Eliminating Tverberg Points, I. An Analogue of the Whitney Trick*

Isaac Mabillard
IST Austria
Am Campus 1
3400 Klosterneuburg
Austria
imabillard@ist.ac.at

Uli Wagner
IST Austria
Am Campus 1
3400 Klosterneuburg
Austria
uli@ist.ac.at

ABSTRACT

Motivated by topological Tverberg-type problems, we consider multiple (double, triple, and higher multiplicity) self-intersection points of maps from finite simplicial complexes (compact polyhedra) into $\mathbb{R}^d$ and study conditions under which such multiple points can be eliminated.

The most classical case is that of embeddings (i.e., maps without *double points*) of a k-dimensional complex K into $\mathbb{R}^{2k}$. For this problem, the work of van Kampen, Shapiro, and Wu provides an efficiently testable *necessary* condition for embeddability (namely, vanishing of the *van Kampen obstruction*). For $k \geq 3$, the condition is also sufficient, and yields a polynomial-time algorithm for deciding embeddability: One starts with an arbitrary map $f: K \to \mathbb{R}^{2k}$, which generically has finitely many double points; if $k \geq 3$ and if the obstruction vanishes then one can successively remove these double points by local modifications of the map f. One of the main tools is the famous Whitney trick that permits eliminating pairs of double points of opposite intersection sign.

We are interested in generalizing this approach to intersection points of higher multiplicity. We call a point $y \in \mathbb{R}^d$ an *r-fold Tverberg point* of a map $f: K^k \to \mathbb{R}^d$ if y lies in the intersection $f(\sigma_1)\cap\ldots\cap f(\sigma_r)$ of the images of r pairwise disjoint simplices of K.

The analogue of (non-)embeddability that we study is the problem $\texttt{Tverberg}^r_{k\to d}$: Given a k-dimensional complex K, does it satisfy a Tverberg-type theorem with parameters r and d, i.e., does every map $f: K^k \to \mathbb{R}^d$ have an r-fold Tverberg point? Here, we show that for fixed r, k and d of the form $d = rm$ and $k = (r-1)m$, $m \geq 3$, there is a polynomial-time algorithm for deciding this (based on the vanishing of a cohomological obstruction, as in the case of embeddings).

Our main tool is an r-fold analogue of the Whitney trick: Given r pairwise disjoint simplices of K such that the intersection of their images contains two r-fold Tverberg points y_+ and y_- of opposite intersection sign, we can eliminate y_+ and y_- by a local isotopy of f.

In a subsequent paper, we plan to develop this further and present a generalization of the classical *Haefliger–Weber Theorem* (which yields a necessary and sufficient condition for embeddability of k-complexes into $\mathbb{R}^d$ for a wider range of dimensions) to intersection points of higher multiplicity.

Categories and Subject Descriptors

F.2.2 [**ANALYSIS OF ALGORITHMS AND PROBLEM COMPLEXITY**]: Nonnumerical Algorithms and Problems

General Terms

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Keywords

Tverberg-type problems, computational topology, simplicial complexes, Whitney trick

1. INTRODUCTION

Graph planarity is a basic theme in graph theory as well as in discrete and computational geometry. The corresponding higher-dimensional questions concerning embeddings of finite simplicial complexes (compact polyhedra) into $\mathbb{R}^d$ are a classical topic in topology (see, e.g., [23, 28] for surveys), and have recently also become the subject of systematic study from a viewpoint of algorithms and computational complexity [18, 17].

Let K be a finite k-dimensional simplicial complex[1] and let $f: K \to \mathbb{R}^d$ be a continuous map. We say that $y \in \mathbb{R}^d$ is an *r-fold intersection point*[2] of f if there are r pairwise distinct points $x_1, \ldots, x_r \in K$ with $f(x_1) = \ldots = f(x_r) = y$. In particular, a map f is an embedding iff it does not have any double (twofold) points.

We aim to generalise classical results concerning embeddings, such as the Whitney trick, and study conditions under

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[1]Slightly abusing notation, we do not scrupulously distinguish between a simplicial complex K and its *underlying topological space* but rely on context to distinguish between the two when necessary.

[2]We will often abbreviate this to *r-intersection point* or *r-fold point.*

which intersection points of higher multiplicity can be eliminated. In this extended abstract, we restrict ourselves to the following subclass of r-fold intersection points.

