$\mathbb{Z}_2$-Genus of Graphs and Minimum Rank of Partial Symmetric Matrices

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Abstract

The *genus* $\mathrm{g}(G)$ of a graph G is the minimum g such that G has an embedding on the orientable surface M_g of genus g. A drawing of a graph on a surface is *independently even* if every pair of nonadjacent edges in the drawing crosses an even number of times. The $\mathbb{Z}_2$-*genus* of a graph G, denoted by $\mathrm{g}_0(G)$, is the minimum g such that G has an independently even drawing on M_g.

By a result of Battle, Harary, Kodama and Youngs from 1962, the graph genus is additive over 2-connected blocks. In 2013, Schaefer and Štefankovič proved that the $\mathbb{Z}_2$-genus of a graph is additive over 2-connected blocks as well, and asked whether this result can be extended to so-called 2-amalgamations, as an analogue of results by Decker, Glover, Huneke, and Stahl for the genus. We give the following partial answer. If $G = G_1 \cup G_2$, G_1 and G_2 intersect in two vertices u and v, and $G - u - v$ has k connected components (among which we count the edge uv if present), then $|\mathrm{g}_0(G) - (\mathrm{g}_0(G_1) + \mathrm{g}_0(G_2))| \le k + 1$. For complete bipartite graphs $K_{m,n}$, with $n \ge m \ge 3$, we prove that $\frac{\mathrm{g}_0(K_{m,n})}{\mathrm{g}(K_{m,n})} = 1 - O(\frac{1}{n})$. Similar results are proved also for the Euler $\mathbb{Z}_2$-genus.

We express the $\mathbb{Z}_2$-genus of a graph using the minimum rank of partial symmetric matrices over $\mathbb{Z}_2$; a problem that might be of independent interest.

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1 Introduction

The *genus* $\mathrm{g}(G)$ of a graph G is the minimum g such that G has an embedding on the orientable surface M_g of genus g. Similarly, the *Euler genus* $\mathrm{eg}(G)$ of G is the minimum g such that G has an embedding on a surface of Euler genus g. We say that two edges in a graph are *independent* (also *nonadjacent*) if they do not share a vertex. The $\mathbb{Z}_2$-*genus* $\mathrm{g}_0(G)$ and *Euler* $\mathbb{Z}_2$-*genus* $\mathrm{eg}_0(G)$ of G are defined as the minimum g such that G has a drawing on M_g and a surface of Euler genus g, respectively, with every pair of independent edges crossing an even number of times. Clearly, $\mathrm{g}_0(G) \le \mathrm{g}(G)$ and $\mathrm{eg}_0(G) \le \mathrm{eg}(G)$.

The definition of the $\mathbb{Z}_2$-genus and Euler $\mathbb{Z}_2$-genus is motivated by the strong Hanani–Tutte theorem [17, 35] stating that a graph is planar if and only if its $\mathbb{Z}_2$-genus is 0. Many variants and extensions of the theorem have been proved [6, 14, 23, 29, 31], and they found various applications in combinatorial and computational geometry; see the survey by Schaefer [28].

It had been a long-standing open problem whether the strong Hanani–Tutte theorem extends to surfaces other than the plane and projective plane, although the problem was first explicitly stated in print by Schaefer and Štefankovič [30] in 2013. They conjectured that the strong Hanani–Tutte theorem extends to every orientable surface, that is, $\mathrm{g}_0(G) = \mathrm{g}(G)$ for every graph G. They proved that a minimal counterexample to their conjecture must be 2-connected; this is just a restatement of their block additivity result, which we discuss later in this section. In a recent manuscript [13], we provided an explicit construction of a graph G for which $\mathrm{g}(G) = 5$ and $\mathrm{g}_0(G) \leq 4$, thereby refuting the conjecture. Nevertheless, the conjecture by Schaefer and Štefankovič [30] that $\mathrm{eg}_0(G) = \mathrm{eg}(G)$ for every graph G might still be true.

The conjecture has been verified only for graphs G with $\mathrm{eg}(G) \leq 1$: Pelsmajer, Schaefer and Stasi [24] proved that the strong Hanani–Tutte theorem extends to the projective plane, using the characterization of projective planar graphs by an explicit list of forbidden minors. Recently, Colin de Verdière et al. [8] gave a constructive proof of the same result.

Schaefer and Štefankovič [30] also formulated a weaker form of their conjecture about the $\mathbb{Z}_2$-genus, stating that there exists a function $f : \mathbb{N} \to \mathbb{N}$ such that $\mathrm{g}(G) \leq f(\mathrm{g}_0(G))$ for every graph G. Assuming the validity of an unpublished Ramsey-type result by Robertson and Seymour, the existence of such f follows as a corollary from our recent result [15] stating that $\mathrm{g}_0(G) = \mathrm{g}(G)$ for the graphs G in the so-called family of Kuratowski minors. Regarding the asymptotics of f, we do not have any explicit upper bound on f, and the existence of G with $\mathrm{g}(G) = 5$ and $\mathrm{g}_0(G) \leq 4$ implies that $f(k) \geq 5k/4$ [13, Corollary 11].

As the next step towards a good understanding of the relation between the (Euler) genus and the (Euler) $\mathbb{Z}_2$-genus we provide further indication of their similarity. We will build upon techniques introduced in [30] and [15], and reduce the problem of estimating the (Euler) $\mathbb{Z}_2$-genus to the problem of estimating the minimum rank of partial symmetric matrices over $\mathbb{Z}_2$.

First, we extend our recent result determining the $\mathbb{Z}_2$-genus of $K_{3,n}$ [15, Proposition 18] in a weaker form to all complete bipartite graphs. A classical result by Ringel [5, 26, 27], [22, Theorem 4.4.7], [16, Theorem 4.5.3] states that for $m, n \geq 2$, we have $\mathrm{g}(K_{m,n}) = \left\lceil \frac{(m-2)(n-2)}{4} \right\rceil$ and $\mathrm{eg}(K_{m,n}) = \left\lceil \frac{(m-2)(n-2)}{2} \right\rceil$.

▶ **Theorem 1.** *If $n \geq m \geq 3$, then*

$$\mathrm{g}_0(K_{m,n}) \geq \frac{(n-2)(m-2)}{4} - \frac{m-3}{2} \quad \textit{and}$$
$$\mathrm{eg}_0(K_{m,n}) \geq \frac{(n-2)(m-2)}{2} - (m-3).$$

Our second result is a $\mathbb{Z}_2$-variant of the results of Stahl [32], Decker, Glover and Huneke [11, 12], Miller [20] and Richter [25] showing that the genus and Euler genus of graphs are almost additive over 2-amalgamations, which we now describe in detail.

We say that a graph G is a *k-amalgamation* of graphs G_1 and G_2 (with respect to vertices $x_1, \ldots, x_k$) if $G = (V(G_1) \cup V(G_2), E(G_1) \cup E(G_2))$ and $V(G_1) \cap V(G_2) = \{x_1, \ldots, x_k\}$, and we write $G = \amalg_{x_1,\ldots,x_k}(G_1, G_2)$. See Figure 1 for an illustration.

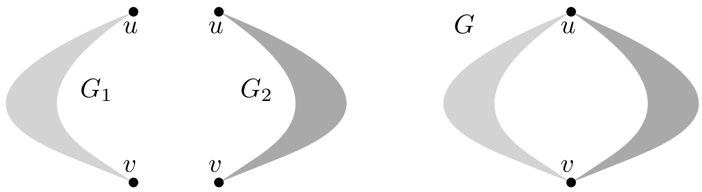

Figure 1 A 2-amalgamation $G = \amalg_{u,v}(G_1, G_2)$.

An old result of Battle, Harary, Kodama and Youngs [4], [22, Theorem 4.4.2], [16, Theorem 3.5.3] states that the genus of a graph is additive over its 2-connected blocks. In other words, if G is a 1-amalgamation of G_1 and G_2 then $\mathrm{g}(G) = \mathrm{g}(G_1) + \mathrm{g}(G_2)$. Stahl and Beinecke [33, Corollary 2], [22, Theorem 4.4.3] and Miller [20, Theorem 1] proved that the same holds for the Euler genus, that is, $\mathrm{eg}(G) = \mathrm{eg}(G_1) + \mathrm{eg}(G_2)$. Neither the genus nor the Euler genus are additive over 2-amalgamations: for example, the nonplanar graph K_5 can be expressed as a 2-amalgamation of two planar graphs in several ways. Nevertheless, the additivity in this case fails only by at most 1 for the genus and by at most 2 for the Euler genus. Formally, Stahl [32] and Decker, Glover and Huneke [11, 12] proved that if G is a 2-amalgamation of G_1 and G_2 then $|\mathrm{g}(G) - (\mathrm{g}(G_1) + \mathrm{g}(G_2))| \leq 1$. For the Euler genus, Miller [20] proved its additivity over edge-amalgamations, which implies $\mathrm{eg}(G_1) + \mathrm{eg}(G_2) \leq \mathrm{eg}(G) \leq \mathrm{eg}(G_1) + \mathrm{eg}(G_2) + 2$. Richter [25] later proved a more precise formula for the Euler genus of 2-amalgamations with respect to a pair of nonadjacent vertices.

