On grounded L-graphs and their relatives

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Abstract

We consider the graph classes Grounded-L and Grounded-{L, ⅃} corresponding to graphs that admit an intersection representation by L-shaped curves (or L-shaped and ⅃-shaped curves, respectively), where additionally the topmost points of each curve are assumed to belong to a common horizontal line. We prove that Grounded-L graphs admit an equivalent characterisation in terms of vertex ordering with forbidden patterns.

We also compare these classes to related intersection classes, such as the grounded segment graphs, the monotone L-graphs (a.k.a. max point-tolerance graphs), or the outer-1-string graphs. We give constructions showing that these classes are all distinct and satisfy only trivial or previously known inclusions.

Mathematics Subject Classifications: 05C62, 05C10, 05C75

1 Introduction

An *intersection representation* of a graph $G = (V, E)$ is a map that assigns to every vertex $x \in V$ a set s_x in such a way that two vertices x and y are adjacent if and only if the two corresponding sets s_x and s_y intersect. The graph G is then the *intersection graph* of

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the set system $\{s_x;\ x \in V\}$. Many natural graph classes can be defined as intersection graphs of sets of a special type.

One of the most general classes of this type is the class of *string graphs*, denoted String. A string graph is an intersection graph of *strings*, which are bounded continuous curves in the plane. All the graph classes we consider in this paper are subclasses of the class of string graphs.

A natural way of restricting a string representation is to impose geometric restrictions on the strings we consider. This leads, for instance, to *segment graphs*, which are intersection graphs of straight line segments, or to L-*graphs*, which are intersection graphs of L-shapes, where an L-shape is a union of a vertical segment and a horizontal segment, in which the bottom endpoint of the vertical segment coincides with the left endpoint of the horizontal one. Apart from L-shapes, we shall also consider ⅃-shapes, which are obtained by reflecting an L-shape along a vertical axis.

Apart from restricting the geometry of the strings, one may also restrict a string representation by imposing conditions on the placement of their endpoints. Following the terminology of Cardinal et al. [4], we will say that a representation is *grounded* if all the strings have one endpoint on a common line (called *grounding line*) and the remaining points of the strings are confined to a single open halfplane with respect to the grounding line. We will usually assume that the grounding line is the x-axis, and the strings extend below the line. The endpoint belonging to the grounding line is the *anchor* of the string.

Similarly, a string representation is an *outer* representation, if all the strings are confined to a disk, and each string has one endpoint on the boundary of the disk. The endpoint on the boundary is again called the *anchor* of the string. One may easily see that a graph admits a grounded string representation if and only if it admits an outer-string representation. Such graphs are known as *outer-string* graphs, and we denote their class by Outer-string.

Our first main result, Theorem 1 in Section 2, is a characterisation of the class of grounded L-graphs by vertex orderings avoiding a pair of forbidden patterns. Our next main result, presented in Section 3, is a collection of constructions providing separations between the classes in Figure 1, showing that there are no nontrivial previously unknown inclusions among them.

Let us now formally introduce the graph classes we are interested in, and briefly review some relevant previously known results.

1-string graphs are the graphs that admit a string representation in which any two distinct strings intersect at most once. The class of 1-string graphs is denoted 1-String.

Outer-1-string graphs (denoted Outer-1-string) are the graphs that have a string intersection representation which is simultaneously a 1-string representation and an outer-string representation. Note that not every graph from 1-string $\cap$ Outer-string is necessarily in Outer-1-string, as we shall see in Section 3.

L-*graphs* (L) are the intersection graphs of L-shapes. This type of representation has received significant amount of interest lately. A notable recent result is a theorem of Gonçalves, Isenmann and Pennarun [9] showing that every planar graph is an L-graph. Since it is known that L-graphs are a subclass of segment graphs [12], this result strength-

