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Emergence of cascading dynamics in interacting tipping elements of ecology and climate

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In ecology, climate and other fields, (sub)systems have been identified that can transition into a qualitatively different state when a critical threshold or tipping point in a driving process is crossed. An understanding of those tipping elements is of great interest given the increasing influence of humans on the biophysical Earth system. Complex interactions exist between tipping elements, e.g. physical mechanisms connect subsystems of the climate system. Based on earlier work on such coupled nonlinear systems, we systematically assessed the qualitative long-term behaviour of interacting tipping elements. We developed an understanding of the consequences of interactions on the tipping behaviour allowing for tipping cascades to emerge under certain conditions. The (narrative) application of these qualitative results to real-world examples of interacting tipping elements indicates that tipping cascades with profound consequences may occur: the interacting Greenland ice sheet and thermohaline ocean circulation might tip before the tipping points of the isolated subsystems are crossed. The eutrophication of the first lake in a lake chain might propagate through the following lakes without a crossing of their individual critical nutrient input levels. The possibility of emerging cascading tipping dynamics calls for the development of a unified theory of interacting tipping elements and the quantitative analysis of interacting real-world tipping elements.

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1. Introduction

Many natural systems exhibit nonlinear dynamics and can undergo a transition into a qualitatively different state when a critical threshold is crossed. Those systems are called tipping elements and the corresponding threshold in terms of a critical parameter is the tipping point of the system. A precise mathematical definition is given in [1]. Examples for tipping elements can be found in ecology as a specific type of regime shifts [2,3] such as the transition of a shallow lake from a clear to a turbid state [4–9]. Furthermore, subsystems of the Earth system [1,10,11] such as the thermohaline circulation [12–18] or the Greenland ice sheet [19,20] have been identified as tipping elements.

The term tipping point among other roots originated from describing the changing prevalence of ethnically diverse population in a US community [21–25] and has been applied to natural systems more recently. However, the idea that systems may show such nonlinear behaviour has already been developed within the frameworks of dynamical systems and catastrophe theory [26–30]. The latter theory received increasing attention and has been applied to several real-world systems in the period after its introduction [31]. Its extensive use has been criticized [32–35] so that it became a mathematical theory without much recent influence [36]. Mostly independently of the results given by catastrophe theory, critical transitions, tipping points and regime shifts have been analysed in ecology [2,9,37–39] using the concepts of multistability and resilience [40,41]. Some first attempts to define a climatic tipping element relating to abrupt climate shifts can be found in [42–44].

Different types of tipping points are discussed in the literature [9,23,37–39,45]. First, a qualitative change of the system's state when a continuously changing control parameter crosses a threshold is called bifurcation-induced tipping [23,46–48]. Noise can induce a transition into an alternative stable state without a change of the system's control parameter [23,38,49]. Furthermore, rate-induced tipping describes the shift to a qualitatively different state when the rate of change of a control parameter crosses a critical threshold [23,45,49,50].

It is known that bifurcation-induced tipping, even though often mentioned, is not the only possible type of tipping [24,45,49,51]. Nevertheless, the response of many natural systems to a control parameter can be described in terms of a double fold bifurcation [48,52,53].

Real-world tipping elements are not independent from each other [52], but there may exist complex interactions between them. Potential interactions through various physical mechanisms were revealed for tipping elements in the climate system [54]. As an example, meltwater influx into the North Atlantic as a result of a tipping of the Greenland ice sheet could weaken the Atlantic meridional overturning circulation [55]. Lake chains or rivers can be seen as an ecological example for coupled tipping elements. Each lake or river section in the chain can undergo a transition from a clear to a turbid state in response to nutrient input [5,6,8]. The single lakes can in reality be connected through small rivers or streams and can therefore not be considered independently [56–58].

The tipping probability of a certain tipping element might be influenced by the behaviour of other interacting tipping elements [54,59]. As a consequence, crossing of a critical threshold of a first tipping element could trigger, as a domino effect, a critical transition in a coupled tipping element or even tipping cascades [37,59–64]. In the following, we use the term tipping cascade for a critical transition triggered by a preceding tipping event of an influencing system. The heterogeneity of the subsystems as well as the coupling strength may be important factors that influence the overall system behaviour [65] and should be considered in the analysis of coupled tipping elements. In the case of interacting climate tipping elements, a tipping cascade may impose a considerable risk on human societies [66].

Different attempts to analyse the influence of coupling between different tipping elements on their tipping behaviour have been followed. The development of critical transitions in lake chains was studied using established models of lake eutrophication [58,67,68]. In analogy to wave propagation in discrete media [69–72], the spread of local disturbances in spatially extended, bistable ecosystems was analysed for explicit ecological examples [73] and more theoretically [74]. In addition, cascades may occur on networks [75–78] and networks of networks [79–83]. [52,84,85] analysed the system behaviour of special cases of coupled cusp catastrophes. The possible appearance of tipping cascades in coupled bifurcational systems and, in particular, in the climate system was supported by results from coupling conceptual models of the Atlantic meridional overturning circulation and El Niño-Southern Oscillation [86].

Consequences of interactions between tipping elements, their nonlinear dynamics as well as the possible development of tipping cascades in systems of interacting tipping elements have not been assessed systematically so far. Here, we make an advance in the theory of interacting tipping elements
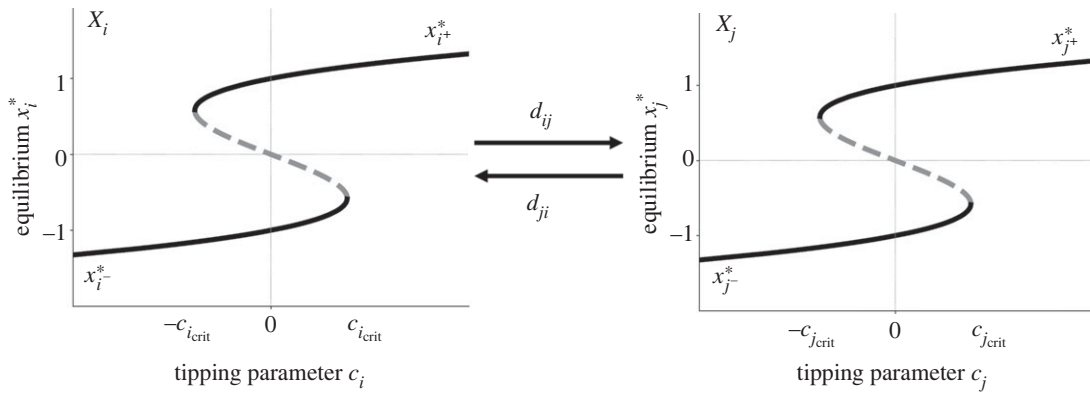
Figure 1. Schematic coupled tipping elements. Two subsystems X_i and X_j are coupled via the coupling functions C_i and C_j with coupling strengths d_{ji} and d_{ij}. The dynamics of both subsystems is given by the normal form of the cusp catastophe. Shown are the equilibria x_i^*, x_j^* depending on the tipping parameter c_i, c_j.

focusing on bifurcation-induced tipping in the form of cusp catastrophes. The special cases presented in [52,85] were extended by looking at uni- and bi-directional coupling and tipping chains consisting of two and three elements. We explored the tipping behaviour of the interacting system extensively under the influence of different coupling types (positive and negative interactions) and the coupling strength and particularly focused on identifying conditions that favour tipping cascades. In addition, we applied our theoretical results to real-world systems to reveal mathematically possible tipping cascades in ecological systems such as lake chains and in the climate system.

2. Model

We use a conceptual model of tipping elements in order to investigate the qualitative long-term behaviour of coupled subsystems each of which exhibits critical transitions. In particular, we consider a continuous dynamical system $\dot{x}_i(t) = f_i(x_1, \ldots, x_n)$ in n-dimensions, where each component $x_i(t) \in \mathbb{R}$ corresponds to a generic tipping element X_i.

The dynamics of the tipping elements is modelled with the topological normal form of the cusp bifurcation [52,85], i.e. the most generic polynomial system exhibiting this type of tipping behaviour (figure 1). The long-term behaviour of many real-world systems in terms of the system's state, such as the strength of the thermohaline circulation [12,15,17,18], the ice thickness of the Greenland ice sheet [19] and the algae density in shallow lakes [6,8], is qualitatively represented by a slice of the cusp catastrophe, a double fold bifurcation, showing bistability, hysteresis properties and transitions to a qualitatively different state when a critical threshold is crossed [87]. In contrast to other possible bifurcations such as the transcritical, pitchfork or Hopf bifurcations, the double fold bifurcation as a 'dangerous' bifurcation [46] captures the catastrophic nature of tipping which is of major interest here. A tipping element X_i is then represented by

$$f_i^0(x_i) = a_i x_i(t) - b_i x_i^3(t) + c_i \quad \text{with } a_i, b_i, c_i \in \mathbb{R}, \tag{2.1}$$

where $a_i, b_i > 0$ and f_i^0 corresponds to the uncoupled case.

The parameter a_i corresponds to the distance between the upper and lower layers of stable equilibria of the cusp and the parameter b_i controls the strength of the nonlinearity in the system. Both parameters are fixed throughout the analysis and chosen to assure bistability of the subsystems for a certain range of the parameter c_i allowing for tipping behaviour. The control parameter c_i is then associated with a tipping parameter the size of which determines whether the system is in this bistable range or not. By leaving the bistability range, a critical transition from one stable state to another may arise (figure 1):

For $-c_{i_{\text{crit}}} < c_i < c_{i_{\text{crit}}}$, equation (2.1) has one negative stable equilibrium $x_{i-}^* < 0$ and a positive stable equilibrium $x_{i+}^* > 0$ as alternative stable states. We call $x_{i-}^* < 0$ and $x_{i+}^* > 0$ the normal and the alternative state, respectively. Increasing the control parameter c_i such that the threshold $c_{i_{\text{crit}}}$ is crossed, the normal state x_{i-}^* disappears and only the alternative state x_{i+}^* exists. If the system resided in the normal state x_{i-}^*, it transitions to the alternative state x_{i+}^* for $c_i > c_{i_{\text{crit}}}$. Analogously, for $c_i < -c_{i_{\text{crit}}}$, only the normal state x_{i-}^* exists and for lowering the control parameter c_i below $-c_{i_{\text{crit}}}$ the system falls

Table 1. Overview of linearly coupled tipping elements studied in the literature.

reference	coupling type	parameter choices
	$n = 2$	
[52]	master–slave system	$b_1 = b_2 = 1$
	with linear coupling	$a_1 = a_2 = 1$
		$C_1(x_1, x_2) = 0$
		$C_2(x_1, x_2) = dx_1$
[85]	Kadyrov style	$a_1 = a_2 = 1$
		$b_1 = b_2 = 1$
		$c_1 = c_2 = 0$
		$C_1(x_1, x_2) = d_{21}x_2$
		$C_2(x_1, x_2) = d_{12}x_1$
		symmetric coupling: $d_{21} = d_{12}$
		asymmetric coupling: $d_{21} \neq d_{12}$
	$n = 3$	
[52]	master–slave–slave system	$b_1 = b_2 = b_3 = 1$
	with linear coupling	$a_1 = a_2 = a_3 = 1$
		$C_1(x_1, x_2, x_3) = 0$
		$C_2(x_1, x_2, x_3) = dx_1$
		$C_3(x_1, x_2, x_3) = dx_2$
	$n > 2$	
[84]	n equations	$x_i = -x_i^3 + A_{ij}x_j$
	coupled in a graph	A_{ij}: matrix of size $N \times N$

from the disappearing alternative state x_{i+}^* to the normal state x_{i-}^* (figure 1). The transition of the uncoupled tipping elements through a changing control parameter c_i at the critical manifold [88] given by the roots of the polynomial can be quantified: depending on the sign of the discriminant $D_i^0 = (b_i c_i/2)^2 - b_i(a_i/3)^3$ there are either one ($D_i^0 > 0$) or two ($D_i^0 \leq 0$) stable equilibria determined by $f_i^0(x_i^*) = 0$.