Schaefer and Štefankovič [30] showed that the $\mathbb{Z}_2$-genus and Euler $\mathbb{Z}_2$-genus are additivite over 2-connected blocks and they asked whether $|\mathrm{g}_0(G) - (\mathrm{g}_0(G_1) + \mathrm{g}_0(G_2))| \leq 1$ if G is a 2-amalgamation of G_1 and G_2, as an analogue of the result by Stahl [32] and Decker, Glover and Huneke [11, 12]. We prove a slightly weaker variant of almost-additivity over 2-amalgamations for both the $\mathbb{Z}_2$-genus and the Euler $\mathbb{Z}_2$-genus.

▶ **Theorem 2.** *Let G be a 2-amalgamation $\amalg_{v,u}(G_1, G_2)$. Let l be the total number of connected components of $G - u - v$ in G. Let $k = l$ if $uv \notin E(G)$ and $k = l + 1$ if $uv \in E(G)$. Then*

a) $\mathrm{g}_0(G_1) + \mathrm{g}_0(G_2) - (k + 1) \leq \mathrm{g}_0(G) \leq \mathrm{g}_0(G_1) + \mathrm{g}_0(G_2) + 1,$ *and*
b) $\mathrm{eg}_0(G_1) + \mathrm{eg}_0(G_2) - (2k - 1) \leq \mathrm{eg}_0(G) \leq \mathrm{eg}_0(G_1) + \mathrm{eg}_0(G_2) + 2.$

Organization

We give basic definitions and tools in Sections 2 and 3. In Section 4, we present linear-algebraic results lying at the heart of our arguments. In Section 5 and 6, we prove Theorem 1 and Theorem 2, respectively. In Section 6, in order to illustrate our techniques in a simpler setting, we first reprove the block additivity result for the Euler $\mathbb{Z}_2$-genus. We finish with concluding remarks in Section 7. Omitted proofs are in the full version.

2 Graphs on surfaces

We refer to the monograph by Mohar and Thomassen [22] for a detailed introduction into surfaces and graph embeddings. By a *surface* we mean a connected compact 2-dimensional topological manifold. Every surface is either *orientable* (has two sides) or *nonorientable* (has only one side). Every orientable surface S is obtained from the sphere by attaching $g \geq 0$ *handles*, and this number g is called the *genus* of S. Similarly, every nonorientable surface S

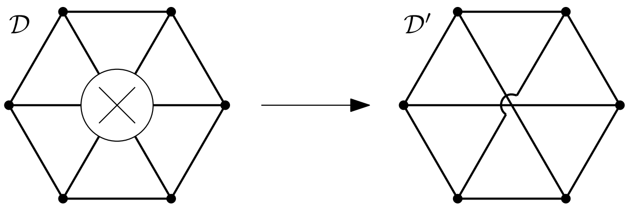

Figure 2 An embedding $\mathcal{D}$ of $K_{3,3}$ in the plane with a single crosscap (left) and its planarization $\mathcal{D}'$ (right).

is obtained from the sphere by attaching $g \geq 1$ *crosscaps*, and this number g is called the *(nonorientable) genus* of S. The simplest orientable surfaces are the sphere (with genus 0) and the torus (with genus 1). The simplest nonorientable surfaces are the projective plane (with genus 1) and the Klein bottle (with genus 2). We denote the orientable surface of genus g by M_g, and the nonorientable surface of genus g by N_g. The *Euler genus* of M_g is $2g$ and the Euler genus of N_g is g.

Let $G = (V, E)$ be a graph or a multigraph with no loops, and let S be a surface. A *drawing* of G on S is a representation of G where every vertex is represented by a unique point in S and every edge e joining vertices u and v is represented by a simple curve in S joining the two points that represent u and v. If it leads to no confusion, we do not distinguish between a vertex or an edge and its representation in the drawing and we use the words "vertex" and "edge" in both contexts. We assume that in a drawing no edge passes through a vertex, no two edges touch, every edge has only finitely many intersection points with other edges and no three edges cross at the same inner point. In particular, every common point of two edges is either their common endpoint or a crossing. Let $\mathcal{D}$ be a drawing of a graph G. We denote by $\mathrm{cr}_{\mathcal{D}}(e, f)$ the number of crossings between the edges e and f in $\mathcal{D}$. A drawing of G on S is an *embedding* if no two edges cross.

A drawing of a graph is *independently even* if every pair of independent edges in the drawing crosses an even number of times. In the literature, the notion of $\mathbb{Z}_2$*-embedding* is used to denote an independently even drawing [30], but also an *even drawing* [6] in which all pairs of edges cross evenly.

3 Topological and algebraic tools

3.1 Combinatorial representation of drawings

Schaefer and Štefankovič [30] used the following combinatorial representation of drawings of graphs on M_g and N_g. First, every drawing of a graph on M_g can be considered as a drawing on the nonorientable surface N_{2g+1}, since M_g minus a point is homeomorphic to an open subset of N_{2g+1}. The surface N_h minus a point can be represented combinatorially as the plane with h *crosscaps*. A crosscap at a point x is a combinatorial representation of a Möbius strip whose boundary is identified with the boundary of a small circular hole centered in x. Informally, the main "objective" of a crosscap is to allow a set of curves intersect transversally at x without counting it as a crossing.

Let $\mathcal{D}$ be a drawing of a graph G in the plane with h crosscaps. To every edge $e \in E(G)$ we assign a vector $y_e^{\mathcal{D}}$ (or simply y_e) from $\mathbb{Z}_2^h$ such that $(y_e^{\mathcal{D}})_i = 1$ if and only if e passes an odd number of times through the ith crosscap.

Given a drawing $\mathcal{D}$ of a graph G in the plane with h crosscaps, the *planarization* of $\mathcal{D}$ is a drawing $\mathcal{D}'$ of G in the plane, obtained from $\mathcal{D}$ as follows; see Figure 2 for an illustration. We turn the crosscaps into holes, fill the holes with discs, reconnect the severed edges of

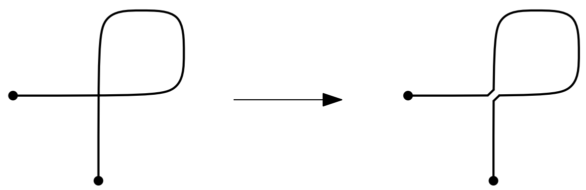

Figure 3 Removing a self-crossing of an edge.

G by simple curves drawn across the filling discs while avoiding creating common crossing points of three and more edges, and finally we eliminate self-crossings of edges by cutting and rerouting the edges at such crossings; see Figure 3. Since $\mathcal{D}$ represents a drawing $\mathcal{D}_h$ on N_h, we denote by $\mathrm{cr}^*_{\mathcal{D}}(e,f)$ the number of crossings between the edges e and f that occur outside crosscaps in $\mathcal{D}$, which is equal to $\mathrm{cr}_{\mathcal{D}_h}(e,f)$. Writing $y_e^\top y_f$ for the scalar product of y_e and y_f, we have

$$\mathrm{cr}^*_{\mathcal{D}}(e,f) \equiv \mathrm{cr}_{\mathcal{D}'}(e,f) + y_e^\top y_f \pmod 2, \tag{1}$$

since $y_e^\top y_f$ has the same parity as the number of new crossings between e and f introduced during the construction of the planarization. If $\mathcal{D}$ represents a drawing on M_g (in the plane with $h = 2g+1$ crosscaps), we say that $\mathcal{D}$ is *orientable*. This is equivalent with every cycle passing through the crosscaps an even number of times.

We will use the first two of the following three lemmata by Schaefer and Štefankovič [30].

