Non co-preservation of the $1/2$ & $1/(2l + 1)$–rational caustics along deformations of circles

V. Kaloshin[*], C. E. Koudjinan[†]

May 30, 2021

Abstract

For any given positive integer l, we prove that every plane deformation of a circle which preserves the $1/2$ and $1/(2l+1)$–rational caustics is trivial *i.e.* the deformation consists only of similarities (rescalings and isometries).

MSC2020 numbers: 37C83, 37E40, 37J51, 70H09

Keywords: Birkhoff billiard, caustics, perturbation theory, analytic functions, smooth deformation of circle.

1   Notations

- $\nabla f(t) := f(t) - f(t - 2\pi/3)$

- $p := p(t)$, $\dot{p} := \dot{p}(t)$, $p^- := p(t^-)$, $\dot{p}^- := \dot{p}(t^-)$, $p^+ := p(t^+)$, $\dot{p}^+ := \dot{p}(t^+)$

- $\mathscr{F}_I(f)(t) := \sum_{k \in I} f_k \, e^{ikt}$, the projection on the Fourier's modes k in $I \subseteq \mathbb{Z}$. If $I = n\mathbb{Z}$ for some $n \in \mathbb{Z}$, we write $\mathscr{F}_n(f) = \mathscr{F}_{n\mathbb{Z}}(f)$.

- Denote by $C_\rho^w(\mathbb{T}, \mathbb{C})$, the set of analytic function on the strip
$$\mathbb{T}_\rho := \{z \in \mathbb{C} \ : \ |\operatorname{Im} z| < \rho\}/2\pi\mathbb{Z},$$
  endowed with the sup–norm $\|f\|_\rho := \sup_{\mathbb{T}_\rho} |f|$.

[*]Vadim Kaloshin: University of Maryland, College Park, MD, USA & Institute of Science and Technology Austria, Am Campus 1, 3400 Klosterneuburg, Austria. *vadim.kaloshin@gmail.com*

[†]Comlan Edmond Koudjinan: Institute of Science and Technology Austria (IST Austria), Am Campus 1, 3400 Klosterneuburg, Austria. *edmond.koudjinan@ist.ac.at*

2   Introduction

A billiard is a mathematical modeling of the dynamic of a confined massless particle without friction and reflecting elastically on the boundary (without friction): the particle moves along a straight line with constant speed till it hits the boundary, then reflects off with reflection angle equals to the angle of incidence and follows the reflected straight line. It was introduced by G.D. Birkhoff [Bir20] in 1920.
A key notion in billiard dynamic is that of caustic.

Definition 1 *A caustic of the billiard dynamic in a domain Ω is a curve $\mathfrak{C}$ with the property that any billiard trajectory that is once tangent to $\mathfrak{C}$ stays tangent to $\mathfrak{C}$ after each reflection on the boundary.*

Mather [Mat82] proves the non–existence of caustics if the curvature of the boundary vanishes at one point. Thus, as far as caustics are concerned, we can focus on billiards in strictly convex domains; such billiards with at least C^3–boundary will be called Birkhoff billiards.[1] However, a caustic, if it exists, need neither be convex nor differentiable.
Nevertheless, according to KAM Theory [Laz73, KP90], a positive measure set of convex differentiable caustics which accumulates on the boundary and on which the motion is smoothly conjugate to a rigid rotation do exists for Birkhoff billiards provided the boundary of the domain is sufficiently smooth. in general, the billiard dynamic induces naturally an orientation preserving circle homeomorphism on each convex caustic, which in particular admits a rotation number, also called rotation number of the caustic. In particular, a caustic is called rational (*resp.* irrational) if its rotation number is rational (*resp.* irrational). In this work, we are mainly concerned with rational caustics.

Definition 2 *Given $m, n \in \mathbb{N}$ with $m \geqslant 2$, we call a caustic n/m–**rational** if all the corresponding tangential billiards trajectories are periodic with the rotation number n/m. We denote by $\Gamma_{n/m}(\Omega)$ the collection of all the n/m–rational caustics of Ω.*

unlike irrational caustics which tends to be robust under perturbation according to KAM Theory, rational caustics tends to be quite rigid and, therefore, break up under perturbation. All the rational caustics may be even destroyed as show by Pinto-de-Carvalho and Ramírez-Ros [PdCRR13] who proved that can perturb an elliptic billiard and destroy all its rational caustics.
In contrast, Kaloshin and Ke Zhang [KZ18] proved that can perturb a Birkhoff billiard table and create a new $1/q$–rational caustic for sufficiently larges q, provide the boundary

---

[1]Observe that if Ω is not convex, then the billiard map is not continuous; in this article we will be interested only in strictly convex domains. Moreover, as pointed out by Halpern [Hal77], if the boundary is not at least C^3, then the flow might not be complete.

of the table is C^r with $r > 4$. However, nothing is known for smalls q. Moreover, it is not also known if one can always perturb a sufficiently smooth Birkhoff billiard table and creates, simultaneously, many rational caustics.

On the other side, a natural question is:

Question 1 *Can one perturb a sufficiently smooth Birkhoff billiard table and (co–)preserve many of its rational caustics?*

The question is still widely open, even in the simplest case of co–preservation of two rational caustics. Actually, the following intriguing conjecture has been made by Tabachnikov over ten years ago:

Conjecture 3 (S. Tabachnikov) *In a sufficiently small C^r ($r = 2, \cdots, \infty, w$) neighborhood of the circle there is no other billiard domain of constant width and preserving $1/3$–caustics.*

It has been proven by J. Zhang [Zha19] that in the class of $\mathbb{Z}_2$–symmetric analytic deformation of the circle with certain Fourier decaying rate, any deformation of the circle that co–preserves $1/2$ and $1/3$-rational caustics is necessarily an isometric transformation.

In the present paper, we settle the analytic deformative case of Conjecture 3. We prove that, for any given positive integer l, if an analytic deformation of the circle co–preserves $1/2$ and $1/(2l + 1)$–rational caustics then this deformation is trivial *i.e.* consists only of circles (see Theorem 9 below).