For given a_i and b_i and setting $D_i^0 = 0$, the critical value for the control parameter $c_{i_{\text{crit}}}(a_i, b_i) = \pm 2\sqrt{\frac{1}{b_i}(\frac{a_i}{3})^3}$ can be calculated, where the transition into a regime with only one equilibrium takes place. We call $c_{i_{\text{crit}}}(a_i, b_i)$ the intrinsic tipping points for an uncoupled tipping element as given by equation (2.1).

In the following, we couple the subsystems with each other using a coupling function $C_i \in \mathbb{R}$ (figure 1). Subsystem X_i then becomes

$$f_i(x_1, x_2, \ldots, x_n) = a_i x_i(t) - b_i x_i^3(t) + c_i + C_i(x_1(t), x_2(t), \ldots, x_n(t)), \tag{2.2}$$

with $a_i, b_i > 0$.

For simplicity, we choose a linear coupling [52,85]. The linear coupling function for subsystem X_i then reads as

$$C_i(x_1(t), x_2(t), \ldots, x_n(t)) = \sum_{j=1}^{n} d_{ji} x_j(t) \quad \text{with } i \neq j, \tag{2.3}$$

where a coupling $d_{ji} \neq 0$ indicates an influence of another subsystem X_j to subsystem X_i. Even though equations (2.2) and (2.3) provide the simplest equations to describe the threshold behaviour of n coupled tipping elements, they can be used for understanding the qualitative features of all systems with the same critical behaviour. Using the concept of topological equivalence [88], the critical behaviour of a class of more complicated real-world systems can be mapped to the system above. Table 1 provides an overview of special cases of coupling between interacting cusp catastrophes

investigated in the literature [52,84,85]. In addition, more theoretical analyses on bifurcations of coupled cell systems (with symmetry properties) have been conducted (e.g. [89]).

For $n = 2$, the corresponding equations read

$$\dot{x}_1(t) = a_1 x_1(t) - b_1 x_1^3(t) + c_1 + d_{21} x_2(t)$$

and

$$\dot{x}_2(t) = a_2 x_2(t) - b_2 x_2^3(t) + c_2 + d_{12} x_1(t). \tag{2.4}$$

with a_i, $b_i > 0$. With $d_{21} = 0$ and $d_{12} \neq 0$, we recover a generic master–slave configuration. The stable equilibria can be determined analogously to the uncoupled case with $f_i(x_1^*, x_2^*, \ldots, x_n^*) = 0$ $\forall i$. The discriminant for the second tipping element X_2 becomes $D_2 = (b_2 (c_2 + d_{12} x_1^*)/2)^2 - b_2 (a_2/3)^3$.

Note that D_2 is a function of the control parameter c_2, the coupling strength d_2 and the equilibrium x_1^*. The number of stable equilibria of subsystem X_2 depends on the sign of the discriminant. For $D_2 \leq 0$, we find two stable equilibria and for $D_2 > 0$, we find one stable equilibrium. The threshold of the control parameter c_2 at which the number of solutions changes is obtained by solving $D_2 = 0$ and is given by

$$c_2 = -d_{12} x_1^* \pm c_{2_{\mathrm{crit}}}(a_2, b_2), \tag{2.5}$$

where

$$c_{2_{\mathrm{crit}}}(a_2, b_2) = 2\sqrt{\frac{1}{b_2}\left(\frac{a_2}{3}\right)^3}, \tag{2.6}$$

as the effective tipping point of the coupled tipping element X_2.

In the following section, we will elaborate on how one can infer the qualitative behaviour of the coupled system using this expression.

3. Results

Different types of tipping behaviour of a coupled system can be derived for the governing system of equations (2.4). For simplicity, let us consider the case $a_i = 1$, $b_i = 1$ (arbitrary b_i can be achieved by rescaling x_i) for $i = 1, 2$ here and thereafter and $d_{21} = 0$, i.e. unidirectional coupling. Subsystem X_2 leaves the bistable range for

$$c_2 + d_{12} x_1^* \geq c_{2_{\mathrm{crit}}}(a_2, b_2), \tag{3.1}$$

following expression (2.5) for the effective tipping point of the coupled tipping element X_2 in the previous §2, possibly giving rise to a critical transition to the alternative state $x_{2+}^* > 0$.

Based on this simple system, rules on the spread of tipping processes in the considered system of coupled tipping elements are formulated in the following. These tipping rules depend on the type of coupling, i.e. whether the subsystems are positively or negatively coupled, as well as on the relation between the control parameters c_1 (determining the possible stable states of subsystem X_1) and c_2 and the absolute value of the coupling strength d_{12}.

Let $d_{12} > 0$, i.e. subsystem X_2 is positively coupled to subsystem X_1. Then:

— **Facilitated tipping** (figure 2, upper panel): Assume that subsystem X_1 is in its alternative state x_{1+}^*. Note that this assumption can be fulfilled if either subsystem X_1 transitions to the alternative state by crossing its intrinsic tipping point $c_{1_{\mathrm{crit}}}$ with an increase of the control parameter c_1 or if subsystem X_1 simply occupies the alternative state (which is in general possible for $c_1 > -c_{1_{\mathrm{crit}}}$). Then subsystem X_2 is pushed towards its tipping point in our model and can undergo a critical transition to its alternative state x_{2+}^* for $c_2 \geq c_{2_{\mathrm{crit}}} - d_{12} x_1^*$. The effective tipping point of subsystem X_2 is lower than its intrinsic tipping point $c_{2_{\mathrm{crit}}}$. The higher the coupling strength, the lower the necessary critical value of the control parameter c_2 for which subsystem X_2 can tip.
— **Impeded tipping** (figure 2, lower panel): If subsystem X_1 is in its normal state x_{1-}^*, subsystem X_2 is pulled away from its tipping point in our model and can undergo a critical transition for $c_2 \geq c_{2_{\mathrm{crit}}} + d_{12}|x_1^*|$. The effective tipping point of subsystem X_2 is higher than its intrinsic tipping point $c_{2_{\mathrm{crit}}}$. The higher the coupling strength, the higher the necessary critical value of the control parameter c_2 for which subsystem X_2 can tip.
— **Back-tipping**: Assume that subsystem X_1 is in its normal state x_{1-}^*. If subsystem X_2 occupies the alternative state x_{2+}^*, subsystem X_2 can tip back to the normal state x_{2-}^* for $c_2 < -c_{2_{\mathrm{crit}}} + d_{12}|x_1^*|$ (figure 3, upper panel). This behaviour especially occurs for a high coupling strength d_{12} and
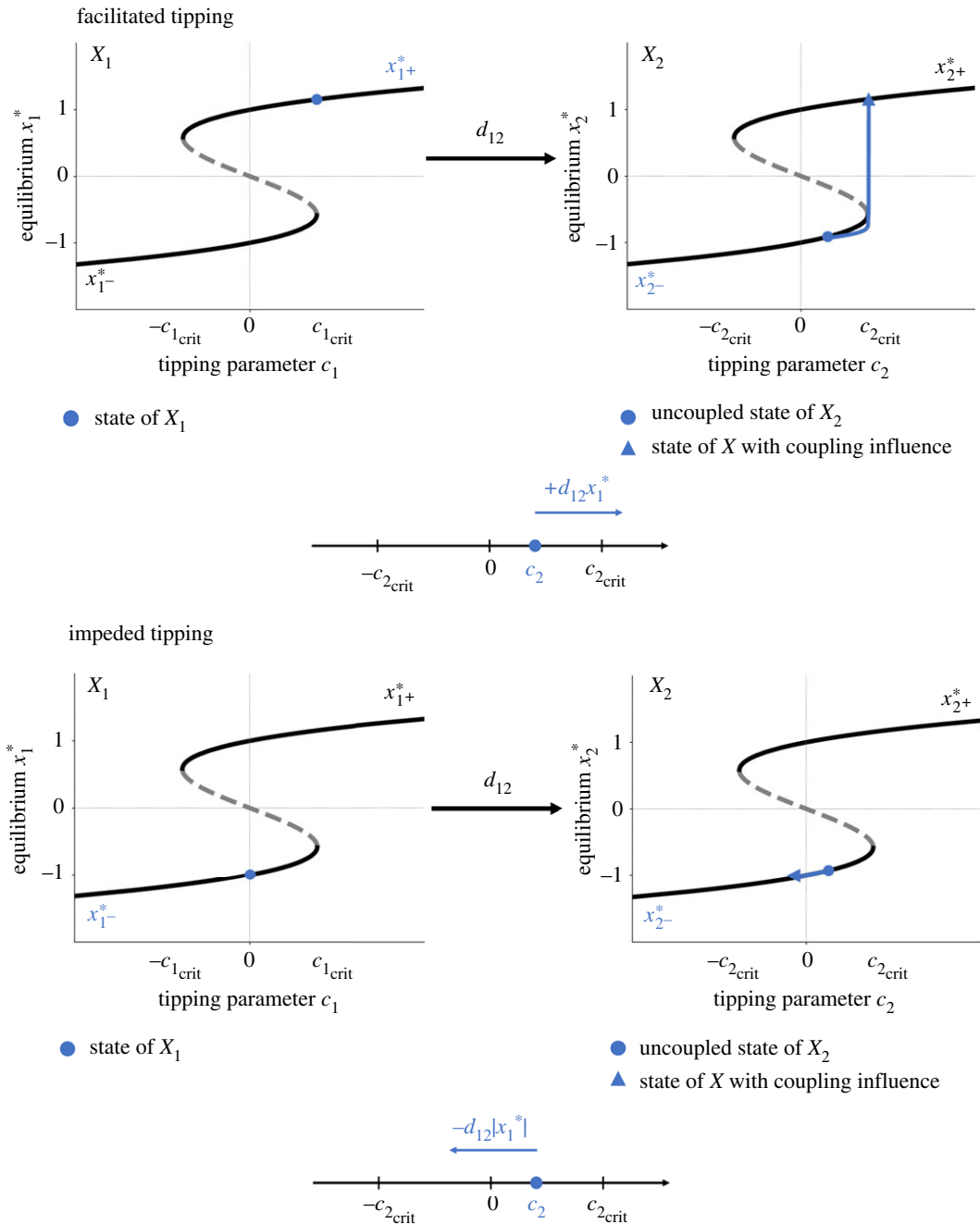
Figure 2. Schematic of the tipping rules of facilitated (upper panel) and impeded tipping (lower panel) for $d_{12} > 0$. The blue dot in the left bifurcation diagram represents a possible state of the master system X_1. The master system X_1 influences the slave system X_2 via a linear coupling with a coupling strength $d_{12} > 0$ and results in the shift of the uncoupled slave system's state (indicated by a blue dot in the right bifurcation diagram) along the blue line.

small values of the control parameter c_2. However, subsystem X_2 is staying in the alternative state x_{2+}^* if $-c_{2\text{crit}} + d_{12}|x_1^*| < c_2$. Here, subsystem X_2 is pushed into the bistable range of the system (figure 3, lower panel). This behaviour especially occurs for a high coupling strength d_{12} and high values of the control parameter c_2.

