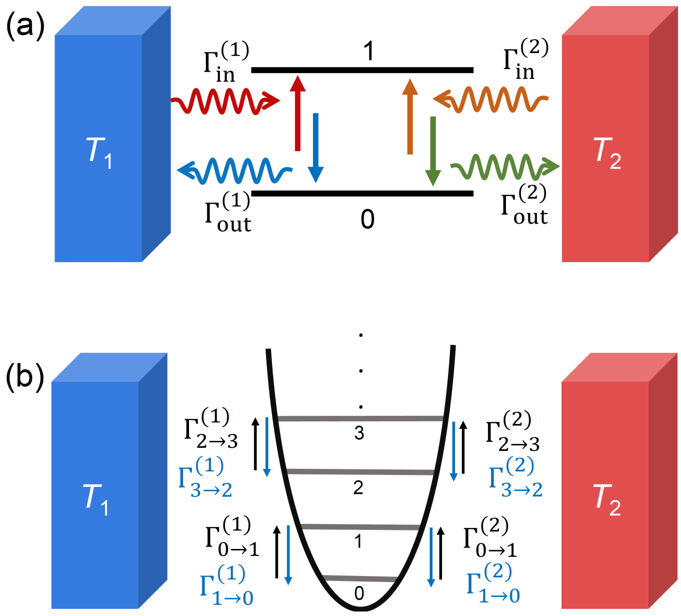
Supplementary Information

Supplementary Note 1: Rectification in a multi-level system

A. Two level System

Consider a qubit coupled to two baths as shown in Supplementary Fig. 1. The transition rates are given by



Supplementary Figure 1: Transition between two thermal baths: (a) A two-level system coupled to two baths at temperatures T_1 and T_2, together with the associated transition rates. (b) The same for a harmonic oscillator.

$$\begin{aligned}\Gamma_{\rm in}^{(1)} &= g_1 \frac{\omega_0}{e^{\beta_1 \hbar\omega_0}-1}, \qquad \Gamma_{\rm in}^{(2)} = g_2 \frac{\omega_0}{e^{\beta_2 \hbar\omega_0}-1} \\ \Gamma_{\rm out}^{(1)} &= g_1 \frac{\omega_0}{1-e^{-\beta_1 \hbar\omega_0}}, \qquad \Gamma_{\rm out}^{(2)} = g_2 \frac{\omega_0}{1-e^{-\beta_2 \hbar\omega_0}}, \end{aligned} \tag{S.1}$$

where g_i is the coupling to bath $i = 1, 2$, $\hbar\omega_0$ denotes the energy level separation of the qubit, and $\beta_i = 1/k_{\rm B}T_i$ is the inverse temperature of each bath. In steady state the population of the excited state, $\rho_e = 1 - \rho_g$ reads

$$\rho_e = \frac{\Gamma_{\rm in}}{\Gamma_{\rm in} + \Gamma_{\rm out}}, \tag{S.2}$$

where $\Gamma_{\rm in,out} = \Gamma_{\rm in,out}^{(1)} + \Gamma_{\rm in,out}^{(2)}$ and ρ_g is the population of the ground state of the qubit. The expression for power to bath i is then

