The Design Space of Plane Elastic Curves Supplemental Material

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In this document, we show two derivations that were omitted in the main article. In Section 1, we derive the moment equilibrium equation of an elastic rod under gravity from the variational form. In Section 2, we derive the adjoint of the equality-constrained Jacobi equations used to determine stability of kinematic elastic rods.

Notation. To keep this document self-contained, we repeat some definitions and notation from the main article. Let $\gamma : [0, l] \to \mathbb{R}^2$ an arc-length parametrized plane curve in C^2. Denote by $\alpha : [0, l] \to \mathbb{R}$ a turning angle function in C^1, such that $\gamma' = (\cos\alpha, \sin\alpha)$, and by $\kappa : [0, l] \to \mathbb{R}$ the curvature of γ, such that $\kappa = \alpha'$. Furthermore, let $K : [0, l] \to \mathbb{R}_{\geq 0}$, the spatially-varying stiffness of an elastic curve.

Variations. We use the δ-operator to describe variations of the primary variable α, and also variations induced in other variables and functionals. To preserve Dirichlet boundary conditions, $\delta\alpha$ satisfies $\delta\alpha(0) = 0 = \delta\alpha(l)$. Induced variations of a quantity $G[\alpha]$ are given by $\delta G[\alpha; \delta\alpha] = (\mathrm{d}/\mathrm{d}\varepsilon)\, G[\alpha + \varepsilon\, \delta\alpha]|_{\varepsilon=0}$, where we will usually omit the dependence on α and $\delta\alpha$. For example, we have

$$\delta\gamma' = \delta \begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix} = R\gamma'\delta\alpha,$$

with $R = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. Variations $\delta\alpha$ are chosen from the same space as α, but we will formally allow distributional $\delta\alpha$ to simplify computations. This can be made more precise with an approximation argument.

1 MOMENT EQUILIBRIUM UNDER GRAVITY

In Section 5.1 of the main article, we give an expression for the energy potential of an elastic curve with constant thickness and spatially-varying width under gravity,

$$W[\alpha] = \int_0^l K \left(\frac{1}{2}\kappa^2 + \langle \gamma, e \rangle \right),$$

where $e \in \mathbb{R}^2$ is a constant dependent on gravity, thickness, and material properties. We are looking for extremals subject to the constraint $\gamma(l) = \gamma_l$, i.e., the endpoint of the curve is fixed to lie at γ_l.

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According to the method of Lagrange multipliers, these extremals are characterized by α and $\lambda \in \mathbb{R}^2$ for which the variation of

$$L[\alpha, \lambda] = \underbrace{\int_0^l \frac{1}{2} K\kappa^2}_{A} + \underbrace{\int_0^l K\langle \gamma, e \rangle}_{B} + \underbrace{\int_0^l \langle \lambda, \gamma' \rangle}_{C}$$

vanishes, and such that $\gamma(l) = \gamma_l$. The variations of A, B, and C are given by

$$\delta A = \int_0^l K\kappa\delta\alpha' = -\int_0^l (K\kappa)'\delta\alpha,$$

$$\delta B = \int_0^l K(s) \left(\int_0^s \langle R\gamma', e \rangle \delta\alpha \right) \mathrm{d}s,$$

$$\delta C = \int_0^l \langle \lambda, R\gamma' \rangle \delta\alpha.$$

To derive the moment equilibrium equation at a point $\varsigma \in (0, l)$, we formally set $\delta\alpha = \delta_\varsigma$, the delta distribution centered at ς. Then, the variations of A and C evaluate to

$$\delta A = -(K\kappa)'(\varsigma) \quad \text{and} \quad \delta C = \langle \lambda, R\gamma'(\varsigma) \rangle.$$

The variation of B is given by

$$\begin{aligned} \delta B &= \int_0^l K(s) \langle R\gamma'(\varsigma), e \rangle \chi_{[0,s]}(\varsigma)\, \mathrm{d}s \\ &= \langle R\gamma'(\varsigma), e \rangle \int_0^l K(s) \chi_{[0,s]}(\varsigma)\, \mathrm{d}s = \langle R\gamma'(\varsigma), e \rangle \int_\varsigma^l K, \end{aligned}$$

where $\chi_{[0,s]}$ denotes the characteristic function on $[0, s]$. Setting the variation of L to zero, we arrive at the Euler–Lagrange equation

$$0 = -(K\kappa)'(\varsigma) + \langle b, \gamma'(\varsigma) \rangle + \langle R^t e, \gamma'(\varsigma) \rangle \int_\varsigma^l K,$$

where we have substituted $\lambda = Rb$.

