# Connectivity of Triangulation Flip Graphs in the Plane (Part II: Bistellar Flips)

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## Abstract

Given a finite point set $P$ in general position in the plane, a *full triangulation* is a maximal straight-line embedded plane graph on $P$. A *partial triangulation* on $P$ is a full triangulation of some subset $P'$ of $P$ containing all extreme points in $P$. A *bistellar flip* on a partial triangulation either flips an edge, removes a non-extreme point of degree 3, or adds a point in $P \setminus P'$ as vertex of degree 3. The *bistellar flip graph* has all partial triangulations as vertices, and a pair of partial triangulations is adjacent if they can be obtained from one another by a bistellar flip. The goal of this paper is to investigate the structure of this graph, with emphasis on its connectivity.

For sets $P$ of $n$ points in general position, we show that the bistellar flip graph is $(n-3)$-connected, thereby answering, for sets in general position, an open questions raised in a book (by De Loera, Rambau, and Santos) and a survey (by Lee and Santos) on triangulations. This matches the situation for the subfamily of regular triangulations (i.e., partial triangulations obtained by lifting the points and projecting the lower convex hull), where $(n-3)$-connectivity has been known since the late 1980s through the secondary polytope (Gelfand, Kapranov, Zelevinsky) and Balinski's Theorem.

Our methods also yield the following results (see the full version [13]): (i) The bistellar flip graph can be covered by graphs of polytopes of dimension $n-3$ (products of secondary polytopes). (ii) A partial triangulation is regular, if it has distance $n-3$ in the Hasse diagram of the partial order of partial subdivisions from the trivial subdivision. (iii) All partial triangulations are regular iff the trivial subdivision has height $n-3$ in the partial order of partial subdivisions. (iv) There are arbitrarily large sets $P$ with non-regular partial triangulations, while every proper subset has only regular triangulations, i.e., there are no small certificates for the existence of non-regular partial triangulations (answering a question by F. Santos in the unexpected direction).

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# 1 Introduction

Throughout this paper we let $P$ denote a finite planar point set in *general position* (no 3 points on a line) with $n \geq 3$ points. The set of *extreme points* of $P$ (i.e., the vertices of the convex hull of $P$) is denoted by $\mathsf{xtr}P$, and $P^\circ := P \setminus \mathsf{xtr}P$ denotes the set of *inner* (i.e., non-extreme) *points* in $P$. We consistently use $h = h(P) := |\mathsf{xtr}P|$ and $n^\circ = n^\circ(P) := |P^\circ| = n - h$. We let $\mathsf{E}_{\mathsf{hull}} = \mathsf{E}_{\mathsf{hull}}(P) \subseteq \binom{P}{2}$ denote the $h$ edges of the convex hull of $P$.

For graphs $G = (P', E)$, $P' \subseteq P$, $E \subseteq \binom{P'}{2}$, on $P'$ we often identify edges $\{p, q\}$ with their corresponding straight line segments $pq$. We let $\mathsf{V}G := P'$ and $\mathsf{E}G := E$.

▶ **Definition 1** (plane). A graph $G$ on $P$ is *plane* if no two straight line segments corresponding to edges in $\mathsf{E}G$ cross (i.e., they are disjoint except for possibly sharing an endpoint).

▶ **Definition 2** (full, partial triangulation). A *full triangulation* of $P$ is a maximal plane graph $T = (P, E)$. A *partial triangulation* of $P$ is a full triangulation $T = (P', E)$ with $\mathsf{xtr}P \subseteq P' \subseteq P$ (hence $\mathsf{E}_{\mathsf{hull}} \subseteq \mathsf{E}T$). Points in $\mathsf{V}^\circ T := P^\circ \cap \mathsf{V}T$ are called *inner points*. Points in $P^\circ \setminus \mathsf{V}^\circ T$ are called *skipped* in $T$. Edges in $\mathsf{E}^\circ T := \mathsf{E}T \setminus \mathsf{E}_{\mathsf{hull}}$ are called *inner edges*. Edges in $\mathsf{E}_{\mathsf{hull}}$ are called *boundary edges*. $\mathcal{T}_{\mathsf{part}}(P)$ denotes the set of all partial triangulations of $P$.

▶ **Convention.** From now on, we will mostly use "triangulation" for "partial triangulation".


**Figure 1** Edge flips and point flips (point removal, left to right; point insertion, right to left).

▶ **Definition 3** (bistellar flip). Let $T$ be a triangulation of $P$. An edge $e \in \mathsf{E}^\circ T$ is called *flippable in* $T$ if removal of $e$ in $T$ creates a convex quadrilateral face $Q$, when $T[e]$ is the triangulation with the other diagonal $\bar{e}$ of $Q$ added instead of $e$, i.e., $\mathsf{V}T[e] := \mathsf{V}T$ and $\mathsf{E}T[e] := \mathsf{E}T \setminus \{e\} \cup \{\bar{e}\}$; we call this an *edge flip*.

A point $p \in P^\circ$ is called *flippable in* $T$ if $p \in P^\circ \setminus \mathsf{V}^\circ T$ or if $p \in \mathsf{V}^\circ T$, of degree 3 in $T$. (a) If $p \in P^\circ \setminus \mathsf{V}^\circ T$ then $T[p]$ is the triangulation with $p$ added as a point of degree 3 (there is a unique way to do so); we call this a *point insertion flip*. (b) If $p \in \mathsf{V}^\circ T$ of degree 3 in $T$ then $T[p]$ is obtained by removing $p$ and its incident edges; we call this a *point removal flip*.

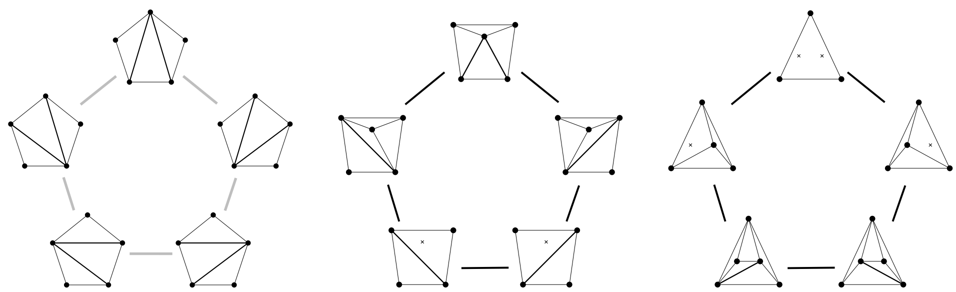

**Figure 2** Bistellar flip graphs for 5 points. Small crosses indicate skipped points in $P$.

Whenever we write $T[x]$ for a triangulation $T$, then $x$ is either a flippable point in $P^\circ$ or a flippable edge in $\mathsf{E}^\circ T$, and we write $T[x, y]$ short for $(T[x])[y]$, etc. The *bistellar flip graph* of $P$ is the graph with vertex set $\mathcal{T}_{\mathsf{part}}(P)$ and edge set $\{\{T, T[x]\} \mid T \in \mathcal{T}_{\mathsf{part}}(P), x \text{ flippable in } T\}$.

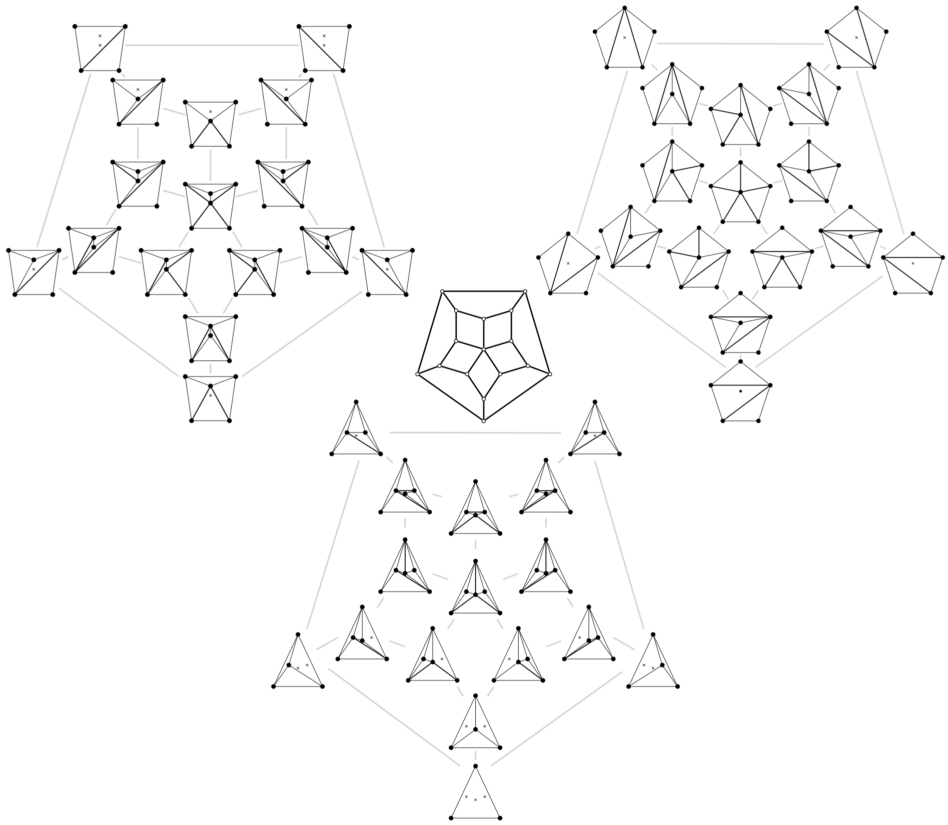

**Figure 3** Sets of 6 points with isomorphic bistellar flip graphs of triangulations. (Points indicated by crosses are points in $P$ skipped in the corresponding triangulation.)

