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CONVERGENCE RATES OF THE ALLEN–CAHN EQUATION TO MEAN CURVATURE FLOW: A SHORT PROOF BASED ON RELATIVE ENTROPIES*

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Abstract. We give a short and self-contained proof for rates of convergence of the Allen–Cahn equation towards mean curvature flow, assuming that a classical (smooth) solution to the latter exists and starting from well-prepared initial data. Our approach is based on a relative entropy technique. In particular, it does not require a stability analysis for the linearized Allen–Cahn operator. As our analysis also does not rely on the comparison principle, we expect it to be applicable to more complex equations and systems.

Key words. mean curvature flow, Allen–Cahn equation, relative entropy method, diffuse interface, reaction-diffusion equations

AMS subject classifications. 53E10, 35A15, 35K57, 53C38, 35B25

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1. Introduction.

The Allen–Cahn equation

$$\frac{d}{dt}u_\varepsilon = \Delta u_\varepsilon - \frac{1}{\varepsilon^2}W'(u_\varepsilon) \tag{1}$$

—with a suitable double-well potential W such as, for instance, $W(s) = c\,(1-s^2)^2$, $c>0$—is the most natural diffuse-interface approximation for (two-phase) mean curvature flow: It is well known that in the limit of vanishing interface width $\varepsilon \to 0$, the solutions u_ε to the Allen–Cahn equation (1) converge to a characteristic function $\chi : \mathbb{R}^d \times [0,T] \to \{-1,1\}$ whose interface evolves by motion by mean curvature. For a proof of this fact in the framework of Brakke solutions to mean curvature flow, we refer the reader to [8]; for the convergence towards the viscosity solution of the level-set formulation under the assumption of nonfattening, see [5]. Provided that the total energy converges in the limit $\varepsilon \to 0$, one may prove that the limit is a distributional solution [10]. For a general compactness statement using the gradient-flow structure of (1) and the identification of the limit in the radially symmetric case, we refer the reader to [2]. Under the assumption of the existence of a smooth limiting evolution, rates of convergence may be derived based on a strategy of matched asymptotic expansions and the stability of the linearized Allen–Cahn operator [3, 4].

The Allen–Cahn equation corresponds to the L^2 gradient flow of the Ginzburg–

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Landau energy functional

$$E_\varepsilon[v] := \int_{\mathbb{R}^d} \frac{\varepsilon}{2}|\nabla v|^2 + \frac{1}{\varepsilon}W(v)\,\mathrm{d}x. \tag{2}$$

Solutions to the Allen–Cahn equation (1) satisfy the energy dissipation estimate

$$\frac{d}{dt}\int_{\mathbb{R}^d} \frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon}W(u_\varepsilon)\,\mathrm{d}x = -\int_{\mathbb{R}^d} \frac{1}{\varepsilon}\left|\varepsilon\Delta u_\varepsilon - \frac{1}{\varepsilon}W'(u_\varepsilon)\right|^2 \mathrm{d}x. \tag{3}$$

In the present work, we pursue a strategy of deriving a quantitative convergence result in the sharp-interface limit $\varepsilon \to 0$ based purely on the energy dissipation structure. In particular, we give a short proof for the following quantitative convergence of solutions of the Allen–Cahn equation towards a smooth solution of mean curvature flow.

THEOREM 1. *Let $d \in \mathbb{N}$. Let $I(t) \subset \mathbb{R}^d$, $t \in [0,T]$, be a compact interface $I(t) = \partial\Omega(t)$ evolving smoothly by mean curvature, and let $\chi : \mathbb{R}^d \times [0,T] \to \{-1,1\}$ be the corresponding phase indicator function*

$$\chi(x,t) := \begin{cases} 1 & \text{if } x \in \Omega(t), \\ -1 & \text{if } x \notin \Omega(t). \end{cases}$$

Let W be a standard double-well potential as described below and denote by θ the corresponding one-dimensional interface profile. Let u_ε be the solution to the Allen–Cahn equation (1) *with initial data given by $u_\varepsilon(x,0) = \theta(\varepsilon^{-1}\mathrm{dist}^\pm(x,I(0)))$, where θ is the equilibrium profile defined in* (5) *and $\mathrm{dist}^\pm(x,I(0))$ is the signed distance function to $I(0)$ with the convention $\mathrm{dist}^\pm(x,I(0)) > 0$ for $x \in \Omega(0)$. Define $\psi_\varepsilon(x,t) := \int_0^{u_\varepsilon(x,t)} \sqrt{2W(s)}\,\mathrm{d}s$. Then the error estimate*

$$\sup_{t\in[0,T]} ||\psi_\varepsilon(\cdot,t) - \chi(\cdot,t)||_{L^1(\mathbb{R}^d)} \le C\big(d,T,(I(t))_{t\in[0,T]}\big)\,\varepsilon \tag{4}$$

holds.

Remark 2. Our arguments also show that the estimate (4) holds for a larger class of solutions to the Allen–Cahn equation (1); we only require solutions u_ε whose initial data satisfies $u_\varepsilon(\cdot,0) \in [-1,1]$ and whose initial relative entropy, defined below in (10), is comparable to that of the optimal transition profile of Theorem 1, i.e., we have $E[u_\varepsilon(\cdot,0)|I(0)] \le C\varepsilon^2$.

We note that this error estimate is of optimal order, as ε is the typical width of the diffuse interface in the Allen–Cahn approximation (i.e., the typical width of the region in which the function ψ_ε takes values in the range $[-1+\delta, 1-\delta]$ for any fixed $\delta > 0$).

The assumptions required for the double-well potential W are standard: We require W to satisfy $W(1) = W(-1) = 0$ and $W(s) \ge c\min\{|s-1|^2, |s+1|^2\}$; furthermore, we require W to be twice continuously differentiable, symmetric around the origin, and subject to the normalization $\int_{-1}^1 \sqrt{2W(s)}\,\mathrm{d}s = 2$. The simplest example is the normalized standard double-well potential $W(s) := \frac{9}{8}(1-s^2)^2$. Under these assumptions, we may define the one-dimensional equilibrium profile $\theta : \mathbb{R} \to \mathbb{R}$ to be the unique odd solution of the ODE

$$\begin{cases} \theta'(s) &= \sqrt{2W(\theta(s))}, \\ \theta(\pm\infty) &= \pm 1; \end{cases} \tag{5}$$

the profile θ then approaches its boundary values ± 1 at $\pm\infty$ with an exponential rate; see [11].

