The Topological Correctness of PL-Approximations of Isomanifolds

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Abstract

Isomanifolds are the generalization of isosurfaces to arbitrary dimension and codimension, i.e. manifolds defined as the zero set of some multivariate vector-valued smooth function $f : \mathbb{R}^d \to \mathbb{R}^{d-n}$. A natural (and efficient) way to approximate an isomanifold is to consider its Piecewise-Linear (PL) approximation based on a triangulation $\mathcal{T}$ of the ambient space $\mathbb{R}^d$. In this paper, we give conditions under which the PL-approximation of an isomanifold is topologically equivalent to the isomanifold. The conditions are easy to satisfy in the sense that they can always be met by taking a sufficiently fine triangulation $\mathcal{T}$. This contrasts with previous results on the triangulation of manifolds where, in arbitrary dimensions, delicate perturbations are needed to guarantee topological correctness, which leads to strong limitations in practice. We further give a bound on the Fréchet distance between the original isomanifold and its PL-approximation. Finally we show analogous results for the PL-approximation of an isomanifold with boundary.

2012 ACM Subject Classification Theory of computation → Computational geometry

Keywords and phrases PL-approximations, isomanifolds, solution manifolds, topological correctness

Digital Object Identifier 10.4230/LIPIcs.SoCG.2020.20

Related Version A full version of this paper is available at `https://hal.archives-ouvertes.fr/hal-02386193`.

Funding The research leading to these results has received funding from the European Research Council (ERC) under the European Union's Seventh Framework Programme (FP/2007-2013) / ERC Grant Agreement No. 339025 GUDHI (Algorithmic Foundations of Geometry Understanding in Higher Dimensions).
Mathijs Wintraecken: Supported by the European Union's Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement No. 754411.

Acknowledgements First and foremost, we acknowledge Siargey Kachanovich for discussions. We thank Herbert Edelsbrunner and all members of his group, all former and current members of the Datashape team (formerly known as Geometrica), and André Lieutier for encouragement. We thank the reviewers for their comments which improved the exposition.

1 Introduction

Isomanifolds (also called solution manifolds) are the generalization of isosurfaces to arbitrary dimension and codimension, i.e. manifolds defined as the zero set of some multivariate vector-valued function $f : \mathbb{R}^d \to \mathbb{R}^{d-n}$. Not all submanifolds of $\mathbb{R}^d$ are isomanifolds although locally we can always write an embedded smooth manifold as the zero set of a smooth function, because it can be parametrized as a function from the tangent space to the manifold itself as a consequence of the implicit function theorem. Isosurfaces play a crucial role in medical imaging, computer graphics and geometry processing. Higher dimensional isomanifolds are also of fundamental importance in many fields like statistics [19], dynamical systems [49], econometrics or mechanics [41].

The most widely used algorithm to approximate an isosurface is the celebrated Marching Cube algorithm (MC) [37, 42]. Extending the MC to isomanifolds of higher dimensions and codimensions leads to major difficulties and it is then preferred to use the so-called Marching Simplex algorithm (MS) where a simplicial triangulation of the ambient space is used instead of the cubical grid [1, 29, 40]. The major advantage is that the PL-approximation $\hat{\mathcal{M}}$ of $\mathcal{M}$ is well defined and easy to compute inside each d-simplex of the triangulation. Moreover, some regular triangulations, like Coxeter or Kuhn-Freudenthal triangulations do not require to be stored and can be used implicitly like the cubical grid. Hence the algorithm can be made efficient for isomanifolds of low dimensions even if they are embedded in high dimensional spaces [16].

Proving that either the MC or the MS produce good approximations has been a long lasting question. Some results were achieved within the computational geometry community in three dimensions. In [10, 44], conditions were given that ensure that $\hat{\mathcal{M}}$ is a PL-manifold close and topology equivalent (homeomorphic) to $\mathcal{M}$. Unfortunately, [10] uses a particular combination of collapses and Morse theory and [44] something akin to normal surface theory [47], both of which are specific to low dimensions. In the case of MS in higher dimensions, weaker results [2, 3] have been known for a while, e.g. bounds on the one-sided Hausdorff distance, on the approximation of tangent spaces, and manifoldness of the approximation (under strong conditions). However, it is only very recently that complete correctness results have been proved for general submanifolds of any dimension [7, 11, 12, 15]. However, the proofs in higher dimensions rely on perturbation schemes that are quite intricate and make the methods of little practical value. This is a major difference with the case of curves and surfaces where no such requirements exist [31], [46, Section 10.2].

In this paper, we restrict our attention to isomanifolds and show that, in this case, no perturbation scheme is required to obtain correct outputs. This is a major achievement with respect to effective computation and applications. Also the techniques used here are different from many of the standard tools. We, in particular, do not rely on Delaunay triangulations [48, 25, 21], nor on the closed ball property [33], Whitney's lemma [13] or collapses [4]. The current paper mainly relies on a version of the implicit function theorem from non-smooth analysis [23] with some Morse theory. Another clear difference with previous methods is that we do not provide lower bounds on the quality of the linear pieces in the Piecewise-Linear (PL) approximation. Although this is an appealing property, it is extremely difficult to satisfy in practice as mentionned above. Here we ask for less but still provide strong guarantees on $\hat{\mathcal{M}}$. Perturbation techniques can be used in an optional postprocessing step to improve the quality of the simplices of the output. They no longer play a critical role in the construction of the approximation.

The rest of this paper is subdivided in two sections. In the first section, we treat closed isomanifolds, i.e. compact manifolds without boundary. We show that, for a fine enough ambient triangulation, the output $\hat{\mathcal{M}}$ of the MS is a manifold that is isotopic to $\mathcal{M}$, close to $\mathcal{M}$ with respect to the Fréchet distance, that approximates well the tangent bundle of $\mathcal{M}$ (Theorem 20 and Corollaries 22 and Proposition 6).

In the second section, we prove similar results for isomanifolds with boundary. Extension to general isostratifolds is briefly discussed in Section 4. All proofs can be found in [17].

2 Isomanifolds (without boundary)

Let $f : \mathbb{R}^d \to \mathbb{R}^{d-n}$ be a smooth (C^2 suffices) function and suppose that 0 is a regular value of f, meaning that at every point x such that $f(x) = 0$, the Jacobian of f is non-degenerate. Then the zero set of f is an n-dimensional manifold as a direct consequence of the implicit function theorem, see for example [30, Section 3.5]. We further assume that $f^{-1}(0)$ is compact. As in [2] we consider a triangulation $\mathcal{T}$ of $\mathbb{R}^d$. The function f_{PL} is a linear interpolation of the values of f at the vertices if restricted to a single simplex $\sigma \in \mathcal{T}$. For any function $g : \mathbb{R}^d \to \mathbb{R}^{d-n}$ we write g^i, with $i = 1, \dots, d-n$, for the components of g.

We prove that, under certain conditions, there is an isotopy from the zero set of f to the zero set of f_{PL}. The proof will be using the Piecewise-Linear (PL) map

$$F_{PL}(x, \tau) = (1 - \tau)f(x) + \tau f_{PL}(x), \tag{1}$$

which interpolates between f and f_{PL} and is based on the generalized implicit function theorem. The isotopy is in fact stronger than just the existence of a homeomorphism from the zero set of f to that of f_{PL}.