The bistellar flip graph is connected (this follows easily from the connectedness of the edge flip graph of full triangulations, as established by Lawson in 1972 [8]). Here, we investigate how well connected the bistellar flip graph is. We refer to standard texts like [2, 6] for basics like the definition of *$k$-vertex connectivity* and *Menger's Theorem*. Our main result is:

▶ **Theorem 4.** *Let $P$ be a set of $n \geq 3$ points in general position in the plane. Then the bistellar flip graph of $P$ is $(n-3)$-vertex connected. (This is tight: Any triangulation of $P$ that skips all inner points has degree $(n-3)$ in the bistellar flip graph.)*

This answers (for points in general position) a question by De Loera, Rambau, and Santos in 2010 [4, Exercise 3.23], and by Lee and Santos in 2017 [9, pg. 442]. A corresponding result, $\lceil \frac{n}{2} - 2 \rceil$-connectedness of the edge flip graph of full triangulations, is proved in [14].

A particular way of obtaining a triangulation of a point set $P$ is to vertically lift the points to $\mathbb{R}^3$ such that no 4 points are coplanar, and then to project the lower convex hull of the lifted points back into the plane. Triangulations obtained in this way are called *regular triangulations* (e.g., [4]). It is well known that point sets may have non-regular triangulations.

Furthermore, we study the partially ordered set of *subdivisions* of $P$ (see Def. 9 below, and, e.g., [4]), in which triangulations are the minimal elements. We introduce the notions of *slack* (Def. 10), *perfect coarsenings* (Def. 20), and *perfect coarseners* (Def. 21), and we prove the so-called *Coarsening Lemma 25.* We consider these our main contributions besides Thm. 4. As consequences, these yield several other results on the structure of the bistellar flip graph and regular triangulations (see abstract); in particular, they allow us to settle, in an unexpected direction, another question by Santos [12] regarding the size of certificates for the existence of non-regular triangulations in the plane. Here, we focus on the proof of the connectivity, and we refer to the full version [13] for these additional results.

If $P$ is in convex position, full, partial, and regular triangulations coincide. It is well-known that there is an $(n-3)$-dimensional convex polytope, the *associahedron*, whose vertices correspond to the triangulations of $P$ and whose edges correspond to flips (Fig. 4, see [3] for a historical account). A classical theorem of Balinski [1], which asserts that the graph of any $d$-dimensional polytope is $d$-connected, immediately implies that the graph of the associahedron is $(n-3)$-connected. More generally, for arbitrary sets in the plane, it is known that there is an $(n-3)$-dimensional polytope, the *secondary polytope* defined by Gelfand et al. [7], whose vertices correspond to the regular triangulations of $P$ and edges correspond to bistellar flips; again, Balinski's Theorem implies $(n-3)$-connectivity. Our result extends this to arbitrary triangulations of arbitrary sets in general position in the plane.

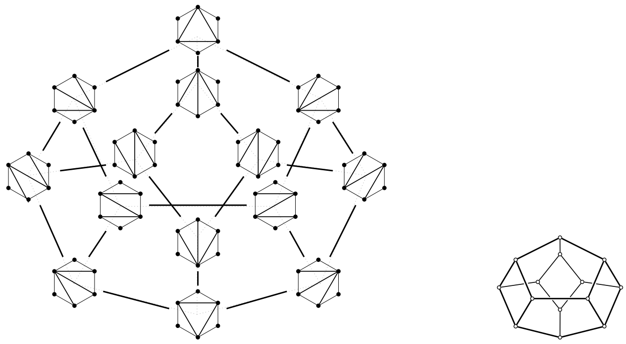

**Figure 4** The flip graph of the convex hexagon, the graph of the order 5 associahedron.

### Approach and Intuition

There is evidence that the bistellar flip graph of partial triangulations does not exhibit a polytopal structure as we see it with regular triangulations [4]. Still, the intuition behind our approach is to "pretend" that such a structure exists, at least locally for the small dimensional features. This will become clearer below, and is made more explicit in the full version [13] where it shown that the bistellar flip graph can be covered by polytopal structures.

## 2 Preliminaries, Terminology, and Notation

▶ **Definition 5** (legal graph; region)**.** For a graph $G = (P', E)$, $P' \subseteq P$, we let $\mathsf{V}^{\mathsf{by}}G$ be the points in $P'$ which are isolated in $G$, called *bystanders* in $G$. $G$ is called *legal* if it is plane, if $\mathsf{E}_{\mathsf{hull}}(P) \subseteq \mathsf{E}G$ (hence $\mathsf{xtr}P \subseteq P'$), and if the graph $(\mathsf{V}G \setminus \mathsf{V}^{\mathsf{by}}G, \mathsf{E}G)$ is 2-edge connected.

Let $G$ be a legal graph. Similar to triangulations, we define $\mathsf{E}^\circ G := \mathsf{E}G \setminus \mathsf{E}_{\mathsf{hull}}$ and $\mathsf{V}^\circ G := \mathsf{V}G \cap P^\circ$. Moreover, we let $\mathsf{V}^{\mathsf{inv}}G := \mathsf{V}^\circ G \setminus \mathsf{V}^{\mathsf{by}}G$ (the *involved points*). Bounded faces of $(\mathsf{V}G \setminus \mathsf{V}^{\mathsf{by}}G, \mathsf{E}G)$ are called *regions* of $G$, i.e., these are bounded connected components in the complement of the straight line embedding of $G$, ignoring its bystanders.

Regions of legal graphs are bounded simply connected polygonal open sets, pairwise disjoint. We state the following well-known facts for ease of reference.

▶ **Lemma 6.** *For a full triangulation $T$ of $P$, $|\mathsf{E}T| = |\mathsf{E}^\circ T| + h = 3n-3-h = 3n^\circ - 3 + 2h$ and the number of regions (which does not include the unbounded face) is $2n-2-h = 2n^\circ - 2 + h$.*

▶ **Definition 7** (locked). In a legal graph $G$ on $P$, an edge $e \in \mathsf{E}G$ is *locked at endpoint* $p$ if the angle obtained at $p$ (between the edges adjacent to $e$ at $p$) after removal of $e$ exceeds $\pi$.

An edge in a triangulation is flippable iff it is locked by none of its endpoints. Edges locked at a common endpoint $p$ have to be consecutive around $p$. There can be at most 3 edges locked at a given point $p$, and 3 edges can be locked at $p$ only if $p$ has degree 3.

Given a legal graph $G$, we consider *partial orientations* $\vec{G}$: These assign orientations to *some* (not all) of the edges in $\mathsf{E}G$, with no edge oriented in both directions, and with the boundary edges not oriented. We need the following [14, Lemma 5.1(i)]:

▶ **Lemma 8** (Unoriented Edges Lemma). *Let $G$ be a legal graph with $\mathsf{V}^{\mathsf{by}}G = \emptyset$, $N := |\mathsf{V}G|$, and $D := 3N - 3 - h - |\mathsf{E}G|$, i.e., the number of edges missing in $G$ towards a full triangulation of $\mathsf{V}G$. For $\vec{G}$ a partial orientation of $G$, let $C_i$ be the number of inner points of $\vec{G}$ with indegree $i$ and suppose $C_i = 0$ for $i \geq 4$. Then the number of unoriented inner edges is at least $N - 3 - C_3 - D$.*

To indicate, how this can be useful in our context, consider $G = T$, $T$ a triangulation, i.e., $D = 0$. Orient every locked inner edge to the endpoint where it is locked. Then $C_i = 0$ for $i \geq 4$, $C_3$ is exactly the number of inner points of degree 3, and the inner unoriented edges are exactly the unlocked, i.e., flippable edges. It follows that there are $C_3$ point removal flips, at least $N - 3 - C_3$ edge flips, and obviously $n - N$ point insertion flips. Altogether, there are at least $n - 3$ flips.