▶ **Lemma 3** ([30, Lemma 5]). *Let G be a graph that has an independently even drawing $\mathcal{D}$ on a surface S and let F be a forest in G. Let $h = 2g+1$ if $S = M_g$ and $h = g$ if $S = N_g$. Then G has a drawing $\mathcal{E}$ in the plane with h crosscaps, such that*

1) *for every pair of independent edges e, f the number $\mathrm{cr}^*_{\mathcal{E}}(e,f)$ is even, and*

2) *every edge f of F passes through each crosscap an even number of times; that is, $y_f^{\mathcal{E}} = 0$.*

We will be using Lemma 3 when G is connected and F is a spanning tree of G.

▶ **Lemma 4** ([30, Lemma 3]). *Let G be a graph that has an orientable drawing $\mathcal{D}$ in the plane with finitely many crosscaps such that for every pair of independent edges e, f the number $\mathrm{cr}^*_{\mathcal{D}}(e,f)$ is even. Let d be the dimension of the vector space generated by the set $\{y_e^{\mathcal{D}};\, e \in E(G)\}$. Then G has an independently even drawing on $M_{\lfloor d/2 \rfloor}$.*

▶ **Lemma 5** ([30, Lemma 4]). *Let G be a graph that has a drawing in the plane with finitely many crosscaps such that for every pair of independent edges e, f the number $\mathrm{cr}^*_{\mathcal{D}}(e,f)$ is even. Let d be the dimension of the vector space generated by the set $\{y_e^{\mathcal{D}};\, e \in E(G)\}$. Then G has an independently even drawing on a surface of Euler genus d.*

3.2 Bounding $\mathbb{Z}_2$-genus by matrix rank

In Proposition 10 we will strengthen Lemma 4 and Lemma 5. An immediate corollary of Proposition 10 (Corollary 11 below) can be thought of as a $\mathbb{Z}_2$-variant of a result of Mohar [21, Theorem 3.1]. Roughly speaking, Proposition 10 says that we can upper bound the $\mathbb{Z}_2$-genus and Euler $\mathbb{Z}_2$-genus of a graph G in terms of the rank of a symmetric matrix A encoding the parity of crossings between independent edges. The entries in A representing the parity of crossings between adjacent edges, and in the case of the Euler $\mathbb{Z}_2$-genus also diagonal elements, can be chosen arbitrarily. The choice of such undetermined entries minimizing the rank of A will play a crucial role in the proof of Theorem 2.

We use the theory of symmetric matrices over the two-element field $\mathbb{F}_2$, developed by Albert [1]. Our goal will be to express a symmetric $n \times n$ matrix A over $\mathbb{F}_2$ as a Gram matrix of n vectors spanning a vector space of minimum possible dimension. This is equivalent to finding an $m \times n$ matrix B of minimum rank such that $A = B^\top B$.

A symmetric matrix over $\mathbb{F}_2$ is *alternate* if its diagonal contains only 0-entries[1]. Two square matrices A and B are *congruent* if there exists an invertible matrix C such that $B = C^\top AC$. We use the following two results by Albert [1], which hold over an arbitrary field.

▶ **Lemma 6** ([1, Theorem 3]). *The rank of an alternate matrix is even.*

▶ **Lemma 7** ([1, Theorem 6]). *Every non-alternate symmetric matrix is congruent to a diagonal matrix.*

MacWilliams [19] gave a concise exposition of the following result of Albert [1].

▶ **Lemma 8** ([19, Theorem 1]). *An invertible symmetric matrix A over $\mathbb{F}_2$ can be factored as $B^\top B$ for some square matrix B if and only if A is not alternate.*

We need to extend the factorization from Lemma 8 to alternate and to non-invertible matrices. In the case of non-alternate matrices we again obtain their rank factorization. We use Lemma 7 to achieve this.

▶ **Lemma 9.** *Let A be a symmetric $n \times n$ matrix over $\mathbb{F}_2$ and let r be the rank of A. If A is non-alternate, then there is an $r \times n$ matrix B of rank r such that $A = B^\top B$. If A is alternate, then there is an $(r+1) \times n$ matrix B of rank r or $r+1$ such that $A = B^\top B$.*

Proof. If A is not alternate, let $A' = A$; otherwise let $A' = (a'_{ij})$ be a symmetric matrix obtained from A by adding a single row and single column, as the first row and the first column of A', with $a'_{11} = 1$ and $a'_{1i} = a'_{i1} = 0$ for $i > 1$. Let r' be the rank of A'. Clearly, if A is alternate then $r' = r + 1$.

By Lemma 7, there are an invertible matrix C and a diagonal matrix D of rank r' such that $A' = C^\top DC$. Since every element of $\mathbb{F}_2$ is a square of itself, we have $D = D^\top D$. Let E be the $r' \times n$ matrix obtained from D by removing all the zero rows from D. Then $D = E^\top E$ is a rank factorization of D, and hence $A' = C^\top E^\top EC = (EC)^\top EC$ is a rank factorization of A'. If A is not alternate, we choose B as EC.

If A is alternate, we obtain B from EC by deleting the first column. By the definition of A', we have $B^\top B = A$. Clearly, we have $r \leq \mathrm{rank}(B) \leq \mathrm{rank}(EC) = r' = r + 1$. ◀

Let $\mathcal{D}$ be a drawing of a graph G in the plane and let $E(G) = \{e_1, \ldots, e_m\}$. We say that a symmetric $m \times m$ matrix $A = (a_{ij})$ over $\mathbb{F}_2$ *represents* $\mathcal{D}$ if for every independent pair of edges e_i, e_j we have $a_{ij} = \mathrm{cr}_{\mathcal{D}}(e_i, e_j) \bmod 2$. In particular, if G has a vertex of degree at least 2 then a matrix representing $\mathcal{D}$ is not unique.

▶ **Proposition 10.** *Let $\mathcal{D}$ be a drawing of a graph G in the plane. If a matrix A represents $\mathcal{D}$ then $\mathrm{eg}_0(G) \leq \mathrm{rank}(A)$. If additionally A has only zeros on the diagonal then $\mathrm{g}_0(G) \leq \mathrm{rank}(A)/2$.*

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[1] Over an arbitrary field, A is an alternate matrix if $A^\top = -A$ and all the diagonal entries of A are 0. The diagonal condition is redundant for the fields of characteristic other than 2. Alternate matrices are precisely coordinate matrices of *alternating* bilinear forms.

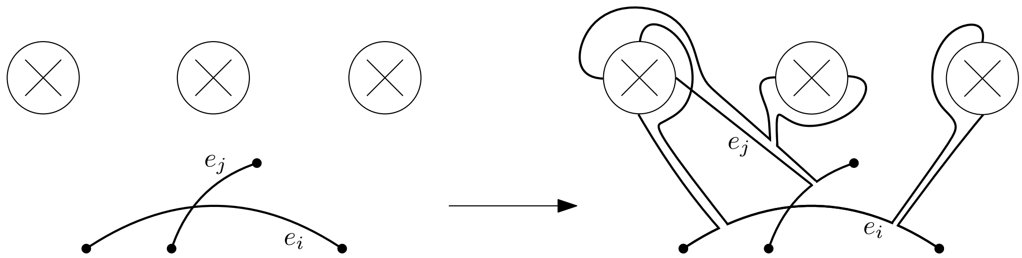

Figure 4 Pulling the edges e_i and e_j with the crosscap vectors $y_{e_i}^\top = (1,0,1)$ and $y_{e_j}^\top = (1,1,0)$, respectively, over crosscaps.

Proof. Let r be the rank of A. If $r = 0$, then $\mathcal{D}$ is independently even, and hence $\text{eg}_0(G) = \text{g}_0(G) = 0$. Now assume that $r > 0$. Let $A = B^\top B$ be the factorization from Lemma 9. The matrix B is an $h \times m$ matrix of rank r or $r+1$, and $r \le h \le r+1$. Moreover, if $h = r+1$, A is an alternate matrix. Write B as $\begin{pmatrix} y_{e_1} & \dots & y_{e_m} \end{pmatrix}$. We will do the following. For every $i \in [m]$, we interpret $y_{e_i} \in \mathbb{Z}_2^h$ as a crosscap vector of the edge e_i of G. Then we construct a drawing $\mathcal{D}_0$ of G in the plane with h crosscaps in which $\text{cr}^*_{\mathcal{D}_0}(e,f)$ is even for every pair of independent edges e,f, and such that $y_{e_i}^{\mathcal{D}_0} = y_{e_i}$ for every $i \in [m]$.

