5.7}$$
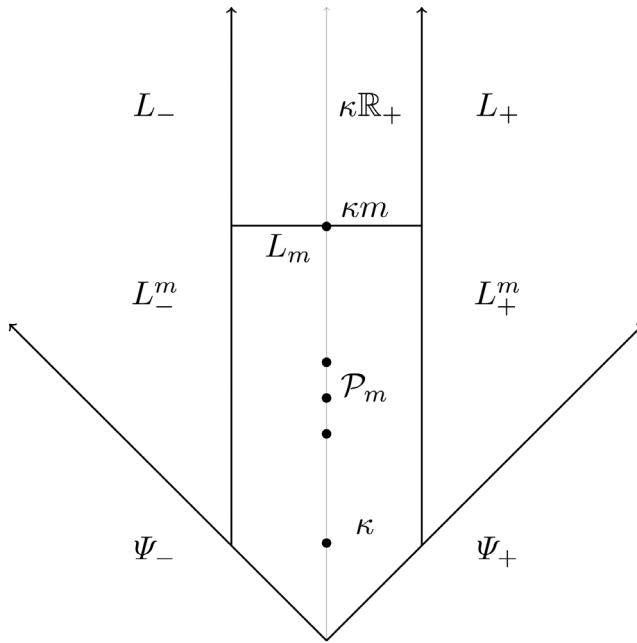
FIGURE 7   The contours $L_\pm, L_\pm^m$ and $\Psi_\pm$ and the subset of poles $\mathcal{P}_m$ drawn rotated, such that $\kappa\mathbb{R}_+$ points straight up

and then we show that the right-hand side of (5.7) can be rewritten as a sum of residues. Let $R > 0$ be a positive constant, and let $R_\pm$ be the arc segment of the circle of radius R, which connects $\Psi_\pm$ and $L_\pm$. Because $L_\pm$ is parallel to $\kappa\mathbb{R}_+$ outside a neighborhood of the origin, there exists a real positive constant $b_0 > 0$ such that every $y \in R_\pm$ is of the form

$$y = \kappa a \pm b, (a, b) \in \mathbb{R}_+ \times [b_0, +\infty)$$

and therefore exists a positive real constant $A > 0$ independent of R, which gives an upper bound

$$\left| \frac{1}{1 - e^{-\kappa y}} \right| < \frac{1}{1 - \exp\left(-\Re(\kappa)b_0\right)} := A > 0$$

for all $y \in R_+$. It follows that we obtain a uniform estimate

$$B(y) = 2^{n-2} \frac{2\kappa}{\pi i} \frac{\prod_{j=1}^n \sinh(\frac{\kappa y}{2p_j})}{\left(e^{\kappa y/2}(1 - e^{-y\kappa})\right)^{n-2}}$$

$$= O\left(e^{-y\kappa/2} \prod_{j=1}^n \sinh\left(\frac{\kappa y}{2p_j}\right)\right) = O(e^{A_1 R})$$

(5.8)

for all $y \in R_+$ for a real constant A_1. For fixed τ, there exists a positive real constant $A_2 > 0$ giving a uniform bound on R_+

$$e^{-y^2/\tau} = O(e^{-A_2 R^2}).$$

(5.9)

By combining the estimates (5.8) and (5.9), and using that the arc length of $R_\pm$ is proportional to R, we obtain the estimate

$$\int_{R_+} \exp(-y^2/\tau)B(y)\,dy = O(Re^{-A_2 R^2 + A_1 R}).$$

By similar reasoning, there exist constants $B_2 > 0, B_1 \in \mathbb{R}$ giving the estimate

$$\int_{R_-} \exp(-y^2/\tau)B(y)\,dy = O(Re^{-B_2 R^2 + B_1 R}).$$

Thus we obtain that

$$\lim_{R\to\infty} \sum_{\epsilon\in\{\pm 1\}} \int_{R_\epsilon} \exp(-y^2/\tau)B(y)\,dy = 0, \tag{5.10}$$

which gives the desired identity (5.7).