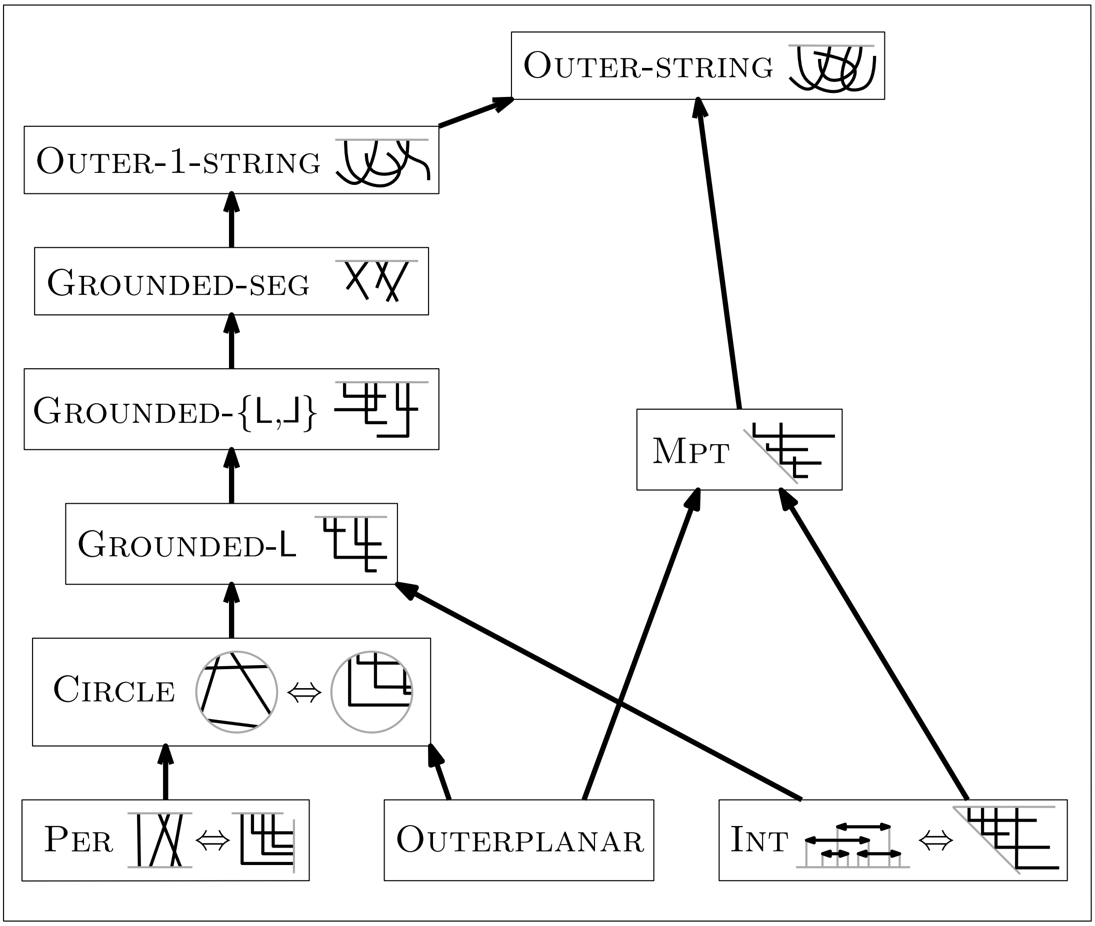

Figure 1: Graph classes considered in this paper. Arrows indicate inclusions. We will see in Section 3 that there are no other inclusions among these classes apart from those implied by the depicted arrows. In particular, the classes are all distinct.

ens an earlier result of Chalopin and Gonçalves [6] showing that all planar graphs are segment graphs.

Max point-tolerance graphs (Mpt), also known as *monotone* L*-graphs*, are the graphs with an L-representation in which all the bends of the L-shapes belong to a common downward-sloping line. This class was independently introduced by Soto and Thraves Caro [15], by Catanzaro et al. [5] and by Ahmed et al. [1]. Apart from the above intersection representation by L-shapes, it admits several other equivalent characterisations. Notably, Mpt graphs can be characterised as graphs that admit a vertex ordering that avoids a certain forbidden pattern [1, 5, 15]. This graph class is known to be a superclass of several important graph classes, such as outerplanar graphs and interval graphs, among others [1, 5, 15].

Grounded segment graphs (Grounded-seg) are the intersection graphs admitting a grounded segment representation. Cardinal et al. [4] proved that these are also precisely the intersection graphs of downward rays in the plane. Note that any grounded segment graph also admits an outer-segment representation, but the converse does not hold, as shown by Cardinal et al. [4]. They also showed that outer-segment graphs form a proper subclass of the class of outer-1-string graphs. This strengthens an earlier result of Cabello and Jejčič [3], who showed that outer-segment graphs are a proper subclass of the class of outer-string graphs.

Grounded L*-graphs* (Grounded-L) are the intersection graphs of grounded L-shapes, that is, L-shapes with top endpoint on the x-axis. This class of graphs was first considered by McGuinness [11], who represented them as intersection graphs of upward-infinite L-shapes. These graphs can also equivalently be represented as intersection graphs of L-shapes inside a disk, with the top endpoint of each L-shape anchored to the boundary of the disk. McGuinness showed that this class is χ-bounded, that is, these graphs have chromatic number bounded from above by a function of their clique number. The χ-boundedness result was later generalized to all outer-string graphs by Rok and Walczak [14].

