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On Manin's Conjecture for Singular del Pezzo Surfaces of Degree 4, I

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1. Introduction

Let $Q_1, Q_2 \in \mathbb{Z}[x_0, \dots, x_4]$ be a pair of quadratic forms whose common zero locus defines a geometrically integral surface $X \subset \mathbb{P}^4$. Then X is a del Pezzo surface of degree 4. We assume henceforth that the set $X(\mathbb{Q}) = X \cap \mathbb{P}^4(\mathbb{Q})$ of rational points on X is nonempty, so that in particular $X(\mathbb{Q})$ is dense in X under the Zariski topology. Given a point $x = [x_0, \dots, x_4] \in \mathbb{P}^4(\mathbb{Q})$ with $x_0, \dots, x_4 \in \mathbb{Z}$ such that $\gcd(x_0, \dots, x_4) = 1$, we let $H(x) = \max_{0 \le i \le 4}|x_i|$. Then $H\colon \mathbb{P}^4(\mathbb{Q}) \to \mathbb{R}_{\ge 0}$ is the height attached to the anticanonical divisor $-K_X$ on X parametrized by the choice of norm $\max_{0 \le i \le 4}|x_i|$. A finer notion of density is provided by analyzing the asymptotic behavior of the quantity

$$N_{U,H}(B) = \#\{x \in U(\mathbb{Q}) : H(x) \le B\},$$

as $B \to \infty$, for appropriate open subsets $U \subseteq X$. Since every quartic del Pezzo surface X contains a line, it is natural to estimate $N_{U,H}(B)$ for the open subset U obtained by deleting the lines from X. The motivation behind this paper is to consider the asymptotic behavior of $N_{U,H}(B)$ for singular del Pezzo surfaces of degree 4.

A classification of quartic del Pezzo surfaces $X \subset \mathbb{P}^4$ can be found in the work of Hodge and Pedoe [8, Book IV, Sec. XIII.11], which shows in particular that there are only finitely many isomorphism classes to consider. Let $\tilde{X}$ denote the minimal desingularization of X, and let $\operatorname{Pic}\tilde{X}$ be the Picard group of $\tilde{X}$. Then Manin has stated a very general conjecture [6] that predicts the asymptotic behavior of counting functions associated to suitable Fano varieties. In our setting this leads us to expect the existence of a positive constant $c_{X,H}$ such that

$$N_{U,H}(B) = c_{X,H}B(\log B)^{\rho-1}(1 + o(1)), \tag{1.1}$$

as $B \to \infty$, where ρ denotes the rank of $\operatorname{Pic}\tilde{X}$. The constant $c_{X,H}$ has received a conjectural interpretation at the hands of Peyre [9], which in turn has been generalized to cover certain other cases by Batyrev and Tschinkel [2] and Salberger [11]. A brief discussion of $c_{X,H}$ will take place in Section 2.

There has been rather little progress towards the Manin conjecture for del Pezzo surfaces of degree 4. The main successes in this direction are to be found in work of Batyrev and Tschinkel [1], covering the case of toric varieties, and in the work of

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Chambert-Loir and Tschinkel [5], covering the case of equivariant compactifications of vector groups. It is our intention to investigate the distribution of rational points in the special case where X is defined by the pair of equations

$$x_0x_1 - x_2^2 = 0,$$
$$x_0x_4 - x_1x_2 + x_3^2 = 0.$$

Then $X \subset \mathbb{P}^4$ is a del Pezzo surface of degree 4 with a unique singular point $[0,0,0,0,1]$ of type $\mathbf{D}_5$. Furthermore, X contains precisely one line $x_0 = x_2 = x_3 = 0$. It turns out that X is an equivariant compactification of $\mathbb{G}_a^2$, so the work of Chambert-Loir and Tschinkel [5, Thm. 0.1] ensures that the asymptotic formula (1.1) holds when $U \subset X$ is taken to be the open subset formed by deleting the unique line from X. Nonetheless, there are several reasons why this problem is still worthy of attention. Firstly, in making an exhaustive study of X it is hoped that a template will be set down for the treatment of other singular del Pezzo surfaces. We actually make no explicit use of the fact that X is an equivariant compactification of $\mathbb{G}_a^2$, and the techniques developed in this paper have already been applied to other surfaces [3; 4]. Second, in addition to improving upon Chambert-Loir and Tschinkel's asymptotic formula for $N_{U,H}(B)$, the results that we obtain lend themselves more readily as a bench test for future refinements of the Manin conjecture, such as that recently proposed by Swinnerton-Dyer [12].

Let $X \subset \mathbb{P}^4$ be the $\mathbf{D}_5$ del Pezzo surface just defined, and let $U \subset X$ be the corresponding open subset. Our first result concerns the height zeta function

$$Z_{U,H}(s) = \sum_{x \in U(\mathbb{Q})} \frac{1}{H(x)^s}, \tag{1.2}$$

which is defined when $\mathrm{Re}(s)$ is sufficiently large. The analytic properties of $Z_{U,H}(s)$ are intimately related to the asymptotic behavior of the counting function $N_{U,H}(B)$. For $\mathrm{Re}(s) > 0$ we define the functions

$$E_1(s+1) = \zeta(6s+1)\zeta(5s+1)\zeta(4s+1)^2\zeta(3s+1)\zeta(2s+1), \tag{1.3}$$

$$E_2(s+1) = \frac{\zeta(14s+3)\zeta(13s+3)^3}{\zeta(10s+2)\zeta(9s+2)\zeta(8s+2)^3\zeta(7s+2)^3\zeta(19s+4)}. \tag{1.4}$$

It is easily seen that $E_1(s)$ has a meromorphic continuation to the entire complex plane with a single pole at $s = 1$. Similarly it is clear that $E_2(s)$ is holomorphic and bounded on the half-plane $\mathrm{Re}(s) \geq 9/10 + \varepsilon$ for any $\varepsilon > 0$. We are now ready to state our main result.

THEOREM 1. *Let $\varepsilon > 0$. Then there exist a constant $\beta \in \mathbb{R}$ and functions $G_1(s), G_2(s)$ that are holomorphic on the half-plane* $\mathrm{Re}(s) \geq 5/6 + \varepsilon$, *such that for* $\mathrm{Re}(s) > 1$,

$$Z_{U,H}(s) = E_1(s)E_2(s)G_1(s) + \frac{12/\pi^2 + 2\beta}{s-1} + G_2(s).$$

In particular, $(s-1)^6 Z_{U,H}(s)$ *has a holomorphic continuation to the half-plane* $\mathrm{Re}(s) > 9/10$. *The function* $G_1(s)$ *is bounded for* $\mathrm{Re}(s) \geq 5/6 + \varepsilon$ *and satisfies* $G_1(1) \neq 0$, *and the function* $G_2(s)$ *satisfies*

$$G_2(s) \ll_\varepsilon (1 + |\mathrm{Im}(s)|)^{6(1-\mathrm{Re}(s))+\varepsilon}$$

on this domain.

An explicit expression for β can be found in (5.24), and the formulas (6.1)–(6.4) can be used to deduce an explicit expression for G_1. There are several features of Theorem 1 that are worthy of remark. The first step in the proof of Theorem 1 is the observation that

$$Z_{U,H}(s) = s \int_1^\infty t^{-s-1} N_{U,H}(t)\,\mathrm{d}t. \tag{1.5}$$

Thus we find ourselves in the situation of establishing a preliminary estimate for $N_{U,H}(B)$ in order to deduce the analytic properties of $Z_{U,H}(s)$ presented in Theorem 1, after which we use this information to deduce an improved estimate for $N_{U,H}(B)$; see Theorem 2. With this order of events in mind we remark that the term $(12/\pi^2)(s-1)^{-1}$ appearing in Theorem 1 corresponds to an isolated conic contained in X. Moreover, whereas the first term $E_1(s)E_2(s)G_1(s)$ in the expression for $Z_{U,H}(s)$ corresponds to the main term in our preliminary estimate for $N_{U,H}(B)$ and arises through the approximation of certain arithmetic quantities by real-valued continuous functions, the term involving β has a more arithmetic interpretation. Indeed, it will be seen to arise purely from the error terms produced by approximating these arithmetic quantities by continuous functions. Finally we observe that, under the assumption of the Riemann hypothesis, $E_2(s)$ is holomorphic for $\mathrm{Re}(s) > 17/20$ and so $Z_{U,H}(s)$ has an analytic continuation to this domain.

We now come to explaining how Theorem 1 can be used to deduce an asymptotic formula for $N_{U,H}(B)$. We shall verify in Section 2 that the following result is in accordance with the Manin conjecture.

THEOREM 2. *Let* $\delta \in (0, 1/12)$. *Then there exists a polynomial* P *of degree* 5 *such that*

$$N_{U,H}(B) = BP(\log B) + O(B^{1-\delta})$$

for any $B \geq 1$. *Moreover the leading coefficient of* P *is equal to*

$$\frac{\tau_\infty}{28800} \prod_p \left(1 - \frac{1}{p}\right)^6 \left(1 + \frac{6}{p} + \frac{1}{p^2}\right),$$

where

$$\tau_\infty = \int_0^1 \int_{-1}^{1/v} \left(\min\{\sqrt{u^3+1}, 1/v^3\} - \sqrt{\max\{u^3-1, 0\}}\right) du\, dv. \tag{1.6}$$

The deduction of Theorem 2 from Theorem 1 will take place in Section 7 and amounts to a routine application of Perron's formula. Although we choose not to give the details here, it is in fact possible to take $\delta \in (0, 1/11)$ in the statement of

Theorem 2 by using more sophisticated estimates for moments of the Riemann zeta function in the critical strip. By expanding the height zeta function $Z_{U,H}(s)$ as a power series in $(s-1)^{-1}$, one may obtain explicit expressions for the lower order coefficients of the polynomial P in Theorem 2. It would be interesting to obtain refinements of Manin's conjecture that admit conjectural interpretations of the lower order coefficients.

The principal tool in the proof of Theorem 1 is a passage to the universal torsor above the minimal desingularization $\tilde{X}$ of X. Although originally introduced to aid in the study of the Hasse principle and weak approximation, universal torsors have recently become endemic in the context of counting rational points of bounded height. In Section 4 we shall establish a bijection between $U(\mathbb{Q})$ and integer points on the universal torsor, which in this setting has the natural affine embedding

$$v_2 y_0^2 y_4 - v_0 y_1^3 y_2^2 + v_3 y_3^2 = 0.$$

Once we have translated the problem to the universal torsor, Theorem 1 will be established by using a range of techniques drawn from classical analytic number theory. In particular, Theorem 1 seems to be the first instance of a height zeta function that has been calculated via a passage to the universal torsor.

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2. Conformity with the Manin Conjecture

In this section we will show that Theorem 2 agrees with the Manin conjecture. For this we need to calculate the value of $c_{U,H}$ and ρ in (1.1). We therefore review some of the geometry of the surface $X \subset \mathbb{P}^4$ as defined by the pair of quadratic forms

$$Q_1(\mathbf{x}) = x_0 x_1 - x_2^2 \quad \text{and} \quad Q_2(\mathbf{x}) = x_0 x_4 - x_1 x_2 + x_3^2, \tag{2.1}$$

where $\mathbf{x} = (x_0, \dots, x_4)$.

Let $\tilde{X}$ denote the minimal desingularization of X, and let $\pi : \tilde{X} \to X$ denote the corresponding blow-up map. We let E_6 denote the strict transform of the unique line contained in X and let $E_1, \dots, E_5$ denote the exceptional curves of π. Then the divisors $E_1, \dots, E_6$ satisfy the Dynkin diagram

$$\begin{tikzcd} & & E_3 \arrow[d, dash] & & \\ E_5 \arrow[r, dash] & E_4 \arrow[r, dash] & E_1 \arrow[r, dash] & E_2 \arrow[r, dash] & E_6 \end{tikzcd}$$

and generate the Picard group of $\tilde{X}$. In particular we have $\rho = 6$ in (1.1), which agrees with Theorem 2.

It remains to discuss the conjectured value of the constant $c_{X,H}$ in (1.1). For this we will follow the presentation of Batyrev and Tschinkel [2, Sec. 3.4]. Let $\Lambda_{\mathrm{eff}}(\tilde{X}) \subset \mathrm{Pic}(\tilde{X}) \otimes_{\mathbb{Z}} \mathbb{R}$ be the cone of effective divisors on $\tilde{X}$, and let

$$\Lambda_{\mathrm{eff}}^{\vee}(\tilde{X}) = \{\mathbf{s} \in \mathrm{Pic}^{\vee}(\tilde{X}) \otimes_{\mathbb{Z}} \mathbb{R} : \langle \mathbf{s}, \mathbf{t} \rangle \geq 0 \ \forall \mathbf{t} \in \Lambda_{\mathrm{eff}}(\tilde{X})\}$$

be the corresponding dual cone, where $\mathrm{Pic}^{\vee} \tilde{X}$ denotes the dual lattice to $\mathrm{Pic}\, \tilde{X}$. Then, letting $\mathrm{d}\mathbf{t}$ denote the Lebesgue measure on $\mathrm{Pic}^{\vee}(\tilde{X}) \otimes_{\mathbb{Z}} \mathbb{R}$, we define

$$\alpha(\tilde{X}) = \int_{\Lambda_{\mathrm{eff}}^{\vee}(\tilde{X})} e^{-\langle -K_{\tilde{X}}, \mathbf{t} \rangle}\, \mathrm{d}\mathbf{t},$$

where $-K_{\tilde{X}}$ is the anticanonical divisor of $\tilde{X}$. Thus $\alpha(\tilde{X})$ measures the volume of the polytope obtained by intersecting $\Lambda_{\mathrm{eff}}^{\vee}(\tilde{X})$ with a certain affine hyperplane.