   We now turn to the computation of $\int_L \exp(-y^2/\tau)B(y)\,dy$. For each $m \in \mathbb{N}$, let L_m be a small line segment with

$$L_m \cap e^{i\pi/4}\mathbb{R}_+ = \{m\kappa\},$$

and which meets $e^{i\pi/4}\mathbb{R}_+$ in a right angle. We can arrange that L_m is of fixed length and that L_m meet $L_\pm$ in a point. Thus we have

$$L_m = L_0 + m\kappa.$$

Let $L_\pm^m$ be the bounded component of $L_\pm \setminus L_m$, and let $\mathcal{P}_m \subset \mathcal{P}$ be the set of poles of B that lie within the bounded component of the complement of the contour

$$\Psi_m = L_+^m \cup L_m \cup L_-^m$$

(see Figure 7). Equip Ψ_m with the counter clockwise orientation. An application of Cauchy's residue theorem now gives

$$2\pi i \sum_{\omega\in\mathcal{P}_m} \mathrm{Res}(e^{-y^2/\tau}B(y), y = \omega) = \int_{\Psi_m} \exp(-y^2/\tau)B(y)\,dy$$

$$= \int_{L_m} \exp(-y^2/\tau)B(y)\,dy + \sum_{\epsilon\in\{\pm 1\}} \epsilon \int_{L_\epsilon^m} \exp(-y^2/\tau)B(y)\,dy. \tag{5.11}$$

Because B is κ periodic as stated in (5.6) and $L_m = L_0 + m\kappa$, we see that that there exists $C > 0$ giving a uniform bound

$$C > \sup\{|B(y)| \mid y \in \cup_{m\in\mathbb{N}} L_m\}.$$

Because of this universal bound, it is easy to see that

$$\lim_{m\to\infty} \int_{L_m} \exp(-y^2/\tau)B(y)\,\mathrm{d}y = 0.$$

It follows that the right-hand side of (5.11) converges to

$$\sum_{\epsilon\in\{\pm 1\}} \int_{L_\epsilon} \exp(-y^2/\tau)B(y)\,\mathrm{d}y = \sum_{\epsilon\in\{\pm 1\}} \epsilon \int_{\Psi_\epsilon} \exp(-y^2/\tau)B(y)\,\mathrm{d}y.$$

This also implies that the sum of residues is convergent.

Let us now recall a simple transformation law for residues. Let $z_0 \in \mathbb{C}$ and let $f \in \mathcal{M}_{z_0}(\mathbb{C})$ be the germ of a meromorphic function with a pole at z_0. Assume $w_0 \in \mathbb{C}$ and that $z \in \mathcal{O}_{w_0}(\mathbb{C})$ satisfies $z(w_0) = z_0, \dot{z}(w_0) \neq 0$. If either z_0 is a simple pole, or $z(w)$ is linear in w, then we have that

$$\mathrm{Res}(f(z(w)), w = w_0) = \frac{\mathrm{Res}(f(z), z = z(w_0))}{\dot{z}(w_0)}. \tag{5.12}$$

Introduce the variable

$$\xi = \frac{\kappa y}{2}.$$

Using the relation (5.5) between B and F, the relation (5.4) between B and $\mathcal{B}(Z_0)$ and the transformation law (5.12) we obtain

$$q^\Delta \hat{Z}_0(q) = \frac{\lambda}{\tau} \int_\Psi \exp\left(-\frac{y^2}{\tau}\right) B(y)\,\mathrm{d}y$$

$$= \frac{2\lambda}{\tau} \left(\int_{\Psi_+} \exp\left(-\frac{y^2}{\tau}\right) B(y)\,\mathrm{d}y - 2\pi i \sum_{\omega\in\mathcal{P}} \mathrm{Res}(e^{-y^2/\tau}B(y), y = \omega) \right)$$

$$= \frac{2\lambda}{\tau} \left(\int_{\Gamma_+} \exp\left(-\frac{x}{\tau}\right) \mathcal{B}(Z_0)(x)\,\mathrm{d}x - \sum_{m=1}^\infty \mathrm{Res}(e^{-g(\xi)/\tau}F(\xi), \xi = 2\pi i m) \right)$$

$$= \mathcal{L}(\tau) + R(\tau).$$

Finally, we prove (5.3) for $\tau \in \mathfrak{h}_+$. First observe that for x and τ in the upper right half-plane we have

$$\mathfrak{R}(x/\tau) > 0. \tag{5.13}$$

Push the contour $\tilde{\Gamma}_+$ to $\mathbb{R}_+$. If the integral is invariant under this deformation of the contour, we obtain the desired identity. To see that the integral is invariant under this deformation of the contour, we apply a limiting argument, together with Cauchy's residue formula. To that end, let $R > 0$ be a positive parameter, and let C_R be the arc segment of the circle of radius R, which connects R to $Re^{i\pi\frac{3}{4}}$ and stays in the upper half-plane. As we are not moving the contour across any singularities

of $\mathcal{B}(Z_0)$, the only difficulty is to show that

$$\lim_{R\to+\infty} \int_{C_R} e^{-x/\tau} \mathcal{B}(Z_0)(x)\,\mathrm{d}x = 0. \tag{5.14}$$