Now we describe the construction of $\mathcal{D}_0$ in more detail. In the complement of $\mathcal{D}$ in the plane we introduce h crosscaps. For every $i \in [m]$ and $j \in [h]$, if $(y_{e_i})_j = 1$, we pull e_i over the jth crosscap (in an arbitrary order). Since every edge is pulled over each crosscap at most once, we can easily avoid creating self-crossings of the edges. See [30, Fig. 1] or Figure 4 for an illustration. Let $\mathcal{D}_0$ be the resulting drawing in the plane with h crosscaps. For every $e_j \in E(G)$, the parity of $\text{cr}^*_{\mathcal{D}_0}(e_i,e_j)$ differs from the parity of $\text{cr}_{\mathcal{D}}(e_i,e_j)$ if and only if $y_{e_j}^\top y_{e_i}$ is odd. By the definition of A, if e_i and e_j are independent, the parity of $\text{cr}_{\mathcal{D}}(e_i,e_j)$ is the same as the parity of $y_{e_j}^\top y_{e_i}$, and so $\text{cr}^*_{\mathcal{D}_0}(e_i,e_j)$ is even as required.

The drawing $\mathcal{D}_0$ represents an independently even drawing on N_h. If $h = r$, the first part of the proposition follows. If $h = r+1$ then A is alternate, and Lemma 6 implies that r is even. The decomposition $A = B^\top B$ now also implies that $y_{e_i}^\top y_{e_i} \bmod 2 = 0$ for every $i \in [m]$, and hence the drawing $\mathcal{D}_0$ is orientable. Therefore, by Lemma 4 we have $\text{g}_0(G) \le \lfloor (r+1)/2 \rfloor = r/2$. Since $\text{eg}_0(G) \le 2\text{g}_0(G)$, we also get $\text{eg}_0(G) \le r$. ◀

An almost immediate corollary of Proposition 10 is the following.

▶ **Corollary 11.** *We have* $\text{eg}_0(G) = \min_{A,\mathcal{D}} \text{rank}(A)$*, where we minimize over symmetric matrices A representing a drawing $\mathcal{D}$ of G in the plane, and* $\text{g}_0(G) = \min_{A,\mathcal{D}} \text{rank}(A)/2$*, where we minimize over alternate matrices A representing a drawing $\mathcal{D}$ of G in the plane.*

4 Minimum rank of partial symmetric matrices

In this section we prove linear-algebraic results that we use to establish Theorem 1 and Theorem 2. We write I_n and J_n for the $n \times n$ identity matrix and all-one matrix, respectively.

It is a basic fact that the matrix rank is subadditive over an arbitrary field, that is, for any two matrices A_1 and A_2 of the same dimensions we have

$$\text{rank}(A_1 + A_2) \le \text{rank}(A_1) + \text{rank}(A_2). \quad (2)$$

4.1 Tournament matrices

All matrices in this subsection are $\{0,1\}$-matrices and all matrix computations are performed over $\mathbb{F}_2$. An $n \times n$ matrix $A = (a_{ij})$ is a *tournament matrix* if $a_{ij} = a_{ji} + 1$ whenever $i \neq j$.

The aim of this subsection is to extend de Caen's [10] lower bound on the rank of tournament matrices to certain block matrices. This extension lies at the heart of the proof of Theorem 1.

De Caen [10] proved that every $n \times n$ tournament matrix A satisfies

$$\operatorname{rank}(A) \geq \left\lceil \frac{n-1}{2} \right\rceil, \tag{3}$$

which can be seen as follows. We have $A + A^\top = J_n + I_n$, and (2) implies that $\operatorname{rank}(I_n + J_n) \geq n-1$. Using (2) again, we get $n - 1 \leq \operatorname{rank}(A) + \operatorname{rank}(A^\top) = 2 \cdot \operatorname{rank}(A)$.

▶ **Lemma 12.** *Let $m, n \geq 2$. Let $A = (A_{ij})$ be an $m \times m$ block matrix, where each block A_{ij} is an $n \times n$ matrix. Let B be an $n \times n$ tournament matrix. Assume that A is symmetric and that for each off-diagonal block A_{ij}, $i \neq j$, one of the matrices $A_{ij} + B$ or $A_{ij} + B + J_n$ is a diagonal $n \times n$ matrix. Then* $\operatorname{rank}(A) \geq \left\lceil \frac{(m-1)(n-1)}{2} \right\rceil - (m-2)$.

4.2 Block symmetric matrices

In this section, we prove minimum rank formulas for certain partial block symmetric matrices that play an important role in the proof of Theorem 2. The study of the rank of partial block matrices was initiated by Cohen et al. [7], Davis [9], and Woerdeman [36]. We adapt previous results to the setting of symmetric matrices.

Let $A(X) = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & X \end{pmatrix}$ be a block matrix over an arbitrary field in which the block X is treated as a variable. Woerdeman [36] and Davis [9] proved that

$$\min_X \operatorname{rank}(A(X)) = \operatorname{rank}\begin{pmatrix} A_{11} & A_{12} \end{pmatrix} + \operatorname{rank}\begin{pmatrix} A_{11} \\ A_{12} \end{pmatrix} - \operatorname{rank}(A_{11}). \tag{4}$$

The following lemma shows that for symmetric $A(X)$, the minimum in (4) is achieved for a symmetric matrix X.

▶ **Lemma 13.** *Let $A_{21} = A_{12}^\top$ and let A_{11} be symmetric. Then*

$$\min_X \operatorname{rank}(A(X)) = 2 \cdot \operatorname{rank}\begin{pmatrix} A_{11} & A_{12} \end{pmatrix} - \operatorname{rank}(A_{11}),$$

where we minimize over symmetric X.

Let $A(X_2, X_3) = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & X_2 & A_{23} \\ A_{31} & A_{32} & X_3 \end{pmatrix}$ be a block matrix over an arbitrary field in which the blocks X_2 and X_3 are treated as variables. For matrices over fields of characteristic different from 2, Cohen et al. [7], see also [34], proved that

$$\min_{X_2, X_3} \operatorname{rank}(A(X_2, X_3)) = \operatorname{rank}\begin{pmatrix} A_{11} & A_{12} & A_{13} \end{pmatrix} + \operatorname{rank}\begin{pmatrix} A_{11} \\ A_{21} \\ A_{31} \end{pmatrix} \tag{5}$$

$$+ \min\left\{ \operatorname{rank}\begin{pmatrix} A_{11} & A_{12} \\ A_{31} & A_{32} \end{pmatrix} - \left(\operatorname{rank}\begin{pmatrix} A_{11} & A_{12} \end{pmatrix} + \operatorname{rank}\begin{pmatrix} A_{11} \\ A_{31} \end{pmatrix} \right), \right.$$

$$\left. \operatorname{rank}\begin{pmatrix} A_{11} & A_{13} \\ A_{21} & A_{23} \end{pmatrix} - \left(\operatorname{rank}\begin{pmatrix} A_{11} & A_{13} \end{pmatrix} + \operatorname{rank}\begin{pmatrix} A_{11} \\ A_{21} \end{pmatrix} \right) \right\},$$

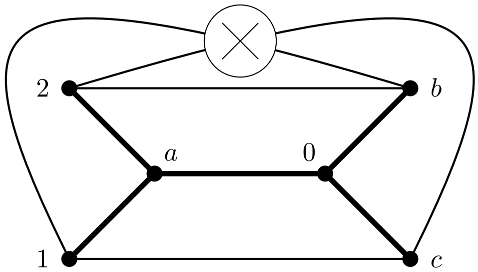

Figure 5 An embedding of $K_{3,3}$ in the plane with a single crosscap. The edges of the spanning tree T are thickened.

In the following lemma, we prove an upper bound on $\min_{X_2,X_3} \operatorname{rank}(A(X_2, X_3))$, which is equal to the right-hand side of (5), if we restrict ourselves to symmetric matrices $A(X_2, X_3)$. The lemma is valid for the symmetric matrices over an arbitrary field.

▶ **Lemma 14.** *Let $A_{21} = A_{21}^\top, A_{31} = A_{13}^\top, A_{32} = A_{23}^\top$, and let A_{11} be symmetric. Then*

$$\min_{X_2,X_3} \operatorname{rank}(A(X_2, X_3)) \leq 2 \cdot \operatorname{rank} \begin{pmatrix} A_{11} & A_{12} & A_{13} \end{pmatrix} + \operatorname{rank} \begin{pmatrix} A_{11} & A_{12} \\ A_{31} & A_{32} \end{pmatrix} - \left(\operatorname{rank} \begin{pmatrix} A_{11} & A_{12} \end{pmatrix} + \operatorname{rank} \begin{pmatrix} A_{11} & A_{13} \end{pmatrix}\right)$$

where we minimize over symmetric matrices X_2 and X_3.

