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No-dimension Tverberg's theorem and its corollaries in Banach spaces of type p

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Abstract

We continue our study of 'no-dimension' analogues of basic theorems in combinatorial and convex geometry in Banach spaces. We generalize some results of the paper (Adiprasito, Bárány and Mustafa, 'Theorems of Carathéodory, Helly, and Tverberg without dimension', *Proceedings of the Thirtieth Annual ACM-SIAM Symposium on Discrete Algorithms* (Society for Industrial and Applied Mathematics, San Diego, California, 2019) 2350–2360) and prove no-dimension versions of the colored Tverberg theorem, the selection lemma and the weak ε-net theorem in Banach spaces of type $p > 1$. To prove these results, we use the original ideas of Adiprasito, Bárány and Mustafa for the Euclidean case, our no-dimension version of the Radon theorem and slightly modified version of the celebrated Maurey lemma.

1. *Introduction*

In [**1**], the authors started a systematic study of what they called 'no-dimension' analogues of basic theorems in combinatorial and convex geometry such as Carathéodory's, Helly's and Tverberg's theorems, and others. All original versions of these theorems state different combinatorial properties of convex sets in $\mathbb{R}^d$. And the results depend on the dimension d (some of these theorems can be used to characterize the dimension). The idea behind these 'no-dimension' or approximate versions of the well-known theorems is to make them independent of the dimension. However, it comes at some cost – the approximation error. For example, Carathéodory's theorem states that any point p in the convex hull of a set $S \subset \mathbb{R}^d$ is a convex combination of at most $d+1$ points of S. In [**1**, Theorem 2.2], the authors proved that the distance between any point p in the convex hull of a bounded set S of a Euclidean space and the k-convex hull is at most $\frac{\operatorname{diam} S}{\sqrt{2k}}$. Here, the k-convex hull of S is the set of all convex combinations of at most k points of S. Clearly, the last statement does not involve the dimension, but it can guarantee only an approximation of a point.

All the proofs in [**1**] exploit the properties of Euclidean metric significantly. And in general, this type of questions were mostly considered in the Euclidean case (see, for example, the survey [**8**] and the references therein). To our knowledge, there is only one exception at the moment. The celebrated Maurey lemma [**14**] is an approximate version of Carathéodory's theorem for Banach spaces that have (Rademacher) type $p > 1$. Recently Barman [**6**] showed that Maurey's lemma is useful for some algorithms (for example, for computing Nash equilibria and for densest bipartite subgraph problem). Moreover, different problems about approximation of operators (see [**9, 11**]) can be reformulated in the language of no-dimension theorems.

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In this paper, we continue our study of no-dimension theorems in Banach spaces started in [10], where the author provided a greedy algorithm proof of Maurey's lemma in a uniformly smooth Banach space. The main results of this paper is the generalization of approximate Tverberg's theorem and its corollaries to Banach spaces of type p.

The famous Tverberg theorem [15] has been a cornerstone of discrete geometry for several decades. It asserts that:

> Given $(r-1)(d+1)+1$ points in $\mathbb{R}^d$, there is a partition of them into r parts whose convex hulls intersect.

There is a large number of generalizations and variations on this important result (a lively overview of recent developments in the area is presented in [5]). As explained in detail in [13], the colored version of Tverberg theorem, first settled in [16], plays a crucial role in applications. We formulate here the following version of colored Tverberg theorem:

> For any integers $r, d \geqslant 2$, there exists an integer t such that given any $t(d+1)$-point set $Y \subset \mathbb{R}^d$ partitioned into $d+1$ color classes $Y_1, \dots, Y_{d+1}$ with t points each, there exist r pairwise disjoint sets $A_1, \dots, A_r$ such that each A_i contains exactly one point of each Y_j, $j \in [d+1]$, and $\bigcap_{i \in [r]} \operatorname{conv} A_i \neq \emptyset$.

Surprisingly enough, both the following dimensionless colored version of the Tverberg theorem and its Euclidean predecessor [2, Theorem 6.1] imply the same type of combinatorial results in the corresponding Banach spaces, namely, the piercing lemma and the weak ε-net theorem, as the original colored Tverberg theorem in the classical setting.

Let us recall the notion of the (Rademacher) type of a Banach space; we refer to the book [12] as an excellent source on this topic. For a fixed finite set $S = \{s_1, \dots, s_{|S|}\}$ in a linear space L, we use $\mathbb{E}\, \varphi(\operatorname{Rad}(S))$ to denote the expected value of a function $\varphi : L \to \mathbb{R}$ taken over all possible $2^{|S|}$ signed sums $\pm s_1 \dots \pm s_{|S|}$ with the uniform distribution. A Banach space X is said to be of *type* p if there exists a constant $T_p(X) < \infty$ so that, for every finite set of vectors $S = \{x_j\}_{j=1}^n$ in X, we have

$$\mathbb{E}\, \|\operatorname{Rad}(S)\| \leqslant T_p(X) \left(\sum_{j=1}^{n} \|x_j\|^p \right)^{1/p}.$$

By the triangle inequality, every Banach space is of type 1. On the other hand, a Banach space cannot be of type p with $p > 2$. In all our statements, we consider Banach spaces of type $p > 1$.

