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Excitation Spectrum of Interacting Bosons in the Mean-Field Infinite-Volume Limit

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Abstract. We consider homogeneous Bose gas in a large cubic box with periodic boundary conditions, at zero temperature. We analyze its excitation spectrum in a certain kind of a mean-field infinite-volume limit. We prove that under appropriate conditions the excitation spectrum has the form predicted by the Bogoliubov approximation. Our result can be viewed as an extension of the result of Seiringer (Commun. Math. Phys. 306:565–578, 2011) to large volumes.

1. Introduction and Main Results

Many physical properties of complicated interacting systems can be derived from simple Hamiltonians involving independent (bosonic or fermionic) *quasiparticles* (see [5] for a detailed discussion of this concept) with appropriately chosen dispersion relation (the dependence of the quasiparticle energy on the momentum). One of such systems is the weakly interacting Bose gas at zero temperature. On the heuristic level, the quasiparticle description of the Bose gas can be derived from the Bogoliubov approximation ([2], see also [4]). The main goal of this paper is a rigorous justification of this approximation for a homogeneous system of N interacting bosons in a certain kind of a mean-field large-volume limit.

Let us state the assumptions on the 2-body potential that we will use throughout the paper. Consider a real function $\mathbb{R}^d \ni \mathbf{x} \mapsto v(\mathbf{x})$, with its Fourier transform defined by

$$\hat{v}(\mathbf{p}) := \int_{\mathbb{R}^d} v(\mathbf{x})\, \mathrm{e}^{-\mathrm{i}\mathbf{p}\mathbf{x}}\, \mathrm{d}\mathbf{x}.$$

We assume that $v(\mathbf{x}) = v(-\mathbf{x})$, and that $v \in L^1(\mathbb{R}^d)$ and $\hat{v} \in L^1(\mathbb{R}^d)$. We also suppose that the potential is positive and positive definite, i.e.

$$v(\mathbf{x}) \geq 0, \quad \mathbf{x} \in \mathbb{R}^d, \qquad \hat{v}(\mathbf{p}) \geq 0, \quad \mathbf{p} \in \mathbb{R}^d.$$

We will consider Bose gas in large but finite volume. To do this, following the standard approach, we replace the infinite space $\mathbb{R}^d$ by the torus $\Lambda =]-L/2, L/2]^d$, that is, the d-dimensional cubic box of side length L. We will always assume that $L \geq 1$.

The original potential v is replaced by its *periodized* version

$$v^L(\mathbf{x}) := \frac{1}{L^d} \sum_{\mathbf{p} \in (2\pi/L)\mathbb{Z}^d} \mathrm{e}^{\mathrm{i}\mathbf{p}\mathbf{x}}\, \hat{v}(\mathbf{p}).$$

Here, $\mathbf{p} \in (2\pi/L)\mathbb{Z}^d$ is the discrete momentum variable. Note that v^L is periodic with respect to the domain Λ and that $v^L(\mathbf{x}) \to v(\mathbf{x})$ as $L \to \infty$. Consider the Hamiltonian

$$-\sum_{i=1}^{N} \Delta_i^L + \lambda \sum_{1 \leq i < j \leq N} v^L(\mathbf{x}_i - \mathbf{x}_j) \tag{1.1}$$

acting on the space $L_{\mathrm{s}}^2(\Lambda^N)$ (the symmetric subspace of $L^2(\Lambda^N)$). The Laplacian is assumed to have periodic boundary conditions.

Let $\rho = N/L^d$ be the density of the gas. The Bogoliubov approximation [2] predicts that the ground-state energy is

$$\frac{1}{2}\lambda\rho\hat{v}(\mathbf{0})(N-1) - \frac{1}{2} \sum_{\mathbf{p} \in \frac{2\pi}{L}\mathbb{Z}^d \backslash \{0\}} \left(|\mathbf{p}|^2 + \rho\lambda\hat{v}(\mathbf{p}) - |\mathbf{p}|\sqrt{|\mathbf{p}|^2 + 2\rho\lambda\hat{v}(\mathbf{p})} \right)$$

and that the low-lying excited states can be derived from the following *elementary excitation spectrum*:

$$|\mathbf{p}|\sqrt{|\mathbf{p}|^2 + 2\rho\lambda\hat{v}(\mathbf{p})}. \tag{1.2}$$

Note that within the Bogoliubov approximation both the ground-state energy and the excitation spectrum depend on ρ and λ only through the product $\rho\lambda$. The dependence on L is very weak:

1. The elementary excitation spectrum (1.2) depends on L only through the spacing of the momentum lattice $\frac{2\pi}{L}\mathbb{Z}^d$.
2. The expression for the ground-state energy divided by the volume L^d converges for $L \to \infty$ to a finite expression

$$\frac{1}{2}\rho^2\lambda\hat{v}(\mathbf{0}) - \frac{1}{2(2\pi)^d} \int \left(|\mathbf{p}|^2 + \rho\lambda\hat{v}(\mathbf{p}) - |\mathbf{p}|\sqrt{|\mathbf{p}|^2 + 2\rho\lambda\hat{v}(\mathbf{p})} \right) \mathrm{d}\mathbf{p}. \tag{1.3}$$

We believe that it is important to understand the Bogoliubov approximation for large L. Important physical properties, such as the phonon group velocity and the description of the Beliaev damping in terms of analyticity properties of Green's functions, have an elegant description when we can view the momentum as a continuous variable, which is equivalent to taking the limit $L \to \infty$.

Note that in our problem there are three a priori uncorrelated parameters: λ, N and L. By the *mean-field limit* one usually understands $N \to \infty$ with $\lambda \simeq \frac{1}{N}$ and $L = \text{const}$. However, when both N and L are large it is natural to consider a somewhat different scaling. In our paper the mean-field limit will correspond to $N \to \infty$ with $\lambda \simeq \frac{1}{\rho} = \frac{L^d}{N}$.

Motivated by the above argument, we will consider a system described by the Hamiltonian

$$H_N^L = -\sum_{i=1}^N \Delta_i^L + \frac{L^d}{N} \sum_{1 \le i < j \le N} v^L(\mathbf{x}_i - \mathbf{x}_j). \tag{1.4}$$

It is translation invariant: it commutes with the total momentum operator

$$P_N^L := -\sum_{i=1}^N \mathrm{i}\partial_{\mathbf{x}_i}^L. \tag{1.5}$$

We will denote by E_N^L the ground-state energy of (1.4). If $\mathbf{p} \in \frac{2\pi}{L}\mathbb{Z}^d \backslash \{\mathbf{0}\}$ let $K_N^{L,1}(\mathbf{p}), K_N^{L,2}(\mathbf{p}), \ldots$ be the eigenvalues of $H_N^L - E_N^L$ of total momentum $\mathbf{p}$ in the order of increasing values, counting the multiplicity. The lowest eigenvalue of $H_N^L - E_N^L$ of total momentum $\mathbf{p} = \mathbf{0}$ is 0 by general arguments [4]. Let $K_N^{L,1}(\mathbf{0}), K_N^{L,2}(\mathbf{0}), \ldots$ be the next eigenvalues of $H_N^L - E_N^L$ of total momentum $\mathbf{0}$, also in the order of increasing values, counting the multiplicity.

We also introduce the *Bogoliubov energy*

$$E_{\mathrm{Bog}}^L := -\frac{1}{2} \sum_{\mathbf{p} \in \frac{2\pi}{L}\mathbb{Z}^d \backslash \{0\}} \left(|\mathbf{p}|^2 + \hat{v}(\mathbf{p}) - |\mathbf{p}|\sqrt{|\mathbf{p}|^2 + 2\hat{v}(\mathbf{p})} \right)$$

and the *Bogoliubov elementary excitation spectrum*

$$e_{\mathbf{p}} = |\mathbf{p}|\sqrt{|\mathbf{p}|^2 + 2\hat{v}(\mathbf{p})}. \tag{1.6}$$

For any $\mathbf{p} \in \frac{2\pi}{L}\mathbb{Z}^d$ we consider the corresponding excitation energies with momentum $\mathbf{p}$:

$$\left\{ \sum_{i=1}^j e_{\mathbf{k}_i} \ : \ \mathbf{k}_1, \ldots, \mathbf{k}_j \in \frac{2\pi}{L}\mathbb{Z}^d \backslash \{\mathbf{0}\}, \ \mathbf{k}_1 + \cdots + \mathbf{k}_j = \mathbf{p}, \quad j = 1, 2, \ldots \right\}.$$

Let $K_{\mathrm{Bog}}^{L,1}(\mathbf{p}), K_{\mathrm{Bog}}^{L,2}(\mathbf{p}), \ldots$ be these excitation energies in the order of increasing values, counting the multiplicity. We will use the term *excitation spectrum in the Bogoliubov approximation* to denote the set of pairs $\left(K_{\mathrm{Bog}}^{L,j}(\mathbf{p}), \mathbf{p}\right) \in \mathbb{R} \times \mathbb{R}^d$. Later on we will see that it coincides with the joint spectrum of commuting operators $H_{\mathrm{Bog}}^L - E_{\mathrm{Bog}}^L$ and P^L with $(0, \mathbf{0})$ removed. (See (6.3) for the definition of H_{Bog}^L.)

Below we present pictures of the excitation spectrum of 1-dimensional Bose gas in the Bogoliubov approximation (Figs. 1, 3) for two potentials, v_1 and v_2 (Figs. 2, 4). Both potentials are appropriately scaled Gaussians. (Note that Gaussians satisfy the assumptions of our main theorem). On both pictures the (black) dot at the origin corresponds to the quasiparticle vacuum, (red) dots correspond to 1-quasiparticle excitations, (blue) triangles correspond to 2-quasiparticles excitations, while (green) squares correspond to n-quasiparticles excitations with $n \ge 3$. We also give the graphs of the Fourier transforms of both potentials.

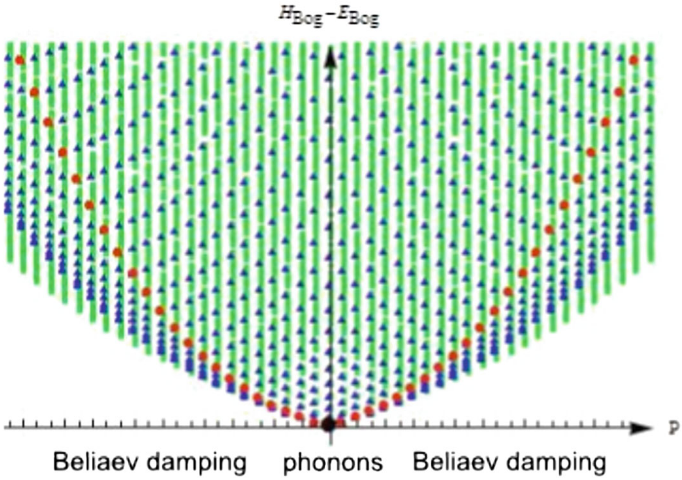

FIGURE 1. Excitation spectrum of 1-dimensional homogeneous Bose gas with potential v_1 in the Bogoliubov approximation

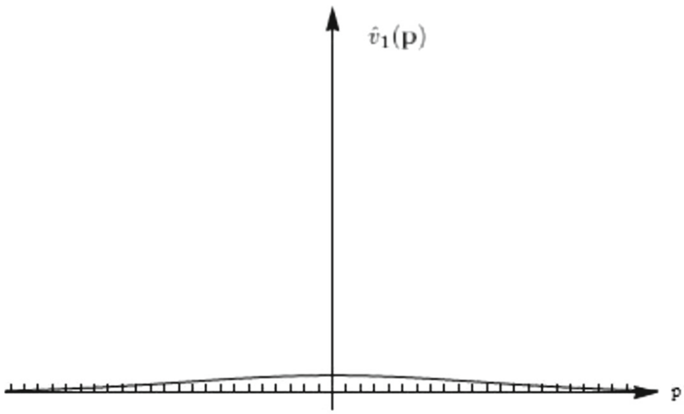

FIGURE 2. $\hat{v}_1(\mathbf{p}) = \frac{\mathrm{e}^{-\mathbf{p}^2/5}}{10}$

Note that all figures are drawn in the same scale, apart from Fig. 4, where the potential had to be scaled down because of space limitations. In our units of length $\frac{2\pi}{L} = \frac{15}{100}$.

Notice that for some total momentum $\mathbf{p}$ many-quasiparticle excitation energies are lower than the elementary excitation spectrum. In particular, for the potential v_1 it happens already for low momenta. Physically this means that the corresponding 1-quasiparticle excitation is not stable: it may decay to m-quasiparticle states, $m \geq 2$, with a lower energy. This phenomenon has been observed experimentally [10] and is called the *Beliaev damping* [1]. If one

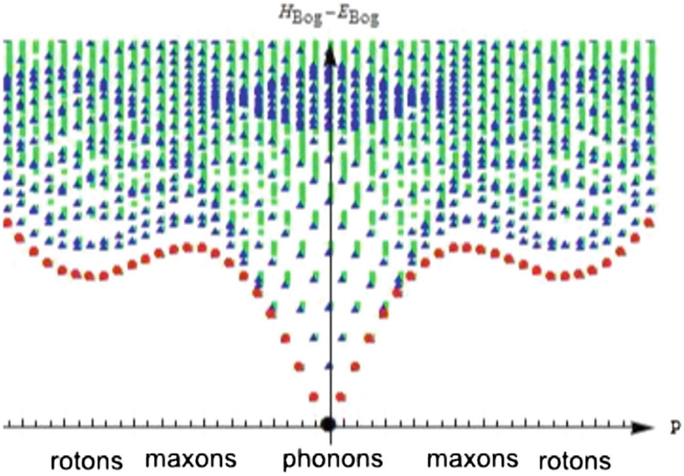

FIGURE 3. Excitation spectrum of 1-dimensional homogeneous Bose gas with potential v_2 in the Bogoliubov approximation

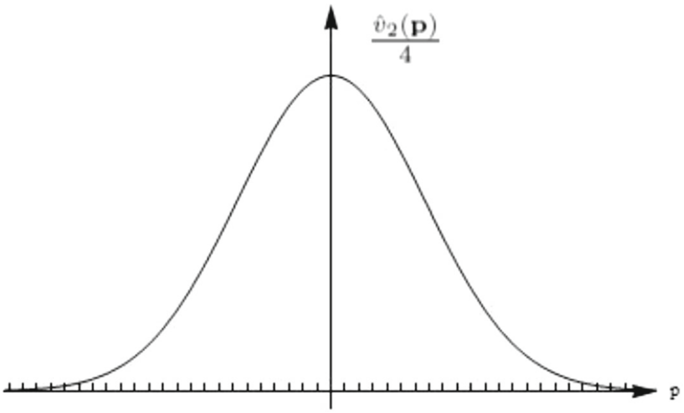

FIGURE 4. $\hat{v}_2(\mathbf{p}) = \frac{15\,\mathrm{e}^{-\mathbf{p}^2/2}}{2}$

can assume that the momentum variable is continuous, the Beliaev damping corresponds to a pole of the Green's function on a non-physical sheet of the energy complex plane. The imaginary part of the position of this pole, computed by Beliaev, is responsible for the rate of decay of quasiparticles.