A bounded convex planar domain may be parametrized in various way, amongst which we have the parametrization with support function.

3   Support function and some facts

Given a bounded convex planar domain Ω with C^1 boundary $\partial\Omega$ such that the origin of the cartesian coordinates is in the interior of Ω, we denote by $p_\Omega \colon [0, 2\pi) \to [0, \infty)$ the support function of $\partial\Omega$. Denoting by $(x(t), y(t))$ the cartesian coordinates of the point on $\partial\Omega$ corresponding to $(t, p_\Omega(t))$, we have[2]

$$\begin{cases} x(t) = p_\Omega(t) \cos t - \dot{p}_\Omega(t) \sin t \\ y(t) = p_\Omega(t) \sin t + \dot{p}_\Omega(t) \cos t, \end{cases} \tag{1}$$

where $\dot{p}_\Omega$ denotes the derivative of p_Ω.

---

[2]We refer the reader to [Res15] for more details.

Given a supporting function p, we associate the generating function, denoted by $\mathscr{E}_p$, of the billiard map in the corresponding domain and given by

$$\mathscr{E}_p(t,t^+) := \sqrt{(x(t) - x(t^+)^2 + (y(t) - y(t^+)^2)} \overset{(1)}{=} \left(p^2 + \dot{p}^2 + p^{+2} + (\dot{p}^+)^2 - 2pp^+ \cos(t - t^+) - \right.$$
$$\left. - 2p\dot{p}^+ \sin(t - t^+) + 2\dot{p}p^+ \sin(t - t^+) - 2\dot{p}\dot{p}^+ \cos(t - t^+)\right)^{1/2}$$

$$(2)$$

We have the following nice characterization of convex domains with $1/2$–rational caustics:

Lemma 4 *A bounded convex domain Ω with C^0 boundary posseses a 2–periodic caustic iff its support function p is of constant width:*

$$p(t) = \frac{\omega}{2} + \sum_{k\in\mathbb{Z}} p^{(k)} e^{i(2k+1)t}, \qquad t \in \mathbb{T},$$

where ω is the average width of Ω and $\{p^{(k)}\}_{k\in\mathbb{Z}} \subseteq \mathbb{C}$.

The following error–function is the basis of the Lagrangian alternative approach proposed by Moser and Levi.

Definition 5 *Given a bounded convex domain Ω with C^1 boundary and support function p, $u \in C^0(\mathbb{T})$ and $m \in \mathbb{N}$, we set*

$$E^m(p,u) := \partial_1 \mathscr{E}_p(u,u^+) + \partial_2 \mathscr{E}_p(u^-,u),\tag{3}$$

where $u^\pm(t) := u(t \pm \frac{2\pi}{m})$. For the sake of simplicity, we shall write E for E^3.

Following Moser–Levi[LM01], we have the characterization:

Lemma 6 *Given $m \in \mathbb{N}\backslash\{1\}$, a bounded convex domain $\Omega \subseteq \mathbb{R}^2$ whose support function $p \in C^1(\mathbb{T})$ admits a $1/m$–periodic caustic iff there is a homeomorphism $u\colon \mathbb{T} \to \mathbb{T}$ such that*

$$E^m(p,u) = \partial_1 \mathscr{E}_p(u,u^+) + \partial_2 \mathscr{E}_p(u^-,u) = 0.\tag{4}$$

Let $\Omega_0 := \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leqslant 1\}$ be the unit disc and consider the one–parameter family Ω_ε of deformation of Ω_0 such that

$$p_{\Omega_\varepsilon}(t) = 1 + \varepsilon p_1(t) + O(\varepsilon^2), \qquad \text{for some} \quad p_1 \in C^1(\mathbb{T}),$$

and let $\mathscr{D} := \{\Omega_\varepsilon : \varepsilon \geqslant 0\}$.

Remark 7 For any $\lambda > 0$, the generating function of the disc $\lambda\Omega_0$ of radius λ is $\mathscr{E}_\lambda(t, t^+) = \lambda \cdot \sqrt{2(1 - \cos(t - t^+))}$. Thus, for any $m \in \mathbb{N}$,

$$E^m(\lambda, \mathtt{id}) = 0,$$

i.e. $\lambda\Omega_0$ possesses a $1/m$–rational caustic.

The following extends Lemma 4 to all natural numbers and is contained in Ramirez–Ros[RR06]. We provide in §5 an alternative proof.

Theorem 8 *Let $m \in \mathbb{N}$ with $m \geqslant 2$ and $\varepsilon > 0$. Assume Ω_ε admits a $1/m$–rational caustic. Then $p_{1,km} = 0$ for all $k \in \mathbb{Z}\backslash\{0\}$.*
Consequently, the set of $\Omega \in \mathscr{D}$ having a $1/m$–rational caustics is a submanifold of $\mathscr{D}$ of infinite codimension.

4 Main result

Denote by $\mathscr{C}$ the set of deformations Ω_ε of the unit disc Ω_ε within the class of strictly convex plane domains, whose support function $p_\varepsilon \in C^3(\mathbb{T})$ and is of the form: $p_\varepsilon(t) = 1 + \varepsilon\,p_1 + O(\varepsilon^2)$ with $p_1 \in C_\rho^w(\mathbb{T}, \mathbb{R})$, for some $\rho > 0$. Then, the following holds.

Theorem 9 *Let $l \in \mathbb{N}$ and $\Omega_\varepsilon \in \mathscr{C}$ be a deformation of the unit disc. Assume, there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in [0, \varepsilon_0)$, Ω_ε possesses a $1/2$–rational and a $1/(2l+1)$–rational caustic. Then, the deformation Ω_ε is trivial: Ω_ε is a disc for any $\varepsilon \in [0, \varepsilon_0)$.*

5 Proof of Theorem 8

The proof of Theorem 8 will be deduce from the following Lemma.