Let $d_{12} < 0$, i.e. subsystem X_2 is negatively coupled to subsystem X_1. Then:

— **Impeded tipping**: Assume that subsystem X_1 is in its alternative state x_{1+}^*. Note that this assumption can be fulfilled if either subsystem X_1 transitions to the alternative state by crossing its intrinsic
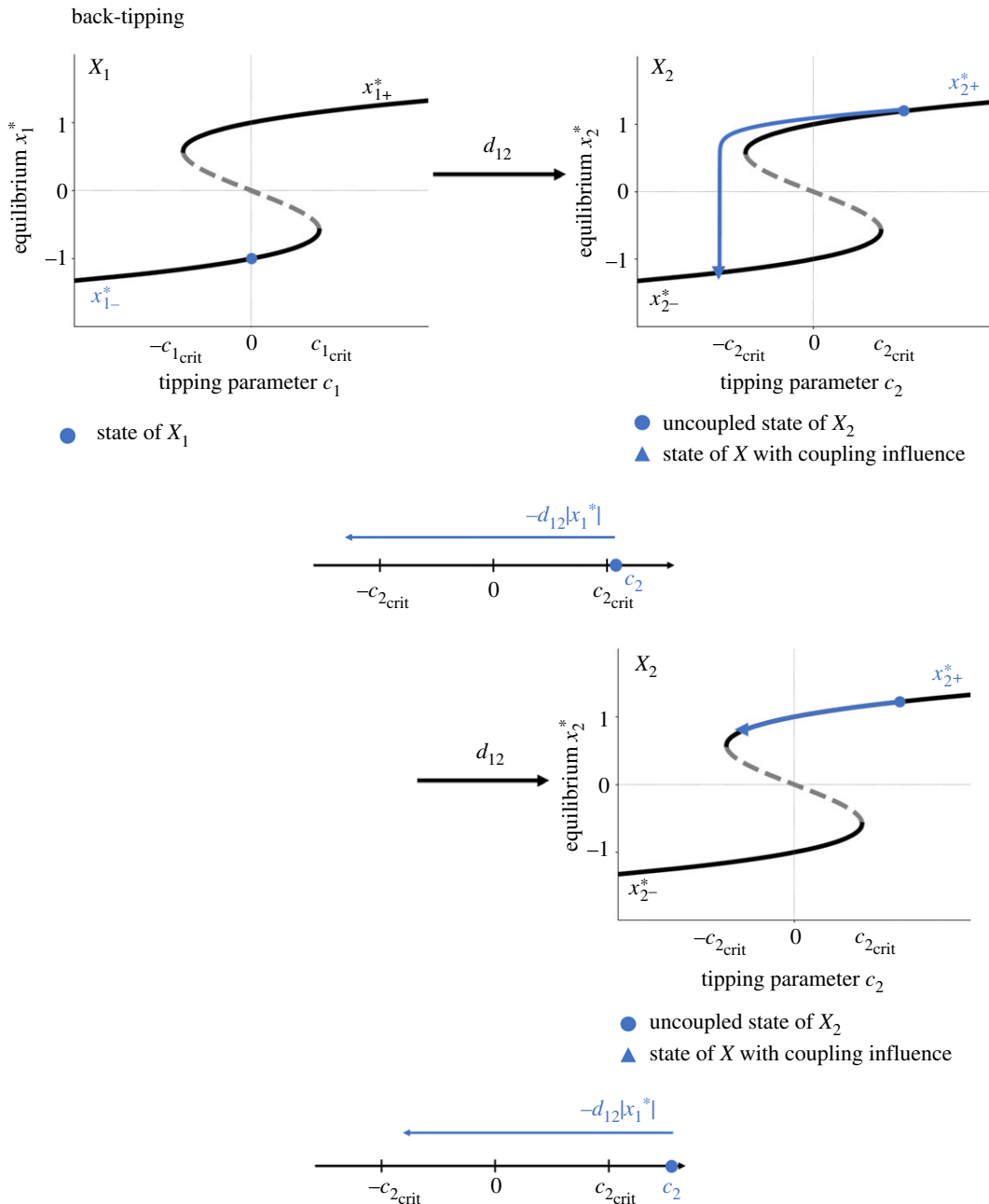
back-tipping

Figure 3. Schematic of the tipping rule of back-tipping for $d_{12} > 0$. The blue dot in the left bifurcation diagram represents a possible state of the master system X_1. The master system X_1 influences the slave system X_2 via a linear coupling with a coupling strength $d_{12} > 0$ and results in the shift of the uncoupled slave system's state (indicated by a blue dot in the right bifurcation diagram) along the blue line.

tipping point $c_{1_{\text{crit}}}$ with an increase of the control parameter c_1 or if subsystem X_1 simply occupies the alternative state (which is in general possible for $c_1 > -c_{1_{\text{crit}}}$). Subsystem X_2 is pulled away from its tipping point in our model and can undergo a critical transition for $c_2 \geq c_{2_{\text{crit}}} + |d_{12}\|x_1^*|$. The effective tipping point of subsystem X_2 is higher than its intrinsic tipping point $c_{2_{\text{crit}}}$. The higher the coupling strength, the higher the necessary critical value of the control parameter c_2 for which subsystem X_2 can tip.

— **Facilitated tipping**: If subsystem X_1 is in its normal state x_{1-}^*, subsystem X_2 is pushed towards its tipping point in our model and can undergo a critical transition to its alternative state x_{2+}^* for $c_2 \geq c_{2_{\text{crit}}} - |d_{12}\|x_1^*|$. The effective tipping point of subsystem X_2 is lower than its intrinsic tipping point $c_{2_{\text{crit}}}$. The higher the coupling strength, the lower the necessary critical value of the control parameter c_2 for which subsystem X_2 can tip.

— **Back-tipping**: Assume that subsystem X_1 is in its normal state x_{1-}^* . If subsystem X_2 occupies the alternative state x_{2+}^*, subsystem X_2 can tip back to the normal state x_{2-}^* if $c_2 < -c_{2_{crit}} - |d_{12}\|x_1^*|$. This behaviour especially occurs for a high coupling strength d_{12} and a low value of the control parameter c_2. However, subsystem X_2 stays in the alternative state if $c_2 \geq -c_{2_{crit}} - |d_{12}\|x_1^*|$. Here, subsystem X_2 is pulled to the bistable area of the system. This behaviour especially occurs for a high coupling strength $|d_{12}|$ and high values of the control parameter c_2.

These tipping rules based on the analytic solution of two unidirectionally coupled tipping elements give an impression of the interplay between certain system parameters and their influence on the tipping process. Going beyond this, using numerical calculations of fixed points and their stability (via the eigenvalues of the system's Jacobian at the respective fixed point), the overall qualitative long-term behaviour of up to three interacting tipping elements with uni- and bi-directional coupling of varying sign has been assessed.

A stability map displays the number of stable equilibria of the system under consideration in the control parameter space for fixed coupling strengths. Multiple stability maps have been calculated for various combinations of the coupling strengths. The stability maps have been arranged in the form of a matrix, where one matrix element corresponds to one stability map. For illustrative purposes, we refer to the example given in figure 4 for two interacting tipping elements.

The system loses or gains stable fixed points through the variation of one or various control parameters c_i for fixed coupling strengths, which is associated with switches between the areas of different number of stable equilibria in the control parameter space by crossing the boundaries between the coloured areas. In addition, the phase space portrait may change.

Depending on the changes in the phase space (in terms of the stable fixed points and the flow) and the occupied state of the system, different types of system behaviour emerge. Combining the stability maps with phase space portraits (example given in figure 5 for illustrative purposes), the different areas in the stability maps can be characterized in terms of the emerging system behaviour, and possible critical transitions can be identified. Depending on which state the system was in, critical transitions can occur or not. For example, if the system resided in an equilibrium which lost stability and disappeared through the variation of one (or multiple) control parameters, the flow in the phase space suggests the state to which the system may transition.

In the following, results for selected examples of coupled tipping elements are shown. First, a simple master–slave system with a unidirectional coupling is presented in example 3.1. In example 3.2, the previous system is extended by an additional negative coupling resulting in a bidirectionally coupled system of two tipping elements. Finally, the propagation in a unidirectionally coupled system consisting of three tipping elements is described in example 3.3. We only analyse the system behaviour for $c_i \geq 0$.

Example 3.1. Master–slave system for $d_{12} > 0$, e.g. a pair of lakes

The behaviour of a master–slave system ($n = 2$) given by equations (2.2) and (2.3) with $i = 1, 2$ and positive coupling $d_{12} > 0$ and $d_{21} = 0$, which was used for the derivation of the tipping rules, is described in more detail in the following. This type of coupling can be seen as an example for a pair of interacting lakes. Each lake may undergo a transition from a clear to a turbid state when some critical magnitude of the nutrient input as the tipping parameter is exceeded [6,8]. It is assumed that the lakes are connected [57,58,67] through a unidirectional water stream. Critical transitions can be derived using the numerically calculated phase portrait in combination with the stability map for a fixed coupling strength $d_{12} > 0$ (figure 5). Note that figure 5 is a zoom into figure 4 at the lower right. The system has four stable equilibria for small values of the control parameters c_1 and c_2 which are separated by four saddles and an unstable node in the centre of the phase space.

With increasing control parameter c_1, a critical transition of subsystem X_1 occurs in our model when its intrinsic tipping point $c_{1_{crit}}$ is crossed (figure 5, moving from lower left to lower right along the green arrow), given that the system occupied one of the stable equilibria which lose stability for $c_1 > c_{1_{crit}}$.

With increasing control parameter c_2, a critical transition of subsystem X_2 occurs in our model even if $c_2 < c_{2_{crit}}$, given that subsystem X_1 is in the alternative state (figure 5, moving upwards from the lower left along the yellow arrow). The coupled subsystem X_2 tips at an effective tipping point lower than its intrinsic tipping point $c_{2_{crit}}$.