As C_R remain at least a fixed distance away from the axis of poles of $\mathcal{B}(Z_0)(x)$, limit (5.14) follows by (5.13) together with arguments similar to the arguments giving limit (5.10). $\qquad\square$

5.1.2 | Asymptotic expansions of q-series with periodic coefficients

Let $B_m(x)$ denote the mth Bernoulli polynomial, that is,

$$\frac{te^{tx}}{e^t - 1} = \sum_{m=0}^{\infty} \frac{B_m(x)}{m!} t^m. \tag{5.15}$$

We recall the following result.

Proposition 15 [62, 74]. *Let $C : \mathbb{Z} \to \mathbb{C}$ be a periodic function with period M and mean value equal to zero*

$$\sum_{m=1}^{M} C(n) = 0.$$

Consider the L-series $L(s,C)$, which for $\Re(s) > 1$ is defined by

$$L(s,C) = \sum_{m=1}^{\infty} \frac{C(m)}{m^s}.$$

This L-series admits an analytic extension to all of $\mathbb{C}$ and for $r \in \mathbb{N}$

$$L(-r,C) = -\frac{M^r}{r+1} \sum_{m=1}^{M} C(m)B_{r+1}\left(\frac{m}{M}\right). \tag{5.16}$$

For any polynomial Q of degree d

$$Q(x) = \sum_{u=0}^{d} q_u x^u \in \mathbb{C}[x]$$

the following asymptotic expansions hold for real and positive t

$$\sum_{m=1}^{\infty} e^{-tm^2} C(m)Q(m) \underset{t\to 0}{\sim} \sum_{u=0}^{d} \sum_{r=0}^{\infty} q_u L(-2r - u, C)\frac{(-t)^r}{r!}. \tag{5.17}$$

Proof. The existence of the analytic extension of the L-series of C, as well as the explicit evaluation (5.16) are proven in [74].

In [62, 74] the following asymptotic expansions are proven

$$
\sum_{m=1}^{\infty} C(m)e^{-tm^2} \underset{t\to 0}{\sim} \sum_{r=0}^{\infty} L(-2r,C)\frac{(-t)^r}{r!},
$$
$$
\sum_{m=1}^{\infty} mC(m)e^{-tm^2} \underset{t\to 0}{\sim} \sum_{r=0}^{\infty} L(-2r-1,C)\frac{(-t)^r}{r!}. \tag{5.18}
$$

We have

$$
\sum_{m=1}^{\infty} e^{-tm^2}C(m)Q(m) = \sum_{j=0}^{\infty} \frac{\partial^j}{\partial(-t)^j} \sum_{m=0}^{\infty} (q_{2j+1}m + q_{2j})e^{-tm^2},
$$

where it is understood that $q_l = 0$ for $l > d$. The expansion (5.17) follows formally from differentiating the expansions given in (5.18). This differentiation is valid because Poincaré asymptotic expansions of analytic functions which are valid on suitable sectors can be termwise differentiated. Clearly, $t \mapsto \sum_{m\geq 0} C(m)Q(m)\exp(-tm^2)$ is an analytic function of t in a small tubular neighborhood of $(0,1]$, and from the proof given in [74] it is clear that the asymptotic expansions (5.18) are valid on such a small sector. $\qquad\square$

Recall the definition of the meromorphic function F given in (2.3). Next we prove that the coefficients of the principal part of F at poles are periodic functions with mean value equal to zero.

Proposition 16. *For $j = 1, 2, \ldots, n - 2$ define $f_j : \mathbb{Z} \to \mathbb{C}$ as the coefficients of the principal part of F at $2\pi im$ for $m \in \mathbb{Z}$, for example, for y near $2\pi im$*

$$
F(y) = \sum_{j=1}^{n-2} f_j(m)(y - 2\pi im)^{-j} + \text{reg}.
$$

Then each f_j is 2P-periodic and if P is even, then we have for each even $k \in \mathbb{Z}$

$$
\sum_{m=1,\ldots,2P} e^{kg(2\pi im)} f_j(m) = 0. \tag{5.19}
$$

Proof. The periodicity of the functions $f_j, j = 1, \ldots, n - 3$ follow directly from the $4\pi iP$-periodicity of F.