Throughout the paper, for a Banach space X of type p, $T_p(X)$ denotes the smallest constant that appears in the definition of type p and $w = \frac{1-p}{p}$. We use $C(X)$ to denote a constant depending only on a space X, $B(q, r)$ denotes the ball with radius r around point q. For a positive integer k, we use the notation $[k] = \{1, \dots, k\}$, and we use $\binom{S}{k}$ to denote the set of the k-element subsets of S. The convex hull of a set S is denoted by $\operatorname{conv} S$.

The following statements are the main results of the paper.

THEOREM 1 (No-dimension colored Tverberg theorem). *Let X be a Banach space of type $p > 1$. Let $Z_1, \dots, Z_r \subset X$ be r pairwise-disjoint sets of points in X and with $|Z_i| = k$ for all $i \in [r]$. Let $S = \bigcup_1^r Z_i$ and $D = \max_i \operatorname{diam} Z_i$. Then there is a point q and a partition $S_1, \dots, S_k$ of S such that $|S_i \cap Z_j| = 1$ for every $i \in [k]$ and every $j \in [r]$ satisfying*

$$\operatorname{dist}(q, \operatorname{conv} S_i) \leqslant C_T(X) r^w D \text{ for every } i \in [k], \tag{1}$$

where $C_T(X) = \frac{2^{1/p}}{1-2^w} T_p(X)$.

THEOREM 2 (No-dimension selection lemma). *Let X be a Banach space of type $p>1$. Given a set P in X with $|P|=n$ and $D=\operatorname{diam} P$ and an integer $r\in[n]$. Then there is a point q such that the ball $B(q,C_S(X)r^wD)$ intersects the convex hull of $r^{-r}\binom{n}{r}$ r-tuples in P, where $C_S(X)=(\frac{2^{1/p}}{1-2^w}+1)T_p(X)$.*

THEOREM 3 (No-dimension weak ε-net theorem). *Let X be a Banach space of type $p>1$. Assume P is a subset of X, $|P|=n$, $D=\operatorname{diam} P$, $r\in[n]$ and $\varepsilon>0$. Then there is a set $F\subset X$ of size at most $r^r\varepsilon^{-r}$ such that for every $Y\subset P$ with $|Y|\geqslant\varepsilon n$*

$$(F+B(0,C_E(X)r^wD))\cap\operatorname{conv}Y\neq\emptyset,$$

where $C_E(X)=(\frac{2^{1/p}}{1-2^w}+1)T_p(X)$.

In fact, we just generalize the averaging technique used in [1] to prove no-dimension versions of colored Tverberg's theorem, the selection lemma and the weak ε-net theorem for the Euclidean space. For this purpose, we prove the following no-dimension version of the Radon theorem.

Given a finite set S in a linear space, we denote by $\mathrm{c}(S)$ the centroid of S, that is,

$$\mathrm{c}(S)=\frac{1}{|S|}\sum_{s\in S}s.$$

THEOREM 4 (No-dimension colorful Radon theorem). *Let X be a Banach space of type $p>1$. Let $Z_1,\dots,Z_r\subset X$ be r pairwise-disjoint sets of points in X and with $|Z_i|=n$ for all $i\in[r]$. Let $S=\bigcup_1^r Z_i$ and $D=\max_i\operatorname{diam}Z_i$. Then there is a partition Q^0,Q^1 of S with $|Q^0\cap Z_j|=\lfloor\frac{n}{2}\rfloor$ and $|Q^1\cap Z_j|=\lceil\frac{n}{2}\rceil$ for every $j\in[r]$ such that*

$$\left\|\mathrm{c}(Q^0)-\mathrm{c}(S)\right\|\leqslant\left\|\mathrm{c}(Q^1)-\mathrm{c}(S)\right\|\leqslant C(X)\left\lceil\frac{n}{2}\right\rceil^w r^wD,$$

where $C(X)=2^{1/p}T_p(X)$.

Using Theorem 4, our proofs of Theorems 1–3 follow the same lines as in [1]. We tried to be as close to the original proofs in the Euclidean case as possible. Our results give the same asymptotic in r as in [1] for the Euclidean case and constants are reasonably close (our constant in Theorems 2 and 3 is $2(\sqrt{2}+1)+1$, which is less than twice the constant obtained in [1] for the Euclidean case).

The space ℓ_1 of sequences whose series is absolutely convergent is a Banach space in which neither the no-dimension colorful Radon theorem nor the no-dimension colored Tverberg theorem hold. Fix integer $k,r\geqslant 2$. We use e_n to denote the sequence of zeros and ones where one is in the nth position and all other entries equal 0. Let $Z_i=\{e_{k(i-1)+1},\dots,e_{ki}\}, i\in[r]$ and $S=\bigcup_1^r Z_i$. We leave it to the reader to verify that for any point $q\in\ell_1$ and any partition $S_1,\dots,S_k$ of S such that $|S_i\cap Z_j|=1$ for every $i\in[k]$ and every $j\in[r]$, we have

$$\max_{i\in[r]}\operatorname{dist}(q,\operatorname{conv}S_i)\geqslant 1.$$

In the next section, we discuss the Maurey lemma and prove Theorem 4. Then, in Section 3, we prove the main results of the paper.