Tverberg points and Tverberg-type problems. An r-fold point y of a map $f\colon K \to \mathbb{R}^d$ is called an *r-fold Tverberg point* (or *r-Tverberg point*, for short) of f if the preimages x_i lie in r pairwise disjoint simplices of K, i.e., $y \in f(\sigma_1) \cap \ldots \cap f(\sigma_r)$ with $\sigma_i \cap \sigma_j = \emptyset$ for $1 \leq i < j \leq r$. Thus, being a Tverberg point depends on the actual complex, not just the underlying space.

The classical geometric Tverberg theorem [30], a cornerstone of convex geometry, can be rephrased as saying that if $N = (d+1)(r-1)$ then any *affine* map from the N-dimensional simplex Δ^N to $\mathbb{R}^d$ has an r-Tverberg point.

The topological Tverberg conjecture asserts that the conclusion should remain true for continuous maps $\Delta^N \to \mathbb{R}^d$. The conjecture is known to be true if r is a prime [3] or a prime power [22, 32], but in general this remains one of the most challenging open problems in topological combinatorics.

A closely related problem (originally motivated [2] by the *k-set problem*, see [19, Ch. 11] for a detailed discussion) is the *Colored Tverberg problem* about Tverberg points of maps $f\colon K \to \mathbb{R}^d$ in the case where $K = V_1 * V_2 * \ldots * V_{d+1}$ is a *join* of $d+1$ discrete point sets (thought of as color classes). It is known that any such map necessarily has an r-Tverberg point if $m = d+1$, each $|V_i|$ has size at least $2r-1$, and r is a prime or a prime power [39, 34]. In a recent breakthrough [4], sharp bounds were obtained for the case that $r+1$ is a prime: in this case, it is sufficient to have $|V_1| = \ldots = |V_{d+1}| = r$.

There are numerous close relatives and other variants of Tverberg-type problems (see, e.g., [5] for further results and references). Here, we propose to investigate the computational complexity of deciding whether a given simplicial complex admits a topological Tverberg theorem.

We define the decision problem $\texttt{Tverberg}^r_{k\to d}$, which has as input a finite k-dimensional simplicial complex K, and the output is `Yes` or `No` depending on whether every map $f\colon K \to \mathbb{R}^d$ has an r-Tverberg point or not.[3]

Throughout this paper, we will work with maps that are *piecewise linear* (*PL*), i.e., simplexwise linear on some subdivision of K. Note that if there exists a continuous map $f : K \to \mathbb{R}^d$ without r-Tverberg point then, by compactness we may slightly perturb f to a PL map without r-Tverberg points, so the PL assumption on f is no loss of generality.

In the special case $r = 2$, the problem $\texttt{Tverberg}^2_{k\to d}$ is closely related to the following problem $\texttt{Embed}_{k\to d}$, introduced in [18]: given a k-complex K, does it admit a (PL) embedding into $\mathbb{R}^d$?More precisely, the answer to $\texttt{Tverberg}^2_{k\to d}$ is `No` if and only if there exists an *almost-embedding* $f\colon K \to \mathbb{R}^d$, i.e., a map such that $f(\sigma_1) \cap f(\sigma_2) = \emptyset$ whenever σ_1 and σ_2 are *disjoint* simplices of K. The classical *Haefliger–Weber theorem* (see below) implies that in the so-called *metastable range* $d \geq 3(k+1)/2$, embeddability and almost-embeddability are equivalent, hence $\texttt{Embed}_{k\to d}$ is equivalent to $\texttt{Tverberg}^2_{k\to d}$ (or, more precisely, to $\neg\texttt{Tverberg}^2_{k\to d}$) in that range.

[3]We emphasize that this is a rather different problem than that of finding a (geometric) Tverberg point (which is guaranteed to exist) for a given set P of $N = (d+1)(r-1)+1$ points in $\mathbb{R}^d$, as considered, e.g., in [1, 21].

As a first step towards a systematic study of the computational complexity of $\texttt{Tverberg}^r_{k\to d}$ for general r, we prove the following positive result.