We now prove (5.19) assuming P is even - which is equivalent to exactly one the p_j being even. Using the definition (2.3) of F we obtain

$$
F(y + 2\pi iP) = \frac{1}{4}\sinh(y/2 + \pi iP)^{2-n} \prod_{j=1}^{n} \sinh(y/p_j + \pi iP/p_j)
$$

$$
= (-1)^{nP}(-1)^{\sum_{j=1}^{n} \frac{P}{p_j}} F(y) = -F(y).
$$

This implies that for each $j = 1, \ldots, n - 3$ and $m = 1, \ldots, P$ we have

$$f_j(m + P) = -f_j(m).$$

On the other hand, we have for integral m

$$g(2\pi i(P + m)) = \frac{i(2\pi i(P + m))^2}{8\pi P} = \frac{-i\pi m^2}{2P} \mp i\pi mP - \frac{i\pi P}{2}$$

$$= g(2\pi i(P \mp m)) \mod \pi i\mathbb{Z}.$$

It follows that for even k we have a pairwise cancellation:

$$\sum_{m=1}^{2P} e^{kg(2\pi im)} f_j(m) = \sum_{m=1}^{P} e^{kg(2\pi im)} f_j(m) + e^{kg(2\pi i(m+P))} f_j(m + P)$$

$$= \sum_{m=1}^{P} e^{kg(2\pi im)} f_j(m) - e^{kg(2\pi im)} f_j(m) = 0.$$

This concludes the proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

5.1.3 | The asymptotic expansion of the GPPV invariant

Recall the definition the functions F and g given in (2.3). Let $m \in \mathbb{N}$. For y close to $2\pi im$ we use the notation of Proposition 16 and write

$$F(y) = \sum_{j=1}^{n-2} f_j(m)(y - 2\pi im)^{-j} + \text{reg.},$$

where each $f_j : \mathbb{Z} \to \mathbb{C}$ is $2P$-periodic. For each $l \in \mathbb{N}$ there exists a uniquely determined polynomial

$$P_l(x, y) \in Q[\pi i, x, y]$$

such that

$$\frac{1}{l!} \frac{\partial^l e^{g(y)/\tau}}{\partial y^l}(2\pi im) = e^{g(2\pi im)/\tau} P_l(\tau^{-1}, m). \qquad (5.20)$$

There exist uniquely determined complex coefficients $p_{l,u,v}$, with

$$P_l(\tau^{-1}, m) = \sum_{u,v} p_{l,u,v} \tau^{-u} m^v.$$

Recall that $B_m(x)$ denotes the mth Bernoulli polynomial and is defined by (5.15). Define for each $\theta \in \mathrm{CS}_\mathbb{C}(X)$ the following polynomials with coefficients in power series

$$R_\theta(k, t) = \sum_{\substack{m=1,\ldots,2P, \\ g(2\pi im) \equiv 2\pi i\theta}} \sum_{j,u,v} f_j(m) k^u p_{j-1,u,v} \sum_{r=0}^{\infty} \frac{(2P)^{2r+v}}{2r + v + 1} B_{2r+v+1}\left(\frac{m}{2P}\right) \frac{(-t)^r}{r!},$$

$$\check{Z}_\theta(k,t) = \sum_{\substack{m=1,\dots,2P,\\ g(2\pi im)\equiv 2\pi i\theta}} \sum_{j,u,v} f_j(m) k^u p_{j-1,u,v} \sum_{r=1}^\infty \frac{(2P)^{2r+v}}{2r+v+1} B_{2r+v+1}\left(\frac{m}{2P}\right)\frac{(-t)^r}{r!} \qquad (5.21)$$

Observe

$$\check{Z}_\theta(k,t) = R_\theta(k,t) - R_\theta(k,0),$$

that is, $\check{Z}_\theta$ is equal to R_θ minus the constant in the parameter t.

We can now prove Theorem 4.

Proof of Theorem 4. Recall the decompsition

$$\frac{\sqrt{\tau}}{\lambda} q^\Delta \hat{Z} = \mathcal{L}(\tau) + R(\tau)$$

given in Equation (5.2) in Lemma 14. Recall the decomposition of the normalized quantum invariant

$$Z_k(X) = Z^I(k) + Z^R(k)$$

given in (2.4). This decomposition together with Equation (1.4), which relates the normalized quantum invariant $\check{Z}_k(X)$ to the WRT invariant $\tau_k(X)$, shows the radial limit identity can be proved by proving the following two limits:

$$Z^I(k) = \lim_{\tau\uparrow 1/k} \frac{\sqrt{\tau}}{2\lambda}\mathcal{L}(\tau), \qquad\qquad Z^R(k) = \lim_{\tau\uparrow 1/k}, \frac{\sqrt{\tau}}{2\lambda}R(\tau).$$