Next we discuss the Tamagawa measure on the closure $\overline{\tilde{X}(\mathbb{Q})}$ of $\tilde{X}(\mathbb{Q})$ in $\tilde{X}(\mathbb{A}_{\mathbb{Q}})$, where $\mathbb{A}_{\mathbb{Q}}$ denotes the adele ring. Write $L_p(s, \mathrm{Pic}\, \tilde{X})$ for the local factors of $L(s, \mathrm{Pic}\, \tilde{X})$. Furthermore, let ω_∞ denote the archimedean density of points on X and let ω_p denote the usual p-adic density of points on X for any prime p. Then we may define the Tamagawa measure

$$\tau_H(\tilde{X}) = \lim_{s \to 1}((s-1)^{\rho} L(s, \mathrm{Pic}\, \tilde{X}))\omega_\infty \prod_p \frac{\omega_p}{L_p(1, \mathrm{Pic}\, \tilde{X})}, \tag{2.2}$$

where ρ denotes the rank of $\mathrm{Pic}\, \tilde{X}$ as before. With these definitions in mind, the conjectured value of the constant in (1.1) is equal to

$$c_{X,H} = \alpha(\tilde{X})\beta(\tilde{X})\tau_H(\tilde{X}), \tag{2.3}$$

where $\beta(\tilde{X}) = \#H^1(\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}), \mathrm{Pic}\, \tilde{X}) = 1$, since $\tilde{X}$ is rational.

We begin by calculating the value of $\alpha(\tilde{X})$, for which we will follow the approach of Peyre and Tschinkel [10, Sec. 5]. We need to determine the cone $\Lambda_{\mathrm{eff}}(\tilde{X})$ and the anticanonical divisor $-K_{\tilde{X}}$. To determine these we may use the Dynkin diagram to write down the following intersection matrix.

	E_1	E_2	E_3	E_4	E_5	E_6
E_1	-2	1	1	1	0	0
E_2	1	-2	0	0	0	1
E_3	1	0	-2	0	0	0
E_4	1	0	0	-2	1	0
E_5	0	0	0	1	-2	0
E_6	0	1	0	0	0	-1

But then it follows that $\Lambda_{\mathrm{eff}}(\tilde{X})$ is generated by the divisors $E_1, \ldots, E_6$. Furthermore, on employing the adjunction formula $-K_{\tilde{X}} \cdot D = 2 + D^2$ for $D \in \mathrm{Pic}\, \tilde{X}$, we easily deduce that

$$-K_{\tilde{X}} = 6E_1 + 5E_2 + 3E_3 + 4E_4 + 2E_5 + 4E_6.$$

It therefore follows that

$$\begin{aligned}\alpha(\tilde{X}) &= \mathrm{Vol}\{(t_1,t_2,t_3,t_4,t_5,t_6)\in\mathbb{R}^6_{\geq 0} : 6t_1+5t_2+3t_3+4t_4+2t_5+4t_6=1\}\\ &= \tfrac{1}{4}\mathrm{Vol}\{(t_1,t_2,t_3,t_4,t_5)\in\mathbb{R}^5_{\geq 0} : 6t_1+5t_2+3t_3+4t_4+2t_5\leq 1\},\end{aligned}$$

whence in fact

$$\alpha(\tilde{X}) = 1/345600. \tag{2.4}$$

It remains to calculate the value of $\tau_H(\tilde{X})$ in (2.3).

LEMMA 1. *We have* $\tau_H(\tilde{X}) = 12\tau_\infty\tau$, *where* τ_∞ *is given by* (1.6) *and*

$$\tau = \prod_p \Big(1-\frac{1}{p}\Big)^6\Big(1+\frac{6}{p}+\frac{1}{p^2}\Big). \tag{2.5}$$

Proof. Recall the definition (2.2) of $\tau_H(\tilde{X})$. Our starting point is the observation that $L(s,\mathrm{Pic}\,\tilde{X}) = \zeta(s)^6$. Hence it easily follows that

$$\lim_{s\to 1}((s-1)^\rho L(s,\mathrm{Pic}\,\tilde{X})) = \lim_{s\to 1}((s-1)^6\zeta(s)^6) = 1.$$

Furthermore, we plainly have

$$L_p(1,\mathrm{Pic}\,\tilde{X})^{-1} = \Big(1-\frac{1}{p}\Big)^6 \tag{2.6}$$

for any prime p.

We proceed by employing the method of Peyre [9] to calculate the value of the archimedean density ω_∞. It will be convenient to parametrize the points via the choice of variables x_0,x_1,x_4, for which we first observe that the Leray form $\omega_L(\tilde{X})$ is given by $(4x_2x_3)^{-1}dx_0dx_1dx_4$ since

$$\det\begin{pmatrix}\frac{\partial Q_1}{\partial x_2} & \frac{\partial Q_2}{\partial x_2}\\ \frac{\partial Q_1}{\partial x_3} & \frac{\partial Q_2}{\partial x_3}\end{pmatrix} = -4x_2x_3.$$

Now in any real solution to the pair of equations $Q_1(\mathbf{x}) = Q_2(\mathbf{x}) = 0$, the components x_0 and x_1 must necessarily share the same sign. Taking into account that $\mathbf{x}$ and $-\mathbf{x}$ represent the same point in $\mathbb{P}^4$, we therefore see that

$$\omega_\infty = 2\int_{\{\mathbf{x}\in\mathbb{R}^5 : Q_1(\mathbf{x})=Q_2(\mathbf{x})=0,\ 0\leq x_0,x_1,x_3,|x_4|\leq 1\}} \omega_L(\tilde{X}).$$

We write $\omega_{\infty,-}$ to denote the contribution to ω_∞ from the case $x_2 = -\sqrt{x_0x_1}$ and write $\omega_{\infty,+}$ for the contribution from the case $x_2 = \sqrt{x_0x_1}$. But then it follows that

$$\omega_{\infty,+} = \frac{1}{2}\iiint \frac{\mathrm{d}x_0\,\mathrm{d}x_1\,\mathrm{d}x_4}{\sqrt{x_0^2x_1(x_0^{-1/2}x_1^{3/2}-x_4)}},$$

where the triple integral is over all $x_0,x_1,x_4\in\mathbb{R}$ such that

$$0\leq x_0,x_1,|x_4|\leq 1,\quad x_1^{3/2}\geq x_0^{1/2}x_4,\quad 1+x_0x_4\geq x_1^{3/2}x_0^{1/2}.$$

The change of variables $u = x_1^{1/2}/x_0^{1/6}$, and then $v = x_0^{1/6}$, therefore yields

$$\omega_{\infty,+} = 6 \iiint \frac{\mathrm{d}u \, \mathrm{d}v \, \mathrm{d}x_4}{\sqrt{u^3 - x_4}},$$

where the triple integral is now over all $u, v, x_4 \in \mathbb{R}$ such that

$$0 \le v \le 1, \quad 0 \le u \le 1/v, \quad \max\{-1, u^3 - 1/v^6\} \le x_4 \le \min\{1, u^3\}.$$

On performing the integration over x_4, a straightforward calculation leads to the equality

$$\omega_{\infty,+} = 12 \int_0^1 \int_0^{1/v} \Big(\min\big\{\sqrt{u^3+1}, 1/v^3\big\} - \sqrt{\max\{u^3 - 1, 0\}}\,\Big) \,\mathrm{d}u \, \mathrm{d}v.$$

The calculation of $\omega_{\infty,-}$ is similar. Following the steps just outlined, one is easily led to the equality

$$\omega_{\infty,-} = 12 \int_0^1 \int_{-1}^0 \min\big\{\sqrt{u^3+1}, 1/v^3\big\} \,\mathrm{d}u \, \mathrm{d}v.$$

Once taken together, these equations combine to show that $\omega_\infty = 12\tau_\infty$, where τ_∞ is given by (1.6).

It remains to calculate the value of $\omega_p = \lim_{r\to\infty} p^{-3r} N(p^r)$, where we have written $N(p^r) = \#\{\mathbf{x} \pmod{p^r} : Q_1(\mathbf{x}) \equiv Q_2(\mathbf{x}) \equiv 0 \pmod{p^r}\}$. To begin with, we write

$$x_0 = p^{k_0} x_0' \quad \text{and} \quad x_1 = p^{k_1} x_1'$$

with $p \nmid x_0' x_1'$. Now we have $p^r \mid x_2^2$ if and only if $k_0 + k_1 \ge r$, and there are at most $p^{r/2}$ square roots of zero modulo p^r. When $k_0 + k_1 < r$, it follows that $k_0 + k_1$ must be even and we may write

$$x_2 = p^{(k_0+k_1)/2} x_2'$$

with $p \nmid x_2'$ and

$$x_0' x_1' - x_2'^2 \equiv 0 \pmod{p^{r-k_0-k_1}}.$$

The number of possible choices for x_0', x_1', x_2' is therefore

$$h_p(r, k_0, k_1) = \begin{cases} \phi(p^{r-k_0}) \phi(p^{r-(k_0+k_1)/2}) p^{k_0} & \text{if } k_0 + k_1 < r, \\ O(p^{5r/2 - k_0 - k_1}) & \text{if } k_0 + k_1 \ge r. \end{cases}$$

It remains to determine the number of solutions x_3, x_4 modulo p^r such that

$$p^{k_0} x_0' x_4 - p^{k_0/2 + 3k_1/2} x_1' x_2' + x_3^2 \equiv 0 \pmod{p^r}. \tag{2.7}$$

In order to do so, we distinguish between three basic cases: either $k_0 + k_1 < r$ and $k_0 \le 3k_1$, or $k_0 + k_1 < r$ and $k_0 > 3k_1$, or else $k_0 + k_1 \ge r$. For the first two of these cases we must take care to sum only over values of k_0, k_1 such that $k_0 + k_1$ is even. We shall denote by $N_i(p^r)$ the contribution to $N(p^r)$ from the ith case $(1 \le i \le 3)$, so that

$$N(p^r) = N_1(p^r) + N_2(p^r) + N_3(p^r). \tag{2.8}$$

We begin by calculating the value of $N_1(p^r)$. For this we write $x_3 = p^{k_3}x_3'$ with $k_3 = \min\{r/2, \lceil k_0/2\rceil\} = \lceil k_0/2\rceil$. The number of possibilities for x_3' is $p^{r-\lceil k_0/2\rceil}$, each one leading to a congruence of the form

$$x_0'x_4 - p^{3k_1/2-k_0/2}x_1'x_2' + p^{2\lceil k_0/2\rceil-k_0}x_3'^2 \equiv 0 \pmod{p^{r-k_0}}.$$

Modulo p^{r-k_0}, there is one choice for x_4 and so there are $p^{r+k_0-\lceil k_0/2\rceil} = p^{r+\lfloor k_0/2\rfloor}$ possibilities for x_3 and x_4. Summing these contributions over all relevant values of k_0, k_1 yields

$$N_1(p^r) = \sum_{\substack{k_0+k_1<r,\, k_0,k_1\geq 0\\ k_0\leq 3k_1,\, 2\mid(k_0+k_1)}} p^{r+\lfloor k_0/2\rfloor}h_p(r,k_0,k_1) = p^{3r-2}(p^2+4p+1)(1+o(1))$$

as $r\to\infty$.

Next we calculate $N_2(p^r)$, for which we shall not use the previous calculation for $h_p(r,k_0,k_1)$. On writing $x_3 = p^{k_3}x_3'$ with

$$k_3 = \min\{r/2, \lceil k_0/4+3k_1/4\rceil\} = \lceil k_0/4+3k_1/4\rceil,$$

we observe that $k_0/2 + 3k_1/2$ must be even because $p\nmid x_1'x_2'$. Hence $k_3 = k_0/4+3k_1/4$ and $p\nmid x_3'$. In this way (2.7) becomes

$$p^{k_0/2-3k_1/2}x_0'x_4 - x_1'x_2' + x_3'^2 \equiv 0 \pmod{p^{r-k_0/2-3k_1/2}},$$

thereby implying that $x_3'^2 \equiv x_1'x_2' \pmod{p^{k_0/2-3k_1/2}}$. At this point we recall the auxiliary congruence $x_2'^2 \equiv x_0'x_1' \pmod{p^{r-k_0-k_1}}$ that is satisfied by $x_0', x_1'x_2'$. We proceed by fixing values of x_2' and x_3', for which there are precisely $(1-1/p)^2p^{r-k_0/2-k_1/2}p^{r-k_3}$ choices. But then x_1' is fixed modulo $p^{k_0/2-3k_1/2}$ and so there are $p^{r-k_1-(k_0/2-3k_1/2)}$ possibilities for x_1'. Finally, we deduce from the remaining two congruences that there are p^{k_1} ways of fixing x_0' and p^{k_0} ways of fixing x_4. Summing over the relevant values of k_0 and k_1 then yields

$$N_2(p^r) = \sum_{\substack{k_0+k_1<r,\, k_0,k_1\geq 0\\ k_0>3k_1,\, 4\mid(k_0-k_1)}} (1-1/p)^2p^{3r-(k_0-k_1)/4} = 2p^{3r-1}(1+o(1))$$

as $r\to\infty$.