THEOREM 1. *Suppose that $r \geq 2$, $d - k \geq 3$, and $rk \leq d(r-1)$. Then there is an algorithm for deciding $\texttt{Tverberg}^r_{k\to d}$. For fixed r, d, and k the algorithm runs in polynomial time in the size of the input complex K.*

We note that for $rk < d(r-1)$, a PL map $f\colon K \to \mathbb{R}^d$ in general position has no r-intersection points, so the interesting case is $rk = d(r-1)$, i.e., $k = (r-1)m$ and $d = rm$ for some $m \geq 3$. The theorem generalizes the classical fact ($r = 2$) that the *van Kampen obstruction* provides an algorithm for $\texttt{Embed}_{k\to 2k}$, $k \geq 3$. For a brief discussion of other ranges of the parameters d, k, r, see the paragraph on future work below.

The first (and standard) tool for the proof of Theorem 1 are *deleted products* and *equivariant obstruction theory*, which yield an efficiently testable *necessary* condition for the existence of a map $f\colon K \to \mathbb{R}^d$ without r-Tverberg points.

Deleted products and the van Kampen obstruction. In the most classical case $r = 2$, the twofold (combinatorial) *deleted product* of a simplicial complex K is the cell complex K^2_Δ whose cells are the products $\sigma_1 \times \sigma_2$ of pairs of disjoint simplices of K. If $f\colon K \to \mathbb{R}^d$ is an almost-embedding then we get a map

$$\widetilde{f}\colon K^2_\Delta \to S^{d-1}, \qquad \widetilde{f}(x_1, x_2) := \frac{f(x_1) - f(x_2)}{\|f(x_1) - f(x_2)\|}, \quad (1)$$

called the *Gauss map*. Moreover, the group $\mathbb{Z}_2$ naturally acts on both K^2_Δ and S^{d-1} (by swapping coordinates and by antipodality, respectively), and the Gauss map is *equivariant* with respect to these actions, i.e., $\widetilde{f}(x_2, x_1) = -\widetilde{f}(x_1, x_2)$. Thus, the existence of a $\mathbb{Z}_2$-equivariant map, denoted $K^2_\Delta \to_{\mathbb{Z}_2} S^{d-1}$ is a necessary condition for the existence of an almost-embedding.

The famous *Haefliger–Weber Theorem* [11, 36] (see also [28] for a modern survey and extensions) states that in the so-called *metastable range* $d \geq 3(k+1)/2$ this necessary condition is also sufficient, even for the existence of an embedding, i.e., a k-complex K embeds into $\mathbb{R}^d$ if and only if there exists an equivariant map $K^2_\Delta \to_{\mathbb{Z}_2} S^{d-1}$. The metastable range assumption is tight, i.e., for every pair (k, d) such that $3 \leq d < 3(k+1)/2$, there exists a k-dimensional complex K such that $K^2_\Delta \to_{\mathbb{Z}_2} S^{d-1}$ but K does not embed into $\mathbb{R}^d$, see [15, 26, 9, 25, 10].

The Haefliger–Weber Theorem generalizes earlier work by van Kampen, Shapiro, and Wu [31, 27, 37] regarding the special case $d = 2k$ ($\geq \dim K^2_\Delta$). In this case, the *van Kampen obstruction*, which is an equivariant cohomology class $\mathfrak{o}_{K^2_\Delta} \in H^d_{\mathbb{Z}_2}(K^2_\Delta; \mathcal{Z})$ (see Section 2.3), is the only obstruction to the existence of such an equivariant map, i.e., $K^2_\Delta \to_{\mathbb{Z}_2} S^{d-1}$ if and only if $\mathfrak{o}_{K^2_\Delta} = 0$. Moreover, if $k \geq 3$, this is the case if and only if there is an embedding $f\colon K \to \mathbb{R}^{2k}$.

The condition $\mathfrak{o}_{K^2_\Delta} = 0$ can be efficiently tested (see, e.g., [18] for details), which yields, for fixed $k \geq 3$, a polynomial-time algorithm for $\texttt{Embed}_{k\to 2k}$. More generally, by recent results in computational homotopy theory [6, 13, 7, 8], there is a polynomial-time algorithm for deciding the existence of an equivariant map $K^2_\Delta \to_{\mathbb{Z}_2} S^{d-1}$, and hence for solving

$\mathtt{Embed}_{k\to d}$, for any (k, d) in the metastable range.