Observe that as X is an integral homology sphere, the only Spin^c structure is $a = 0$, and therefore the radial limit conjecture reduces (up to a scalar factor) to Equation (1.13). We first focus on the integral part $\mathcal{L}(\tau)$. For every $k \in \mathbb{N}^*$ the integral part $\mathcal{L}(\tau)$ extends continuously to $\tau = 1/k$ and it follows from Equations (2.4), (3.6) and (5.3) that

$$(2\lambda\sqrt{k})^{-1}\mathcal{L}(1/k) = \mathcal{L}_{\mathbb{R}_+}(\mathcal{B}(Z_0))(k) = Z^I(k).$$

Now recalling that $\tau_{k,t} = (k - i\frac{2Pt}{\pi})^{-1}$, we see that the non-trivial parts left in order to prove the asymptotic expansion (1.12) stated in Theorem 4 are the asymptotic expansion

$$\frac{\sqrt{\tau_{k,t}}}{2\lambda}R(\tau_{k,t}) \underset{t\to 0}{\sim} \sum_{\theta\in\mathrm{CS}_\mathbb{C}(X)} e^{2\pi ik\theta} R_\theta(k,t), \qquad (5.22)$$

and the identity

$$\sum_{\theta\in\mathrm{CS}_\mathbb{C}(X)} e^{2\pi ik\theta} R_\theta(k,0) = Z^R(k). \qquad (5.23)$$

We start with the expansion (5.22). To ease notation we set

$$g(2\pi i m) = \theta_m.$$

We have

$$\operatorname{Res}(F(y)e^{g(y)/\tau}, 2\pi i m) = e^{\theta_m/\tau} \sum_{j=1}^{n-3} f_j(m) P_{j-1}(\tau^{-1}, m).$$

and accordingly

$$\frac{\sqrt{\tau}}{2\lambda} R(\tau) = - \sum_{m=0}^{\infty} e^{\theta_m/\tau} \sum_{j=1}^{n-3} f_j(m) P_{j-1}(\tau^{-1}, m).$$

Note that $\tau_{k,t} \in \mathfrak{h}$ for $t \in (0,1)$. The function g has the following transformation property

$$g(y + 4\pi i m P) = \frac{i(y + 4\pi i m P)^2}{8\pi P} = g(y) - my - 2\pi i m^2 P.$$

Because of this, and the $2P$ periodicity of the functions f_j, we can write

$$\frac{\sqrt{\tau}}{2\lambda} R(\tau) = - \sum_{m=0}^{\infty} e^{-tm^2} e^{kg(2\pi i m)} \sum_{j=1}^{n-3} f_j(m) P_{j-1}(\tau^{-1}, m)$$

$$= - \sum_{j=1}^{n-2} \sum_{u,v} \sum_{m=0}^{\infty} e^{-tm^2} e^{k\theta_m} f_j(m) p_{j-1,u,v} \tau^{-u} m^v.$$

Now for each $j, u = 0, \ldots, n-3$ we can apply Proposition 15 to the $2P$-periodic function of mean value zero given by

$$C_j(m) = e^{k\theta_m} f_j(m)$$

and the polynomial

$$Q_{j,u}(m) = \sum_v p_{j-1,u,v} m^v.$$

The fact that each C_j is of mean value equal to zero follows from Proposition 16. The result of applying Proposition 15 is

$$\frac{\sqrt{\tau}}{2\lambda} R(\tau) \underset{t \to 0}{\sim} - \sum_{j,u,v} \sum_{r=0}^{\infty} k^u p_{j-1,u,v} L(-2r - v, C_j) \frac{(-t)^r}{r!}$$

$$= \sum_{j,u,v} \sum_{r=0}^{\infty} k^u p_{j-1,u,v} \frac{(2P)^{2r+v}}{2r + v + 1} \sum_{m=1}^{2P} C_j(m) B_{2r+v+1}\left(\frac{m}{2P}\right) \frac{(-t)^r}{r!}$$

$$= \sum_{m=1}^{2P} \sum_{j,u,v} e^{k\theta_m} f_j(m) k^u p_{j-1,u,v} \sum_{r=0}^{\infty} \frac{(2P)^{2r+v}}{2r + v + 1} B_{2r+v+1}\left(\frac{m}{2P}\right) \frac{(-t)^r}{r!}.$$

This establishes the asymptotic expansion (5.22).