Grounded $\{\mathsf{L}, \lrcorner\}$*-graphs* (Grounded-$\{\mathsf{L}, \lrcorner\}$) are analogous to grounded L-graphs, but their representation may use both L-shapes and ⅃-shapes. An argument of Middendorf and Pfeiffer [12] shows that Grounded-$\{\mathsf{L}, \lrcorner\}$ is a subclass of Grounded-Seg.

Circle graphs (Circle) are the intersection graphs of chords inside a circle, or equivalently, the intersection graphs of L-shapes drawn inside a circle, so that both endpoints of each L-shape touch the circle. Circle graphs include all outerplanar graphs [16].

Interval graphs (Int) are the intersection graphs of intervals on the real line. Equivalently, we may easily observe that these are exactly the graphs with an intersection representation which is simultaneously an Mpt-representation and a Grounded-L-representation. But note that not every graph from the intersection of Mpt and Grounded-L is an interval graph, as witnessed, for example, by any cycle C_n of length $n \geqslant 4$.

Permutation graphs (Per) are the intersection graphs of segments between a pair of parallel lines, with each segment having one endpoint on each of the two lines. Equivalently, we may observe that these are exactly the graphs admitting an L-representation in which the top endpoints of all the L-shapes are on a common horizontal line and the right endpoints are on a common vertical line.

We will always assume implicitly that the intersection representations we deal with satisfy certain non-degeneracy assumptions. In particular, we will assume that the strings have no self-intersections, that any two strings intersect in at most finitely many points (except for interval representations), and that any intersection of two strings is a proper crossing. In particular, an endpoint of a string does not belong to another string. Moreover, we will assume that every segment in a segment representation is non-degenerate, that is, it has distinct endpoints. This also applies to horizontal and vertical segments forming an L-shape or ⅃-shape. These assumptions imply, in particular, that in any $\{\mathsf{L}, \lrcorner\}$-representation, each intersection is realized as a crossing of a horizontal segment with a vertical one.

Note that in any grounded representation with a horizontal grounding line, the left-to-right ordering of the anchors on the grounding line defines a linear order on the vertex set of the represented graph. We say that this linear order is *induced* by the representation.

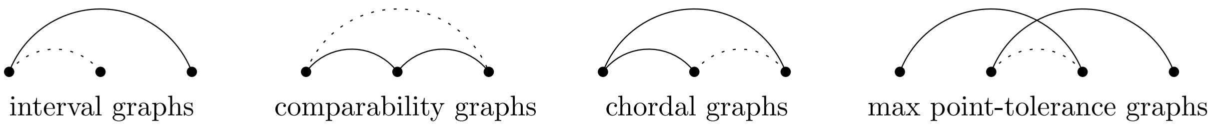

Figure 2: Forbidden order patterns for various graph classes [2, 5, 7]. The solid arcs denote compulsory edges and the dotted arcs are forbidden edges.

Figure 3: The two forbidden ordering patterns for the class Grounded-L.

Similarly, for an Mpt representation, we can define the induced order by following the top-left to bottom-right order of the bends along their common supporting line. Induced vertex orders play an important role both in characterising graphs in a given class and in separating different classes.

2 Vertex orders with forbidden patterns

Our main result is a characterisation of grounded L-graphs as graphs that admit vertex orderings avoiding a pair of four-vertex patterns. Let us begin by formalising the key notions.

An *ordered graph* is a pair $(G, <)$, where $G = (V, E)$ is a graph and $<$ is a linear order on V. A *pattern of order k* is a triple $P = (W, C, F)$ where W is the set $\{1, 2, \ldots, k\}$ while C and F are two disjoint subsets of $\binom{W}{2}$. The set W is the *vertex set* of the pattern P, C is the set of *compulsory edges* of P, and F is the set of *forbidden edges*.

For an ordered graph $(G, <)$ with $G = (V, E)$, we say that $(G, <)$ *contains* a pattern $P = (W, C, F)$ of order k if G contains k distinct vertices $x_1 < x_2 < \cdots < x_k$ such that for every $\{i, j\} \in C$ the vertices x_i and x_j are adjacent in G, while for every $\{i, j\} \in F$, x_i and x_j are non-adjacent in G. If $(G, <)$ does not contain P, we say that it *avoids* P. For simplicity, we will often write an edge $\{i, j\}$ as ij.

Many important graph classes can be characterised in terms of vertex orderings with forbidden patterns, that is, for a class $\mathcal{C}$ there is a pattern $P_{\mathcal{C}}$ such that a graph $G = (V, E)$ is in $\mathcal{C}$ if and only if it admits a linear order $<$ such that $(G, <)$ avoids $P_{\mathcal{C}}$; see Figure 2 for examples of classes with their forbidden patterns. The forbidden pattern characterisation of Mpt was found independently by at least three groups of authors [1, 5, 15].

As our first main result, we show that Grounded-L is characterised by a pair of forbidden patterns.