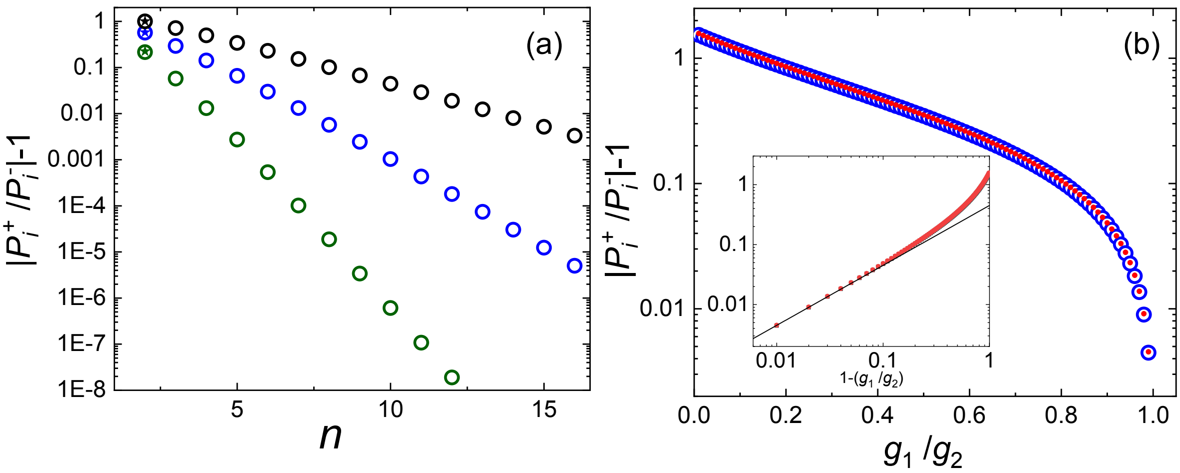
$$P_i = \hbar\omega_0(\rho_e \Gamma_{\rm out}^{(i)} - \rho_g \Gamma_{\rm in}^{(i)}). \tag{S.3}$$

The thermal rectification by definition is given by

$$\mathcal{R} = \left| \frac{P_i^+}{P_i^-} \right|, \tag{S.4}$$

where $\pm$ refers to the sign of the temperature bias. Consider case when one of the bath temperatures is much smaller than the other with $k_{\rm B}T = 1/\beta$ the higher temperature. In this case the rectification ratio is given by

$$\mathcal{R} = \frac{g_1 + g_2 \coth(\frac{\beta\hbar\omega_0}{2})}{g_1 \coth(\frac{\beta\hbar\omega_0}{2}) + g_2}. \tag{S.5}$$



Supplementary Figure 2: Rectification ratio. (a) $\mathcal{R}-1$ versus number of levels n. The parameters are: $g_1/g_2 = 1/3$, black circles $\beta_1\hbar\omega_0 = 0.4$ and $\beta_2\hbar\omega_0 = 4.8$, blue circles $\beta_1\hbar\omega_0 = 0.8$ and $\beta_2\hbar\omega_0 = 4.8$, and green circles $\beta_1\hbar\omega_0 = 1.6$ and $\beta_2\hbar\omega_0 = 4.8$. Stars at $n = 2$ are from Equation S5. (b) $\mathcal{R}-1$ versus asymmetry for $n = 2$. Blue circles are from the numerics and red ones from Equation S5. The parameters are: $\beta_1\hbar\omega_0 = 0.8$ and $\beta_2\hbar\omega_0 = 4.8$ The inset replots this with the horizontal index instead showing $1-(g_1/g_2)$.

For small asymmetry $\delta = 1 - g_1/g_2$, $|\delta| \ll 1$, one can expand the rectification ratio into

$$\mathcal{R} - 1 = e^{-\beta\hbar\omega_0}\delta. \tag{S.6}$$

The inset of Supplementary Figure 2b shows this result by solid line for the corresponding temperature.

B. Multilevel system

Consider n-level system with constant energy spacing $\hbar\omega_0$. In the limit of $n\to\infty$ it represents a linear harmonic oscillator as shown in Supplementary Figure 1b. The transition rates between levels k and $k\pm1$ are given by

$$\Gamma^{(i)}_{k\to k\pm1} = \frac{1}{\hbar^2}|\langle k|\hat{Q}|k\pm1\rangle|^2 S_i(\mp\omega_0). \tag{S.7}$$