2. *An averaging technique*

2.1. *Maurey's lemma*

Maurey's lemma [**14**] is an approximate version of the Carathéodory theorem for Banach spaces of type $p > 1$. We can formulate Maurey's lemma as follows (see also [**7**, Lemma D]).

Let S be a bounded set in a Banach space X of type $p > 1$ and $a \in \operatorname{conv} S$. Then there exists a sequence $\{x_i\}_1^k \subset S$ such that for vectors $a_k = \frac{1}{k} \sum\limits_{i\in[k]} x_i$, the following inequality holds

$$\|a - a_k\| \leqslant T_p(X) k^w \operatorname{diam} S.$$

As we need a slightly more general colored statement, we provide the proof of the following version of Maurey's lemma, which trivially follows from the original one.

Lemma 2.1. *Let $P_1, \dots, P_r$ be r sets in a Banach space X of type $p > 1$, $D = \max_{i\in[r]} \operatorname{diam} P_i$ and $\eta > 0$, and $k \in \mathbb{Z}^+$. Assume that $B(a,\eta) \cap \operatorname{conv} P_i \neq \emptyset$ for every $i \in [r]$. Then there exist r sequences $\{x_j^i\}_{j=1}^k \subset P_i$ such that for vectors $a_k^i = \frac{1}{k}\sum_{j\in[k]} x_j^i$ the following inequality holds*

$$\left\|a - \operatorname{c}\left(a_k^1, \dots, a_k^r\right)\right\| \leqslant T_p(X) k^w r^w D + \eta.$$

Proof. Let x_i be a point of $B(a,\eta) \cap \operatorname{conv} P_i$, $i \in [r]$. By the triangle inequality, it is enough to show that there are k-element multisubsets Q_i of P_i satisfying

$$\|\operatorname{c}(\{Q_1, \dots, Q_r\}) - \operatorname{c}(\{x_1, \dots, x_r\})\| \leqslant T_p(X) k^w r^w D.$$

Since $x_i \in \operatorname{conv} P_i$, there exist $y_1^i, \dots, y_{N_i}^i \in P_i$ and positive scalars $\lambda_1^i, \dots, \lambda_{N_i}^i$, $\sum_{j\in N_i} \lambda_j^i$, such that $x_i = \sum_{j\in N_i} \lambda_j^i y_j^i$. Let F_i be a P_i-valued random variable which takes y_j^i with probability λ_j^i. Let $F_i(1), \dots, F_i(k)$ and $F_i'(1), \dots, F_i'(k)$ be a series of independent copies of F_i,$i \in [r]$. Then

$$\mathop{\mathbb{E}}_{F_1,\dots,F_r} \left\| \sum_{j\in[k]} \sum_{i\in[r]} (F_i(j) - x_i) \right\| \overset{\text{(Ave)}}{\leqslant} \mathop{\mathbb{E}}_{F_1,\dots,F_r} \mathop{\mathbb{E}}_{F_1',\dots,F_r'} \left\| \sum_{j\in[k]} \sum_{i\in[r]} (F_i(j) - F_i'(j)) \right\| \overset{\text{(S)}}{=}$$

$$\mathop{\mathbb{E}}_{F_1,\dots,F_r} \mathbb{E} \left\| \operatorname{Rad}\Big(\{F_i(j) - F_i'(j)\}_{i\in[r]}^{j\in[k]} \Big) \right\| \leqslant T_p(X) k^{1/p} r^{1/p} D,$$

where in step (Ave) we use identity $\mathop{\mathbb{E}}_{F_i}(F_i - x_i) = 0$, in step (S) we use that functions $F_i(j) - F_i'(j)$ are symmetric and independent of each other, and the last inequality is a direct consequence of the definition of type p. Dividing by kr, we see that randomly chosen $Q_1, \dots, Q_r$ satisfy the desired bound and complete the proof. □

In other words, we approximate a point in the convex hull by the centroid of a multiset, that is, one element might be counted several times. But in our proofs, we need to choose different points or a subset of cardinality k. And, as the following simple example shows, not every point of the convex hull of a set can be approximated by the centroids of its subsets.

Example. Let points of a set S of a Banach space X be 'concentrated' around a unit vector q. Adding $-q$ to S, we may assume that $0 \in \operatorname{conv} S$. It is easy to see that the centroid of any $Q \in \binom{S}{k}$ is at constant distance from the origin for $k \geqslant 4$.

However, as will be shown in the next section, we can always approximate the centroid of a set by the centroids of its k-element subset in a Banach space of type $p > 1$.