5 Estimating the $\mathbb{Z}_2$-genus and the Euler $\mathbb{Z}_2$-genus of $K_{m,n}$

We prove Theorem 1, whose proof is based on our previous result [15, Lemma 17], which we present next. All matrices and vectors in this subsection are $\{0, 1\}$-matrices and all matrix and vector computations are performed over $\mathbb{F}_2$.

In 1976, Kleitman [18] proved that every drawing of $K_{3,3}$ in the plane contains an odd number of unordered pairs of independent edges crossings an odd number of times. Let $\{a, b, c\}$ and $\{0, 1, 2\}$ be the two maximal independent sets in $K_{3,3}$ and let T be the spanning tree of $K_{3,3}$ containing all the edges incident to a and 0. Let $\mathcal{D}$ be a drawing of $K_{3,3}$ in the plane with finitely many crosscaps in which $\operatorname{cr}^*_{\mathcal{D}}(e, f)$ is even for every pair of independent edges e, f, and $y_e = 0$ for every $e \in E(T)$; see Figure 5 for an illustration. The result of Kleitman implies the following lemma, restating [15, Lemma 17].

▶ **Lemma 15.** *In the drawing $\mathcal{D}$, $y_{b1}^\top y_{c2} + y_{c1}^\top y_{b2} = 1$.*

Proof. Let $\mathcal{D}'$ be the planarization of $\mathcal{D}$. By (1), $\operatorname{cr}_{\mathcal{D}'}(e, f) = y_e^\top y_f$ for every pair of independent edges e and f in $K_{3,3}$, since $\mathcal{D}$ is independently even. Using Kleitman's result, $1 = \sum_{e,f} \operatorname{cr}_{\mathcal{D}'}(e, f) = \sum_{e,f} y_e^\top y_f$, where we sum over unordered independent pairs. Hence, $\sum_{e,f} y_e^\top y_f = y_{b1}^\top y_{c2} + y_{c1}^\top y_{b2}$ concludes the proof. ◀

Proof of Theorem 1. We denote the vertices of $K_{m,n}$ in one part by $u_0, \dots, u_{m-1}$ and in the other part by $v_0, \dots, v_{n-1}$.

Let $\mathcal{D}$ be the combinatorial representation of an independently even drawing of $K_{m,n}$ on a surface S in the plane with finitely many crosscaps (see Section 3.1). Let $y_e = y_e^{\mathcal{D}}$ be the crosscap vector of $e \in E(K_{m,n})$ associated with $\mathcal{D}$. For $i_1, i_2 \in \{1, \dots, m-1\} = [m-1]$, let $A_{i_1 i_2} = (a_{j_1 j_2})$ be the $(n-1) \times (n-1)$ matrix with entries $a_{j_1 j_2} = y_{i_1 j_1}^\top y_{i_2 j_2}$. Let $A = (A_{i_1 i_2})$ be the $(m-1) \times (m-1)$ block matrix composed of the previously defined $A_{i_1 i_2}$'s. By Lemma 3, we assume that $y_e = 0$ for $e \in E' = \{u_0v_0, \dots, u_0v_{n-1}, v_0u_1, \dots, v_0u_{m-1}\}$. Hence, we can let $A' = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$ be a matrix representing $\mathcal{D}$, where the rows and columns of the three all-zero blocks correspond to the edges in E'. For every i_1, i_2, j_1 and j_2, where $i_1 \neq i_2$ and $j_1 \neq j_2$ we then apply Lemma 15 to the drawing of $K_{3,3}$ induced by the vertices $u_0, v_0, u_{i_1}, u_{i_2}, v_{j_1}, v_{j_1}$ in $\mathcal{D}$ and obtain that $a_{j_1j_2} + a_{j_2j_1} = 1$. In other words, $A_{i_1i_2}$ is a tournament matrix. We show that either $A_{i_1i_2} = B + D_{i_1i_2}$ or $A_{i_1i_2} = B + J_{n-1} + D_{i_1i_2}$, where B is a fixed tournament matrix and $D_{i_1i_2}$ is a diagonal matrix.

If the previous claim holds then Lemma 12 applies to A. Thus, $\mathrm{rank}(A') = \mathrm{rank}(A) \geq \left\lceil \frac{(m-2)(n-2)}{2} \right\rceil - (m-3)$. By Corollary 11 or just by observing that $\mathrm{rank}(A')$ is upper bounded by the dimension of the space generated by the crosscap vectors associated with $\mathcal{D}$, $\mathrm{eg}_0(K_{m,n}) \geq \left\lceil \frac{(m-2)(n-2)}{2} \right\rceil - (m-3)$ as desired. Similarly, $2 \cdot \mathrm{g}_0(K_{m,n}) \geq \left\lceil \frac{(m-2)(n-2)}{2} \right\rceil - (m-3)$ and the claimed lower bound for $\mathrm{g}_0(K_{m,n})$ follows as well.

It remains to prove the claim. To this end we apply the argument that was used to prove [15, Lemma 17]. In the drawing of $K_{3,3}$ induced by the vertices $u_0, v_0, u_{i_1}, u_{i_2}, v_{j_1}, v_{j_1}$ in $\mathcal{D}$ we locally deform $\mathcal{D}$ in a close neighborhood of u_0, so that the edges $u_0v_0, u_0v_{j_1}$ and $u_0v_{j_2}$ cross one another an even number of times, while keeping $\mathcal{D}$ independently even. It is easy to see that this is indeed possible. Similarly, we adjust the drawing in a close neighborhood of v_0, so that the edges $v_0u_0, v_0u_{i_1}$ and $v_0u_{i_2}$ cross one another an even number of times. Let $\mathcal{D}'$ be the resulting modification of $\mathcal{D}$.

We will prove below that

(*) In the block A_{i_1,i_2}, for $j_1 \neq j_2$, the value $a_{j_1,j_2} = 1$ if and only if up to the choice of orientation the edges $u_0v_0, u_0v_{j_1}, u_0v_{j_2}$ and $v_0u_0, v_0u_{i_1}, v_0u_{i_2}$ appear in the rotation at u_0 and v_0, respectively, in this order clockwise.

Hence, suppose that (*) holds and that $v_0u_0, v_0u_{i_1}, v_0u_{i_2}$ appear in the rotation at v_0 in $\mathcal{D}'$ in this order clockwise. For $i_1', i_2' \in [m-1]$, $i_1' \neq i_2'$, we adjust the drawing in a close neighborhood of v_0, so that the edges $v_0u_0, v_0u_{i_1'}$ and $v_0u_{i_2'}$ cross one another an even number of times. Let $\mathcal{D}''$ be the resulting drawing. By (*), $A_{i_1'i_2'} = A_{i_1i_2} + D_{i_1'i_2'}$, where $D_{i_1i_2}$ is a diagonal matrix, if $v_0u_0, v_0u_{i_1'}, v_0u_{i_2'}$ in $\mathcal{D}''$ appear in the rotation at v_0 in this order clockwise; and $A_{i_1'i_2'} = A_{i_1i_2} + D_{i_1'i_2'} + J_{n-1}$, if $v_0u_0, v_0u_{i_1'}, v_0u_{i_2'}$ appear in the rotation at v_0 in $\mathcal{D}''$ in this order counterclockwise. It remains to prove (*).

Let $\gamma_{i,j}$ be the closed curve representing the cycle traversing vertices u_0, v_0, u_i and v_j in $\mathcal{D}$. The condition that characterizes when $a_{j_1j_2} = 1$, for $j_1 \neq j_2$, follows by considering a slightly perturbed drawing of γ_{i_1,j_1} and γ_{i_2,j_2}, in which all their intersections become proper edge crossings. Note that $a_{j_1j_2} = 1$ if and only if $u_{i_1}v_{j_i}$ and $u_{i_2}v_{j_2}$ have an odd number of intersections at crosscaps. Furthermore, γ_{i_1,j_1} and γ_{i_2,j_2} must have an even number of intersections in total. Therefore as $\mathcal{D}$ is an independently even drawing, $a_{j_1j_2} = 1$ if and only if in $\mathcal{D}'$ the edge u_0v_0 is a transversal intersection of γ_{i_1,j_1} and γ_{i_2,j_2}; in other words, up to the choice of orientation $u_0v_0, u_0v_{j_1}, u_0v_{j_2}$ and $v_0u_0, v_0u_{i_1}, v_0u_{i_2}$ appear in the rotation at u_0 in this order clockwise. ◀

6 Amalgamations

All matrices and vectors in this subsection are $\{0,1\}$-matrices and all matrix and vector computations are performed over $\mathbb{F}_2$.