Our result in particular implies that the zero set of f_{PL} is a manifold. The fact that the zero set of f_{PL} is a manifold was proved (under strong condition) by Allgower and Georg [3, Theorem 15.4.1], without a homeomorphism with the zero set of f. The conditions here are weaker, because we do not require that the zero set avoids simplices that have dimension less than the codimension, see [3, Definition 12.2.2] and the text above [3, Theorem 15.4.1]. The idea to avoid these low dimensional simplices originates with Whitney [50], apparently unbeknownst to Allgower and George [3, 2]. Very heavy perturbation schemes for the vertices of the ambient triangulation $\mathcal{T}$ are required for the manifold to be sufficiently far from simplices in $\mathcal{T}$ that have dimension less than the codimension of the manifold [50, 15]. Various techniques have been developed to compute such perturbations with guarantees. They typically consist in perturbing the position of the sample points or in assigning weights to the points. Complexity bounds are then obtained using volume arguments. See, for example [20, 14, 11, 9]. However, these techniques suffer from several drawbacks. The constants in the complexity depend exponentially on the ambient dimension. Moreover the analysis assumes that the probability of the simplices of dimension less than the codimension to intersect the manifold is zero, which is not true when dealing with finite precision. As a result, the actual implementations we are aware of fail to work well in practice except in very simple cases.

We are, by definition, only interested in $f^{-1}(0)$ so we can ignore points that are sufficiently far from this zero set. More precisely, we observe the following: If $f^i(x)$ is positive for all x in a geometric simplex σ then so is $f^i_{PL}(x)$, because $f^i_{PL}(x)$ is a convex combination of the (positive) values at the vertices. This in turn implies that $F^i_{PL}(x, \tau)$ is positive on $\sigma \times [0, 1]$, as for each τ it is convex combination of positive numbers. The same argument holds for negative values. So we see that

▶ Remark 1. Write Σ_0 for the set of all $\sigma \in \mathcal{T}$, such that $(f^i)^{-1}(0) \cap \sigma \neq \emptyset$ for all i. Then for all τ, $\{x \mid F_{PL}(x, \tau) = 0\} \subset \Sigma_0$.

The results will be expressed using constants defined in terms of f and the ambient triangulation $\mathcal{T}$.

▶ **Definition 2.** *We define*

$$\gamma_0 = \inf_{x\in\Sigma_0} |\det(\mathrm{grad}(f^i)\cdot\mathrm{grad}(f^j))_{i,j}| \tag{2}$$

$$\gamma_1 = \sup_{x\in\Sigma_0} \max_i |\mathrm{grad}(f^i)| \tag{3}$$

$$\alpha = \sup_{x\in\Sigma_0} \max_i \|Hes(f^i)(x)\|_2 = \sup_x \max_i \|(\partial_k\partial_l f^i(x))_{k,l}\|_2, \tag{4}$$

$$D \ : \ \textit{the longest edge length of a simplex in } \Sigma_0 \tag{5}$$

$$T \ : \ \textit{the smallest thickness of a simplex in } \Sigma_0. \tag{6}$$

Here $\mathrm{grad}(f^i) = (\partial_j f_i)_j$ *denotes the gradient of component* f^i*,* $\det(\mathrm{grad}(f^i)\cdot\mathrm{grad}(f^j))_{i,j}$ *denotes the determinant of the matrix with entries* $\mathrm{grad}(f^i)\cdot\mathrm{grad}(f^j)$*,* $\|\cdot\|_2$ *the operator* 2*-norm, and* $(\partial_k\partial_l f_i)_{k,l}$ *the matrix of second order derivatives, that is the Hessian (Hes). We recall the definition of the operator norm:* $\|A\|_p = \max_{x\in\mathbb{R}^n} \frac{|Ax|_p}{|x|_p}$*, with* $|\cdot|_p$ *the p-norm on* $\mathbb{R}^n$*. The thickness is the ratio of the height over the longest edge length.*

We will assume that $\gamma_0, \gamma_1, \alpha, D, T \in (0,\infty)$. The constant γ_0 quantifies how close 0 to not being a regular value of f. The thickness is a measure of how well shaped the simplices are. A good choice for $\mathcal{T}$ is the Coxeter triangulation of type A_d, see [24, 22], or the related Freudenthal triangulations, see [34, 36, 32, 49], which can be defined for different values of D while keeping T constant. In this paper, we will thus think of all the above quantities as well as d and n as constants except D and our results will hold for D small enough.

The result

We are going to construct an ambient isotopy based on (1). The zero set of $F_{PL}(x,0)$ (or $f(x)$) gives the smooth isosurface, while the zero set of $F_{PL}(x,1)$ (or $f_{PL}(x)$) gives the PL approximation, that is the triangulation of the isosurface after possible barycentric subdivision. The map $\tau \mapsto \{x \mid F_{PL}(x,\tau) = 0\}$ in fact gives an isotopy. Without too much extra work we will also bound the Fréchet distance between $f(x)$ and $f_{PL}(x)$.

Proving isotopy consists of two technical steps, which consume most of the space in the proof below, as well as the use of a standard observation from Morse theory/gradient flow in the third step. The technical steps are

- Let $\sigma \in \mathcal{T}$. We first show that $\{(x,\tau) \mid F_{PL}(x,\tau) = 0\} \cap (\sigma \times [0,1])$ is a smooth manifold, under certain conditions (Corollary 8).
- We prove that $F_{PL}^{-1}(0)$ is a manifold, under certain conditions, using techniques from nonsmooth analysis (Corollary 19).

Along the way we shall also see that $F_{PL}^{-1}(0)$ is never tangent to the $\tau = c$ planes, where c is a constant. The gradient of $(x,\tau), \mapsto \tau$ in the ambient space is $(0,1)$. Projecting this vector onto the tangent space of $F_{PL}^{-1}(0)$ gives the gradient of $(x,\tau), \mapsto \tau$ restricted to $F_{PL}^{-1}(0)$. Because of the non-tangency property, this projection is non-zero. So the gradient field of the function $(x,\tau), \mapsto \tau$ restricted to $F_{PL}^{-1}(0)$, is piecewise smooth (because $F_{PL}^{-1}(0)$ is piecewise smooth) and never vanishes.

Now we arrive at the third step, which is similar to a standard observation in Morse theory [38, 39], with the exception that we now consider piecewise-smooth instead of smooth vector fields. We refer to Milnor [38] for an excellent introduction, see Lemma 2.4 and Theorem 3.1 in particular.