The excitation spectrum for potential v_2 has a very different shape—it has local maxima and local minima away from the zero momentum. On the picture we show traditional names of quasiparticles—*phonons* in the low-momentum region, where the dispersion relation is approximately linear, *maxons* near the local maximum and *rotons* near the local minimum of the elementary excitation spectrum (see [9] for details).

From now on, we will drop the superscript L. Let us state our main result. It is slightly different for the upper and lower bounds:

Theorem 1.1. 1. *Let $c > 0$. Then there exists C such that*

(a) *if*

$$L^{2d+2} \leq cN, \tag{1.7}$$

then

$$E_N \geq \frac{1}{2}\hat{v}(\mathbf{0})(N-1) + E_{\mathrm{Bog}} - CN^{-1/2}L^{2d+3}; \tag{1.8}$$

(b) *if in addition*

$$K_N^j(\mathbf{p}) \leq cNL^{-d-2}, \tag{1.9}$$

then

$$\begin{aligned} E_N + K_N^j(\mathbf{p}) \geq &\frac{1}{2}\hat{v}(\mathbf{0})(N-1) + E_{\mathrm{Bog}} + K_{\mathrm{Bog}}^j(\mathbf{p}) \\ &- CN^{-1/2}L^{d/2+3}\left(K_N^j(\mathbf{p}) + L^d\right)^{3/2}. \end{aligned} \tag{1.10}$$

2. *Let $c > 0$. Then there exists $c_1 > 0$ and C such that*

(a) *if*

$$L^{2d+1} \leq cN \tag{1.11}$$

$$\textit{and} \qquad L^{d+1} \leq c_1 N, \tag{1.12}$$

then

$$E_N \leq \frac{1}{2}\hat{v}(\mathbf{0})(N-1) + E_{\mathrm{Bog}} + CN^{-1/2}L^{2d+3/2}; \tag{1.13}$$

(b) *if in addition*

$$K_{\mathrm{Bog}}^j(\mathbf{p}) \leq cNL^{-d-2} \tag{1.14}$$

$$\textit{and} \qquad K_{\mathrm{Bog}}^j(\mathbf{p}) \leq c_1 NL^{-2}, \tag{1.15}$$

then

$$\begin{aligned} E_N + K_N^j(\mathbf{p}) \leq &\frac{1}{2}\hat{v}(\mathbf{0})(N-1) + E_{\mathrm{Bog}} + K_{\mathrm{Bog}}^j(\mathbf{p}) \\ &+ CN^{-1/2}L^{d/2+3}(K_{\mathrm{Bog}}^j(\mathbf{p}) + L^{d-1})^{3/2}. \end{aligned} \tag{1.16}$$

Let us stress that the constants C and c_1 that appear in the theorem depend only the potential v, the dimension d, and the constant c, but *do not* depend on N, j and L. Note also that both in (1), resp. (2) we can deduce (a) from (b) by setting $K_N^j(\mathbf{p}) = 0$, resp. $K_{\mathrm{Bog}}^j(\mathbf{p}) = 0$.

Theorem 1.1 expresses the idea that the Bogoliubov approximation becomes exact for large N and L provided that the volume does not grow too fast. This may appear not very transparent, since the error terms in the theorem depend on two parameters L and N as well as on the excitation energy. Therefore, we give some consequences of our theorem, where the error term depends only on N. They generalize the corresponding remarks of [18].

Corollary 1.2. *Let $b > 1, -1 - \frac{1}{2d+1} < \alpha \leq 1$ and $L^{4d+6} \leq bN^{1-\alpha}$. Then there exists M such that if $N > M$, then*

1. $E_N = \frac{1}{2}\hat{v}(\mathbf{0})(N-1) + E_{\mathrm{Bog}} + O(N^{-\alpha/2})$;
2. *if* $\min\left(K_N^j(\mathbf{p}), K_{\mathrm{Bog}}^j(\mathbf{p})\right) \leq (bN^{1-\alpha}L^{-d-6})^{1/3}$, *then*
$$K_N^j(\mathbf{p}) = K_{\mathrm{Bog}}^j(\mathbf{p}) + O(N^{-\alpha/2});$$
3. *if* $0 < \alpha \leq 1$ *and* $\min\left(K_N^j(\mathbf{p}), K_{\mathrm{Bog}}^j(\mathbf{p})\right) \leq bN^{1-\alpha}L^{-d-6}$, *then*
$$K_N^j(\mathbf{p}) = K_{\mathrm{Bog}}^j(\mathbf{p}) + \left(1 + K_{\mathrm{Bog}}^j(\mathbf{p})\right) O(N^{-\alpha/2}).$$

The proof that Theorem 1.1 implies Corollary 1.2 is given in Appendix.

Remark 1.3.
1. The case $\alpha = 1$, $L = 1$ of Corollary 1.2 corresponds directly to the result of [18].
2. In part (3) of Corollary 1.2 one can also include the case $\alpha = 0$ provided that L is sufficiently large.

Thus, for large N within a growing range of the volume, the low-lying energy–momentum spectrum of the homogeneous Bose gas is well described by the Bogoliubov approximation. In the infinite-volume limit momentum becomes a continuous variable, which is important when we want to consider the so-called *critical velocity* and *phase velocity* introduced by Landau. They play a crucial role in his theory of superfluidity ([11,12], see also [4,20]).

Mathematically, the Bogoliubov approximation has been studied mostly in the context of the ground-state energy ([6,7,15,16,19,21], see also [14]). This makes the work of Seiringer [18], Grech and Seiringer [8] and more recently by Lewin et al. [13] even more notable, since they are devoted to a rigorous study of the excitation spectrum of a Bose gas.

Seiringer [18] proves that for a system of N bosons on a flat unit torus $\mathbb{T}^d$ which interacts with a two-body interaction $v(\mathbf{x})/(N-1)$, the excitation spectrum up to an energy κ is formed by elementary excitations of momentum $\mathbf{p}$ with a corresponding energy of the form (1.2) up to an error term of the order $O(\kappa^{3/2}N^{-1/2})$. Also in [8] and [13] the authors are concerned with finite systems in the large particle number limit.

Our result can be considered as an extension of Seiringer's result to systems of arbitrary volume. The ultimate goal would be to prove similar results in the thermodynamic limit with a fixed coupling constant. Since this is at the moment out of reach, we try to pass to some other limits, which involve convergence of the volume to infinity.

The rest of this paper is devoted to a proof of Theorem 1.1. It uses partly the methods presented in [18]. Note, however, that naive mimicking leads to a much weaker result, which involves assuming that $N \geq C\,\mathrm{e}^{cL^{d/2}}$ to ensure that the error terms tend to zero when taking the infinite-volume limit. This can be easily seen by looking for example at equation (24) of [18]. In this equation one of the constants is given by the expression e^{C_2} where C_2 is given by

$$\sqrt{\frac{64N}{N-1}\sum_{\mathbf{p}\neq\mathbf{0}}\beta_{\mathbf{p}}^2}.$$

In the infinite-volume limit the sum could be replaced by an integral which one can compensate by the factor $L^{d/2}$. This leads to a factor $\mathrm{e}^{cL^{d/2}}$ in the estimates.

Our proof uses certain identities that allow us to simplify the algebraic computations involved in the proof. We use the method of second quantization, working in the Fock space containing all N-particle spaces at once. We embed this space in the so-called *extended space*, which contains non-physical states with a negative number of zero modes. This method leads to relatively simple algebraic calculations, which is helpful when we want to control the volume dependence. Note also that our method yields the same results as in [18] if one takes $L = 1$.

Strangely, we have never seen the method of the extended space in the literature. Some authors (starting with Bogoliubov [3]) introduce the operator $a_0^\dagger(\mathbb{1}+N_0)^{-1/2}$, which coincides with our operator $U^\dagger$ on the physical space. Both operators increase the number of zeroth modes by one. The operator $U^\dagger$, however, acts on the extended space and is unitary, whereas $a_0^\dagger(\mathbb{1}+N_0)^{-1/2}$ acts on the physical space and is only isometric.

One can also see some similarity of our method with that of [13] where, however, states with a negative number of modes do not appear.

2. Miscellanea

Let us describe some notation and basic facts from operator theory used in our paper.

If A, B are operators, then the following inequality will be often used:

$$-A^\dagger A - B^\dagger B \leq A^\dagger B + B^\dagger A \leq A^\dagger A + B^\dagger B. \tag{2.1}$$

We will write $A + \mathrm{hc}$ for $A + A^\dagger$.

If A is a self-adjoint operator and Ω a Borel subset of the spectrum of A, then $\mathbb{1}_\Omega(A)$ will denote the spectral projection of A onto Ω.

Let A be a bounded from below self-adjoint operator on Hilbert space $\mathcal{H}$. For simplicity, let us assume that it has only discrete spectrum.

We define

$$\overrightarrow{\mathrm{sp}}(A) := (E_1, E_2, \ldots),$$

where $E_1, E_2, \ldots$ are the eigenvalues of A in the order of increasing values, counting the multiplicity. If $\dim\mathcal{H} = n$, then we set $E_{n+1} = E_{n+2} = \cdots = \infty$.

We will use repeatedly two consequences of the *min-max principle* [17]:

$$A \leq B \ \text{ implies } \overrightarrow{\mathrm{sp}}(A) \leq \overrightarrow{\mathrm{sp}}(B),$$

and the so-called *Rayleigh-Ritz principle*: if $\mathcal{K}$ is a closed subspace of $\mathcal{H}$, let $P_{\mathcal{K}}$ be the projection onto $\mathcal{K}$. Then we have

$$\overrightarrow{\mathrm{sp}}(A) \leq \overrightarrow{\mathrm{sp}}\left(P_{\mathcal{K}} A P_{\mathcal{K}}\Big|_{\mathcal{K}}\right). \tag{2.2}$$

3. Second Quantization

As discussed in the introduction, the main object of our paper, the Hamiltonian H_N is defined on the N-particle bosonic space

$$\mathcal{H}_N := L^2_{\mathrm{s}}(\Lambda^N).$$

We will work most of the time in the momentum representation, in which the 1-particle space $L^2(\Lambda)$ is represented as $l^2\left(\frac{2\pi}{L}\mathbb{Z}^d\right)$, thus

$$\mathcal{H}_N \simeq \otimes_{\mathrm{s}}^N l^2\left(\frac{2\pi}{L}\mathbb{Z}^d\right).$$

It is convenient to consider simultaneously the direct sum of the N-particle spaces, the *bosonic Fock space*

$$\mathcal{H} := \bigoplus_{N=0}^{\infty} \mathcal{H}_N = \Gamma_{\mathrm{s}}\left(l^2\left(\frac{2\pi}{L}\mathbb{Z}^d\right)\right). \tag{3.1}$$

The direct sum of the Hamiltonians H_N will be denoted H. Using the notation of the second quantization it can be written as

$$H := \bigoplus_{N=0}^{\infty} H_N = \sum_{\mathbf{p}} \mathbf{p}^2 a_{\mathbf{p}}^\dagger a_{\mathbf{p}} + \frac{1}{2N}\sum_{\mathbf{p},\mathbf{q},\mathbf{k}} \hat{v}(\mathbf{k}) a_{\mathbf{p}+\mathbf{k}}^\dagger a_{\mathbf{q}-\mathbf{k}}^\dagger a_{\mathbf{q}} a_{\mathbf{p}}.$$

If A is an operator on the one-particle space, then by its *second quantization* we will mean the operator that on the N-particle space equals

$$\sum_{i=1}^{N} A_i.$$

If we use an orthonormal basis, say, $|\mathbf{p}\rangle$, $\mathbf{p} \in \frac{2\pi}{L}\mathbb{Z}^d$, then this operator written in the second quantized notation equals

$$\frac{1}{2}\sum_{\mathbf{p}_1,\mathbf{p}_2} \langle \mathbf{p}_1|A|\mathbf{p}_2\rangle a_{\mathbf{p}_1}^\dagger a_{\mathbf{p}_2}.$$

Let us introduce some special notation for various operators and their second quantization.

Let P be the projection onto the constant function in $L^2(]L/2, L/2]^d)$, and $Q = 1 - P$. The operator that counts the number of particles in, resp. outside the zero momentum mode will be denoted by N_0, resp. $N^>$, i.e.

$$N_0 = \sum_{i=1}^{N} P_i, \quad N^> = \sum_{i=1}^{N} Q_i. \tag{3.2}$$

In the second quantization notation,

$$N_0 = a_{\mathbf{0}}^\dagger a_{\mathbf{0}}, \quad N^> = \sum_{\mathbf{p}\neq\mathbf{0}} a_{\mathbf{p}}^\dagger a_{\mathbf{p}}.$$

For N-particle bosonic wave functions Ψ, Φ we have

$$\langle\Psi|N^>|\Phi\rangle = N\langle\Psi|Q_1|\Phi\rangle, \tag{3.3}$$

$$\langle\Psi|N^>(N^>-1)|\Phi\rangle = N(N-1)\langle\Psi|Q_1Q_2|\Phi\rangle. \tag{3.4}$$

The symbol T will denote the kinetic energy of the system: $T=-\sum_{i=1}^N \Delta_i$. For further reference, note that

$$\langle\Psi|N^>|\Psi\rangle \leq \frac{L^2}{(2\pi)^2}\langle\Psi|T|\Psi\rangle. \tag{3.5}$$

We will also need the notion of the second quantization of certain 2-body operators. More precisely, let w be an operator on the symmetrized 2-particle space. Then by its second quantization, we will mean the operator that restricted to the N-particle space equals

$$\sum_{1\leq i<j\leq N} w_{ij}.$$

If w is an operator on the unsymmetrized 2-particle space, then we can also speak about its second quantization, but now its restriction to the N-particle space equals

$$\frac{1}{2}\sum_{1\leq i\neq j\leq N} w_{ij}.$$

In the momentum basis this operator written in the second quantized language equals

$$\frac{1}{2}\sum_{\mathbf{p}_1,\mathbf{p}_2,\mathbf{p}_3,\mathbf{p}_4} \langle\mathbf{p}_1,\mathbf{p}_2|w|\mathbf{p}_3,\mathbf{p}_4\rangle a_{\mathbf{p}_1}^\dagger a_{\mathbf{p}_2}^\dagger a_{\mathbf{p}_3} a_{\mathbf{p}_4}.$$