Lemma 10 *Let $m > 2$, $p_1^* := \sum_{k\in\mathbb{Z}} p_{1,k}^* e^{ikt} \in C^2(\mathbb{T})$ and $u_1^* := \sum_{k\in\mathbb{Z}} u_{1,k}^* e^{ikt} \in L^2(\mathbb{T})$. Then,*

$$E^m(1 + \varepsilon p_1^*, \mathtt{id} + \varepsilon u_1^*) = O(\varepsilon^2), \tag{5}$$

iff for any $k \in \mathbb{Z}\backslash m\mathbb{Z}$,

$$u_{1,k}^* = a_{m,k}\,p_{1,k}^* \quad and \quad \mathscr{F}_{m\mathbb{Z}\backslash\{0\}}(p_1^*) = 0, \tag{6}$$

where

$$a_{m,k} := i\left(k\cot^2\left(\frac{\pi k}{m}\right) - \cot\left(\frac{\pi}{m}\right)\cot\left(\frac{\pi k}{m}\right)\right).$$

5

Proof By Lemma A.1 (see §A.1), $E^m(1 + \varepsilon p_1^*, \mathtt{id} + \varepsilon u_1^*) = O(\varepsilon^2)$ *iff*

$$\sin\left(\frac{\pi}{m}\right)(\dot{p}_1^+ + \dot{p}_1^- + 2\dot{p}_1 + u_1^+ + u_1^- - 2u_1) + (p_1^- - p_1^+)\cos\left(\frac{\pi}{m}\right) = 0$$

i.e. for any $k \in \mathbb{Z}$,

$$-2i\sin\left(\frac{\pi}{m}\right)\sin^2\left(\frac{\pi k}{m}\right)u_{1,k} = \left(2k\sin\left(\frac{\pi}{m}\right)\cos^2\left(\frac{\pi k}{m}\right) - \cos\left(\frac{\pi}{m}\right)\sin\left(\frac{2\pi k}{m}\right)\right)p_{1,k},$$

which, in turn, is equivalent to (6) as, for all $k \in m\mathbb{Z}\backslash\{0\}$,

$$2k\sin\left(\frac{\pi}{m}\right)\cos^2\left(\frac{\pi k}{m}\right) - \cos\left(\frac{\pi}{m}\right)\sin\left(\frac{2\pi k}{m}\right) = 2k\sin\left(\frac{\pi}{m}\right) \neq 0.$$

∎

Proof of Theorem 9 By assumption and Lemma 6, there exists $u_1 \in C^0(\mathbb{T})$ such that $E^m(1 + \varepsilon p_1 + O(\varepsilon^2), \mathtt{id} + \varepsilon u_1) = 0$. Hence, $0 = E^m(1 + \varepsilon p_1 + O(\varepsilon^2), \mathtt{id} + \varepsilon u_1) = E^m(1 + \varepsilon p_1, \mathtt{id} + \varepsilon u_1) + O(\varepsilon^2)$. Thus, $E^m(1 + \varepsilon p_1, \mathtt{id} + \varepsilon u_1) = O(\varepsilon^2)$. Now, applying Lemma 10, we obtain $\mathscr{F}_{m\mathbb{Z}\backslash\{0\}}(p_1) = 0$. ∎

6 Proof of Theorems 9

We start setting up some notation. We shall denote

- $t^{\pm} := t \pm \dfrac{2\pi}{3}, \quad p_n^{\pm} := p_n(t^{\pm}), \quad u_n^{\pm} := u_n(t^{\pm}), \quad P_N := \displaystyle\sum_{n=0}^{N}\varepsilon^n p_n \quad U_N := \sum_{n=0}^{N}\varepsilon^n u_n,$

where $p_0 := 1$ and $u_0 := \mathtt{id}$.

- $E^m(P_N, U_N) = \displaystyle\sum_{k=-N}^{\infty} E_{N,k}^m\, \varepsilon^{N+k}$

where

$$E_{N,k}^m := \frac{1}{(N+k)!}\frac{d^{N+k}}{d\varepsilon^{N+k}}E^m(P_N, U_N)\bigg|_{\varepsilon=0}.$$

For $m = 3$, here and henceforth, we will drop the superscript m and write E for E^3. The following Lemma will be needed.

Lemma 11 *Let $f \in C_\rho^\omega(\mathbb{T}, \mathbb{C})$, for some $\rho > 0$. If*

$$\sum_{k \in \mathbb{Z}} f_k \overline{f_{k-n}} = 0, \quad \text{for all } n \in \mathbb{Z} \backslash \{0\}, \tag{7}$$

then $f \equiv f_0$.

Proof Set $g(z) := f(z)\overline{f(z)}$ and consider the usual scalar product on $L^2(\mathbb{T})$:

$$\langle u, v \rangle := \int_{\mathbb{T}} u\bar{v}.$$

Fix $0 < \rho' < \rho$. Then, for all $k \in \mathbb{Z}$,

$$|f_k| \leq \|f\|_\rho \, e^{-\rho|k|}, \tag{8}$$

so that,

$$\sup_{z \in \mathbb{T}_{\rho'}} \sum_{k \in \mathbb{Z}} |f_k \, e^{ikz}| \leq \sum_{k \in \mathbb{Z}} |f_k| \, e^{\rho'|k|} \overset{(8)}{\leq} \|f\|_\rho \sum_{k \in \mathbb{Z}} e^{(\rho'-\rho)|k|} < \infty. \tag{9}$$

Thus,

$$f(z) = \sum_{k \in \mathbb{Z}} f_k \, e^{ikz}, \qquad \text{on} \qquad \mathbb{T}_{\rho'}, \tag{10}$$

and, therefore,

$$g(z) = \sum_{k \in \mathbb{Z}} g_k \, e^{ikz}, \qquad \text{on} \qquad \mathbb{T}_{\rho'}. \tag{11}$$

Moreover, for any $n \in \mathbb{Z} \backslash \{0\}$,

$$\begin{aligned}
g_n &= \left\langle g, \, e^{int} \right\rangle \\
&= \int_{\mathbb{T}} \left(\sum_{k \in \mathbb{Z}} f_k \, e^{ikt} \right) \left(\sum_{m \in \mathbb{Z}} \overline{f}_m \, e^{-imt} \right) e^{-int} \, dt \\
&= \sum_{k \in \mathbb{Z}} f_k \overline{f_{k-n}} \\
&\overset{(7)}{=} 0.
\end{aligned}$$