As our quantitative convergence analysis does not rely on the comparison principle, it may be applicable to more complex models, such as systems of Navier–Stokes–Allen–Cahn type [1]; note that a weak-strong uniqueness theorem for the two-fluid free boundary problem for the Navier–Stokes equation (i.e., the corresponding sharp-interface model) has already been obtained in [6]. We note that a relative entropy concept related to the one in [6] had already been employed by Jerrard and Smets [9] to deduce weak-strong uniqueness of solutions to binormal curvature flow. In the forthcoming work [7], we employ an energy-based strategy to deduce a weak-strong uniqueness theorem for multiphase mean curvature flow.

2. Definition of the relative entropy and Gronwall estimate.

2.1. Extending the unit normal vector field of the surface evolving by mean curvature. Let $I = I(t)$ be a surface that evolves smoothly by motion by mean curvature. Let $P_{I(t)} : \mathbb{R}^d \to I(t)$ be the nearest point projection to $I(t)$ and fix $r_c > 0$ small enough depending on $(I(t))_{t\in[0,T]}$ so that for all $t \in [0,T]$ the map $P_{I(t)}$ is smooth in a tubular neighborhood of $I(t)$ of width r_c; for example, one may take the minimum over $t \in [0,T]$ of the normal injectivity radius of I_t. For each $t \in [0,T]$, we extend the inner unit normal n_I of the surface $I(t)$ to a vector field on $\mathbb{R}^d$ by defining

$$\xi(x) := \eta(\mathrm{dist}^{\pm}(x, I))\mathrm{n}_I(P_I(x)), \tag{6}$$

where η is a cutoff for all $s \in \mathbb{R}$ satisfying $\eta(s) \geq 0$ and

$$\eta(0) = 1, \qquad \eta(s) = 0 \quad \text{for } |s| \geq \frac{r_c}{2}, \tag{7a}$$

$$\eta(s) \leq \max\{1 - c r_c^{-2} s^2, 0\}, \tag{7b}$$

$$|\eta'(s)| \leq C \min\{r_c^{-1}, r_c^{-2}|s|\}. \tag{7c}$$

Furthermore, we will consider a standard cutoff $\tilde\eta$ satisfying $\tilde\eta(s) = 1$ for $|s| \leq \frac{r_c}{4}$, $\tilde\eta(s) = 0$ for $|s| \leq \frac{r_2}{2}$, and $|\eta'(s)| \leq C r_c^{-1}$, in which case one may take $\eta(s) := (1 - c r_c^{-2} s^2)\tilde\eta(s)$.

The extended unit normal vector field ξ and mean curvature vector $\mathrm{H}_I(x) := \mathrm{H}_I(P_I x)\tilde\eta(\mathrm{dist}(x, I))$ then satisfy the PDEs

$$\frac{d}{dt}\xi = -(\mathrm{H}_I \cdot \nabla)\xi - (\nabla \mathrm{H}_I)^T \xi + O(\mathrm{dist}(x, I)), \tag{8a}$$

$$\frac{d}{dt}|\xi|^2 = -(\mathrm{H}_I \cdot \nabla)|\xi|^2 + O(\mathrm{dist}^2(x, I)), \tag{8b}$$

and

$$-\nabla \cdot \xi = \mathrm{H}_I \cdot \xi + O(\mathrm{dist}(x, I)), \tag{8c}$$

where the constant implicit in the O-notation depends on the interface I. Furthermore, we have the estimate

$$|\nabla \xi| + |\mathrm{H}_I| + |\nabla \mathrm{H}_I| \leq C(I(t)). \tag{8d}$$

To see that (8a) and (8b) hold, one makes use of the formulas $\mathrm{n}_I(x) = \nabla \mathrm{dist}^{\pm}(x, I)$ and $\partial_t \mathrm{dist}^{\pm}(x, I) = -\mathrm{H}_I \cdot \mathrm{n}_I(P_I x)$ valid in a neighborhood of $I(t)$. Formula (8c) is an immediate consequence of the equality $\mathrm{H}_I = -(\nabla \cdot \mathrm{n}_I)\mathrm{n}_I$ valid on the interface $I(t)$ and the Lipschitz continuity of both sides of the equation.

2.2. The relative entropy inequality. Our argument is based on a relative entropy method. As the Modica–Mortola trick will play an important role in the definition of the relative entropy, we introduce the function

$$\psi_\varepsilon(x,t) := \int_0^{u_\varepsilon(x,t)} \sqrt{2W(s)}\,\mathrm{d}s. \tag{9}$$

Given a smooth solution u_ε to the Allen–Cahn equation (1) and a surface $I(t)$ which evolves smoothly by mean curvature flow, we define the relative entropy $E[u_\varepsilon|I]$ as

$$E[u_\varepsilon|I] := \int_{\mathbb{R}^d} \frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon}W(u_\varepsilon) - \xi\cdot\nabla\psi_\varepsilon\,\mathrm{d}x, \tag{10}$$

where for reasons of tradition we use the term “relative entropy” as opposed to the perhaps more accurate term “relative energy.” Introducing the shorthand notation

$$\mathrm{n}_\varepsilon := \frac{\nabla u_\varepsilon}{|\nabla u_\varepsilon|} \tag{11a}$$

(with $\mathrm{n}_\varepsilon(x,t)\in\mathbb{S}^{d-1}$ arbitrary but fixed in case $|\nabla u_\varepsilon|=0$) and writing

$$E[u_\varepsilon|I] = \int_{\mathbb{R}^d} \frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon}W(u_\varepsilon) - |\nabla\psi_\varepsilon|\,\mathrm{d}x + \int_{\mathbb{R}^d}(1-\xi\cdot\mathrm{n}_\varepsilon)|\nabla\psi_\varepsilon|\,\mathrm{d}x,$$

we see that the relative entropy consists of two contributions: The first term

$$\int_{\mathbb{R}^d} \frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon}W(u_\varepsilon) - |\nabla\psi_\varepsilon|\,\mathrm{d}x = \int_{\mathbb{R}^d} \frac{1}{2}\Big|\sqrt{\varepsilon}|\nabla u_\varepsilon| - \frac{1}{\sqrt{\varepsilon}}\sqrt{2W(u_\varepsilon)}\Big|^2\,\mathrm{d}x$$

controls the local lack of equipartition of energy between the terms $\frac{\varepsilon}{2}|\nabla u_\varepsilon|^2$ and $\frac{1}{\varepsilon}W(u_\varepsilon)$, while the second term

$$\int_{\mathbb{R}^d}(1-\xi\cdot\mathrm{n}_\varepsilon)|\nabla\psi_\varepsilon|\,\mathrm{d}x \geq \frac{1}{2}\int_{\mathbb{R}^d}|\mathrm{n}_\varepsilon-\xi|^2|\nabla\psi_\varepsilon|\,\mathrm{d}x$$

controls the local deviation of the normals n_ε and n_I. Note that the latter term also controls the distance to the interface $I(t)$ (since $|\xi|\leq\max\{1-cr_c^{-2}\operatorname{dist}^2(x,I),0\}$).