## 3 Partial Subdivisions – Slack and Order

We now define *partial subdivisions*, which form a poset in which the partial triangulations of $P$ are the minimal elements.

▶ **Definition 9** (full, partial subdivision). A *partial subdivision* $S$ on $P$ is a legal graph with all of its regions convex. For a region $r$ of $S$, let $\mathsf{V}r := \overline{r} \cap \mathsf{V}S$ ($\overline{r}$ the closure of $r$). $S_{\mathsf{triv}} = S_{\mathsf{triv}}(P) := (P, \mathsf{E}_{\mathsf{hull}})$ is called the *trivial subdivision of* $P$. If $\mathsf{V}S = P$ and $\mathsf{V}^{\mathsf{by}}S = \emptyset$, then $S$ is called a *full subdivision* on $P$.

▶ **Convention.** From now on, we will mostly use "subdivision" for "partial subdivision".

$\mathsf{V}S$ is essential in the definition of a subdivision, it is *not* simply the set of endpoints of edges in $S$, there are also bystanders; e.g., for $T$ a triangulation of $P$, all graphs $(P', \mathsf{E}T)$, $\mathsf{V}T \subseteq P' \subseteq P$, are subdivisions of $P$, all different. $\mathsf{V}S$ partitions into boundary points, involved points, and bystanders, i.e., $\mathsf{V}S = \mathsf{xtr}P \,\dot{\cup}\, \mathsf{V}^{\mathsf{inv}}S \,\dot{\cup}\, \mathsf{V}^{\mathsf{by}}S$. Moreover there are the skipped points, $P \setminus \mathsf{V}S$.

A first important example of a subdivision is obtained from a triangulation $T$ and an element $x$ flippable in $T$, i.e., $\{T, T[x]\}$ is an edge of the bistellar flip graph:

$$T_{\pm x} := (\mathsf{V}T \cup \mathsf{V}T[x], \mathsf{E}T \cap \mathsf{E}T[x])$$

If $x = e$ is a flippable edge, then $T_{\pm e}$ has one convex quadrilateral region $Q$; all other regions are triangular. We obtain $T$ and $T[e]$ from $T_{\pm e}$ by adding one or the other of the 2 diagonals of $Q$ to $T_{\pm e}$. If $x = p$ is a flippable point, then $T_{\pm p}$ is almost a triangulation, all regions are triangular, except that $p \in \mathsf{V}T_{\pm p}$ is a bystander. We obtain $T$ and $T[p]$ by either removing this point from $T_{\pm p}$ or by adding the 3 edges from $p$ to the points of the triangular region in which $p$ lies. The subdivision $T_{\pm x}$ is close to a triangulation and, in a sense, represents the flip between $T$ and $T[x]$. To formalize and generalize this we introduce the following notion:

▶ **Definition 10** (slack). Given a subdivision $S$ of $P$, we call a region of $S$ *active* if it is not triangular or if it contains at least one point in $\mathsf{V}S$ (necessarily a bystander) in its interior.

For a region $r$ of $S$, we define its *slack* $\mathsf{sl}\,r := |\mathsf{V}r| - 3$. The *slack of* $S$, $\mathsf{sl}\,S$, is the sum of slacks of its regions.

▶ **Lemma 11.** *For a subdivision $S$ with $s$ bystanders we have*

$$\mathsf{sl}S = 3(|\mathsf{V}S| - s) - 3 - h - |\mathsf{E}S| + s = 3|\mathsf{V}S| - 3 - h - |\mathsf{E}S| - 2s \ .$$

**Proof.** The slack of a region $r$ equals the number of edges it takes to triangulate $r$ (ignoring bystanders) plus the number of bystanders. Thus, $\mathsf{sl}\,S$ is the number of edges it takes to triangulate $(\mathsf{V}S \setminus \mathsf{V}^{\mathsf{by}}S, \mathsf{E}S)$ plus $|\mathsf{V}^{\mathsf{by}}S|$. Now the claim follows from Lemma 6. ◀

▶ **Observation 12.** *Let $S$ be a subdivision. (i) $\mathsf{sl}S = 0$ iff $S$ is a triangulation iff $S$ has no active region. (ii) $\mathsf{sl}S = 1$ iff $S$ has exactly one active region of slack 1; this region is either a convex quadrilateral, or a triangular region with one bystander in its interior. (iii) $\mathsf{sl}S = 2$ iff $S$ has either (a) exactly 2 active regions, both of slack 1, or (b) exactly one active region of slack 2, where this region is either a convex pentagon, or a convex quadrilateral with one bystander in its interior, or a triangular region with 2 bystanders in its interior (Fig. 2).*

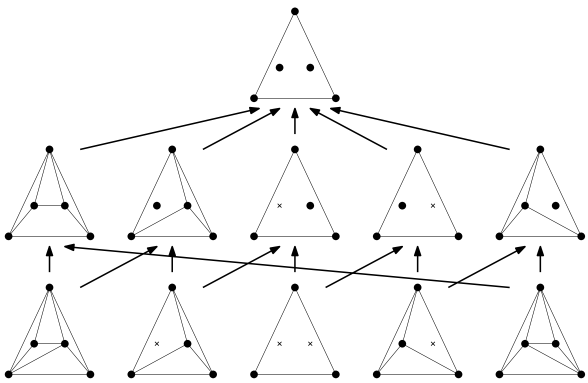

**Figure 5** Hasse diagram of the partial order $\preceq$ for a set of 5 points.

▶ **Definition 13** (coarsening, refinement). For subdivisions $S_1$ and $S_2$ of $P$, $S_2$ *coarsens* $S_1$, in symbols $S_2 \succeq S_1$, if $\mathsf{V}S_2 \supseteq \mathsf{V}S_1$, and $\mathsf{E}S_2 \subseteq \mathsf{E}S_1$. We also say that $S_1$ *refines* $S_2$, $(S_1 \preceq S_2)$.

The example in Fig. 5 hides some of the intricacies of the partial order $\preceq$; e.g., in general, it is not true that all paths from a triangulation to $S_{\mathsf{triv}}$ have the same length $n - 3$. $S_{\mathsf{triv}}$ is the unique coarsest (maximal) element. The triangulations (i.e., subdivisions of slack 0) are the minimal elements.

▶ **Definition 14** (set of refining triangulations). For a subdivision $S$ of $P$ we let $\mathcal{T}_{\mathsf{part}}\langle S\rangle := \{T \in \mathcal{T}_{\mathsf{part}}(P) \,|\, T \preceq S\}$.

Note that $\mathcal{T}_{\mathsf{part}}\langle S_{\mathsf{triv}}\rangle = \mathcal{T}_{\mathsf{part}}(P)$ and for $x$ flippable in $T$, $\mathcal{T}_{\mathsf{part}}\langle T_{\pm x}\rangle = \{T, T[x]\}$.

▶ **Observation 15.** *(i) Any subdivision $S$ of slack 1 of $P$ equals $T_{\pm x}$ for some triangulation $T \preceq S$ and some $x$ flippable in $T$. (ii) Let $S$ be a subdivision of slack 2 of $P$. If there are exactly 2 active regions in $S$ (of slack 1 each), then $\mathcal{T}_{\mathsf{part}}\langle S\rangle$ has cardinality 4, spanning a 4-cycle in the bistellar flip graph of $P$ (Fig. 6). If there is exactly 1 active region in $S$ (of slack 2), then $\mathcal{T}_{\mathsf{part}}\langle S\rangle$ has cardinality 5, spanning a 5-cycle (Fig. 2).*

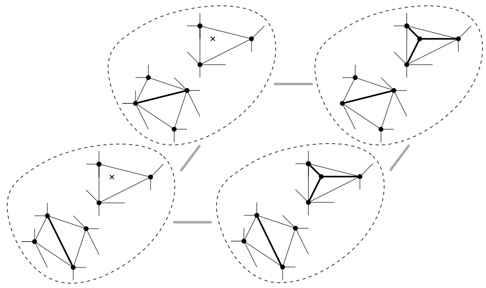

**Figure 6** A subdivision $S$ with 2 active regions of slack 1 each with $\mathcal{T}_{\mathsf{part}}\langle S\rangle$ spanning a 4-cycle.