Finally we calculate the value of $N_3(p^r)$. In this case we write $x_2 = p^{r/2}x_2'$; but then a similar calculation ultimately shows that $N_3(p^r) = o(p^{3r})$ as $r\to\infty$. On combining our estimates for $N_1(p^r), N_2(p^r), N_3(p^r)$ into (2.8), we therefore deduce that

$$N(p^r) = p^{3r}\left(1+\frac{6}{p}+\frac{1}{p^2}\right)(1+o(1))$$

as $r\to\infty$, whence

$$\omega_p = \lim_{r\to\infty} p^{-3r}N(p^r) = 1+\frac{6}{p}+\frac{1}{p^2}$$

for any prime p. We combine this with (2.6), in the manner indicated by (2.2), in order to deduce (2.5). □

We end this section by combining (2.4) and Lemma 1 in (2.3) to deduce that the conjectured value of the constant in (1.1) is

$$c_{X,H} = \frac{1}{28800}\tau_\infty \tau,$$

where τ_∞ is given by (1.6) and τ is given by (2.5). This agrees with the value of the leading coefficient obtained in Theorem 2.

3. Congruences

In this section we collect together some of the basic facts concerning congruences that will be needed in the proof of Theorem 1. We begin by discussing the case of quadratic congruences. For any integers a, q such that $q > 0$, we define the arithmetic function $\eta(a; q)$ to be the number of positive integers $n \le q$ such that $n^2 \equiv a \pmod{q}$. When q is odd it follows that

$$\eta(a; q) = \sum_{d \mid q} |\mu(d)| \left(\frac{a}{d}\right),$$

where $\left(\frac{a}{d}\right)$ is the usual Jacobi symbol. On noting that $\eta(a; 2^\nu) \le 4$ for any $\nu \in \mathbb{N}$, it easily follows that

$$\eta(a; q) \le 2^{\omega(q)+1} \tag{3.1}$$

for any $q \in \mathbb{N}$. Here $\omega(q)$ denotes the number of distinct prime factors of q.

Turning to the case of linear congruences, let $\kappa \in [0, 1]$ and let ϑ be any arithmetic function such that

$$\sum_{d=1}^{\infty} \frac{|(\vartheta * \mu)(d)|}{d^\kappa} < \infty, \tag{3.2}$$

where $(f * g)(d) = \sum_{e \mid d} f(e) g(d/e)$ is the usual Dirichlet convolution of any two arithmetic functions f, g. Then, for any coprime integers a, q such that $q > 0$ and any $t \ge 1$, we deduce that

$$\begin{aligned}
\sum_{\substack{n \le t \\ n \equiv a \ (\mathrm{mod}\, q)}} \vartheta(n) &= \sum_{\substack{d=1 \\ \gcd(d,q)=1}}^{\infty} (\vartheta * \mu)(d) \sum_{\substack{m \le t/d \\ md \equiv a \ (\mathrm{mod}\, q)}} 1 \\
&= \frac{t}{q} \sum_{\substack{d=1 \\ \gcd(d,q)=1}}^{\infty} \frac{(\vartheta * \mu)(d)}{d} + O\left(t^\kappa \sum_{d=1}^{\infty} \frac{|(\vartheta * \mu)(d)|}{d^\kappa}\right),
\end{aligned}$$

on using the equality $\vartheta = (\vartheta * \mu) * 1$ and the trivial estimate $\lfloor x \rfloor = x + O(x^\kappa)$ for any $x > 0$. We summarize this estimate in the following result.

LEMMA 2. *Let $\kappa \in [0, 1]$, let ϑ be any arithmetic function such that* (3.2) *holds, and let $a, q \in \mathbb{Z}$ be such that $q > 0$ and $\gcd(a, q) = 1$. Then*

$$\sum_{\substack{n \le t \\ n \equiv a \ (\mathrm{mod}\, q)}} \vartheta(n) = \frac{t}{q} \sum_{\substack{d=1 \\ \gcd(d,q)=1}}^{\infty} \frac{(\vartheta * \mu)(d)}{d} + O\left(t^\kappa \sum_{d=1}^{\infty} \frac{|(\vartheta * \mu)(d)|}{d^\kappa}\right).$$

Define the real-valued function $\psi(t) = \{t\} - 1/2$, where $\{t\}$ denotes the fractional part of $t \in \mathbb{R}$. Then ψ is periodic with period 1. When $\vartheta(n) = 1$ for all $n \in \mathbb{N}$ we are able to refine Lemma 2 considerably.

LEMMA 3. *Let $a, q \in \mathbb{Z}$ be such that $q > 0$, and let $t_1, t_2 \in \mathbb{R}$ be such that $t_2 \geq t_1$. Then*

$$\#\{t_1 < n \leq t_2 : n \equiv a \pmod q\} = \frac{t_2 - t_1}{q} + r(t_1, t_2; a, q),$$

where

$$r(t_1, t_2; a, q) = \psi\Big(\frac{t_1 - a}{q}\Big) - \psi\Big(\frac{t_2 - a}{q}\Big).$$

Proof. Write $a = b + qc$ for some integer $0 \leq b < q$. Then it is clear that

$$\#\{t_1 < n \leq t_2 : n \equiv b \pmod q\} = \Big\lfloor \frac{t_2 - b}{q} \Big\rfloor - \Big\lfloor \frac{t_1 - b}{q} \Big\rfloor,$$

whence

$$\#\{t_1 < n \leq t_2 : n \equiv a \pmod q\} - \frac{t_2 - t_1}{q} = r(t_1, t_2; b, q).$$

We complete the proof of Lemma 3 by noting that $r(t_1, t_2; b, q) = r(t_1, t_2; a, q)$, since ψ has period 1. $\square$

We shall also need to know something about the average order of the function ψ. We proceed by demonstrating the following result.

LEMMA 4. *Let $\varepsilon > 0$, $t \geq 0$, and $X \geq 1$. Then, for any $b, q \in \mathbb{Z}$ such that $q > 0$ and $\gcd(b, q) = 1$,*

$$\sum_{0 \leq x < X} \psi\Big(\frac{t - bx^2}{q}\Big) \ll_\varepsilon (qX)^\varepsilon \Big(\frac{X}{q^{1/2}} + q^{1/2}\Big).$$

Proof. Throughout this proof we will write $e(t) = e^{2\pi i t}$ and $e_q(t) = e^{2\pi i t/q}$. In order to establish Lemma 4, we expand the function $f(k) = \psi((t-k)/q)$ as a Fourier series. Thus we have

$$f(k) = \sum_{0 \leq l < q} a(l) e_q(kl)$$

for any $k \in \mathbb{Z}$, where the coefficients $a(l)$ are given by

$$a(l) = \frac{1}{q} \sum_{0 \leq j < q} f(j) e_q(-jl).$$

Let $\|\alpha\|$ denote the distance from $\alpha \in \mathbb{R}$ to the nearest integer. We proceed by proving the estimates

$$a(l) \ll \begin{cases} q^{-1}, & l = 0, \\ q^{-1}\|l/q\|^{-1}, & l \neq 0. \end{cases} \tag{3.3}$$

This is straightforward. To verify the estimate for $a(0)$, we simply note that

$$\begin{aligned}
a(0) &= \frac{1}{q}\sum_{0\le j<q}\left(\left\{\frac{t-j}{q}\right\}-\frac{1}{2}\right)\\
&= \frac{1}{q}\sum_{0\le j\le t}\left(\frac{t-j-q/2}{q}\right)+\frac{1}{q}\sum_{t<j<q}\left(\frac{t-j+q/2}{q}\right) \ll \frac{1}{q}.
\end{aligned}$$

Similarly, when $l\neq 0$ we have

$$\begin{aligned}
a(l) &= \frac{1}{q}\sum_{0\le j\le t}\left(\frac{t-j-q/2}{q}\right)e_q(-jl)+\frac{1}{q}\sum_{t<j<q}\left(\frac{t-j+q/2}{q}\right)e_q(-jl)\\
&= \frac{1}{q}\sum_{0\le j<q}\frac{-j}{q}e_q(-jl)-\frac{1}{2q}\sum_{0\le j\le t}e_q(-jl)+\frac{1}{2q}\sum_{t<j<q}e_q(-jl) \ll \frac{1}{q\|l/q\|}
\end{aligned}$$

as required.

The foregoing argument now yields

$$\begin{aligned}
\sum_{0\le x<X}\psi\left(\frac{t-bx^2}{q}\right) &= \sum_{0\le l<q}a(l)\sum_{0\le x<X}e_q(lbx^2)\\
&= a(0)\lceil X\rceil+\sum_{m\mid q}\sum_{\substack{1\le l'<q/m\\ \gcd(l',q/m)=1}}a(l'm)\sum_{0\le x<X}e_{q/m}(l'x^2).
\end{aligned}$$

But here the inner sum can plainly be estimated using Weyl's inequality and so has size

$$\ll_\varepsilon X^\varepsilon\left(\frac{m^{1/2}X}{q^{1/2}}+\frac{q^{1/2}}{m^{1/2}}\right).$$

Employing (3.3), we therefore deduce that

$$\begin{aligned}
\sum_{0\le x<X}\psi\left(\frac{t-bx^2}{q}\right) &\ll_\varepsilon \frac{X}{q}+\sum_{m\mid q}m^{1/2}X^\varepsilon\sum_{1\le l'<q/m}\frac{q^{-1/2}X+q^{1/2}}{q\|l'm/q\|}\\
&\ll_\varepsilon (qX)^{2\varepsilon}\left(\frac{X}{q^{1/2}}+q^{1/2}\right),
\end{aligned}$$

which completes the proof of Lemma 4. □

Let $\varepsilon>0$ and $t\ge 0$. Then, for any $b,q\in\mathbb{Z}$ such that $q>0$ and $\gcd(b,q)=1$, we may deduce from Lemma 4 that

$$\sum_{0\le x<q}\psi\left(\frac{t-bx^2}{q}\right)\ll_\varepsilon q^{1/2+\varepsilon}. \tag{3.4}$$

It follows from an application of Möbius inversion that

$$\begin{aligned}
\sum_{\substack{0\le x<q\\ \gcd(x,q)=1}}\psi\left(\frac{t-bx^2}{q}\right) &= \sum_{n\mid q}\mu(n)\sum_{0\le x'<q/n}\psi\left(\frac{t/n-bnx'^2}{q/n}\right)\\
&= \sum_{\substack{n\mid q\\ m=\gcd(n,q/n)}}\mu(n)m\sum_{0\le x'<q/(mn)}\psi\left(\frac{t/(mn)-bnx'^2/m}{q/(mn)}\right),
\end{aligned}$$

whence (3.4) yields

$$\sum_{\substack{0\le x<q\\ \gcd(x,q)=1}} \psi\left(\frac{t-bx^2}{q}\right) \ll_\varepsilon q^{1/2+\varepsilon}\sum_{n\mid q}\left(\frac{\gcd(n,q/n)}{n}\right)^{1/2} \ll_\varepsilon q^{1/2+2\varepsilon}.$$

This establishes the following result once we re-define the choice of ε.

LEMMA 5. *Let $\varepsilon > 0$ and $t \ge 0$. Then, for any $b,q \in \mathbb{Z}$ such that $q > 0$ and* $\gcd(b,q) = 1$,

$$\sum_{\substack{0\le x<q\\ \gcd(x,q)=1}} \psi\left(\frac{t-bx^2}{q}\right) \ll_\varepsilon q^{1/2+\varepsilon}.$$

4. Preliminary Steps

We begin this section by introducing some notation. For any $n \ge 2$ we let Z^{n+1} denote the set of primitive vectors in $\mathbb{Z}^{n+1}$, where $\mathbf{v} = (v_0,\dots,v_n) \in \mathbb{Z}^{n+1}$ is said to be *primitive* if $\gcd(v_0,\dots,v_n) = 1$. Moreover, we let $\mathbb{Z}_*^{n+1}$ (resp. Z_*^{n+1}) denote the set of vectors $\mathbf{v} \in \mathbb{Z}^{n+1}$ (resp. $\mathbf{v} \in Z^{n+1}$) such that $v_0\cdots v_n \ne 0$. Finally we emphasize that, throughout this paper, $\mathbb{N}$ is always taken to denote the set of positive integers.

The proof of Theorem 1 rests upon establishing a preliminary asymptotic formula for the counting function $N_{U,H}(B)$. Recall the definition (2.1) of the quadratic forms Q_1, Q_2. Our first task in this section is to relate $N_{U,H}(B)$ to the quantity

$$N(Q_1,Q_2;B) = \#\{\mathbf{x} \in Z_*^5 : 0 < x_0,x_1,x_3,|x_4| \le B,\ Q_1(\mathbf{x}) = Q_2(\mathbf{x}) = 0\}.$$

In fact, we shall establish the following result rather easily.