The last step in deriving the moment equilibrium equation is to find an antiderivative of the function above. For the first two summands, an antiderivative is given by $-K\kappa + \langle b, \gamma \rangle$. For the last summand, we use the identity

$$\frac{\mathrm{d}}{\mathrm{d}s} \left(f(s) \int_s^l g - \int_s^l fg \right) = f'(s) \int_s^l g$$

to conclude that an antiderivative is given by

$$\langle R^t e, \gamma(\varsigma) \rangle \int_\varsigma^l K - \int_\varsigma^l \langle R^t e, \gamma \rangle K.$$

In its integrated form, the moment equilibrium equation is given by

$$-K(\varsigma)\kappa(\varsigma) + \langle b, \gamma(\varsigma) \rangle + \langle R^t e, \gamma(\varsigma) \rangle \int_\varsigma^l K - \int_\varsigma^l \langle R^t e, \gamma \rangle K + a = 0$$

with the integration constant a.

2 ADJOINT STABILITY EQUATIONS

In Section 6.2 of the main article, we summarize the isoperimetric Jacobi criterion [Manning et al. 1998], which can be used to determine whether an extremal of a variational problem with multiple equality constraints is a minimum. In this section, we derive the corresponding adjoint equations, which are also listed in Appendix B of the main article.

2.1 The Primal Equations

For easier reference, we repeat all primal equations that need to be taken into account when deriving the adjoint.

$$
\begin{aligned}
-(K\alpha')' + \langle \lambda, R\gamma' \rangle &= 0, & \alpha(0) &= \alpha_0, & \alpha(l) &= \alpha_l, \\
\text{subject to} \quad \gamma(l) &= \gamma_l, \\
-(K\zeta')' - \langle \lambda, \gamma' \rangle \zeta &= 0, & \zeta(0) &= 0, & \zeta'(0) &= 1, \\
-(K\eta_i')' - \langle \lambda, \gamma' \rangle \eta_i &= \langle R\gamma', e_i \rangle, & \eta_i(0) &= 0, & \eta_i'(0) &= 1, \\
M_i' &= \langle R\gamma', e_i \rangle \zeta, & M_i(0) &= 0, \\
N_{ij}' &= \langle R\gamma', e_i \rangle \eta_j, & N_{ij}(0) &= 0, \\
Z &= \begin{pmatrix} \zeta & \eta_1 & \eta_2 \\ M_1 & N_{11} & N_{12} \\ M_2 & N_{21} & N_{22} \end{pmatrix}, \\
\mathcal{Z} &= \det Z, \\
F &= \int \varphi \mathcal{Z}.
\end{aligned}
$$

The first two lines define an extremal of the bending energy with kinematic boundary conditions. Lines 3–8 define the "stability indicator" $\mathcal{Z}$, which can be checked for zero crossings on the interval $(0, l)$. The existence of $\sigma \in (0, l)$ with $\mathcal{Z}(\sigma) = 0$ indicates that the extremal is a saddle point and not a minimum. The stability recovery algorithm, which is detailed in the main article, depends on computing the variational derivative of F with respect to K. Here, φ can be thought of as either the delta distribution δ_σ centered at σ, or an approximation of δ_σ through a function with unit mass.

2.2 Céa's Method

To derive the adjoint equations, we apply Céa's method [Sharp 2019]. We will first illustrate the method on a short example, and then apply it to the full set of equations.