▶ **Lemma 3** (Gradient flow induced isotopies)**.** *The flow of a non-vanishing piecewise-smooth gradient vector field of a function* τ *on a compact manifold generates an isotopy from* $\tau = c_1$ *to* $\tau = c_2$*, where* c_1 *and* c_2 *are constants.*

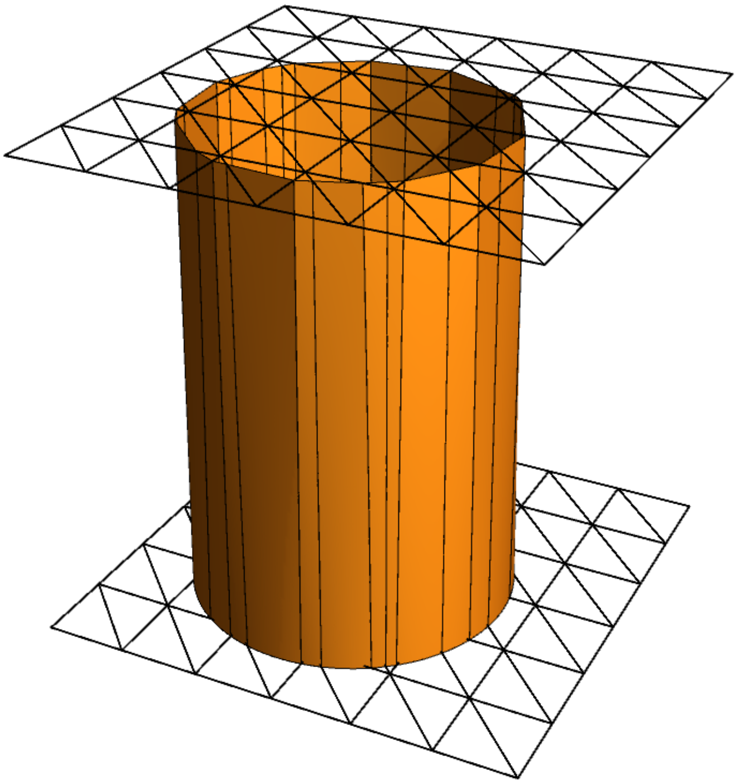

Figure 1 A pictorial overview of the proof. The τ-direction goes upwards. Similarly to Morse theory we find that $f_{PL}^{-1}(0)$ (top) and $f^{-1}(0)$ (bottom) are homeomorphic if the function τ restricted to $F_{PL}^{-1}(0)$ does not encounter a Morse critical point.

Bounds on the gradient of τ on the manifold give a bound on the Fréchet distance, which is defined as follows:

▶ **Definition 4** (Fréchet distance for embedded manifolds). *Let $\mathcal{M}$ and $\mathcal{M}'$ be two homeomorphic, compact submanifolds of $\mathbb{R}^d$. Write $\mathcal{H}$ for the set of all homeomorphisms from $\mathcal{M}$ to $\mathcal{M}'$. The Fréchet distance between $\mathcal{M}$ and $\mathcal{M}'$ is*

$$d_F(\mathcal{M}, \mathcal{M}') = \inf_{h \in \mathcal{H}} \sup_{x \in \mathcal{M}} d(x, h(x)).$$

2.1 Estimates for a single simplex

We now first concentrate on a single simplex σ and write f_L for the linear function whose values on the vertices of σ coincide with f, that is f_L is the linear extension of the interpolation of f. Note that f_L coincides with f_{PL} within the geometric simplex σ (but not necessarily outside).

2.1.1 Preliminaries and variations of know results

We need a simple estimate similar to Proposition 2.1 of Allgower and George [2].

▶ **Lemma 5.** *Let $\sigma \subset \Sigma_0$ and let f_L be as described above. Then $|f_L^i(x) - f^i(x)| \leq 2D^2\alpha$ for all $x \in \sigma$.*

We will also be using an estimate similar to Proposition 2.2 of Allgower and George [2].

▶ **Proposition 6.** *Let $\sigma \subset \Sigma_0$ and let f_L be as described above. Then*

$$|\mathrm{grad} f_L^i - \mathrm{grad} f^i| = \sqrt{\sum_j (\partial_j f_L^i(x) - \partial_j f^i(x))^2} \leq \frac{4dD\alpha}{T},$$

for all x in the geometric simplex σ.

2.1.2 Estimates on the gradient inside a single simplex

We write

$$F_L(x,\tau) = (1-\tau)f(x) + \tau f_L(x). \tag{7}$$

We note that F_L extends smoothly outside σ, that is we can think of $F_L : \mathbb{R}^d \to \mathbb{R}^{d-n}$. Here and throughout we restrict ourselves to the setting where $\tau \in [0,1]$.

We now find the following

▶ **Lemma 7.** *If we write $\mathrm{grad}_{(x,\tau)}$ for the gradient that includes the τ component, we have*

$$|\det(\mathrm{grad}_{(x,\tau)}(F_L^i) \cdot \mathrm{grad}_{(x,\tau)}(F_L^j))_{i,j}| > \gamma_0 - g_1(D), \tag{8}$$

with $g_1(D) = \mathcal{O}(D)$. See Appendix A of [17] for the exact expression of g_1.

▶ **Corollary 8** ($F_L^{-1}(0)$ is a manifold in a neighbourhood of $\sigma \times [0,1]$)**.** *If $\gamma_0 > g_1(D)$ the implicit function theorem applies to $F_L(x,\tau)$ inside $\sigma \times [0,1]$. (In fact it applies to an open neighbourhood of this set). In particular, we have proven the first of our two technical steps, $\{(x,\tau) \mid F_{PL}(x,\tau) = 0\} \cap (\sigma \times [0,1])$ is a smooth manifold.*

2.1.3 Transversality with regard to the τ-direction

We will also prove the main result which we need for the third step, that is the gradient of τ restricted to $F_{PL}^{-1}(0)$, is piecewise smooth and never vanishes. We now prove that inside each $\sigma \times [0,1]$ the gradient of τ on $F_L^{-1}(0)$ is smooth and does not vanish.

We first give a simple lower bound on the lengths of vectors $v^1, \ldots, v^{d-n}$, assuming that the norms $|v^i|$ are upper bounded and the determinant of the Gram matrix is lower bounded.

▶ **Lemma 9.** *Let $v^1, \ldots, v^{d-n} \in \mathbb{R}^d$, $|v^i| \leq \gamma_1$, for all i, and assume that $\det(v^i \cdot v^j)_{i,j} > \gamma_0$. Then $|v^i| \geq \sqrt{\gamma_0}/\gamma_1^{d-n-1}$.*

We also need to bound the angle of the vectors $\mathrm{grad}_{(x,\tau)}(F_L^i)$ and the x plane, that is $\mathbb{R}^d \subset \mathbb{R}^{d+1}$. We recall the definition. If $v \in \mathbb{R}^{d+1}$ is a vector and $\Xi = \mathbb{R}^d \subset \mathbb{R}^{d+1}$, is the space spanned by the d basis vectors corresponding to the x-directions, the angle between v and Ξ is $\angle(v,\Xi) = \inf_{w\in\Xi} \angle(v,w)$.

▶ **Lemma 10.** *Let Ξ be as above. We have*

$$\tan \angle(\mathrm{grad}_{(x,\tau)}(F_L^i), \Xi) \leq \frac{2D^2\alpha}{\sqrt{\gamma_0}/\gamma_1^{d-n-1} - \frac{4dD\alpha}{T}}.$$

In particular the manifold $F_L^{-1}(0)$ inside $\sigma \times [0,1]$ is never tangent to the $\tau = c$ planes, where c is a constant.

Combining Lemma 10 and Corollary 8 gives:

▶ **Corollary 11.** *If $\gamma_0 > g_1(D)$, and $\sqrt{\gamma_0}/\gamma_1^{d-n-1} > \frac{4dD\alpha}{T}$, then inside each $\sigma \times [0,1]$ the gradient of τ on $F_L^{-1}(0)$ is smooth and does not vanish.*

2.2 Global result

We are now going to prove the global result. For this, we need to recall some definitions and results from non-smooth analysis. We refer to [23] for an extensive introduction.