6.1 1-amalgamations

In order to ease up the readability, as a warm-up we first reprove the result of Schaefer and Štefankovič for the Euler genus. The proof of our result for 2-amalgamations follows the same blueprint, but the argument gets slightly more technical.

▶ **Theorem 16** ([30]). *Let G_1 and G_2 be graphs. Let $G = \amalg_v(G_1, G_2)$. Then* $\mathrm{eg}_0(G_1) + \mathrm{eg}_0(G_2) = \mathrm{eg}_0(G)$.

Proof. The Euler $\mathbb{Z}_2$-genus of a graph is the sum of the Euler $\mathbb{Z}_2$-genera of its connected components [30, Lemma 7]. Thus, we assume that both G_1 and G_2 are connected.

We start the argument similarly as in [30] by choosing an appropriate spanning tree T in G and fixing an independently even drawing of G on N_g, in which each edge in $E(T)$ passes an even number of times through each crosscap. Nevertheless, the rest of the proof differs considerably, one of the key differences being the use of Proposition 10 rather than Lemma 5 to bound the Euler $\mathbb{Z}_2$-genus of involved graphs.

The following claim is rather easy to prove.

▷ Claim 17. We have

$$\mathrm{eg}_0(G) \le \mathrm{eg}_0(G_1) + \mathrm{eg}_0(G_2). \tag{6}$$

It remains to prove the opposite inequality. We first choose a spanning tree T of G with the following property. Recall that v is a fixed cut vertex. For every $e \in E(G) \setminus E(T)$ it holds that if $v \notin e$ then the unique cycle in $T \cup e$ does not pass through v. The desired spanning tree T is obtained as the exploration tree of a Depth-First-Search in G starting at v. We consider an independently even drawing of G on a surface S witnessing its Euler genus. By Lemma 3, we obtain a drawing $\mathcal{D}$ of G in the plane with finitely many crosscaps in which $\mathrm{cr}^*_{\mathcal{D}}(e, f)$ is even for every pair of independent edges e, f, and every edge of T passes through each crosscap an even number of times. In the following we will write y_e for $y_e^{\mathcal{D}}$.

First, a few words on the strategy of the rest of the proof. Let $B' = \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix}$ be a matrix representing the planarization of $\mathcal{D}$, where the rows and columns of the three all-zero blocks correspond to the edges in T. For a pair of edges e and f in G this parity is given by $y_e^\top y_f$. By introducing an appropriate block structure on B, and using Lemma 13 we show that the rank of B can be lower bounded by $\mathrm{eg}_0(G_1) + \mathrm{eg}_0(G_2)$. This will conclude the proof since the rank of B is easily upper bounded by $\mathrm{eg}_0(G)$.

Let E_1 and E_2 be the set of edges in $E(G_1) \setminus E(T)$ and $E(G_2) \setminus E(T)$, respectively, that are not incident to v. Let F_1 and F_2 be the set of edges in $E(G_1) \setminus E(T)$ and $E(G_2) \setminus E(T)$, respectively, that are incident to v.

Let $\alpha, \beta \in \{E_1, E_2, F_1, F_2\}$. Let $\alpha = \{e_1, \ldots, e_{|\alpha|}\}$. Let $\beta = \{e'_1, \ldots, e'_{|\beta|}\}$. Let $A_{\alpha,\beta} = (a_{ij})$ be the $|\alpha| \times |\beta|$ matrix over $\mathbb{Z}_2$ such that $a_{ij} = y_{e_i}^\top y_{e'_j}$. Let $B = (B_{ij})$ be a 4×4 block matrix such that $B_{ij} = A_{\alpha_i,\alpha_j}$, where $\alpha_1 = E_1, \alpha_2 = F_1, \alpha_3 = F_2$ and $\alpha_4 = E_2$. Clearly, B' represents the planarization of $\mathcal{D}$.

In what follows we collect some properties of B and its submatrices, whose combination establishes the result. The rank of B', and therefore also B, is at most the dimension of the space generated by the crosscap vectors of $\mathcal{D}$. The latter is at most $\mathrm{eg}_0(G)$ since crosscap vectors have $\mathrm{eg}_0(G)$ or $\mathrm{eg}_0(G) + 1$ coordinates depending on whether the original drawing of G is on N_g or M_g, but in the latter we loose one dimension since every crosscap vector has an even number of ones. Hence, we have

$$\mathrm{eg}_0(G) \ge \mathrm{rank}(B). \tag{7}$$

If we arbitrarily change blocks A_{F_1,F_1} and A_{F_2,F_2} of B, B' will still represent the planarization of $\mathcal{D}$. Let $B_1(X) = \begin{pmatrix} A_{E_1,E_1} & A_{E_1,F_1} \\ A_{F_1,E_1} & X \end{pmatrix}$ and $B_2(X) = \begin{pmatrix} X & A_{F_2,E_2} \\ A_{E_2,F_2} & A_{E_2,E_2} \end{pmatrix}$. Then by Proposition 10,

$$\mathrm{eg}_0(G_i) \leq \min_X \begin{pmatrix} \mathrm{rank}(B_i(X)) & 0 \\ 0 & 0 \end{pmatrix} = \min_X \mathrm{rank}(B_i(X)), \tag{8}$$

where we minimize over symmetric matrices X. By Lemma 13,

$$\min_X \mathrm{rank}(B_i(X)) = 2 \cdot \mathrm{rank}\begin{pmatrix} A_{E_i,E_i} & A_{E_i,F_i} \end{pmatrix} - \mathrm{rank}(A_{E_i,E_i}). \tag{9}$$

The last ingredient in the proof is the following claim which holds due to the careful choice of the spanning tree T.

▷ Claim 18. We have

$$2 \cdot \left(\mathrm{rank}\begin{pmatrix} A_{E_1,E_1} & A_{E_1,F_1} \end{pmatrix} + \mathrm{rank}\begin{pmatrix} A_{E_2,E_2} & A_{E_2,F_2} \end{pmatrix}\right) - (\mathrm{rank}(A_{E_1,E_1}) + \mathrm{rank}(A_{E_2,E_2})) \leq \mathrm{rank}(B).$$

We are done by the following chain of inequalities.

$$\begin{aligned} \mathrm{eg}_0(G) \overset{(6)}{\leq} \mathrm{eg}_0(G_1) + \mathrm{eg}_0(G_2) &\overset{(8)}{\leq} \min_X \mathrm{rank}(B_1(X)) + \min_X \mathrm{rank}(B_2(X)) \\ &\overset{(9)}{=} 2 \cdot \left(\mathrm{rank}\begin{pmatrix} A_{E_1,E_1} & A_{E_1,F_1} \end{pmatrix} + \mathrm{rank}\begin{pmatrix} A_{E_2,E_2} & A_{E_2,F_2} \end{pmatrix}\right) \\ &\quad - (\mathrm{rank}(A_{E_1,E_1}) + \mathrm{rank}(A_{E_2,E_2})) \\ &\leq \mathrm{rank}(B) \overset{(7)}{\leq} \mathrm{eg}_0(G) \end{aligned}$$ ◀

6.2 2-amalgamations

Proof of Theorem 2. We will prove the parts a) and b) in parallel. We assume that $G-u-v$ has precisely 2 connected components and that $uv \notin E(G)$ (the general case is treated in the full version). By the block additivity result [30], we assume that none of u and v is a cut vertex of G, and by the additivity over connected components [30, Lemma 7] that G is connected. We follow the line of thought analogous to the proof of Theorem 16.

It is easy to prove the following claim.

▷ Claim 19.

$$\mathrm{g}_0(G) \leq \mathrm{g}_0(G_1) + \mathrm{g}_0(G_2) + 1 \quad \text{and} \quad \mathrm{eg}_0(G) \leq \mathrm{eg}_0(G_1) + \mathrm{eg}_0(G_2) + 2 \tag{10}$$

It remains to prove the opposite inequalities of a) and b). We choose an appropriate spanning tree T of G and fix an independently even drawing of G on N_g, in which each edge of T passes an even number of times through each crosscap. To this end we first choose a spanning tree T' of $G - v$ with the following property. For every $e \in E(G-v) \setminus E(T')$, if $u \notin e$ then the unique cycle in $T' \cup e$ does not pass through u. The desired spanning T' is obtained as the exploration tree of a Depth-First-Search in $G - v$ starting at u. Let u_i be an arbitrary vertex such that $vu_i \in E(G_i)$, for $i = 1, 2$. We obtain T as $T' \cup vu_1$.