LEMMA 6. *Let $B \ge 1$. Then*

$$N_{U,H}(B) = 2N(Q_1,Q_2;B) + \frac{12}{\pi^2}B + O(B^{2/3}).$$

Proof. It is clear that any solution to the pair of equations $Q_1(\mathbf{x}) = Q_2(\mathbf{x}) = 0$ that satisfies $x_0 = 0$ must correspond to a point on the line $x_0 = x_2 = x_3 = 0$ contained in X. Noting that $\mathbf{x}$ and $-\mathbf{x}$ represent the same point in projective space, we therefore deduce that

$$N_{U,H}(B) = \frac{1}{2}\#\{\mathbf{x} \in Z^5 : \|\mathbf{x}\| \le B,\ Q_1(\mathbf{x}) = Q_2(\mathbf{x}) = 0,\ x_0 \ne 0\},$$

where $\|\mathbf{x}\| = \max_{0\le i\le 4}|x_i|$. We proceed to consider the contribution from the vectors $\mathbf{x} \in Z^5$ for which $\|\mathbf{x}\| \le B$ and

$$Q_1(\mathbf{x}) = Q_2(\mathbf{x}) = 0, \quad x_1x_2x_3x_4 = 0. \tag{4.1}$$

Note first that $x_1 = 0$ if and only if $x_2 = 0$ in (4.1), since $x_0 \ne 0$. Thus, if we consider the contribution from those vectors for which $x_1x_2 = 0$, it follows that we must count integers $|x_0|, |x_3|, |x_4| \le B$ for which $\gcd(x_0, x_3, x_4) = 1$ and $x_0 x_4 + x_3^2 = 0$. Now either $x_3 = 0$, in which case $\mathbf{x} = (1, 0, 0, 0, 0)$ since $\mathbf{x}$ is primitive and $x_0 \ne 0$, or else the primitivity of $\mathbf{x}$ implies that $\mathbf{x} = (a^2, 0, 0, \pm ab, -b^2)$ for coprime nonzero integers a, b. Hence the overall contribution from this case is clearly $12B/\pi^2 + O(B^{1/2})$.

Suppose now that $x_3 = 0$ and $x_1 x_2 \ne 0$ in (4.1). Then we must count the number of mutually coprime nonzero integers x_0, x_1, x_2, x_4, with modulus at most B, such that $x_0 x_1 = x_2^2$ and $x_0 x_4 = x_1 x_2$. We are interested only in an upper bound and so it clearly suffices to count nonzero integers x_0, x_1, x_4, with modulus at most B, such that $\gcd(x_0, x_1, x_4) = 1$ and $x_0 x_4^2 = x_1^3$. But then it follows that $(x_0, x_1, x_4) = \pm(a^3, ab^2, b^3)$ for coprime nonzero integers a, b, whence the overall contribution is $O(B^{2/3})$. Finally the case $x_4 = 0$ and $x_1 x_2 \ne 0$ in (4.1) is handled in much the same way, now via the parametrization $\mathbf{x} = \pm(a^4, b^4, a^2 b^2, ab^3, 0)$. This establishes that

$$N_{U,H}(B) = \frac{1}{2}\#\{\mathbf{x} \in Z_*^5 : \|\mathbf{x}\| \le B,\ Q_1(\mathbf{x}) = Q_2(\mathbf{x}) = 0\} + \frac{12}{\pi^2} B + O(B^{2/3}).$$

We complete the proof of Lemma 6 by choosing $x_0 > 0$ and $x_3 > 0$. This then forces the inequality $x_1 > 0$, whence $\|\mathbf{x}\| = \max\{x_0, x_1, x_3, |x_4|\}$. □

We now turn to the task of establishing a bijection between the points counted by $N(Q_1, Q_2; B)$ and integral points on the universal torsor above the minimal desingularization of X. Let $\mathbf{x} \in Z_*^5$ be any vector counted by $N(Q_1, Q_2; B)$. In particular it follows that x_0, x_1, x_3 are positive. We begin by considering solutions to the equation $Q_1(\mathbf{x}) = 0$. It is easy to see that there is a bijection between the set of integers x_0, x_1, x_2 such that $x_0, x_1 > 0$ and $x_0 x_1 = x_2^2$ and the set of integers x_0, x_1, x_2 such that

$$x_0 = z_0^2 z_2, \quad x_1 = z_1^2 z_2, \quad x_2 = z_0 z_1 z_2$$

for nonzero integers z_0, z_1, z_2 such that $z_0, z_2 > 0$ and

$$\gcd(z_0, z_1) = 1. \tag{4.2}$$

We now substitute these values into the equation $Q_2(\mathbf{x}) = 0$ in order to obtain

$$x_4 z_0^2 z_2 - z_0 z_1^3 z_2^2 + x_3^2 = 0. \tag{4.3}$$

It is clear that $z_0 z_2$ divides x_3^2. Hence we write

$$z_0 = v_0 v_3 y_0''^2 \quad \text{and} \quad z_2 = v_2 v_3 y_2''^2$$

for $v_0, v_2, v_3, y_0'', y_2'' \in \mathbb{N}$ such that the products $v_0 v_3$ and $v_2 v_3$ are square-free, with $\gcd(v_0, v_2) = 1$. In particular, the product $v_0 v_2 v_3$ is clearly square-free. We easily deduce that $v_0 v_2 v_3 y_0'' y_2''$ must divide x_3, whence there exists $y_3'' \in \mathbb{N}$ such that $x_3 = v_0 v_2 v_3 y_0'' y_2'' y_3''$. Combining the various coprimality conditions arising from (4.2) and the definitions of v_0, v_2, and v_3, we obtain

$$|\mu(v_0 v_2 v_3)| = 1 \quad \text{and} \quad \gcd(v_0 v_3 y_0'', z_1) = 1, \tag{4.4}$$

where $\mu(n)$ denotes the Möbius function. After making the appropriate substitutions into (4.3), we deduce that

$$v_0 v_3 x_4 y_0''^2 - v_2 v_3 y_2''^2 z_1^3 + v_0 v_2 y_3''^2 = 0. \tag{4.5}$$

At this point it is convenient to deduce a further coprimality condition that follows from the assumption (made at the outset) that $\gcd(x_0, \dots, x_4) = 1$. Recalling the various changes of variables that we have made so far, it is easily checked that $\gcd(x_0, x_1, x_2, x_3) = v_2 v_3 y_2'' \gcd(y_2'', v_0 y_0'' y_3'')$. Hence we must have

$$\gcd(v_2 v_3 y_2'', x_4) = 1. \tag{4.6}$$

Now it follows from (4.5) that v_0 divides $v_2 v_3 y_2''^2 z_1^3$. But then, since v_0 is square-free, we may conclude from (4.4) that $v_0 \mid y_2''$. Similarly we deduce from (4.4) and (4.6) that $v_2 \mid y_0''$ and $v_3 \mid y_3''$. Hence there exist $y_0', y_2', y_3' \in \mathbb{N}$ and $y_1, y_4 \in \mathbb{Z}_*$ such that

$$y_0'' = v_2 y_0', \quad z_1 = y_1, \quad y_2'' = v_0 y_2', \quad y_3'' = v_3 y_3', \quad x_4 = y_4.$$

Substituting these terms into (4.5) gives

$$v_2 y_0'^2 y_4 - v_0 y_1^3 y_2'^2 + v_3 y_3'^2 = 0. \tag{4.7}$$

Moreover, we may combine (4.4) and (4.6) to get

$$|\mu(v_0 v_2 v_3)| = 1, \quad \gcd(v_0 v_2 v_3 y_0', y_1) = \gcd(v_0 v_2 v_3 y_2', y_4) = 1. \tag{4.8}$$

Finally we write $v_1 = \gcd(y_0', y_2', y_3')$. Hence there exist $y_0, y_2, y_3 \in \mathbb{N}$ such that

$$y_0' = v_1 y_0, \quad y_2' = v_1 y_2, \quad y_3' = v_1 y_3,$$

and we obtain the final equation

$$v_2 y_0^2 y_4 - v_0 y_1^3 y_2^2 + v_3 y_3^2 = 0. \tag{4.9}$$

It remains to collect together the coprimality conditions that have arisen from this last change of variables. However, we first take a moment to deduce three further coprimality conditions:

$$\gcd(y_0, y_2) = 1, \quad \gcd(y_0, y_3) = 1, \quad \gcd(y_2, y_3) = 1. \tag{4.10}$$

To do so we simply use the obvious fact that $\gcd(y_0, y_2, y_3) = 1$. Suppose that p is any prime divisor of y_2 and y_3. Then we clearly have $p^2 \mid v_2 y_0^2 y_4$ in (4.9), which is impossible by (4.8) and the fact that $\gcd(y_0, y_2, y_3) = 1$. From this we may establish the second relation in (4.10). Indeed, if $p \mid y_0, y_3$ then clearly $p^2 \mid v_0 y_1^3 y_2^2$, which is impossible by (4.8) and the fact that $\gcd(y_2, y_3) = 1$. One checks the first relation in (4.10) in a similar fashion. Combining (4.8) with (4.10), we therefore deduce that

$$\gcd(y_3, y_0 y_2) = \gcd(y_4, v_0 v_1 v_2 v_3 y_2) = 1 \tag{4.11}$$

and

$$|\mu(v_0 v_2 v_3)| = 1, \quad \gcd(y_1, v_0 v_1 v_2 v_3 y_0) = \gcd(y_0, y_2) = 1. \tag{4.12}$$

In fact, it will be necessary to reformulate these coprimality conditions somewhat. We claim that, once taken together with (4.9), the relations (4.11) and (4.12) are equivalent to

$$\gcd(y_3, v_0 y_0 y_2) = \gcd(y_4, v_1 v_2) = 1 \tag{4.13}$$

and

$$\gcd(y_1, v_0 v_1 v_2 v_3 y_0) = 1, \tag{4.14}$$

$$|\mu(v_0 v_2 v_3)| = 1, \quad \gcd(v_2 v_3 y_0, y_2) = \gcd(v_0 v_3, y_0) = 1. \tag{4.15}$$

We first show how (4.9), (4.11), and (4.12) imply (4.9), (4.13), (4.14), and (4.15). Suppose that p is any prime divisor of v_0 and y_3. Then (4.9) implies that $p \mid v_2 y_0^2 y_4$, which is easily seen to be impossible via (4.11) and (4.12). Thus $\gcd(y_3, v_0) = 1$. Now suppose that p is a prime divisor of v_3 and y_2. Then $p \mid v_2 y_0^2 y_4$, which is also impossible and so $\gcd(v_3, y_2) = 1$. The supplementary conditions $\gcd(v_2, y_2) = \gcd(v_0 v_3, y_0) = 1$ easily follow from the relations $\gcd(v_0 y_2, v_3 y_3) = \gcd(v_3, y_1) = 1$ together with (4.9). The converse is established along similar lines.

At this point we may summarize our argument as follows. Let $\mathcal{T} \subset \mathbb{Z}_*^9$ denote the set of $(\mathbf{v}, \mathbf{y}) = (v_0, v_1, v_2, v_3, y_0, \dots, y_4) \in \mathbb{N}^4 \times \mathbb{Z}_*^5$ such that $y_0, y_2, y_3 > 0$, (4.9), and (4.13)–(4.15) hold. Then, for any $\mathbf{x} \in Z_*^5$ counted by $N(Q_1, Q_2; B)$, we have shown that there exists $(\mathbf{v}, \mathbf{y}) \in \mathcal{T}$ such that

$$\begin{aligned}
x_0 &= v_0^4 v_1^6 v_2^5 v_3^3 y_0^4 y_2^2,\\
x_1 &= v_0^2 v_1^2 v_2 v_3 y_1^2 y_2^2,\\
x_2 &= v_0^3 v_1^4 v_2^3 v_3^2 y_0^2 y_1 y_2^2,\\
x_3 &= v_0^2 v_1^3 v_2^2 v_3^2 y_0 y_2 y_3,\\
x_4 &= y_4.
\end{aligned}$$

Conversely, given any $(\mathbf{v}, \mathbf{y}) \in \mathcal{T}$, any such point $\mathbf{x}$ will be a solution of the equations $Q_1(\mathbf{x}) = Q_2(\mathbf{x}) = 0$ with $\mathbf{x} \in Z_*^5$. To see the primitivity of $\mathbf{x}$ we first recall that, once taken together with (4.9), the coprimality relations (4.13)–(4.15) are equivalent to (4.11) and (4.12). But then it follows that $\gcd(x_0, x_1, x_2, x_3)$ divides $v_0^2 v_1^2 v_2 v_3 y_2^2$. Finally, an application of (4.11) and (4.12) yields

$$\gcd(x_0, \dots, x_4) = \gcd(\gcd(x_0, x_1, x_2, x_3), x_4) \le \gcd(v_0^2 v_1^2 v_2 v_3 y_2^2, y_4) = 1$$

as claimed.

Define the function $\Psi : \mathbb{R}^9 \to \mathbb{R}_{\ge 0}$ as given by

$$\Psi(\mathbf{v}, \mathbf{y}) = \max\{v_0^4 v_1^6 v_2^5 v_3^3 y_0^4 y_2^2, v_0^2 v_1^2 v_2 v_3 y_1^2 y_2^2, v_0^2 v_1^3 v_2^2 v_3^2 y_0 y_2 y_3, |y_4|\}.$$

We have thus established the following result.