▶ **Definition 12** (Generalized Jacobian, Definition 2.6.1 of [23]). *Let $F : \mathbb{R}^{d+1} \to \mathbb{R}^{d-n}$, where F is assumed to be just Lipschitz. The generalized Jacobian of F at x_0 denoted by $J_F(x_0)$, is the convex hull of all $(d-n) \times (d+1)$-matrices B obtained as the limit of a sequence of the form $J_F(x_i)$, where $x_i \to x_0$ and F is differentiable at x_i.*

Following [23, page 253] we also define:

▶ **Definition 13.** *The generalized Jacobian $J_F(x_0)$ is said to be of maximal rank provided every matrix in $J_F(x_0)$ is of maximal rank.*

Write $\mathbb{R}^{d+1} = \mathbb{R}^{n+1} \times \mathbb{R}^{d-n}$ and denote the coordinates of $\mathbb{R}^{d+1}$ by (x, y) accordingly. Fix a point (a, b), with $F(a, b) = 0 \in \mathbb{R}^{d-n}$. We now write:

▶ **Notation 14** ([23, page 256]). *$J_F(x_0, y_0)|_y$ is the set of all $(n+1) \times (n+1)$-matrices M such that, for some $(n+1) \times (d-n)$-matrix N, the $(n+1) \times (d+1)$-matrix $[N, M]$ belongs to $J_F(x_0, y_0)$.*

With these definitions and notations we now have:

▶ **Theorem 15** (The generalized implicit function theorem [23, page 256]). *Suppose that $J_F(a, b)|_y$ is of maximal rank. Then there exists an open set $U \subset \mathbb{R}^{n+1}$ containing a such that there exists a Lipschitz function $g : U \to \mathbb{R}^{d-n}$, such that $g(a) = b$ and $F(x, g(x)) = 0$ for all $x \in U$.*

We recall the definition of F_{PL},

$$F_{PL}(x, \tau) = (1-\tau)f(x) + \tau f_{PL}(x). \tag{1}$$

Further recall that the closed star of a vertex v in a simplicial complex is the closure of all simplices in the complex that contain v.

Because of the definition of α, see (4), and Proposition 6, we have that $\mathrm{grad}_{(x,\tau)} F_{PL}(x, \tau)$ and $\mathrm{grad}_{(x,\tau)} F_{PL}(\tilde{x}, \tau)$ are close if x and $\tilde{x}$ are. In particular,

▶ **Lemma 16.** *Let v be a vertex in $\mathcal{T}$, $x_1, x_2 \in star(v)$, and $\tau_1, \tau_2 \in [0, 1]$, such that $\mathrm{grad}_{(x,\tau)} F^i_{PL}(x_1, \tau_1)$ and $\mathrm{grad}_{(x,\tau)} F^i_{PL}(x_2, \tau_2)$ are well defined, then*

$$|\mathrm{grad}_{(x,\tau)} F^i_{PL}(x_1, \tau_1) - \mathrm{grad}_{(x,\tau)} F^i_{PL}(x_2, \tau_2)| \leq \frac{10d^2 D\alpha}{T} + 4\gamma_1 D + 4D^2\alpha.$$

We now immediately have the same bound on points in the convex hull of a number of such vectors:

▶ **Corollary 17.** *Suppose we are in the setting of Lemma 16 and $x_0, x_1, \ldots, x_m \in star(v)$, $\tau_0, \ldots, \tau_m \in [0, 1]$, and suppose that $\mu_1, \ldots, \mu_m$ are positive weights such that $\mu_1 + \cdots + \mu_m = 1$ then,*

$$\left| \mathrm{grad}_{(x,\tau)} F^i_{PL}(x_0, \tau_0) - \sum_{k=1}^{m} \mu_k \mathrm{grad}_{(x,\tau)} F^i_{PL}(x_k, \tau_k) \right| \leq \frac{10d^2 D\alpha}{T} + 4\gamma_1 D + 4D^2\alpha.$$

Using Lemma 7, we see

▶ **Lemma 18.** *Let v be a vertex in $\mathcal{T}$, $x_1, \dots, x_m \in star(v)$, and $\tau_1, \dots, \tau_m \in [0,1]$, such that $\operatorname{grad}_{(x,\tau)} F^i_{PL}(x_k, \tau_k)$, $k = 0, \dots, m$ are well defined. If we moreover assume $D \leq 1$, and $\frac{6dD\alpha}{T} \leq \gamma_1$ we have that*

$$\left| \det \left(\left(\sum_{k=1}^{m} \mu_k \operatorname{grad}_{(x,\tau)} F^i_{PL}(x_k, \tau_k) \right) \cdot \left(\sum_{k=1}^{m} \mu_k \operatorname{grad}_{(x,\tau)} F_{PL}(x_k, \tau_k) \right) \right)_{i,j} \right| \geq \gamma_0 - g_2(D),$$

with $g_2(D) = \mathcal{O}(D)$. See Appendix A of [17] for the exact expression of g_2.

From the previous lemma, we immediately have that

▶ **Corollary 19** ($\{(x,\tau) \mid F_{PL}(x,\tau) = 0\}$ is a manifold)**.** *If $D \leq 1$, $\frac{6dD\alpha}{T} \leq \gamma_1$, and $\gamma_0 > g_2(D)$ the generalized implicit function theorem, Theorem 15, applies to $F_{PL}(x,\tau) = 0$. In particular, $\{(x,\tau) \mid F_{PL}(x,\tau) = 0\}$ is a manifold.*

We notice that this bound is stronger than the bound in Corollary 8, that is $g_1(D) \leq g_2(D)$. This means that $F^{-1}_{PL}(0)$ is a Piecewise-Smooth manifold if the conditions of Corollary 19 hold. The second technical step of the proof is now also completed.

The fact that $F_L(x,\tau) = 0$ is a Piecewise-Smooth manifold and Corollary 11 give that the gradient of τ is a Piecewise-Smooth vector field whose flow we can integrate to give a isotopy ι from the zero set of f to that of f_{PL}.

We summarize in a theorem:

▶ **Theorem 20.** *If, $D \leq 1$, $\frac{6dD\alpha}{T} \leq \gamma_1$, $\sqrt{\gamma_0}/\gamma_1^{d-n-1} > \frac{4dD\alpha}{T}$, and $\gamma_0 > g_2(D)$ then the zero set of f_{PL} is a manifold isotopic to the zero set of f. We stress that one can satisfy all conditions by choosing D sufficiently small.*

2.2.1 Fréchet distance

To bound the Fréchet distance (d_F) between the zero sets of $f(x)$ and f_{PL}, it suffices to bound the angle that the gradient of τ, as restricted to $F^{-1}_{PL}(0)$), makes with the (ambient) τ-direction.

For this we will use the angle bound of Lemma 10, together with some estimates that are similar in spirit to those in [8, Lemma C.13].

▶ **Lemma 21.** *Let $v^1, \dots, v^{d-n} \in \mathbb{R}^{d+1}$, $|v^i| \leq \tilde{\gamma}_1$, for all i, and assume that $\det(v^i \cdot v^j)_{i,j} > \tilde{\gamma}_0 > 0$. Let e_τ be a unit vector. If for all i, $\cos(\angle v^i, e_\tau) \leq \phi_0$, then for any $w \in \operatorname{span}(v^1, \dots, v^{d-n})$*

$$\cos \angle(w, e_\tau) \leq \frac{(d-n) d^{d-n-1} \phi_0 \tilde{\gamma}_1^{d-n}}{\sqrt{\tilde{\gamma}_0}}.$$

Let e_τ be the τ direction and let g_τ be the gradient of τ restricted to $F^{-1}_{PL}(0)$, whenever it exists. We want to bound the angle of g_τ and the τ-direction. Because the isotopy ι is given by integrating the gradient flow and we have a bound on the norm of the gradient, the Fréchet distance is bounded (by the norm of the gradient because the time of the flow is 1).