We furthermore introduce the notation

$$\mathrm{H}_\varepsilon := -\Big(\varepsilon\Delta u_\varepsilon - \frac{1}{\varepsilon}W'(u_\varepsilon)\Big)\frac{\nabla u_\varepsilon}{|\nabla u_\varepsilon|}, \tag{11b}$$

motivated by the fact that H_ε will play the role of a curvature vector.

The key step in our analysis is the following Gronwall-type estimate for the relative entropy.

THEOREM 3. *Let $I(t)$, $t\in[0,T]$, be an interface evolving smoothly by mean curvature. Let u_ε be a solution to the Allen–Cahn equation* (1) *with initial data given by $u_\varepsilon(x,0)=\theta(\varepsilon^{-1}\operatorname{dist}^\pm(x,I(0)))$. Then for any $t\in[0,T]$ the estimate*

$$\begin{aligned}\frac{d}{dt}E[u_\varepsilon|I] + \int_{\mathbb{R}^d}\frac{1}{4\varepsilon}\big|\mathrm{H}_\varepsilon - \mathrm{H}_I\,\varepsilon|\nabla u_\varepsilon|\big|^2 + \frac{1}{4\varepsilon}\big|\mathrm{n}_\varepsilon\cdot\mathrm{H}_\varepsilon - (-\nabla\cdot\xi)\sqrt{2W(u_\varepsilon)}\big|^2\,\mathrm{d}x \\ \leq C(d,(I(s))_{s\in[0,t]})E[u_\varepsilon|I]\end{aligned}$$

holds.

2.3. Coercivity properties of the relative entropy functional. For the proof of the Gronwall-type inequality of Theorem 3, we shall need the following coercivity properties of the relative entropy.

LEMMA 4. *We have the estimates*

$$\int_{\mathbb{R}^d}\left(\sqrt{\varepsilon}|\nabla u_\varepsilon|-\frac{1}{\sqrt{\varepsilon}}\sqrt{2W(u_\varepsilon)}\right)^2\mathrm{d}x\le 2E[u_\varepsilon|I], \tag{12a}$$

$$\int_{\mathbb{R}^d}|\mathrm{n}_\varepsilon-\xi|^2|\nabla\psi_\varepsilon|\,\mathrm{d}x\le 2E[u_\varepsilon|I], \tag{12b}$$

$$\int_{\mathbb{R}^d}|\mathrm{n}_\varepsilon-\xi|^2\varepsilon|\nabla u_\varepsilon|^2\,\mathrm{d}x\le 12E[u_\varepsilon|I], \tag{12c}$$

$$\int_{\mathbb{R}^d}\min\{\mathrm{dist}^2(x,I),1\}\left(\frac{\varepsilon}{2}|\nabla u_\varepsilon|^2+\frac{1}{\varepsilon}W(u_\varepsilon)\right)\mathrm{d}x\le C(I)E[u_\varepsilon|I]. \tag{12d}$$

Proof. We complete the square to get

$$E[u_\varepsilon|I]=\int_{\mathbb{R}^d}\frac{1}{2}\left(\sqrt{\varepsilon}|\nabla u_\varepsilon|-\frac{1}{\sqrt{\varepsilon}}\sqrt{2W(u_\varepsilon)}\right)^2+(1-\xi\cdot\mathrm{n}_\varepsilon)|\nabla\psi_\varepsilon|\,\mathrm{d}x.$$

In particular, we directly obtain (12a) and (12b) by $|\xi|\le 1$. By the property (7b) of the cutoff η (and hence $1-\xi\cdot\mathrm{n}_\varepsilon\ge\min\{cr_c^{-2}\,\mathrm{dist}^2(x,I),1\}$), we deduce (12d) with $|\nabla\psi_\varepsilon|$ instead of the energy density, which we may replace upon using (12a).

Employing Young's inequality in the form of

$$\begin{aligned}\varepsilon|\nabla u_\varepsilon|^2&=|\nabla\psi_\varepsilon|+\sqrt{\varepsilon}|\nabla u_\varepsilon|\left(\sqrt{\varepsilon}|\nabla u_\varepsilon|-\frac{1}{\sqrt{\varepsilon}}\sqrt{2W(u_\varepsilon)}\right)\\&\le|\nabla\psi_\varepsilon|+\frac{1}{2}\varepsilon|\nabla u_\varepsilon|^2+\frac{1}{2}\left(\sqrt{\varepsilon}|\nabla u_\varepsilon|-\frac{1}{\sqrt{\varepsilon}}\sqrt{2W(u_\varepsilon)}\right)^2,\end{aligned} \tag{13}$$

absorption and $|\mathrm{n}_\varepsilon-\xi|\le 2$ yield

$$\begin{aligned}&\int_{\mathbb{R}^d}|\mathrm{n}_\varepsilon-\xi|^2\varepsilon|\nabla u_\varepsilon|^2\,dx\\&\le 2\int_{\mathbb{R}^d}|\mathrm{n}_\varepsilon-\xi|^2|\nabla\psi_\varepsilon|\,\mathrm{d}x+4\int_{\mathbb{R}^d}\left(\sqrt{\varepsilon}|\nabla u_\varepsilon|-\frac{1}{\sqrt{\varepsilon}}\sqrt{2W(u_\varepsilon)}\right)^2\mathrm{d}x.\end{aligned}$$

By (12a) and (12b), this shows (12c). ∎

2.4. Time evolution of the relative entropy functional. The main step in the proof of Theorem 3 is the derivation of the following formula; by estimating the right-hand side using the above-mentioned coercivity properties and equations (8a)–(8c), we will derive the Gronwall-type inequality of Theorem 3.