Lemma 7. *Let $B \ge 1$. Then*

$$N(Q_1, Q_2; B) = \#\{(\mathbf{v}, \mathbf{y}) \in \mathcal{T} : \Psi(\mathbf{v}, \mathbf{y}) \le B\}.$$

It will become clear in subsequent sections that equation (4.9) is a crucial ingredient in our proof of Theorem 1. In fact, (4.9) is an affine embedding of the universal torsor above the minimal desingularization of X. Thus Derenthal, in work to appear, has established the isomorphism

$$\operatorname{Cox}(\tilde{X}) = \operatorname{Spec}(\mathbb{Q}[\mathbf{v},\mathbf{y}]/(v_2y_0^2y_4 - v_0y_1^3y_2^2 + v_3y_3^2)),$$

where $\operatorname{Cox}(\tilde{X})$ is the Cox ring of $\tilde{X}$.

5. The Final Count

In this section we estimate $N(Q_1, Q_2; B)$, which will then be combined with Lemma 6 to provide an initial estimate for $N_{U,H}(B)$. Before proceeding with this task, it may be helpful to outline our strategy. In view of (4.9) it is clear that, for any $(\mathbf{v},\mathbf{y}) \in \mathcal{T}$, the inequality $|y_4| \le B$ is equivalent to

$$-Bv_2y_0^2 \le v_3y_3^2 - v_0y_1^3y_2^2 \le Bv_2y_0^2. \tag{5.1}$$

We henceforth write $\Phi(\mathbf{v},\mathbf{y})$ to denote the condition obtained by replacing the term $|y_4|$ by $|(v_3y_3^2 - v_0y_1^3y_2^2)/(v_2y_0^2)|$ in the definition of $\Psi(\mathbf{v},\mathbf{y})$.

The basic idea behind our method is simply to view equation (4.9) as a congruence

$$v_3y_3^2 \equiv v_0y_1^3y_2^2 \pmod{v_2y_0^2}.$$

Since we will have $\gcd(v_0y_1^3y_2^2, v_2y_0^2) = 1$ when $(\mathbf{v},\mathbf{y}) \in \mathcal{T}$, it follows from (4.9), (4.13), and (4.14) that there exists a unique positive integer $\varrho \le v_2y_0^2$ such that

$$\gcd(\varrho, v_2y_0^2) = 1, \quad v_3\varrho^2 \equiv v_0y_1 \pmod{v_2y_0^2},$$

and

$$y_3 \equiv \varrho y_1y_2 \pmod{v_2y_0^2}.$$

That y_3 and y_4 satisfy the coprimality conditions (4.13) complicates matters slightly and makes it necessary to carry out a Möbius inversion first.

Next we analyze the inequality $\Phi(\mathbf{v},\mathbf{y}) \le B$. In doing so it will be convenient to define the quantities

$$V_1 = \left(\frac{B}{v_0^4v_2^5v_3^3y_0^4y_2^2}\right)^{1/6} \tag{5.2}$$

and

$$Y_1 = \left(\frac{Bv_2y_0^2}{v_0y_2^2}\right)^{1/3}, \quad Y_2 = \left(\frac{B}{v_0^4v_1^6v_2^5v_3^3y_0^4}\right)^{1/2}, \quad Y_3 = \left(\frac{Bv_2y_0^2}{v_3}\right)^{1/2}. \tag{5.3}$$

Moreover, we will need to define the real-valued functions

$$f_-(u,v) = \sqrt{\max\{u^3-1, 0\}}, \qquad f_+(u,v) = \min\{\sqrt{u^3+1}, 1/v^3\},$$

and

$$f(u,v) = f_+(u,v) - f_-(u,v). \tag{5.4}$$

In view of the inequality $v_0^2v_1^3v_2^2v_3^2y_0y_2y_3 \le B$ that is implied by $\Phi(\mathbf{v},\mathbf{y}) \le B$, we plainly have $y_3 \le V_1^3Y_3/v_1^3$. A little thought reveals that, once this is combined with the inequalities in (5.1), the result is

$$Y_3 f_-(y_1/Y_1, v_1/V_1) \le y_3 \le Y_3 f_+(y_1/Y_1, v_1/V_1). \tag{5.5}$$

Using the inequality $v_0^2v_1^2v_2v_3y_1^2y_2^2 \le B$ and deducing from (5.1) that $y_1 > -Y_1$, we also see that

$$-Y_1 < y_1 \le \frac{V_1Y_1}{v_1}. \tag{5.6}$$

Next, it follows from the inequality $\Phi(\mathbf{v},\mathbf{y}) \le B$ that

$$v_0^4v_1^6v_2^5v_3^3y_0^4y_2^2 \le B, \tag{5.7}$$

whence

$$1 \le y_2 \le Y_2 \tag{5.8}$$

and $1 \le v_1 \le V_1$. In particular we must have $V_1 \ge 1$, so we may deduce the further inequality

$$V_1Y_1 \le V_1^3Y_1 = \frac{B^{5/6}}{v_0^{7/3}v_2^{13/6}v_3^{3/2}y_0^{4/3}y_2^{5/3}}. \tag{5.9}$$

This will turn out to be useful at the end of Section 5.1.

After taking care of the contribution S from the variables y_3 and y_4 in Section 5.1, we will proceed in Section 5.2 by summing S over nonzero integers y_1 such that (5.6) holds and over positive integers y_2 such that (5.8) holds, subject to certain conditions. We shall denote this contribution by S'. Finally, in Section 5.3, we obtain an estimate for $N_{U,H}(B)$ by summing S' over the remaining values of v_0, v_1, v_2, v_3, y_0, subject to certain constraints, and then applying Lemma 6. During the course of the ensuing argument, in which we establish estimates for S, S', and finally $N_{U,H}(B)$, it will be convenient to handle the overall contribution from the error term in each estimate as we proceed.

5.1. Summation over the Variables y_3 and y_4

We begin by summing over the variables y_3, y_4. Let $(\mathbf{v}, y_0, y_1, y_2) \in \mathbb{N}^4 \times \mathbb{Z}_*^3$ satisfy (4.14) and (4.15) and be constrained to lie in the region defined by the inequalities (5.6), (5.7), and $y_0, y_2 > 0$. As already indicated, we shall denote the double summation over y_3 and y_4 by S. In order to take care of the coprimality condition $\gcd(y_4, v_1v_2) = 1$ in (4.13), we apply a Möbius inversion to get

$$S = \sum_{k_4 \mid v_1v_2} \mu(k_4) S_{k_4},$$

where the definition of S_{k_4} is as for S but with the extra condition $k_4 \mid y_4$ and without the coprimality condition $\gcd(y_4, v_1v_2) = 1$. Hence S_{k_4} is equal to the number of integers y_3 contained in the region (5.5) such that $\gcd(y_3, v_0y_0y_2) = 1$ and

$$v_3y_3^2 \equiv v_0y_1^3y_2^2 \pmod{k_4v_2y_0^2}.$$

Now it is straightforward to deduce from (4.9), (4.14), (4.15), and the coprimality relation $\gcd(y_3, v_0y_0y_2) = 1$ that

$$\begin{aligned}\gcd(v_0y_1^3y_2^2, k_4v_2y_0^2) &= \gcd(v_0y_1^3y_2^2, k_4)\\ &= \gcd(v_0y_1^3y_2^2, v_1v_2, v_3y_3^2)\\ &= \gcd(\gcd(v_0y_2^2, v_1), v_3y_3^2) = 1\end{aligned}$$

for any k_4 dividing v_1v_2 and y_4. Similarly one sees that $\gcd(v_3, k_4v_2y_0^2) = 1$ for any such k_4. We are therefore interested in summing only over divisors $k_4 \mid v_1v_2$ for which $\gcd(k_4, v_0v_3y_1y_2) = 1$. It actually suffices to sum over all divisors $k_4 \mid v_1v_2$ for which $\gcd(k_4, v_0v_3y_2) = 1$, since any divisor of v_1v_2 is coprime to y_1 by (4.14). Under this understanding it is now clear that there exists a unique positive integer ϱ, with $\varrho \le k_4v_2y_0^2$ and $\gcd(\varrho, k_4v_2y_0^2) = 1$, such that

$$v_3\varrho^2 \equiv v_0y_1 \pmod{k_4v_2y_0^2}, \qquad y_3 \equiv \varrho y_1y_2 \pmod{k_4v_2y_0^2}.$$

Our investigation has therefore led to the equality

$$S = \sum_{\substack{k_4 \mid v_1v_2\\ \gcd(k_4, v_0v_3y_2)=1}} \mu(k_4) \sum_{\substack{\varrho \le k_4v_2y_0^2\\ v_3\varrho^2 \equiv v_0y_1 \pmod{k_4v_2y_0^2}\\ \gcd(\varrho, k_4v_2y_0^2)=1}} S_{k_4}(\varrho),$$

where

$$\begin{aligned}&S_{k_4}(\varrho)\\ &\quad = \#\{y_3 \in \mathbb{Z}_* : \gcd(y_3, v_0y_2) = 1,\ (5.5) \text{ holds},\ y_3 \equiv \varrho y_1y_2 \pmod{k_4v_2y_0^2}\}.\end{aligned}$$

Here the coprimality relation $\gcd(y_0, y_3) = 1$ follows from the relations (4.14), (4.15), and $\gcd(\varrho, k_4v_2y_0^2) = 1$.

In view of the fact that $\gcd(k_4, v_0v_3y_2) = 1$, it follows from (4.14) and (4.15) that $\gcd(\varrho y_1y_2, k_4v_2y_0^2) = 1$ in the definition of $S_{k_4}(\varrho)$. In order to estimate $S_{k_4}(\varrho)$ we may therefore employ Lemma 2 with $\kappa = 0$ and the characteristic function

$$\chi(n) = \begin{cases} 1 & \text{if } \gcd(n, v_0y_2) = 1,\\ 0 & \text{otherwise.}\end{cases}$$

Now it is easy to see that

$$\sum_{\substack{d=1\\ \gcd(d, k_4v_2y_0^2)=1}}^{\infty} \frac{(\chi * \mu)(d)}{d} = \prod_{\substack{p \mid v_0y_2\\ p \nmid k_4v_2y_0}} \Big(1 - \frac{1}{p}\Big) = \prod_{p \mid v_0y_2} \Big(1 - \frac{1}{p}\Big),$$

whence

$$S_{k_4}(\varrho) = \phi^*(v_0y_2) \frac{Y_3 f(y_1/Y_1, v_1/V_1)}{k_4v_2y_0^2} + O(2^{\omega(v_0y_2)}).$$

Here, as throughout, we use the notation

$$\phi^*(n) = \frac{\phi(n)}{n} = \prod_{p \mid n} \Big(1 - \frac{1}{p}\Big) \tag{5.10}$$

for any $n \in \mathbb{N}$. Note that the number of positive integers $\varrho \le k_4 v_2 y_0^2$ such that $\gcd(\varrho, k_4 v_2 y_0^2) = 1$ and

$$v_3 \varrho^2 \equiv v_0 y_1 \pmod{k_4 v_2 y_0^2}$$

is at most $\eta(v_0 v_3 y_1; k_4 v_2 y_0^2) \le 2^{\omega(k_4 v_2 y_0)+1} \le 2^{\omega(v_1 v_2 y_0)+1}$ by (3.1). We have thus established the following result.

LEMMA 8. *Let* $(\mathbf{v}, y_0, y_1, y_2) \in \mathbb{N}^5 \times \mathbb{Z}_* \times \mathbb{N}$ *satisfy* (4.14), (4.15), (5.6), *and* (5.7). *Then, for any* $B \ge 1$, *we have*

$$S = \frac{Y_3 f(y_1/Y_1, v_1/V_1)}{v_2 y_0^2} \Sigma(\mathbf{v}, y_0, y_1, y_2) + O(2^{\omega(v_0 y_2)} 4^{\omega(v_1 v_2 y_0)}),$$

where

$$\Sigma(\mathbf{v}, y_0, y_1, y_2) = \phi^*(v_0 y_2) \sum_{\substack{k_4 \mid v_1 v_2 \\ \gcd(k_4, v_0 v_3 y_2)=1}} \frac{\mu(k_4)}{k_4} \sum_{\substack{\varrho \le k_4 v_2 y_0^2 \\ v_3 \varrho^2 \equiv v_0 y_1 \pmod{k_4 v_2 y_0^2} \\ \gcd(\varrho, k_4 v_2 y_0^2)=1}} 1. \quad (5.11)$$

We close this section by showing that, once it is summed over all $(\mathbf{v}, y_0, y_1, y_2) \in \mathbb{N}^5 \times \mathbb{Z}_* \times \mathbb{N}$ satisfying (5.6) and (5.7), the error term in Lemma 8 is satisfactory. For this we will make use of the familiar estimate

$$\sum_{n \le x} a^{\omega(n)} \ll_a x(\log x)^{a-1},$$

for any $a \in \mathbb{N}$, in addition to estimates that follow from applying partial summation to it. We thereby obtain the overall contribution

$$\begin{aligned}
&\ll \sum_{v_0, v_1, v_2, v_3, y_0, y_2} \sum_{-Y_1 < y_1 \le V_1 Y_1 / v_1} 2^{\omega(v_0 y_2)} 4^{\omega(v_1 v_2 y_0)} \\
&\ll \sum_{v_0, v_1, v_2, v_3, y_0, y_2} 2^{\omega(v_0 y_2)} 4^{\omega(v_1 v_2 y_0)} \frac{V_1 Y_1}{v_1} \\
&\ll (\log B)^4 \sum_{v_0, v_2, v_3, y_0, y_2} 2^{\omega(v_0 y_2)} 4^{\omega(v_2 y_0)} V_1 Y_1.
\end{aligned}$$

But now we may employ (5.9) to bound this quantity by

$$\ll B^{5/6} (\log B)^4 \sum_{v_0, v_2, v_3, y_0, y_2} \frac{2^{\omega(v_0 y_2)} 4^{\omega(v_2 y_0)}}{v_0^{7/3} v_2^{13/6} v_3^{3/2} y_0^{4/3} y_2^{5/3}} \ll B^{5/6} (\log B)^4.$$

We shall see that this is satisfactory.