For an increase of the control parameter c_1 above the intrinsic tipping point $c_{1_{crit}}$ and a slight increase of the control parameter c_2, a tipping cascade, starting from the normal states of X_1 and X_2, with a critical
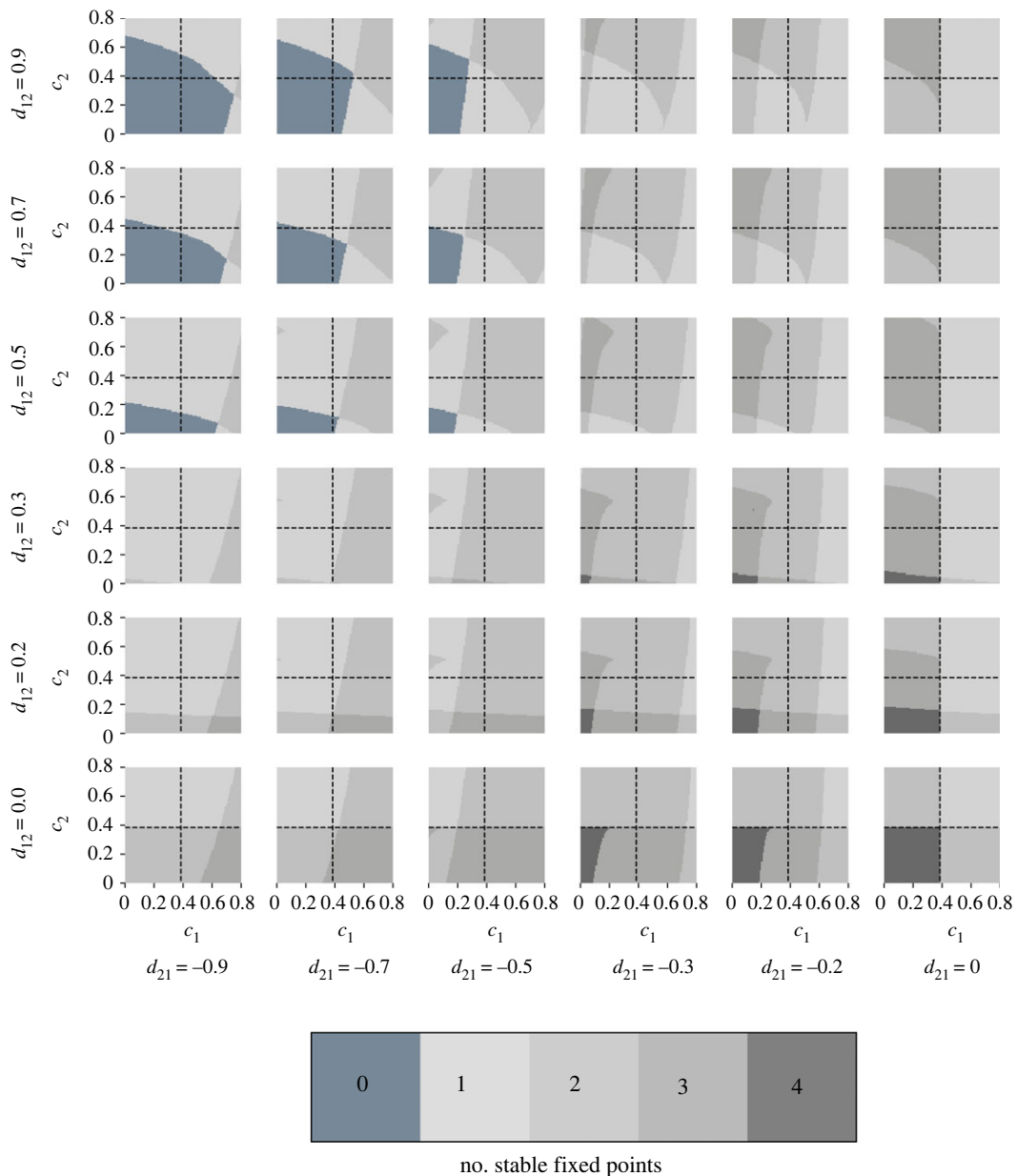
Figure 4. Number of stable fixed points of the system consisting of two bidirectionally coupled tipping elements depending on the control parameters c_1 and c_2 and the coupling strengths $d_{21} \leq 0$ and $d_{12} \geq 0$ in a matrix of stability maps. A stability map shows the number of stable fixed points in the (c_1, c_2)–space for a specific coupling strength, where a certain number of stable fixed points is associated with a specific colour. Note that different areas in the control parameter space with the same colour have the same number of stable fixed point but they do not necessarily have the same phase portrait. The dashed lines represent the intrinsic tipping point of the respective subsystem. The position of a stability map in the matrix is determined by the coupling strength. In the blue–grey region for high coupling strengths with opposite sign but same magnitude indicating the absence of stable equilibria a stable limit cycle (Kadyrov oscillations in [85]) can be observed.

transition in subsystem X_1 and a following transition in subsystem X_2 arises in our model (figure 5, moving from the lower left to the right column along the pink arrow). Note that the cascade occurs before the intrinsic tipping point of subsystem X_2 is crossed.

There is a change in the system behaviour for an increasing coupling strength (electronic supplementary material, figure S1). The previously described area with only one stable fixed point of the two subsystems in the alternative state for $c_1 > c_{1_{\text{crit}}}$ exists for extremely low values of $c_2 \ll c_{2_{\text{crit}}}$. Therefore, a tipping cascade can occur for even lower values of the control parameter $c_2 \ll c_{2_{\text{crit}}}$ than for a system with lower coupling strength. For $c_1 < c_{1_{\text{crit}}}$ and low values of the control parameter c_2,
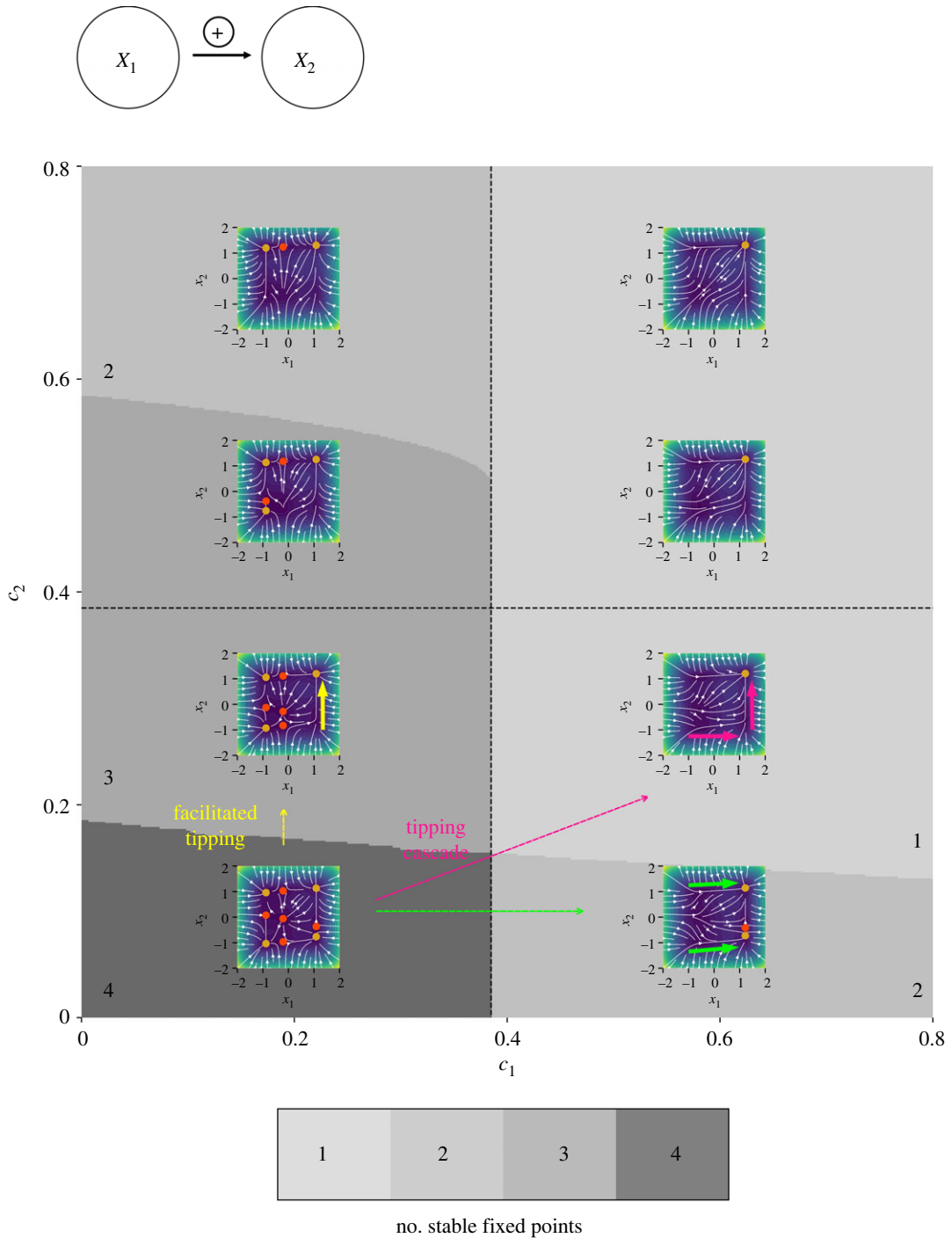
Figure 5. Number of stable fixed points and phase space portraits in a master–slave system with a low positive coupling strength $d_{12} = 0.2 > 0$ depending on the control parameters c_1 and c_2. The dashed lines represent the intrinsic tipping point of the respective subsystem. The phase space portraits allow to derive the possible critical transitions in the master–slave system. Within the phase space portraits stable fixed points are shown in orange, while unstable fixed points are shown in red. The background colour indicates the normalized speed $v = \sqrt{\dot{x}_1^2 + \dot{x}_2^2}/v_{\max}$ going from close to zero (purple) to fast (yellow–green).

subsystem X_2 either transitions to the alternative state for $c_2 < c_{2_{\mathrm{crit}}}$, given that subsystem X_1 is in the alternative state, or subsystem X_2 tips back from the alternative state to the normal state, given that subsystem X_1 has not tipped. For $c_1 < c_{1_{\mathrm{crit}}}$ and an increased control parameter c_2, the critical transition of subsystem X_2 to the alternative state, given that subsystem X_1 occupies the alternative state, is the only transition that can be observed.

Example 3.2. Bidirectional interaction of two tipping elements, e.g. Greenland ice sheet and Atlantic meridional overturning circulation

Consider a system consisting of two ($n = 2$) tipping elements given by equations (2.2) and (2.3) with $i = 1, 2$ and a bidirectional coupling where $d_{21} < 0$ and $d_{12} > 0$. This type of coupling can for instance be found in the interaction of the Greenland ice sheet (GIS) and the Atlantic meridional overturning circulation (AMOC), whose long-term behaviour may be represented by a double fold as suggested by (simple) models [14,17,90]: increased meltwater influx into the North Atlantic due to tipping of the GIS could lead to a weakening or even shutdown (tipping) of the AMOC [55], i.e. introducing a positive coupling. At the same time, a slowdown of the AMOC leads to a relative cooling around Greenland and hence corresponds to a negative coupling [54]. The system is analysed for a low and a high coupling strength (where d_{21} and d_{12} have opposite signs but the same magnitude) due to a substantial change of the qualitative behaviour towards higher coupling strength. There currently is not sufficient knowledge on the strength of the interaction between the GIS and the AMOC so that neither a low nor a high coupling strength can be excluded for certain. The number of stable equilibria and possible critical transitions for different parameter settings are given in figure 6 as zoom into figure 4 for low coupling strengths (lower right). A starting point of the analysis of the system behaviour for the low coupling strength is the area of four stable equilibria for low values of the control parameters c_1 and c_2 in figure 6.

With increasing control parameter c_1, a critical transition of the GIS as subsystem X_1 is possible for $c_1 < c_{1_{crit}}$ in our model, given that the AMOC as subsystem X_2 is in the normal state (figure 6, moving from lower left to the right along the yellow arrow). The GIS might tip at an effective tipping point which is lower than the intrinsic tipping point of the isolated subsystem. A critical transition of the GIS for the AMOC being in the alternative state is possible with a further increase of the control parameter c_1 for $c_1 \gg c_{1_{crit}}$ (figure 6, moving from lower left to the right along the second yellow arrow).