Assume we are given the boundary-value problem $u'' = g$ with $u(0) = 0$ and $u'(1) = 1$, and are interested in how the quantity $F = \int_0^1 u^2$ changes with g. That is, we want to identify $\delta F[g; \delta g]$. The goal of the adjoint method is to avoid computing intermediate quantities such as δu in the process.

Céa's method achieves this by rewriting F as

$$F = \int_0^1 u^2 + \int_0^1 \bar{u}(u'' - g),$$

in which we can choose the function $\bar{u}$ arbitrarily without changing F, because $u'' = g$. If we apply the δ-operator to F, this will result in the appearance of terms involving δu at first. But, as we will see shortly, $\bar{u}$ can be chosen to cancel out δu, which simplifies the computation of δF.

The variation of F is given by

$$
\begin{aligned}
\delta F &= \int_0^1 (2u\,\delta u + \bar{u}(\delta u'' - \delta g)) \\
&= (\bar{u}\,\delta u' - \bar{u}'\delta u)\big|_0^1 + \int_0^1 \left((2u + \bar{u}'')\,\delta u - \bar{u}\,\delta g\right) \\
&= -\bar{u}'(1)\,\delta u(1) - \bar{u}(0)\,\delta u'(0) + \int_0^1 \left((2u + \bar{u}'')\,\delta u - \bar{u}\,\delta g\right),
\end{aligned}
$$

where he have first used integration by parts twice, and then the boundary conditions $\delta u(0) = 0$ and $\delta u'(1) = 0$. We can see that choosing $\bar{u}$ as the solution to the *adjoint equation*

$$\bar{u}'' = -2u, \quad \text{s.t.} \quad \bar{u}(0) = 0, \quad \bar{u}'(1) = 0,$$

simplifies $\delta F = -\int_0^1 \bar{u}\,\delta g$, and δu no longer appears. This means that we can compute δF by first solving the adjoint equation for $\bar{u}$, and then integrating it against δg. Equivalently, $-\bar{u}$ is the L^2-gradient of F, because $\delta F = \langle -\bar{u}, \delta g \rangle_{L^2}$.

The general recipe of Céa's method is to append to the objective function F a sum of inner products between quantities known to be zero (expressions involving differential operators, constraints, etc.) and adjoint variables, which we denote by overbars. For differential equations, such as $u'' = g$, adjoint variables take the form of functions, and the L^2-inner product is used. For integral constraints, adjoint variables are scalars, and we use the standard Euclidean inner product. Then, we manipulate δF in order to isolate the adjoint equations, which we can solve in order to simplify δF.

2.3 The Adjoint Equations

In the following, indices always run from 1 to 2, and integrals from 0 to l. We introduce the adjoint variables $\bar{\alpha}$, $\bar{\lambda}$, $\bar{\zeta}$, $\bar{\eta}_i$, $\bar{M}_i$, and $\bar{N}_{ij}$, corresponding to lines 1–6 of the primal equations. Furthermore, denote by $C_{(\cdot)}$ the cofactors of Z, e.g., C_ζ is the cofactor associated with the top-left entry ζ in the matrix.

The variation of F, before appending additional terms, reads

$$\delta F = \int \varphi \left(C_\zeta \delta\zeta + \textstyle\sum_i C_{\eta_i} \delta\eta_i + \sum_i C_{M_i} \delta M_i + \sum_{ij} C_{N_{ij}} \delta N_{ij} \right).$$

Next, let us account for the adjoint terms resulting from expressions of the form $-(Ku')'$, where $u \in \{\alpha, \zeta, \eta_i\}$. This results in

$$
\begin{aligned}
\delta \textstyle\int -\bar{u}(Ku')' &= \left(-\bar{u}u'\delta K + K\bar{u}'\delta u - K\bar{u}\,\delta u'\right)\big|_0^l \\
&\quad + \textstyle\int (\bar{u}'u'\delta K - (K\bar{u}')'\delta u).
\end{aligned}
$$