4. Bounds on Interaction

The potential v can be interpreted as an operator of multiplication by $v(\mathbf{x}_1-\mathbf{x}_2)$ on $L_{\mathrm{s}}^2(\Lambda^2)$. Following [18], we would like to estimate this 2-body operator by simpler, 1-body operators. As a preliminary step we record the following bound:

Lemma 4.1. *Let* $\epsilon>0$. *Then*

$$\begin{aligned} v \geq\ & P\otimes PvP\otimes P + P\otimes PvQ\otimes Q + Q\otimes QvP\otimes P \\ & +(1-\epsilon)(P\otimes Q+Q\otimes P)v(P\otimes Q+Q\otimes P) \\ & +(1-\epsilon^{-1})Q\otimes QvQ\otimes Q, \end{aligned}$$

$$\begin{aligned} v \leq\ & P\otimes PvP\otimes P + P\otimes PvQ\otimes Q + Q\otimes QvP\otimes P \\ & +(1+\epsilon)(P\otimes Q + Q\otimes P)v(P\otimes Q+Q\otimes P) \\ & +(1+\epsilon^{-1})Q\otimes QvQ\otimes Q. \end{aligned}$$

Proof. Using the translation invariance of v we obtain

$$\begin{aligned} v =\ & (P\otimes P + Q\otimes Q)v(P\otimes P+Q\otimes Q) \\ & +(P\otimes Q+Q\otimes P)v(P\otimes Q+Q\otimes P) \\ & +(P\otimes Q+Q\otimes P)vQ\otimes Q + Q\otimes Qv(P\otimes Q+Q\otimes P). \end{aligned}$$

Then we apply the Schwarz inequality to the last two terms. □

Let us now identify the second quantization of various terms on the r.h.s of the estimates of Lemma 4.1:

$$\begin{aligned} & P\otimes PvP\otimes P && \frac{1}{2L^d}\hat{v}(\mathbf{0})a_{\mathbf{0}}^\dagger a_{\mathbf{0}}^\dagger a_{\mathbf{0}}a_{\mathbf{0}} = \frac{1}{2L^d}\hat{v}(\mathbf{0})N_0(N_0-1), \\ & P\otimes PvQ\otimes Q && \frac{1}{2L^d}\sum_{\mathbf{p}\neq\mathbf{0}}\hat{v}(\mathbf{p})a_0^\dagger a_0^\dagger a_{\mathbf{p}}a_{-\mathbf{p}}, \\ & Q\otimes QvP\otimes P && \frac{1}{2L^d}\sum_{\mathbf{p}\neq\mathbf{0}}\hat{v}(\mathbf{p})a_{\mathbf{p}}^\dagger a_{-\mathbf{p}}^\dagger a_0 a_0, \\ & P\otimes QvQ\otimes P,\ Q\otimes PvP\otimes Q && \frac{1}{2L^d}\sum_{\mathbf{p}\neq\mathbf{0}}\hat{v}(\mathbf{0})a_{\mathbf{p}}^\dagger a_{\mathbf{p}}N_0, \\ & P\otimes QvP\otimes Q,\ Q\otimes PvQ\otimes P && \frac{1}{2L^d}\sum_{\mathbf{p}\neq\mathbf{0}}\hat{v}(\mathbf{p})a_{\mathbf{p}}^\dagger a_{\mathbf{p}}N_0. \end{aligned}$$

The second quantization of $Q\otimes QvQ\otimes Q$ can be bounded from above by

$$v(\mathbf{0})\sum_{1\leq i<j\leq N} Q_iQ_j = v(\mathbf{0})\frac{1}{2}N^>(N^>-1).$$

Introduce the family of *estimating Hamiltonians*

$$\begin{aligned} H_{N,\epsilon} :=\ & \frac{1}{2}\hat{v}(\mathbf{0})(N-1) + \sum_{\mathbf{p}\neq\mathbf{0}}\left(|\mathbf{p}|^2+\hat{v}(\mathbf{p})\right)a_{\mathbf{p}}^\dagger a_{\mathbf{p}} \\ & +\frac{1}{2N}\sum_{\mathbf{p}\neq\mathbf{0}}\hat{v}(\mathbf{p})\left(a_{\mathbf{0}}^\dagger a_{\mathbf{0}}^\dagger a_{\mathbf{p}}a_{-\mathbf{p}} + a_{\mathbf{p}}^\dagger a_{-\mathbf{p}}^\dagger a_{\mathbf{0}}a_{\mathbf{0}}\right) \\ & -\frac{1}{N}\sum_{\mathbf{p}\neq\mathbf{0}}\left(\hat{v}(\mathbf{p})+\frac{\hat{v}(\mathbf{0})}{2}\right)a_{\mathbf{p}}^\dagger a_{\mathbf{p}}N^> + \frac{\hat{v}(\mathbf{0})}{2N}N^> \\ & +\frac{\epsilon}{N}\sum_{\mathbf{p}\neq\mathbf{0}}\left(\hat{v}(\mathbf{p})+\hat{v}(\mathbf{0})\right)a_{\mathbf{p}}^\dagger a_{\mathbf{p}}N_0 \\ & +(1+\epsilon^{-1})\frac{1}{2N}v(\mathbf{0})L^dN^>(N^>-1). \end{aligned}$$

The operators $H_{N,\epsilon}$ preserve the N-particle sectors. By the above calculations we obtain the following estimates on the Hamiltonian:

$$H_N \geq H_{N,-\epsilon}, \quad 0 < \epsilon \leq 1; \tag{4.1}$$

$$H_N \leq H_{N,\epsilon}, \quad 0 < \epsilon. \tag{4.2}$$

5. Extended Space

So far we used the *physical Hilbert space* (3.1). By the exponential property of Fock spaces we have the identification

$$\mathcal{H} \simeq \Gamma_{\mathrm{s}}(\mathbb{C}) \otimes \Gamma_{\mathrm{s}}\left(l^2\left(\frac{2\pi}{L}\mathbb{Z}^d \backslash \{\mathbf{0}\}\right)\right). \tag{5.1}$$

Let us embed the space of *zero modes* $\Gamma_{\mathrm{s}}(\mathbb{C}) = l^2(\{0, 1, \ldots\})$ in a larger space $l^2(\mathbb{Z})$. Thus we obtain the *extended Hilbert space*

$$\mathcal{H}^{\mathrm{ext}} := l^2(\mathbb{Z}) \otimes \Gamma_{\mathrm{s}}\left(l^2\left(\frac{2\pi}{L}\mathbb{Z}^d \backslash \{\mathbf{0}\}\right)\right). \tag{5.2}$$

The physical space (5.1) is spanned by vectors of the form $|n_0\rangle \otimes \Psi^{>}$, where $|n_0\rangle$ represents n_0 zero modes ($n_0 \geq 0$) and $\Psi^{>}$ represents a vector outside the zero mode.

The space (5.2) is also spanned by vectors of this form, where now the relation $n_0 \geq 0$ is not imposed. The orthogonal complement of $\mathcal{H}$ in $\mathcal{H}^{\mathrm{ext}}$ will be denoted by $\mathcal{H}^{\mathrm{nph}}$ (for "non-physical").

On $\mathcal{H}^{\mathrm{ext}}$ we have a self-adjoint operator N_0^{ext} such that $N_0^{\mathrm{ext}}|n_0\rangle \otimes \Psi^{>} = n_0|n_0\rangle \otimes \Psi^{>}$. Its spectrum equals $\mathbb{Z}$. Clearly

$$N_0^{\mathrm{ext}}\Big|_{\mathcal{H}} = N_0, \quad \mathcal{H} = \mathrm{Ran}\mathbb{1}_{[0,\infty[}(N_0^{\mathrm{ext}}), \quad \mathcal{H}^{\mathrm{nph}} = \mathrm{Ran}\mathbb{1}_{]-\infty,0[}(N_0^{\mathrm{ext}}).$$

If $N \in \mathbb{Z}$, we will write $\mathcal{H}_N^{\mathrm{ext}}$ for the subspace of $\mathcal{H}^{\mathrm{ext}}$ corresponding to $N^{>} + N_0^{\mathrm{ext}} = N$.

We have also a unitary operator

$$U|n_0\rangle \otimes \Psi^{>} = |n_0 - 1\rangle \otimes \Psi^{>}.$$

Notice that both U and $U^\dagger$ commute with both $a_{\mathbf{p}}$ and $a_{\mathbf{p}}^\dagger$ with $\mathbf{p} \neq \mathbf{0}$. We now define for $\mathbf{p} \neq \mathbf{0}$ the following operator on $\mathcal{H}^{\mathrm{ext}}$:

$$b_{\mathbf{p}} := a_{\mathbf{p}} U^\dagger.$$

Operators $b_{\mathbf{p}}$ and $b_{\mathbf{q}}^\dagger$ satisfy the same CCR as $a_{\mathbf{p}}$ and $a_{\mathbf{q}}^\dagger$.

The extended space is useful in the study of N-body Hamiltonians. To illustrate this, on $\mathcal{H}_N^{\mathrm{ext}}$ let us introduce the *extended Hamiltonian*

$$\begin{aligned} H_N^{\mathrm{ext}} = & \sum_{\mathbf{p}\neq\mathbf{0}} \left(\mathbf{p}^2 + \frac{N_0^{\mathrm{ext}}}{N}(\hat{v}(\mathbf{p}) + \hat{v}(\mathbf{0})\right) b_{\mathbf{p}}^\dagger b_{\mathbf{p}} \\ & + \frac{1}{2} \sum_{\mathbf{p}\neq\mathbf{0}} \left(\hat{v}(\mathbf{p}) \frac{\sqrt{N_0^{\mathrm{ext}}(N_0^{\mathrm{ext}} - 1)}}{N} b_{\mathbf{p}} b_{-\mathbf{p}} + \mathrm{hc}\right) \end{aligned}$$

$$+\frac{1}{N}\sum_{\mathbf{k},\mathbf{p}\neq\mathbf{0}}\left(\hat{v}(\mathbf{k})b_{\mathbf{k}}^{\dagger}b_{\mathbf{p}-\mathbf{k}}^{\dagger}b_{\mathbf{p}}\sqrt{\max(N_0^{\mathrm{ext}},0)}+\mathrm{hc}\right)$$
$$+\frac{1}{2N}\sum_{\mathbf{p},\mathbf{q},\mathbf{k}\neq\mathbf{0}}\hat{v}(\mathbf{k})b_{\mathbf{p}+\mathbf{k}}^{\dagger}b_{\mathbf{q}-\mathbf{k}}^{\dagger}b_{\mathbf{q}}b_{\mathbf{p}}.$$

It is easy to see that H_N^{ext} preserves the N-particle physical space $\mathcal{H}_N$ and on $\mathcal{H}_N$ it coincides with H_N.

In our paper we will use the *extended estimating Hamiltonian*, which is the following operator on $\mathcal{H}_N^{\mathrm{ext}}$:

$$\begin{aligned}
H_{N,\epsilon}^{\mathrm{ext}} := & \frac{1}{2}\hat{v}(\mathbf{0})(N-1)+\sum_{\mathbf{p}\neq\mathbf{0}}\left(|\mathbf{p}|^2+\hat{v}(\mathbf{p})\right)b_{\mathbf{p}}^{\dagger}b_{\mathbf{p}}\\
&+\frac{1}{2}\sum_{\mathbf{p}\neq\mathbf{0}}\hat{v}(\mathbf{p})\left(\frac{\sqrt{(N_0^{\mathrm{ext}}-1)N_0^{\mathrm{ext}}}}{N}b_{\mathbf{p}}b_{-\mathbf{p}}+\mathrm{hc}\right)\\
&-\frac{1}{N}\sum_{\mathbf{p}\neq\mathbf{0}}\left(\hat{v}(\mathbf{p})+\frac{\hat{v}(\mathbf{0})}{2}\right)b_{\mathbf{p}}^{\dagger}b_{\mathbf{p}}N^{>}+\frac{\hat{v}(\mathbf{0})}{2N}N^{>}\\
&+\frac{\epsilon}{N}\sum_{\mathbf{p}\neq\mathbf{0}}\left(\hat{v}(\mathbf{p})+\hat{v}(\mathbf{0})\right)b_{\mathbf{p}}^{\dagger}b_{\mathbf{p}}N_0^{\mathrm{ext}}\\
&+(1+\epsilon^{-1})\frac{1}{2N}v(\mathbf{0})L^dN^{>}(N^{>}-1).
\end{aligned}$$

Note that $H_{N,\epsilon}^{\mathrm{ext}}$ preserves $\mathcal{H}_N$ and restricted to $\mathcal{H}_N$ coincides with $H_{N,\epsilon}$.

6. Bogoliubov Hamiltonian

Consider the operator

$$\sum_{\mathbf{p}\neq\mathbf{0}}\left(|\mathbf{p}|^2+\hat{v}(\mathbf{p})\right)b_{\mathbf{p}}^{\dagger}b_{\mathbf{p}}+\frac{1}{2}\sum_{\mathbf{p}\neq\mathbf{0}}\hat{v}(\mathbf{p})\left(b_{\mathbf{p}}b_{-\mathbf{p}}+b_{\mathbf{p}}^{\dagger}b_{-\mathbf{p}}^{\dagger}\right).$$

acting on $\mathcal{H}^{\mathrm{ext}}$. It commutes with $N_0+N^{>}$ and U. In particular, it preserves $\mathcal{H}_N^{\mathrm{ext}}$. Its restriction to $\mathcal{H}_N^{\mathrm{ext}}$ will be denoted $H_{\mathrm{Bog},N}$.