With increasing control parameter c_2, a critical transition of the AMOC as subsystem X_2 to a state with weakened strength (i.e. the alternative state) is possible in our model for $c_2 < c_{2_{crit}}$, given that the GIS as subsystem X_1 has already tipped (figure 6, moving from lower left upwards along the yellow arrow). The AMOC might tip at an effective tipping point which is lower than the intrinsic tipping point of the isolated subsystem. Given that the GIS is in its normal state, a critical transition of the AMOC is possible with a further increase of the control parameter c_2 for $c_2 \gg c_{2_{crit}}$ at an effective tipping point higher than its intrinsic tipping point (figure 6, moving from lower left upwards along the second yellow arrow). As a result of the model formulation, the GIS would still pull the thermohaline circulation (THC) away from its tipping point even though it has already started to melt but has not tipped to x_{1+}^* (i.e. $0 > x_1^* > -1$). It would be possible to adjust the coupling function or the dynamics of each tipping element (e.g. [76]) so that already a slight change of the GIS state towards the alternative state without a complete critical transition to full loss of the ice sheet would push the AMOC towards its own tipping point.

For a slight increase of both control parameters c_1 and c_2, a critical transition of the GIS as well as the AMOC to the alternative state is possible in our model for $c_1 < c_{1_{crit}}$ and $c_2 < c_{2_{crit}}$ before their respective intrinsic tipping points are crossed (figure 6, moving from lower left along the pink arrow). In contrast to the previous example 3.1, the additional negative coupling results in the tipping of both interacting subsystems at an effective tipping point below their intrinsic tipping points. In a master–slave system (example 3.1), the master system X_1 needs to tip through an increase of its control parameter above its intrinsic tipping point $c_1 > c_{1_{crit}}$ to trigger a critical transition in the slave system X_2 at an effective tipping point $c_2 < c_{2_{crit}}$.

With increasing coupling strengths d_{21} and d_{12} the system behaviour changes. The system has one unstable fixed point and Kadyrov oscillations [85] occur for a wide range of the control parameters in the considered part of the (c_1, c_2)–parameter space (upper left of figure 4).

Example 3.3. Master–slave–slave system, e.g. propagation of critical transitions in lake chains

Consider a system consisting of three ($n = 3$) unidirectionally coupled tipping elements given by equations (2.2) and (2.3) with $i = 1, 2, 3$; $d_{12}, d_{23} > 0$ and $d_{21}, d_{32}, d_{31}, d_{13} = 0$. This type of coupling corresponds to the behaviour of a lake chain subject to an external input of nutrients as a control parameter (as e.g. in [57]). As in example 3.1, it is assumed that the lakes are connected through a unidirectional water stream [57,58,67,91]. The behaviour of subsystem X_1 corresponds to the behaviour of an uncoupled tipping element. Therefore, the eutrophication of the first lake in the lake chain, i.e. the tipping of subsystem X_1, is possible with an increase of its control parameter $c_1 > c_{1_{crit}}$. For $c_1 > c_{1_{crit}}$ only stable fixed points with subsystem X_1 in the alternative state exist. Additionally increasing the control parameters c_2, c_3 or both results in the loss of further stable fixed points and allows for critical transitions in the slave systems X_2 and X_3 (figure 7). In the following, the system
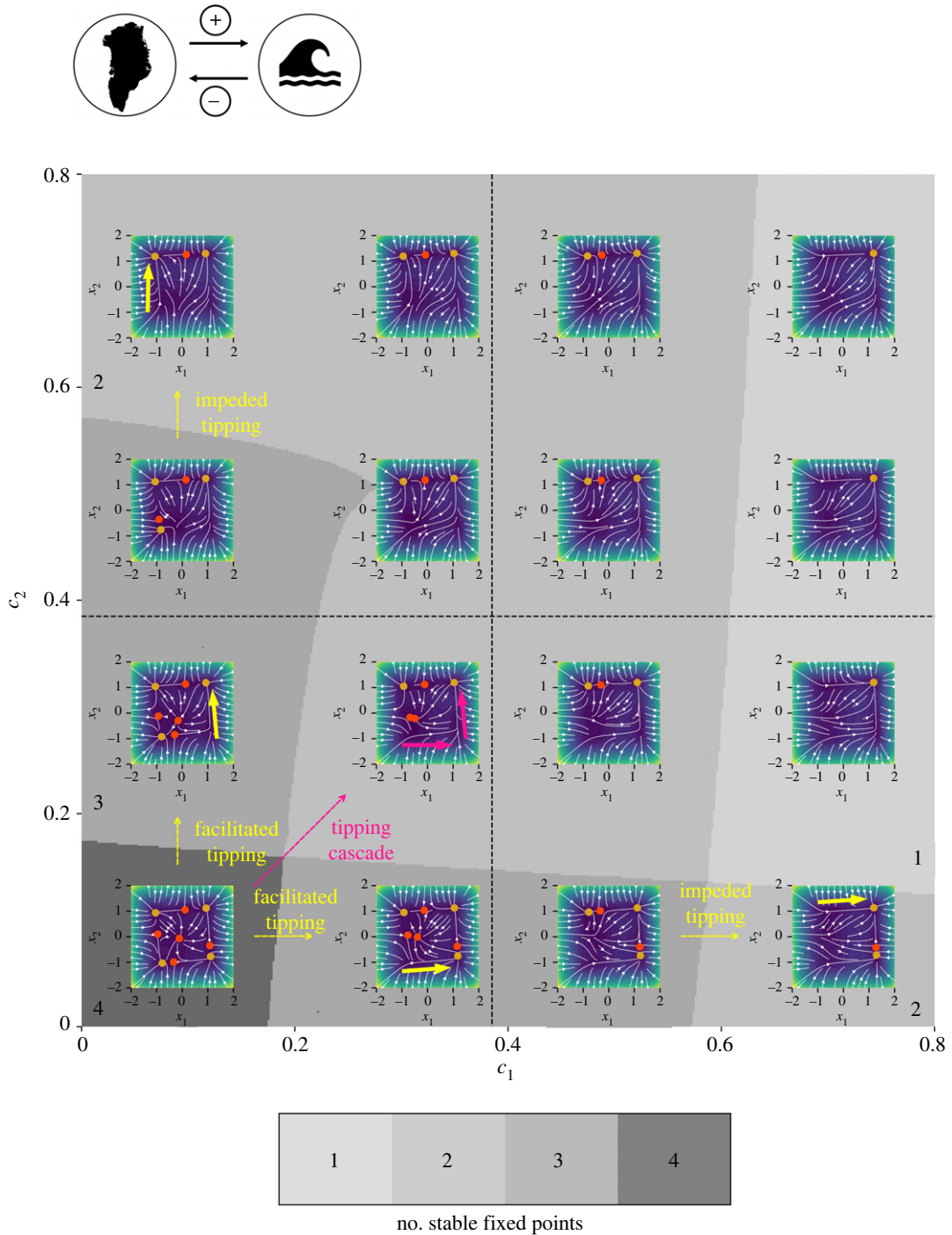
Figure 6. Number of stable fixed points and phase space portraits for two bidirectionally coupled tipping elements with $d_{21} = -0.2 < 0$ and $d_{12} = 0.2 > 0$ and low coupling strengths where $|d_{21}| = |d_{12}|$ depending on the control parameters c_1 and c_2. The dashed lines represent the intrinsic tipping point of the respective subsystem. The phase space portraits allow to derive the possible critical transitions in the master–slave system. Within the phase portraits stable fixed points are shown in orange, while unstable fixed points are shown in red. The background colour indicates the normalized speed $v = \sqrt{\dot{x}_1^2 + \dot{x}_2^2}/v_{max}$ going from close to zero (purple) to fast (yellow–green).

behaviour with $c_1 > c_{1_{crit}}$ for low coupling strengths $d_{12}, d_{23} > 0$ is analysed (see figure 7, lower left and electronic supplementary material, figure S3 for a zoom-in).

With an increasing control parameter c_2 or c_3, a critical transition in the corresponding subsystems is possible in our model for $c_2 < c_{2_{crit}}$ (electronic supplementary material, figure S4) or $c_3 < c_{3_{crit}}$ (electronic supplementary material, figure S5), given that the preceding subsystem occupies the alternative state or undergoes a transition into the alternative state through a continuously changing control parameter. Lake
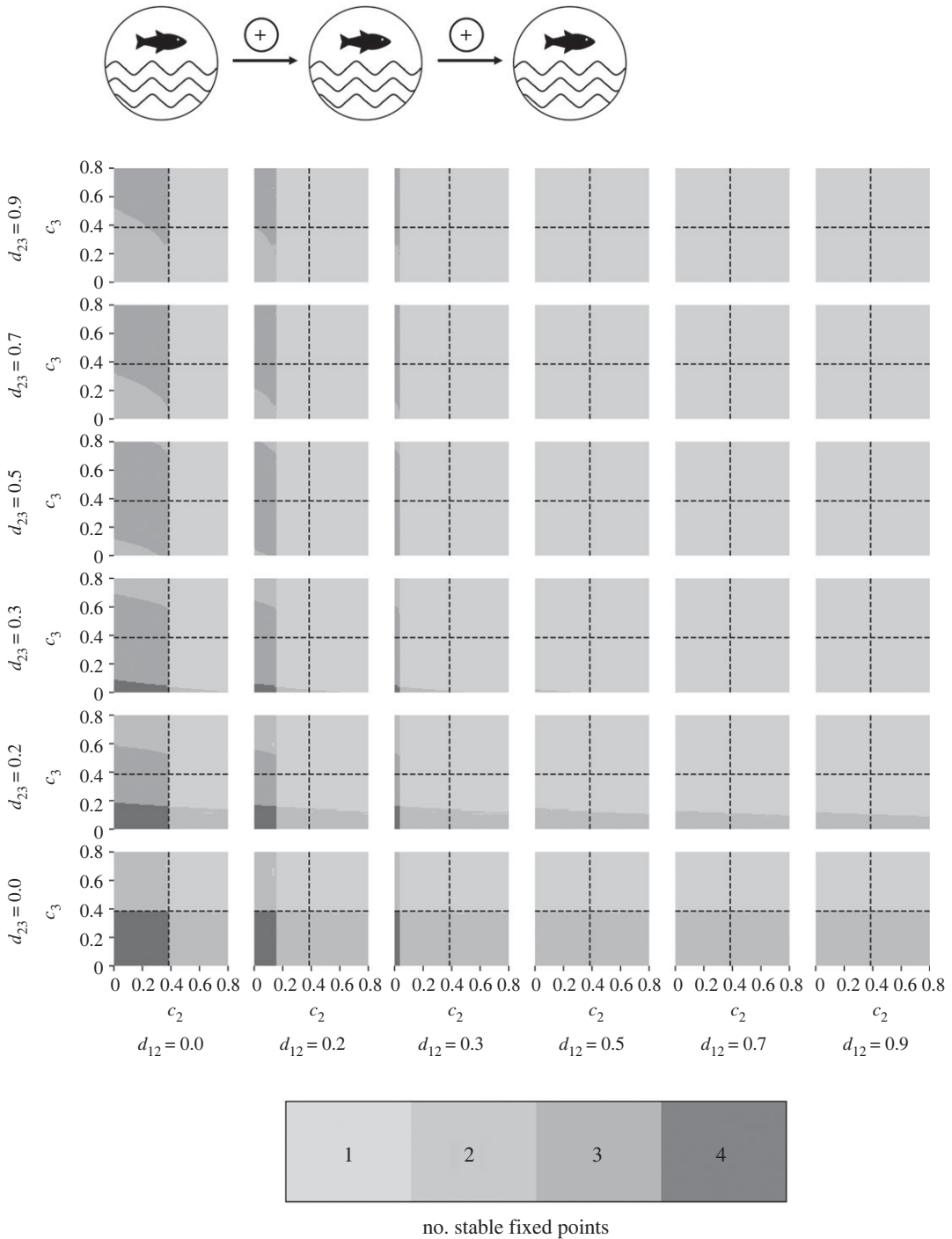
Figure 7. Number of stable fixed points of the system consisting of three unidirectionally coupled tipping elements for $c_1 = 0.4 > c_{1_{crit}}$ depending on the control parameters c_2 and c_3 and the coupling strengths $d_{12} \geq 0$ and $d_{23} \geq 0$ in a matrix of stability maps. A stability map shows the number of stable fixed points in the (c_2, c_3)–space for a specific coupling strength, where a certain number of stable fixed points is associated with a specific colour. Note that different areas in the control parameter space with the same colour have the same number of stable fixed points but they do not necessarily have the same phase portrait. The dashed lines represent the intrinsic tipping point of the respective subsystem. The position of a stability map in the matrix is determined by the coupling strength.