5.2. *Summation over the Variables y_1 and y_2*

Our next task is to sum S over all nonzero integers y_1 that satisfy (4.14) and (5.6) and over all positive integers y_2 that satisfy $\gcd(y_2, v_2 v_3 y_0) = 1$ and (5.8). We therefore write

$$S' = \frac{Y_3}{v_2 y_0^2} \sum_{\substack{y_2 \le Y_2 \\ \gcd(y_2, v_2 v_3 y_0)=1}} \sum_{\substack{-Y_1 < y_1 \le V_1 Y_1/v_1 \\ \gcd(y_1, v_0 v_1 v_2 v_3 y_0)=1}} f\left(\frac{y_1}{Y_1}, \frac{v_1}{V_1}\right) \Sigma(\mathbf{v}, y_0, y_1, y_2),$$

where $\Sigma(\mathbf{v}, y_0, y_1, y_2)$ is given by (5.11).

Let $t > 0$. We begin by establishing asymptotic formulas for the two quantities

$$\mathcal{S}(\pm t) = \phi^*(v_0 y_2) \sum_{\substack{k_4 \mid v_1 v_2 \\ \gcd(k_4, v_0 v_3 y_2)=1}} \frac{\mu(k_4)}{k_4} \sum_{\substack{\varrho \le k_4 v_2 y_0^2 \\ \gcd(\varrho, k_4 v_2 y_0^2)=1}} S'_{k_4}(\varrho; \pm t),$$

where

$$\begin{aligned} &S'_{k_4}(\varrho; +t) \\ &\quad = \#\{0 \le y_1 \le t : \gcd(y_1, v_0 v_1 v_2 v_3 y_0) = 1,\ v_3 \varrho^2 \equiv v_0 y_1 \pmod{k_4 v_2 y_0^2}\}, \\ &S'_{k_4}(\varrho; -t) \\ &\quad = \#\{-t \le y_1 \le 0 : \gcd(y_1, v_0 v_1 v_2 v_3 y_0) = 1,\ v_3 \varrho^2 \equiv v_0 y_1 \pmod{k_4 v_2 y_0^2}\}. \end{aligned}$$

Now it is clear that we have

$$\gcd(v_3 \varrho^2, k_4 v_2 y_0^2) = 1 \tag{5.12}$$

in the definition of $S'_{k_4}(\varrho; \pm t)$, since $\gcd(k_4, v_3) = 1$. In particular, it follows that we may replace the coprimality relation appearing in $S'_{k_4}(\varrho; \pm t)$ with $\gcd(y_1, v_0 v_1 v_3) = 1$. After treating this coprimality condition with a Möbius inversion, we find that $\mathcal{S}(\pm t)$ is equal to

$$\phi^*(v_0 y_2) \sum_{\substack{k_4 \mid v_1 v_2 \\ \gcd(k_4, v_0 v_3 y_2)=1}} \frac{\mu(k_4)}{k_4} \sum_{\substack{k_1 \mid v_0 v_1 v_3 \\ \gcd(k_1, k_4 v_2 y_0)=1}} \mu(k_1) \sum_{\substack{\varrho \le k_4 v_2 y_0^2 \\ \gcd(\varrho, k_4 v_2 y_0^2)=1}} S'_{k_1,k_4}(\varrho; \pm t),$$

where

$$\begin{aligned} S'_{k_1,k_4}(\varrho; +t) &= \#\{0 \le y_1 \le t/k_1 : v_3 \varrho^2 \equiv k_1 v_0 y_1 \pmod{k_4 v_2 y_0^2}\}, \\ S'_{k_1,k_4}(\varrho; -t) &= \#\{-t/k_1 \le y_1 \le 0 : v_3 \varrho^2 \equiv k_1 v_0 y_1 \pmod{k_4 v_2 y_0^2}\}. \end{aligned}$$

Here we have used (5.12) to deduce that we must sum only over those values of $k_1 \mid v_0 v_1 v_3$ for which $\gcd(k_1, k_4 v_2 y_0) = 1$.

Let $(\mathbf{v}, y_0) \in \mathbb{N}^5$ satisfy the constraints

$$|\mu(v_0 v_2 v_3)| = \gcd(v_0 v_3, y_0) = 1 \quad \text{and} \quad v_0^4 v_1^6 v_2^5 v_3^3 y_0^4 \le B \tag{5.13}$$

that follow from (4.15) and (5.7). We let $b_\pm \le k_4 v_2 y_0^2$ be the unique positive integer such that

$$b_\pm k_1 v_0 \equiv \pm v_3 \pmod{k_4 v_2 y_0^2}.$$

In particular it follows from (5.12) that $\gcd(b_\pm, k_4 v_2 y_0^2) = 1$, and we may therefore employ Lemma 3 to deduce that

$$S'_{k_1,k_4}(\varrho; \pm t) = \frac{t}{k_1 k_4 v_2 y_0^2} + r(\pm t; b_\pm \varrho^2),$$

where

$$r(\pm t; b_\pm \varrho^2) = \psi\left(\frac{-b_\pm \varrho^2}{k_4 v_2 y_0^2}\right) - \psi\left(\frac{t/k_1 - b_\pm \varrho^2}{k_4 v_2 y_0^2}\right). \tag{5.14}$$

Recall the definition (5.10) of ϕ^* and observe that $\phi^*(ab)\phi^*(\gcd(a,b)) = \phi^*(a)\phi^*(b)$ for any $a, b \in \mathbb{N}$. We define

$$\vartheta(\mathbf{v}, y_0, y_2) = \begin{cases} \dfrac{\phi^*(v_0 v_1 v_2 y_2)\phi^*(v_0 v_1 v_2 v_3 y_0)}{\phi^*(\gcd(v_1, v_3))} & \text{if (4.15) holds,} \\ 0 & \text{otherwise.} \end{cases} \tag{5.15}$$

Then a straightforward calculation reveals that

$$\mathcal{S}(\pm t) = \vartheta(\mathbf{v}, y_0, y_2) t + \mathcal{R}(\pm t) \tag{5.16}$$

for any nonzero $t > 0$, where

$$\mathcal{R}(\pm t) = \phi^*(v_0 y_2) \sum_{\substack{k_4 \mid v_1 v_2 \\ \gcd(k_4, v_0 v_3 y_2)=1}} \frac{\mu(k_4)}{k_4} \sum_{\substack{k_1 \mid v_0 v_1 v_3 \\ \gcd(k_1, k_4 v_2 y_0)=1}} \mu(k_1) \sum_{\substack{\varrho \le k_4 v_2 y_0^2 \\ \gcd(\varrho, k_4 v_2 y_0^2)=1}} r(\pm t; b_\pm \varrho^2).$$

Here $r(\pm t; b_\pm \varrho^2)$ is given by (5.14), and the positive integers b_-, b_+ are uniquely determined by fixed choices of $k_1, k_4, v_0, v_2, v_3, y_0$ as previously outlined.

We may now apply partial summation to estimate S'. It is clear that $S' = S'_- + S'_+$, where S'_- denotes the contribution from y_1 in the interval $(-Y_1, 0]$ and S'_+ denotes the contribution from y_1 in the interval $(0, V_1 Y_1 / v_1]$. We begin by estimating S'_-, for which we first deduce from (5.2) and (5.3) that

$$\frac{v_1}{V_1} = \left(\frac{y_2}{Y_2}\right)^{1/3} \quad \text{and} \quad Y_1 = \left(\frac{B v_2 y_0^2}{v_0 y_2^2}\right)^{1/3} = \frac{Y_1'}{y_2^{2/3}},$$

say. We may now apply (5.16), in conjunction with partial summation, to deduce that

$$S'_- = \sum_{\substack{y_2 \le Y_2 \\ \gcd(y_2, v_2 v_3 y_0)=1}} \left(\frac{\vartheta(\mathbf{v}, y_0, y_2) Y_1 Y_3}{v_2 y_0^2} \int_{-1}^{0} f\left(\frac{u, v_1}{V_1}\right) \mathrm{d}u\right) + R'_-,$$

where

$$\begin{aligned} R'_- &= \frac{Y_3}{v_2 y_0^2} \sum_{\substack{y_2 \le Y_2 \\ \gcd(y_2, v_2 v_3 y_0)=1}} \int_0^1 f'\left(-u, \left(\frac{y_2}{Y_2}\right)^{1/3}\right) \mathcal{R}\left(-u \frac{Y_1'}{y_2^{2/3}}\right) \mathrm{d}u \\ &= \frac{Y_3}{v_2 y_0^2} \sum_{\substack{k_4 \mid v_1 v_2 \\ \gcd(k_4, v_0 v_3)=1}} \frac{\mu(k_4)}{k_4} \sum_{\substack{k_1 \mid v_0 v_1 v_3 \\ \gcd(k_1, k_4 v_2 y_0)=1}} \mu(k_1) \sum_{\substack{\varrho \le k_4 v_2 y_0^2 \\ \gcd(\varrho, k_4 v_2 y_0^2)=1}} \\ &\qquad \sum_{\substack{y_2 \le Y_2 \\ \gcd(y_2, k_4 v_2 v_3 y_0)=1}} \phi^*(v_0 y_2) \int_0^1 f'\left(-u, \left(\frac{y_2}{Y_2}\right)^{1/3}\right) r\left(-u \frac{Y_1'}{y_2^{2/3}}; b_- \varrho^2\right) \mathrm{d}u. \end{aligned}$$

Define the arithmetic function

$$\phi^\dagger(n) = \prod_{p\mid n}\Big(1+\frac{1}{p}\Big)^{-1}.$$

We estimate R'_- via an application of Lemma 2 with $a = 0$, $q = 1$, and $\kappa = \varepsilon$. This gives

$$\sum_{\substack{y_2\le t\\ \gcd(y_2,k_4v_2v_3y_0)=1}} \phi^*(v_0y_2) = \frac{6}{\pi^2}\phi^\dagger(k_4v_0v_2v_3y_0)t + O(2^{\omega(v_1v_2v_3y_0)}t^{\varepsilon}).$$

Indeed, the corresponding Dirichlet series is equal to

$$\phi^*(v_0)\zeta(s)\prod_{p\nmid k_4v_0v_2v_3y_0}\Big(1-\frac{1}{p^{s+1}}\Big)\prod_{p\mid k_4v_2v_3y_0}\Big(1-\frac{1}{p^{s}}\Big).$$

An application of partial summation therefore yields the estimate

$$\begin{aligned} R'_- &= \frac{\varphi_-(\mathbf{v},y_0)Y_2Y_3}{v_2y_0^2} + O(2^{\omega(v_1v_2)+\omega(v_0v_1v_3)+\omega(v_1v_2v_3y_0)}Y_2^{\varepsilon}Y_3)\\ &= \frac{\varphi_-(\mathbf{v},y_0)Y_2Y_3}{v_2y_0^2} + O_\varepsilon(B^{\varepsilon}Y_3), \end{aligned} \tag{5.17}$$

where

$$\varphi_-(\mathbf{v},y_0) = \frac{18}{\pi^2}\sum_{\substack{k_4\mid v_1v_2\\ \gcd(k_4,v_0v_3)=1}}\frac{\mu(k_4)\phi^\dagger(k_4v_0v_2v_3y_0)}{k_4}\sum_{\substack{k_1\mid v_0v_1v_3\\ \gcd(k_1,k_4v_2y_0^2)=1}}\mu(k_1)$$
$$\int_0^1\int_0^1\Bigg(t^2f'(-u,t)\sum_{\substack{\varrho\le k_4v_2y_0^2\\ \gcd(\varrho,k_4v_2y_0^2)=1}} r\Big(-\frac{v_0v_1^2v_2^2v_3y_0^2u}{t^2};b_-\varrho^2\Big)\Bigg)\,\mathrm{d}u\,\mathrm{d}t.$$