We can write

$$H_{N,\epsilon}^{\mathrm{ext}} = \frac{1}{2}\hat{v}(\mathbf{0})(N-1)+H_{\mathrm{Bog},N}+R_{N,\epsilon}, \tag{6.1}$$

$$\begin{aligned}
R_{N,\epsilon} := & \frac{1}{2}\sum_{\mathbf{p}\neq\mathbf{0}}\hat{v}(\mathbf{p})\left(\left(\frac{\sqrt{(N_0^{\mathrm{ext}}-1)N_0^{\mathrm{ext}}}}{N}-1\right)b_{\mathbf{p}}b_{-\mathbf{p}}+\mathrm{hc}\right)\\
&-\frac{1}{N}\sum_{\mathbf{p}\neq\mathbf{0}}\left(\hat{v}(\mathbf{p})+\frac{\hat{v}(\mathbf{0})}{2}\right)b_{\mathbf{p}}^{\dagger}\mathrm{b}_{\mathbf{p}}N^{>}+\frac{\hat{v}(\mathbf{0})}{2N}N^{>}\\
&+\frac{\epsilon}{N}\sum_{\mathbf{p}\neq\mathbf{0}}\left(\hat{v}(\mathbf{p})+\hat{v}(\mathbf{0})\right)b_{\mathbf{p}}^{\dagger}b_{\mathbf{p}}N_0^{\mathrm{ext}}\\
&+(1+\epsilon^{-1})\frac{1}{2N}v(\mathbf{0})L^dN^{>}(N^{>}-1).
\end{aligned} \tag{6.2}$$

Clearly, all $H_{\mathrm{Bog},N}$ are unitarily equivalent to one another: $UH_{\mathrm{Bog},N}U^\dagger = H_{\mathrm{Bog},N-1}$. It is easy to see that they are all unitarily equivalent to what we can call the *standard Bogoliubov Hamiltonian*:

$$H_{\mathrm{Bog}} = \sum_{\mathbf{p}\neq\mathbf{0}} \left(|\mathbf{p}|^2 + \hat{v}(\mathbf{p})\right) a_{\mathbf{p}}^\dagger a_{\mathbf{p}} + \frac{1}{2}\sum_{\mathbf{p}\neq\mathbf{0}} \hat{v}(\mathbf{p}) \left(a_{\mathbf{p}}a_{-\mathbf{p}} + a_{\mathbf{p}}^\dagger a_{-\mathbf{p}}^\dagger\right). \tag{6.3}$$

H_{Bog} acts on $\Gamma_{\mathrm{s}}\left(l^2\left(\frac{2\pi}{L}\mathbb{Z}^d\backslash\{\mathbf{0}\}\right)\right)$.

We would now like to find a unitary transformation diagonalizing H_{Bog}. To this end set

$$A_{\mathbf{p}} := |\mathbf{p}|^2 + \hat{v}(\mathbf{p}), \quad B_{\mathbf{p}} := \hat{v}(\mathbf{p}).$$

Introduce also $\alpha_{\mathbf{p}}, \beta_{\mathbf{p}}, c_{\mathbf{p}}$ and $s_{\mathbf{p}}$ by

$$\begin{aligned}
\alpha_{\mathbf{p}} &= \frac{1}{B_{\mathbf{p}}}\left(A_{\mathbf{p}} - \sqrt{A_{\mathbf{p}}^2 - B_{\mathbf{p}}^2}\right) \;=\; \tanh(2\beta_{\mathbf{p}}),\\
c_{\mathbf{p}} &= \frac{1}{\sqrt{1-\alpha_{\mathbf{p}}^2}} \;=\; \cosh(2\beta_{\mathbf{p}}),\\
s_{\mathbf{p}} &= \frac{\alpha_{\mathbf{p}}}{\sqrt{1-\alpha_{\mathbf{p}}^2}} \;=\; \sinh(2\beta_{\mathbf{p}}).
\end{aligned}$$

Now let $S = \mathrm{e}^{-X}$, where

$$X = \sum_{\mathbf{p}\neq\mathbf{0}} \beta_{\mathbf{p}}\left(a_{\mathbf{p}}^\dagger a_{-\mathbf{p}}^\dagger - a_{\mathbf{p}}a_{-\mathbf{p}}\right). \tag{6.4}$$

Then using the Lie formula

$$\begin{aligned}
\mathrm{e}^{-X}\, a_{\mathbf{q}}\, \mathrm{e}^{X} &= \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} \underbrace{[X, \ldots [X, a_{\mathbf{q}}] \ldots]}_{j \text{ times}}\\
&= 1 + 2\beta_{\mathbf{q}} a_{-\mathbf{q}}^\dagger + \frac{1}{2}4\beta_{\mathbf{q}}^2 a_{\mathbf{q}} + \cdots
\end{aligned}$$

we get

$$S a_{\mathbf{q}} S^\dagger = c_{\mathbf{q}} a_{\mathbf{q}} + s_{\mathbf{q}} a_{-\mathbf{q}}^\dagger. \tag{6.5}$$

Therefore,

$$\begin{aligned}
H_{\mathrm{Bog}} &= \sum_{\mathbf{p}\neq 0} \frac{1}{2}\left(A_{\mathbf{p}}(a_{\mathbf{p}}^\dagger a_{\mathbf{p}} + a_{-\mathbf{p}}^\dagger a_{-\mathbf{p}}) + B_{\mathbf{p}}(a_{\mathbf{p}}^\dagger a_{-\mathbf{p}}^\dagger + a_{\mathbf{p}}a_{-\mathbf{p}})\right)\\
&= -\frac{1}{2}\sum_{\mathbf{p}\neq 0}\left(A_{\mathbf{p}} - \sqrt{A_{\mathbf{p}}^2 - B_{\mathbf{p}}^2}\right) && (6.6)\\
&\quad + \sum_{\mathbf{p}\neq 0}\sqrt{A_{\mathbf{p}}^2 - B_{\mathbf{p}}^2}\,(c_{\mathbf{p}}a_{\mathbf{p}}^\dagger + s_{\mathbf{p}}a_{-\mathbf{p}})\left(c_{\mathbf{p}}a_{\mathbf{p}} + s_{\mathbf{p}}a_{-\mathbf{p}}^\dagger\right) && (6.7)\\
&= E_{\mathrm{Bog}} + S\left(\sum_{\mathbf{p}\neq 0} e_{\mathbf{p}} a_{\mathbf{p}}^\dagger a_{\mathbf{p}}\right) S^\dagger, && (6.8)
\end{aligned}$$

where $e_{\mathbf{p}}$ and E_{Bog} are defined in the introduction. Thus the spectrum of $H_{\text{Bog}} - E_{\text{Bog}}$ equals

$$\left\{ \sum_{i=1}^{j} e_{\mathbf{k}_i} \ : \ \mathbf{k}_1, \ldots, \mathbf{k}_j \in \frac{2\pi}{L}\mathbb{Z}^d \backslash \{\mathbf{0}\}, \quad j = 0, 1, 2, \ldots \right\}.$$

For further reference note the following identities:

$$\begin{aligned}
\alpha_{\mathbf{p}} &= \frac{\hat{v}(\mathbf{p})}{|\mathbf{p}|^2 + \hat{v}(\mathbf{p}) + |\mathbf{p}|\sqrt{2\hat{v}(\mathbf{p}) + |\mathbf{p}|^2}}, \\
(c_{\mathbf{p}} - s_{\mathbf{p}})^2 = \frac{1 - \alpha_{\mathbf{p}}}{1 + \alpha_{\mathbf{p}}} &= \frac{|\mathbf{p}|}{\sqrt{|\mathbf{p}|^2 + 2\hat{v}(\mathbf{p})}}, \qquad (6.9) \\
s_{\mathbf{p}}(c_{\mathbf{p}} - s_{\mathbf{p}}) = \frac{\alpha_{\mathbf{p}}}{1 + \alpha_{\mathbf{p}}} &= \frac{\hat{v}(\mathbf{p})}{|\mathbf{p}|^2 + 2\hat{v}(\mathbf{p}) + |\mathbf{p}|\sqrt{|\mathbf{p}|^2 + 2\hat{v}(\mathbf{p})}}, \qquad (6.10) \\
2s_{\mathbf{p}}c_{\mathbf{p}}(c_{\mathbf{p}} - s_{\mathbf{p}})^2 = \frac{\alpha_{\mathbf{p}}}{(1 + \alpha_{\mathbf{p}})^2} &= \frac{\hat{v}(\mathbf{p})}{|\mathbf{p}|^2 + 2\hat{v}(\mathbf{p})}.
\end{aligned}$$

We note also an alternative formula for the Bogoliubov energy:

$$E_{\text{Bog}} = -\frac{1}{2} \sum_{\mathbf{p} \in \frac{2\pi}{L}\mathbb{Z}^d \backslash \{0\}} \frac{\hat{v}(\mathbf{p})^2}{|\mathbf{p}|^2 + \hat{v}(\mathbf{p}) + |\mathbf{p}|\sqrt{|\mathbf{p}|^2 + 2\hat{v}(\mathbf{p})|\mathbf{p}|}}.$$

7. Lower Bound

In this section, we prove the lower bound part of Theorem 1.1. Using the notation introduced in the previous sections it follows from the following statement:

Theorem 7.1. *Let $c > 0$. Then there exists C such that for any $\kappa \geq 0$ with*

$$L^{d+2}(L^d + \kappa) \leq cN \qquad (7.1)$$

we have

$$\begin{aligned}
\overrightarrow{\mathrm{sp}} \left(\mathbb{1}_{[0,\kappa]}(H_N - E_N) H_N \right) \geq \frac{1}{2}\hat{v}(\mathbf{0})(N-1) + \overrightarrow{\mathrm{sp}} \left(H_{\text{Bog}} \right) \\
-CN^{-1/2}L^{d/2+3}(\kappa + L^d)^{3/2}.
\end{aligned}$$

The proof of the lower bound starts with estimates analogous to Lemmas 1 and 2 of [18]. Note that in these estimates all operators involve the physical Hilbert space.

Lemma 7.2. *The ground-state energy E_N of H_N satisfies the bounds*

$$0 \geq E_N - \frac{1}{2}(N-1)\,\hat{v}(\mathbf{0}) \geq \frac{1}{2}\left(\hat{v}(\mathbf{0}) - L^d v(\mathbf{0})\right). \qquad (7.2)$$

Proof. The upper bound to the ground-state energy follows by using a constant trial wave function $\Psi = L^{-Nd/2}$, which gives

$$\frac{1}{2}(N-1)\hat{v}(\mathbf{0}) \geq E_N. \tag{7.3}$$

Using $\hat{v}(\mathbf{p}) \geq 0$ for every $\mathbf{p} \in \frac{2\pi}{L}\mathbb{Z}^d$ we obtain $\sup_{\mathbf{x}} v(\mathbf{x}) = v(\mathbf{0})$. Moreover,

$$\frac{1}{2L^d} \sum_{\mathbf{p}\in\frac{2\pi}{L}\mathbb{Z}^d\setminus\{\mathbf{0}\}} \hat{v}(\mathbf{p}) \left|\sum_{i=1}^{N} \mathrm{e}^{\mathrm{i}\mathbf{p}\mathbf{x}_j}\right|^2 \geq 0.$$

This is equivalent to

$$\sum_{1\leq i<j\leq N} v(\mathbf{x}_i - \mathbf{x}_j) \geq \frac{N^2}{2L^d}\hat{v}(\mathbf{0}) - \frac{N}{2}v(\mathbf{0}). \tag{7.4}$$

Hence,

$$H_N \geq T + \frac{L^d}{N}\left(\frac{N^2}{2L^d}\hat{v}(\mathbf{0}) - \frac{N}{2}v(\mathbf{0})\right), \tag{7.5}$$

and so

$$E_N \geq \frac{L^d}{N}\left(\frac{N^2}{2L^d}\hat{v}(\mathbf{0}) - \frac{N}{2}v(\mathbf{0})\right).$$

□

Let $\kappa \geq 0$. For brevity, we introduce the following notation for the spectral projection onto the spectral subspace of H_N corresponding to the energy less than or equal to $E_N + \kappa$:

$$\mathbb{1}_\kappa^N := \mathbb{1}_{[0,\kappa]}(H_N - E_N).$$

$\mathbb{1}_\kappa^N$ can be understood as a projection acting on the extended space with range in the physical space.

Lemma 7.3. *There exists C such that*

$$N^> \leq CL^2(H_N - E_N + L^d). \tag{7.6}$$

Consequently,

$$\mathbb{1}_\kappa^N N^> \mathbb{1}_\kappa^N \leq CL^2\left(L^d + \kappa\right). \tag{7.7}$$

Proof. Using first (7.5) and (7.3) we obtain

$$\begin{aligned} T &\leq H_N - E_N - \frac{1}{2}\hat{v}(\mathbf{0}) + \frac{1}{2}L^d v(\mathbf{0}) \\ &\leq C(H_N - E_N + L^d). \end{aligned}$$

By (3.5) this implies (7.6). □

Lemma 7.4. *We have*

$$\mathbb{1}_\kappa^N (N^>)^2 \mathbb{1}_\kappa^N \leq CL^4\left(L^d + \kappa\right)^2. \tag{7.8}$$

Proof. Let $\mathbb{1}_\kappa^N \Psi = \Psi$. As in [18],

$$\langle \Psi | N^> T | \Psi \rangle = \left\langle \Psi | N^> \left(H_N - E_N - \frac{1}{2}\kappa \right) | \Psi \right\rangle \tag{7.9}$$

$$+ N \left\langle \Psi | Q_1 \left(E_N + \frac{1}{2}\kappa - \frac{L^d}{N} \sum_{2 \le i < j \le N} v(\mathbf{x}_i - \mathbf{x}_j) \right) | \Psi \right\rangle \tag{7.10}$$

$$- L^d \left\langle \Psi | Q_1 \sum_{2 \le j \le N} v(\mathbf{x}_1 - \mathbf{x}_j) | \Psi \right\rangle. \tag{7.11}$$

Using Schwarz's inequality, the first term can be bounded as

$$\begin{aligned} |(7.9)| &\le \| N^> \Psi \| \left\| H_N - E_N - \frac{1}{2}\kappa \right\| \\ &\le \frac{\kappa}{2} \langle \Psi | (N^>)^2 \Psi \rangle^{1/2}. \end{aligned}$$

Let us estimate the second term. Using (7.4) we get

$$\begin{aligned} &E_N - \frac{L^d}{N} \sum_{2 \le i < j \le N} v(\mathbf{x}_i - \mathbf{x}_j) \\ &\le \frac{1}{2}(N-1)\hat{v}(\mathbf{0}) + \frac{L^d}{2N}(N-1)v(\mathbf{0}) - \frac{1}{2N}(N-1)^2 \hat{v}(\mathbf{0}) \\ &= \frac{1}{2}\frac{N-1}{N} \big(\hat{v}(\mathbf{0}) + L^d v(\mathbf{0}) \big). \end{aligned}$$

Hence,

$$\begin{aligned} (7.10) &\le \left(\frac{\kappa}{2} + \frac{1}{2}\frac{N-1}{N} \big(\hat{v}(\mathbf{0}) + L^d v(\mathbf{0}) \big) \right) N \langle \Psi | Q_1 | \Psi \rangle \\ &\le \left(\frac{\kappa}{2} + \frac{1}{2} \big(\hat{v}(\mathbf{0}) + L^d v(\mathbf{0}) \big) \right) \langle \Psi | N^> | \Psi \rangle. \end{aligned}$$