X_2 and X_3 in the lake chain can, therefore, become eutrophic before the intrinsic critical level of nutrient input of an isolated lake is crossed, given that the preceding lake has already become eutrophic.

For a simultaneous, slight increase of the control parameters c_2 and c_3 of both subsystems X_2 and X_3, a critical transition in both subsystems X_2 and X_3 is possible in our model for $c_2 < c_{2_{crit}}$ and $c_3 < c_{3_{crit}}$. As a result, a tipping cascade can be observed, given subsystem X_1 has already tipped or tips with $c_1 > c_{1_{crit}}$
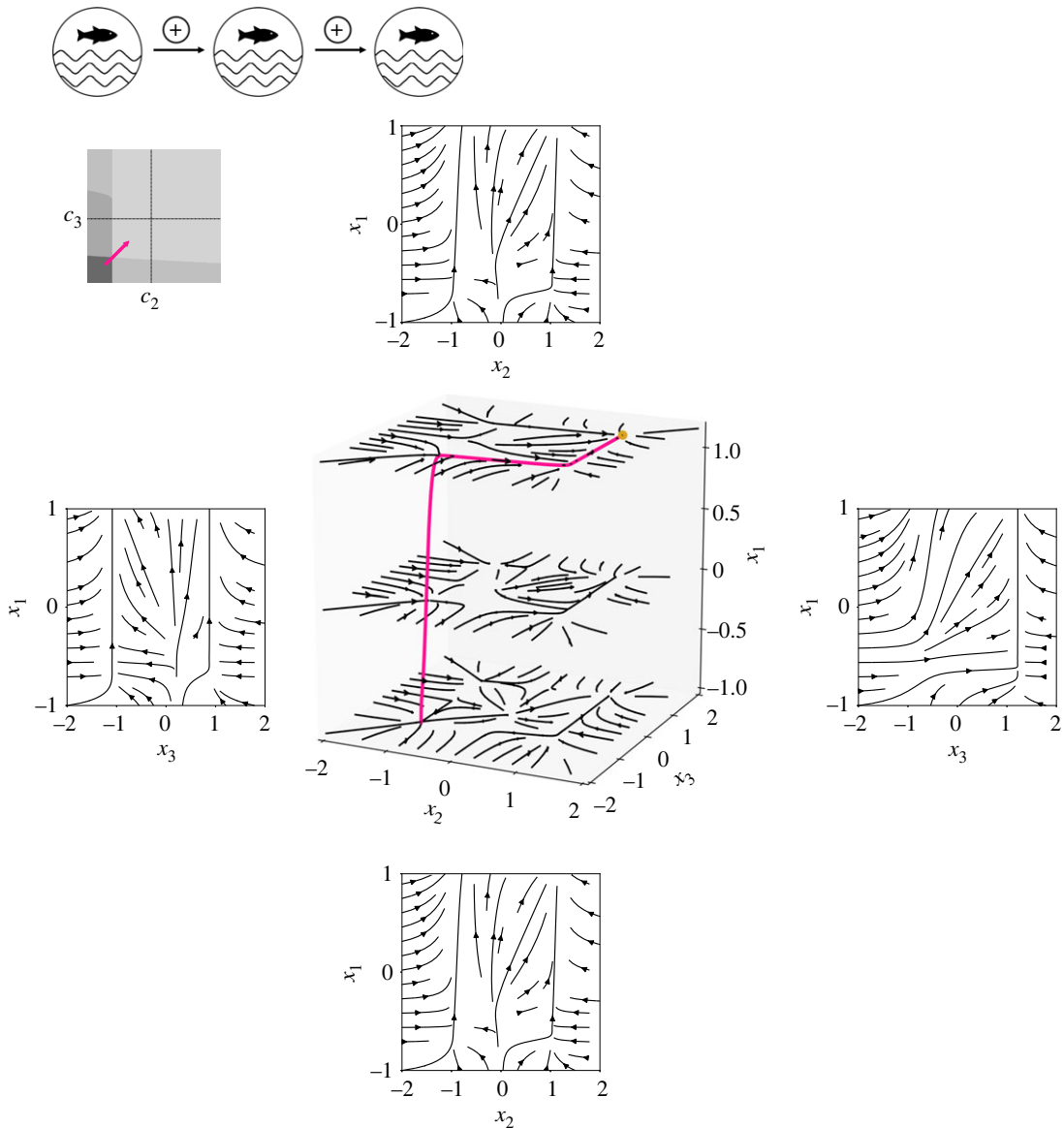
Figure 8. Tipping cascade in a system of three unidirectionally coupled tipping elements for an increase of the control parameters $c_2 \ll c_{2_{crit}}$ and $c_3 \ll c_{3_{crit}}$ and $c_1 > c_{1_{crit}}$ (as indicated by the pink arrow in the stability map, upper left panel). The central cube shows the flow in the (x_2, x_3)-space as part of the three-dimensional phase space and the remaining stable fixed point (in orange) for $c_1 = 0.4 > c_{1_{crit}}$, $c_2 = c_3 = 0.2$ with $d_{12} = 0.2 > 0$ and $d_{23} = 0.2 > 0$. The tipping cascade is highlighted by the pink trajectory. Two-dimensional plots arranged around the central cube show the flow in the (x_2, x_1)- and (x_3, x_1)-space corresponding to the lateral surfaces of the cube.

(pink trajectory in figure 8). Consequently, after the eutrophication of the first lake, a critical transition to the turbid state of a lake can spread in the lake chain even if the intrinsic critical value of nutrient input known from an isolated lake is not crossed.

4. Discussions and conclusion

The qualitative long-term behaviour of interacting, cusp-like tipping elements has been assessed in a simple analytic and an extensive numerical analysis of a conceptual model. Depending on the type of coupling and the coupling strength, qualitatively different behaviours of the systems of interacting tipping elements were observed. In particular, tipping cascades where a critical transition in one subsystem triggers the tipping of a coupled subsystem may occur under certain conditions.

Simple analytic calculations resulted in the formulation of tipping rules for the spread of tipping processes in systems of interacting tipping elements: a shift in the threshold value of the control parameter, at which an interacting tipping element undergoes a transition into a qualitatively different state, can occur. We call the threshold value of the isolated subsystem the intrinsic tipping point of the tipping element. If an interaction with another tipping element exists, the tipping process takes place when the so-called effective tipping point is crossed. Depending on the coupling direction, the effective tipping point can occur at either lower (facilitated tipping) or higher (impeded tipping) values of the control parameter than the intrinsic tipping point.

We have generalized and extended existing studies of special cases of coupled cusp-like tipping elements [52,85] through an extensive numerical analysis of two and three interacting tipping elements with one- or bidirectional coupling of varying direction. The behaviour of the special cases including a window with both subsystems of a simple master–slave system [52] in the alternative state, a tipping cascade in a positively coupled master–slave–slave system [52] and the Kadyrov oscillator [85] for two bidirectionally coupled tipping elements for a high coupling strength of same magnitude but with opposite signs is consistent with the system behaviour observed in our analysis.

In addition, our extensive analysis allowed to identify types of coupling that favour critical tipping scenarios. Conditions in terms of coupling strength and control parameters of the subsystem under which the tipping scenarios occur have been determined. Cascades of tipping processes that occur before the crossing of intrinsic tipping points, i.e. where the effective tipping point lies at lower values than the intrinsic tipping point of the uncoupled tipping element, are of special interest. In a simple master–slave system with positive coupling, a critical transition in the master system due to a crossing of its intrinsic tipping point triggers a critical transition of the slave system at an effective tipping point lower than its intrinsic tipping point. By contrast, a negative coupling would prevent a facilitated tipping of the slave system in the case of a master system being in the alternative state. In a system of two tipping elements with bidirectional coupling, a tipping cascade is favoured if one of the coupling terms is negative. In a master–slave–slave system with $d_{12} > 0$ and $d_{23} > 0$, the initial tipping of the master system can trigger cascading tipping processes in the following subsystems before the intrinsic threshold of the corresponding control parameter is crossed. Such a tipping cascade before the crossing of the corresponding intrinsic thresholds cannot be observed after the introduction of a negative coupling (results not shown here). Tipping processes are suppressed instead in this case and do not spread into all subsystems.

Applying the qualitative system behaviour to selected interacting real-world tipping elements revealed possible tipping scenarios, which are relevant for the future development of the Earth system, and in addition, due to the consequences of tipping, such as sea-level rise [92,93], for the economy, infrastructure and society more broadly. In particular, the analysis of the qualitative long-term system behaviour of two bidirectionally coupled tipping elements with opposite sign but same magnitude suggests that the Greenland ice sheet and the AMOC might tip before their intrinsic tipping points are reached. In other words, the meltdown of the Greenland ice sheet and the slowdown of the AMOC might begin before the intrinsic threshold ranges identified for isolated tipping elements [94] is crossed. The possible existence of such tipping cascades increases the risks that anthropogenic climate change poses to human societies, since the intrinsic threshold ranges of some climatic tipping elements including the Greenland ice sheet are assumed to lie even within the 1.5–2°C target range of the Paris agreement [94].

When it comes to the application of tipping behaviour to real-world systems, it should be noted that tipping elements were described in an idealized way using the normal form of the cusp catastrophe onto which, by the concept of topological equivalence [88], the critical behaviour of a class of real-world systems can be mapped. The proposed model of interacting tipping elements, therefore, shows a hypothetical, but mathematically possible system behaviour. It was motivated by its catastrophic features [46,87] in contrast to other bifurcational systems allowing the transition into a qualitatively different state by the variation of a bifurcation parameter and the appearance of the double fold bifurcation in many real world systems [6,8,12,15,17–19,95]. However, processes which are not taken into account in the conceptual representation of tipping elements, but are present in the real world, might influence the system and its tipping behaviour. In addition to a direct coupling of tipping elements in the climate system [54], an indirect, 'diffusive' interaction through e.g. the global mean temperature [1] could be considered as a potential coupling mechanism. Furthermore, chains and pairs of tipping elements have been analysed isolated from the larger network of interacting climatic tipping elements [54], i.e. possible interactions with other climatic tipping elements have been neglected. Here, we focused on bifurcation-induced tipping assuming that the control parameter

varies sufficiently slowly for the system to keep track with the stable states. It should be noted that a change of the control parameter with a high rate is likely, given the increasing influence of humans on the Earth system, possibly giving rise to rate-induced tipping [45,50].