Consequently, $g \overset{(11)}{\equiv} 0$ on $\mathbb{T}_{\rho'}$ *i.e.* $|f|^2 \big|_{\mathbb{T}_{\rho'}} \equiv g_0$, and this holds for all $0 < \rho' < \rho$. Thus, $|f|^2 \equiv g_0$ on $\mathbb{T}_\rho$. Then, the open mapping theorem yields $f \equiv f_0$. ∎

Proof of Theorem 9
• **Case** $n = 1$: We argue by contradiction. Let $\Omega \in \mathscr{D}_{2,3}$ with support function $p(t) =$

$1 + \varepsilon\, p_1 + \varepsilon^2 p_2 + O(\varepsilon^3)$, where $p_1 \in C_{\tilde{\rho}}^w(\mathbb{T}, \mathbb{R})$, $p_2 \in C^3(\mathbb{T})$, for some $\tilde{\rho} > 0$ and for ε close to 0. Without loss of generality, we can assume that

$$p_{1,-1} = p_{1,1} = 0, \qquad \text{and} \qquad p_1 \not\equiv 0. \tag{12}$$

Then, by Lemma 6, there exists $u(t) = t + \sum_{n=1}^{\infty} \varepsilon^n u_n(t)$, with $\{u_n\}_{n \geqslant 1} \subseteq C^0(\mathbb{T})$ such that

$$E(p, u) = 0. \tag{13}$$

Also, observe that, by Lemma 4, we have[3]

$$\mathscr{F}_{2\mathbb{Z}}(p_n) = 0, \qquad \forall\, n \geqslant 1. \tag{14}$$

We have $0 = E(p, u) = E(1 + \varepsilon p_1, \mathtt{id} + \varepsilon u_1) + O(\varepsilon^2)$, so that $E(1 + \varepsilon p_1, \mathtt{id} + \varepsilon u_1) = O(\varepsilon^2)$. Thus, by Lemma 10, we have, for any $k \in \mathbb{Z} \backslash 3\mathbb{Z}$,

$$u_{1,k} = a_{3,k}\, p_{1,k} \quad \text{and} \quad \mathscr{F}_{3\mathbb{Z}\backslash\{0\}}(p_1) = 0. \tag{15}$$

Therefore, Lemma A.1 yields

$$E(1 + \varepsilon p_1, \mathtt{id} + \varepsilon u_1) = E_{1,1}\, \varepsilon^2 + O(\varepsilon^3). \tag{16}$$

Now, using Lemma A.2, we have

$$E(P_2, U_2) \overset{(A.8)}{=} E(P_1, U_1) + \widetilde{E}_{2,0}\, \varepsilon^2 + O(\varepsilon^3) \overset{(16)}{=} (E_{1,1} + \widetilde{E}_{2,0})\varepsilon^2 + O(\varepsilon^3), \tag{17}$$

where

$$\widetilde{E}_{2,0} = \frac{1}{4}(-p_2^+ + p_2^-) + \frac{\sqrt{3}}{4}(\dot{p}_2^+ + \dot{p}_2^- + 2\dot{p}_2) + \frac{\sqrt{3}}{4}(u_2^+ + u_2^- - 2u_2)\,. \tag{18}$$

Hence,

$$0 = E(u, p) = E(P_2, U_2) + O(\varepsilon^3) \overset{(17)}{=} (E_{1,1} + \widetilde{E}_{2,0})\varepsilon^2 + O(\varepsilon^3), \tag{19}$$

which implies

$$E_{1,1} + \widetilde{E}_{2,0} = 0. \tag{20}$$

Thus, $\mathscr{F}_6(E_{1,1}) = -\mathscr{F}_6(\widetilde{E}_{2,0}) \overset{(18),(14)}{=} 0$. Then, specializing (A.2) to $m = 1$, we obtain, for all $n \in \mathbb{Z}\backslash\{0\}$,

$$\sum_{k \in \mathbb{Z}} k(k - n)p_{1,6k+1}\, p_{1,6(n-k)-1} = 0. \tag{21}$$

---

[3]By making the normalization $\mathscr{F}_0(p_n) = 0$, $n \geqslant 1$.

Now, consider the auxiliary function $f(z) := \sum_{k \in \mathbb{Z}} f_k \, e^{ikz}$ with $f_k := k \, p_{1,6k+1}$. Then $f \in C_\rho^\omega(\mathbb{T}, \mathbb{R})$, where $\rho := \tilde{\rho}/2$. Moreover, as $\overline{f_{k-n}} := (k-n) \, \overline{p_{1,6(k-n)+1}} = (k-n) \, p_{1,6(n-k)-1}$, the last relation in (21) then reads: $\sum_{k \in \mathbb{Z}} f_k \, \overline{f_{k-n}} = 0$, for all $n \in \mathbb{Z}\backslash\{0\}$. Therefore, Lemma 11 yields $f \equiv f_0 = 0$ *i.e.* $p_{1,6k+1} = 0$ for all $k \in \mathbb{Z}\backslash\{0\}$. But then, as $p_{1,-1} = p_{1,1} \overset{(12)}{=} 0$, we would get $p_1 \equiv 0$, which contradicts (12).