Finally, let us consider the third term:

$$\begin{aligned} \langle \Psi | Q_1 v(\mathbf{x}_1 - \mathbf{x}_2) | \Psi \rangle &= \langle \Psi | Q_1 Q_2 v(\mathbf{x}_1 - \mathbf{x}_2) | \Psi \rangle \\ &\quad + \langle \Psi | Q_1 P_2 v(\mathbf{x}_1 - \mathbf{x}_2) Q_2 | \Psi \rangle \\ &\quad + \langle \Psi | Q_1 P_2 v(\mathbf{x}_1 - \mathbf{x}_2) Q_2 | \Psi \rangle, \\ | \langle \Psi | Q_1 Q_2 v(\mathbf{x}_1 - \mathbf{x}_2) | \Psi \rangle | &\le v(\mathbf{0}) \langle \Psi | Q_1 Q_2 | \Psi \rangle^{1/2}, \\ | \langle \Psi | Q_1 P_2 v(\mathbf{x}_1 - \mathbf{x}_2) Q_2 | \Psi \rangle | &\le v(\mathbf{0}) \langle \Psi | Q_1 | \Psi \rangle, \\ \langle \Psi | Q_1 P_2 v(\mathbf{x}_1 - \mathbf{x}_2) P_2 | \Psi \rangle | &= \hat{v}(\mathbf{0}) \langle \Psi | Q_1 P_2 | \Psi \rangle \ge 0. \end{aligned}$$

Therefore, using (3.3) and (3.4)

$$\begin{aligned} |(7.11)| &\le v(\mathbf{0}) L^d \left(\sqrt{\frac{N-1}{N}} \langle \Psi | (N^> - 1) N^> | \Psi \rangle^{1/2} + \frac{N-1}{N} \langle \Psi | N^> | \Psi \rangle \right) \\ &\le v(\mathbf{0}) L^d \left(\langle \Psi | (N^>)^2 \Psi \rangle^{1/2} + \langle \Psi | N^> | \Psi \rangle \right). \end{aligned}$$

Now

$$\langle\psi|(N^>)^2|\psi\rangle \leq \frac{L^2}{(2\pi)^2}\langle\psi|N^>T|\psi\rangle. \tag{7.12}$$

We can add the three estimates, use (7.12) and obtain

$$\begin{aligned}\langle\Psi|N^>T|\Psi\rangle &\leq C(\kappa+L^d)\left(\langle\Psi|(N^>)^2|\Psi\rangle^{1/2}+\langle\Psi|N^>|\Psi\rangle\right)\\ &\leq CL^2(\kappa+L^d)^2\\ &\quad+CL(\kappa+L^d)\langle\Psi|N^>T|\Psi\rangle^{1/2}.\end{aligned}$$

Setting $X:=\langle\psi|N^>T|\psi\rangle^{1/2}$ we can rewrite this as $X^2<c+aX$ in the obvious notation. Solving this inequality we get that

$$X^2 \leq \frac{a^2}{2}+c+\sqrt{a^2+4c}.$$

This implies

$$\mathbb{1}_\kappa^N N^>T\mathbb{1}_\kappa^N \leq CL^2\left(L^d+\kappa\right)^2. \tag{7.13}$$

If in addition we use (7.12), we obtain (7.8). □

Lemma 7.5.

$$\sup_{0<\epsilon\leq 1}\mathbb{1}_\kappa^N R_{N,-\epsilon}\mathbb{1}_\kappa^N \geq -CN^{-1/2}L^{d/2+3}(L^d+\kappa)^{3/2}. \tag{7.14}$$

Proof.

$$\begin{aligned}\mathbb{1}_\kappa^N R_{N,-\epsilon}\mathbb{1}_\kappa^N \geq{}& \mathbb{1}_\kappa^N\frac{1}{2}\sum_{\mathbf{p}\neq\mathbf{0}}\hat{v}(\mathbf{p})\left(\left(\frac{\sqrt{(N_0-1)N_0}}{N}-1\right)b_{\mathbf{p}}b_{-\mathbf{p}}+\mathrm{hc}\right)\mathbb{1}_\kappa^N\\ &-\mathbb{1}_\kappa^N\frac{1}{N}\sum_{\mathbf{p}\neq\mathbf{0}}\left(\hat{v}(\mathbf{p})+\frac{\hat{v}(\mathbf{0})}{2}\right)b_{\mathbf{p}}^\dagger b_{\mathbf{p}}N^>\mathbb{1}_\kappa^N\\ &-\epsilon\mathbb{1}_\kappa^N\frac{1}{N}\sum_{\mathbf{p}\neq\mathbf{0}}\left(\hat{v}(\mathbf{p})+\hat{v}(\mathbf{0})\right)b_{\mathbf{p}}^\dagger b_{\mathbf{p}}N_0\mathbb{1}_\kappa^N\\ &-\epsilon^{-1}\mathbb{1}_\kappa^N\frac{1}{2N}v(\mathbf{0})L^d(N^>)^2\mathbb{1}_\kappa^N.\end{aligned} \tag{7.15}$$

Note that the range of $\mathbb{1}_\kappa^N$ is inside the physical space, so whenever possible we replaced N_0^{ext} by N_0. It is easy to estimate from below various terms on the right of (7.15) by expressions involving $N^>$. The first term requires more work than the others. We have

$$\begin{aligned}N-\sqrt{(N_0-1)N_0} &= \frac{2NN^>-(N^>)^2+N-N^>}{N+\sqrt{(N-N^>-1)(N-N^>)}}\\ &\leq 2N^>+1.\end{aligned}$$

Then we use

$$\begin{aligned}&\left(\sqrt{(N_0-1)N_0}-N\right)\sum_{\mathbf{p}\neq\mathbf{0}}\hat{v}(\mathbf{p})b_{\mathbf{p}}b_{-\mathbf{p}}+\mathrm{hc}\\&\quad\geq-\left(\sum_{\mathbf{p}\neq\mathbf{0}}\hat{v}(\mathbf{p})b_{\mathbf{p}}b_{-\mathbf{p}}\right)^{\dagger}\sum_{\mathbf{p}\neq\mathbf{0}}\hat{v}(\mathbf{p})b_{\mathbf{p}}b_{-\mathbf{p}}\\&\qquad-\left(\sqrt{(N_0-1)N_0}-N\right)^2\\&\quad\geq-C(N^{>})^2-(2N^{>}+1)^2\\&\quad\geq-C_1\left((N^{>})^2+1\right).\end{aligned}$$

To bound the third term we use $N_0\leq N$. We obtain

$$\begin{aligned}\mathbb{1}_\kappa^N R_{N,-\epsilon}\mathbb{1}_\kappa^N\geq&-C\mathbb{1}_\kappa^N\frac{(N^{>})^2+1}{N}\mathbb{1}_\kappa^N\\&-C\mathbb{1}_\kappa^N\frac{(N^{>})^2}{N}\mathbb{1}_\kappa^N\\&-\epsilon C\mathbb{1}_\kappa^N N^{>}\mathbb{1}_\kappa^N\\&-\epsilon^{-1}C\mathbb{1}_\kappa^N L^d\frac{(N^{>})^2}{N}\mathbb{1}_\kappa^N.\end{aligned}$$

Using that $0\leq\epsilon\leq1$ and $L\geq1$, we can partly absorb the first two terms in the fourth:

$$\geq-\frac{C}{N}\mathbb{1}_\kappa^N-\epsilon C\mathbb{1}_\kappa^N N^{>}\mathbb{1}_\kappa^N-\epsilon^{-1}C\mathbb{1}_\kappa^N L^d\frac{(N^{>})^2}{N}\mathbb{1}_\kappa^N.$$

By (7.6) and (7.8), this can be estimated by

$$\geq-CN^{-1}-\epsilon CL^2(L^d+\kappa)-\epsilon^{-1}CN^{-1}L^{d+4}(L^d+\kappa)^2. \tag{7.16}$$

Setting $\epsilon=c^{-1/2}L^{d/2+1}(L^d+\kappa)^{1/2}N^{-1/2}$ in (7.16), which by Condition (7.1) is less than 1, we bound it by

$$\geq\ -CN^{-1}-CN^{-1/2}L^{d/2+3}(L^d+\kappa)^{3/2}.$$

Using $L\geq1$, we can absorb the first term in the second. □

Proof of Theorem 7.1. Recall inequality (4.1), which implies for $0<\epsilon\leq1$

$$\mathbb{1}_\kappa^N H_N\mathbb{1}_\kappa^N\geq\mathbb{1}_\kappa^N\left(\frac{1}{2}\hat{v}(\mathbf{0})(N-1)+H_{\mathrm{Bog},N}+R_{N,-\epsilon}\right)\mathbb{1}_\kappa^N. \tag{7.17}$$

Thus, it suffices to apply Lemma 7.5 and the min-max principle. □

Proof of Theorem 1.1 (1). First set $\kappa=0$. Then Condition (7.1) becomes Condition (1.7) and we obtain Theorem 1.1 (1a).

Next set $\kappa=K_N^j(\mathbf{p})$. Then Condition (7.1) is equivalent to the conjunction of Conditions (1.7) and (1.9). We obtain Theorem 1.1 (1b). □

8. Upper Bound

In this section we prove the following theorem, which implies the upper bound of Theorem 1.1:

Theorem 8.1. *Let $c > 0$. Then there exist $c_1 > 0$ and C such that if $\kappa \geq 0$ and*

$$L^{d+2}(\kappa + L^{d-1}) \leq cN, \tag{8.1}$$

$$L^2(\kappa + L^{d-1}) \leq c_1 N \tag{8.2}$$

then

$$\overrightarrow{\mathrm{sp}}\,(H_N) \leq \frac{1}{2}\hat{v}(\mathbf{0})(N-1) + \overrightarrow{\mathrm{sp}}\,\big(\mathbb{1}_{[0,\kappa]}(H_{\mathrm{Bog}} - E_{\mathrm{Bog}})H_{\mathrm{Bog}}\big)$$
$$+CN^{-1/2}L^{d/2+3}(\kappa + L^{d-1})^{3/2}.$$

For brevity, we set

$$\mathbb{1}_\kappa^{\mathrm{Bog}} := \mathbb{1}_{[0,\kappa]}(H_{\mathrm{Bog},N} - E_{\mathrm{Bog}}).$$

From now on, to simplify the notation we will also write H_{Bog} instead of $H_{\mathrm{Bog},N}$, even though this is an abuse of notation. ($H_{\mathrm{Bog},N}$ is unitarily equivalent, but strictly speaking distinct from (6.3)).

We also set

$$d_{\mathbf{p}} := S b_{\mathbf{p}} S^\dagger$$

where S is defined as in (6.4) with operators a's replaced by b's. Clearly,

$$d_{\mathbf{p}} = c_{\mathbf{p}} b_{\mathbf{p}} + s_{\mathbf{p}} b_{-\mathbf{p}}^\dagger, \quad d_{\mathbf{p}}^\dagger = c_{\mathbf{p}} b_{\mathbf{p}}^\dagger + s_{\mathbf{p}} b_{-\mathbf{p}}.$$

Lemma 8.2. *There exist C_1, C_2 such that*

$$H_{\mathrm{Bog}} - E_{\mathrm{Bog}} \geq C_1 L^{-2} N^> - C_2 L^{d-1}. \tag{8.3}$$

Consequently,

$$\mathbb{1}_\kappa^{\mathrm{Bog}} N^> \mathbb{1}_\kappa^{\mathrm{Bog}} \leq CL^2(L^{d-1} + \kappa). \tag{8.4}$$

Proof. Using (6.6) we have that

$$H_{\mathrm{Bog}} - E_{\mathrm{Bog}} = \sum_{\mathbf{p}\neq 0} e_{\mathbf{p}} S b_{\mathbf{p}}^\dagger b_{\mathbf{p}} S^\dagger$$
$$\geq \sum_{\mathbf{p}\neq 0} \frac{\pi\sqrt{8\hat{v}(\mathbf{0})}}{L} S b_{\mathbf{p}}^\dagger b_{\mathbf{p}} S^\dagger = \frac{\pi\sqrt{8\hat{v}(\mathbf{0})}}{L} S N^> S^\dagger.$$

Now

$$SN^>S^\dagger = \sum_{\pm\mathbf{p}\neq\mathbf{0}} \left(d_{\mathbf{p}}^\dagger d_{\mathbf{p}} + d_{-\mathbf{p}}^\dagger d_{-\mathbf{p}}\right) = \sum_{\pm\mathbf{p}\neq\mathbf{0}} \Big((c_{\mathbf{p}}^2 + s_{\mathbf{p}}^2)\left(b_{\mathbf{p}}^\dagger b_{\mathbf{p}} + b_{-\mathbf{p}}^\dagger b_{-\mathbf{p}}\right)$$
$$+2c_{\mathbf{p}} s_{\mathbf{p}}\left(b_{\mathbf{p}}^\dagger b_{-\mathbf{p}}^\dagger + b_{\mathbf{p}} b_{-\mathbf{p}}\right) + 2s_{\mathbf{p}}^2\Big).$$

(When we write $\pm\mathbf{p}$ under the summation symbol, we sum over all pairs $\{\mathbf{p}, -\mathbf{p}\}$). Using

$$b_{\mathbf{p}}^\dagger b_{-\mathbf{p}}^\dagger + b_{\mathbf{p}} b_{-\mathbf{p}} \geq -\left(b_{\mathbf{p}}^\dagger b_{\mathbf{p}} + b_{-\mathbf{p}}^\dagger b_{-\mathbf{p}} + 1\right)$$

we obtain

$$\begin{aligned}&\sum_{\pm\mathbf{p}\neq\mathbf{0}} \left(d_{\mathbf{p}}^\dagger d_{\mathbf{p}} + d_{-\mathbf{p}}^\dagger d_{-\mathbf{p}}\right)\\ &\quad\geq \sum_{\pm\mathbf{p}\neq\mathbf{0}} \left((c_{\mathbf{p}} - s_{\mathbf{p}})^2 \left(b_{\mathbf{p}}^\dagger b_{\mathbf{p}} + b_{-\mathbf{p}}^\dagger b_{-\mathbf{p}}\right) - 2s_{\mathbf{p}}(c_{\mathbf{p}} - s_{\mathbf{p}})\right). \end{aligned} \tag{8.5}$$

By (6.9) we know that $\inf_{\mathbf{p}\neq 0}(c_{\mathbf{p}} - s_{\mathbf{p}})^2 \geq \frac{\sqrt{2}\pi}{\sqrt{\hat{v}(0)}L}$. Also, (6.10) yields

$$\frac{1}{L^d} \sum_{\pm\mathbf{p}\neq\mathbf{0}} s_{\mathbf{p}}(c_{\mathbf{p}} - s_{\mathbf{p}}) < \infty,$$

uniformly in L. Thus

$$\begin{aligned} H_{\mathrm{Bog}} - E_{\mathrm{Bog}} &\geq \frac{C}{L} S N^> S^\dagger \\ &\geq \frac{C_1}{L^2} \sum_{\pm\mathbf{p}\neq\mathbf{0}} \left(b_{\mathbf{p}}^\dagger b_{\mathbf{p}} + b_{-\mathbf{p}}^\dagger b_{-\mathbf{p}}\right) - \frac{C_2 L^{d-1}}{L^d} \sum_{\pm\mathbf{p}\neq\mathbf{0}} 2s_{\mathbf{p}}(c_{\mathbf{p}} - s_{\mathbf{p}}) \\ &= C_1 L^{-2} N^> - C_2 L^{d-1}. \end{aligned}$$