LEMMA 5. *Let u_ε be a solution to the Allen–Cahn equation* (1)*, and let $I=I(t)$ be a smooth solution to mean curvature flow. Let ξ be as defined in* (6)*. The time* *evolution of the relative entropy is then given by*

$$
\begin{aligned}
\frac{d}{dt}E[u_\varepsilon|I] = &-\int_{\mathbb{R}^d} \frac{1}{2\varepsilon}\big|\mathrm{H}_\varepsilon - \mathrm{H}_I\,\varepsilon|\nabla u_\varepsilon|\big|^2 + \frac{1}{2\varepsilon}\big|\mathrm{n}_\varepsilon \cdot \mathrm{H}_\varepsilon - (-\nabla\cdot\xi)\sqrt{2W(u_\varepsilon)}\big|^2 \,\mathrm{d}x \\
&+ \int_{\mathbb{R}^d} |\mathrm{H}_I|^2 \frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 + |\nabla\cdot\xi|^2\frac{1}{\varepsilon}W(u_\varepsilon) + \mathrm{H}_\mathrm{I}\cdot \mathrm{n}_\varepsilon(\nabla\cdot\xi)|\nabla\psi_\varepsilon|\,\mathrm{d}x \\
&+ \int_{\mathbb{R}^d} \nabla\cdot\mathrm{H}_I\left(\frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon}W(u_\varepsilon) - |\nabla\psi_\varepsilon|\right)\mathrm{d}x \\
&- \int_{\mathbb{R}^d} \nabla\mathrm{H}_I : \mathrm{n}_\varepsilon \otimes \mathrm{n}_\varepsilon(\varepsilon|\nabla u_\varepsilon|^2 - |\nabla\psi_\varepsilon|)\,\mathrm{d}x \\
&- \int_{\mathbb{R}^d} \nabla\mathrm{H}_I : (\mathrm{n}_\varepsilon - \xi)\otimes(\mathrm{n}_\varepsilon-\xi)|\nabla\psi_\varepsilon|\,\mathrm{d}x \qquad (14)\\
&+ \int_{\mathbb{R}^d} \nabla\cdot\mathrm{H}_I(1-\xi\cdot\mathrm{n}_\varepsilon)|\nabla\psi_\varepsilon|\,\mathrm{d}x \\
&- \int_{\mathbb{R}^d} |\nabla\psi_\varepsilon|(\mathrm{n}_\varepsilon - \xi)\cdot\left(\frac{d}{dt}\xi + (\mathrm{H}_I\cdot\nabla)\xi + (\nabla\mathrm{H}_I)^T\xi\right)\mathrm{d}x \\
&- \int_{\mathbb{R}^d} |\nabla\psi_\varepsilon|\xi\cdot\left(\frac{d}{dt}\xi + (\mathrm{H}_I\cdot\nabla)\xi\right)\mathrm{d}x.
\end{aligned}
$$

Proof. By direct computation, we obtain

$$
\begin{aligned}
\frac{d}{dt}E[u_\varepsilon|I] &= \frac{d}{dt}\int_{\mathbb{R}^d}\frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 + \frac{1}{\varepsilon}W(u_\varepsilon) - \xi\cdot\nabla\psi_\varepsilon\,\mathrm{d}x \\
&\overset{(3),(1)}{=} -\int_{\mathbb{R}^d}\frac{1}{\varepsilon}\left|\varepsilon\Delta u_\varepsilon - \frac{1}{\varepsilon}W'(u_\varepsilon)\right|^2\mathrm{d}x \\
&\quad - \int_{\mathbb{R}^d}\nabla\psi_\varepsilon\cdot\frac{d}{dt}\xi\,\mathrm{d}x + \int_{\mathbb{R}^d}\sqrt{2W(u_\varepsilon)}\left(\Delta u_\varepsilon - \frac{1}{\varepsilon^2}W'(u_\varepsilon)\right)\nabla\cdot\xi\,\mathrm{d}x.
\end{aligned}
$$

With the definitions (11a) and (11b), we obtain

$$
\begin{aligned}
\frac{d}{dt}E[u_\varepsilon|I] = &\int_{\mathbb{R}^d} -\frac{1}{\varepsilon}|\mathrm{H}_\varepsilon|^2 + \mathrm{n}_\varepsilon\cdot\mathrm{H}_\varepsilon\,(-\nabla\cdot\xi)\;\frac{1}{\varepsilon}\sqrt{2W(u_\varepsilon)}\,\mathrm{d}x \\
&+ \int_{\mathbb{R}^d}\nabla\mathrm{H}_I : \xi\otimes\mathrm{n}_\varepsilon|\nabla\psi_\varepsilon|\,\mathrm{d}x \\
&+ \int_{\mathbb{R}^d}(\mathrm{H}_I\cdot\nabla)\xi\;\cdot\nabla\psi_\varepsilon\,\mathrm{d}x \\
&- \int_{\mathbb{R}^d}\nabla\psi_\varepsilon\cdot\left(\frac{d}{dt}\xi + (\mathrm{H}_I\cdot\nabla)\xi + (\nabla\mathrm{H}_I)^T\xi\right)\mathrm{d}x.
\end{aligned}
$$

We exploit the symmetry of the Hessian $\nabla^2\psi_\varepsilon$,

$$
\begin{aligned}
&\int_{\mathbb{R}^d}(\mathrm{H}_I\cdot\nabla)\xi\;\cdot\nabla\psi_\varepsilon\,\mathrm{d}x \\
&= -\int_{\mathbb{R}^d}\mathrm{H}_I\otimes\xi : \nabla^2\psi_\varepsilon\,\mathrm{d}x - \int_{\mathbb{R}^d}\nabla\cdot\mathrm{H}_I\;\xi\cdot\nabla\psi_\varepsilon\,\mathrm{d}x \\
&= \int_{\mathbb{R}^d}(\xi\cdot\nabla)\mathrm{H}_I\cdot\nabla\psi_\varepsilon\,\mathrm{d}x + \int_{\mathbb{R}^d}(\nabla\cdot\xi\;\mathrm{H}_I - \nabla\cdot\mathrm{H}_I\;\xi)\cdot\nabla\psi_\varepsilon\,\mathrm{d}x,
\end{aligned}
$$

which yields

$$\begin{aligned}\frac{d}{dt}E[u_\varepsilon|I] = &\int_{\mathbb{R}^d} -\frac{1}{\varepsilon}|\mathrm{H}_\varepsilon|^2 + \mathrm{n}_\varepsilon \cdot \mathrm{H}_\varepsilon\,(-\nabla\cdot\xi)\;\frac{1}{\varepsilon}\sqrt{2W(u_\varepsilon)}\,\mathrm{d}x\\
&+\int_{\mathbb{R}^d} \nabla \mathrm{H}_I : \xi\otimes \mathrm{n}_\varepsilon|\nabla\psi_\varepsilon|\,\mathrm{d}x\\
&+\int_{\mathbb{R}^d} (\xi\cdot\nabla)\mathrm{H}_I\cdot \mathrm{n}_\varepsilon|\nabla\psi_\varepsilon|\,\mathrm{d}x\\
&+\int_{\mathbb{R}^d} (\nabla\cdot\xi\;\mathrm{H}_I - \nabla\cdot \mathrm{H}_I\;\xi)\cdot\nabla\psi_\varepsilon\,\mathrm{d}x\\
&-\int_{\mathbb{R}^d} \nabla\psi_\varepsilon\cdot\left(\frac{d}{dt}\xi + (\mathrm{H}_I\cdot\nabla)\xi + (\nabla \mathrm{H}_I)^T\xi\right)\mathrm{d}x.\end{aligned}$$