There is one subtlety, because the manifold is only Piecewise-Smooth, we need to take into account the points where g_τ is not uniquely defined. Because for each simplex σ, F_L extends to a neighbourhood of $\sigma \times [0,1]$, there exists a limit of $g_\tau(x_i, \tau_i)$ for any sequence (x_i, τ_i) that lies in $\operatorname{int}(\sigma) \times [0,1]$, where int denotes the interior. This means that if we bound g_τ for each simplex we also bound its limits, where the limits are as just described.

We are now ready to combine Lemmas 10, 21, and Theorem 20.

▶ **Corollary 22** (Bound on the Fréchet distance). *Suppose that the conditions of Theorem 20 are satisfied. Then,* $d_F(f^{-1}(0), f_{PL}^{-1}(0)) \leq \tan \arcsin g_3(D)$*, with* $g_3(D) = \mathcal{O}(D^2)$*, where we think of* γ_0*,* γ_1*,* d*,* n*,* T *and* α *as constants. See Appendix A of [17] for the exact expression of* g_2*.*

The most important thing to observe is that $\tan(\arcsin(x)) = \frac{x}{\sqrt{1-x^2}}$, so that we find that $d_F(f^{-1}(0), f_{PL}^{-1}(0)) = \mathcal{O}(D^2)$, where we think of γ_0, γ_1, d, n, T and α as constants.

3 Isomanifolds with boundary

We will now consider isomanifolds with boundary. By this we mean that on top of the function $f : \mathbb{R}^d \to \mathbb{R}^{d-n}$, we'll have another function $f_\partial : \mathbb{R}^d \to \mathbb{R}$ and the set we consider is $M = f^{-1}(0) \cap f_\partial^{-1}([0,\infty))$. This is a manifold with boundary if the gradients of f^i span a $(d-n)$-dimensional space at each point of $f^{-1}(0)$ and the gradients of f^i and f_∂ span a $(d-n+1)$-dimensional space at each point of $\partial M = f^{-1}(0) \cap f_\partial^{-1}(0)$, as a consequence of the submersion theorem. We will again write f_{PL} for the PL interpolation of f. Similarly we write $f_{\partial,PL}$ for the PL interpolation of f_∂. We prove that, under certain conditions, there is an isotopy from $f^{-1}(0) \cap f_\partial^{-1}([0,\infty))$ to $f_{PL}^{-1}(0) \cap f_{\partial,PL}^{-1}([0,\infty))$. The conditions are very similar to the conditions we have before, but of course we need to include bounds on the gradient of $f_{\partial,PL}$.

Overview of the proof

We will again construct an isotopy, but in this case it will consist of two steps.

- In the **first step**, we isotope the part of $f^{-1}(0)$ that is far from $f_\partial^{-1}(0)$ to its piecewise linear approximation, while leaving the part of $f^{-1}(0)$ that is close to $f_\partial^{-1}(0)$ smooth. We will denote the result by $M_1 = (F_{PL,1}(\cdot,1))^{-1}(0)$, see (9).
- In the **second step**, we consider a (small) tubular neighbourhood around $f_\partial^{-1}(0)$ as restricted to M_1 by looking at all $f_\partial^{-1}(\epsilon)$ for $|\epsilon|$ sufficiently small.[1] We then isotope $M_1 \cap f_\partial^{-1}(\epsilon)$ to its piecewise linear approximation. Again the isotopy is chosen in such a way that for ϵ relatively large (for the points such that M_1 is already Piecewise-Linear) it leaves $M_1 \cap f_\partial^{-1}(\epsilon)$ invariant. This gives an isotopy of a tubular neighbourhood of $\partial M_1 = M_1 \cap f_\partial^{-1}(0)$ to its Piecewise-Linear approximation.

We will first partition the manifold in two parts using a smooth bump function $\phi : \mathbb{R} \to [0,1]$ that is zero in a neighbourhood of zero and $\phi(y) = 1$ if $|y| > y_0$, for some $y_0 > 0$. Such bump functions can be easily constructed, see for example [35, Section 2.2]. We will be using the function $\phi\left(\sum_i (f^i)^2 + f_\partial^2\right)$.

The first step will be using the zero set of the following function:

$$F_{PL,1}(x,\tau) = \left(1 - \tau\phi\left(\sum_i (f^i)^2 + f_\partial^2\right)\right) f(x) + \tau\phi\left(\sum_i (f^i)^2 + f_\partial^2\right) f_{PL}(x), \tag{9}$$

on which we'll apply the same gradient flow argument as before.

The resulting set M_1 is the same zero set of f_{PL} as before if we stay sufficiently far away from ∂M and the isotopy leaves the manifold invariant close to ∂M. In particular, $\partial M_1 = \partial M$.

[1] We stress that ϵ may be negative.

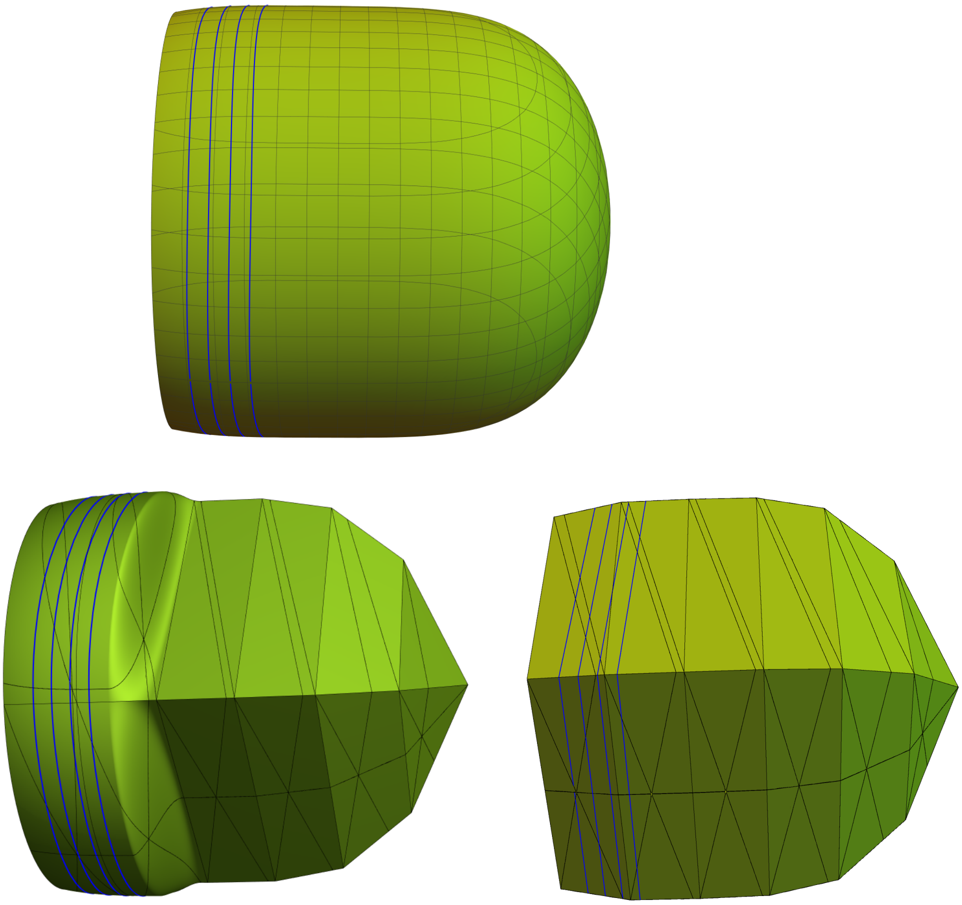

Figure 2 Top: we see the original isosurface with $f_\partial^{-1}(-1/10)$, $f_\partial^{-1}(0)$, $f_\partial^{-1}(1/10)$, and $f_\partial^{-1}(2/10)$ indicated in blue. Bottom left: we see that at the end of Step 1 the neighbourhood of the boundary is intact, while the rest has been isotoped to a Piecewise-Linear approximation. Bottom right: we have also isotoped the neighbourhood of the boundary to a Piecewise-Linear approximation by isotoping $f_\partial^{-1}(\epsilon)$, to its Piecewise-Linear approximation for all sufficiently small ϵ.