2.2. *Approximation by the centroids*

Let $S = \{s_1, \dots, s_{|S|}\}$ be the disjoint union of sets (considered colors) $Z_1, \dots, Z_r$, and each Z_j has size $n \geqslant 2$. For any subset Q of S, we use Q_i to denote $Q \cap Z_i$. Let $d \in [n-1]$. The set of all $(d \times r)$-element subsets Q of S such that $|Q_i| = d$ is denoted as $\binom{S}{d/r}$. We use $(\Omega_d(S), P)$ to denote a probability space on $\binom{S}{d/r}$ with the uniform distribution. That is, the probability of choosing $Q \in \binom{S}{d/r}$ is

$$\frac{1}{P^d_{(n,r)}}, \text{ where } P^d_{(n,r)} = \binom{n}{d}^r.$$

We use $\sigma(K)$ to denote the sum of all elements of a set K.

The following statement is the key tool in the proof of Theorem 4, we phrase it in a general form. Apart from the cumbersome double counting, we use only Jensen's inequality to prove the following lemma.

Lemma 2.2. *Under the above conditions, let additionally $S = \bigcup_{i \in [r]} Z_i$ be a subset of points in a linear space L with $\sigma(Z_i) = 0$ for all $i \in [r]$. Let φ be a convex function $L \to \mathbb{R}$. Then*

$$\mathop{\mathbb{E}}_{(\Omega_d(S),P)} \varphi(\gamma\sigma(Q)) \leqslant \mathbb{E}\, \varphi(\operatorname{Rad}(S)), \tag{2}$$

where γ is real number from interval (1,2) *that depends only on n and d.*

Proof. First, we explain the idea of the proof and then proceed with the technical details. For a fixed $Q \in \binom{S}{d/r}$, we group all summands in the right-hand side of (2) such that a signed sum $\pm s_1 \cdots \pm s_{|S|}$ can be obtained from $\sigma(Q) - \sigma(S \setminus Q)$ by changing some signs either in the set Q_i or in $Z_i \setminus Q_i$ for each $i \in [r]$. Then we apply Jensen's inequality for every group of summands that corresponds to a fixed $Q \in \binom{S}{d/r}$. Using the symmetry, we understand that the argument of φ looks like $a\sigma(Q) - b\sigma(S \setminus Q)$. Calculating the coefficients and using identity $\sigma(Q) = -\sigma(S \setminus Q)$, we prove the lemma.

For the sake of convenience, let $Q_i(+) = Q_i$ and $Q_i(-) = Z_i \setminus Q_i$, $\delta(+) = d$ and $\delta(-) = n - d$. Let $\operatorname{Cext}_i(Q_i)$ be the set of all subsets Y_i of Z_i such that either Y_i is a non-empty subset of $Q_i(+)$ or $Y_i \subset Q_i(-)$. We use $\operatorname{Cext}(Q)$ to denote the set of all sets $Y \subset S$ such that $Y_i \in \operatorname{Cext}_i(Q_i)$. For every $Y \in \operatorname{Cext}(Q)$, we define $\operatorname{sgn} Y_i$ to be $+$ or $-$ whenever $Y_i \subset Q_i(+)$ or $Y_i \subset Q_i(-)$, respectively. Let

$$W(Y) = \prod_{i \in [r]} \binom{\delta(-\operatorname{sgn}(Y_i)) + |Y_i|}{|Y_i|},$$

be the weight of $Y \in \operatorname{Cext}(Q)$.

Fix a choice of signs $\varepsilon_1, \dots, \varepsilon_{|S|} \in \{-1, 1\}$. Then $\varepsilon_1 s_1 + \cdots + \varepsilon_{|S|} s_{|S|}$ can be represented as

$$\sigma(Q) - \sigma(S \setminus Q) - 2 \sum_{i \in [r]} \operatorname{sgn} Y_i \cdot \sigma(Y_i),$$

for some $Q \in \binom{S}{d/r}$ and $Y \in \operatorname{Cext} Q$. In this representation, neither Q_i nor Y_i are unique. However, $\operatorname{sgn} Y_i$, $|Y_i|$ are uniquely determined. Indeed, for each $i \in [r]$, there are two cases. First, if d_i, the number of $+$ signs in Z_i, is strictly less than $\delta(+)$, then Q_i must contain those elements in Z_i with a $+$ sign, and an additional $d - d_i$ elements, which may be chosen in $\binom{n-d_i}{\delta(+)-d_i}$ ways. The set of this additional elements will be Y_i, and clearly, $\operatorname{sgn} Y_i = +$ and $|Y_i| = \delta(+) - d_i$. If $d_i \geqslant \delta(+)$, then Q_i must contain d of those elements of Z_i that have a $+$ sign, that is, Q_i may be chosen in $\binom{d_i}{d}$ ways. The remaining elements of Z_i with a $+$ sign will be Y_i, and clearly, $|Y_i| = d_i - d$ and $\operatorname{sgn} Y_i = -$.