▶ **Lemma 16.** *Any proper refinement of a subdivision of slack 2 has slack at most 1.*

**Proof.** Let $\mathsf{sl}\, S' = 2$ and let $S$ be a proper refinement of $S'$. For a refinement we add $m$ edges, thereby involving $s'$ bystanders, and we remove $s''$ bystanders (some of these parameters may be 0, but not all, since the refinement is assumed to be proper). We have $\mathsf{sl}\, S = \mathsf{sl}\, S' - (m - 2s' + s'')$ (easy consequence of Lemma 11) and want to show $m - 2s' + s'' > 0$.

Since $\mathsf{sl}\, S' = 2$, $S'$ has at most 2 bystanders and thus $s' \leq 2$. If $s' = 0$, then $m - 2s' + s'' > 0$ holds, since some of the 3 parameters have to be positive. If $s' = 1$, we observe that we need at least 3 edges to involve a bystander and $m - 2s' \geq 3 - 2 \cdot 1$. If $s' = 2$, we need at least 5 edges to involve 2 bystanders and $m - 2s' \geq 5 - 2 \cdot 2$. ◀

For $D \geq 3$, a proper refinement of a subdivision of slack $D$ can have slack $D$ or even higher (Fig. 7). The proof fails, since we can involve 3 bystanders with 6 edges.

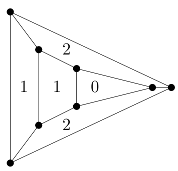


**Figure 7** 8 points, with a subdivision of slack 6, a refinement of $S_{\mathsf{triv}}$ of slack $8 - 3 = 5$.

Intuitively, as briefly alluded to at the end of Sec. 1, one can think of the subdivisions as the faces of a higher-dimensional geometric structure behind the bistellar flip graph, with the slack playing the role of dimension, somewhat analogous to the secondary polytope for regular triangulations. (For the edge flip graph of full triangulations, an analogous higher-dimensional *flip complex* is treated in [11, 10], and provides a similar geometric intuition for the arguments in [14].) The following lemma shows that – for slack at most 2 – we have the property corresponding to the fact that faces of dimension $d$ are either equal, or intersect in a common face of smaller dimension (possibly empty).

▶ **Lemma 17.**

**(i)** *For subdivisions $S_1$ and $S_2$ of slack 2, $\mathcal{T}_{\mathsf{part}}\langle S_1\rangle \cap \mathcal{T}_{\mathsf{part}}\langle S_2\rangle$ is either* (a) *empty,* (b) *equals $\{T\}$ for some triangulation $T$,* (c) *equals $\{T, T[x]\}$ for some triangulation $T$ and some flippable element $x$, or* (d) *$S_1 = S_2$.*

**(ii)** *Let $x$ and $y$ be two distinct flippable elements in triangulation $T$. If there is a subdivision $S$ of slack 2 with $\{T[x], T, T[y]\} \subseteq \mathcal{T}_{\mathsf{part}}\langle S\rangle$, then this $S$ is unique.*

**Proof.** If $\mathcal{T}_{\mathsf{part}}\langle S_1\rangle \cap \mathcal{T}_{\mathsf{part}}\langle S_2\rangle$ contains some triangulation, then we easily see that $S_1 \wedge S_2 := (\mathsf{V}S_1 \cap \mathsf{V}S_2, \mathsf{E}S_1 \cup \mathsf{E}S_2)$ is a subdivision, and $\mathcal{T}_{\mathsf{part}}\langle S_1 \wedge S_2\rangle = \mathcal{T}_{\mathsf{part}}\langle S_1\rangle \cap \mathcal{T}_{\mathsf{part}}\langle S_2\rangle$.

(i) If (a) does not apply, let $S := S_1 \wedge S_2$, a subdivision with $\mathcal{T}_{\mathsf{part}}\langle S\rangle = \mathcal{T}_{\mathsf{part}}\langle S_1\rangle \cap \mathcal{T}_{\mathsf{part}}\langle S_2\rangle$. If $\mathsf{sl}\, S = 0$ we have property (b), if $\mathsf{sl}\, S = 1$ we have property (c). In the remaining case $\mathsf{sl}\, S \geq 2$, $S$ is a refinement of $S_1$ and of $S_2$. Lemma 16 tells us that $S$ cannot be a proper refinement of $S_1$, hence $S = S_1$; similarly, $S = S_2$, hence $S_1 = S_2$.

(ii) Suppose $S_1$ and $S_2$ are subdivisions of slack 2 with $\{T[x], T, T[y]\} \subseteq \mathcal{T}_{\mathsf{part}}\langle S_1\rangle \cap \mathcal{T}_{\mathsf{part}}\langle S_2\rangle$. Since options (a-c) above cannot apply, we are left with $S_1 = S_2$. ◀

Two edges incident to a vertex of a polytope may span a 2-face, or not; same here:

▶ **Definition 18** (compatible elements)**.** Two distinct flippable elements $x, y \in \mathsf{V}^\circ T \cup \mathsf{E}^\circ T$ are called *compatible in* $T$, in symbols $x \diamond y$, if there is a subdivision $T_{\pm x,y} \succeq T$ of slack 2, s.t. $\{T[x], T, T[y]\} \subseteq \mathcal{T}_{\mathsf{part}}\langle T_{\pm x,y}\rangle$. (Note that $T_{\pm x,y}$ is unique, by Lemma 17(ii).) Otherwise, $x$ and $y$ are called *incompatible in* $T$, in symbols $x \not\diamond\, y$.

This needs some time to digest. In particular, if two flippable edges $e$ and $f$ share a common endpoint of degree 4, then they are compatible, see Fig. 8 (bottom left), quite contrary to the situation for full triangulations as treated in [14]. The configurations of 2 flippable but incompatible are shown in Fig. 8, rightmost examples: (a) Two flippable edges $e$ and $f$ whose removal creates a nonconvex pentagon and whose common endpoint $q$ has degree at least 5. (b) A flippable edge $e$ and a flippable point $p$ of degree 3 whose removal creates a nonconvex quadrilateral region whose reflex point $q$ has degree at least 5 in the triangulation.

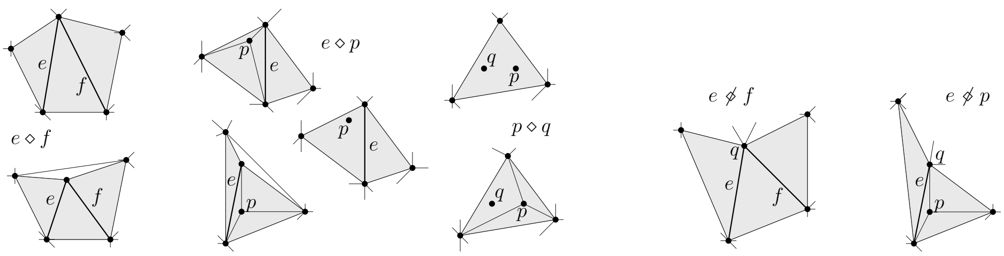


**Figure 8** Compatible elements (with overlapping incident regions, all contained in a 5-cycle, see Fig. 2 and incompatible elements (two rightmost, where $q$ is assumed to have degree at least 5). Shaded areas are unions of incident regions of flippable elements (not the active region in $T_{\pm x,y}$!).

What is essential for us is that whenever $x$ and $y$ are compatible in a triangulation $T$, then there is a cycle of length 4 or 5 containing $(T[x], T, T[y])$, and therefore, apart from the path $(T[x], T, T[y])$, there exists a $T$-avoiding $T[x]$-$T[y]$-path of length 2 or 3.

▶ **Observation 19.** *Let $T \in \mathcal{T}_{\mathsf{part}}(P)$. (i) A skipped point $p \in P^\circ \setminus \mathsf{V}^\circ T$ is compatible with every flippable element of $T$. (ii) Any two flippable points $p, q \in P^\circ$ are compatible.*

## 4 Coarsening Partial Subdivisions

As in [14] for full triangulations, the existence of many coarsenings is essential for our connectivity result. In order to motivate the definitions below, note that for full subdivisions (as employed in [14]), if $S_1 \preceq S_2$, then $(S_1, S_2)$ is an edge in the Hasse-diagram of the partial order $\preceq$ iff $\mathsf{sl}\, S_2 = \mathsf{sl}\, S_1 + 1$. For partial subdivisions, this is not the case (Fig. 9).