• **General case** $n \in \mathbb{N}$ **:** The proof in the general case is completely identical, up to two minor adjustments. The first one is the Cohomological equation (21) which, according to (A.2), is in general:

$$-\frac{i}{16} \sum_{k \in \mathbb{Z}} \sum_{r=1}^{4m+1} \mathscr{P}_r^m(n,k) \, p_{1,2(2m+1)k+r} \, p_{1,2(2m+1)(n-k)-r} = 0, \quad \forall \, n \in \mathbb{Z}\backslash\{0\}. \qquad (22)$$

But, each of the polynomials $\mathscr{P}_r^m(n,k)$ splits:

$$\mathscr{P}_r^m(n,k) = -16i \, (c_{m,r}^*)^2 (k - c_{m,r}^{**})(n - k + c_{m,r}^{**}),$$

where

$$c_{m,r}^* = \sqrt{(2m+1)^3 \sin\left(\frac{\pi}{2m+1}\right) \cot^2\left(\frac{\pi r}{2m+1}\right)} \quad \text{and} \quad c_{m,r}^{**} := \frac{\cot\left(\frac{\pi}{2m+1}\right) \tan\left(\frac{\pi r}{2m+1}\right) - r}{4m+2}.$$

Hence, the auxiliary function $f(z) := \sum_{k \in \mathbb{Z}} f_k \, e^{ikz}$ should be defined by:[4] $f_k := c_{m,r}^*(k - z_{m,r}) \, p_{1,6k+r}$. ∎

Appendix

A Reccurent formula for p_n and u_n and Taylor's series expansion of $E^m(P_N, U_N)$

A.1 Expansion of $E^m(P_1, U_1)$

Let $p_1 := \sum_{k \in \mathbb{Z}} p_{1,k} \, e^{ikt} \in C^2(\mathbb{T})$ and $u_1 := \sum_{k \in \mathbb{Z}} u_{1,k} \, e^{ikt} \in C^0(\mathbb{T})$. Set $P_1 := 1 + \varepsilon p_1$, $U_1 := \mathtt{id} + \varepsilon u_1$ and $U_1^\pm(t) := U_1(t \pm \frac{2\pi}{m})$.

Lemma A.1 *Given any integer* $m \geqslant 2$, *we have*

$$E^m(1 + \varepsilon p_1, \mathtt{id} + \varepsilon u_1) = E_{1,0}^m \, \varepsilon + E_{1,1}^m \, \varepsilon^2 + O(\varepsilon^3), \qquad (A.1)$$

with

---

[4]This is the second adjustment.

$$E_{1,0}^m := \frac{1}{2}\left(\sin\left(\frac{\pi}{m}\right)(\dot{p}_1^+ + \dot{p}_1^- + 2\dot{p}_1 + u_1^+ + u_1^- - 2u_1) + (p_1^- - p_1^+)\cos\left(\frac{\pi}{m}\right)\right),$$

$$E_{1,1}^m := -\frac{\csc^2\left(\frac{\pi}{m}\right)}{32}\left(-5\cos\left(\frac{\pi}{m}\right)(p_1^+)^2 + \cos\left(\frac{3\pi}{m}\right)(p_1^+)^2 + 6\sin\left(\frac{\pi}{m}\right)u_1 p_1^+ - 2\sin\left(\frac{3\pi}{m}\right)u_1 p_1^- -\right.$$

$$-6\sin\left(\frac{\pi}{m}\right)u_1^+ p_1^+ + 2\sin\left(\frac{3\pi}{m}\right)u_1^+ p_1^+ + 10\sin\left(\frac{\pi}{m}\right)\dot{p}_1 p_1^+ + 2\sin\left(\frac{3\pi}{m}\right)\dot{p}_1 p_1^+ +$$

$$+6\sin\left(\frac{\pi}{m}\right)\dot{p}_1^+ p_1^+ - 2\sin\left(\frac{3\pi}{m}\right)\dot{p}_1^+ p_1^+ + 4\cos\left(\frac{\pi}{m}\right)\ddot{p}_1 p_1^+ - 4\cos\left(\frac{3\pi}{m}\right)\ddot{p}_1 p_1^+ -$$

$$-\left(\cos\left(\frac{3\pi}{m}\right) - 5\cos\left(\frac{\pi}{m}\right)\right)(p_1^-)^2 - \cos\left(\frac{\pi}{m}\right)(u_1^+)^2 + \cos\left(\frac{3\pi}{m}\right)(u_1^+)^2 +$$

$$+\cos\left(\frac{\pi}{m}\right)(u_1^-)^2 - \cos\left(\frac{3\pi}{m}\right)(u_1^-)^2 + \cos\left(\frac{\pi}{m}\right)(\dot{p}_1^+)^2 - \cos\left(\frac{3\pi}{m}\right)(\dot{p}_1^+)^2 -$$

$$-\cos\left(\frac{\pi}{m}\right)(\dot{p}_1^-)^2 + \cos\left(\frac{3\pi}{m}\right)(\dot{p}_1^-)^2 + 2\cos\left(\frac{\pi}{m}\right)u_1 u_1^+ - 2\cos\left(\frac{3\pi}{m}\right)u_1 u_1^+ -$$

$$-2\cos\left(\frac{\pi}{m}\right)u_1 u_1^- + 2\cos\left(\frac{3\pi}{m}\right)u_1 u_1^- - 2\cos\left(\frac{\pi}{m}\right)u_1^+ \dot{p}_1 + 2\cos\left(\frac{3\pi}{m}\right)u_1^+ \dot{p}_1 +$$

$$+2\cos\left(\frac{\pi}{m}\right)u_1^- \dot{p}_1 - 2\cos\left(\frac{3\pi}{m}\right)u_1^- \dot{p}_1 + 2\cos\left(\frac{\pi}{m}\right)u_1 \dot{p}_1^+ - 2\cos\left(\frac{3\pi}{m}\right)u_1 \dot{p}_1^+ +$$

$$+2\cos\left(\frac{\pi}{m}\right)u_1^+ \dot{p}_1^+ - 2\cos\left(\frac{3\pi}{m}\right)u_1^+ \dot{p}_1^+ - 2\cos\left(\frac{\pi}{m}\right)\dot{p}_1 \dot{p}_1^+ + 2\cos\left(\frac{3\pi}{m}\right)\dot{p}_1 \dot{p}_1^+ +$$

$$+2p_1\left(-\left(\cos\left(\frac{3\pi}{m}\right) - 5\cos\left(\frac{\pi}{m}\right)\right)p_1^+ + \left(\cos\left(\frac{3\pi}{m}\right) - 5\cos\left(\frac{\pi}{m}\right)\right)p_1^- -\right.$$

$$-2\sin\left(\frac{\pi}{m}\right)\left(-4u_1\sin^2\left(\frac{\pi}{m}\right) + 2u_1^-\sin^2\left(\frac{\pi}{m}\right) - \cos\left(\frac{2\pi}{m}\right)u_1^+ + u_1^+ +\right.$$