We obtain that

$$\mathbb{E}\,\varphi(\mathrm{Rad}(S)) = \frac{1}{2^{nr}} \sum_{Q\in\binom{S}{d/r}} \sum_{Y\in\mathrm{Cext}(Q)} \frac{1}{W(Y)}\varphi\left(\sigma(Q) - \sigma(S\setminus Q) - 2\sum_{i\in[r]} \operatorname{sgn} Y_i \cdot \sigma(Y_i)\right) \quad (3)$$

By the symmetry, the total sum $\frac{1}{2^{nr}}\sum_{Y\in\mathrm{Cext}(Q)} \frac{1}{W(Y)}$ of coefficients at φ in the right-hand side of (3) is independent of a choice of $Q \in \binom{S}{d/r}$. Since the sum of coefficients at φ in the left-hand side of (3) is one, we have

$$\frac{1}{2^{nr}} \sum_{Y\in\mathrm{Cext}(Q)} \frac{1}{W(Y)} = \frac{1}{P^d_{(n,r)}}, \quad (4)$$

for a fixed $Q \in \binom{S}{d/r}$.

Using this and Jensen's inequality for each $Q \in \binom{S}{d/r}$ in the right-hand side of (3), we get that

$$\begin{aligned}&\mathbb{E}\,\varphi(\mathrm{Rad}(S))\\ &\geqslant \frac{1}{P^d_{(n,r)}} \sum_{Q\in\binom{S}{d/r}} \varphi\left(\sum_{Y\in\mathrm{Cext}(Q)} \frac{P^d_{(n,r)}}{2^{nr}W(Y)}\left[\sigma(Q) - \sigma(S\setminus Q) - 2\sum_{i\in[r]}\operatorname{sgn} Y_i\cdot\sigma(Y_i)\right]\right) \quad (5)\end{aligned}$$

Let us carefully calculate the argument of φ in the right-hand side of the last inequality for a fixed Q. This argument is the sum of elements of S with some coefficients. By the symmetry, for a fixed $i \in [r]$, the coefficients at elements of $Q_i(+)$ are the same, analogously, the coefficients at elements of $Q_i(-)$ coincide. That is, the argument of φ is $\sum_{i\in[r]} \alpha_i^+\sigma(Q_i(+)) - \sum_{i\in[r]} \alpha_i^-\sigma(Q_i(-))$, where α_i^+ and α_i^-, $i \in [r]$, are some coefficients. Again, by the symmetry, we have that $\alpha_1^+ = \cdots = \alpha_r^+$ and $\alpha_1^- = \cdots = \alpha_r^-$. Hence, it is enough only to calculate $\alpha_1^+\sigma(Q_1(+)) - \alpha_1^-\sigma(Q_1(-))$. We denote this expression by $A_1(Q)$.

By (4), the part of the argument containing the elements of the first color is

$$A_1(Q) = \sigma(Q_1(+)) - \sigma(Q_1(-)) - 2\frac{P^d_{(n,r)}}{2^{nr}} \sum_{Y_1^0\in\mathrm{Cext}_1(Q_1)} \operatorname{sgn} Y_1^0\cdot\sigma\left(Y_1^0\right)\left[\sum_{\substack{Y\in\mathrm{Cext}(Q);\\ Y_1=Y_1^0}} \frac{1}{W(Y)}\right]. \quad (6)$$

Denote $W_1(Y) = W_1(Y_1) = W_1(\operatorname{sgn} Y_1, |Y_1|) = \binom{\delta(-\operatorname{sgn}(Y_1))+|Y_1|}{|Y_1|}$. Then identity (6) can be rewritten as

$$\begin{aligned}A_1(Q) &= \sigma(Q_1(+)) - \sigma(Q_1(-))\\ &\quad -2\frac{P^d_{(n,1)}}{2^n} \sum_{Y_1^0\in\mathrm{Cext}_1(Q_1)} \frac{\operatorname{sgn} Y_1^0\cdot\sigma\left(Y_1^0\right)}{W_1(Y_1^0)}\left[\frac{P^d_{(n,r-1)}}{2^{n(r-1)}}\sum_{\substack{Y\in\mathrm{Cext}(Q);\\ Y_1=Y_1^0}}\frac{W_1(Y)}{W(Y)}\right]\end{aligned}$$

Using (4) for $r \to r-1$ and since the signs at the elements of different colors are chosen independently, we have

$$\frac{1}{2^{n(r-1)}} \sum_{\substack{Y\in\mathrm{Cext}(Q);\\ Y_1=Y_1^0}} \frac{W_1(Y)}{W(Y)} = \frac{1}{P^d_{(n,r-1)}},$$

for a fixed $Y_1^0 \in \mathrm{Cext}_1(Q_1)$.