Here $i = 1, 2$ refers to the baths, $\hat{Q} = i\sqrt{\frac{\hbar}{2Z_0}}(\hat{a}^\dagger - \hat{a})$ and $S_i(\omega) = 2R_i\frac{\hbar\omega}{1-e^{-\beta_i\hbar\omega}}$ are charge operator and voltage noise when applied to a circuit, respectively. Here, Z_0 is a characteristic impedance of the system, $\hat{a}$ ($\hat{a}^\dagger$) is annihilation (creation) operator of the ladder of the system, and R_i is the resistance of the bath. In this case only the transitions between the nearest levels are allowed according to

$$\begin{aligned}\Gamma^{(i)}_{k\to k-1} &= k\Gamma^{(i)}_{\rm out}\\ \Gamma^{(i)}_{k-1\to k} &= k\Gamma^{(i)}_{\rm in}.\end{aligned} \tag{S.8}$$

The steady state population of each level reads $\rho_i = S_i/S$, where

$$S_0 = \prod_{k=1}^{n}\Gamma_{k\to k-1},\quad S_j = \prod_{k=1}^{j}\Gamma_{k-1\to k}\prod_{k=j+1}^{n}\Gamma_{k\to k-1},\quad S_n = \prod_{k=1}^{n}\Gamma_{k-1\to k},\quad S = \sum_{i=0}^{n}S_i, \tag{S.9}$$

where $1 \le j \le n-1$ and $\Gamma_{k\to k\pm1} = \Gamma^{(1)}_{k\to k\pm1} + \Gamma^{(2)}_{k\to k\pm1}$. The expression of power to bath i is then given by

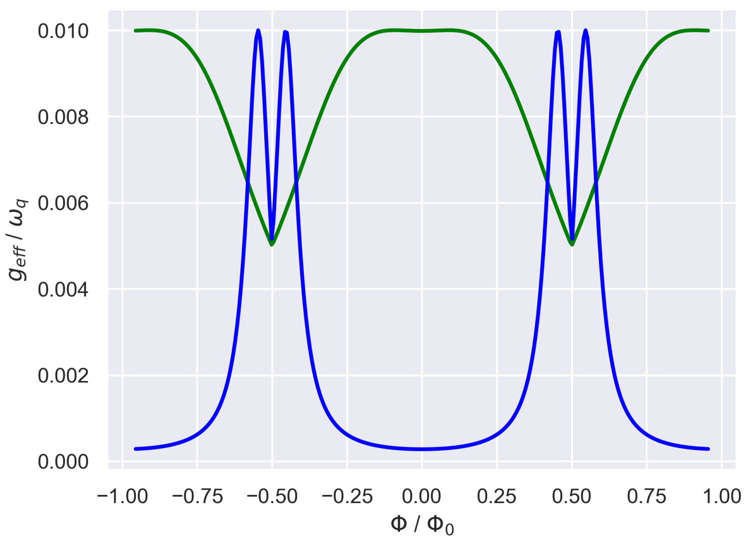
$$P_i = \hbar\omega_0\sum_{k=1}^{n}(\Gamma^{(i)}_{k\to k-1}\rho_k - \Gamma^{(i)}_{k-1\to k}\rho_{k-1}). \tag{S.10}$$

Supplementary Note 2: Introducing an effective qubit-reservoir coupling

We consider a situation as in the experiment, where a qubit with tunable frequency $\omega(\Phi) \approx \omega_q \sqrt{|\cos(\pi\Phi/\Phi_0)|}$ is coupled to resonators with frequencies $\omega_1 < \omega_2$ and bare coupling constants $g_1 \approx g_2 \ll \omega_1, \omega_2$. The resonators interact with heat baths (ohmic spectral densities) at temperatures $T_1 \ll T_2$. For simplicity, we put $T_1 = 0$ so that Eq. (2) from the paper applies. The spectral density of bath i as seen from the qubit is that of a damped harmonic oscillator with central frequency ω_i. This provides an effective coupling between the qubit with reservoir i as (see also Motz et al. NJP 20 (2018))