In the second step, we define an isotopy that will act only on a small neighbourhood of ∂M. Consider the sets $B_1(\epsilon) = M_1 \cap f_\partial^{-1}(\epsilon)$ and, for each ϵ, define the function

$$\begin{aligned} F_{PL,2,\epsilon}(x,\tau) = &\left(1 - \tau\psi\left(\sum_i (f^i)^2 + f_\partial^2\right)\right)(F_{PL,1}(x,1), f_\partial(x) - \epsilon) \\ &+ \tau\psi\left(\sum_i (f^i)^2 + f_\partial^2\right)(f_{PL}(x), f_{\partial,PL}(x) - \epsilon), \end{aligned} \tag{10}$$

where $\psi : \mathbb{R} \to [0,1]$ is now a smooth bump function that is 1 in a sufficiently large neighbourhood of zero (somewhat larger than y_0) and zero outside some compact set. We stress that $F_{PL,2,\epsilon}$ is a mapping from $\mathbb{R}^d \times [0,1]$ to $\mathbb{R}^{d-n+1}$. Using the result for isomanifolds (with some modifications), we can prove that each individual set $B_1(\epsilon)$ is isotopic to $f_{PL}^{-1}(0) \cap f_{\partial,PL}^{-1}(\epsilon)$ for small ϵ while, for sufficiently large ϵ, it leaves the set invariant.

3.1 Step 1

The proof closely follows the proof for the case without boundary in Section 2. The main technical difficulty will be to provide bounds that serve as the counterparts of Lemmas 7 and 18. To be able to do so, we first need to discuss bounds on the bump functions ϕ and ψ.

3.1.1 Bump functions

Following [35, Section 2.2], we write

$$\zeta_1(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ e^{-1/x} & \text{if } x > 0 \end{cases}$$

For $0 < y_1 < y_2$ we write $\zeta_2(x) = \zeta_1(x - y_1)\zeta_1(y_2 - x)$. Then we define $\phi_l : \mathbb{R} \to [0, 1]$ by $\phi_l(x) = \int_x^{y_2} \zeta_2(x')\mathrm{d}x' \Big/ \int_{y_1}^{y_2} \zeta_2(x')\mathrm{d}x'$. Finally define $\phi_b : \mathbb{R} \to [0, 1]$ by $\phi_b(x) = \phi_l(|x|)$, and let $\phi(x) = 1 - \phi_b(x)$.

▶ **Lemma 23.** *We have $\phi_b(x) \in [0, 1]$ and, writing $2y_1 = y_2 = y_0$,*

$$\partial_x(\phi_l(x)) \leq 2\frac{e^{\frac{4}{3(y_2-y_1)}}}{y_2 - y_1} = 4\frac{e^{\frac{2}{3y_0}}}{y_0} = \gamma_\phi. \tag{11}$$

3.1.2 Inside a single simplex

Similarly to Lemma 7, we now give a condition that ensures that the zero set of $F^i_{PL,1}(x, \tau)$ is smooth inside $\sigma \times [0, 1]$. In fact, similarly to (7), we define

$$\begin{aligned} F^i_{L,1}(x, \tau) &= \left(1 - \tau\phi\left(\sum_l (f^l)^2 + f_\partial^2\right)\right) f^i(x) + \tau\phi\left(\sum_l (f^l)^2 + f_\partial^2\right) f^i_L(x) \\ &= f(x) + \tau\phi\left(\sum_i (f^i)^2 + f_\partial^2\right)(f^i_L(x) - f^i(x)), \end{aligned}$$

where ϕ is as defined above. Observe that $F^i_{L,1}(x, \tau)$ can be extended to a neighbourhood of $\sigma \times [0, 1]$.

▶ Remark 24. For the constants, it is better if y_0 can be chosen as large as possible, but we need y_1 to be quite a bit larger than y_0. In turn, we cannot choose y_1 arbitrarily large because this would mean that the gradient field $\mathrm{grad} f_\partial|_{f^{-1}(0)}$ (seen as restricted on $f^{-1}(0)$) would never vanish. The latter is in general impossible thanks to the hairy ball theorem [18].

We introduce the following definition that complements Definition 2:

▶ **Definition 25.**

$$\gamma_2 = \sup_{x \in \Sigma_0} \left| \mathrm{grad}\left(\sum_l (f^l)^2 + f_\partial^2\right)\right| = 2 \sup_{x \in \Sigma_0} \left|\sum_l f^l \mathrm{grad} f^l + f_\partial \, \mathrm{grad} f_\partial\right| \tag{12}$$

We have then the analog of Lemma 7:

▶ **Lemma 26.** *We have :*

$$|\det(\mathrm{grad}_{(x,\tau)} F^i_{L,1}(x, \tau) \cdot \mathrm{grad}_{(x,\tau)} F^j_{L,1}(x, \tau))_{i,j}| > \gamma_0 - g_4(D),$$

with $g_4(D) = \mathcal{O}(D)$. The exact expression of g_4 is given in [17, Appendix A].

The following corollary is then the analog of Corollary 8:

▶ **Corollary 27** ($F^{-1}_{L,1}(0)$ is a manifold)**.** *If $\gamma_0 > g_4(D)$, where $g_4(D) = O(D)$ is as in Lemma 26, then $F^{-1}_{L,1}(0)$ is a smooth manifold inside an ϵ neighbourhood of $\sigma \times [0, 1]$.*

3.1.3 Transversality with regard to the τ-direction

We note that, similarly to Lemma 10, we have

▶ **Lemma 28.** *Let Ξ be as in Lemma 10 and γ_ϕ as in (11).*

$$\tan\angle(\mathrm{grad}_{(x,\tau)}(F_{L,1}),\Xi) \leq \frac{2D^2\alpha}{\sqrt{\gamma_0}/\gamma_1^{d-n-1} - \gamma_2\gamma_\phi 2D^2\alpha - \frac{4dD\alpha}{T}}.$$

In particular, if $\sqrt{\gamma_0}/\gamma_1^{d-n-1} > \gamma_2\gamma_\phi 2D^2\alpha + \frac{4dD\alpha}{T}$, $F_{L,1}^{-1}(0)$ (if it is a manifold) is never tangent to the $\tau = c$ planes, where c is a constant.