By this, we see that

$$A_1(Q) = \sigma(Q_1(+)) - \sigma(Q_1(-)) - 2\frac{P^d_{(n,1)}}{2^n} \sum_{Y_1 \in \operatorname{Cext}_1 Q_1} \frac{\operatorname{sgn} Y_1 \cdot \sigma(Y_1)}{W_1(Y_1)}.$$

There are $\binom{\delta(\operatorname{sgn} Y_1)}{|Y_1|}$ possible sets Y_1 for fixed $\operatorname{sgn} Y_1$ and $|Y_1|$. These sets cover set $Q_1(\operatorname{sgn} Y_1)$ uniformly. Grouping all Y_1 with fixed $\operatorname{sgn} Y_1$ and $|Y_1|$ together, we obtain

$$A_1(Q) = \sigma(Q_1(+)) - \sigma(Q_1(-)) - 2\frac{P^d_{(n,1)}}{2^n} \sum_{\operatorname{sgn},q} \operatorname{sgn} \cdot \sigma(Q_1(\operatorname{sgn})) \frac{M(\operatorname{sgn},q)}{W_1(\operatorname{sgn},q)}, \tag{7}$$

where $M(\operatorname{sgn},q) = \frac{q}{\delta(\operatorname{sgn})}\binom{\delta(\operatorname{sgn})}{q}$ and the summation is over all sgn and q such that there exists $Y_1 \in \operatorname{Cext}_1(Q_1)$ with $|Y_1| = q$ and $\operatorname{sgn} Y_1 = \operatorname{sgn}$. Using (4) for $r = 1$, we obtain

$$\sum \frac{P^d_{(n,1)}}{2^n} \frac{\binom{d(\operatorname{sgn})}{q}}{W_1(\operatorname{sgn},q)} = 1,$$

where the summation is over all sgn and q such that there exists $Y_1 \in \operatorname{Cext}_1(Q_1)$ with $|Y_1| = q$ and $\operatorname{sgn} Y_1 = \operatorname{sgn}$. Using this and identity $\sigma(Q_1(+)) = -\sigma(Q_1(-))$, we have that $A_1(Q) = 2\gamma\sigma(Q_1(+))$, where

$$\gamma = \frac{P^d_{(n,1)}}{2^n} \sum_0^{\delta(-)} \left(1 - \frac{j}{\delta(-)}\right) \frac{\binom{\delta(-)}{j}}{\binom{\delta(+)+j}{j}} + \frac{P^d_{(n,1)}}{2^n} \sum_1^{\delta(+)} \left(1 - \frac{j}{\delta(+)}\right) \frac{\binom{\delta(+)}{j}}{\binom{\delta(-)+j}{j}}.$$

After simple transformations, we get

$$\gamma = \frac{1}{2^n}\left[\sum_0^{\delta(-)} \left(1 - \frac{j}{\delta(-)}\right)\binom{n}{\delta(-)-j} + \sum_1^{\delta(+)} \left(1 - \frac{j}{\delta(+)}\right)\binom{n}{\delta(+)-j}\right].$$

Since $n - (\delta(+) - j) = \delta(-) + j$ and $\binom{n}{k} = \binom{n}{n-k}$, we have that $\gamma < 1$. As for a lower bound on γ, after simple transformations, we have

$$2^n\gamma = \frac{n}{\delta(-)} \sum_2^{\delta(-)} \binom{n-1}{\delta(-)-j} + \binom{n-1}{\delta(-)-1} + \binom{n-1}{\delta(-)} + \frac{n}{\delta(+)} \sum_1^{\delta(+)} \binom{n-1}{\delta(-)+j}.$$

Clearly, the right-hand side here is strictly bigger than 2^{n-1} whenever $\delta(+) \in [n-1]$. Therefore, $2\gamma \in (1,2)$. Returning to (5), we obtain

$$\mathbb{E}\,\varphi(\operatorname{Rad}(S)) \geqslant \frac{1}{P^d_{(n,r)}} \sum_{Q \in \binom{S}{d/r}} \varphi\left(\gamma \sum_{i \in [r]} \sigma(Q_i)\right) = \mathop{\mathbb{E}}_{(\Omega_d(S),P)} \varphi(\gamma\sigma(Q)).$$

This completes the proof. □

Proof of Theorem 4. Denote $d = \lceil \frac{n}{2} \rceil$.
The first inequality follows from identity

$$d(\mathrm{c}(Q) - \mathrm{c}(S)) + (n-d)(\mathrm{c}(S \setminus Q) - \mathrm{c}(S)) = 0,$$

where $Q \in \binom{S}{d/r}$.

By Lemma 2.2 for $d = \lceil \frac{n}{2} \rceil$ and sets $Z_i - \mathrm{c}(Z_i)$ and by the definition of type p, we see that there exists Q^1 such that

$$\|\sigma(Q^1) - d\,\mathrm{c}(S)\| \leqslant \gamma\|\sigma(Q^1) - d\,\mathrm{c}(S)\| \leqslant T_p(X)(nr)^{1/p}D.$$

Dividing the last inequality by $d = \lceil \frac{n}{2} \rceil$, we get the needed inequality. Clearly, $C(X)$ can be chosen to be $2^{1/p} T_p(X)$. $\square$

3. *Proofs of the main results*

Theorem 1 is a direct consequence of Theorem 4 and its proof only requires the validity of the no-dimension colorful Radon theorem. Theorems 2 and 3 follows from the no-dimension colorful Radon theorem and the Maurey lemma. We think that is better to separate the statements that require some geometric properties of a Banach space from their combinatorial corollaries.