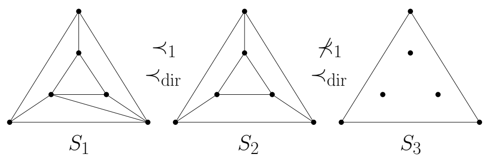


**Figure 9** $\mathsf{sl}\, S_1 = 2$, $\mathsf{sl}\, S_2 = 3$, $\mathsf{sl}\, S_3 = 3$. Note that $S_2 \prec_{\text{dir}} S_3$ but $S_2 \not\prec_1 S_3$, and that $S_1 \preceq S_3$ with $\mathsf{sl}\, S_3 = \mathsf{sl}\, S_1 + 1$ but $S_1 \not\prec_1 S_3$.

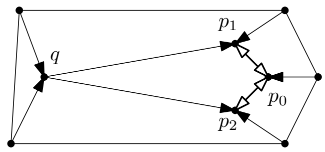


**Figure 10** A subdivision, edges are oriented to endpoints where locked (not what we called a partial orientation, since some edges are doubly oriented). Removing the 3 edges incident to $p_0$ does not yield a subdivision, since a reflex angle occurs at $p_1$ and $p_2$. The edges incident to $\{p_0, p_1, p_2\}$ are not looked outside this set, but removing all incident edges creates a reflex angle at point $q$.

▶ **Definition 20** (direct, perfect coarsening)**.** Let $S_1$ and $S_2$ be subdivisions. (i) We call $S_2$ a *direct coarsening of* $S_1$ (and $S_1$ a *direct refinement of* $S_2$), in symbols $S_1 \prec_{\text{dir}} S_2$, if $S_1 \preceq S_2$ and any subdivision $S$ with $S_1 \preceq S \preceq S_2$ satisfies $S \in \{S_1, S_2\}$ (equivalently, if $(S_1, S_2)$ is an edge in the Hasse diagram of $\preceq$). (ii) We call $S_2$ a *perfect coarsening of* $S_1$ ($S_1$ a *perfect refinement of* $S_2$), in symbols $S_1 \prec_1 S_2$, if $S_1 \prec_{\text{dir}} S_2$ and $\mathsf{sl}\, S_2 = \mathsf{sl}\, S_1 + 1$. (iii) $\prec_1^*$ is the reflexive transitive closure of $\prec_1$.

The reflexive transitive closure of $\prec_{\text{dir}}$ is exactly $\preceq$, while $\prec_1^* \subseteq \preceq$ and, in general, the inclusion is proper.

To motivate the upcoming definitions, let us discuss a few possibilities of coarsenings, direct coarsenings and perfect coarsenings. There are the simple operations of removing an unlocked edge, and of adding a point $p \in P \setminus \mathsf{V}S$ as a bystander. For a triangulation, we can isolate a point of degree 3. How does this generalize to subdivisions? Removing the edges incident to a point of degree 3 does not work if some incident edge might be locked at its other endpoint (e.g., $p_0$ in Fig. 10). If, however, no edge incident to a given point $p$ (of any degree) is locked at the respective other endpoint, then we can isolate this point for a coarsening $S'$. Unless $p$ has degree 3, $S'$ is not a direct coarsening of $S$, though. If $p$ has degree at least 4, one of the incident edges, say $e$, is not locked at $p$, thus not locked at all, and therefore, $S \preceq S'' \preceq S'$ for $S'' := (\mathsf{V}S, \mathsf{E}S \setminus \{e\})$. Finally, suppose we want to isolate all points in a set $U$ of points for obtaining a coarsening $S'$. For this to work, it is necessary that no edge $e$ connecting $U$ with the outside is locked at the endpoint of $e$ not in $U$. However, this is not a sufficient condition, because several edges connecting $U$ with a point not in $U$ can collectively create a reflex vertex by their removal (e.g., $U = \{p_0, p_1, p_2\}$ in Fig. 10). Moreover, for $S \prec_{\text{dir}} S'$ to hold, $U$ cannot be incident to unlocked edges, and no nonempty subset of $U$ can be suitable for such an isolation operation.

▶ **Definition 21** (prime, perfect coarsener; increment)**.** Let $S$ be a subdivision and let $U \subseteq \mathsf{V}S \cap P^\circ$. (i) $U$ is called a *coarsener*, if (a) $U$ is incident to at least one edge in $S$, and (b) removal of the set $E_U$ of all edges incident to $U$ in $S$ yields a subdivision. (ii) If $U$ is a coarsener, the *increment of* $U$, $\mathsf{inc}\, U$, is defined as $|E_U| - 2|U|$. (iii) $U$ is called a *prime coarsener*, if (a) $U$ is a coarsener, (b) $U$ is a minimal coarsener, i.e., no proper subset of $U$ is a coarsener, and (c) all edges incident to $U$ are locked. (iv) $U$ is called a *perfect coarsener*, if (a) $U$ is a prime coarsener, and (b) $\mathsf{inc}\, U = 1$.

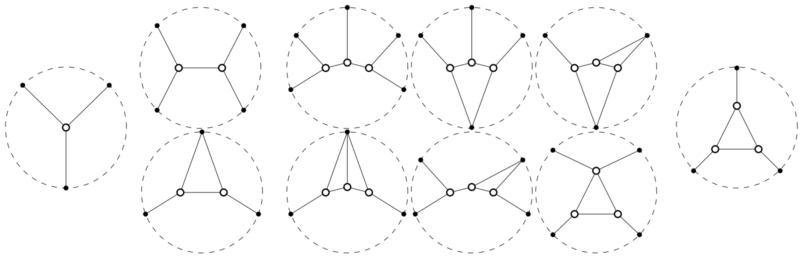

**Figure 11** Prime coarseners, all perfect, except for the rightmost one (with $\mathsf{inc} = 0$).

The following observation, a simple consequence of Lemma 11, explains the term "increment".

▶ **Observation 22.** *Let $S$ be a subdivision with coarsener $U$, and let $S'$ be the subdivision obtained from $S$ by removing all edges incident to $U$. Then $\mathsf{sl}S' = \mathsf{sl}S + \mathsf{inc}U$.*

▶ **Observation 23.**

**(i)** *Every subdivision $S$ with $\mathsf{E}^\circ S \neq \emptyset$ has a coarsener (the set $\mathsf{V}^\circ S$).*

**(ii)** *If $U_1$ and $U_2$ are coarseners, then $U_1 \cap U_2$ is a coarsener, unless there is no edge of $S$ incident to $U_1 \cap U_2$.*

**(iii)** *If $U_1$ and $U_2$ are prime coarseners, then $U_1 = U_2$ or $U_1 \cap U_2 = \emptyset$.*

**(iv)** *If $U$ is a prime coarsener, then the subgraph of $S$ induced by $U$ is connected.*

The following observation lists all ways of obtaining direct and perfect coarsenings.

▶ **Observation 24.** *Let $S = (V, E)$ and $S'$ be subdivisions.*

**(i)** *$S'$ is a direct coarsening of $S$ iff it is obtained from $S$ by one of the following.*
Adding a single point. *For $p \in P \setminus V$, $S' = (V \cup \{p\}, E)$ (with $\mathsf{sl}S' = \mathsf{sl}S + 1$).*
Removing a single unlocked edge. *For $e \in E$, not locked by either of its two endpoints, $S' = (V, E \setminus \{e\})$ (with $\mathsf{sl}S' = \mathsf{sl}S + 1$).*
Isolating a prime coarsener. *For $U$ a prime coarsener in $S$, $S'$ is obtained from $S$ by removal of the set, $E_U$, of all edges incident to points in $U$, i.e., $S' = (V, E \setminus E_U)$ (with $\mathsf{sl}S' = \mathsf{sl}S + \mathsf{inc}U$).*

**(ii)** *$S'$ is a perfect coarsening of $S$ iff it is obtained from $S$ by adding a single point, removing a single unlocked edge, or by isolating a perfect coarsener.*

▶ **Lemma 25** (Coarsening Lemma for Partial Subdivisions). *Every subdivision of slack $D$ has at least $n - 3 - D$ perfect coarsenings (i.e., direct coarsenings of slack $D + 1$).*

**Proof.** We start with the case $D = 0$, i.e., we have a triangulation $T$ and we want to show that there are at least $n - 3$ direct coarsenings of slack 1. Let $N := |\mathsf{V}T|$. We orient inner locked edges to their locking endpoints (recall that in a triangulation there is at most one such endpoint for each inner edge). Let $C_i$, $i \in \mathbb{N}_0$, be the number of points $p \in \mathsf{V}^\circ T$ with indegree $i$. The number of unoriented, thus unlocked edges is at least $N - 3 - C_3$ (Lemma 8).