Now, similarly to Corollary 11, we find that

▶ **Corollary 29** (Transversality with respect to τ for Step 1). *Suppose that $\gamma_0 > g_4(D)$ and that $\sqrt{\gamma_0}/\gamma_1^{d-n-1} > \gamma_2\gamma_\phi 2D^2\alpha + \frac{4dD\alpha}{T}$. Then, inside each $\sigma \times [0,1]$, the gradient of τ on $F_{L,1}^{-1}(0)$ is smooth and does not vanish.*

3.1.4 Global result

We now have to prove that $F_{PL,1}^{-1}(0)$ is a manifold. For this, we shall use a bound similar to the one given in Lemma 18, so that we are able to apply the generalized implicit function theorem if this bound is satisfied. But first of all, we need the following bound, which is similar to Lemma 16.

▶ **Lemma 30.** *Assuming that the gradients are well defined, we have $|\mathrm{grad}_{(x,\tau)}F^i_{PL,1}(x_1,\tau_1) - \mathrm{grad}_{(x,\tau)}F^i_{PL,1}(x_2,\tau_2)| \leq g_5(D)$, with $g_5(D) = \mathcal{O}(D)$. The expression for g_5 is given in [17, Appendix A].*

Just as in Corollary 17, we immediately have the same bound on points in the convex hull of a number of such vectors:

▶ **Corollary 31.** *Suppose we are in the setting of Lemma 30 and $x_0, x_1, \dots, x_m \in star(v)$, $\tau_0, \dots, \tau_m \in [0,1]$, such that $\mathrm{grad}_{(x,\tau)}F^i_{PL,1}(x_i,\tau_i)$ is well defined for all i. Further assume that $\mu_1, \dots, \mu_m$ are positive weights such that $\mu_1 + \dots + \mu_m = 1$. Then,*

$$\left|\mathrm{grad}_{(x,\tau)}F^i_{PL,1}(x_0,\tau_0) - \sum_{k=1}^m \mu_k \mathrm{grad}_{(x,\tau)}F^i_{PL,1}(x_k,\tau_k)\right| \leq g_5(D).$$

▶ **Lemma 32.** *Under the same conditions as in Lemma 18,*

$$\det\left(\left(\sum_{k=1}^m \mu_k\,\mathrm{grad}_{(x,\tau)}F^i_{PL,1}(x_k,\tau_k)\right)\cdot\left(\sum_{k=1}^m \mu_k\,\mathrm{grad}_{(x,\tau)}F^j_{PL,1}(x_k,\tau_k)\right)\right)_{i,j}$$
$$\geq \gamma_0 - g_4(D) - g_6(D),$$

with $g_6(D) = \mathcal{O}(D)$. The exact expression of g_6 is given in [17, Appendix A].

Lemma 32 immediately yields that

▶ **Corollary 33** ($F_{PL,1}^{-1}(0)$ is a manifold). *If, $\gamma_0 > g_4(D) + g_6(D)$ the generalized implicit function theorem, Theorem 15, applies to $F_{PL,1}(x,\tau) = 0$. In particular $F_{PL,1}^{-1}(0)$ is a manifold.*

We stress again that inside the set $\{x|\phi\left(\sum_i (f^i)^2(x) + f_\partial^2(x)\right) = 1\}$ the zero set of $F_{PL,1}(x,1)$ coincides with the zero set of $f_{PL}(x)$.

3.2 Step 2

Before we can proceed we have to specify the bump function ψ. We suppose that

$$\psi(x) = \begin{cases} 1 & \text{if } |x| \leq \frac{101}{100} y_0 \\ 0 & \text{if } |x| \geq 2y_0. \end{cases}$$

In particular we pick $\psi(x) = \phi_b(x)$, with the choice $y_1 = \frac{101}{100} y_0$ and $y_2 = 2y_0$.

First we stress that the zero set of $F_{PL,2,\epsilon}(x, 1)$ coincides with the zero set of $(f_{PL}(x), f_{\partial,PL}(x) - \epsilon)$, provided that $\psi(\sum_i f_i(x)^2 + f_\partial(x)^2) = 1$.

Secondly, we now claim the following:

▶ **Lemma 34.** *The zero set of $F_{PL,2,\epsilon}(x, 1)$ is a subset of the zero set of $f_{PL}(x)$, for each ϵ.*

The technical result that remains to be proven is the counterpart of Theorem 20 for $F_{PL,2,\epsilon}(x, \tau)$ and for each sufficiently small ϵ. To be precise it suffices for $\epsilon \leq 2y_0$. We remark that it is likely that this bound on ϵ can be improved.

We again follow the same path to prove this result. That is we first concentrate on a single simplex and prove that inside that simplex the zero set of $F_{PL,2,\epsilon}$ is a smooth manifold on which the gradient of τ as restricted to the manifold does not vanish. We then prove that is the zero set of $F_{PL,2,\epsilon}$ is globally a manifold.

3.2.1 Assumptions and notations

Because we are now faced with both $f(x)$ and $f_\partial(x)$ we need to introduce a bound on how far the gradients of all there are from being colinear. We write

$$f_B(x) = (f(x), f_\partial(x)). \tag{13}$$

Before we were only interested in the set Σ_0, similarly here we sometimes concentrate on a neighbourhood of the zero set of both f_∂ and f. Therefore we write B_ν for all $\sigma \in \mathcal{T}$ such that $(\sum_l (f^l)^2 + (f_\partial)^2)^{-1}([-2y_0, 2y_0]) \cap \sigma \neq \emptyset$.

We define γ_0^B in terms of the determinant of the Gram matrix of the gradients, that is

$$\gamma_0^B = \inf_{x \in B_\nu \cap \Sigma_0} |\det(\mathrm{grad}(f_B^i) \cdot \mathrm{grad}(f_B^j))_{i,j}|. \tag{14}$$

We note that because we take the gradients we can just ignore the ϵ constant. For the lengths of the gradients of f_B we define,

$$\gamma_1^B = \sup_{x \in \Sigma_0} \max_i |\mathrm{grad}(f_B^i)|, \tag{15}$$

for all $1 \leq i \leq d - n + 1$. Similarly to α, we define β as the bound on the operator 2-norm of all Hessians of f_B, that is

$$\beta = \sup_{x \in \Sigma_0} \max_i \|\mathrm{Hes}(f_B^i)\|_2 = \sup_{x \in \Sigma_0} \max_i \|(\partial_k \partial_l f_B^i)_{k,l}\|_2. \tag{16}$$

We stress that we have chosen our definitions such that $\alpha \leq \beta$.

We use the same notation for the ambient triangulation $\mathcal{T}$, the lower bound on the thickness of the simplices T and upper bound on the longest edge length D. We also need to introduce a bound on the differential of the bump function ψ. Similarly to (11) we define,

$$\gamma_\psi = 2\frac{e^{\frac{4}{3(y_2 - y_1)}}}{y_2 - y_1} = 2\frac{e^{\frac{4}{3(2y_0 - \frac{101}{100}y_0)}}}{2y_0 - \frac{101}{100}y_0} = \frac{200}{99}\frac{e^{\frac{400}{297y_0}}}{y_0}, \tag{17}$$

because we picked $y_1 = \frac{101}{100} y_0$ and $y_2 = 2y_0$, for ψ.