We say that *the no-dimension colorful Radon theorem with constants $C(X)$ and $w \in (0,1)$ holds in a Banach space X* if for any $r \in \mathbb{Z}^+$ and any r pairwise disjoint subsets $Z_1, \dots, Z_r$ of X with $|Z_i| = n$ for all $i \in [r]$, there exists a partition Q^0, Q^1 of $\bigcup_1^r Z_i$ with $|Q^0 \cap Z_j| = \lfloor \frac{n}{2} \rfloor$ and $|Q^1 \cap Z_j| = \lceil \frac{n}{2} \rceil$ for every $j \in [r]$ such that

$$\left\| \mathrm{c}(Q^0) - \mathrm{c}\left(\bigcup_1^r Z_i \right) \right\| \leqslant \left\| \mathrm{c}(Q^1) - \mathrm{c}\left(\bigcup_1^r Z_i \right) \right\| \leqslant C(X) \left\lceil \frac{n}{2} \right\rceil^w r^w \max_i \operatorname{diam} Z_i. \tag{8}$$

We say that *the no-dimension colorful Carathéodory theorem with constants $C(X)$ and $w \in (0,1)$ holds in a Banach space X* if for any $r \in \mathbb{Z}^+$ and any r subsets $P_1, \dots, P_r$ of X and point $q \in X$ such that $B(q, \eta) \cap \operatorname{conv} P_i \neq \emptyset$ for every $i \in [r]$, there exist points $s_i \in P_i$, $i \in [r]$, such that

$$\operatorname{dist}(q - \operatorname{conv}\{s_1, \dots, s_r\}) \leqslant C(X) r^w \max_i \operatorname{diam} P_i + \eta. \tag{9}$$

THEOREM 1^*. *Let the no-dimension colorful Radon theorem with constants $C_1(X)$ and $w \in (0,1)$ hold in a Banach space X. Let $Z_1, \dots, Z_r \subset X$ be under the same condition as in Theorem 1, let $S = \bigcup_1^r Z_i$ and $D = \max_i \operatorname{diam} Z_i$. Then there is a point q and a partition $S_1, \dots, S_k$ of S such that $|S_i \cap Z_j| = 1$ for every $i \in [k]$ and every $j \in [r]$ satisfying*

$$\operatorname{dist}(q, \operatorname{conv} S_i) \leqslant C(X) r^w D \text{ for every } i \in [k], \tag{10}$$

where $C(X) = \frac{1}{1 - 2^w} C_1(X)$.

Proof. We build an incomplete binary tree. Its root is S and its vertices are subsets of S. The children of S are Q^0, Q^1 from the no-dimensional colorful Radon theorem, the children of Q^0, respectively, Q^1 are Q^{00}, Q^{01} and Q^{10}, Q^{11} obtained again by applying the no-dimensional colorful Radon theorem to Q^0 and Q^1.

We split the resulting sets into two parts of as equal sizes as possible the same way, and repeat. We stop when the set $Q^{\delta_1 \dots \delta_h}$ contains exactly one element from each color class. In the end, we have sets $S_1, \dots, S_r$ at the leaves. They form a partition of S with $|S_i \cap Z_j| = 1$ for every $j \in [r]$ and $i \in [k]$. We have to estimate $\| \mathrm{c}(S_i) - \mathrm{c}(S) \|$. Let $S, Q^{\delta_1}, \dots, Q^{\delta_1 \dots \delta_h}, S_i$ be the sets in the tree on the path from the root to S_i. Using inequality (8) gives

$$\begin{aligned} \|\mathrm{c}(S) - \mathrm{c}(S_i)\| &\leqslant \left\|\mathrm{c}(S) - \mathrm{c}(Q^{\delta_1})\right\| + \left\|\mathrm{c}(Q^{\delta_1}) - \mathrm{c}(Q^{\delta_1 \delta_2})\right\| + \dots + \left\|\mathrm{c}(Q^{\delta_1 \dots \delta_h}) - \mathrm{c}(S_i)\right\| \\ &\leqslant C_1(X) \sum_0^\infty \left(2^i\right)^w r^w D \leqslant \frac{C_1(X)}{1 - 2^\omega} r^w D. \end{aligned}$$ $\square$

As in [1], Theorem $1*$ implies the following statement.

COROLLARY 3.1. *Let the no-dimension colorful Radon theorem with constants $C_1(X)$ and $w \in (0,1)$ hold in a Banach space X. Given a set P of n points in X and an integer $k \in [n]$, there exists a point q and a partition of P into k sets $P_1, \ldots, P_k$ such that*

$$\operatorname{dist}(q, \operatorname{conv} P_i) \leqslant C(X) \left(\frac{n}{k}\right)^w \operatorname{diam} P$$

for every $i \in [k]$.