This proves (8.3), which can be rewritten as

$$N^> \leq C_1^{-1} L^2 (H_{\mathrm{Bog}} - E_{\mathrm{Bog}} + C_2 L^{d-1}), \tag{8.6}$$

which implies (8.4). □

Lemma 8.3. *Set*

$$\begin{aligned} M &:= \sum_{\mathbf{p}\neq\mathbf{0}} (c_{\mathbf{p}} - s_{\mathbf{p}})^2 b_{\mathbf{p}}^\dagger b_{\mathbf{p}}, \\ A_1 &:= \sum_{\mathbf{p}\neq\mathbf{0}} 2s_{\mathbf{p}}(c_{\mathbf{p}} - s_{\mathbf{p}}), \\ A_2 &:= \sum_{\mathbf{p}\neq 0} 4(c_{\mathbf{p}} - s_{\mathbf{p}})^2 s_{\mathbf{p}}^2. \end{aligned}$$

Then

$$(SN^> S^\dagger + A_1)^2 \geq M^2 - A_2. \tag{8.7}$$

Proof.

$$\begin{aligned}&\left(\sum_{\pm\mathbf{p}\neq\mathbf{0}} \left(d_{\mathbf{p}}^\dagger d_{\mathbf{p}} + d_{-\mathbf{p}}^\dagger d_{-\mathbf{p}} + 2s_{\mathbf{p}}(c_{\mathbf{p}} - s_{\mathbf{p}})\right)\right)^2 \\ &\quad= \sum_{\pm\mathbf{p},\pm\mathbf{q}\neq\mathbf{0}} \left(d_{\mathbf{p}}^\dagger \left(d_{\mathbf{q}}^\dagger d_{\mathbf{q}} + d_{-\mathbf{q}}^\dagger d_{-\mathbf{q}} + 2s_{\mathbf{q}}(c_{\mathbf{q}} - s_{\mathbf{q}})\right) d_{\mathbf{p}}\right.\end{aligned}$$

$$
\begin{aligned}
&+d_{-\mathbf{p}}^{\dagger}\left(d_{\mathbf{q}}^{\dagger}d_{\mathbf{q}}+d_{-\mathbf{q}}^{\dagger}d_{-\mathbf{q}}+2s_{\mathbf{q}}(c_{\mathbf{q}}-s_{\mathbf{q}})\right)d_{-\mathbf{p}}\Big)\\
&+\sum_{\pm\mathbf{p},\pm\mathbf{q}\neq\mathbf{0}}2s_{\mathbf{p}}(c_{\mathbf{p}}-s_{\mathbf{p}})\left(d_{\mathbf{q}}^{\dagger}d_{\mathbf{q}}+d_{-\mathbf{q}}^{\dagger}d_{-\mathbf{q}}+2s_{\mathbf{q}}(c_{\mathbf{q}}-s_{\mathbf{q}})\right)\\
&+\sum_{\pm\mathbf{p}\neq\mathbf{0}}\left(d_{\mathbf{p}}^{\dagger}d_{\mathbf{p}}+d_{-\mathbf{p}}^{\dagger}d_{-\mathbf{p}}\right).
\end{aligned}
$$

Using (8.5) we bound this from below by

$$
\begin{aligned}
&\sum_{\pm\mathbf{p},\pm\mathbf{q}\neq\mathbf{0}}(c_{\mathbf{q}}-s_{\mathbf{q}})^2\left(d_{\mathbf{p}}^{\dagger}\left(b_{\mathbf{q}}^{\dagger}b_{\mathbf{q}}+b_{-\mathbf{q}}^{\dagger}b_{-\mathbf{q}}\right)d_{\mathbf{p}}+d_{-\mathbf{p}}^{\dagger}\left(b_{\mathbf{q}}^{\dagger}b_{\mathbf{q}}+b_{-\mathbf{q}}^{\dagger}b_{-\mathbf{q}}\right)d_{-\mathbf{p}}\right)\\
&\quad+\sum_{\pm\mathbf{p},\pm\mathbf{q}\neq\mathbf{0}}2s_{\mathbf{p}}(c_{\mathbf{p}}-s_{\mathbf{p}})(c_{\mathbf{q}}-s_{\mathbf{q}})^2\left(b_{\mathbf{q}}^{\dagger}b_{\mathbf{q}}+b_{-\mathbf{q}}^{\dagger}b_{-\mathbf{q}}\right)\\
&\quad+\sum_{\pm\mathbf{p}\neq\mathbf{0}}\left((c_{\mathbf{p}}-s_{\mathbf{p}})^2\left(b_{\mathbf{p}}^{\dagger}b_{\mathbf{p}}+b_{-\mathbf{p}}^{\dagger}b_{-\mathbf{p}}\right)-2s_{\mathbf{p}}(c_{\mathbf{p}}-s_{\mathbf{p}})\right)\\
&\quad=\sum_{\pm\mathbf{p},\pm\mathbf{q}\neq\mathbf{0}}(c_{\mathbf{q}}-s_{\mathbf{q}})^2\left(b_{\mathbf{q}}^{\dagger}\left(d_{\mathbf{p}}^{\dagger}d_{\mathbf{p}}+d_{-\mathbf{p}}^{\dagger}d_{-\mathbf{p}}\right)b_{\mathbf{q}}+b_{-\mathbf{q}}^{\dagger}\left(d_{\mathbf{p}}^{\dagger}d_{\mathbf{p}}+d_{-\mathbf{p}}^{\dagger}d_{-\mathbf{p}}\right)b_{-\mathbf{p}}\right)\\
&\quad+\sum_{\pm\mathbf{p}\neq\mathbf{0}}(c_{\mathbf{p}}-s_{\mathbf{p}})^2\left(2s_{\mathbf{p}}^2\left(b_{\mathbf{p}}^{\dagger}b_{\mathbf{p}}+b_{-\mathbf{p}}^{\dagger}b_{-\mathbf{p}}\right)+2c_{\mathbf{p}}s_{\mathbf{p}}\left(b_{\mathbf{p}}^{\dagger}b_{-\mathbf{p}}^{\dagger}+b_{\mathbf{p}}b_{-\mathbf{p}}\right)+2s_{\mathbf{p}}^2\right)\\
&\quad+\sum_{\pm\mathbf{p},\pm\mathbf{q}\neq\mathbf{0}}2s_{\mathbf{p}}(c_{\mathbf{p}}-s_{\mathbf{p}})(c_{\mathbf{q}}-s_{\mathbf{q}})^2\left(b_{\mathbf{q}}^{\dagger}b_{\mathbf{q}}+b_{-\mathbf{q}}^{\dagger}b_{-\mathbf{q}}\right)\\
&\quad+\sum_{\pm\mathbf{p}\neq\mathbf{0}}\left((c_{\mathbf{p}}-s_{\mathbf{p}})^2\left(b_{\mathbf{p}}^{\dagger}b_{\mathbf{p}}+b_{-\mathbf{p}}^{\dagger}b_{-\mathbf{p}}\right)-2s_{\mathbf{p}}(c_{\mathbf{p}}-s_{\mathbf{p}})\right)\\
&\quad=\sum_{\pm\mathbf{p},\pm\mathbf{q}\neq\mathbf{0}}(c_{\mathbf{q}}-s_{\mathbf{q}})^2\left(b_{\mathbf{q}}^{\dagger}\left(d_{\mathbf{p}}^{\dagger}d_{\mathbf{p}}+d_{-\mathbf{p}}^{\dagger}d_{-\mathbf{p}}+2s_{\mathbf{p}}(c_{\mathbf{p}}-s_{\mathbf{p}})\right)b_{\mathbf{q}}\right.\\
&\quad\left.+b_{-\mathbf{q}}^{\dagger}\left(d_{\mathbf{p}}^{\dagger}d_{\mathbf{p}}+d_{-\mathbf{p}}^{\dagger}d_{-\mathbf{p}}+2s_{\mathbf{p}}(c_{\mathbf{p}}-s_{\mathbf{p}})\right)b_{-\mathbf{p}}\right)\\
&\quad+\sum_{\pm\mathbf{p}\neq\mathbf{0}}(c_{\mathbf{p}}-s_{\mathbf{p}})^2\left(\left(2s_{\mathbf{p}}^2+1\right)\left(b_{\mathbf{p}}^{\dagger}b_{\mathbf{p}}+b_{-\mathbf{p}}^{\dagger}b_{-\mathbf{p}}\right)+2c_{\mathbf{p}}s_{\mathbf{p}}\left(b_{\mathbf{p}}^{\dagger}b_{-\mathbf{p}}^{\dagger}+b_{\mathbf{p}}b_{-\mathbf{p}}\right)\right)\\
&\quad+\sum_{\pm\mathbf{p}\neq\mathbf{0}}\left((c_{\mathbf{p}}-s_{\mathbf{p}})^2 2s_{\mathbf{p}}^2-2s_{\mathbf{p}}(c_{\mathbf{p}}-s_{\mathbf{p}})\right).
\end{aligned}
$$

Using (8.5) one more time, we bound this from below by

$$
\begin{aligned}
&\sum_{\pm\mathbf{p},\pm\mathbf{q}\neq\mathbf{0}}(c_{\mathbf{q}}-s_{\mathbf{q}})^2(c_{\mathbf{p}}-s_{\mathbf{p}})^2\\
&\qquad\times\left(b_{\mathbf{q}}^{\dagger}\left(b_{\mathbf{p}}^{\dagger}b_{\mathbf{p}}+b_{-\mathbf{p}}^{\dagger}b_{-\mathbf{p}}\right)b_{\mathbf{q}}+b_{-\mathbf{q}}^{\dagger}\left(b_{\mathbf{p}}^{\dagger}b_{\mathbf{p}}+b_{-\mathbf{p}}^{\dagger}b_{-\mathbf{p}}\right)b_{-\mathbf{q}}\right)\\
&\qquad+\sum_{\pm\mathbf{p}\neq\mathbf{0}}(c_{\mathbf{p}}-s_{\mathbf{p}})^2(2s_{\mathbf{p}}^2-2c_{\mathbf{p}}s_{\mathbf{p}}+1)\left(b_{\mathbf{p}}^{\dagger}b_{\mathbf{p}}+b_{-\mathbf{p}}^{\dagger}b_{-\mathbf{p}}\right)
\end{aligned}
$$

$$+\sum_{\pm\mathbf{p}\neq\mathbf{0}}\left((-2c_{\mathbf{p}}s_{\mathbf{p}}+2s_{\mathbf{p}}^2)(c_{\mathbf{p}}-s_{\mathbf{p}})^2-2s_{\mathbf{p}}(c_{\mathbf{p}}-s_{\mathbf{p}})\right)$$

$$=\left(\sum_{\pm\mathbf{p}\neq 0}(c_{\mathbf{p}}-s_{\mathbf{p}})^2\left(b_{\mathbf{p}}^\dagger b_{\mathbf{p}}+b_{-\mathbf{p}}^\dagger b_{-\mathbf{p}}\right)\right)^2-\sum_{\pm\mathbf{p}\neq 0}4(c_{\mathbf{p}}-s_{\mathbf{p}})^2 c_{\mathbf{p}}s_{\mathbf{p}}.$$

□

Lemma 8.4. *There exist C_1, C_2 such that*

$$(H_{\mathrm{Bog}}-E_{\mathrm{Bog}})^2\geq C_1L^{-4}(N^>)^2-C_2L^{2d-2}. \tag{8.8}$$

Therefore,

$$\mathbb{1}_\kappa^{\mathrm{Bog}}(N^>)^2\mathbb{1}_\kappa^{\mathrm{Bog}}\leq CL^4(L^{d-1}+\kappa)^2. \tag{8.9}$$

Proof. As in the proof of Lemma 8.2,

$$(H_{\mathrm{Bog}}-E_{\mathrm{Bog}})^2\geq\frac{\left(\pi\sqrt{8\hat{v}(\mathbf{0})}\right)^2}{L^2}\left(SN^>S^\dagger\right)^2. \tag{8.10}$$

For any $\delta>0$, Lemma 8.3 implies

$$(1+\delta)(SN^>S^\dagger)^2+(1+\delta^{-1})A_1^2\geq M^2-A_2.$$

Moreover, the limits $\lim_{L\to\infty}\frac{A_1}{L^d}$ and $\lim_{L\to\infty}\frac{A_2}{L^d}$ exist. Therefore,

$$\left(SN^>S^\dagger\right)^2\geq M^2-CL^{2d}.$$

Using (8.10) and $M\geq C_1L^{-1}N^>$, we easily conclude that (8.8) holds. Hence

$$(N^>)^2\leq C_2^{-1}L^4\left((H_{\mathrm{Bog}}-E_{\mathrm{Bog}})^2+C_3L^{2d-2}\right),$$

which easily implies (8.9). □

Suppose now that G is a smooth nonnegative function on $[0,\infty[$ such that

$$G(s)=\begin{cases}1, & \text{if } s\in[0,\frac{1}{3}]\\ 0, & \text{if } s\in[1,\infty[.\end{cases} \tag{8.11}$$

Set

$$A_N:=G(N^>/N),\quad A_N^{\mathrm{nph}}:=\mathbb{1}-A_N.$$

The operator A_N will serve as a smooth approximation to the projection onto the physical space. Set

$$Y_\kappa:=\mathbb{1}_\kappa^{\mathrm{Bog}}A_N.$$

Lemma 8.5. *We have*

$$\mathbb{1}_\kappa^{\mathrm{Bog}}-Y_\kappa Y_\kappa^\dagger=O\left(L^2(\kappa+L^{d-1})N^{-1}\right).$$

Proof. We have

$$\begin{aligned}\mathbb{1}_\kappa^{\mathrm{Bog}}-Y_\kappa Y_\kappa^\dagger&=\mathbb{1}_\kappa^{\mathrm{Bog}}\left(1-G(N^>/N)^2\right)\mathbb{1}_\kappa^{\mathrm{Bog}}\\&=\mathbb{1}_\kappa^{\mathrm{Bog}}(N^>/N)^{1/2}\left(\left(1-G(N^>/N)^2\right)(N^>/N)^{-1}\right)(N^>/N)^{1/2}\mathbb{1}_\kappa^{\mathrm{Bog}}.\end{aligned}$$

But

$$\| \left(1 - G(N^>/N)^2\right) (N^>/N)^{-1} \| = \sup_s \{ |(1 - G(s)^2)s^{-1}| \} < \infty,$$

and by (8.4)

$$(N^>/N)^{-1/2} \mathbb{1}_\kappa^{\mathrm{Bog}} = O\left(L(\kappa + L^{d-1})^{1/2} N^{-1/2} \right).$$

□

Let $0 < c_0 < 1$. If

$$\| \mathbb{1}_\kappa^{\mathrm{Bog}} - Y_\kappa Y_\kappa^\dagger \| \leq c_0, \tag{8.12}$$

then $Y_\kappa Y_\kappa^\dagger$ is invertible on $\mathrm{Ran}\mathbb{1}_\kappa^{\mathrm{Bog}}$. We will denote by $\left(Y_\kappa Y_\kappa^\dagger\right)^{-1}$ the corresponding inverse. We set

$$X_\kappa := \left(Y_\kappa Y_\kappa^\dagger\right)^{-1/2}.$$

On the orthogonal complement of $\mathrm{Ran}\mathbb{1}_\kappa^{\mathrm{Bog}}$ we extend it by 0.