We now turn to the identity (5.23). Set

$$R_0(k) = \sum_{\theta \in CS_{\mathbb{C}}(X)} e^{2\pi i k \theta} R_\theta(k, 0) = \sum_{m=1}^{2P} \sum_{j,u,v} e^{k\theta_m} f_j(m) p_{j-1,u,v} k^u \frac{(2P)^v}{v+1} B_{v+1}\left(\frac{m}{2P}\right). \quad (5.24)$$

Let x be a complex coordinate near 0 and set $y_m = x + 2\pi i m$ for $m = 1, 2, \ldots, 2P$. Recall that

$$Z^R(k) = -\sum_{m=1}^{2P-1} \operatorname{Res}\left(\frac{F(y)\exp(kg(y))}{1 - \exp(-ky)}, y = 2\pi i m\right).$$

We have that

$$g(y_m) = g(x + 2\pi i m) = \frac{i(x + 2\pi i m)^2}{8\pi P} = g(2\pi i m) + g(x) - \frac{xm}{2P},$$

and accordingly

$$\exp(kg(y_m)) = \exp(k\theta_m + kg(x)) \cdot \exp\left(-k\frac{xm}{2P}\right). \quad (5.25)$$

Recall that the polynomials P_l defined in (5.20) as the coefficients of the Taylor series of $e^{kg(y)}$. Therefore it follows from Cauchy's formula for multiplication of power series, the formula for the Taylor expansion of the exponential and the identity (5.25) that the following holds for all k, m:

$$P_c(k, m) = \sum_{a+b=c} P_a(k, 0)\left(\frac{-mk}{2P}\right)^b \frac{1}{b!}.$$

Writing this out in terms of coefficients gives

$$\sum_{c,u,v} p_{c,u,v} k^u m^v = \sum_{a+b=c} \sum_{s} p_{a,s,0}\left(\frac{(-1)}{2P}\right)^b \frac{1}{b!} k^{s+b} m^b.$$

This is equivalent to the identities

$$p_{j,u,0}\left(\frac{-1}{2P}\right)^v \frac{1}{v!} = p_{j+v,u+v,v}. \quad (5.26)$$

Recall the definition (5.15) of the Bernoulli polynomials B_m. Write

$$-\frac{F(y_m)\exp(kg(y_m))}{1 - \exp(-ky_m)} = F(y_m)\exp(k(g(x) + \theta_m))\frac{\exp\left(\frac{-kxm}{2P}\right)}{\exp(-kx) - 1}$$

$$= \left(\sum_{j=1}^{n-3} f_j(m)x^{-j} + \text{reg.}\right)\left(e^{k\theta_m}\sum_{a=0}^{\infty} P_a(k, 0)x^a\right)\left(\sum_{b=0}^{\infty} \frac{B_b\left(\frac{m}{2P}\right)}{b!}(-kx)^{b-1}\right).$$

By comparing with Equation (5.24) and using the facts that F has a multiple order zero at multiples of $4\pi i P$ and that we know that the k^{-1} term cancels in $Z^R(k)$, we obtain the desired identity

$$
Z^R(k) = \sum_{m=1}^{2P} \sum_{j=1}^{n-2} \sum_{\substack{a+b=j, \\ a \geqslant 0, b \geqslant 1}} e^{k\theta_m} f_j(m) P_a(k,0) \frac{B_b\left(\frac{m}{2P}\right)}{b!} (-k)^{b-1}
$$

$$
= \sum_{m=1}^{2P} \sum_{j=1}^{n-2} \sum_{\substack{a+b=j, \\ a \geqslant 0, b \geqslant 1}} e^{k\theta_m} f_j(m) P_a(k,0) \left(\frac{-1}{2P}\right)^{b-1} \frac{k^{b-1}}{(b-1)!} \frac{(2P)^{b-1}}{b} B_b\left(\frac{m}{2P}\right)
$$

$$
= \sum_{m=1}^{2P} \sum_{j=1}^{n-2} \sum_{\substack{a+b=j, \\ a \geqslant 0, b \geqslant 1}} \sum_{s} e^{k\theta_m} f_j(m) P_{a,s,0} \left(\frac{-1}{2P}\right)^{b-1} \frac{k^{s+b-1}}{(b-1)!} \frac{(2P)^{b-1}}{b} B_b\left(\frac{m}{2P}\right)
$$

$$
= \sum_{m=1}^{2P} \sum_{j=1}^{n-2} \sum_{\substack{s,a+b=j, \\ a \geqslant 0, b \geqslant 1}} \sum_{s} e^{k\theta_m} f_j(m) P_{a+b-1,s+b-1,b-1} k^{s+b-1} \frac{(2P)^{b-1}}{b} B_b\left(\frac{m}{2P}\right) \tag{5.27}
$$

$$
= \sum_{m=1}^{2P} \sum_{j=1}^{n-2} \sum_{u,v} e^{k\theta_m} f_j(m) p_{j-1,u,v} k^u \frac{(2P)^v}{v+1} B_{v+1}\left(\frac{m}{2P}\right) \tag{5.28}
$$

$$
= R_0(k).
$$

In (5.27) we used the identity (5.26), and in (5.28) we set $u = s + b - 1$ and $v = b - 1$. This finishes the proof. $\qquad\square$