Here we have used the trivial inequality $2^{\omega(n)} = O_\varepsilon(n^\varepsilon)$ for any $n\in\mathbb{N}$. An application of Lemma 5 clearly reveals that

$$\varphi_-(\mathbf{v},y_0) \ll_\varepsilon (v_2y_0^2)^{1/2+\varepsilon}2^{\omega(v_1v_2)+\omega(v_0v_1v_3)}$$

for any $\varepsilon > 0$. Our estimate (5.17) for R'_- isn't terribly good when Y_2 is small. Fortunately, we may invert the order of summation over ϱ and y_2 and then use Lemma 5 to deduce the alternative estimate

$$\begin{aligned} R'_- &= \frac{\varphi_-(\mathbf{v},y_0)Y_2Y_3}{v_2y_0^2} + O_\varepsilon\Big(2^{\omega(v_1v_2)+\omega(v_0v_1v_3)}\frac{Y_2Y_3}{(v_2y_0^2)^{1/2-\varepsilon}}\Big)\\ &= \frac{\varphi_-(\mathbf{v},y_0)Y_2Y_3}{v_2y_0^2} + O_\varepsilon\Big(B^{\varepsilon}\frac{Y_2Y_3}{(v_2y_0^2)^{1/2}}\Big). \end{aligned} \tag{5.18}$$

Note here that the main term is dominated by the error term. On combining (5.17) and (5.18), however, we obtain the estimate

$$R'_- = \frac{\varphi_-(\mathbf{v}, y_0)Y_2Y_3}{v_2 y_0^2} + O_\varepsilon\left(B^\varepsilon Y_3 \min\left\{1, \frac{Y_2}{(v_2 y_0^2)^{1/2}}\right\}\right).$$

By arguing in a similar fashion, it is straightforward to deduce that

$$S'_+ = \sum_{\substack{y_2 \le Y_2 \\ \gcd(y_2, v_2v_3y_0)=1}} \left(\frac{\vartheta(\mathbf{v}, y_0, y_2)Y_1Y_3}{v_2 y_0^2} \int_0^{V_1/v_1} f\left(u, \frac{v_1}{V_1}\right) \mathrm{d}u\right) + R'_+,$$

where

$$R'_+ = \frac{\varphi_+(\mathbf{v}, y_0)Y_2Y_3}{v_2 y_0^2} + O_\varepsilon\left(B^\varepsilon Y_3 \min\left\{1, \frac{Y_2}{(v_2 y_0^2)^{1/2}}\right\}\right).$$

Here one finds that

$$\varphi_+(\mathbf{v}, y_0) = \frac{18}{\pi^2} \sum_{\substack{k_4 \mid v_1v_2 \\ \gcd(k_4, v_0v_3)=1}} \frac{\mu(k_4)\phi^\dagger(k_4v_0v_2v_3y_0)}{k_4} \sum_{\substack{k_1 \mid v_0v_1v_3 \\ \gcd(k_1, k_4v_2y_0^2)=1}} \mu(k_1)$$
$$\int_0^1 \int_0^{1/t} \left(t^2 f'(u,t) \sum_{\substack{\varrho \le k_4v_2y_0^2 \\ \gcd(\varrho, k_4v_2y_0^2)=1}} r\left(\frac{v_0v_1^2v_2^2v_3y_0^2u}{t^2}; b_+\varrho^2\right)\right) \mathrm{d}u\, \mathrm{d}t$$

with $\varphi_+(\mathbf{v}, y_0) \ll_\varepsilon (v_2y_0^2)^{1/2+\varepsilon} 2^{\omega(v_1v_2)+\omega(v_0v_1v_3)}$.

We may now complete our estimate for S'. Recall the definition (5.4) of the function $f(u,v)$, and define

$$g(v) = \int_{-1}^{1/v} f(u,v)\, \mathrm{d}u. \tag{5.19}$$

Then g is a bounded differentiable function whose derivative is also bounded on the interval $[0, \infty)$. Moreover, let

$$\varphi(\mathbf{v}, y_0) = \varphi_-(\mathbf{v}, y_0) + \varphi_+(\mathbf{v}, y_0). \tag{5.20}$$

Then combining our various estimates allows us to establish the following result.

LEMMA 9. *Let* $(\mathbf{v}, y_0) \in \mathbb{N}^5$ *satisfy* (5.13). *Then, for any* $B \ge 1$, *we have*

$$S' = \sum_{\substack{y_2 \le Y_2 \\ \gcd(y_2, v_2v_3y_0)=1}} \left(\frac{\vartheta(\mathbf{v}, y_0, y_2)Y_1Y_3 g(v_1/V_1)}{v_2y_0^2}\right) + \frac{\varphi(\mathbf{v}, y_0)Y_2Y_3}{v_2y_0^2}$$
$$+ O_\varepsilon\left(B^\varepsilon Y_3 \min\left\{1, \frac{Y_2}{(v_2y_0^2)^{1/2}}\right\}\right),$$

where $\vartheta(\mathbf{v}, y_0, y_2)$ *is given by* (5.15), g *is given by* (5.19), *and* $\varphi(\mathbf{v}, y_0)$ *is given by* (5.20) *and satisfies*

$$\varphi(\mathbf{v}, y_0) \ll_\varepsilon (v_2y_0^2)^{1/2+\varepsilon} 2^{\omega(v_1v_2)+\omega(v_0v_1v_3)} \tag{5.21}$$

for any $\varepsilon > 0$.

We end this section by showing that, once it is summed over all $(\mathbf{v}, y_0) \in \mathbb{N}^5$ satisfying (5.12), the error term in Lemma 9 is satisfactory. Recalling the definition (5.13) of Y_2 and Y_3 and then summing over y_0, we easily obtain the satisfactory overall contribution

$$\begin{aligned}
&\ll_\varepsilon B^{1/2+\varepsilon} \sum_{v_0, v_1, v_2, v_3, y_0} \frac{(v_2 y_0^2)^{1/2}}{v_3^{1/2}} \min\left\{1, \frac{B^{1/2}}{v_0^2 v_1^3 v_2^3 v_3^{3/2} y_0^3}\right\} \\
&\ll_\varepsilon B^{5/6+\varepsilon} \sum_{v_0, v_1, v_2, v_3} \frac{1}{v_0^{4/3} v_1^2 v_2^{3/2} v_3^{3/2}} \ll_\varepsilon B^{5/6+\varepsilon}.
\end{aligned}$$

5.3. Summation over the Remaining Variables

In this section we complete our preliminary estimate for $N_{U,H}(B)$. It is clear from Lemma 9 that we have two distinct terms to deal with. We begin by deducing from (5.3) that

$$\frac{Y_1 Y_3}{v_2 y_0^2} = \frac{B^{5/6} n^{1/6}}{v_0 v_1 v_2 v_3 y_0 y_2}$$

in the statement of Lemma 9, where $n = v_0^4 v_1^6 v_2^5 v_3^3 y_0^4 y_2^2$. Define the arithmetic function

$$\Delta(n) = B^{-5/6} \sum_{\substack{\mathbf{v}, y_0, y_2 \\ v_0^4 v_1^6 v_2^5 v_3^3 y_0^4 y_2^2 = n}} \frac{\vartheta(\mathbf{v}, y_0, y_2) Y_1 Y_3}{v_2 y_0^2}, \tag{5.22}$$

where $\vartheta(\mathbf{v}, y_0, y_2)$ is given by (5.15). Recall the definition (5.19) of the function g and that of the counting function $N(Q_1, Q_2; B)$ appearing in the statement of Lemma 6. Let $\varepsilon > 0$. We now establish the existence of a constant $\beta \in \mathbb{R}$ for which

$$N(Q_1, Q_2; B) = B^{5/6} \sum_{n \le B} \Delta(n) g\left(\left(\frac{n}{B}\right)^{1/6}\right) + \beta B + O_\varepsilon(B^{5/6+\varepsilon}). \tag{5.23}$$

This follows rather easily from Lemma 9. Define the sum

$$T(B) = \sum_{\substack{\mathbf{v}, y_0 \\ \text{(5.13) holds}}} \frac{\varphi(\mathbf{v}, y_0) Y_2 Y_3}{v_2 y_0^2}$$

for any $B \ge 1$. Then, in view of the error terms that we have estimated along the way in Sections 5.1 and 5.2, it is clearly enough to establish the existence of a constant $\beta \in \mathbb{R}$ for which

$$T(B) = \beta B + O(B^{5/6}).$$

On recalling (5.3), we see that

$$\frac{Y_2 Y_3}{v_2 y_0^2} = \frac{B}{v_0^2 v_1^3 v_2^3 v_3^2 y_0^3}.$$

Therefore, if we take $\varepsilon < 1/3$ then it follows from (5.21) that

$$
\begin{aligned}
T(B) - \beta B &\ll_\varepsilon B \sum_{\substack{\mathbf{v}, y_0 \\ v_0^4 v_1^6 v_2^5 v_3^3 y_0^4 > B}} \frac{(v_0 v_1 v_2 v_3 y_0)^\varepsilon}{v_0^2 v_1^3 v_2^{5/2} v_3^2 y_0^2} \\
&\ll_\varepsilon B^{5/6} \sum_{\mathbf{v}, y_0} \frac{(v_0 v_1 v_2 v_3 y_0)^\varepsilon}{v_0^{4/3} v_1^2 v_2^{5/3} v_3^{3/2} y_0^{4/3}} \ll B^{5/6}
\end{aligned}
$$

with

$$
\beta = \sum_{\substack{\mathbf{v}, y_0 \\ \gcd(v_0 v_3, y_0) = 1}} \frac{|\mu(v_0 v_2 v_3)| \varphi(\mathbf{v}, y_0)}{v_0^2 v_1^3 v_2^3 v_3^2 y_0^3}. \tag{5.24}
$$

This completes the proof of (5.23). Inserting this estimate into Lemma 6 then yields the following result.

LEMMA 10. *Let $\varepsilon > 0$. Then, for any $B \geq 1$, we have*

$$
N_{U,H}(B) = 2B^{5/6} \sum_{n \leq B} \Delta(n) g\left(\left(\frac{n}{B}\right)^{1/6}\right) + \left(\frac{12}{\pi^2} + 2\beta\right) B + O_\varepsilon(B^{5/6+\varepsilon}),
$$

where g is given by (5.19), *Δ is given by* (5.22), *and β is given by* (5.24).

6. The Height Zeta Function

For $\mathrm{Re}(s) > 1$ we recall the definition of the height zeta function (1.2) and the identity (1.5). Thus it follows from Lemma 10 that $Z_{U,H}(s) = Z_1(s) + Z_2(s)$, where

$$
Z_1(s) = 2s \int_1^\infty t^{-s-1/6} \sum_{n \leq t} \Delta(n) g\left(\left(\frac{n}{t}\right)^{1/6}\right) \mathrm{d}t,
$$

$$
Z_2(s) = \frac{12/\pi^2 + 2\beta}{s-1} + G_2(s),
$$

and

$$
G_2(s) = s \int_1^\infty t^{-s-1} R(t)\, \mathrm{d}t
$$

for some function $R(t)$ such that $R(t) \ll_\varepsilon t^{5/6+\varepsilon}$ for any $\varepsilon > 0$. But then it easily follows that $G_2(s)$ is holomorphic on the half-plane $\mathrm{Re}(s) \geq 5/6 + \varepsilon$ and satisfies $G_2(s) \ll 1 + |\mathrm{Im}(s)|$ on this domain. Finally, an application of the Phragmén–Lindelöf theorem yields the finer upper bound

$$
G_2(s) \ll_\varepsilon (1 + |\mathrm{Im}(s)|)^{6(1-\mathrm{Re}(s))+\varepsilon}
$$

on this domain.

To establish Theorem 1, it therefore remains only to analyze the function $Z_1(s)$. Recall the definition (5.22) of Δ and define the corresponding Dirichlet series

$$
F(s) = \sum_{n=1}^\infty \frac{\Delta(n)}{n^s}.
$$

Then it is easily seen that

$$Z_1(s) = 2sF\left(s-\frac{5}{6}\right)\int_1^\infty t^{-s-1/6}g\left(\frac{1}{t^{1/6}}\right)\mathrm{d}t = F\left(s-\frac{5}{6}\right)G_{1,1}(s),$$

where

$$G_{1,1}(s) = 12s\int_0^1 v^{6s-6}g(v)\,\mathrm{d}v. \tag{6.1}$$

Recall the definition (5.19) of g. A simple calculation reveals that $G_{1,1}(1) = 12\tau_\infty$ in the notation of (1.6). Moreover, an application of partial integration yields

$$G_{1,1}(s) = \frac{12s}{6s-5}\left(g(1) - \int_0^1 v^{6s-5}g'(v)\,\mathrm{d}v\right),$$

whence it is clear that $G_{1,1}(s)$ is holomorphic and bounded on the half-plane $\mathrm{Re}(s) \geq 5/6+\varepsilon$ for any $\varepsilon > 0$.

We proceed by analyzing the Dirichlet series $F(s-5/6)$ in more detail. Define the function

$$G_{1,2}(s) = \frac{F(s-5/6)}{E_1(s)E_2(s)} \tag{6.2}$$

for $\mathrm{Re}(s) > 5/6$, and let $\varepsilon > 0$. Here $E_1(s)$ and $E_2(s)$ are given by (1.3) and (1.4), respectively. In order to complete the proof of Theorem 1 with

$$G_1(s) = G_{1,1}(s)G_{1,2}(s), \tag{6.3}$$

it remains to establish that $G_{1,2}(1) \neq 0$ and that $G_{1,2}(s)$ is holomorphic and bounded for $\mathrm{Re}(s) \geq 5/6+\varepsilon$. This we achieve via the following result.