Theorem 1. *Consider the two patterns* $P_1 = (\{1,2,3,4\}, \{13,24\}, \{12,23\})$ *and* $P_2 = (\{1,2,3,4\}, \{12,14,23\}, \{13\})$*; see Figure 3. A graph* $G = (V, E)$ *is a grounded* L*-graph if and only if it has a vertex ordering that avoids both* P_1 *and* P_2*. In fact, a linear order* $<$ *on* V *avoids the two patterns* P_1 *and* P_2 *if and only if* G *has a grounded* L*-representation which induces the linear order* $<$.

Proof. Suppose first that G has a grounded L-representation. Let $\ell_1, \ell_2, \dots, \ell_n$ be the L-shapes used in the representation, ordered left to right according to the positions of their anchors. Let h_i and v_i denote, respectively, the horizontal and vertical segment of ℓ_i. Let x_i be the vertex represented by ℓ_i. We will show that the vertex ordering $x_1 < x_2 < \dots < x_n$ avoids the two patterns P_1 and P_2.

Assume that $(G, <)$ contains P_1, and let $x_p < x_q < x_r < x_s$ be the four vertices forming a copy of P_1. Since $x_q x_s$ is an edge, the two L-shapes ℓ_q and ℓ_s intersect. Let R be the rectangle whose vertices are the anchors of ℓ_q and ℓ_s, the bend of ℓ_q and the intersection of ℓ_q and ℓ_s. Since neither x_p nor x_r is adjacent to x_q, we see that ℓ_p is completely outside of R, while v_r is inside R. It follows that ℓ_p and ℓ_r are disjoint, and fail to represent the compulsory edge 13 of P_1.

Suppose now that $(G, <)$ contains P_2, and let $x_p < x_q < x_r < x_s$ now be the four vertices forming a copy P_2. Since $x_p x_s$ is an edge, the segment h_p intersects v_s. Moreover, v_q intersects h_p, while v_r does not intersect h_p, and in particular, ℓ_q and ℓ_r fail to represent the compulsory edge 23 of P_2. We conclude that any grounded L-representation of G induces a vertex order that avoids P_1 and P_2.

To prove the converse, assume that G is a graph with a vertex ordering $x_1 < x_2 < \dots < x_n$ which avoids both P_1 and P_2. We will construct a grounded L-representation $\ell_1, \ell_2, \dots, \ell_n$ of G inducing the order $<$, with ℓ_i being the L-shape representing the vertex x_i.

We fix the anchor of ℓ_i to be the point $(i, 0)$ on the horizontal axis. Next, we process the vertices left to right, and for a vertex x_i we define the representing shape ℓ_i, assuming $\ell_1, \ell_2, \dots, \ell_{i-1}$ have already been defined, and assuming further that for any $j < i$ such that $x_j x_i$ is an edge of G, the horizontal segment h_j of ℓ_j reaches to the right of the point $(i, 0)$.

To define ℓ_i, we first describe its vertical segment v_i. Let N_i^- be the set of vertices x_j such that $j < i$ and $x_j x_i \in E$. If N_i^- is empty, choose the vertical segment v_i to be shorter than any of $v_1, \dots, v_{i-1}$. In particular, v_i will not intersect any of the L-shapes constructed in previous steps. If N_i^- is nonempty, let x_p be a vertex from N_i^- chosen so that v_p is as long as possible (and therefore h_p is as low as possible). Then define v_i to be slightly longer than v_p, so that v_i intersects h_p (recall that h_p reaches to the right of $(i, 0)$) but does not intersect any L-shape whose horizontal segment is below h_p. This choice of v_i guarantees that v_i intersects h_j for any $x_j \in N_i^-$.

It remains to define the segment h_i. Let j be the largest index such that $j > i$ and $x_i x_j \in E$. If no such j exists, set $j = i$. The horizontal segment h_i then has length $j - i + \frac{1}{2}$, and in particular, its right endpoint has horizontal coordinate $j + \frac{1}{2}$.

Having defined the L-shapes $\ell_1, \dots, \ell_n$ as above, let us verify that their intersection graph is G. If $x_j x_i$ is an edge of G with $j < i$, then the definitions of v_i and h_j guarantee