The obstruction for r-fold Tverberg points. More generally, one can define the *r-fold deleted product* K^r_Δ as the subcomplex of the Cartesian product K^r whose cells are products $\sigma_1 \times \ldots \times \sigma_r$ of r-tuples of pairwise disjoint simplices of K. The symmetric group $\mathfrak{S}_r$ acts on K^r_Δ by permuting components. One easily constructs an analogue of the Gauss map (1) to get (see Section 2.3):

LEMMA 2. *For $r \geq 2$, d, $k \geq 1$, if there is a map $f\colon K \to \mathbb{R}^d$ without r-Tverberg point then there is an equivariant map (with respect to a suitable action of $\mathfrak{S}_r$ on $S^{d(r-1)-1}$)*

$$\widetilde{f}\colon K^r_\Delta \to_{\mathfrak{S}_r} S^{d(r-1)-1}. \quad (2)$$

If $d(r-1) \geq \dim(K^r_\Delta)$ then equivariant obstruction theory yields a generalized *r-fold van Kampen obstruction* $\mathfrak{o}_{K^r_\Delta} \in H^{d(r-1)}_{\mathfrak{S}_r}(K^r_\Delta; \mathcal{Z})$, such that $K^r_\Delta \to_{\mathfrak{S}_r} S^{d(r-1)-1}$ if and only if $\mathfrak{o}_{K^r_\Delta} = 0$, see Section 2.3. (the coefficent ring $\mathcal{Z}$ denotes the integers with a suitable $\mathfrak{S}_r$-action.)

For example, Özaydin [22] and Volovikov [32] showed that for $r = p^\ell$ a prime power and $K = \Delta^N$ the N-simplex, $N = (d+1)(r-1)$, there is no equivariant map $(\Delta^N)^r_\Delta \to_{\mathfrak{S}_r} S^{d(r-1)-1}$, thus proving the topological Tverberg conjecture in this case.

On the other hand, Özaydin [22] also showed that for every r that is *not* a prime power, there *does* exist an equivariant map $g\colon (\Delta^N)^r_\Delta \to_{\mathfrak{S}_r} S^{d(r-1)-1}$. The construction does not give a map of the form $g = \widetilde{f}$ for a map $f\colon \Delta^N \to \mathbb{R}^d$, so it does not provide a counterexample to the general topological Tverberg conjecture, but it shows that the conjecture cannot be proved by applying Lemma 2 (at least not in a straightforward way, with $K = \Delta^N$). It also raises the question if, at least under suitable additional hypotheses, one may prove a converse to Lemma 2. As a first step in this direction, we prove the following:

THEOREM 3. *Suppose that $r \geq 2$, $d - k \geq 3$, and $d(r-1) \geq rk$, and let K be a k-dimensional simplicial complex. Then there exists a map $f : K \to \mathbb{R}^d$ without r-Tverberg point if and only if there exists an equivariant map*

$$g\colon K^r_\Delta \to_{\mathfrak{S}_r} S^{d(r-1)-1}, \quad (3)$$

and this happens if and only if the r-fold generalized van Kampen obstruction vanishes, i.e., $\mathfrak{o}_{K^r_\Delta} = 0$.

Moreover, the condition $\mathfrak{o}_{K^r_\Delta} = 0$ can be efficiently tested, and hence Theorem 3 implies Theorem 1:

LEMMA 4. *There is an algorithm for deciding $\mathfrak{o}_{K^r_\Delta} = 0$, which for fixed r, k and d runs in polynomial time in the size of the input K.*

Strategy of the proof. The proof of Theorem 3 is modeled after the classical proof for the completeness of the van Kampen obstruction [9] and is structured as follows (the remaining definitions will be given in the subsequent sections). As remarked before, the interesting case is $rk = d(r-1)$, i.e., we will work under the assumptions

$$k = (r-1)m, \qquad d = rm, \qquad m \geq 3. \quad (4)$$

The proof consists of the following steps:

(a) Let K be a k-dimensional simplicial complex. The obstruction cohomology class $\mathfrak{o}_{K^r_\Delta}$ can be computed geometrically as follows: Take a PL map $f\colon K \to \mathbb{R}^d$ in general position. Then for each r-tuple of pairwise disjoint k-simplices of K, their images intersect transversely in finitely many r-intersection points $y \in f(\sigma_1) \cap \ldots \cap f(\sigma_r)$. For each such y, one can define an *intersection sign* $\mathrm{sign}_y(f(\sigma_1), \ldots, f(\sigma_r))$, and by setting $\varphi_f(\sigma_1 \times \ldots \times \sigma_r) = \sum_y \mathrm{sign}_y(f(\sigma_1), \ldots, f(\sigma_r))$, one gets an equivariant *intersection cocycle* $\varphi_f \in Z^{d(r-1)}_{\mathfrak{S}_r}(K^r_\Delta; \mathcal{Z})$ such that $\mathfrak{o}_{K^r_\Delta} = [\pm\varphi_f]$. In particular, $[\varphi_f]$ is independent of f, and if f has no Tverberg points then $\varphi_f = 0$ and hence $\mathfrak{o}_{K^r_\Delta} = [\varphi_f] = 0$ (Sections 2.2 and 2.3).