By Lemma 8.5 and Condition (8.2) with a sufficiently small c_1, we can guarantee that (8.12) holds with, say, $c_0 \leq 1/2$. Therefore, in what follows X_κ is well defined.

Lemma 8.6.

$$\mathbb{1}_\kappa^{\mathrm{Bog}} - X_\kappa = O\left(L^2(\kappa + L^{d-1}) N^{-1} \right). \tag{8.13}$$

Proof.

$$\| \mathbb{1}_\kappa^{\mathrm{Bog}} - \left(Y_\kappa Y_\kappa^\dagger\right)^{-1} \| \leq c_0 (1 - c_0)^{-1}$$

by the convergent Neumann series. This is $O\left(L^2(\kappa + L^{d-1})N^{-1}\right)$. This implies (8.13) by the spectral theorem. □

Lemma 8.7.

$$X_\kappa [A_N, [A_N, H_{\mathrm{Bog}}]] X_\kappa = O\left(N^{-2} L^2 (\kappa + L^{d-1}) \right).$$

Proof. We have

$$[N^>, [N^>, H_{\mathrm{Bog}}]] = 2 \sum \hat{v}(\mathbf{p}) (b_{\mathbf{p}} b_{-\mathbf{p}} + b_{\mathbf{p}}^\dagger b_{-\mathbf{p}}^\dagger).$$

Using

$$-b_{\mathbf{p}}^\dagger b_{\mathbf{p}} - b_{-\mathbf{p}}^\dagger b_{-\mathbf{p}} - 1 \leq b_{\mathbf{p}} b_{-\mathbf{p}} + b_{\mathbf{p}}^\dagger b_{-\mathbf{p}}^\dagger \leq b_{\mathbf{p}}^\dagger b_{\mathbf{p}} + b_{-\mathbf{p}}^\dagger b_{-\mathbf{p}} + 1$$

we obtain

$$-C(N^> + L^d) \leq [N^>, [N^>, H_{\mathrm{Bog}}]] \leq C(N^> + L^d).$$

This implies

$$\left\| (N^> + L^d)^{-1/2} \left[N^>, [N^>, H_{\mathrm{Bog}}] \right] (N^> + L^d)^{-1/2} \right\| \leq C. \tag{8.14}$$

Now we use one of the well-known methods for dealing with functions of operators, for instance, the representation

$$A_N = G(N^>/N) = \frac{1}{2\pi}\int \hat{G}(t)\,\mathrm{e}^{\mathrm{i}tN^>/N}\,\mathrm{d}t.$$

To this end, note that for operators S and T one has

$$\left[\mathrm{e}^{\mathrm{i}tS}, T\right] = \int_0^t \frac{\mathrm{d}}{\mathrm{d}u}\,\mathrm{e}^{\mathrm{i}uS}\,T\,\mathrm{e}^{-\mathrm{i}uS}\,\mathrm{d}u\,\mathrm{e}^{\mathrm{i}tS} = \int_0^t \mathrm{i}\mathrm{e}^{\mathrm{i}uS}[S,T]\,\mathrm{e}^{\mathrm{i}(t-u)S}\,\mathrm{d}u$$

which together with the representation mentioned above yields

$$\begin{aligned}[A_N,[A_N,H_{\mathrm{Bog}}]] = \frac{-1}{4\pi^2N^2}\int \mathrm{d}p \int_0^p \mathrm{d}s \int \mathrm{d}t \int_0^t \mathrm{d}u \hat{G}(p)\hat{G}(t)\\ \times\,\mathrm{e}^{\mathrm{i}(s+u)\frac{N^>}{N}}\left[N^>,[N^>,H_{\mathrm{Bog}}]\right]\mathrm{e}^{\mathrm{i}(t-u+p-s)\frac{N^>}{N}}.\end{aligned}$$

Therefore,

$$\begin{aligned}&\|X_\kappa[A_N,[A_N,H_{\mathrm{Bog}}]]X_\kappa\|\\ &\le \frac{1}{4\pi^2N^2}\int \mathrm{d}p\int \mathrm{d}t|p\hat{G}(p)t\hat{G}(t)|\|X_\kappa(N^>+L^d)^{1/2}\|\\ &\quad\times\|(N^>+L^d)^{-1/2}[N^>,[N^>,H_{\mathrm{Bog}}]](N^>+L^d)^{-1/2}\|\|(N^>+L^d)^{1/2}X_\kappa\|.\end{aligned}$$

Now, by Lemma 8.2,

$$\begin{aligned}\|X_\kappa(N^>+L^d)^{1/2}\|\|(N^>+L^d)^{1/2}X_\kappa\| &= \|X_\kappa(N^>+L^d)X_\kappa\|\\ &\le C\left(\|\mathbb{1}_\kappa^{\mathrm{Bog}}N^>\mathbb{1}_\kappa^{\mathrm{Bog}} + L^d\right)\\ &\le 2CL^2(L^{d-1}+\kappa).\end{aligned}$$

Besides, $\hat{G}$ decays fast. Thus it is enough to use (8.14) to complete the proof. □

We define

$$Z_\kappa := X_\kappa A_N = \left(\mathbb{1}_\kappa^{\mathrm{Bog}}A_N^2\mathbb{1}_\kappa^{\mathrm{Bog}}\right)^{-1/2}A_N.$$

Clearly, Z_κ is a partial isometry with initial space $\mathrm{Ran}(A_N\mathbb{1}_\kappa^{\mathrm{Bog}})$ and final space $\mathrm{Ran}(\mathbb{1}_\kappa^{\mathrm{Bog}})$.

Lemma 8.8.

$$\begin{aligned}\mathbb{1}_\kappa^{\mathrm{Bog}}(H_{\mathrm{Bog}}-E_{\mathrm{Bog}})\mathbb{1}_\kappa^{\mathrm{Bog}} &= Z_\kappa(H_{\mathrm{Bog}}-E_{\mathrm{Bog}})Z_\kappa^\dagger\\ &\quad+O\left(L^2(L^{d-1}+\kappa)\kappa N^{-1}\right)\\ &\quad+O\left(L^2(L^{d-1}+\kappa)N^{-2}\right).\end{aligned}$$

Proof. We have

$$\begin{aligned}\mathbb{1}_\kappa^{\mathrm{Bog}}(H_{\mathrm{Bog}}-E_{\mathrm{Bog}})\mathbb{1}_\kappa^{\mathrm{Bog}} &= \left(\mathbb{1}_\kappa^{\mathrm{Bog}}-X_\kappa\right)(H_{\mathrm{Bog}}-E_{\mathrm{Bog}})\mathbb{1}_\kappa^{\mathrm{Bog}} &(8.15)\\ &\quad+X_\kappa(H_{\mathrm{Bog}}-E_{\mathrm{Bog}})\left(\mathbb{1}_\kappa^{\mathrm{Bog}}-X_\kappa\right) &(8.16)\\ &\quad+X_\kappa(H_{\mathrm{Bog}}-E_{\mathrm{Bog}})X_\kappa;\\ X_\kappa(H_{\mathrm{Bog}}-E_{\mathrm{Bog}})X_\kappa &= -X_\kappa A_N^{\mathrm{nph}}(H_{\mathrm{Bog}}-E_{\mathrm{Bog}})A_N^{\mathrm{nph}}X_\kappa\\ &\quad+X_\kappa(H_{\mathrm{Bog}}-E_{\mathrm{Bog}})A_N^{\mathrm{nph}}X_\kappa &(8.17)\end{aligned}$$

$$+ X_\kappa A_N^{\mathrm{nph}}(H_{\mathrm{Bog}} - E_{\mathrm{Bog}})X_\kappa \tag{8.18}$$
$$+ X_\kappa A_N(H_{\mathrm{Bog}} - E_{\mathrm{Bog}})A_N X_\kappa;$$

$$-X_\kappa A_N^{\mathrm{nph}}(H_{\mathrm{Bog}} - E_{\mathrm{Bog}})A_N^{\mathrm{nph}} X_\kappa = -\frac{1}{2}X_\kappa(A_N^{\mathrm{nph}})^2(H_{\mathrm{Bog}} - E_{\mathrm{Bog}})X_\kappa \tag{8.19}$$
$$-\frac{1}{2}X_\kappa(H_{\mathrm{Bog}} - E_{\mathrm{Bog}})(A_N^{\mathrm{nph}})^2 X_\kappa \tag{8.20}$$
$$+\frac{1}{2}X_\kappa\left[A_N^{\mathrm{nph}}, [A_N^{\mathrm{nph}}, H_{\mathrm{Bog}}]\right]X_\kappa. \tag{8.21}$$

The error term in the lemma equals the sum of (8.15), ... ,(8.21). By (8.13),

$$(8.15), (8.16) = O\left(L^2(L^{d-1} + \kappa)\kappa N^{-1}\right).$$

By (8.9),

$$(8.17), \ldots, (8.20) = O\left(L^2(L^{d-1} + \kappa)\kappa N^{-1}\right).$$

By Lemma 8.7,

$$(8.21) = O\left(L^2(L^{d-1} + \kappa)N^{-2}\right).$$

□

Lemma 8.9. *Assume* (8.1). *Then*

$$\inf_{0<\epsilon\le 1} Z_\kappa R_{N,\epsilon} Z_\kappa^\dagger \le C L^{d/2+3}(L^{d-1} + \kappa)^{3/2} N^{-1/2}. \tag{8.22}$$

Proof.

$$\begin{aligned}
Z_\kappa R_{N,\epsilon} Z_\kappa^\dagger \le\ & Z_\kappa \frac{1}{2}\sum_{\mathbf{p}\neq\mathbf{0}} \hat{v}(\mathbf{p})\left(\left(\frac{\sqrt{(N_0-1)N_0}}{N} - 1\right)\mathrm{b}_{\mathbf{p}}\mathrm{b}_{-\mathbf{p}} + \mathrm{hc}\right)Z_\kappa^\dagger \\
& + Z_\kappa \frac{\hat{v}(\mathbf{0})}{2N} N^> Z_\kappa^\dagger \\
& + \epsilon Z_\kappa \frac{1}{N}\sum_{\mathbf{p}\neq\mathbf{0}}\left(\hat{v}(\mathbf{p}) + \hat{v}(\mathbf{0})\right)\mathrm{b}_{\mathbf{p}}^\dagger \mathrm{b}_{\mathbf{p}} N_0^{\mathrm{ext}} Z_\kappa^\dagger \\
& + (1+\epsilon^{-1})Z_\kappa \frac{1}{2N} v(\mathbf{0}) L^d N^>(N^> - 1) Z_\kappa^\dagger \\
\le\ & \mathbb{1}_\kappa^{\mathrm{Bog}} C\frac{(N^>)^2 + 1}{N}\mathbb{1}_\kappa^{\mathrm{Bog}} \\
& + \mathbb{1}_\kappa^{\mathrm{Bog}} C\frac{N^>}{N}\mathbb{1}_\kappa^{\mathrm{Bog}} \\
& + \epsilon \mathbb{1}_\kappa^{\mathrm{Bog}} C N^> \mathbb{1}_\kappa^{\mathrm{Bog}} \\
& + (1+\epsilon^{-1})\mathbb{1}_\kappa^{\mathrm{Bog}} C\frac{L^d(N^>)^2}{N}\mathbb{1}_\kappa^{\mathrm{Bog}}.
\end{aligned}$$

Using $\epsilon \le 1$, we can simplify the bound as follows:

$$\le \mathbb{1}_\kappa^{\mathrm{Bog}}\frac{C}{N} + \epsilon\mathbb{1}_\kappa^{\mathrm{Bog}} C N^> \mathbb{1}_\kappa^{\mathrm{Bog}} + \epsilon^{-1}\mathbb{1}_\kappa^{\mathrm{Bog}} C\frac{L^d(N^>)^2}{N}\mathbb{1}_\kappa^{\mathrm{Bog}}, \tag{8.23}$$

By (8.4) and (8.9). this can be estimated by

$$CN^{-1} + \epsilon CL^2(L^{d-1} + \kappa) + \epsilon^{-1}CL^{d+4}(L^{d-1} + \kappa)^2 N^{-1}.$$

Setting $\epsilon = c^{-1/2}N^{-1/2}L^{d/2+1}(L^{d-1}+\kappa)^{1/2}$, which is less than 1 by Condition (8.1), we obtain

$$CN^{-1} + CL^{d/2+3}(L^{d-1} + \kappa)^{3/2}N^{-1/2}. \tag{8.24}$$

By changing C, the second term can obviously absorb CN^{-1}. □

Proof of Theorem 8.1. Z_κ is a partial isometry with the initial space contained in the physical space and the final projection $\mathbb{1}_\kappa^{\mathrm{Bog}}$. Therefore,

$$\begin{aligned}\overrightarrow{\mathrm{sp}} H_N &\le \overrightarrow{\mathrm{sp}}\left(Z_\kappa^\dagger Z_\kappa H_N Z_\kappa^\dagger Z_\kappa \Big|_{\mathrm{Ran} Z_\kappa^\dagger}\right)\\ &= \overrightarrow{\mathrm{sp}}\left(Z_\kappa H_N Z_\kappa^\dagger \Big|_{\mathrm{Ran}\mathbb{1}_\kappa^{\mathrm{Bog}}}\right).\end{aligned}$$

$$\begin{aligned}Z_\kappa H_N Z_\kappa^\dagger &\le Z_\kappa H_{N,\epsilon} Z_\kappa^\dagger\\ &= \frac{1}{2}\hat{v}(\mathbf{0})(N-1)\mathbb{1}_\kappa^{\mathrm{Bog}} + H_{\mathrm{Bog}}\mathbb{1}_\kappa^{\mathrm{Bog}}\\ &\quad + Z_\kappa(H_{\mathrm{Bog}} - E_{\mathrm{Bog}})Z_\kappa^\dagger - (H_{\mathrm{Bog}} - E_{\mathrm{Bog}})\mathbb{1}_\kappa^{\mathrm{Bog}} \qquad (8.25)\\ &\quad + Z_\kappa R_{N,\epsilon} Z_\kappa^\dagger. \qquad (8.26)\end{aligned}$$

By Lemma 8.8,

$$\begin{aligned}(8.25) &\le CL^2(L^{d-1} + \kappa)\kappa N^{-1} \qquad (8.27)\\ &\quad + CL^2(L^{d-1} + \kappa)N^{-2}. \qquad (8.28)\end{aligned}$$

Using $\kappa < \kappa + L^{d-1}$ and later (8.1) we have

$$\begin{aligned}(8.27) &\le CL^2(L^{d-1} + \kappa)^2 N^{-1}\\ &\le CL^{-d/2+1}(L^{d-1} + \kappa)^{3/2}N^{-1/2}.\end{aligned}$$

Thus (8.27) can be absorbed in $O(L^{d/2+3}(L^{d-1} + \kappa)^{3/2}N^{-1/2})$.