Proof. Write $|P| = n = kr + s$ with $k \in \mathbb{N}$ so that $0 \leqslant s \leqslant k-1$. Then delete s elements from P and split the remaining set into sets (colors) $C_1, \ldots, C_r$, each of size k. Apply the colored version, that is Theorem 1∗, and add back the deleted elements (anywhere you like). The outcome is the required partition. $\square$

THEOREM 2∗. *Let the no-dimension colorful Radon theorem with constants $C_1(X)$ and $w \in (0,1)$ hold in a Banach space X, and let the no-dimension colorful Carathéodory theorem with constants $C_2(X)$ and w hold in X. Given a set P in X with $|P| = n$ and $D = \operatorname{diam} P$ and an integer $r \in [n]$. Then there is a point q such that the ball $B(q, C(X) r^w D)$ intersects the convex hull of $r^{-r}\binom{n}{r}$ r-tuples in P, where $C(X) = \frac{1}{1-2^w} C_1(X) + C_2(X)$.*

Proof. This is a combination of the no-dimension colorful Carathéodory and the no-dimension colored Tverberg theorem, like in [**4**]. We assume that $n = kr$ (k is an integer) by discarding at most $r-1$ points of P. Set $\gamma = \frac{1}{1-2^w} C_1(X) r^w D$. The no-dimension colored Tverberg theorem implies that P has a partition $\{P_1, \ldots, P_k\}$ such that $|P_i| = r$ and $\operatorname{conv} P_i$ intersects the ball $B(q, \gamma)$ for every $i \in [k]$, where $q \in X$ is a suitable point.

Next choose a sequence $1 \leqslant j_1 \leqslant j_2 \leqslant \ldots \leqslant j_r \leqslant k$ (repetitions allowed) and, by the no-dimension Carathéodory theorem, there are points $s_i \in P_{j_i}$, $i \in [r]$, such that inequality (9) holds for $P_{j_1}, \ldots, P_{j_r}$ and $\eta = \gamma$. If at this step we have chosen some points several times, we add other arbitrary chosen points of the set P_j such that we use the number of appearances of j in $1 \leqslant j_1 \leqslant j_2 \leqslant \ldots \leqslant j_r \leqslant k$ elements of P_j for each $j \in [r]$. This gives a transversal $T_{j_1 \ldots j_r}$ of $P_{j_1}, \ldots, P_{j_r}$ whose convex hull intersects the ball

$$B(q, \gamma + C_2(X) r^w D).$$

So the convex hull of all of these transversals intersects $B(q, \gamma + C_2(X) r^w D)$. They are all distinct r-element subsets of P and their number is

$$\binom{k+r-1}{r} = \binom{\frac{n-s}{r} + r - 1}{r} \geqslant r^{-r} \binom{n}{r}. \qquad \square$$

THEOREM 3∗. *Let the no-dimension colorful Radon theorem with constants $C_1(X)$ and $w \in (0,1)$ hold in a Banach space X, and let the no-dimension colorful Carathéodory theorem with constants $C_2(X)$ and w hold in X. Assume P is a subset of X, $|P| = n$, $D = \operatorname{diam} P$, $r \in [n]$ and $\varepsilon > 0$. Then there is a set $F \subset X$ of size at most $r^r \varepsilon^{-r}$ such that for every $Y \subset P$ with $|Y| \geqslant \varepsilon n$*

$$(F + B(0, C(X) r^w D)) \cap \operatorname{conv} Y \neq \emptyset,$$

where $C(X) = \frac{1}{1-2^w} C_1(X) + C_2(X)$.

Proof. The proof is an algorithm that goes along the same lines as in the original weak ε-net theorem [**3**]. Set $F := \emptyset$, $C(X) = \frac{1}{1-2^w} C_1(X) + C_2(X)$ and let $\mathcal{H}$ be the family of all r-tuples of P. On each iteration we will add a point to F and remove r-tuples from $\mathcal{H}$.

If there is $Y \subset P$ with $(F + B(0, C(X)r^w D)) \cap \operatorname{conv} Y = \emptyset$, then apply Theorem 2∗ to that Y resulting in a point $q \in X$ such that the convex hull of at least

$$\frac{1}{r^r}\binom{\varepsilon n}{r}$$

r-tuples from Y intersect $B(q, C(X)r^w D)$. Add the point q to F and delete all r-tuples $Q \subset Y$ from $\mathcal{H}$ whose convex hull intersects $B(q, C(X)r^w D)$. On each iteration, the size of F increases by one, and at least $r^{-r}\binom{\varepsilon n}{r}$ r-tuples are deleted from $\mathcal{H}$. So after

$$\frac{\binom{n}{r}}{\frac{1}{r^r}\binom{\varepsilon n}{r}} \leqslant \frac{r^r}{\varepsilon^r}$$

iterations the algorithm terminates as there cannot be any further $Y \subset P$ of size εn with $(F + B(q, C(X)r^w D)) \cap \operatorname{conv} Y = \emptyset$. Consequently the size of F is at most $r^r \varepsilon^{-r}$. □

By Lemmas 2.1 and 2.2, the no-dimension colorful Radon theorem with constants $2^{1/p}T_p(X)$ and $w = \frac{1-p}{p}$ and the no-dimension colorful Carathéodory theorem with constants $T_p(X)$ and $w = \frac{1-p}{p}$ hold in a Banach space X of type $p > 1$. Therefore, Theorems 1∗–3∗ imply Theorems 1–3.

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