3.2.2 Inside a single simplex

Similarly to Lemma 26, we now give a condition that ensure that the zero set of $F_{PL,2,\epsilon}(x,\tau)$ is smooth inside $\sigma \times [0,1]$. In fact similarly to (7), we define

$$F_{L,2,\epsilon}(x,\tau) = \left(1 - \tau\psi\left(\sum_i (f^i)^2 + f_\partial^2\right)\right)(F_{L,1}(x,1), f_\partial(x) - \epsilon) + \tau\psi\left(\sum_i (f^i)^2 + f_\partial^2\right)(f_L(x), f_{\partial,L}(x) - \epsilon),$$

which can be extended to a neighbourhood of $\sigma \times [0,1]$.

▶ **Lemma 35.** *For all ϵ,* $\det(\mathrm{grad}_{(x,\tau)} F^i_{L,2,\epsilon}(x,\tau) \cdot \mathrm{grad}_{(x,\tau)} F^j_{L,2,\epsilon}(x,\tau))_{i,j} \geq \gamma_0^B - g_7(D)$ *with $g_7(D) = \mathcal{O}(D)$. The exact expression of g_7 is given in [17, Appendix A].*

▶ **Corollary 36** ($F^{-1}_{L,2,\epsilon}(0)$ is a manifold)**.** *We have that $F^{-1}_{L,2,\epsilon}(0)$ is a smooth manifold inside an small neighbourhood of $\sigma \times [0,1]$ provided $\gamma_0^B > g_7(D)$, with $g_7(D)$ as in Lemma 35. As usual this can always be satisfied by choosing the triangulation fine enough, that is D sufficiently small.*

3.2.3 Transversality with regard to the τ-direction

Once more similarly to Lemma 10, we have

▶ **Lemma 37.** *Let Ξ be as in Lemma 10. We have*

$$\tan\angle(\mathrm{grad}_{(x,\tau)}(F_{L,2,\epsilon}), \Xi) \leq \frac{2D^2\beta}{\sqrt{\gamma_0^B}/(\gamma_1^B)^{d-n-2} - (\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta - \frac{12dD\beta}{T}}.$$

In particular the manifold $F^{-1}_{L,2,\epsilon}(0)$ inside $\sigma \times [0,1]$, if well defined, is never tangent to the $\tau = c$ planes, where c is a constant, if

$$\sqrt{\gamma_0^B}/(\gamma_1^B)^{d-n-2} > (\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T}.$$

Now similary to Corollary 11, we find that

▶ **Corollary 38** (Transversality with respect to τ for Step 2)**.** *Suppose that the conditions of Corollary 36 are satisfied. If moreover*

$$\sqrt{\gamma_0^B}/(\gamma_1^B)^{d-n-2} > (\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T},$$

then inside each $\sigma \times [0,1]$ the gradient of τ on $F^{-1}_{L,2,\epsilon}(0)$ is smooth and does not vanish.

3.2.4 Global result

We now have to prove that $F^{-1}_{PL,2,\epsilon}(0)$ is a manifold, for all sufficiently small ϵ. For this we shall use a bound similar to the one given in Lemma 18, so that we are able to apply the generalized implicit function theorem if this bound is satisfied. For this, we first need the following bound, which is similar to Lemma 30.

▶ **Lemma 39.** *Let v be a vertex in $\mathcal{T}$, $x_1, x_2 \in$ star(v), and $\tau_1, \tau_2 \in [0,1]$, such that* $\mathrm{grad}_{(x,\tau)} F^i_{PL,2,\epsilon}(x_1,\tau_1)$ *and* $\mathrm{grad}_{(x,\tau)} F^i_{PL,2,\epsilon}(x_2,\tau_2)$ *are well defined, then*

$$|\mathrm{grad}_{(x,\tau)} F^i_{PL,2,\epsilon}(x_1,\tau_1) - \mathrm{grad}_{(x,\tau)} F^i_{PL,2,\epsilon}(x_2,\tau_2)| \leq g_8(D),$$

with $g_8(D) = \mathcal{O}(D)$.The exact expression of g_8 is given in [17, Appendix A].

Just as in Corollary 17, we immediately have the same bound on points in the convex hull of a number of such vectors:

▶ **Corollary 40.** *Suppose $x_0, x_1, \dots, x_m \in star(v)$, $\tau_0, \dots, \tau_m \in [0,1]$, such that $\mathrm{grad}_{(x,\tau)} F^i_{PL,2,\epsilon}(x_i, \tau_i)$ is well defined for all i. Further assume that $\mu_1, \dots, \mu_m$ are positive weights such that $\mu_1 + \cdots + \mu_m = 1$. Then,*

$$\left| \mathrm{grad}_{(x,\tau)} F^i_{PL,2,\epsilon}(x_0, \tau_0) - \sum_{k=1}^{m} \mu_k \mathrm{grad}_{(x,\tau)} F^i_{PL,2,\epsilon}(x_k, \tau_k) \right| \leq g_8(D).$$

▶ **Lemma 41.** *Under the same conditions as in Corollary 40,*

$$\det \left(\left(\sum_{k=1}^{m} \mu_k \, \mathrm{grad}_{(x,\tau)} F^i_{PL,2,\epsilon}(x_k, \tau_k) \right) \cdot \left(\sum_{k=1}^{m} \mu_k \, \mathrm{grad}_{(x,\tau)} F^j_{PL,2,\epsilon}(x_k, \tau_k) \right) \right)_{i,j}$$
$$\geq \gamma_0^B - g_7(D) - g_9(D),$$

where $g_9(D) = \mathcal{O}(D)$. See Appendix A of [17] for the g_9.

Lemma 41 immediately yields that

▶ **Corollary 42** (The generalized implicit function theorem in Step 2)**.** *If, $\gamma_0^B > g_7(D) + g_9(D)$ the generalized implicit function theorem, Theorem 15, applies to $F_{PL,1}(x, \tau) = 0$. In particular $F^{-1}_{PL,1}(0)$ is a manifold.*

We stress that this condition only needs to be satisfied in a when $\sum_l (f^l)^2 + (f_\partial)^2$ is small, outside this neighbourhood the isotopy leaves the zero set invariant.

▶ **Theorem 43.** *If,*

$$\sqrt{\gamma_0}/\gamma_1^{d-n-1} > \gamma_2 \gamma_\phi 2D^2 \alpha + \frac{4dD\alpha}{T} \qquad \text{(Corollary 29)}$$
$$\gamma_0 > g_4(D) + g_6(D) \qquad \text{(Corollaries 27 and 33)}$$
$$\sqrt{\gamma_0^B}/(\gamma_1^B)^{d-n-2} > (\gamma_2(2\gamma_\phi + \gamma_\psi) + 1)2D^2\beta + \frac{12dD\beta}{T} \qquad \text{(Corollary 38)}$$
$$\gamma_0^B > g_7(D) + g_9(D), \qquad \text{(Corollaries 36 and 42)}$$

then $f^{-1}(0) \cap f_\partial^{-1}([0,\infty))$ is isotopic to $f^{-1}_{PL}(0) \cap f^{-1}_{\partial,PL}([0,\infty))$. We stress that one can satisfy all conditions by choosing D sufficiently small. See Appendix A of [17] for the $g_i(D)$.

4 Isostratifolds

There is no obstruction in principle that prevents us from extending the approach above to isostratifolds. By isostratifolds we mean stratifolds that are given by the zero sets of functions and inequalities. See [17], for a short discussion of the approach. However, finding the precise constants involved would become prohibitively lengthy. Apart from some Delaunay based work on triangulations of stratifolds in three dimensions [43, 45, 28, 27, 26], we are not aware of similar results. Significant effort did go in the detection of strata, in this case in arbitrary dimension, see for example [6, 5].

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