(b) Conversely, if $\mathfrak{o}_{K^r_\Delta} = [\varphi_g] = 0$, for some map $g\colon K \to \mathbb{R}^d$ then φ_g is an equivariant coboundary and thus can be expressed as a sum of *elementary equivariant coboundaries.* Adding a single elementary coboundary to φ_g corresponds geometrically to modifying g by a generalized *r-fold van Kampen finger move.* Starting with an arbitrary initial map and applying a finite number of such moves, one concludes (Section 2.4):

LEMMA 5. *If $\mathfrak{o}_{K^r_\Delta} = 0$ then there is a PL map $f\colon K \to \mathbb{R}^d$ in general position with $\varphi_f = 0$.*

(c) If $f\colon K \to \mathbb{R}^d$ is a PL map in general position with $\varphi_f = 0$ as a cocycle, then for every cell $\sigma_1 \times \ldots \times \sigma_r$ of K^r_Δ of maximal dimension $rk = d(r-1)$, the sum of intersection signs is zero, i.e., we can partition the r-Tverberg points in $f(\sigma_1) \cap \ldots \cap f(\sigma_r)$ into pairs $\{y_+, y_-\}$ of opposite sign. By repeatedly applying the following r-fold generalization of the Whitney trick (proved in Section 3), we can remove these pairs successively until we get a map $K \to \mathbb{R}^d$ without r-Tverberg points. This is the key step in the proof of Theorem 3.

THEOREM 6 (r-FOLD WHITNEY TRICK). *Let $\sigma_1, \ldots, \sigma_r$ be (oriented) k-simplices, and let*

$$f : \sigma_1 \sqcup \cdots \sqcup \sigma_r \to \mathbb{R}^d$$

be a PL map in general position defined on their disjoint union, with k, d as in (4). Suppose that the intersection $f(\sigma_1) \cap f(\sigma_2) \cap \cdots \cap f(\sigma_r)$ contains a pair $\{y_+, y_-\}$ of r-intersection points of opposite intersection signs. Then there exist $r-1$ "local" PL isotopies[4] $H^2_t, \ldots, H^r_t : \mathbb{R}^d \to \mathbb{R}^d$, $0 \leq t \leq 1$, with

$$f(\sigma_1) \cap H^2_1(f(\sigma_2)) \cap \cdots \cap H^r_1(f(\sigma_r)) = (f(\sigma_1) \cap f(\sigma_2) \cap \cdots \cap f(\sigma_r)) \setminus \{y_+, y_-\}$$

Here, "local" means that we can make each isotopy H^2_t, …, H^r_t have support inside a small d-dimensional PL ball B that intersects each $f(\sigma_i)$ in a small neighborhood of a path λ_i in the interior of $f(\sigma_i)$ connecting y_+ and y_- (we get to choose the paths λ_i and the ball B).

[4]An *(ambient) PL isotopy* in $\mathbb{R}^d$ is a PL homeomorphism $H\colon \mathbb{R}^d \times [0,1] \to \mathbb{R}^d \times [0,1]$ that preserves *levels*, i.e., every (y, t) is mapped to some point $(H_t(y), t)$, and so we can think of H as a continuous family of PL homeomorphisms $H_t\colon \mathbb{R}^d \to \mathbb{R}^d$, $0 \leq t \leq 1$. Furthermore, we require that $H_0 = \mathrm{id}_{\mathbb{R}^d}$ is the identity. We say that the *support of H* is contained in some subset $U \subseteq \mathbb{R}^d$ if H fixes the complement of U, i.e., $H_t(y) = y$ for all $y \in \mathbb{R}^d \setminus U$ and $0 \leq t \leq 1$.

We remark that the condition on the dimension of the σ_i in Thm. 6 can easily be relaxed to $\sum_i(d - \dim(\sigma_i)) = d$ and $d - \dim(\sigma_i) \geq 3$ for all i. In particular, they do not need to be all of the same dimension k, but for simplicity of exposition, we restrict our proof to this specific case.