APPENDIX: RESURGENCE AND RESUMMATION

A.1 | Resurgent functions and the Borel transform

The theory of resurgence was originally developed by Écalle in [40, 41]. See [80] for an introduction to the mathematical theory of resurgence and see [39] for an introduction to the general use of resurgence in quantum field theory. Garoufalidis [48] and Witten [95] were the pioneers of the use of resurgence in quantum Chern–Simons theory.

Definition A1. For a Riemann surface C with universal covering space

$$
\pi : \tilde{C} \to C
$$

the group of resurgent functions is $\mathcal{R}(C) = \mathcal{M}(\tilde{C})$.

One source of resurgent functions are the Borel transforms of Laplace integrals. We now introduce the Borel transform. Let $\Gamma \in \mathcal{M}(\mathbb{C})$ be the Gamma function, which for $z \in \mathbb{C}$ with $\operatorname{Re}(z) > 0$ is defined by

$$
\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} \, \mathrm{d}t.
$$

Definition A2. Let $\{\alpha_j\}_{j=0}^{\infty}$ be an increasing sequence of positive real numbers, $\{\beta_j\}_{j=0}^{\infty}$ a sequence of non-negative integers and $\{c_j\}_{j=0}^{\infty}$ a sequence of complex numbers. Consider the formal series

$$\tilde{\varphi}(\lambda) = \sum_{j=0}^{\infty} c_j \lambda^{-\alpha_j} \log(\lambda)^{\beta_j}.$$

The Borel transform of $\tilde{\varphi}(\lambda)$ is given by the formal series

$$\mathcal{B}(\tilde{\varphi})(\zeta) = \sum_{j=0}^{\infty} c_j (-1)^{\beta_j} \frac{\partial^{\beta_j}}{\partial \alpha_j^{\beta_j}} \left(\frac{\zeta^{\alpha_j - 1}}{\Gamma(\alpha_j)} \right).$$

A.2 | Borel–Laplace resummation

We now discuss in more detail the relation between the Borel transform $\mathcal{B}$ and the Laplace transform, which we now introduce. Let $\gamma \subset \mathbb{C}$ be an oriented contour. Let g be a measurable function defined in a neighborhood of γ. Denote by $\mathcal{L}_\gamma(g)$ the Laplace transform given by

$$\mathcal{L}_\gamma(g)(\lambda) = \int_\gamma \exp(-\lambda \cdot z) g(z) \, dz, \tag{A1}$$

for all $\zeta \in \mathbb{C}$ such that the integral is absolutely convergent. Here we think of $\lambda \in \mathbb{C}^*$ as a large modulus asymptotics parameter. For any $\alpha \in \mathbb{C}^*$ we let the contour $\alpha \mathbb{R}_+$ be oriented in the direction of α unless we state otherwise.

That the transforms $\mathcal{B}$ and $\mathcal{L}_{\mathbb{R}_+}$ are formally inverses of each other should be understood as follows. If $\alpha \in \mathbb{C}$ satisfies $\mathrm{Re}(\alpha) > -1$ and $l \in \mathbb{N}$ then

$$\mathcal{L}_{\mathbb{R}_+}\left(\zeta^\alpha \log(\zeta)^l\right) = \frac{d^l}{d\alpha^l} \left(\frac{\Gamma(\alpha+1)}{\lambda^{\alpha+1}} \right).$$

We may introduce a polynomial $Q_{\alpha,l} \in \mathbb{C}[x]$ of degree l such that

$$\mathcal{L}_{\mathbb{R}_+}\left(\zeta^\alpha \log(\zeta)^l\right) = \lambda^{-\alpha-1} Q_{\alpha,l}(\log(\lambda)). \tag{A2}$$