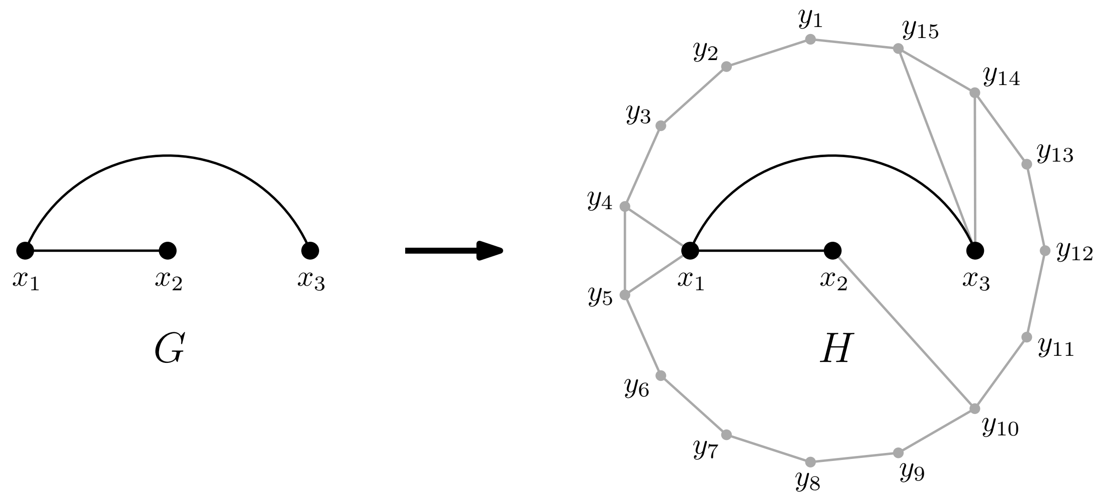

Figure 4: An ordered graph G, and an example of its cycle extension H.

that v_i intersects h_j, and therefore the two L-shapes ℓ_j and ℓ_i intersect.

To prove the converse, suppose for contradiction that for some $j < i$ the two L-shapes ℓ_j and ℓ_i intersect while $x_j x_i$ is not an edge of G. Choose such a pair i, j so that i is the smallest possible. There must be an index $k > i$ such that $x_j x_k$ is an edge of G, otherwise h_j would be too short to intersect v_i. Similarly, there must be an index $m < i$ such that $x_m x_i$ is an edge of G, and v_m is longer than v_j, otherwise v_i would not be long enough to intersect h_j.

We now distinguish two cases depending on the relative position of m and j. If $m < j$, then ℓ_m and ℓ_j are disjoint (recall that v_m is longer than v_j) and hence $x_m x_j$ is not an edge of G. It follows that the four vertices $x_m < x_j < x_i < x_k$ form the pattern P_1, a contradiction. Suppose now that $j < m$. It follows that ℓ_j intersects ℓ_m, and therefore $x_j x_m$ is an edge of G, by the minimality of i. Thus, the four vertices $x_j < x_m < x_i < x_k$ form the pattern P_2, which is again a contradiction. □

3 Separations between classes

Consider again the classes in Figure 1. The inclusions indicated by arrows are either easy to observe or follow from known results that we have pointed out in the introduction. Our goal now is to argue that there are no other inclusions among these classes except those that follow by transitivity from the depicted arrows. In particular, the classes are all distinct.

As our main tool, we will use a lemma which is a slight modification of the 'Cycle Lemma' of Cardinal et al. [4]. The lemma allows us to prescribe the cyclic order of a subset of vertices in an outer-1-string representation of a graph. Let $G = (V_G, E_G)$ be a graph on n vertices $x_1, x_2, \dots, x_n$, and let $<$ be the linear order $x_1 < x_2 < \cdots < x_n$. The *cyclic shift* of $<$ is the linear order $<_s$ defined as $x_n <_s x_1 <_s x_2 <_s \cdots <_s x_{n-1}$. The *reversal* of $<$, denoted $<_r$, is defined as $x_n <_r x_{n-1} <_r \cdots <_r x_1$. We say that two linear

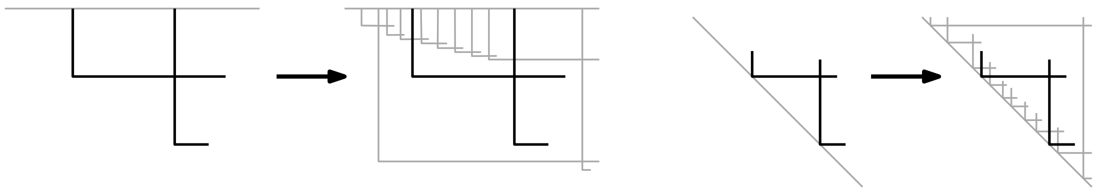

Figure 5: Extending the representation of G into a representation of a cycle extension for grounded L-representations (left) and MPT representations (right).

orders of V are *equivalent* if one can be obtained from the other by a sequence of cyclic shifts and reversals.