$$g_{\rm eff,i}(\Phi) = \frac{g_i}{Q_i} \frac{\omega_i^4}{\left[\omega_i^2 - \omega(\Phi)^2\right]^2 + \omega_i^4/Q_i^2} \tag{S.11}$$

where Q_i is the quality factor of the corresponding resonator. With respect to the resonant behavior of the effective resonator-reservoir couplings, the low frequency oscillator $\omega_1 < \omega_q$ has maxima near $\Phi/\Phi_0 = \pm(k+1/2), k = 0, 1, 2, \ldots$. For the high frequency oscillator $\omega_2 \approx \omega_q$ the coupling is maximal near $\Phi/\Phi_0 = \pm k, k = 0, 1, 2 \ldots$. Near a resonance with respect to oscillator i one has $g_{\rm eff,i} = g_i Q_i$ while for the other one $g_eff, j \approx (g_j/Q_j)1/(1 - \omega_i^2/\omega_j^2)^2$. At $\Phi/\Phi_0 = 1/2$ (and for identical Q-factor and bare couplings g) both effective couplings coincide to give $g_{\rm eff,1}(\pi/2) = g_{\rm eff,2}(\pi/2) = gQ/(1+Q^2)$. Results are shown in Fig. 3; by way of example, we assume: $\omega_1/\omega_q = 0.38, \omega_2/\omega_q = 0.98, Q_1 \approx Q_2 \approx 1, g_1/\omega_q \approx g_2/\omega_q = 0.01$, and $\omega_q \hbar\beta/2 = 0.4$ (corresponding to T = 400K). Note that for the minimal couplings we have $g_{\rm eff,2}(\Phi \approx \Phi_0/2) \gg g_{\rm eff,1}(\Phi \approx 0)$.

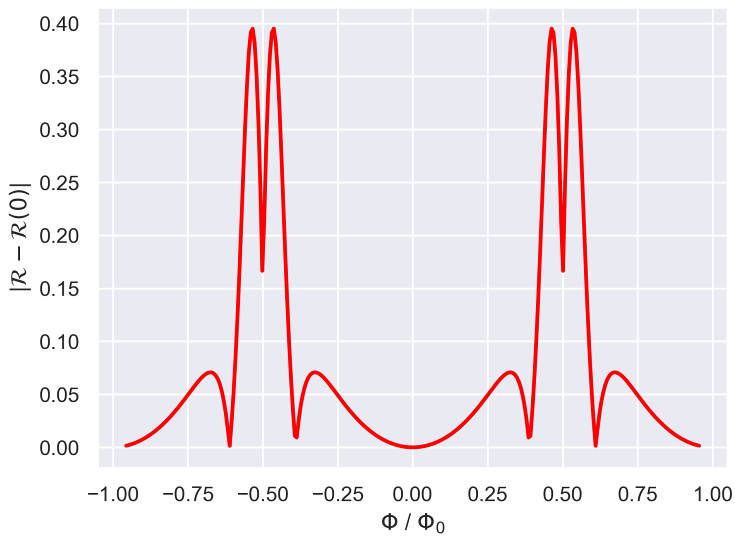


Supplementary Figure 3: Effective qubit-reservoir couplings $g_{\rm eff,2}(\Phi)$ (green, high frequency oscillator) and $g_{\rm eff,1}(\phi)$ (blue, low frequency oscillator) according to Equation S.11 for parameters specified in the text.

Accordingly, the expression for the rectification coefficient derived from a two level system coupled directly to heat reservoirs (see Nitzan paper) can be applied according to [see Eq. (2) in the paper]

$$\mathcal{R} = \frac{g_{\rm eff,1}(\Phi) + g_{\rm eff,2}(\Phi)\coth[\omega(\Phi)\hbar\beta/2]}{g_{\rm eff,1}(\Phi)\coth[\omega(\Phi)\hbar\beta/2] + g_{\rm eff,2}(\Phi)} \tag{S.12}$$

The flux dependence of the rectification coefficient is shown in Supplementary Figure 4. While the absolute value of the rectification coefficient is much smaller in the experiment, qualitatively the above description provides the observed behaviour. To find a better agreement one should account also for the finite temperature of reservoir 1.