Together with $\xi\otimes\mathrm{n}_\varepsilon + \mathrm{n}_\varepsilon\otimes\xi = -(\mathrm{n}_\varepsilon-\xi)\otimes(\mathrm{n}_\varepsilon-\xi)+\mathrm{n}_\varepsilon\otimes\mathrm{n}_\varepsilon+\xi\otimes\xi$ the computation (15) below then implies

$$\begin{aligned}\frac{d}{dt}E[u_\varepsilon|I] = &\int_{\mathbb{R}^d} -\frac{1}{\varepsilon}|\mathrm{H}_\varepsilon|^2 + \mathrm{H}_\varepsilon\cdot\mathrm{H}_I\,|\nabla u_\varepsilon| + \mathrm{n}_\varepsilon \cdot \mathrm{H}_\varepsilon\,(-\nabla\cdot\xi)\;\frac{1}{\varepsilon}\sqrt{2W(u_\varepsilon)}\,\mathrm{d}x\\
&+\int_{\mathbb{R}^d} \nabla\cdot\mathrm{H}_I|\nabla\psi_\varepsilon|\,\mathrm{d}x\\
&+\int_{\mathbb{R}^d} \nabla\cdot\mathrm{H}_I\left(\frac{\varepsilon}{2}|\nabla u_\varepsilon|^2+\frac{1}{\varepsilon}W(u_\varepsilon)-|\nabla\psi_\varepsilon|\right)\mathrm{d}x\\
&-\int_{\mathbb{R}^d} \nabla\mathrm{H}_I : \mathrm{n}_\varepsilon\otimes\mathrm{n}_\varepsilon(\varepsilon|\nabla u_\varepsilon|^2-|\nabla\psi_\varepsilon|)\,\mathrm{d}x\\
&-\int_{\mathbb{R}^d} \nabla\mathrm{H}_I : (\mathrm{n}_\varepsilon-\xi)\otimes(\mathrm{n}_\varepsilon-\xi)|\nabla\psi_\varepsilon|\,\mathrm{d}x\\
&+\int_{\mathbb{R}^d} (\xi\cdot\nabla)\mathrm{H}_I\cdot\xi|\nabla\psi_\varepsilon|\,\mathrm{d}x\\
&+\int_{\mathbb{R}^d} (\nabla\cdot\xi\;\mathrm{H}_I - \nabla\cdot \mathrm{H}_I\;\xi)\cdot\nabla\psi_\varepsilon\,\mathrm{d}x\\
&-\int_{\mathbb{R}^d} \nabla\psi_\varepsilon\cdot\left(\frac{d}{dt}\xi + (\mathrm{H}_I\cdot\nabla)\xi + (\nabla \mathrm{H}_I)^T\xi\right)\mathrm{d}x.\end{aligned}$$

Completing the squares and adding zero, we obtain (14). ∎

2.5. Auxiliary computation. In the above computation, we have made use of the formula

$$\begin{aligned}&\int_{\mathbb{R}^d}\nabla\mathrm{H}_I : \mathrm{n}_\varepsilon\otimes\mathrm{n}_\varepsilon|\nabla\psi_\varepsilon|\,\mathrm{d}x\\
&\quad=\int_{\mathbb{R}^d}\mathrm{H}_\varepsilon\cdot\mathrm{H}_I|\nabla u_\varepsilon|\,\mathrm{d}x+\int_{\mathbb{R}^d}\nabla\cdot\mathrm{H}_I\left(\frac{\varepsilon}{2}|\nabla u_\varepsilon|^2+\frac{1}{\varepsilon}W(u_\varepsilon)\right)\mathrm{d}x\\
&\qquad-\int_{\mathbb{R}^d}\nabla\mathrm{H}_I : \mathrm{n}_\varepsilon\otimes\mathrm{n}_\varepsilon(\varepsilon|\nabla u_\varepsilon|^2-|\nabla\psi_\varepsilon|)\,\mathrm{d}x.\end{aligned}\tag{15}$$

Indeed, due to definition (11b) we have

$$-\int_{\mathbb{R}^d}\mathrm{H}_\varepsilon\cdot\mathrm{H}_I|\nabla u_\varepsilon|\,\mathrm{d}x = \int_{\mathbb{R}^d}\left(\varepsilon\Delta u_\varepsilon - \frac{W'(u_\varepsilon)}{\varepsilon}\right)\mathrm{H}_I\cdot\nabla u_\varepsilon\,\mathrm{d}x.$$

Using the identity $\sum_{i=1}^d \partial_i\partial_i u_\varepsilon \partial_j u_\varepsilon = \sum_{i=1}^d (\partial_i(\partial_i u_\varepsilon \partial_j u_\varepsilon)) - \frac{1}{2}\partial_j|\nabla u_\varepsilon|^2$ we calculate

$$\begin{aligned}
&\int_{\mathbb{R}^d} \left(\varepsilon\Delta u_\varepsilon - \frac{W'(u_\varepsilon)}{\varepsilon}\right) \mathrm{H}_I \cdot \nabla u_\varepsilon \,\mathrm{d}x \\
&\quad = \int_{\mathbb{R}^d} \sum_{i,j=1}^d (\varepsilon\partial_i\partial_i u_\varepsilon \partial_j u_\varepsilon \mathrm{H}_{I,j}) - \frac{1}{\varepsilon}\mathrm{H}_I \cdot \nabla\left(W(u_\varepsilon)\right) \mathrm{d}x \\
&\quad = \int_{\mathbb{R}^d} \sum_{i,j=1}^d (-\varepsilon\partial_i \mathrm{H}_{I,j}\partial_i u_\varepsilon \partial_j u_\varepsilon) + \nabla\cdot \mathrm{H}_I \left(\frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 + \frac{W(u_\varepsilon)}{\varepsilon}\right) \mathrm{d}x.
\end{aligned}$$

Recalling the abbreviation $\mathrm{n}_\varepsilon = \frac{\nabla u_\varepsilon}{|\nabla u_\varepsilon|}$ we get

$$\begin{aligned}
&-\int_{\mathbb{R}^d} \mathrm{H}_\varepsilon \cdot \mathrm{H}_I |\nabla u_\varepsilon| \,\mathrm{d}x \\
&\quad = \int_{\mathbb{R}^d} \nabla\cdot \mathrm{H}_I \left(\frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 + \frac{W(u_\varepsilon)}{\varepsilon}\right) - \nabla \mathrm{H_I} : (\mathrm{n}_\varepsilon \otimes \mathrm{n}_\varepsilon)\,\varepsilon|\nabla u_\varepsilon|^2 \,\mathrm{d}x.
\end{aligned} \tag{16}$$

With the goal of replacing the expressions $\frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 + \frac{W(u_\varepsilon)}{\varepsilon}$ and $\varepsilon|\nabla u_\varepsilon|^2$ by $|\nabla\psi_\varepsilon|$ we rewrite the identity (16) as (15).