A *cycle extension* of the ordered graph $(G, <)$ is an (unordered) graph $H = (V_H, E_H)$ with the following properties (see Figure 4):

- V_H is the disjoint union of the sets $V_G = \{x_1, \ldots, x_n\}$ and $V_C = \{y_1, \ldots, y_{5n}\}$. The vertices V_G induce the graph G (in particular, $E_G \subseteq E_H$), and V_C induce a cycle of length $5n$ with edges $y_1y_2, y_2y_3, \ldots, y_{5n-1}y_{5n}, y_{5n}y_1$.
- For each vertex $x_i \in V_G$, either x_i is adjacent to y_{5i} and has no other neighbors in V_C, or x_i is adjacent to y_{5i-1} and y_{5i} and has no other neighbors in V_C.

For the classes of graphs we consider, an intersection representation of a graph G inducing an order $<$ can always be extended into a representation of a cycle extension of G, without modifying the curves representing G. This is formalised by the next lemma.

Lemma 2. *Given a graph class $\mathcal{C} \in \{$*GROUNDED-L, GROUNDED-{L, ⅃}, GROUNDED-SEG, MPT, OUTER-1-STRING*$\}$, for every $\mathcal{C}$-representation of a graph G inducing an order $<$ on V_G there is a cycle extension H of $(G, <)$ such that a $\mathcal{C}$-representation of H can be constructed by adding the curves representing the vertices of $V_H \setminus V_G$ into the given $\mathcal{C}$-representation of G.*

Proof. Suppose we are given a $\mathcal{C}$-representation of G. It is easy to see that we can add the curves representing the cycle V_C close enough to the grounding line; see Figure 5. Note that for MPT-representations, each original L-shape may have to be intersected by two consecutive L-shapes from the added cycle. In all the other types of representations, each vertex x_i of G will have a unique neighbor y_{5i} among the V_C. □

Recall that two linear orders are equivalent if one can be obtained from the other by a sequence of cyclic shifts and reversals. The key property of cycle extensions of $(G, <)$ is that they restrict the possible vertex orders of the G-part to an order equivalent to $<$, as shown by the next lemma.

Lemma 3. *If $(G, <)$ is an ordered graph with a cycle extension H, then in every grounded 1-string representation of H, the order of the vertices of G induced by the representation is equivalent to the order $<$.*

Proof. The proof follows the same ideas as the proof of the Cycle Lemma of Cardinal et al. [4, Lemma 4].

Suppose $(G,<)$ is an ordered graph with vertex set $V_G = \{x_1 < x_2 < \cdots < x_n\}$ and edge set E_G, and H is its cycle extension, with vertices $V_H = V_G \cup V_C$ as in the definition of cycle extension and $V_C = \{y_1, \ldots, y_{5n}\}$. When working with the indices of the vertices in V_C, we will assume that arithmetic operations are performed modulo $5n$, so $5n+1=1$, etc. Suppose that H has a grounded 1-string representation. We may transform this representation into an outer-1-string representation, while preserving the induced vertex order up to equivalence. Suppose then that an outer-1-string representation of H is given, inside a disk whose boundary is a circle B. Let c_j be the string representing y_j, and let $p_{j,j+1}$ be the intersection point of c_j and c_{j+1}. The subcurve of c_j between the two intersection points $p_{j-1,j}$ and $p_{j,j+1}$ is the *central part* of c_j, denoted $\mathrm{center}(j)$. The part of c_j between the anchor and the first point of $\mathrm{center}(j)$ is the *initial part* of c_j, denoted $\mathrm{start}(j)$. Let p_j be the common endpoint of $\mathrm{start}(j)$ and $\mathrm{center}(j)$. Note that p_j is equal to $p_{j-1,j}$ or to $p_{j,j+1}$. The sequence of curves $\mathrm{center}(1), \mathrm{center}(2), \ldots, \mathrm{center}(5n)$ forms a closed Jordan curve, denoted by C. Note that C contains all the points $p_{k,k+1}$ for $k = 1, \ldots, 5n$. Let R_C be the interior region of C.

Consider now a vertex x_i, represented by a string s_i. Note that s_i can intersect the curve C only at a point of $\mathrm{center}(5i)$ or possibly $\mathrm{center}(5i-1)$. Let R_i be the planar region bounded by the union of the following four curves: $\mathrm{start}(5i-3)$, $\mathrm{start}(5i+2)$, the arc of C between p_{5i-3} and p_{5i+2} that contains $\mathrm{center}(5i-1) \cup \mathrm{center}(5i)$, and the arc of B between the anchors of c_{5i-3} and c_{5i+2} that contains the anchors of c_{5i-1} and c_{5i}.