There are $n - N$ subdivisions obtained from $T$ by adding a single point, there are at least $N - 3 - C_3$ subdivisions obtained from $T$ by removing a single unlocked edge, and there are $C_3$ direct coarsenings obtained from $T$ by isolating an inner point of degree 3. Adding up these numbers gives at least $n - 3$ perfect coarsenings of $T$.

We let $S$ be a subdivision of slack $D \geq 1$ assuming the assertion holds for slack less than $D$.

Case 1. *There is a bystander $p_0 \in \mathsf{V}S \cap P^\circ$.* Then $(\mathsf{V}S \setminus \{p_0\}, \mathsf{E}S)$ is a subdivision of slack $D-1$ of $P \setminus \{p_0\}$ with at least $(n-1)-3-(D-1) = n-3-D$ perfect coarsenings of slack $D$. For each such perfect coarsening $S'$, the subdivision $(\mathsf{V}S' \cup \{p_0\}, \mathsf{E}S')$ is a direct coarsening of $S$ of slack $D+1$, thus a perfect coarsening.

Case 2. *There is no bystander in $S$.* Again we employ a partial orientation of $S$. The choice of the orientation is somewhat more intricate and we will proceed in three phases (Fig. 12). We keep the invariant that the unoriented inner edges are exactly the unlocked inner edges.

In a *first phase*, we orient all locked inner edges to *all* of their locking endpoints, i.e., we temporarily allow edges to be directed to both ends (to be corrected in the third phase); edges directed to both endpoints are called *mutual edges*. We can give the following interpretation to an edge directed from $p$ to $q$: If we decide to isolate $p$ (i.e., remove all incident edges of $p$) for a coarsening of $S$, then $q$ becomes a reflex point of some region and we have to isolate $q$ as well (i.e., every coarsener containing $p$ must contain $q$ as well). In particular, if $\{p, q\}$ is a mutual edge, then either both or none of the points $p$ and $q$ will be isolated. In fact, if we consider the graph $G$ on $\mathsf{V}^\circ S$ with all mutual edges in the current orientation, then in any coarsening of $S$ either all points in a connected component of $G$ are isolated, or none.

A connected component $K$ of $G$ is called a *candidate component*, (a) if all edges connecting $K$ with points outside are directed towards $K$, (b) no point in $K$ is incident to an unoriented edge, (c) all points in $K$ have indegree 3, and (d) the mutual edges in $K$ do not form any cycle (i.e., they have to form a spanning tree of $K$). It follows that if $K$ has $k$ points then the number of edges is $3k-(k-1) = 2k+1$. The term "candidate" refers to the fact that removing all edges incident to $K$ *seems* like a direct coarsening step with incrementing the slack by 1 (Lemma 11); however, while individual edges connecting $K$ to the rest of the graph are not locked at their endpoints outside $K$, some of these edges collectively may actually create a reflex vertex in this way (see $K$ and $q$ in Fig. 12, left). So $K$ is only a candidate for a perfect coarsener.

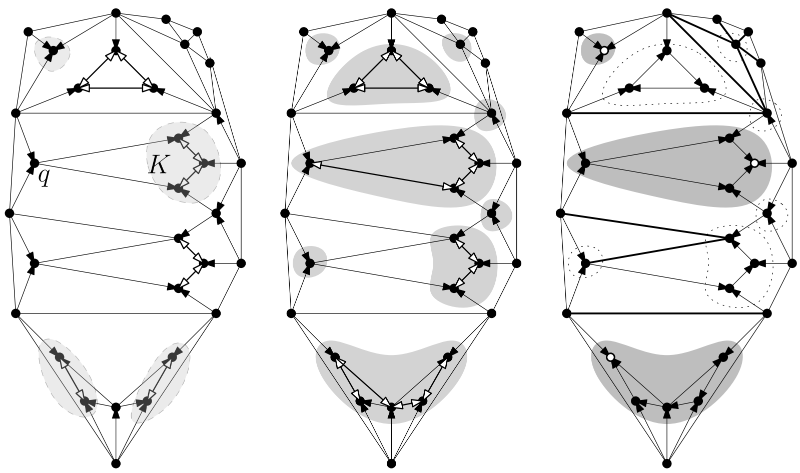


**Figure 12** Left: orientation after phase 1, with candidate components shaded; middle: after phase 2, with the connected components of $G^*$; right: after phase 3, with unoriented edges bold (each of these can be removed for a coarsening of slack 1 larger), and with the candidate components with a leader shaded (perfect coarseners).

We start the *second phase* of orienting edges further. In the spirit of our remarks about candidate components of $G$, suppose $q$ is an inner point outside a candidate $K$ of $G$ (thus all edges connecting $q$ to $K$ are directed from $q$ to $K$), such that removing the edges connecting

$q$ to $K$ creates a reflex angle at $q$. Then we orient one (and only one) of the edges connecting $q$ to $K$, say $\{p, q\}$, also to $q$ (thereby making this edge mutual). We call all the edges connecting $K$ to $q$, except for $\{p, q\}$, the *witnesses of the extra new orientation of* $\{p, q\}$ *from* $p$ *to* $q$. We successively proceed orienting edges, with the graph $G$ of mutual edges evolving in this way (and candidate components growing or disappearing).[1] The process will clearly stop at some point when the second phase is completed. We freeze $G$ and denote it by $G^*$.

Before we start the third phase, let us make a few crucial observations:

**(i)** If $p, q$ are inner points in the same connected component of $G^*$, then any coarsener contains both or none (i.e., if a connected component is a coarsener, then it is prime). This holds after phase 1, and whenever we expand a connected component, it is maintained.

**(ii)** During the second phase, an edge can be witness only once, and it is and will never be directed to the endpoint where it witnesses. Why? (a) Before it becomes a witness, it connects different connected components of $G$, after that it is and stays in a connected component of $G$. (b) Before it becomes a witness, it is not directed to the endpoint to which it witnesses an orientation, after that it is and stays in a connected component of $G$ and can therefore not get an extra direction. (An unoriented edge can never get an orientation and it can never be a witness.)

**(iii)** If we remove, conceptually, for each incoming edge of a point $q$ the witnesses (which direct away from $q$) for the orientation of this edge to $q$, then among remaining incident edges, all the incoming edges are locked at $q$ (an incoming edge that was oriented already in the first phase to $q$ has no witness). In particular, the indegree of $q$ cannot exceed 3, and if $q$ is incident to some not ingoing edge which is not a witness for any edge incoming at $q$, then the indegree of $q$ is at most 2. (We might generate incoming edges to a point $q$ that are not consecutive around $q$.)

**(iv)** If an unoriented edge $e$ connects two points of the same connected component of $G^*$, then both endpoints have indegree at most 2 (recall that this edge $e$ cannot be a witness at its endpoints). If an edge $e$ is directed from a connected component $K$ of $G^*$ to a point outside $K$, then the tail of this edge $e$ has indegree at most 2 (recall that $e$ cannot be a witness at all, since its endpoints are in different connected components if $G^*$).

**(v)** A candidate component $K$ of $G^*$ is a perfect coarsener. It is a coarsener (otherwise, we would have expanded it further), it is a prime coarsener (see (i) above) and $\mathsf{inc}\, K = 1$ (we have argued before that a candidate component increases the slack by exactly 1).

The *third phase* will make sure that each mutual edge loses exactly one direction. Our goal is to have in every connected component $K$ of $G^*$ at most one point with indegree 3. To be more precise, only candidate components have exactly one point with indegree 3, others don't. Consider a connected component $K$.

**(a)** If the mutual edges form cycles in $K$, choose such a cycle $c$ and keep for each edge on $c$ one orientation so that we have a directed cycle, counterclockwise, say. All other mutual edges in $K$ keep the direction in decreasing distance in $G^*$ to $c$, ties broken arbitrarily. This completed, no point in $K$ has indegree 3, since there is always a mutual edge incident that decreases the distance to $c$ and the incoming direction of this edge will be removed.

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[1] The reader will correctly observe that our approach is very conservative towards prime coarseners, but by what we observed and by what will follow, since we are interested only in perfect coarseners, we can afford to leave alone connected components other than the candidate components.

**(b)** If $K$ has points of indegree at most 2, choose one such point $p$ with indegree at most 2, orient all mutual edges in $K$ in decreasing distance in $G^*$ to $p$, ties broken arbitrarily. Again, this completed, no point in $K$ will have indegree 3.