The qualitative and theoretically possible system behaviours studied here and their application to real-world systems, therefore, introduces further research questions regarding tipping elements and their interactions in ecology, climate science and other fields. The conceptual approach should be extended to networks of tipping elements as already suggested in [84] and motivated in [54]. Networks of interacting tipping elements can be analysed using methods of statistical mechanics [96]. Critical transitions may spread across a whole network of tipping elements depending on the clustering and the spatial organization of the network [76–78]. Taking the important interactions of climatic tipping elements into account in a network approach, realistic complex models must be used for quantitatively approximating the effective tipping point. In addition, interacting tipping elements with heterogeneous intrinsic threshold and varying internal time scales should be considered [65] as, for example, the critical nutrient input of lakes varies with their depth [7]. Finally, generic early warning signals for tipping cascades comparable to already existing indicators for critical transitions of isolated tipping elements [97,98] are desirable to forecast cascading tipping events and counteract undesired consequences of tipping (cascades). A first step towards early warning indicators of tipping cascades has been presented only recently [86]. However, up to now, it remains an open question whether early warning signals for tipping cascades based on critical slowing down [97,98] exist.

References

1. Lenton TM, Held H, Kriegler E, Hall JW, Lucht W, Rahmstorf S, Schellnhuber HJ. 2008 Tipping elements in the Earth's climate system. Proc. Natl Acad. Sci. USA 105, 1786–1793. (doi:10.1073/pnas.0705414105)

2. Folke C, Carpenter S, Walker B, Scheffer M, Elmqvist T, Gunderson L, Holling CS. 2004 Regime shifts, resilience, and biodiversity in ecosystem management. Annu. Rev. Ecol., Evol. Syst. 35, 557–581. (doi:10.1146/annurev.ecolsys.35.021103.105711)

3. Lees K, Pitois S, Scott C, Frid C, Mackinson S. 2006 Characterizing regime shifts in the marine environment. Fish Fish. 7, 104–127. (doi:10.1111/j.1467-2979.2006.00215.x)

4. Carpenter SR. 2003 Regime shifts in lake ecosystems: pattern and variation, 15th edn. Oldendorf/Luhe, Germany: International Ecology Institute.

5. Carpenter SR, Ludwig D, Brock WA. 1999 Management of eutrophication for lakes subject to potentially irreversible change. Ecol. Appl. 9, 751–771. (doi:10.1890/1051-0761(1999)009[0751:MOEFLS]2.0.CO;2)

6. Scheffer M. 1989 Alternative stable states in eutrophic, shallow freshwater systems: a minimal model. Hydrobiol. Bull. 23, 73–83. (doi:10.1007/BF02286429)

7. Scheffer M, van Nes EH. 2007 Shallow lakes theory revisited: various alternative regimes driven by climate, nutrients, depth and lake size. Hydrobiologia 584, 455–466. (doi:10.1007/s10750-007-0616-7)

8. Scheffer M, Hosper S, Meijer M, Moss B, Jeppesen E. 1993 Alternative equilibria in shallow lakes. Trends Ecol. Evol. 8, 275–279. (doi:10.1016/0169-5347(93)90254-M)

9. Scheffer M, Carpenter S, Foley JA, Folke C, Walker B. 2001 Catastrophic shifts in ecosystems. Nature 413, 591–596. (doi:10.1038/35098000)

10. Levermann A et al. 2012 Potential climatic transitions with profound impact on Europe. Clim. Change 110, 845–878. (doi:10.1007/s10584-011-0126-5)

11. Lenton TM, Rockström J, Gaffney O, Rahmstorf S, Richardson K, Steffen W, Schellnhuber HJ. 2019 Climate tipping points—too risky to bet against. Nature 575, 592–595. (doi:10.1038/d41586-019-03595-0)

12. Hofmann M, Rahmstorf S. 2009 On the stability of the Atlantic meridional overturning circulation. Proc. Natl Acad. Sci. USA 106, 20 584–20 589. (doi:10.1073/pnas.0909146106)

13. Rahmstorf S. 1995 Bifurcations of the Atlantic thermohaline circulation in response to changes in the hydrological cycle. Nature 378, 145–149. (doi:10.1038/378145a0)

14. Rahmstorf S. 1996 On the freshwater forcing and transport of the Atlantic thermohaline circulation. Clim. Dyn. 12, 799–811. (doi:10.1007/s003820050144)

15. Rahmstorf S et al. 2005 Thermohaline circulation hysteresis: a model intercomparison. Geophys. Res. Lett. 32, L23605. (doi:10.1029/2005GL023655)

16. Stocker TF, Wright DG. 1991 Rapid transitions of the ocean's deep circulation induced by changes in surface water fluxes. Nature 351, 729–732. (doi:10.1038/351729a0)

17. Stommel H. 1961 Thermohaline convection with two stable regimes of flow. *Tellus* **13**, 224–230. (doi:10.3402/tellusa.v13i2.9491)

18. Titz S, Kuhlbrodt T, Rahmstorf S, Feudel U. 2002 On freshwater-dependent bifurcations in box models of the interhemispheric thermohaline circulation. *Tellus A: Dyn. Meteorol. Oceanogr.* **54**, 89–98. (doi:10.3402/tellusa.v54i1.12126)

19. Robinson A, Calov R, Ganopolski A. 2012 Multistability and critical thresholds of the Greenland ice sheet. *Nat. Clim. Change* **2**, 429–432. (doi:10.1038/nclimate1449)

20. Toniazzo T, Gregory J, Huybrechts P. 2004 Climatic impact of a Greenland deglaciation and its possible irreversibility. *J. Clim.* **17**, 21–33. (doi:10.1175/1520-0442(2004)017<0021:CIOAGD>2.0.CO;2)

21. Grodzins M. 1957 Metropolitan segregation. *Sci. Am.* **197**, 33–41. (doi:10.1038/scientificamerican1057-33)

22. Kopp RE, Shwom RL, Wagner G, Yuan J. 2016 Tipping elements and climate–economic shocks: pathways toward integrated assessment. *Earth's Future* **4**, 346–372. (doi:10.1002/2016EF000362)

23. Lenton TM. 2013 Environmental tipping points. *Annu. Rev. Environ. Resour.* **38**, 1–29. (doi:10.1146/annurev-environ-102511-084654)

24. van Nes EH, Arani BM, Staal A, van der Bolt B, Flores BM, Bathiany S, Scheffer M. 2016 What do you mean,'tipping point'?*Trends Ecol. Evol.* **31**, 902–904. (doi:10.1016/j.tree.2016.09.011)

25. Milkoreit M *et al.* 2018 Defining tipping points for social-ecological systems scholarship—an interdisciplinary literature review. *Environ. Res. Lett.* **13**, 033005. (doi:10.1088/1748-9326/aaaa75)

26. Arnold VI. 1992 *Catastrophe theory*, 3rd edn. Berlin, Germany: Springer.

27. Arnold VI. 1994 *Dynamical Systems V: bifurcation theory and catastrophe theory*, 1st edn. Berlin, Germany: Springer.

28. Golubitsky M. 1978 An introduction to catastrophe theory and its applications. *SIAM Rev.* **20**, 352–387. (doi:10.1137/1020043)

29. Stewart I. 1981 Applications of catastrophe theory to the physical sciences. *Physica D: Nonlinear Phenomena* **2**, 245–305. (doi:10.1016/0167-2789(81)90012-9)

30. Stewart I. 1982 Catastrophe theory in physics. *Rep. Progress Phys.* **45**, 185–221. (doi:10.1088/0034-4885/45/2/002)

31. Zeeman EC. 1976 Catastrophe theory. *Sci. Am.* **234**, 65–83. (doi:10.1038/scientificamerican0476-65)

32. Guckenheimer J. 1978 The catastrophe controversy. *Math. Intelligencer* **1**, 15–20. (doi:10.1007/BF03023037)

33. Kolata GB. 1977 Catastrophe theory: the emperor has no clothes. *Science* **196**, 287–351. (doi:10.1126/science.196.4287.287)

34. Sussmann HJ. 1976 Catastrophe theory: a preliminary critical study. In *PSA: Proc. of the Biennial Meeting of the Philosophy of Science Association*, vol. 1976, pp. 256–286.

35. Sussmann HJ, Zahler RS. 1978 Catastrophe theory as applied to the social and biological sciences: a critique. *Synthese* **37**, 117–216. (doi:10.1007/BF00869575)

36. Rosser Jr JB. 2007 The rise and fall of catastrophe theory applications in economics: was the baby thrown out with the bathwater? *J. Econ. Dyn. Control* **31**, 3255–3280. (doi:10.1016/j.jedc.2006.09.013)

37. Biggs R, Blenckner T, Folke C, Gordon L, Norström A, Nyström M *et al.* 2012 Regime shifts. In *Encyclopedia of Theoretical Ecology* (eds A Hastings, L Gross). Ewing, NJ: University of California Press.

38. Beisner BE, Haydon DT, Cuddington K. 2003 Alternative stable states in ecology. *Front. Ecol. Environ.* **1**, 376–382. (doi:10.1890/1540-9295(2003)001[0376:ASSIE]2.0.CO;2)

39. Scheffer M, Carpenter SR. 2003 Catastrophic regime shifts in ecosystems: linking theory to observation. *Trends Ecol. Evol.* **18**, 648–656. (doi:10.1016/j.tree.2003.09.002)

40. Holling CS. 1973 Resilience and stability of ecological systems. *Annu. Rev. Ecol. Syst.* **4**, 1–23. (doi:10.1146/annurev.es.04.110173.000245)

41. May RM. 1977 Thresholds and breakpoints in ecosystems with a multiplicity of stable states. *Nature* **269**, 471–477. (doi:10.1038/269471a0)

42. Alley RB *et al.* 2003 Abrupt climate change. *Science* **299**, 2005–2010. (doi:10.1126/science.1081056)

43. Lockwood J. 2001 Abrupt and sudden climatic transitions and fluctuations: a review. *Int. J. Climatol.* **21**, 1153–1179. (doi:10.1002/joc.630)

44. Rial JA *et al.* 2004 Nonlinearities, feedbacks and critical thresholds within the Earth's climate system. *Clim. Change* **65**, 11–38. (doi:10.1023/B:CLIM.0000037493.89489.3f)

45. Ashwin P, Wieczorek S, Vitolo R, Cox P. 2012 Tipping points in open systems: bifurcation, noise-induced and rate-dependent examples in the climate system. *Phil. Trans. R. Soc. A* **370**, 1166–1184. (doi:10.1098/rsta.2011.0306)

46. Thompson JMT, Stewart H, Ueda Y. 1994 Safe, explosive, and dangerous bifurcations in dissipative dynamical systems. *Phys. Rev. E* **49**, 1019–1027. (doi:10.1103/PhysRevE.49.1019)