Note that s_i is the only string among the strings representing V_G that can intersect the boundary of R_i. Note also that the string c_{5i} cannot intersect the boundary of R_k for $k \neq i$, and therefore c_{5i} is contained in $R_i \cup R_C$. Since s_i intersects c_{5i}, and since s_i also cannot cross the boundary of R_k for $k \neq i$, it follows that s_i is also contained in $R_C \cup R_i$, and in particular, the anchor of s_i is in $R_i \cap B$. Therefore, the anchors of $s_1, \ldots, s_n$ appear on B in the order which, up to equivalence, corresponds to the order $<$ on V_G. □

We will now use Lemmas 2 and 3 to construct graphs that have no representation in a given intersection class. Our goal is to show that there are no inclusions missing in Figure 1. The classes Int, Circle, Outerplanar and Per are well studied [2], and simple examples show that there are no inclusions among them other than those depicted in Figure 1.

Catanzaro et al. [5, Observation 6.9] observed that the graph $K_{2,2,2}$ (the octahedron) is a permutation graph not in Mpt, and therefore neither Per nor any superclass of Per is contained in Mpt. Cardinal et al. [4] showed that Grounded-seg is a proper subclass of Outer-1-string. To complete the hierarchy, we only need the following separations.

Theorem 4. *The following properties hold.*

(i) The class Grounded-$\{\mathsf{L}, \mathsf{⅃}\}$ *is not a subclass of* Grounded-L.

(ii) The class Grounded-seg *is not a subclass of* Grounded-$\{\mathsf{L}, \mathsf{⅃}\}$.

(iii) The class Mpt *is not a subclass of* Outer-1-string.

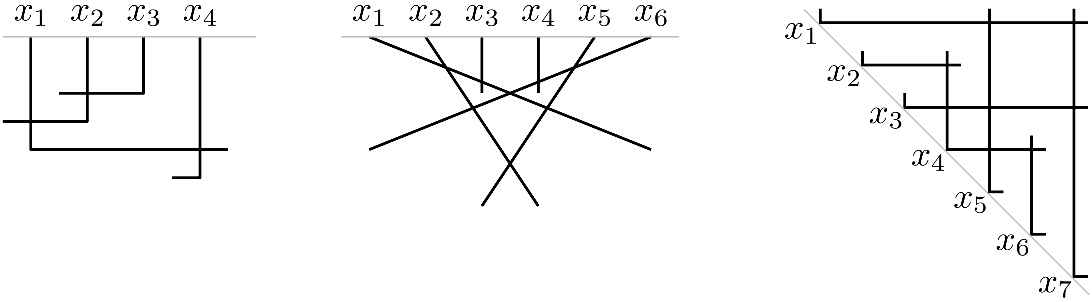

Figure 6: The three intersection representations used to prove Theorem 4. In each case, a representation cannot be replaced by a representation from a smaller class while preserving the induced vertex order. Left: a GROUNDED-{L, ⅃} representation which cannot be replaced by a GROUNDED-L one. Middle: a GROUNDED-SEG representation which cannot be replaced by a GROUNDED-{L, ⅃} one. Right: an MPT representation which cannot be replaced by an OUTER-1-STRING one.

Proof. We first prove part (i) of the theorem. Consider the graph $G = (V, E)$ with $V = \{x_1, x_2, x_3, x_4\}$ and $E = \{x_1x_2, x_2x_3, x_1x_4\}$. Figure 6 (left) shows a grounded {L, ⅃}-representation of G which induces the order $<$ defined as $x_1 < x_2 < x_3 < x_4$ on V. Note that there is no grounded L-representation of G that would induce the vertex order $<$, because $(G, <)$ contains the pattern P_2 of Theorem 1.

Let $(G', <')$ be the ordered graph obtained by putting $(G, <)$ and the mirror image of $(G, <)$ side by side. Formally, $(G', <')$ has vertex set $V' = \{x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4\}$, edge set $E' = \{x_1x_2, x_2x_3, x_1x_4, y_1y_2, y_2y_3, y_1y_4\}$ and vertex order $x_1 <' x_2 <' x_3 <' x_4 <' y_4 <' y_3 <' y_2 <' y_1$. Finally, let $(G'', <'')$ be the ordered graph obtained by placing two disjoint copies of $(G', <')$ side by side. Clearly G'' has a grounded {L, ⅃}-representation which induces the vertex order $<''$. However, G'' has no grounded L-representation inducing a vertex order equivalent with $<''$, since in any vertex order equivalent with $<''$ there are four consecutive vertices forming a copy of P_2.

By Lemma 2, the ordered graph $(G'', <'')$ has a cycle extension H that admits a grounded {L, ⅃}-representation. By Lemma 3, any grounded 1-string representation (and therefore any grounded L-representation) of H induces on V'' an order which is equivalent with $<''$. It follows that H has no grounded L-representation, and therefore GROUNDED-{L, ⅃} is not a subclass of GROUNDED-L, as claimed.