**(c)** If none of the above applies, the mutual edges of $K$ form a spanning tree and all points in $K$ have indegree 3. Moreover, all edges connecting $K$ with points outside are directed towards $K$ and no edge within $K$ is unoriented (violation of these properties force a point of indegree at most 2). So this is a candidate component. We choose an arbitrary point $p$ in $K$, call it the *leader of* $K$, and for all mutual edges keep the orientation of decreasing distance in $G^*$ to $p$ (ties cannot occur, mutual edges form a tree). Now the leader $p$ is the only point of $K$ with indegree 3, all other points in $K$ have indegree exactly 2.

Phase 3 is completed. Let us denote the obtained partial orientation on $S$ as $\vec{S}^*$. It has identified certain connected components of $G^*$ which have a leader of indegree 3. In fact, every point of indegree 3 after phase 3 is part of a perfect coarsener (probably of size 1).

We can now describe a sufficient supply of perfect coarsenings of $S$. Let $N := |\mathsf{V}S|$ and let $C_3$ be the number of points of indegree 3 in $\vec{S}^*$. We know that there are at least $N - 3 - D - C_3$ unoriented inner edges (Lemma 8).

**(I)** There are $n - N$ perfect coarsenings obtained by adding a single point $p \in P \setminus \mathsf{V}S$.

**(II)** There are at least $N - 3 - D - C_3$ perfect coarsenings obtained by removing a single unoriented inner edge in $\vec{S}^*$.

**(III)** And there are $C_3$ perfect coarsenings obtained by isolating all points in a candidate component in $G^*$ (with a leader of indegree 3).

In this way we have identified at least $n - 3 - D$ perfect coarsenings. ◀

▶ **Corollary 26.** *Let $T$ be a triangulation.* (i) *$T$ has at least $n-3$ flippable elements.* (ii) *For every $x$ flippable in $T$ there are at least $n-4$ elements compatible with $x$.*

Part (i) of the corollary was proved, without general position assumption, in [5, Thm. 2.1].

## 5 The Link of a Triangulation – Proof of $(n-3)$-Connectivity

To complete the proof of the connectivity bound for the bistellar flip graph, we need two further ingredients. The first is the following variant of Menger's Theorem [14, Lemma 3.1].

▶ **Lemma 27** (Local Menger). *Let $k \geq 2$ be an integer and let $G$ be a* connected *simple undirected graph. Then $G$ is $k$-vertex connected iff $G$ has at least $k+1$ vertices and for any pair of vertices $u$ and $v$* at distance 2 *there are $k$ pairwise internally vertex disjoint $u$-$v$-paths.*

The second ingredient are *links* of triangulations, which are graphs that represent the compatibility relation among flippable elements (Def. 18). Recall that if $x$ is a flippable element in a triangulation $T$ then $T_{\pm x}$ denotes the subdivision with $\mathcal{T}_{\mathsf{part}}\langle T_{\pm x}\rangle = \{T, T[x]\}$, and if $y$ is compatible with $x$, denoted $x \diamond y$, then $T_{\pm x,y}$ denotes the unique coarsening of slack 2 of $T$ with $\{T[x], T, T[y]\} \subseteq \mathcal{T}_{\mathsf{part}}\langle T_{\pm x,y}\rangle$ (Def. 18).

▶ **Definition 28** (link). For $T \in \mathcal{T}_{\mathsf{part}}(P)$, the *link of* $T$, denoted $\mathsf{Lk}T$, is the edge-weighted graph with vertices $\mathsf{F}T := \{x \in \mathsf{V}^\circ T \cup \mathsf{E}^\circ T \mid x \text{ flippable in } T\}$ and edge set $\{\{x, y\} \in \binom{\mathsf{F}T}{2} \mid x \diamond y\}$. The *weight of* an edge $\{x, y\}$ is $|\mathcal{T}_{\mathsf{part}}\langle T_{\pm x,y}\rangle| - 2$ (which is 2 or 3).

We will see that it is enough to prove $(n-4)$-vertex connectivity of all links. Again, the intuition can be explained for polytopes: Recall that for a vertex $v$ in a $d$-polytope $\mathcal{P}$, its vertex figure is the $(d-1)$-polytope $\mathcal{P}'$ obtained by intersecting $\mathcal{P}$ with a hyperplane that separates $v$ from the remaining vertices of the polytope. Vertices of $\mathcal{P}'$ correspond to edges of $\mathcal{P}$, edges in the graph of $\mathcal{P}'$ correspond to 2-faces of $\mathcal{P}$. There is a natural way of mapping paths in the graph of $\mathcal{P}'$ to paths in the graph of $\mathcal{P}$. This can be easily made an inductive proof of Balinski's Theorem, as mentioned in Sec. 1 (using the Local Menger Lemma 27). We follow exactly this line of thought in our setting, except that we will not need induction – the link is a dense graph which directly yields $(n-4)$-vertex connectivity.

Note that, indeed, the following lemma implies that the complement of the link is sparse, hence the link is dense.

▶ **Lemma 29.** *The complement of $\mathsf{Lk}T$ has no cycle of length 4, i.e., if $(x_0, x_1, x_2, x_3)$ are flippable elements in $T$, then there exists $i \in \{0, 1, 2, 3\}$ such that $x_i \diamond x_{i+1 \bmod 3}$.*

**Proof.** Recall that all $p \in P^\circ \setminus \mathsf{V}^\circ T$ are flippable and compatible with every flippable element (Obs. 19), hence let us assume $\{x_0, x_1, x_2, x_3\} \subseteq \mathsf{V}^\circ T \cup \mathsf{E}^\circ T$. Moreover, if $p, q \in \mathsf{V}^\circ T$ are two distinct points flippable in $T$, then $p \diamond q$. Hence, we assume that no two consecutive elements in the cyclic sequence $(x_0, x_1, x_2, x_3)$ are points; w.l.o.g. let $x_0 = e$ and $x_2 = f$ be edges.

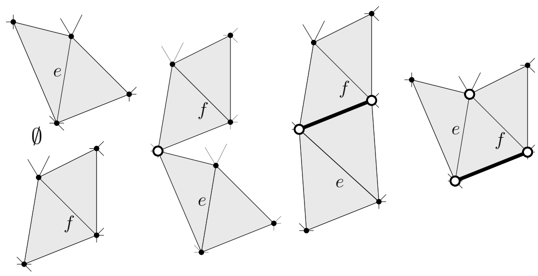


**Figure 13** Intersections of boundaries of territories of two flippable edges.

For an inner edge $e$ in a triangulation $T$, we define its *territory* $\mathsf{terr}\, e = \mathsf{terr}_T\, e$, as the interior of the closure of the union of the two regions in $T$ incident to $e$. Obviously, $e$ is flippable in $T$ iff the quadrilateral $\mathsf{terr}_T\, e$ is convex. Note that for an element $x$ to be incompatible with edge $e$, $x$ must appear on the boundary of $\mathsf{terr}\, e$, and analogously elements incompatible with $f$ must appear on the boundary of $\mathsf{terr}\, f$.

We show that there is at most one flippable element in the intersection of the boundaries of $\mathsf{terr}\, e$ and $\mathsf{terr}\, f$ (Fig. 13). This is obvious, if $\overline{\mathsf{terr}\, e} \cap \overline{\mathsf{terr}\, f}$ is empty or a single point (recall that $\overline{A}$ denotes the closure of $A \subseteq \mathbb{R}^2$). If this intersection is an edge and its two endpoints, we observe that among any edge and its two incident points, at most one element can be flippable (inner degree 3 points cannot be adjacent and cannot be incident to a flippable edge). This covers already all possibilties if $\mathsf{terr}\, e$ and $\mathsf{terr}\, f$ are disjoint (since they are convex). Finally, $\overline{\mathsf{terr}\, e} \cap \overline{\mathsf{terr}\, f}$ can be a triangle, in which case the common boundary consists of the common endpoint of $e$ and $f$, clearly not flippable, and an edge with its two endpoints; again, only one of these three can be flippable. ◀

▶ **Lemma 30.** *Given a triangulation $T$ with $x$ and $y$ flippable elements, $x \neq y$, every $x$-$y$-path of weight $w$ in $\mathsf{Lk}T$ induces a $T$-avoiding $T[x]$-$T[y]$-path of length $w$ in the bistellar flip graph. Interior vertex disjoint $x$-$y$-paths in the link induce interior vertex disjoint $T[x]$-$T[y]$-paths.*