$$\left.\left.+2\cos\left(\frac{2\pi}{m}\right)\dot{p}_1 + 6\dot{p}_1 - \cos\left(\frac{2\pi}{m}\right)\dot{p}_1^+ + \dot{p}_1^+ - \cos\left(\frac{2\pi}{m}\right)\dot{p}_1^- + \dot{p}_1^-\right)\right) -$$

$$-2\cos\left(\frac{\pi}{m}\right)u_1 \dot{p}_1^- + 2\cos\left(\frac{3\pi}{m}\right)u_1 \dot{p}_1^- - 2\cos\left(\frac{\pi}{m}\right)u_1^- \dot{p}_1^- + 2\cos\left(\frac{3\pi}{m}\right)u_1^- \dot{p}_1^- +$$

$$+2\cos\left(\frac{\pi}{m}\right)\dot{p}_1 \dot{p}_1^- - 2\cos\left(\frac{3\pi}{m}\right)\dot{p}_1 \dot{p}_1^- - 12\sin\left(\frac{\pi}{m}\right)u_1^+ \ddot{p}_1 + 4\sin\left(\frac{3\pi}{m}\right)u_1^+ \ddot{p}_1 -$$

$$-12\sin\left(\frac{\pi}{m}\right)u_1^- \ddot{p}_1 + 4\sin\left(\frac{3\pi}{m}\right)u_1^- \ddot{p}_1 - 24\sin\left(\frac{\pi}{m}\right)\dot{p}_1 \ddot{p}_1 + 8\sin\left(\frac{3\pi}{m}\right)\dot{p}_1 \ddot{p}_1 -$$

$$-12\sin\left(\frac{\pi}{m}\right)\dot{p}_1^+ \ddot{p}_1 + 4\sin\left(\frac{3\pi}{m}\right)\dot{p}_1^+ \ddot{p}_1 - 12\sin\left(\frac{\pi}{m}\right)\dot{p}_1^- \ddot{p}_1 + 4\sin\left(\frac{3\pi}{m}\right)\dot{p}_1^- \ddot{p}_1 +$$

$$+4p_1^-\sin\left(\frac{\pi}{m}\right)\left(2u_1\sin^2\left(\frac{\pi}{m}\right) - 2u_1^-\sin^2\left(\frac{\pi}{m}\right) + \cos\left(\frac{2\pi}{m}\right)\dot{p}_1 + 3\dot{p}_1 - \cos\left(\frac{2\pi}{m}\right)\dot{p}_1^- +\right.$$

$$
+\dot{p}_1^- - 2\sin\left(\frac{2\pi}{m}\right)\ddot{p}_1\Big) - 12\sin\left(\frac{\pi}{m}\right)u_1^+\ddot{p}_1^+ + 4\sin\left(\frac{3\pi}{m}\right)u_1^+\ddot{p}_1^+ - 12\sin\left(\frac{\pi}{m}\right)u_1^-\ddot{p}_1^- +
$$

$$
+4\sin\left(\frac{3\pi}{m}\right)u_1^-\ddot{p}_1^-\Big).
$$

In particular, if $\mathscr{F}_{(2m+1)\mathbb{Z}\backslash\{0\}}(p_1) = 0$ and $u_{1,k} = a_{2m+1,k}\,p_{1,k}$, for all $k \in \mathbb{Z}\backslash(2m+1)\mathbb{Z}$, then

$$
\mathscr{F}_{2(2m+1)}(E_{1,1}^{2m+1}) = \sum_{n\in\mathbb{Z}} n\, e^{i2(2m+1)nt} \sum_{k\in\mathbb{Z}} \sum_{r=1}^{4m+1} \mathscr{P}_r^m(n,k)\, p_{1,2(2m+1)k+r}\, p_{1,2(2m+1)(n-k)-r} \cdot
$$

$$
\tag{A.2}
$$

$$
\mathscr{P}_r^m(n,k) := c_{m,r}\left(\left(-1 + e^{\frac{2i\pi r}{2m+1}}\right)\cos\left(\frac{\pi}{2m+1}\right) - i\left(1 + e^{\frac{2i\pi r}{2m+1}}\right)\sin\left(\frac{\pi}{2m+1}\right)(k(4m+2)+r)\right)
$$

$$
\left(\left(1 + e^{\frac{2i\pi r}{2m+1}}\right)\sin\left(\frac{\pi}{2m+1}\right)((4m+2)(n-k)-r) - i\left(-1 + e^{\frac{2i\pi r}{2m+1}}\right)\cos\left(\frac{\pi}{2m+1}\right)\right),
$$

$$
c_{m,r} := \frac{-4(2m+1)\csc\left(\frac{\pi}{2m+1}\right)}{\left(-1 + e^{\frac{2i\pi r}{2m+1}}\right)^2}.
$$

Proof For the sake of simplicity, we shall give the proof for $m = 1$; the general case follows word–by–word the same lines.

(i) Indeed,

$$
\mathscr{E}_{P_1}^2(U_1, U_1^+) = 2(1 - \cos\frac{2\pi}{m}) + 2\varepsilon\Big((1 - \cos\frac{2\pi}{m})(p_1 + p_1^+) - \sin\frac{2\pi}{m}(u_1 - u_1^+) -
$$

$$
- \sin\frac{2\pi}{m}(\dot{p}_1 - \dot{p}_1^+)\Big) + O(\varepsilon^2), \tag{A.3}
$$