For the other two parts of the theorem, the argument is analogous, the main difference is in the choice of the initial ordered graph $(G, <)$. To prove part (ii), consider the graph G on six vertices whose GROUNDED-SEG representation is in the middle of Figure 6, and let $<$ be the vertex order induced by the depicted representation.

Let us argue that G has no grounded {L, ⅃}-representation inducing the vertex order $<$. For contradiction, suppose that such a representation exists, and let ℓ_i denote the L-shape or ⅃-shape representing x_i in this representation. Let h_i and v_i be the horizontal and vertical segments of ℓ_i, respectively. Assume, without loss of generality, that v_1 is longer than v_6. Since ℓ_1 and ℓ_6 intersect, h_6 must intersect v_1, and ℓ_6 is a ⅃-shape. Since ℓ_2

intersects both ℓ_1 and ℓ_6, v_2 must be longer than v_6, and v_2 intersects h_6. But this means that ℓ_3 must intersect either ℓ_2 or ℓ_6 in order to intersect ℓ_1, a contradiction.

Note that the graph $(G, <)$ is isomorphic to its reversal. Consider the ordered graph $(G', <')$ obtained by placing two copies of $(G, <)$ side by side: note that in any vertex order equivalent to $<'$, G' contains a copy of $(G, <)$, and therefore there is no grounded $\{\mathsf{L}, \mathsf{⅃}\}$-representation of G' inducing a vertex order equivalent to $<'$. We apply Lemmas 2 and 3 to $(G', <')$ and obtain its cycle extension H, which is in Grounded-seg but not in Grounded-$\{\mathsf{L}, \mathsf{⅃}\}$.

To prove part (iii), consider the graph G whose Mpt-representation is depicted in the right part of Figure 6, and let $<$ be the vertex order induced by the representation. We claim that there is no grounded 1-string representation of G inducing the order $<$. For contradiction, suppose that such a representation exists, and let s_i be the string representing the vertex x_i. Additionally, let a_i denote the anchor of s_i, and for a pair of intersecting strings s_i, s_j let p_{ij} denote their intersection.

Assume, without loss of generality, that when we follow s_4 starting at a_4, we encounter p_{24} before we encounter p_{46}. Let C be the closed Jordan curve obtained as the union of the subcurve of s_1 between a_1 and p_{17}, the subcurve of s_7 between p_{17} and p_{37}, the subcurve of s_3 between p_{37} and a_3, and the segment a_1a_3 of the grounding line. Note that s_2 is inside C (except a_2, which lies on C), and both a_4 and s_6 are outside C. Therefore, s_4 must intersect C at least twice: once between a_4 and p_{24}, and once between p_{24} and p_{46}. However, s_4 can only intersect C in the point p_{34}, a contradiction.

To complete the proof, we first observe that G has no grounded 1-string representation inducing a vertex order equivalent with $<$, since such a representation could be trivially transformed into a grounded 1-string representation inducing $<$. We apply Lemmas 2 and 3 to G, to obtain a graph H which is in Mpt but not in Outer-1-string. □

Note that these results imply that Outer-string is a proper superclass of both Mpt and Outer-1-string.

We remark that Mpt is clearly a subclass of 1-string and of Outer-string, but it is not a subclass of Outer-1-string, as we just saw.

4 Concluding remarks

We have seen that the vertex orders induced by grounded L-representations can be characterised by a pair of forbidden patterns. Previously, a characterisation by a single forbidden pattern has been found for vertex orders induced by Mpt representations [1, 5, 15]. Another related result, due to Pach and Tomon [13], provides a characterisation by a single forbidden pattern for the so-called semi-comparability graphs, which include the complements of the intersection graphs of grounded y-monotone curves. It is an open problem whether a characterisation by forbidden patterns can be obtained for other similar grounded intersection classes, such as the class Grounded-$\{\mathsf{L}, \mathsf{⅃}\}$.

Another problem concerns the recognition complexity of the graph classes we considered. Recognition of max point-tolerance graphs is mentioned as a prominent open problem by Ahmed et al. [1], by Catanzaro et al. [5], as well as by Soto and Thraves Caro [15]. For the classes Grounded-L and Grounded-{L, ⅃}, recognition is open as well. On the other hand, the recognition problem for Grounded-seg is known to be $\exists\mathbb{R}$-complete, as shown by Cardinal et al. [4]. In particular, Grounded-seg cannot be characterised by finitely many forbidden vertex order patterns, unless $\exists\mathbb{R}$ is equal to NP.

The characterisation of Grounded-L by forbidden vertex order patterns might conceivably be helpful in designing a polynomial recognition algorithm, but note that even a graph class characterised by a forbidden vertex order pattern may have NP-hard recognition [8], although it is known that recognition is decidable in polynomial time for all classes described by a set of forbidden patterns of order at most three [10].

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