We consider an independently even drawing of G on a surface S witnessing its genus (respectively, Euler genus). By Lemma 3, we obtain a drawing $\mathcal{D}$ in the plane with finitely many crosscaps in which $\mathrm{cr}^*_{\mathcal{D}}(e, f)$ is even for every pair of independent edges e, f, and every edge of T passes through each crosscap an even number of times. In the following we will write y_e for $y_e^{\mathcal{D}}$.

First, a few words on the strategy of the rest of the proof. Let $B' = \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix}$ be a matrix representing the planarization of $\mathcal{D}$, where the rows and columns of the three all-zero blocks correspond to the edges in T. For a pair of edges e and f in G this parity is given by $y_e^\top y_f$. By introducing an appropriate block structure on B, and using Lemma 14 we show that the rank of B can be lower bounded by $\mathrm{g}_0(G_1) + \mathrm{g}_0(G_2) - 7/2$ (respectively, $\mathrm{eg}_0(G_1) + \mathrm{eg}_0(G_2) - 3$). This will conclude the proof in this case, since the rank of B is easily upper bounded by $2 \cdot \mathrm{g}_0(G)$ (respectively, $\mathrm{eg}_0(G)$).

Let E_1 and E_2 be the set of edges in $E(G_1) \setminus E(T)$ and $E(G_2) \setminus E(T)$, respectively, that are incident neither to v nor to u. Let F_1 and F_2 be the set of edges in $E(G_1) \setminus E(T)$ and $E(G_2) \setminus E(T)$, respectively, that are incident to u. Let H_1 and H_2 be the set of edges in $E(G_1) \setminus E(T)$ and $E(G_2) \setminus E(T)$, respectively, that are incident to v. Thus, we have that $E(T), E_1, E_2, F_1, F_2, H_1$ and H_2 form a partition of $E(G)$.

Let $\alpha, \beta \in \{E_1, E_2, F_1, F_2, H_1, H_2\}$. Let $\alpha = \{e_1, \dots, e_{|\alpha|}\}$. Let $\beta = \{e'_1, \dots, e'_{|\beta|}\}$. Let $A_{\alpha,\beta} = (a_{ij})$ be the $|\alpha| \times |\beta|$ matrix over $\mathbb{Z}_2$ such that $a_{ij} = y_{e_i}^\top y_{e'_j}$. Let $B = (B_{ij})$ be a 6 by 6 block matrix such that $B_{ij} = A_{\alpha_i,\alpha_j}$, where $\alpha_1 = E_1, \alpha_2 = F_1, \alpha_3 = H_1, \alpha_4 = H_2, \alpha_5 = F_2$ and $\alpha_6 = E_2$. Clearly, B' represents the planarization of $\mathcal{D}$.

In what follows we collect some properties of B and its submatrices, whose combination establishes the result. Since the rank of B', and therefore also B, is at most the dimension of the space generated by the crosscap vectors of $\mathcal{D}$, we have the following

$$2 \cdot \mathrm{g}_0(G) \geq \mathrm{rank}(B) \quad (\text{respectively,} \quad \mathrm{eg}_0(G) \geq \mathrm{rank}(B)). \tag{11}$$

Let $B_1(X_2, X_3) = \begin{pmatrix} A_{E_1,E_1} & A_{E_1,F_1} & A_{E_1,H_1} \\ A_{F_1,E_1} & X_2 & A_{F_1,H_1} \\ A_{H_1,E_1} & A_{H_1,F_1} & X_3 \end{pmatrix}$, and
let $B_2(X_1, X_2) = \begin{pmatrix} X_1 & A_{H_2,F_2} & A_{H_2,E_2} \\ A_{F_2,H_2} & X_2 & A_{F_2,E_2} \\ A_{E_2,H_2} & A_{E_2,F_2} & A_{E_2,E_2} \end{pmatrix}$.

Since changing blocks A_{F_i,F_i} and A_{H_i,H_i} in B, for $i = 1, 2$, except for the diagonal, does not affect the property that B' represents the planarization of $\mathcal{D}$, by Proposition 10,

$$2 \cdot \mathrm{g}_0(G_i) \leq \min_{X_2,X_3} \mathrm{rank}(B_i(X_2, X_3)) + 2 \quad (\text{respectively,} \quad \mathrm{eg}_0(G_i) \leq \min_{X_2,X_3} \mathrm{rank}(B_i(X_2, X_3))), \tag{12}$$

where we minimize over symmetric matrices. We add 2 on the right hand side in the first inequality due to the orientability. In particular, it can happen that X_2 or X_3 minimizing the rank has a 1-entry on the diagonal. If this is the case, in the corresponding independently even drawing, as constructed in the proof of Proposition 10, there exists an edge e incident to u or v such that $y_e^\top y_e = 1$. In order to make $y_e^\top y_e = 0$, we introduce a crosscap, and pull the edge e over it. The introduced crosscap can be shared by the edges incident to v and by the edges incident to u. Therefore adding 2 crosscaps is sufficient. By Lemma 14,

$$\begin{aligned} \min_{X_2,X_3} \mathrm{rank}(B_i(X_2, X_3)) \leq 2 \cdot \mathrm{rank} \begin{pmatrix} A_{E_i,E_i} & A_{E_i,F_i} & A_{E_i,H_i} \end{pmatrix} + \mathrm{rank} \begin{pmatrix} A_{E_i,E_i} & A_{E_i,F_i} \\ A_{H_i,E_i} & A_{H_i,F_i} \end{pmatrix} \\ - \left(\mathrm{rank} \begin{pmatrix} A_{E_i,E_i} & A_{E_i,F_i} \end{pmatrix} + \mathrm{rank} \begin{pmatrix} A_{E_i,E_i} & A_{E_i,H_i} \end{pmatrix} \right) \end{aligned} \tag{13}$$

The inequality (13) implies the last ingredient in the proof which is stated next. The claim holds due to the careful choice of the spanning tree T.

▷ Claim 20. $\min_{X_2,X_3} \operatorname{rank}(B_1(X_2,X_3)) + \min_{X_1,X_2} \operatorname{rank}(B_2(X_1,X_2)) \le \operatorname{rank}(B)+3$, where we minimize over symmetric matrices.

We are done by the following two chains of (in)equalities.

$$\begin{aligned}
-2 + 2\cdot \mathrm{g}_0(G) &\overset{(10)}{\le} 2\cdot \mathrm{g}_0(G_1) + 2\cdot \mathrm{g}_0(G_2)\\
&\overset{(12)}{\le} \min_{X_2,X_3} \operatorname{rank}(B_1(X_2,X_3)) + \min_{X_1,X_2} \operatorname{rank}(B_2(X_1,X_2)) + 4\\
&\le \operatorname{rank}(B) + 7\\
&\overset{(11)}{\le} 2\cdot \mathrm{g}_0(G) + 7,
\end{aligned}$$

$$\begin{aligned}
-2 + \mathrm{eg}_0(G) &\overset{(10)}{\le} \mathrm{eg}_0(G_1) + \mathrm{eg}_0(G_2)\\
&\overset{(12)}{\le} \min_{X_2,X_3} \operatorname{rank}(B_1(X_2,X_3)) + \min_{X_1,X_2} \operatorname{rank}(B_2(X_1,X_2))\\
&\le \operatorname{rank}(B) + 3\\
&\overset{(11)}{\le} \mathrm{eg}_0(G) + 3.
\end{aligned}$$ ◀

7 Conclusion

Theorem 1 does not determine the (Euler) $\mathbb{Z}_2$-genus of $K_{m,n}$ precisely for $m \ge 4$, and we find the problem of computing the precise values interesting already for $m = 4$. We also leave as an open problem whether in Theorem 2, the dependence of the upper bounds on k can be removed. Let G be a k-amalgamation of G_1 and G_2 for some $k \ge 3$. On the one hand, the result of Miller [20] and Richter [25] was extended by Archdeacon [2] to k-amalgamations, for $k \ge 3$, with the error term $\mathrm{eg}(G_1) + \mathrm{eg}(G_2) - \mathrm{eg}(G)$ being at most quadratic in k. On the other hand, in a follow-up paper [3] Archdeacon showed that for $k \ge 3$, the genus of a graph is not additive over k-amalgamations, in a very strong sense. In particular, the value of $\mathrm{g}(G_1) + \mathrm{g}(G_2) - \mathrm{g}(G)$ can be arbitrarily large even for $k = 3$. We wonder if the $\mathbb{Z}_2$-genus and the Euler $\mathbb{Z}_2$-genus behave in a similar way.