Thus,

$$
\mathscr{E}_{P_1}^{-1}(U_1, U_1^+) \overset{\text{(A.3)}}{=} \left(2\sin\frac{2\pi}{m}\right)^{-1} - \varepsilon\left(2\sin\frac{2\pi}{m}\right)^{-3}\Big((1 - \cos\frac{2\pi}{m})(p_1 + p_1^+) -
$$

$$
- \sin\frac{2\pi}{m}(u_1 - u_1^+) - \sin\frac{2\pi}{m}(\dot{p}_1 - \dot{p}_1^+)\Big) + O(\varepsilon^2), \tag{A.4}
$$

and, substituting t by $t - 2\pi/m$ in (A.4), we obtain

$$
\mathscr{E}_{P_1}^{-1}(U_1^-, U_1) = \left(2\sin\frac{2\pi}{m}\right)^{-1} - \varepsilon\left(2\sin\frac{2\pi}{m}\right)^{-3}\Big((1 - \cos\frac{2\pi}{m})(p_1^- + p_1) -
$$

$$
- \sin\frac{2\pi}{m}(u_1^- - u_1) - \sin\frac{2\pi}{m}(\dot{p}_1^- - \dot{p}_1)\Big) + O(\varepsilon^2). \tag{A.5}
$$

11

Moreover,

$$2\mathscr{E}_{P_1}(U_1,U_1^+)\partial_1\mathscr{E}_{P_1}(U_1,U_1^+) = -2\sin\frac{2\pi}{m} + 2\varepsilon\bigg(\dot{p}_1 - \dot{p}_1^+\cos\frac{2\pi}{m} + (u_1 - u_1^+)\cos\frac{2\pi}{m} -$$
$$- (p_1^+ + p_1 + \ddot{p}_1)\sin\frac{2\pi}{m}\bigg) + O(\varepsilon^2) \quad \text{(A.6)}$$

and

$$2\mathscr{E}_{P_1}(U_1^-,U_1)\partial_2\mathscr{E}_{P_1}(U_1^-,U_1) = 2\sin\frac{2\pi}{m} + 2\varepsilon\bigg(\dot{p}_1 - \dot{p}_1^-\cos\frac{2\pi}{m} - (u_1^- - u_1)\cos\frac{2\pi}{m} +$$
$$+ (p_1^- + p_1 + \ddot{p}_1)\sin\frac{2\pi}{m}\bigg) + O(\varepsilon^2). \quad \text{(A.7)}$$

Therefore, writing $E^m(P_1,U_1) = \mathscr{E}_{P_1}^{-1}(U_1,U_1^+)\mathscr{E}_{P_1}(U_1,U_1^+)\partial_1\mathscr{E}_{P_1}(U_1,U_1^+) + + \mathscr{E}_{P_1}^{-1}(U_1^-,U_1)\mathscr{E}_{P_1}(U_1^-,U_1)\partial_2\mathscr{E}_{P_1}(U_1^-,U_1)$ and using (A.4)–(A.7), we obtain the formula of the first order term $E_{1,0}^m$ in (A.1).
Similarly, expanding (A.3)–(A.7) up to the second order, one gets the formula of the second order term $E_{1,1}^m$ in (A.1). Then, simple computations yields the formula (A.2).   ∎

A.2   Recurrent formula for $E^m(P_N, U_N)$ for $N \geqslant 2$

We adopt the same notations as in §6

Lemma A.2 *Let $m \geqslant 2$, $N \geqslant 1$, $p_1, \cdots, p_{N+1} \in C^2(\mathbb{T})$, and $u_1, \cdots, u_{N+1} \in L^2(\mathbb{T})$. Then, we have*

$$E^m(P_{N+1}, U_{N+1}) = E^m(P_N, U_N) + \widetilde{E}_{N+1,0}^m \varepsilon^{N+1} + O(\varepsilon^{N+2}), \quad \text{(A.8)}$$

with

$$\widetilde{E}_{N+1,0}^m = \frac{1}{2}\bigg(\sin\left(\frac{\pi}{m}\right)\left(\dot{p}_{N+1}^+ + \dot{p}_{N+1}^- + 2\dot{p}_{N+1} + u_{N+1}^+ + u_{N+1}^- - 2u_{N+1}\right) +$$
$$+ \cos\left(\frac{\pi}{m}\right)\left(p_{N+1}^- - p_{N+1}^+\right)\bigg). \quad \text{(A.9)}$$

Proof For the sake of simplicity, we shall give the proof for $m = 3$; the general case follows word–by–word the same lines.

We have on one hand,

$$\mathscr{E}^2_{P_{N+1}}(U_{N+1}, U^+_{N+1}) = P^2_{N+1}(U_{N+1}) + \dot{P}^2_{N+1}(U_{N+1}) + P^2_{N+1}(U^+_{N+1}) + \dot{P}^2_{N+1}(U^+_{N+1}) -$$

$$- 2\left(P_{N+1}(U_{N+1})P_{N+1}(U^+_{N+1}) + \dot{P}_{N+1}(U_{N+1})\dot{P}_{N+1}(U^+_{N+1}) \right) \cos(U_N - U^+_N) +$$

$$+ 2\left(\dot{P}_{N+1}(U_{N+1})P_{N+1}(U^+_{N+1}) - P_{N+1}(U_{N+1})\dot{P}_{N+1}(U^+_{N+1}) \right) \sin(U_N - U^+_N) =$$

$$= \mathscr{E}^2_{P_N}(U_N, U^+_N) + \varepsilon^{N+1}\left(3(p_{N+1} + p^+_{N+1}) - \sqrt{3}(u_{N+1} - u^+_{N+1}) - \sqrt{3}(\dot{p}_{N+1} - \dot{p}^+_{N+1}) \right) +$$

$$+ O(\varepsilon^{N+2}). \tag{A.10}$$

Therefore,

$$\mathscr{E}^{-1}_{P_{N+1}}(U_{N+1}, U^+_{N+1}) = \mathscr{E}^{-1}_{P_N}(U_N, U^+_N) - \frac{1}{2}\mathscr{E}^{-3}_{P_N}(U_N, U^+_N)\varepsilon^{N+1}\left(3(p_{N+1} + p^+_{N+1}) - \right.$$

$$\left. - \sqrt{3}(u_{N+1} - u^+_{N+1}) - \sqrt{3}(\dot{p}_{N+1} - \dot{p}^+_{N+1}) \right) + O(\varepsilon^{N+2})$$