47. Kuehn C. 2011 A mathematical framework for critical transitions: bifurcations, fast–slow systems and stochastic dynamics. *Physica D: Nonlinear Phenomena* **240**, 1020–1035. (doi:10.1016/j.physd.2011.02.012)

48. Thompson JMT, Sieber J. 2011 Predicting climate tipping as a noisy bifurcation: a review. *Int. J. Bifurcation Chaos* **21**, 399–423. (doi:10.1142/S0218127411028519)

49. Lenton TM. 2012 Arctic climate tipping points. *AMBIO* **41**, 10–22. (doi:10.1007/s13280-011-0221-x)

50. Wieczorek S, Ashwin P, Luke CM, Cox PM. 2011 Excitability in ramped systems: the compost-bomb instability. *Proc. R. Soc. A* **467**, 1243–1269. (doi:10.1098/rspa.2010.0485)

51. Boettiger C, Ross N, Hastings A. 2013 Early warning signals: the charted and uncharted territories. *Theor. Ecol.* **6**, 255–264. (doi:10.1007/s12080-013-0192-6)

52. Brummitt CD, Barnett G, D'Souza RM. 2015 Coupled catastrophes: sudden shifts cascade and hop among interdependent systems. *J. R. Soc. Interface* **12**, 20150712. (doi:10.1098/rsif.2015.0712)

53. Ditlevsen PD, Johnsen SJ. 2010 Tipping points: early warning and wishful thinking. *Geophys. Res. Lett.* **37**, L19703. (doi:10.1029/2010GL044486)

54. Kriegler E, Hall JW, Held H, Dawson R, Schellnhuber HJ. 2009 Imprecise probability assessment of tipping points in the climate system. *Proc. Natl Acad. Sci. USA* **106**, 5041–5046. (doi:10.1073/pnas.0809117106)

55. Caesar L, Rahmstorf S, Robinson A, Feulner G, Saba V. 2018 Observed fingerprint of a weakening Atlantic Ocean overturning circulation. *Nature* **556**, 191–196. (doi:10.1038/s41586-018-0006-5)

56. Carpenter SR, Lathrop RC. 2014 Phosphorus loading, transport and concentrations in a lake chain: a probabilistic model to compare management options. *Aquatic Sci.* **76**, 145–154. (doi:10.1007/s00027-013-0324-5)

57. Scheffer M. 2004 *Ecology of shallow lakes*, 22. Population and Community Biology Series. Dordrecht, The Netherlands: Springer.

58. Van Gerven LP, Kuiper JJ, Janse JH, Janssen AB, Jeuken M, Mooij WM, De Klein JJ. 2017 How regime shifts in connected aquatic ecosystems are affected by the typical downstream increase of water flow. *Ecosystems* **20**, 733–744. (doi:10.1007/s10021-016-0061-4)

59. Kinzig AP, Ryan P, Etienne M, Allison H, Elmqvist T, Walker BH. 2006 Resilience and regime shifts: assessing cascading effects. *Ecol. Soc.* **11**, 20. (doi:10.5751/ES-01678-110120)

60. Barnosky AD *et al.* 2012 Approaching a state shift in Earth's biosphere. *Nature* **486**, 52–58. (doi:10.1038/nature11018)

61. Hughes TP. 1994 Catastrophes, phase shifts, and large-scale degradation of a Caribbean coral reef. *Science* **265**, 1547–1551. (doi:10.1126/science.265.5178.1547)

62. Lenton TM, Williams HT. 2013 On the origin of planetary-scale tipping points. *Trends Ecol. Evol.* **28**, 380–382. (doi:10.1016/j.tree.2013.06.001)

63. Rocha JC, Peterson GD, Biggs R. 2015 Regime shifts in the Anthropocene: drivers, risks, and resilience. *PLoS ONE* **10**, 1–16. (doi:10.1371/journal.pone.0134639)

64. Rocha JC, Peterson G, Bodin Ö, Levin S. 2018 Cascading regime shifts within and across scales. *Science* **362**, 1379–1383. (doi:10.1126/science.aat7850)

65. Scheffer M *et al.* 2012 Anticipating critical transitions. *Science* **338**, 344–348. (doi:10.1126/science.1225244)

66. Cai Y, Lenton TM, Lontzek TS. 2016 Risk of multiple interacting tipping points should encourage rapid CO_2 emission reduction. *Nat. Clim. Change* **6**, 520–525. (doi:10.1038/nclimate2964)

67. Hilt S, Köhler J, Kozerski HP, van Nes EH, Scheffer M. 2011 Abrupt regime shifts in space and time along rivers and connected lake systems. *Oikos* **120**, 766–775. (doi:10.1111/j.1600-0706.2010.18553.x)

68. van Nes EH, Scheffer M. 2005 Implications of spatial heterogeneity for catastrophic regime shifts in ecosystems. *Ecology* **86**, 1797–1807. (doi:10.1890/04-0550)

69. Erneux T, Nicolis G. 1993 Propagating waves in discrete bistable reaction-diffusion systems. *Physica D: Nonlinear Phenomena* **67**, 237–244. (doi:10.1016/0167-2789(93)90208-I)

70. Fáth G. 1997 Propagation failure of traveling waves in a discrete bistable medium. *Physica D: Nonlinear Phenomena* **116**, 176–190. (doi:10.1016/s0167-2789(97)00251-0)

71. Keener JP. 1987 Propagation and its failure in coupled systems of discrete excitable cells. *SIAM J. Appl. Math.* **47**, 556–572. (doi:10.1137/0147038)

72. Zinner B. 1992 Existence of traveling wavefront solutions for the discrete Nagumo equation. *J. Differ. Equ.* **96**, 1–27. (doi:10.1016/0022-0396(92)90142-A)

73. van de Leemput IA, van Nes EH, Scheffer M. 2015 Resilience of alternative states in spatially extended ecosystems. *PLoS ONE* **10**, 1–17. (doi:10.1371/journal.pone.0116859)

74. Bel G, Hagberg A, Meron E. 2012 Gradual regime shifts in spatially extended ecosystems. *Theor. Ecol.* **5**, 591–604. (doi:10.1007/s12080-011-0149-6)

75. Watts DJ. 2002 A simple model of global cascades on random networks. *Proc. Natl Acad. Sci. USA* **99**, 5766–5771. (doi:10.1073/pnas.082090499)

76. Krönke J, Wunderling N, Winkelmann R, Staal A, Stumpf B, Tuinenburg OA, Donges JF. 2020 Dynamics of tipping cascades on complex networks. *Phys. Rev. E* **101**, 042311. (doi:10.1103/PhysRevE.101.042311).

77. Wunderling N, Donges JF, Kurths J, Winkelmann R. Submitted. Interacting tipping elements increase risk of climate domino effects under global warming. *Earth Syst. Dynam. Discuss.* (doi:10.5194/esd-2020-18)

78. Wunderling N, Stumpf B, Krönke J, Staal A, Tuinenburg OA, Winkelmann R, Donges JF. 2020 How motifs condition critical thresholds for tipping cascades in complex networks: linking micro- to macro-scales. *Chaos* **30**, 043129. (doi:10.1063/1.5142827)

79. Brummitt CD, D'Souza RM, Leicht EA. 2012 Suppressing cascades of load in interdependent networks. *Proc. Natl Acad. Sci. USA* **109**, E680–E689. (doi:10.1073/pnas.1110586109)

80. Buldyrev SV, Parshani R, Paul G, Stanley HE, Havlin S. 2010 Catastrophic cascade of failures in interdependent networks. *Nature* **464**, 1025–1028. (doi:10.1038/nature08932)

81. Hu Y, Ksherim B, Cohen R, Havlin S. 2011 Percolation in interdependent and interconnected networks: abrupt change from second- to first-order transitions. *Phys. Rev. E* **84**, 066116. (doi:10.1103/PhysRevE.84.066116)

82. Parshani R, Buldyrev SV, Havlin S. 2010 Interdependent networks: reducing the coupling strength leads to a change from a first to second order percolation transition. *Phys. Rev. Lett.* **105**, 048701. (doi:10.1103/PhysRevLett.105.048701)

83. Reis SD, Hu Y, Babino A, Andrade Jr JS, Canals S, Sigman M, Makse HA. 2014 Avoiding catastrophic failure in correlated networks of networks. *Nat. Phys.* **10**, 762–767. (doi:10.1038/nphys3081)

84. Abraham R. 1990 Cuspoidal nets. In *Towards a just society for future generations*. Pemona, CA: International Society for the System Sciences, California State Polytechnic University. pp. 667–678.

85. Abraham R, Keith A, Koebbe M, Mayer-Kress G. 1991 Computational unfolding of double-cusp models of opinion formation. *Int. J. Bifurcation Chaos* **1**, 417–430. (doi:10.1142/S0218127491000324)

86. Dekker MM, Heydt AS, Dijkstra HA. 2018 Cascading transitions in the climate system. *Earth System Dyn.* **9**, 1243–1260. (doi:10.5194/esd-9-1243-2018)

87. Gilmore R. 1993 *Catastrophe theory for scientists and engineers*. New York, NY: Dover Publications.

88. Kuznetsov YA. 2004 *Elements of applied bifurcation theory*. Applied Mathematical Sciences 112. New York, NY: Springer Science & Business Media.

89. Golubitsky M, Pivato M, Stewart I. 2004 Interior symmetry and local bifurcation in coupled cell networks. *Dyn. Syst.* **19**, 389–407. (doi:10.1080/14689360512331318006)

90. Levermann A, Winkelmann R. 2016 A simple equation for the melt elevation feedback of ice sheets. *The Cryosphere* **10**, 1799–1807. (doi:10.5194/tc-10-1799-2016)

91. Soranno PA, Cheruvelil KS, Webster KE, Bremigan MT, Wagner T, Stow CA. 2010 Using landscape limnology to classify freshwater ecosystems for multi-ecosystem management and conservation. *BioScience* **60**, 440–454. (doi:10.1525/bio.2010.60.6.8)

92. Gregory JM, Huybrechts P, Raper SC. 2004 Threatened loss of the Greenland ice-sheet. *Nature* **428**, 616. (doi:10.1038/428616a)

93. Levermann A et al. 2014 Projecting Antarctic ice discharge using response functions from SeaRISE ice-sheet models. *Earth System Dyn.* **5**, 271–293. (doi:10.5194/esd-5-271-2014)

94. Schellnhuber HJ, Rahmstorf S, Winkelmann R. 2016 Why the right climate target was agreed in Paris. *Nat. Clim. Change* **6**, 649–653. (doi:10.1038/nclimate3013)

95. Hirota M, Holmgren M, Van Nes EH, Scheffer M. 2011 Global resilience of tropical forest and savanna to critical transitions. *Science* **334**, 232–235. (doi:10.1126/science.1210657)

96. Albert R, Barabási AL. 2002 Statistical mechanics of complex networks. *Rev. Modern Phys.* **74**, 47. (doi:10.1103/RevModPhys.74.47)

97. Scheffer M et al. 2009 Early-warning signals for critical transitions. *Nature* **461**, 53–59. (doi:10.1038/nature08227)

98. Lenton TM. 2011 Early warning of climate tipping points. *Nat. Clim. Change* **1**, 201–209. (doi:10.1038/nclimate1143)