References

1. A. Adrian Albert. Symmetric and alternate matrices in an arbitrary field, I. *Trans. Amer. Math. Soc.*, 43(3):386–436, 1938. doi:10.2307/1990068.
2. Dan Archdeacon. The nonorientable genus is additive. *J. Graph Theory*, 10(3):363–383, 1986. doi:10.1002/jgt.3190100313.
3. Dan Archdeacon. The orientable genus is nonadditive. *J. Graph Theory*, 10(3):385–401, 1986. doi:10.1002/jgt.3190100314.
4. Joseph Battle, Frank Harary, Yukihiro Kodama, and J. W. T. Youngs. Additivity of the genus of a graph. *Bull. Amer. Math. Soc.*, 68:565–568, 1962. doi:10.1090/S0002-9904-1962-10847-7.
5. André Bouchet. Orientable and nonorientable genus of the complete bipartite graph. *J. Combin. Theory Ser. B*, 24(1):24–33, 1978. doi:10.1016/0095-8956(78)90073-4.
6. G. Cairns and Y. Nikolayevsky. Bounds for generalized thrackles. *Discrete Comput. Geom.*, 23(2):191–206, 2000. doi:10.1007/PL00009495.
7. Nir Cohen, Charles R. Johnson, Leiba Rodman, and Hugo J. Woerdeman. Ranks of completions of partial matrices. In *The Gohberg anniversary collection, Vol. I (Calgary, AB, 1988)*, volume 40 of *Oper. Theory Adv. Appl.*, pages 165–185. Birkhäuser, Basel, 1989.

8 Éric Colin de Verdière, Vojtěch Kaluža, Pavel Paták, Zuzana Patáková, and Martin Tancer. A direct proof of the strong Hanani–Tutte theorem on the projective plane. *J. Graph Algorithms Appl.*, 21(5):939–981, 2017. doi:10.7155/jgaa.00445.

9 Chandler Davis. Completing a matrix so as to minimize the rank. In *Topics in operator theory and interpolation*, volume 29 of *Oper. Theory Adv. Appl.*, pages 87–95. Birkhäuser, Basel, 1988.

10 D. de Caen. The ranks of tournament matrices. *Amer. Math. Monthly*, 98(9):829–831, 1991. doi:10.2307/2324270.

11 R. W. Decker, H. H. Glover, and J. P. Huneke. The genus of the 2-amalgamations of graphs. *J. Graph Theory*, 5(1):95–102, 1981. doi:10.1002/jgt.3190050107.

12 R. W. Decker, H. H. Glover, and J. P. Huneke. Computing the genus of the 2-amalgamations of graphs. *Combinatorica*, 5(4):271–282, 1985. doi:10.1007/BF02579241.

13 R. Fulek and J. Kynčl. Counterexample to an extension of the Hanani–Tutte theorem on the surface of genus 4. Accepted to Combinatorica, 2017. arXiv:1709.00508.

14 Radoslav Fulek and Jan Kynčl. Hanani–Tutte for approximating maps of graphs. In *34th International Symposium on Computational Geometry*, volume 99 of *LIPIcs. Leibniz Int. Proc. Inform.*, pages 39:1–39:15. Schloss Dagstuhl. Leibniz-Zent. Inform., Wadern, 2018. doi:10.4230/LIPIcs.SoCG.2018.39.

15 Radoslav Fulek and Jan Kynčl. The $\mathbb{Z}_2$-genus of Kuratowski minors. In *34th International Symposium on Computational Geometry*, volume 99 of *LIPIcs. Leibniz Int. Proc. Inform.*, pages 40:1–40:14. Schloss Dagstuhl. Leibniz-Zent. Inform., Wadern, 2018. doi:10.4230/LIPIcs.SoCG.2018.40.

16 Jonathan L. Gross and Thomas W. Tucker. *Topological graph theory.* Dover Publications, Inc., Mineola, NY, 2001. Reprint of the 1987 original [Wiley, New York; MR0898434 (88h:05034)] with a new preface and supplementary bibliography.

17 Haim Hanani. Über wesentlich unplättbare Kurven im drei-dimensionalen Raume. *Fundamenta Mathematicae*, 23:135–142, 1934.

18 D. J. Kleitman. A note on the parity of the number of crossings of a graph. *J. Combinatorial Theory Ser. B*, 21(1):88–89, 1976. doi:10.1016/0095-8956(76)90032-0.

19 Jessie MacWilliams. Orthogonal matrices over finite fields. *Amer. Math. Monthly*, 76:152–164, 1969. doi:10.2307/2317262.

20 Gary L. Miller. An additivity theorem for the genus of a graph. *J. Combin. Theory Ser. B*, 43(1):25–47, 1987. doi:10.1016/0095-8956(87)90028-1.

21 Bojan Mohar. An obstruction to embedding graphs in surfaces. *Discrete Math.*, 78(1-2):135–142, 1989. doi:10.1016/0012-365X(89)90170-2.

22 Bojan Mohar and Carsten Thomassen. *Graphs on surfaces.* Johns Hopkins Studies in the Mathematical Sciences. Johns Hopkins University Press, Baltimore, MD, 2001.

23 János Pach and Géza Tóth. Which crossing number is it anyway? *J. Combin. Theory Ser. B*, 80(2):225–246, 2000. doi:10.1006/jctb.2000.1978.

24 Michael J. Pelsmajer, Marcus Schaefer, and Despina Stasi. Strong Hanani-Tutte on the projective plane. *SIAM J. Discrete Math.*, 23(3):1317–1323, 2009. doi:10.1137/08072485X.

25 R. Bruce Richter. On the Euler genus of a 2-connected graph. *J. Combin. Theory Ser. B*, 43(1):60–69, 1987. doi:10.1016/0095-8956(87)90030-X.

26 Gerhard Ringel. Das Geschlecht des vollständigen paaren Graphen. *Abh. Math. Sem. Univ. Hamburg*, 28:139–150, 1965. doi:10.1007/BF02993245.

27 Gerhard Ringel. Der vollständige paare Graph auf nichtorientierbaren Flächen. *J. Reine Angew. Math.*, 220:88–93, 1965. doi:10.1515/crll.1965.220.88.

28 Marcus Schaefer. Hanani-Tutte and related results. In *Geometry—intuitive, discrete, and convex*, volume 24 of *Bolyai Soc. Math. Stud.*, pages 259–299. János Bolyai Math. Soc., Budapest, 2013. doi:10.1007/978-3-642-41498-5_10.

29 Marcus Schaefer. Toward a theory of planarity: Hanani-Tutte and planarity variants. *J. Graph Algorithms Appl.*, 17(4):367–440, 2013. doi:10.7155/jgaa.00298.

30 Marcus Schaefer and Daniel Štefankovič. Block additivity of $\mathbb{Z}_2$-embeddings. In *Graph drawing*, volume 8242 of *Lecture Notes in Comput. Sci.*, pages 185–195. Springer, Cham, 2013. doi:10.1007/978-3-319-03841-4_17.

31 Mikhail Skopenkov. On approximability by embeddings of cycles in the plane. *Topology Appl.*, 134(1):1–22, 2003. doi:10.1016/S0166-8641(03)00069-5.

32 Saul Stahl. Permutation-partition pairs: a combinatorial generalization of graph embeddings. *Trans. Amer. Math. Soc.*, 259(1):129–145, 1980. doi:10.2307/1998149.

33 Saul Stahl and Lowell W. Beineke. Blocks and the nonorientable genus of graphs. *J. Graph Theory*, 1(1):75–78, 1977. doi:10.1002/jgt.3190010114.

34 Yongge Tian. The minimum rank of a 3×3 partial block matrix. *Linear Multilinear Algebra*, 50(2):125–131, 2002. doi:10.1080/03081080290019531.

35 W. T. Tutte. Toward a theory of crossing numbers. *J. Combinatorial Theory*, 8:45–53, 1970. doi:10.1016/S0021-9800(70)80007-2.

36 H. J. Woerdeman. The lower order of lower triangular operators and minimal rank extensions. *Integral Equations Operator Theory*, 10(6):859–879, 1987. doi:10.1007/BF01196124.