After considering the boundary conditions on α, ζ, and η_i, we get

$$
\begin{aligned}
\delta \textstyle\int -\bar{\alpha}(K\alpha')' &= \left(-\bar{\alpha}\alpha'\delta K - K\bar{\alpha}\,\delta\alpha'\right)\big|_0^l \\
&\quad + \textstyle\int (\bar{\alpha}'\alpha'\delta K - (K\bar{\alpha}')'\delta\alpha), \\
\delta \textstyle\int -\bar{\zeta}(K\zeta')' &= -\bar{\zeta}\zeta'\delta K\big|_0^l + \left(K\bar{\zeta}'\delta\zeta - K\bar{\zeta}\,\delta\zeta'\right)\big|_l \\
&\quad + \textstyle\int \left(\bar{\zeta}'\zeta'\delta K - (K\bar{\zeta}')'\delta\zeta\right), \\
\delta \textstyle\int -\bar{\eta}_i(K\eta_i')' &= -\bar{\eta}_i\eta_i'\delta K\big|_0^l + \left(K\bar{\eta}_i'\delta\eta_i - K\bar{\eta}_i\,\delta\eta_i'\right)\big|_l \\
&\quad + \textstyle\int \left(\bar{\eta}_i'\eta_i'\delta K - (K\bar{\eta}_i')'\delta\eta_i\right).
\end{aligned}
$$

The remaining term in line 1 and the equality constraint in line 2 yield

$$\begin{aligned}\delta \int \bar{\alpha}\langle\lambda, R\gamma'\rangle &= \int \bar{\alpha}\left(\langle\delta\lambda, R\gamma'\rangle - \langle\lambda, \gamma'\delta\alpha\rangle\right)\\ &= \langle\delta\lambda, \int R\gamma'\bar{\alpha}\rangle - \int \bar{\alpha}\langle\lambda, \gamma'\rangle\delta\alpha,\\ \delta\langle\bar{\lambda}, \int \gamma' - \gamma_l\rangle &= \int\langle\bar{\lambda}, R\gamma'\rangle\delta\alpha.\end{aligned}$$

The remaining term in line 3 yields

$$\delta \int -\bar{\zeta}\langle\lambda, \gamma'\rangle\zeta = -\langle\delta\lambda, \int \bar{\zeta}\zeta\gamma'\rangle - \int \bar{\zeta}\zeta\langle\lambda, R\gamma'\rangle\delta\alpha - \int \bar{\zeta}\langle\lambda, \gamma'\rangle\delta\zeta,$$

and the ones in line 4,

$$\begin{aligned}&\delta \int -\bar{\eta}_i\left(\langle\lambda, \gamma'\rangle\eta_i + \langle R\gamma', e_i\rangle\right)\\ &= -\langle\delta\lambda, \int \bar{\eta}_i\eta_i\gamma'\rangle + \int \bar{\eta}_i\langle Re_i - \lambda\eta_i, R\gamma'\rangle\delta\alpha - \int \bar{\eta}_i\langle\lambda, \gamma'\rangle\delta\eta_i.\end{aligned}$$

Similarly, for lines 5 and 6,

$$\begin{aligned}&\delta \int \bar{M}_i\left(M_i' - \langle R\gamma', e_i\rangle\zeta\right)\\ &= \bar{M}_i\,\delta M_i\big|_l - \int \bar{M}_i'\delta M_i + \int \bar{M}_i\zeta\langle\gamma', e_i\rangle\delta\alpha - \int \bar{M}_i\langle R\gamma', e_i\rangle\delta\zeta,\end{aligned}$$

$$\begin{aligned}&\delta \int \bar{N}_{ij}\left(N_{ij}' - \langle R\gamma', e_i\rangle\eta_j\right)\\ &= \bar{N}_{ij}\,\delta N_{ij}\big|_l - \int \bar{N}_{ij}'\delta N_{ij} + \int \bar{N}_{ij}\eta_j\langle\gamma', e_i\rangle\delta\alpha - \int \bar{N}_{ij}\langle R\gamma', e_i\rangle\delta\eta_j.\end{aligned}$$