2.6. Derivation of the Gronwall inequality.

Proof of Theorem 3. Using the estimates of Lemma 4 we can control the terms on the right-hand side of the identity (14). Using (8a), (8b), and the bound $||\nabla H_I||_{L^\infty} \le C(I(t))$, the last four lines of (14) may be estimated by

$$C(I(t)) \int_{\mathbb{R}^d} \min\{\mathrm{dist}^2(x,I),1\}|\nabla\psi_\varepsilon| + |\mathrm{n}_\varepsilon - \xi|^2|\nabla\psi_\varepsilon| + (1-\mathrm{n}_\varepsilon\cdot\xi)|\nabla\psi_\varepsilon| \,\mathrm{d}x,$$

which by (12b) and (12d) is bounded by $C(I(t))E[u_\varepsilon|I]$.

The third line on the right-hand side of (14) can be estimated as

$$\int_{\mathbb{R}^d} \left|\nabla\cdot \mathrm{H}_I \left(\frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 + \frac{1}{2\varepsilon}W(u_\varepsilon) - |\nabla\psi_\varepsilon|\right)\right| \mathrm{d}x \le \|\nabla\cdot \mathrm{H}_I\|_\infty E[u_\varepsilon|I]. \tag{17}$$

Thus, it only remains to estimate the second and fourth terms on the right-hand side of (14).

Concerning the fourth term, we use the fact that $(\xi\cdot\nabla)\mathrm{H}_I \equiv 0$ holds in a neighborhood of $I(t)$, Young's inequality, and (9) to deduce

$$\begin{aligned}
&\int_{\mathbb{R}^d} \left|\nabla \mathrm{H}_I : \mathrm{n}_\varepsilon \otimes \mathrm{n}_\varepsilon(\varepsilon|\nabla u_\varepsilon|^2 - |\nabla\psi_\varepsilon|)\right| \mathrm{d}x \\
&= \int_{\mathbb{R}^d} \left|\nabla \mathrm{H}_I : \mathrm{n}_\varepsilon \otimes (\mathrm{n}_\varepsilon - \xi)\left(\varepsilon|\nabla u_\varepsilon|^2 - |\nabla\psi_\varepsilon|\right)\right| \mathrm{d}x \\
&\quad + C\int_{\mathbb{R}^d} \min\{\mathrm{dist}^2(x,I),1\}\big(\varepsilon|\nabla u_\varepsilon|^2 + |\nabla\psi_\varepsilon|\big)\,\mathrm{d}x \\
&\le \|\nabla \mathrm{H}_I\|_\infty \left(\int_{\mathbb{R}^d} |\mathrm{n}_\varepsilon - \xi|^2 \varepsilon|\nabla u_\varepsilon|^2 \,dx\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^d} \left(\sqrt{\varepsilon}|\nabla u_\varepsilon| - \frac{1}{\sqrt{\varepsilon}}\sqrt{2W(u_\varepsilon)}\right)^2 \mathrm{d}x\right)^{\frac{1}{2}} \\
&\quad + C\int_{\mathbb{R}^d} \min\{\mathrm{dist}^2(x,I),1\}\big(\varepsilon|\nabla u_\varepsilon|^2 + |\nabla\psi_\varepsilon|\big)\,\mathrm{d}x.
\end{aligned}$$

Consequently, Lemma 4 implies that the fourth line on the right-hand side of (14) is bounded by $CE[u_\varepsilon|I]$.

It only remains to bound the term on the second line on the right-hand side of (14). To this aim, we complete the square and estimate

$$\begin{aligned}
&\int_{\mathbb{R}^d} |\mathrm{H}_I|^2 \frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 + |\nabla\cdot\xi|^2\frac{1}{\varepsilon}W(u_\varepsilon) + \mathrm{H}_I\cdot \mathrm{n}_\varepsilon \nabla\cdot\xi|\nabla\psi_\varepsilon|\,\mathrm{d}x\\
&= \int_{\mathbb{R}^d} \frac{1}{2}\Big|\sqrt{\varepsilon}|\nabla u_\varepsilon|\mathrm{H}_I + \frac{1}{\sqrt{\varepsilon}}\nabla\cdot\xi\sqrt{2W(u_\varepsilon)}\mathrm{n}_\varepsilon\Big|^2\,\mathrm{d}x\\
&\leq \frac{3}{2}\int_{\mathbb{R}^d}\Big|(\nabla\cdot\xi)\mathrm{n}_\varepsilon\Big(\sqrt{\varepsilon}|\nabla u_\varepsilon| - \frac{1}{\sqrt{\varepsilon}}\sqrt{2W(u_\varepsilon)}\Big)\Big|^2\,\mathrm{d}x\\
&\quad + \frac{3}{2}\int_{\mathbb{R}^d}\Big|(\nabla\cdot\xi)(\xi-\mathrm{n}_\varepsilon)\sqrt{\varepsilon}|\nabla u_\varepsilon|\Big|^2\,\mathrm{d}x\\
&\quad + \frac{3}{2}\int_{\mathbb{R}^d}\Big|\big(\mathrm{H}_I + (\nabla\cdot\xi)\xi\big)\sqrt{\varepsilon}|\nabla u_\varepsilon|\Big|^2\,\mathrm{d}x.
\end{aligned}$$

Inserting the estimates (8c) and (8d) and using the fact that $H_I = (H_I\cdot\xi)\xi + O(\mathrm{dist}(x,I))$, we obtain

$$\begin{aligned}
&\int_{\mathbb{R}^d} |\mathrm{H}_I|^2 \frac{\varepsilon}{2}|\nabla u_\varepsilon|^2 + |\nabla\cdot\xi|^2\frac{1}{\varepsilon}W(u_\varepsilon) + \mathrm{H}_I\cdot \mathrm{n}_\varepsilon \nabla\cdot\xi|\nabla\psi_\varepsilon|\,\mathrm{d}x\\
&\leq C\int_{\mathbb{R}^d}\Big|\sqrt{\varepsilon}|\nabla u_\varepsilon| - \frac{1}{\sqrt{\varepsilon}}\sqrt{2W(u_\varepsilon)}\Big|^2\,\mathrm{d}x\\
&\quad + C\int_{\mathbb{R}^d}|\mathrm{n}_\varepsilon-\xi|^2\varepsilon|\nabla u_\varepsilon|^2 + \min\{\mathrm{dist}^2(x,I),1\}\varepsilon|\nabla u_\varepsilon|^2\,\mathrm{d}x.
\end{aligned}$$