We easily check that the same is true in the case of (8.28). To bound (8.26) we use Lemma 8.9. □

Proof of Theorem 1.1 (2). First set $\kappa = 0$. Then Condition (8.1) becomes Condition (1.11) and Condition (8.2) becomes Condition (1.12). We obtain Theorem 1.1 (2a).

Next set $\kappa = K_{\mathrm{Bog}}^j(\mathbf{p})$. Then Condition (8.1) is equivalent to the conjunction of Conditions (1.11) and (1.14). Condition (8.2) is equivalent to the conjunction of Conditions (1.12) and (1.15). This shows Theorem 1.1 (2b). □

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Appendix A. Proof of Corollary 1.2

The proof of the corollary is based on the following lemma:

Lemma A.1. 1. *Let $b > 1$, $-1 - \frac{1}{d+1} \leq \alpha \leq 1$ and $L^{4d+6} \leq bN^{1-\alpha}$. Then*

(a) $\frac{1}{2}\hat{v}(\mathbf{0})(N-1) + E_{\mathrm{Bog}} \leq E_N + O(N^{-\alpha/2})$;

(b) *if $K_N^j(\mathbf{p}) \leq (bN^{1-\alpha}L^{-d-6})^{1/3}$, then*

$$\frac{1}{2}\hat{v}(\mathbf{0})(N-1) + E_{\mathrm{Bog}} + K_{\mathrm{Bog}}^j(\mathbf{p}) \leq E_N + K_N^j(\mathbf{p}) + O(N^{-\alpha/2});$$

(c) *if $0 \leq \alpha \leq 1$ and $K_N^j(\mathbf{p}) \leq bN^{1-\alpha}L^{-d-6}$, then*

$$\frac{1}{2}\hat{v}(\mathbf{0})(N-1) + E_{\mathrm{Bog}} + K_{\mathrm{Bog}}^j(\mathbf{p}) \leq E_N + K_N^j(\mathbf{p}) + \left(1 + K_N^j(\mathbf{p})\right) O(N^{-\alpha/2}).$$

2. *Let $b > 1, -1 - \frac{1}{2d+1} < \alpha \leq 1$ and $L^{4d+3} \leq bN^{1-\alpha}$. Then there exists M such that if $N > M$, then*

(a) $E_N \leq \frac{1}{2}\hat{v}(\mathbf{0})(N-1) + E_{\mathrm{Bog}} + O(N^{-\alpha/2})$;

(b) *if $K_{\mathrm{Bog}}^j(\mathbf{p}) \leq (bN^{1-\alpha}L^{-d-6})^{1/3}$, then*

$$E_N + K_N^j(\mathbf{p}) \leq \frac{1}{2}\hat{v}(\mathbf{0})(N-1) + E_{\mathrm{Bog}} + K_{\mathrm{Bog}}^j(\mathbf{p}) + O(N^{-\alpha/2});$$

(c) *if $0 < \alpha \leq 1$ and $K_{\mathrm{Bog}}^j(\mathbf{p}) \leq bN^{1-\alpha}L^{-d-6}$, then*

$$E_N + K_N^j(\mathbf{p}) \leq \frac{1}{2}\hat{v}(\mathbf{0})(N-1) + E_{\mathrm{Bog}} + K_{\mathrm{Bog}}^j(\mathbf{p}) + \left(1 + K_{\mathrm{Bog}}^j(\mathbf{p})\right) O(N^{-\alpha/2}).$$

Proof. To prove (1), resp. (2) we use Theorem 1.1 (1), resp. (2). We give a proof of the latter part, since it is slightly more involved (because of the parameter c_1).

(2a): First we check Condition (1.11):

$$L^{2d+1} = \left(L^{4d+3}\right)^{\frac{2d+1}{4d+3}} \leq \left(bN^{1-\alpha}\right)^{\frac{2d+1}{4d+3}}. \tag{A.1}$$

For $-1 - \frac{1}{2d+1} \leq \alpha$ we have (A.1)$\leq cN$.

Next,

$$L^{d+1} = \left(L^{4d+3}\right)^{\frac{d+1}{4d+3}} \leq \left(bN^{1-\alpha}\right)^{\frac{d+1}{4d+3}}. \tag{A.2}$$

We have (A.2)$\leq cNN^{\frac{-3d-2-\alpha(d+1)}{4d+3}}$. Therefore, for $-1-\frac{1}{2d+1}\leq\alpha$, Condition (1.12) is satisfied for large enough N.

Then we apply Theorem 1.1 (2)(a), using

$$N^{-1/2}L^{2d+\frac{3}{2}}\leq N^{-1/2}(bN^{1-\alpha})^{1/2}=O(N^{-\alpha/2}).$$

(2b): We check Condition (1.14):

$$\begin{aligned}K^j_{\mathrm{Bog}}(\mathbf{p})&\leq\left(bN^{1-\alpha}L^{-d-6}\right)^{1/3}\\&\leq\left(bN^{1-\alpha}L^{-d-6}\right)^{1/3}\left(bN^{1-\alpha}L^{-4d-3}\right)^{\frac{2d}{12d+9}}\\&=CN^{\frac{(1-\alpha)(2d+1)}{12d+9}}L^{-d-2}.\end{aligned}\tag{A.3}$$

For $-1-\frac{1}{2d+1}\leq\alpha$ we have (A.3)$\leq CNL^{-d-2}$.

Also

$$A.3=O\left(N^{\frac{-(2+\alpha)(4d+3)+2d(1-\alpha)}{12d+9}}\right)NL^{-2-d}$$

which implies

$$K^j_{\mathrm{Bog}}(\mathbf{p})\leq O\left(N^{\frac{-(2+\alpha)(4d+3)+2d(1-\alpha)}{12d+9}}\right)NL^{-2}.$$

Therefore, if $-1-\frac{1}{2d+1}<\alpha$, Condition (1.15) is satisfied for large enough N.

We clearly have

$$N^{-1/2}L^{d/2+3}\left(K^j_{\mathrm{Bog}}(\mathbf{p})+L^{d-1}\right)^{3/2}\leq 2^{3/2}N^{-1/2}L^{d/2+3}K^j_{\mathrm{Bog}}(\mathbf{p})^{3/2}\tag{A.4}$$
$$+2^{3/2}N^{-1/2}L^{2d+\frac{3}{2}}.\tag{A.5}$$

We already know that (A.5) is $O(N^{-\alpha/2})$. Thus to apply Theorem 1.1 (2b) we need only to bound (A.4):

$$N^{-1/2}L^{d/2+3}K^j_{\mathrm{Bog}}(\mathbf{p})^{3/2}\leq N^{-1/2}L^{d/2+3}\left(bN^{1-\alpha}L^{-d-6}\right)^{1/2}=O(N^{-\alpha/2}).$$

(2c): Condition (1.14) is trivially satisfied, since for $L\geq 1, N\geq 1$ and $\alpha>0$

$$K^j_{\mathrm{Bog}}(\mathbf{p})\leq bN^{1-\alpha}L^{-d-6}\leq bNL^{-d-2}.$$

We have

$$\begin{aligned}K^j_{\mathrm{Bog}}(\mathbf{p})&\leq bN^{1-\alpha}L^{-d-6}L^{d+4}\\&=O(N^{-\alpha})NL^{-2}.\end{aligned}$$

Therefore, Condition (1.15) is satisfied for large enough N.

To apply Theorem 1.1 (2b) we bound (A.4):

$$\begin{aligned}N^{-1/2}L^{d/2+3}K^j_{\mathrm{Bog}}(\mathbf{p})^{3/2}&=b^{1/2}N^{-\alpha/2}\left(b^{-1}N^{-(1-\alpha)}L^{d+6}\right)^{1/2}K^j_{\mathrm{Bog}}(\mathbf{p})^{3/2}\\&\leq O(N^{-\alpha/2})K^j_{\mathrm{Bog}}(\mathbf{p}).\end{aligned}$$

□

Proof of Corollary 1.2. Part (1) follows directly from Lemma A.1 (1a) and (2a).

Let us prove (2). To simplify notation we drop $\mathbf{p}$ from $K_N^j(\mathbf{p})$ and $K_{\mathrm{Bog}}^j(\mathbf{p})$.

Assume first that $K_N^j \le K_{\mathrm{Bog}}^j$. By Lemma A.1 (1b) for some $C > 0$

$$\begin{aligned}\frac{1}{2}\hat{v}(\mathbf{0})(N-1) + E_{\mathrm{Bog}} + K_{\mathrm{Bog}}^j &\le E_N + K_N^j + CN^{-\alpha/2}\\ &\le E_N + K_{\mathrm{Bog}}^j + CN^{-\alpha/2}.\end{aligned}$$

Thus

$$\frac{1}{2}\hat{v}(\mathbf{0})(N-1) + E_{\mathrm{Bog}} - E_N + K_{\mathrm{Bog}}^j - CN^{-\alpha/2} \le K_N^j \le K_{\mathrm{Bog}}^j.$$

By Lemma A.1 (2a),

$$-CN^{-\alpha/2} \le \frac{1}{2}\hat{v}(\mathbf{0})(N-1) + E_{\mathrm{Bog}} - E_N.$$

Hence the statement follows.

Assume now that $K_{\mathrm{Bog}}^j \le K_N^j$. Then we use Lemma A.1 (2b) and obtain

$$K_{\mathrm{Bog}}^j \le K_N^j \le \frac{1}{2}\hat{v}(\mathbf{0})(N-1) + E_{\mathrm{Bog}} - E_N + K_{\mathrm{Bog}}^j + CN^{-\alpha/2}.$$

By Lemma A.1 (1a),

$$\frac{1}{2}\hat{v}(\mathbf{0})(N-1) + E_{\mathrm{Bog}} - E_N \le CN^{-\alpha/2}.$$

The statement follows again. This ends the proof of part (2).

The proof of part (3) is similar, except that one uses Lemma A.1 (1c) and (2c). □

References

[1] Beliaev, S.T.: Energy spectrum of a non-ideal Bose gas. Sov. Phys. JETP **7**, 299 (1958) [reprinted in D. Pines "The Many-Body Problem" (W.A. Benjamin, New York, 1962)]

[2] Bogoliubov, N.N.: On the theory of superfluidity. J. Phys. (U.S.S.R.) **11**, 23–32 (1947)

[3] Bogoliubov, N.N.: Energy levels of the imperfect Bose–Einstein gas. Bull. Moscow State Univ. **7**, 43–56 (1947)

[4] Cornean, H.D., Dereziński, J., Ziń, P.: On the infimum of the energy–momentum spectrum of a homogeneous Bose gas. J. Math. Phys. **50**, 062103 (2009)

[5] Dereziński, J., Napiórkowski, M., Meissner, K.A.: On the infimum of the energy–momentum spectrum of a homogeneous Fermi gas. Ann. Henri Poincaré **14**, 1–36 (2013)

[6] Erdös, L., Schlein, B., Yau, H.-T.: Ground-state energy of a low-density Bose gas: a second-order upper bound. Phys. Rev. A **78**, 053627 (2008)

[7] Giuliani, A., Seiringer, R.: The ground state energy of the weakly interacting Bose gas at high density. J. Stat. Phys. **135**, 915–934 (2009)

[8] Grech, P., Seiringer, R.: The excitation spectrum for weakly interacting bosons in a trap. Commun. Math. Phys. **332**, 559–591 (2013)

[9] Griffin, A.: Excitations in a Bose-condensed liquid. Cambridge University Press, Cambridge (1993)

[10] Hodby, E., Maragó, O.M., Hechenblaikner, G., Foot, C.J.: Experimental observation of beliaev coupling in a Bose–Einstein condensate. Phys. Rev. Lett. **86**, 2196–2199 (2001)

[11] Landau, L.D.: The theory of superfuidity of Helium II. J. Phys. (USSR) **5**, 71 (1941)

[12] Landau, L.D.: On the theory of superfuidity of Helium II. J. Phys. (USSR) **11**, 91 (1947)

[13] Lewin, M., Nam, P.T., Serfaty, S., Solovej, J.P.: Bogoliubov spectrum of interacting Bose gases. Commun. Pure Appl. Math. arXiv:1211.2778 [math-ph]

[14] Lieb, E.H., Seiringer, R., Solovej, J.P., Yngvason, J.: The mathematics of the Bose gas and its condensation. In: Oberwolfach Seminars, vol. 34. Birkhäuser, Basel (2005)

[15] Lieb, E.H., Solovej, J.P.: Ground state energy of the one-component charged Bose gas. Commun. Math. Phys. **217**, 127–163 (2001). Errata **225**, 219–221 (2002)

[16] Lieb, E.H., Solovej, J.P.: Ground state energy of the two-component charged Bose gas. Commun. Math. Phys. **252**, 485–534 (2004)

[17] Reed, M., Simon, B.: Methods of Modern Mathematical Physics. In: Analysis of Operators, vol. 4. Academic Press, New York (1978)

[18] Seiringer, R.: The excitation spectrum for weakly interacting bosons. Commun. Math. Phys. **306**, 565–578 (2011)

[19] Solovej, J.P.: Upper bounds to the ground state energies of the one- and two-component charged Bose gases. Commun. Math. Phys. **266**, 797–818 (2006)

[20] Zagrebnov, V.A., Bru, J.B.: The Bogoliubov model of weakly imperfect Bose gas. Phys. Rep. **350**, 291 (2001)

[21] Yau, H.-T., Yin, J.: The second order upper bound for the ground energy of a Bose gas. J. Stat. Phys. **136**, 453–503 (2009)

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