$$= \mathscr{E}^{-1}_{P_N}(U_N, U^+_N) - \frac{\sqrt{3}}{18}\varepsilon^{N+1}\left(3(p_{N+1} + p^+_{N+1}) - \sqrt{3}(u_{N+1} - u^+_{N+1}) - \sqrt{3}(\dot{p}_{N+1} - \dot{p}^+_{N+1}) \right) +$$

$$+ O(\varepsilon^{N+2})$$

and, replacing t by $t - 2\pi/3$ in the above formula, we get

$$\mathscr{E}^{-1}_{P_{N+1}}(U^-_{N+1}, U_{N+1}) = \mathscr{E}^{-1}_{P_N}(U^-_N, U_N) - \frac{\sqrt{3}}{18}\varepsilon^{N+1}\left(3(p^-_{N+1} + p_{N+1}) - \sqrt{3}(u^-_{N+1} - u_{N+1}) - \right.$$

$$\left. - \sqrt{3}(\dot{p}^-_{N+1} - \dot{p}_{N+1}) \right) + O(\varepsilon^{N+2}).$$

Furthermore,

$$2\mathscr{E}_{P_{N+1}}(U_{N+1}, U^+_{N+1})\, \partial_1\mathscr{E}_{P_{N+1}}(U_{N+1}, U^+_{N+1}) = \left(P_{N+1}(U_{N+1}) + \ddot{P}_{N+1}(U_{N+1}) \right) \times$$

$$\times \left(2\dot{P}_{N+1}(U_{N+1}) - 2\dot{P}_{N+1}(U^+_{N+1})\cos(U_{N+1} - U^+_{N+1}) + 2P_{N+1}(U^+_{N+1})\sin(U_{N+1} - U^+_{N+1}) \right)$$

$$= \mathscr{E}_{P_N}(U_N, U^+_N)\partial_1\mathscr{E}_{P_N}(U_N, U^+_N) + \varepsilon^{N+1}\left(2\dot{p}_{N+1} + \dot{p}^+_{N+1} - \sqrt{3}\left(p_{N+1} + p^+_{N+1} + \ddot{p}_{N+1} \right) + \right.$$

$$\left. + u^+_{N+1} - u_{N+1} \right) + O(\varepsilon^{N+2}), \tag{A.11}$$

13

and

$$2\mathscr{E}_{P_{N+1}}(U^-_{N+1}, U_{N+1})\, \partial_2 \mathscr{E}_{P_{N+1}}(U^-_{N+1}, U_{N+1}) = \big(P_{N+1}(U_{N+1}) + \ddot{P}_{N+1}(U_{N+1})\big)\times$$

$$\times \left(2\dot{P}_{N+1}(U_{N+1}) - 2\dot{P}_{N+1}(U^-_{N+1})\cos(U^-_{N+1} - U_{N+1}) - 2P_{N+1}(U^-_{N+1})\sin(U^-_{N+1} - U_{N+1})\right)$$

$$= \mathscr{E}_{P_N}(U^-_N, U_N)\partial_2\mathscr{E}_{P_N}(U^-_N, U_N) + \varepsilon^{N+1}\left(2\dot{p}_{N+1} + \dot{p}^-_{N+1} - \sqrt{3}\big(-p_{N+1} + p^-_{N+1} - \ddot{p}_{N+1}\big) + \right.$$

$$\left. + u^-_{N+1} - u_{N+1}\right) + O(\varepsilon^{N+2}). \tag{A.12}$$

Thus, writing

$$E(P_{N+1}, U_{N+1}) = \mathscr{E}^{-1}_{P_{N+1}}(U_{N+1}, U^+_{N+1})\cdot \mathscr{E}_{P_{N+1}}(U_{N+1}, U^+_{N+1})\partial_1\mathscr{E}_{P_{N+1}}(U_{N+1}, U^+_{N+1}) +$$
$$+ \mathscr{E}^{-1}_{P_{N+1}}(U^-_{N+1}, U_{N+1})\cdot \mathscr{E}_{P_{N+1}}(U^-_{N+1}, U_{N+1})\partial_2\mathscr{E}_{P_{N+1}}(U^-_{N+1}, U_{N+1}),$$

one gets the (A.8).

References

[Bir20]   George D Birkhoff. On the periodic motions of dynamical systems. In *Hamiltonian Dynamical Systems*, pages 154–174. CRC Press, 2020.

[Hal77]   Benjamin Halpern. Strange billiard tables. *Transactions of the American mathematical society*, 232:297–305, 1977.

[KP90]   Valery Kovachev and Georgi Popov. Invariant tori for the billiard ball map. *Transactions of the American Mathematical Society*, 317(1):45–81, 1990.

[KZ18]   Vadim Kaloshin and Ke Zhang. Density of convex billiards with rational caustics. *Nonlinearity*, 31(11):5214, 2018.

[Laz73]   Vladimir F Lazutkin. The existence of caustics for a billiard problem in a convex domain. *Mathematics of the USSR-Izvestiya*, 7(1):185, 1973.

[LM01]   M Levi and J Moser. A lagrangian proof of the invariant curve theorem for twist mappings. In *Proceedings of Symposia in Pure Mathematics*, volume 69, pages 733–748. Providence, RI; American Mathematical Society; 1998, 2001.

[Mat82]   John N Mather. Glancing billiards. *Ergodic theory and dynamical systems*, 2(3-4):397–403, 1982.

[PdCRR13] Sonia Pinto-de Carvalho and Rafael Ramírez-Ros. Non-persistence of resonant caustics in perturbed elliptic billiards. *Ergodic Theory and Dynamical Systems*, 33(6):1876–1890, 2013.

[Res15]   Howard L Resnikoff. On curves and surfaces of constant width. *arXiv preprint arXiv:1504.06733*, 2015.

[RR06]   Rafael Ramírez-Ros. Break-up of resonant invariant curves in billiards and dual billiards associated to perturbed circular tables. *Physica D: Nonlinear Phenomena*, 214(1):78–87, 2006.

[Zha19]   Jianlu Zhang. Coexistence of period 2 and 3 caustics for deformative nearly circular billiard maps. *Discrete & Continuous Dynamical Systems-A*, 39(11):6419, 2019.