By Lemma 4, we see that these terms are estimated by $CE[u_\varepsilon|I]$. ∎

3. Estimate for the interface error. We now derive the interface error estimate of Theorem 1.

Proof of Theorem 1. *Step* 1: *Estimate for the relative entropy.* In view of Theorem 3, in order to prove

(18)
$$\begin{aligned}
&\sup_{t\in[0,T]} E[u_\varepsilon|I] + \int_0^T\int_{\mathbb{R}^d}\frac{1}{\varepsilon}\big|\mathrm{H}_\varepsilon - \mathrm{H}_I\,\varepsilon|\nabla u_\varepsilon|\big|^2 + \frac{1}{\varepsilon}\big|\mathrm{n}_\varepsilon\cdot\mathrm{H}_\varepsilon - (-\nabla\cdot\xi)\sqrt{2W(u_\varepsilon)}\big|^2\,\mathrm{d}x\,\mathrm{d}t\\
&\qquad \leq C(d,T,(I(t))_{t\in[0,T]})\varepsilon^2
\end{aligned}$$

it only remains to show that the initial relative entropy satisfies the estimate $E[u_\varepsilon|I](0) \leq C(d,I(0))\varepsilon^2$. To this end, we compute using $u_\varepsilon(x,0) = \theta(\varepsilon^{-1}\mathrm{dist}^\pm(x,I(0)))$ and the fact that $\nabla\mathrm{dist}^\pm(x,I(0))\cdot\xi = |\nabla\mathrm{dist}^\pm(x,I(0))||\xi| \geq |\xi|^2$

$$\begin{aligned}
E[u_\varepsilon|I](0) \leq &\int_{\mathbb{R}^d}\frac{|\xi|^2}{2\varepsilon}\Big|\theta'\Big(\frac{1}{\varepsilon}\mathrm{dist}^\pm(x,I(0))\Big)\Big|^2 + \frac{|\xi|^2}{\varepsilon}W\Big(\theta\Big(\frac{1}{\varepsilon}\mathrm{dist}^\pm(x,I(0))\Big)\Big)\\
&\qquad - \frac{1}{\varepsilon}\sqrt{2W\Big(\theta\Big(\frac{1}{\varepsilon}\mathrm{dist}^\pm(x,I(0))\Big)\Big)}\,\theta'\Big(\frac{1}{\varepsilon}\mathrm{dist}^\pm(x,I(0))\Big)|\xi|^2\,\mathrm{d}x\\
&+\int_{\mathbb{R}^d}\frac{1}{\varepsilon}(1-|\xi|^2)\Big(\frac{1}{2}\Big|\theta'\Big(\frac{1}{\varepsilon}\mathrm{dist}^\pm(x,I(0))\Big)\Big|^2 + W\Big(\theta\Big(\frac{1}{\varepsilon}\mathrm{dist}^\pm(x,I(0))\Big)\Big)\Big)\,\mathrm{d}x.
\end{aligned}$$

Using the defining equation $\theta'(s) = \sqrt{2W(\theta(s))}$ as well as the fact that $|\theta'(s)|$ decays exponentially in s and that $|\xi|^2 \geq 1 - c\,\mathrm{dist}^2(x,I)$, we deduce $E[u_\varepsilon|\chi](0) \leq C(d,I(0))\varepsilon^2$.

Step 2: *Interface error estimate.* We now perform an additional computation to obtain a more explicit control on the interface error. We may write

$$\partial_t\psi_\varepsilon = \sqrt{2W(u_\varepsilon)}\partial_t u_\varepsilon \overset{(1),(11b)}{=} -\varepsilon^{-1}\sqrt{2W(u_\varepsilon)}\mathrm{H}_\varepsilon\cdot\mathrm{n}_\varepsilon.$$

We choose $\tau:\mathbb{R}\to[-1,1]$ to be a smooth monotone truncation of the identity map (with $\tau(s)\geq\min\{s,\frac12\}$ for $s>0$, $\tau(s)\leq\max\{s,-\frac12\}$ for $s<0$, and $\tau(s)=\mathrm{sign}(s)$ for $|s|\geq 1$). Fixing $s_0>0$ to be determined later and observing that the measure-function pairing $(\frac{d}{dt}\chi)\tau\big(\frac{1}{s_0}\mathrm{dist}^\pm(x,I)\big)$ vanishes, we obtain

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^d}(\chi-\psi_\varepsilon)\tau\Big(\frac{1}{s_0}\mathrm{dist}^\pm(x,I)\Big)\,\mathrm{d}x\\
&=\int_{\mathbb{R}^d}\varepsilon^{-1}\sqrt{2W(u_\varepsilon)}\mathrm{H}_\varepsilon\cdot\mathrm{n}_\varepsilon\tau\Big(\frac{1}{s_0}\mathrm{dist}^\pm(x,I)\Big)\,\mathrm{d}x\\
&\quad+\int_{\mathbb{R}^d}(\chi-\psi_\varepsilon)\frac{1}{s_0}\tau'\Big(\frac{1}{s_0}\mathrm{dist}^\pm(x,I)\Big)\partial_t\mathrm{dist}^\pm(x,I)\,\mathrm{d}x\\
&=\int_{\mathbb{R}^d}\varepsilon^{-1}\sqrt{2W(u_\varepsilon)}\mathrm{H}_\varepsilon\cdot\mathrm{n}_\varepsilon\tau\Big(\frac{1}{s_0}\mathrm{dist}^\pm(x,I)\Big)\,\mathrm{d}x\\
&\quad-\int_{\mathbb{R}^d}(\chi-\psi_\varepsilon)\mathrm{H}_I\cdot\nabla\Big(\tau\Big(\frac{1}{s_0}\mathrm{dist}^\pm(x,I)\Big)\Big)\,\mathrm{d}x\\
&\quad+\int_{\mathbb{R}^d}(\chi-\psi_\varepsilon)\frac{1}{s_0}\tau'\Big(\frac{1}{s_0}\mathrm{dist}^\pm(x,I)\Big)\big(\partial_t\mathrm{dist}^\pm(x,I)+\mathrm{H}_I\cdot\nabla\mathrm{dist}^\pm(x,I)\big)\,\mathrm{d}x\\
&=\int_{\mathbb{R}^d}(\varepsilon^{-1}\sqrt{2W(u_\varepsilon)}\mathrm{H}_\varepsilon\cdot\mathrm{n}_\varepsilon-\nabla\psi_\varepsilon\cdot\mathrm{H}_I)\tau\Big(\frac{1}{s_0}\mathrm{dist}^\pm(x,I)\Big)\,\mathrm{d}x\\
&\quad+\int_{\mathbb{R}^d}(\chi-\psi_\varepsilon)\tau\Big(\frac{1}{s_0}\mathrm{dist}^\pm(x,I)\Big)\nabla\cdot\mathrm{H}_I\,\mathrm{d}x\\
&\quad+\int_{\mathbb{R}^d}(\chi-\psi_\varepsilon)\frac{1}{s_0}\tau'\Big(\frac{1}{s_0}\mathrm{dist}^\pm(x,I)\Big)\big(\partial_t\mathrm{dist}^\pm(x,I)+H_I\cdot\nabla\mathrm{dist}^\pm(x,I)\big)\,\mathrm{d}x,
\end{aligned}$$