Let $z \in \mathbb{C}$ with $\mathrm{Re}(z) > 0$ and let $m \in \mathbb{N}$. We then have that

$$\mathcal{L}_{\mathbb{R}_+} \circ \mathcal{B}(\lambda^{-z} \log(\lambda)^m) = \lambda^{-z} \log(\lambda)^m,$$
$$\mathcal{B} \circ \mathcal{L}_{\mathbb{R}_+}(\zeta^{z-1} \log(\zeta)^m) = \zeta^{z-1} \log(\zeta)^m. \tag{A3}$$

Lemma A1. *Let $B : \mathbb{R}_+ \to \mathbb{C}$ be a measurable function and assume the integral defining $\mathcal{L}_{\mathbb{R}_+}(B)(\lambda)$ is absolutely convergent for $\mathrm{Re}(\lambda) > 0$. Assume that there exists an increasing sequence $\{\alpha_j\}_{j=0}^{\infty}$ of real numbers strictly greater than -1 and a sequence $\{\beta_j\}_{j=0}^{\infty}$ of positive integers giving an asymptotic expansion*

$$B(t) \sim_{t \to 0} \sum_{j=0}^{\infty} b_j t^{\alpha_j} \log(t)^{\beta_j}.$$

Then the following hold:

(i) *there exists for large λ an asymptotic expansion $\tilde{\varphi}(\lambda)$ of the form*

$$\mathcal{L}_{\mathbb{R}_+}(B)(\lambda) \sim_{\lambda \to \infty} \tilde{\varphi}(\lambda),$$

where

$$\tilde{\varphi}(\lambda) = \sum_{\alpha,\beta} b_j \lambda^{-\alpha_j-1} Q_{\alpha_j,\beta_j}(\log(\lambda))$$

and $Q_{\alpha_j,\beta_j} \in \mathbb{C}[x]$ is the degree β_j polynomial introduced in (A2);

(ii) *the Borel transform of $\tilde{\varphi}$ is equal to the expansion of B*

$$\mathcal{B}(\tilde{\varphi})(t) = \sum_{j=0}^{\infty} b_j t^{\alpha_j} \log(t)^{\beta_j}.$$

Proof. The lemma is an elementary consequence of Equations (A2) and (A3). □

The following theorem explains Borel–Laplace resummation. The content of Theorem A2 is standard in resurgence, and a proof can be found in, for example, [90].

Theorem A2. *Let*

$$\tilde{\varphi}(\lambda) = \sum_{j=0}^{\infty} c_j \lambda^{-\alpha_j} \log(\lambda)^{\beta_j} \tag{A4}$$

be a formal series as in Definition A2. Assume that

- *there exists a sector $S \subset \mathbb{C}$, such that for all $\zeta \in S$ of sufficiently small modulus the Borel transform $\mathcal{B}(\tilde{\varphi})(\zeta)$ converges to a holomorphic function $\hat{\varphi}(\zeta)$ (possibly upon choosing a branch of $\log(\zeta)$ defined on S), and that*
- *the function $\hat{\varphi}$ extends by analytic continuation along a half axis $\Gamma(\theta) = e^{2\pi i\theta}\mathbb{R}_+ \subset S$ (for some $\theta \in [0,1]$ say) and there exists a constant $C > 0$ such that in a neighborhood of $\Gamma(\theta)$ we have $\hat{\varphi}(z) = O(\exp(C|z|))$.*

Then the following hold.

(i) *The Laplace transform $\mathcal{L}_{\Gamma(\theta)}(\hat{\varphi})$ is holomorphic on the open unbounded set $\{\lambda \in \mathbb{C} \mid |\lambda| > C, \forall s \in S \setminus \{0\}\ \mathrm{Re}(\lambda s) > 0\}$.*

(ii) *The Laplace transform $\mathcal{L}_{\Gamma(\theta)}(\hat{\varphi})$ has $\tilde{\varphi}$ as its large λ asymptotic expansion*

$$\mathcal{L}_{\Gamma(\theta)}(\hat{\varphi})(\lambda) \sim_{\lambda \to \infty} \tilde{\varphi}(\lambda).$$

One of the goals of Ecalle's theory [40, 41] is to consider the case where the formal series $\tilde{\varphi}$ (A4) is obtained as a formal solution to some dynamical problem, which can be for instance an ODE or a difference equation (with a singularity at $\lambda^{-1} = 0$). In such situations, the function $\mathcal{L}_{\Gamma(\theta)} \circ \mathcal{B}(\tilde{\varphi})$ will be a holomorphic solution, and resurgence is developed as a tool to analyze the monodromy (known as Stokes phenomena), which occur upon varying the choice of direction θ in

which the Laplace transform is performed. For future studies related to the radial limit conjecture (Conjecture 2), we also mention the work of Marmi and Sauzin [77].

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