**Proof.** Given an $x$-$y$-path in $\mathsf{Lk}T$, we replace every edge $\{z', z''\}$ on this path by $\mathsf{path}_T(z', z'') = (T[z'], \ldots, T[z''])$ (of length 2 or 3) which draws its (1 or 2) interior vertices from $\mathcal{T}_{\mathsf{part}}\langle T_{\pm z', z''}\rangle \setminus \{T[z'], T, T[z'']\}$ (Fig. 14); these vertices must have distance 2 from $T$ in the flip graph, while $T[z']$ and $T[z'']$ have distance 1. In the resulting $T[x]$-$T[y]$-path,

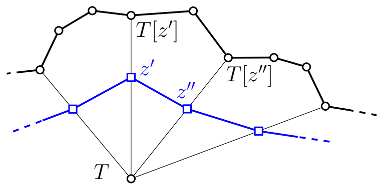


**Figure 14** From a path in the link to a path in the bistellar flip graph.

all interior vertices adjacent to $T$ (i.e., of the form $T[z]$) are distinct from interior vertices at other paths by assumption on the initial paths in the link. For vertices at distance 2, suppose $T_1 \in \mathcal{T}_{\mathsf{part}}\langle T_{\pm z_1',z_1''}\rangle$ coincides with $T_2 \in \mathcal{T}_{\mathsf{part}}\langle T_{\pm z_2',z_2''}\rangle$, both at distance 2 from $T$. Since $\mathsf{sl}\,T_{\pm z_1',z_1''} = \mathsf{sl}\,T_{\pm z_2',z_2''} = 2$, we have that $\mathcal{T}_{\mathsf{part}}\langle T_{\pm z_1',z_1''}\rangle \cap \mathcal{T}_{\mathsf{part}}\langle T_{\pm z_2',z_2''}\rangle$ either (a) equals $\{T\}$, (b) equals $\{T, T[z]\}$ for some $z$, or (c) $T_{\pm z_1',z_1''} = T_{\pm z_2',z_2''}$ (Lemma 17). In (a-b) $T_{\pm z_1',z_1''}$ and $T_{\pm z_2',z_2''}$ cannot possibly share a vertex at distance 2 from $T$. Thus (c) holds. $T_{\pm z_1',z_1''} = T_{\pm z_2',z_2''}$ implies $\{z_1', z_1''\} = \{z_2', z_2''\}$. ◀

▶ **Lemma 31.** *The link $\mathsf{Lk}T$ satisfies:* (i) *There are at least $n-3$ vertices.* (ii) *Every vertex has degree at least $n-4$.* (iii) *Every pair of non-adjacent vertices has at least $n-4$ connecting interior vertex disjoint paths (all of length at most 3).* (iv) *It is $(n-4)$-vertex connected.*

**Proof.** (i) $x$ is a vertex in $\mathsf{Lk}T$ iff $x$ is flippable in $T$ iff $T_{\pm x}$ is a subdivision of slack 1, a perfect coarsening of $T$. Lemma 25 ensures the existence of at least $n-3$ such perfect coarsenings.

(ii) Let $x$ be a vertex of $\mathsf{Lk}T$. $T_{\pm x}$, a subdivision of slack 1, has at least $n-4$ perfect coarsenings of slack 2 (Lemma 25). Each such coarsening equals $T_{\pm x,y}$ for some $y \diamond x$, i.e., $y$ is a neighbor of $x$ in $\mathsf{Lk}T$.

(iii) Let $x$ and $y$ be non-adjacent vertices of $\mathsf{Lk}T$, i.e., $x \not\diamond y$. Let $z_1, z_2, \ldots, z_\ell$ be all flippable elements in $T$ compatible with both $x$ and $y$. Each such element $z_i$ gives rise to a path $(x, z_i, y)$ of length 2 in the link. If $\ell \geq n-4$, we are done. Otherwise, there is an extra supply of elements $x_1, x_2, \ldots, x_{n-4-\ell}$ compatible with $x$ but not with $y$, and elements $y_1, y_2, \ldots, y_{n-4-\ell}$ compatible with $y$ but not with $x$. For all $i = 1, 2, \ldots, n-4-\ell$, $y_i \not\diamond x$, $x \not\diamond y$, and $y \not\diamond x_i$. By Lemma 29, $x_i \diamond y_i$, hence we have a path $(x, x_i, y_i, y)$ of length 3 in the link. Obviously, these paths of length 3 are pairwise internally vertex disjoint, and also internally vertex disjoint from all $x$-$y$-paths of length 2 (interior vertices on length 2 paths are connected to $x$ and $y$, interior vertices on the constructed length 3 paths are not).

(iv) We apply the Local Menger Lemma 27. Indeed, $\mathsf{Lk}T$ has at least $(n-4)+1 = n-3$ vertices (see (i)), and every pair of vertices at distance 2 has at least $n-4$ internally vertex disjoint paths (see (iii)). Hence, the link is $(n-4)$-vertex connected. ◀

## $(n-3)$-Connectivity of the Bistellar Flip Graph – Proof of Thm. 4

**Proof of Thm. 4.** If $n \leq 4$, $(n-3)$-vertex connectivity can be easily checked according to the definition of $k$-vertex connectivity. For $n \geq 5$, we employ the Local Menger Lemma 27. Thus (apart from the presence of at least $n-2$ vertices), we have to show that for any triangulation $T$ and flippable elements $x$ and $y$, there are at least $n-3$ internally vertex disjoint $T[x]$-$T[y]$-paths in the bistellar flip graph. We know that in $\mathsf{Lk}T$ has at least $n-4$ internally vertex disjoint $x$-$y$-paths (Menger's Theorem, [2, 6]). Therefore, there are at least $n-4$ interior vertex disjoint $T[x]$-$T[y]$-paths disjoint from $T$ (Lemma 30). Together with the path $(T[x], T, T[y])$, the claim is established. ◀

## References


1 M. L. Balinski. On the graph structure of convex polyhedra in $n$-space. *Pacific J. Math.*, 11(2):431–434, 1961. URL: `https://projecteuclid.org:443/euclid.pjm/1103037323`.

2 Béla Bollobás. *Modern graph theory (Graduate texts in mathematics)*. New York: Springer, 1998.

3 Cesar Ceballos, Francisco Santos, and Günter M. Ziegler. Many non-equivalent realizations of the associahedron. *Combinatorica*, 35(5):513–551, October 2015. `doi:10.1007/s00493-014-2959-9`.

4 Jesús A De Loera, Jörg Rambau, and Francisco Santos. *Triangulations Structures for algorithms and applications*. Springer, 2010.

5 Jesús A. De Loera, Francisco Santos, and Jorge Urrutia. The number of geometric bistellar neighbors of a triangulation. *Discrete & Computational Geometry*, 21(1):131–142, 1999. `doi:10.1007/PL00009405`.

6 Reinhard Diestel. *Graph theory*. Springer, 1997.

7 Israel M. Gelfand, Mikhail M. Kapranov, and Andrei V. Zelevinsky. Newton polyhedra and principal $A$-determinants. *Soviet Math. Dokl.*, 40:278–281, 1990.

8 Charles L. Lawson. Transforming triangulations. *Discrete Math.*, 3(4):365–372, January 1972. `doi:10.1016/0012-365X(72)90093-3`.

9 Carl W. Lee and Francisco Santos. Subdivisions and triangulations of polytopes. In Csaba D. Toth, Jacob E. Goodman, and Joseph O'Rourke, editors, *Handbook of Discrete and Computational Geometry, 3rd Edition*, pages 415–477. Chapman and Hall/CRC, New York, USA, 2017.

10 Anna Lubiw, Zuzana Masárová, and Uli Wagner. A proof of the orbit conjecture for flipping edge-labelled triangulations. *Discrete & Computational Geometry*, 61(4):880–898, June 2019. `doi:10.1007/s00454-018-0035-8`.

11 David Orden and Francisco Santos. The polytope of non-crossing graphs on a planar point set. *Discrete & Computational Geometry*, 33(2):275–305, February 2005. `doi:10.1007/s00454-004-1143-1`.

12 Francisco Santos Leal, 2019. personal communication.

13 Uli Wagner and Emo Welzl. Connectivity of triangulation flip graphs in the plane. *CoRR*, abs/2003.13557, 2020. `arXiv:2003.13557`.

14 Uli Wagner and Emo Welzl. Connectivity of triangulation flip graphs in the plane (Part I: Edge flips). In *Proceedings of the 31st Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 2823–2841. ACM-SIAM, 2020.