where in the last step we have used integration by parts and $\tau(\mathrm{dist}^\pm(x,I(t)))=0$ on $\mathrm{supp}\,\nabla\chi(\cdot,t)$.

This may be rewritten using the definition of ψ_ε and n_ε as

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^d}(\chi-\psi_\varepsilon)\tau\Big(\frac{1}{s_0}\mathrm{dist}^\pm(x,I)\Big)\,\mathrm{d}x\\
&=\int_{\mathbb{R}^d}\varepsilon^{-1}\sqrt{2W(u_\varepsilon)}(\mathrm{H}_\varepsilon-\mathrm{H}_I\varepsilon|\nabla u_\varepsilon|)\cdot\mathrm{n}_\varepsilon\tau\Big(\frac{1}{s_0}\mathrm{dist}^\pm(x,I)\Big)\,\mathrm{d}x\\
&\quad+\int_{\mathbb{R}^d}(\chi-\psi_\varepsilon)\tau\Big(\frac{1}{s_0}\mathrm{dist}^\pm(x,I)\Big)\nabla\cdot\mathrm{H}_I\,\mathrm{d}x\\
&\quad+\int_{\mathbb{R}^d}(\chi-\psi_\varepsilon)\frac{1}{s_0}\tau'\Big(\frac{1}{s_0}\mathrm{dist}^\pm(x,I)\Big)\big(\partial_t\mathrm{dist}^\pm(x,I)+\mathrm{H}_I\cdot\nabla\mathrm{dist}^\pm(x,I)\big)\,\mathrm{d}x.
\end{aligned}$$

Since $\partial_t\mathrm{dist}^\pm(x,I)=-\mathrm{H}_I\cdot\nabla\mathrm{dist}^\pm(x,I)$ holds in a neighborhood of the interface, the last integral vanishes identically if we choose $s_0>0$ sufficiently small, e.g., $s_0=\frac{r_c}{4}$.

Using Cauchy–Schwarz we deduce

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}^d}(\chi-\psi_\varepsilon)\tau\Big(\frac{1}{s_0}\mathrm{dist}^\pm(x,I)\Big)\,\mathrm{d}x\\
&\leq \int_{\mathbb{R}^d}\varepsilon^{-1}\big|\mathrm{H}_\varepsilon-\mathrm{H}_I\varepsilon|\nabla u_\varepsilon|\big|^2\,\mathrm{d}x+\int_{\mathbb{R}^d}\varepsilon^{-1}2W(u_\varepsilon)\Big|\tau\Big(\frac{1}{s_0}\mathrm{dist}^\pm(x,I)\Big)\Big|^2\,\mathrm{d}x\\
&\quad+||(\nabla\cdot\mathrm{H}_I)_+||_{L^\infty}\int_{\mathbb{R}^d}|\psi_\varepsilon-\chi|\Big|\tau\Big(\frac{1}{s_0}\mathrm{dist}^\pm(x,I)\Big)\Big|\,\mathrm{d}x,
\end{aligned}$$

where $(\nabla\cdot\mathrm{H}_I)_+$ denotes the positive part of $\nabla\cdot\mathrm{H}_I$. In order to be able to apply the Gronwall inequality, we note that $\psi_\varepsilon\in[-1,1]$. The most natural way of ensuring this is by using the maximum principle, although also a purely energetic proof by means of the minimizing movements scheme and a truncation argument is available. By the Gronwall inequality, (18), (12d), this shows that

$$\sup_{t\in[0,T]}\int_{\mathbb{R}^d}|\psi_\varepsilon-\chi|\min\{\mathrm{dist}(x,I),1\}\,\mathrm{d}x\leq C(d,T,(I(t))_{t\in[0,T]})\varepsilon^2. \tag{19}$$

In order to pass to an unweighted norm we use the following elementary estimate for $f\in L^\infty(0,\frac{r_c}{2})$: Applying Fubini's theorem after splitting the square $[0,\frac{r_c}{2}]^2$ into two isosceles triangles yields

$$\Big(\int_0^{\frac{r_c}{2}}|f(y)|\,\mathrm{d}y\Big)^2\leq 2\|f\|_\infty\int_0^{\frac{r_c}{2}}|f(y)|y\,\mathrm{d}y.$$

This allows us to estimate for the $\frac{r_c}{2}$-neighborhood of $I(t)$

$$\begin{aligned}
&\Big(\int_{I(t)+B_{\frac{r_c}{2}}}|\psi_\varepsilon(x,t)-\chi(x,t)|\,\mathrm{d}x\Big)^2\\
&\leq C(d,I(t))\Big(\int_{I(t)}\int_0^{\frac{r_c}{2}}|\psi_\varepsilon(w+y\mathrm{n}_I(w),t)-\chi(w+y\mathrm{n}_I(w),t)|\,\mathrm{d}y\\
&\qquad\qquad+\int_0^{\frac{r_c}{2}}|\psi_\varepsilon(w-y\mathrm{n}_I(w),t)-\chi(w-y\mathrm{n}_I(w),t)|\,\mathrm{d}y\,\mathrm{d}S(w)\Big)^2\\
&\leq C(d,I(t))\int_{I(t)}\int_{-\frac{r_c}{2}}^{\frac{r_c}{2}}|\psi_\varepsilon(w+y\mathrm{n}_I(w),t)-\chi(w+y\mathrm{n}_I(w),t)|\\
&\qquad\qquad\times\mathrm{dist}(w+y\mathrm{n}_I(w),I(t))\,\mathrm{d}y\,\mathrm{d}S(w)\\
&\leq C(d,I(t))\int_{I(t)+B_{\frac{r_c}{2}}}|\psi_\varepsilon(x,t)-\chi(x,t)|\,\mathrm{dist}(x,I)\,\mathrm{d}x,
\end{aligned}$$

which in view of (19) yields Theorem 1. ∎

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