Supplementary Figure 4: Rectification coefficient $|\mathcal{R}(\Phi) - \mathcal{R}(0)|$ according to Equation S12 for $\theta = 0.4$ corresponding to $T_{\rm hot} = 400$mK [$T_{\rm cold} = 0$K].

Supplementary Note 3: Energy levels of the experimental device

The Hamiltonian $\hat{H}$ is given by

$$\hat{H} = \hbar\omega_{\rm L}\hat{a}_{\rm L}^\dagger\hat{a}_{\rm L} + \hbar\omega_{\rm q}\hat{a}_{\rm q}^\dagger\hat{a}_{\rm q} + \hbar\omega_{\rm R}\hat{a}_{\rm R}^\dagger\hat{a}_{\rm R} + g(\hat{a}_{\rm q}\hat{a}_{\rm L}^\dagger + \hat{a}_{\rm q}^\dagger\hat{a}_{\rm L} + \hat{a}_{\rm q}\hat{a}_{\rm R}^\dagger + \hat{a}_{\rm q}^\dagger\hat{a}_{\rm R}) + \tilde{g}(\hat{a}_{\rm L}\hat{a}_{\rm R}^\dagger + \hat{a}_{\rm L}^\dagger\hat{a}_{\rm R}). \tag{S.13}$$

Here, $\hbar\omega_{\rm L}$, $\hbar\omega_{\rm q}$, and $\hbar\omega_{\rm R}$ are the energies of the left resonator, qubit and the right resonator, respectively, g is the common coupling constant of the qubit to the two resonators, and $\tilde{g}$ is the cross-coupling between the resonators. In the eleven-level basis of $|000\rangle$, $|100\rangle$, $|010\rangle$, $|001\rangle$, $|110\rangle$, $|101\rangle$, $|011\rangle$, $|111\rangle$, $|200\rangle$, $|210\rangle$, $|300\rangle$, where the entries in each state refer to the left resonator, the qubit, and the right resonator, respectively, the matrix form of the Hamiltonian can be written

$$H = \hbar\omega_0 \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1-a/2 & \gamma & \tilde{\gamma} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \gamma & r & \gamma & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \tilde{\gamma} & \gamma & 1+a/2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1-a/2+r & \gamma & \tilde{\gamma} & 0 & \sqrt{2}\gamma & 0 & 0 \\ 0 & 0 & 0 & 0 & \gamma & 2 & \gamma & 0 & \sqrt{2}\tilde{\gamma} & 0 & 0 \\ 0 & 0 & 0 & 0 & \tilde{\gamma} & \gamma & r+1+a/2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2+r & 0 & \sqrt{2}\tilde{\gamma} & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2}\gamma & \sqrt{2}\tilde{\gamma} & 0 & 0 & 2-a & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{2}\tilde{\gamma} & 0 & 2-a+r & \sqrt{3}\gamma \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{3}\gamma & 3-3a/2 \end{pmatrix} \tag{S.14}$$

where $\omega_0 = \frac{\omega_{\rm R}+\omega_{\rm L}}{2}$, $a = \frac{\omega_{\rm R}-\omega_{\rm L}}{\omega_0}$, $\gamma = \frac{g}{\hbar\omega_0}$, $\tilde{\gamma} = \frac{\tilde{g}}{\hbar\omega_0}$, and $r = \frac{\hbar\omega_{\rm q}}{\hbar\omega_0}$. Here $\omega_{\rm L} = 2\pi \times 2.8$ GHz and $\omega_{\rm R} = 2\pi \times 6.5$ GHz are constant, and $\hbar\omega_q = \sqrt{8E_JE_C|\cos(\pi\Phi/\Phi_0)|} - E_{\rm C}$ like in the anharmonic Josephson potential.