Next, we gather the expressions multiplying variations that we do not want to evaluate, i.e., $\delta\alpha$, $\delta\lambda$, $\delta\zeta$, $\delta\eta_i$, δM_i, and δN_{ij}. We will do this process in reverse order, because this corresponds to the natural order in which the adjoint equations need to be solved. For δM_i and δN_{ij}, we have

$$\begin{aligned}\bar{M}_i' &= \varphi C_{M_i}, & \bar{M}_i(l) &= 0,\\ \bar{N}_{ij}' &= \varphi C_{N_{ij}}, & \bar{N}_{ij}(l) &= 0.\end{aligned}$$

For $\delta\zeta$ and $\delta\eta_i$, we have

$$\begin{aligned}-(K\bar{\zeta}')' - \langle\lambda, \gamma'\rangle\bar{\zeta} &= \bar{M}_i\langle R\gamma', e_i\rangle - \varphi C_\zeta, & \bar{\zeta}(l) = 0, \quad \bar{\zeta}'(l) = 0,\\ -(K\bar{\eta}_i')' - \langle\lambda, \gamma'\rangle\bar{\eta}_i &= \bar{N}_{ij}\langle R\gamma', e_i\rangle - \varphi C_{\eta_i}, & \bar{\eta}_i(l) = 0, \quad \bar{\eta}_i'(l) = 0.\end{aligned}$$

Collecting terms multiplying $\delta\alpha$ yields

$$\begin{aligned}(K\bar{\alpha}')' = &-\bar{\alpha}\langle\lambda, \gamma'\rangle + \langle\bar{\lambda}, R\gamma'\rangle\\ &+ (\bar{\zeta}\zeta + \textstyle\sum_i \bar{\eta}_i\eta_i)\langle R\lambda, \gamma'\rangle\\ &+ \textstyle\sum_i(\bar{\eta}_i + \bar{M}_i\zeta + \sum_j \bar{N}_{ij}\eta_j)\langle\gamma', e_i\rangle, \quad \bar{\alpha}(0) = 0,\ \bar{\alpha}(l) = 0,\end{aligned}$$

and for $\delta\lambda$, the two-component equation

$$\int R\gamma'\bar{\alpha} = \int(\bar{\zeta}\zeta + \textstyle\sum_i \bar{\eta}_i\eta_i)\gamma'.$$

Putting the equations for $\delta\alpha$ and $\delta\lambda$ together, we see that they correspond to the constrained Euler–Lagrange equations of a quadratic variational problem with linear integral constraints in $\bar{\alpha}$, where $\bar{\lambda}$ is used as a Lagrange multiplier:

$$\begin{aligned}&\int_0^l \frac{1}{2}\left(K\bar{\alpha}'^2 - \langle\lambda, \gamma'\rangle\bar{\alpha}^2\right)\\ &\quad + \left[(\bar{\zeta}\zeta + \textstyle\sum_i \bar{\eta}_i\eta_i)\langle R\lambda, \gamma'\rangle\right.\\ &\qquad \left.+ \textstyle\sum_i(\bar{\eta}_i + \bar{M}_i\zeta + \sum_j \bar{N}_{ij}\eta_j)\langle\gamma', e_i\rangle\right]\bar{\alpha}\\ \text{s.t.}\quad &\begin{matrix}\bar{\alpha}(0) = 0,\\ \bar{\alpha}(l) = 0,\end{matrix} \quad \text{and} \quad \int_0^l R\gamma'\bar{\alpha} = \int_0^l(\bar{\zeta}\zeta + \textstyle\sum_i \bar{\eta}_i\eta_i)\gamma'.\end{aligned}$$

Finally, the variational derivative of F can be assembled from all terms involving δK. This gives

$$\delta F[K;\delta K] = \left(\bar{\zeta} + \textstyle\sum_i \bar{\eta}_i\right)\delta K\big|_0 + \int_0^l\left(\bar{\alpha}'\alpha' + \bar{\zeta}'\zeta' + \textstyle\sum_i \bar{\eta}_i'\eta_i'\right)\delta K.$$

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