LEMMA 11. *Let $\varepsilon > 0$. Then $G_{1,2}(s+1)$ is holomorphic and bounded on the half-plane $\mathcal{H} = \{s \in \mathbb{C} : \mathrm{Re}(s) \geq -1/6+\varepsilon\}$.*

Proof. On writing

$$G_{1,2}(s+1) = \prod_p G_p(s+1),$$

it will clearly suffice to show that $G_p(s+1) = 1 + O_\varepsilon(1/p^{1+\varepsilon})$ uniformly on $\mathcal{H}$. We begin the proof of Lemma 11 by observing that

$$F\left(s+\frac{1}{6}\right) = \sum_{\substack{(\mathbf{v},y_0,y_2)\in\mathbb{N}^6\\ \gcd(v_2v_3y_0,y_2)=1\\ \gcd(v_0v_3,y_0)=1}} \frac{|\mu(v_0v_2v_3)|\phi^*(v_0v_1v_2y_2)\phi^*(v_0v_1v_2v_3y_0)}{\phi^*(\gcd(v_1,v_3))v_0^{4s+1}v_1^{6s+1}v_2^{5s+1}v_3^{3s+1}y_0^{4s+1}y_2^{2s+1}}.$$

After writing $F(s+1/6) = \prod_p F_p(s+1/6)$ as a product of local factors, a straightforward calculation reveals that $F_p(s+1/6)$ is equal to

$$\begin{aligned}1 &+ \frac{1-1/p}{p^{2s+1}-1} + \frac{1-1/p}{p^{4s+1}-1} + \frac{(1-1/p)^2}{p^{6s+1}-1}\left(\frac{p^{2s+1}}{p^{2s+1}-1} + \frac{1}{p^{4s+1}-1}\right)\\ &+ \frac{p^{4s+1}(1-1/p)^2}{(p^{2s+1}-1)(p^{6s+1}-1)} + \frac{p^{5s+1}(1-1/p)^2}{(p^{4s+1}-1)(p^{6s+1}-1)} + \frac{p^{3s}(1-1/p)}{p^{6s+1}-1}\end{aligned} \tag{6.4}$$

for any prime p. Collecting together factors of $(p^{2s+1}-1)^{-1}$ and $(p^{4s+1}-1)^{-1}$ then gives

$$\begin{aligned} F_p\Big(s+\frac{1}{6}\Big)\Big(1-\frac{1}{p^{6s+1}}\Big) = 1 &- \frac{1}{p^{6s+1}} + \frac{1}{p^{3s+1}} \\ &+ \frac{1}{p^{2s+1}-1}\Big(1+\frac{1}{p^{2s}}+\frac{1}{p^{4s}}-\frac{1}{p^{6s+1}}\Big) \\ &+ \frac{1}{p^{4s+1}-1}\Big(1+\frac{1}{p^{s}}\Big) + O_\varepsilon\Big(\frac{1}{p^{1+\varepsilon}}\Big) \end{aligned}$$

on $\mathcal{H}$. We now record the obvious estimates

$$\begin{aligned} \frac{1}{p^{2s+1}-1} &= \frac{1}{p^{2s+1}} + \frac{1}{p^{4s+2}} + O\Big(\frac{1}{p^{2+6\varepsilon}}\Big), \\ \frac{1}{p^{4s+1}-1} &= \frac{1}{p^{4s+1}} + \frac{1}{p^{8s+2}} + \frac{1}{p^{12s+3}} + O_\varepsilon\Big(\frac{1}{p^{4/3+16\varepsilon}}\Big), \end{aligned}$$

and

$$1+\frac{1}{p^{2s}}+\frac{1}{p^{4s}}-\frac{1}{p^{6s+1}} \ll_\varepsilon p^{2/3-4\varepsilon}, \qquad 1+\frac{1}{p^{s}} \ll_\varepsilon p^{1/6-\varepsilon},$$

which all hold on $\mathcal{H}$. Combining these estimates then allows us to deduce that

$$\begin{aligned} F_p\Big(s+\frac{1}{6}\Big)\Big(1-\frac{1}{p^{6s+1}}\Big) = 1 &+ \frac{1}{p^{2s+1}} + \frac{1}{p^{3s+1}} + \frac{2}{p^{4s+1}} + \frac{1}{p^{5s+1}} \\ &+ \frac{1}{p^{8s+2}} + \frac{1}{p^{9s+2}} + \frac{1}{p^{13s+3}} + O_\varepsilon\Big(\frac{1}{p^{1+\varepsilon}}\Big). \end{aligned}$$

Write $E_{1,p}(s+1)$ for the Euler factor of (1.3) and write $E_{2,p}(s+1)$ for the Euler factor of (1.4). It is now a routine matter to deduce that

$$\begin{aligned} \frac{F_p(s+1/6)}{E_{1,p}(s+1)} &= 1 - \frac{3}{p^{7s+2}} - \frac{3}{p^{8s+2}} - \frac{1}{p^{9s+2}} - \frac{1}{p^{10s+2}} + \frac{3}{p^{13s+3}} \\ &\quad + \frac{1}{p^{14s+3}} + O\Big(\frac{1}{p^{1+\varepsilon}}\Big) \\ &= E_{2,p}(s+1)\Big(1+O\Big(\frac{1}{p^{1+\varepsilon}}\Big)\Big) \end{aligned}$$

on $\mathcal{H}$, which completes the proof of Lemma 11. □

It remains to combine the expression (6.4) for $F_p(s+1/6)$ with (1.3) and (6.2) in order to deduce that

$$E_2(1)G_{1,2}(1) = \prod_p \Big(1-\frac{1}{p}\Big)^6\Big(1+\frac{6}{p}+\frac{1}{p^2}\Big) \neq 0.$$

Thus we have completed the proof of Theorem 1.

7. Deduction of Theorem 2

In this section we will deduce Theorem 2 from Theorem 1 and Lemma 10. Let $\varepsilon > 0$ and $T \in [1, B]$. Then an application of Perron's formula yields

$$N_{U,H}(B) - \Big(\frac{12}{\pi^2} + 2\beta\Big)B = \frac{1}{2\pi i}\int_{1+\varepsilon-iT}^{1+\varepsilon+iT} E_1(s)E_2(s)G_1(s)\frac{B^s}{s}\,\mathrm{d}s + O_\varepsilon\Big(\frac{B^{11/6+\varepsilon}}{T}\Big). \tag{7.1}$$

We apply Cauchy's residue theorem to the rectangular contour $\mathcal{C}$ joining the points $\kappa - iT, \kappa + iT, 1 + \varepsilon + iT$, and $1 + \varepsilon - iT$ for any $\kappa \in [11/12, 1)$. We must calculate the residue of $E_1(s)E_2(s)G_1(s)B^s/s$ at $s = 1$. For $\mathrm{Re}(s) > 9/10$, Theorem 1 implies that the product $E_2(s)G_1(s)$ is holomorphic and bounded. In view of (1.3), we see that

$$E_1(s) = \frac{1}{2880(s-1)^6} + O\Big(\frac{1}{(s-1)^5}\Big)$$

as $s \to 1$. Hence it follows that

$$\mathrm{Res}_{s=1}\Big\{E_1(s)E_2(s)G_1(s)\frac{B^s}{s}\Big\} = \frac{E_2(1)G_1(1)}{5!\,2880}BQ_1(\log B)$$

for some monic polynomial Q_1 of degree 5. Recall from (6.3) that $G_1 = G_{1,1}G_{1,2}$. We have already seen in the previous section that $G_1(1) = 12\tau_\infty\tau$ in the notation of (1.6) and (2.5). Putting all of this together, we have thus shown that

$$\frac{1}{2\pi i}\int_{\mathcal{C}} E_1(s)E_2(s)G_1(s)\frac{B^s}{s}\,\mathrm{d}s = \frac{\tau\tau_\infty}{28800}BQ_2(\log B)$$

for some monic polynomial Q_2 of degree 5. Define the difference

$$E(B) = N_{U,H}(B) - \frac{\tau\tau_\infty}{28800}BQ_2(\log B) - \Big(\frac{12}{\pi^2} + 2\beta\Big)B.$$

Then, given (7.1) and that the product $E_2(s)G_1(s)$ is holomorphic and bounded for $\mathrm{Re}(s) > 9/10$, we deduce that

$$E(B) \ll_\varepsilon \frac{B^{11/6+\varepsilon}}{T} + \Big(\int_{\kappa-iT}^{\kappa+iT} + \int_{\kappa-iT}^{1+\varepsilon-iT} + \int_{1+\varepsilon+iT}^{\kappa+iT}\Big)\Big|E_1(s)\frac{B^s}{s}\Big|\,\mathrm{d}s \tag{7.2}$$

for any $\kappa \in [11/12, 1)$ and any $T \in [1, B]$.

We begin by estimating the contribution from the horizontal contours. Recall the well-known convexity bound

$$\zeta(\sigma + it) \ll_\varepsilon |t|^{(1-\sigma)/3+\varepsilon},$$

which is valid for any $\sigma \in [1/2, 1]$ and $|t| \geq 1$. Then it follows that

$$E_1(\sigma + it) \ll_\varepsilon |t|^{8(1-\sigma)+\varepsilon} \tag{7.3}$$

for any $\sigma \in [11/12, 1]$ and $|t| \geq 1$. This estimate allows us to deduce that

$$\begin{aligned}\int_{\kappa-iT}^{1+\varepsilon-iT}\left|E_1(s)\frac{B^s}{s}\right|\mathrm{d}s &\ll_\varepsilon \int_\kappa^{1+\varepsilon} B^\sigma T^{7-8\sigma+\varepsilon}\,\mathrm{d}\sigma\\ &\ll_\varepsilon \frac{B^{1+\varepsilon}T^\varepsilon}{T}+B^\kappa T^{7-8\kappa+\varepsilon}.\end{aligned} \tag{7.4}$$

One may obtain the same estimate for the contribution from the remaining horizontal contour joining $\kappa+iT$ to $1+\varepsilon+iT$.

We now turn to the size of the integral

$$\int_{\kappa-iT}^{\kappa+iT}\left|E_1(s)\frac{B^s}{s}\right|\mathrm{d}s \ll B^\kappa\int_{-T}^{T}\frac{|E_1(\kappa+it)|}{1+|t|}\,\mathrm{d}t = B^\kappa I(T), \tag{7.5}$$

say. For given $0<U\ll T$, we begin by estimating the contribution to $I(T)$ from each integral:

$$\int_U^{2U}\frac{|E_1(\kappa+it)|}{1+|t|}\,\mathrm{d}t \ll \frac{1}{U}\int_U^{2U}|E_1(\kappa+it)|\,\mathrm{d}t = \frac{J(U)}{U},$$

say. Let $\varepsilon>0$ and $k\in\mathbb{N}$. Then we define σ_k to be the infimum of σ such that

$$\frac{1}{T}\int_1^T|\zeta(\sigma+it)|^{2k}\,\mathrm{d}t = O_\varepsilon(T^\varepsilon).$$

It follows from the mean value theorem in [13, Sec. 7.8] that

$$\int_U^{2U}|\zeta(\sigma+it)|^{2k}\,\mathrm{d}t \ll_\varepsilon U^{1+\varepsilon} \tag{7.6}$$

for any $\sigma\in(\sigma_k,1]$ and any $U\geq 1$. We shall apply this estimate in the cases $k=2$ and $k=4$, for which we combine a result due to Heath-Brown [7] with well-known estimates for the fourth moment of $|\zeta(1/2+it)|$ in order to deduce that

$$\sigma_k \leq \begin{cases} 1/2 & \text{if } k=2,\\ 5/8 & \text{if } k=4.\end{cases} \tag{7.7}$$

Returning to our estimate for $J(U)$, for fixed $0<U\ll T$ and any $\kappa\in[11/12,1)$ we define $J(U;c)=\int_U^{2U}|\zeta(c\kappa-c+1+cit)|^4\,\mathrm{d}t$. Then we may apply Hölder's inequality to deduce that

$$J(U)\leq J(U;6)^{1/4}J(U;5)^{1/4}J(U;4)^{1/4}J(U;3)^{1/8}J(U;2)^{1/8}.$$

Combining (7.6), (7.7), and the fact that $\kappa\in[11/12,1)$, we deduce that $J(U)\ll_\varepsilon U^{1+\varepsilon}$ after re-defining ε. Summing over dyadic intervals for $0<U\ll T$ then yields

$$\int_0^T\frac{|E_1(\kappa+it)|}{1+|t|}\,\mathrm{d}t \ll_\varepsilon T^\varepsilon.$$

We obtain the same estimate for the integral over the interval $[-T,0]$ and so it follows that $I(T)\ll_\varepsilon T^\varepsilon$. We may insert this estimate into (7.5), and then combine it with (7.4) in (7.2), in order to conclude that

$$E(B)\ll_\varepsilon \frac{B^{11/6+\varepsilon}T^\varepsilon}{T}+B^\kappa T^\varepsilon$$

for any $T\in[1,B]$. We thus complete the proof of Theorem 2 by taking $T=B$.

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