Motivation & Future Work Our general goal is to draw analogies between the well-studied theory of embeddings and intersection points of higher multiplicity.

A fundamental question is whether there exists a counterexample to the topological Tverberg conjecture. One initial motivation of studying the elimination of higher-multiplicity intersection points is Özaydin's result that for non-prime powers there there *does* exist an equivariant map $(\Delta^N)^r_\Delta \to_{\mathfrak{S}_r} S^{d(r-1)-1}$ and the formal similarity of this fact to the assumptions of the Haefliger–Weber theorem. We stress, however, that the available techniques rely in an essential way on the assumption of *large codimension*, $d-k \geq 3$, which we cannot assume for the classical topological Tverberg problem (it suffices to consider the d-skeleton[5] of Δ^N, but lower skeleta are not enough).

Generalizing the Haefliger–Weber Theorem. One very clear goal that seems realistic with the current techniques is to generalize the results obtained here further and to obtain a generalization of the Haefliger–Weber theorem to Tverberg points. Such a generalization should state that for a k-dimensional complex K the existence of a map $K \to \mathbb{R}^d$ without r-Tverberg points is equivalent to the existence of a $\mathfrak{S}_r$-equivariant map $K^r_\Delta \to S^{d(r-1)-1}$ inside a suitable *r-fold metastable range*, tentatively $rd \geq (r+1)k+3$, i.e., roughly $d \geq \frac{r+1}{r}k$. We plan to investigate this in a follow-up paper. Ideally and optimistically, this might also lead to algorithms for solving $\texttt{Tverberg}^r_{k\to d}$ in this range of dimensions. As a caveat, we should point out, however, that the computational results for equivariant maps [8] usually assume that the group acts *freely* on both spaces, and adapting this to the non-free action of $\mathfrak{S}_r$ on $S^{d(r-1)-1}$ may be a nontrivial task.

Incompleteness of $\mathfrak{o}_{K^r_\Delta}$ and hardness of $\texttt{Tverberg}^r_{k\to d}$? For embeddability, there are examples that show that the deleted product criterion for embeddability is incomplete outside the metastable range [15, 26, 9, 25, 10], and these were used in [18] to prove that $\texttt{Embed}_{k\to d}$ is intractable for $4 \leq d < 3(k+1)/2$. It is a natural question if this can be generalized to r-Tverberg points. The obvious first case we plan to study is $d = rm$, $k = (r-1)m$ for $m = 2$, $r \geq 3$.

Non-Tverberg Multiple Points. The methods presented here can be extended to eliminate r-intersection points that are not Tverberg points, i.e., with preimages in an r-tuple of simplices of K that are not pairwise disjoint. This will be discussed in the forthcoming full version of this paper.

2. DELETED PRODUCTS, INTERSECTION NUMBERS, AND OBSTRUCTIONS

We review the well-known *Configuration Space/Test Map* scheme (see [33, 34] and [16, Ch. 6]) in our context.

In this paper, we work with deleted products, which, in general, provide necessary conditions for the existence of maps into $\mathbb{R}^d$ without r-intersection points that are at least as strong as those provided by *deleted joins* [20, Sec. 3.3].

2.1 Deleted Products and Equivariant Maps

Definitions. Let K be a finite k-dimensional simplicial complex. For an $r \geq 2$, let $K^r = K \times \ldots \times K$ (r factors) denote the r-fold Cartesian product of K; it is a CW complex whose cells are products of simplices of K. We define the *r-fold deleted product* K^r_Δ as the subcomplex of K^r whose cells are all the products $\sigma_1 \times \ldots \times \sigma_r$ of r *pairwise disjoint* simplices of K. We also consider the Cartesian product $(\mathbb{R}^d)^r$ and the *thin diagonal* $\Delta^r_{\mathbb{R}^d} := \{(x,\ldots,x) \mid x \in \mathbb{R}^d\} \subset (\mathbb{R}^d)^r$.

The symmetric group $\mathfrak{S}_r$ acts[6] on the Cartesian products K^r and $(\mathbb{R}^d)^r$ by permuting components. By restriction, $\mathfrak{S}_r$ also acts on the subspaces K^r_Δ, $\Delta^r_{\mathbb{R}^d}$ and $(\mathbb{R}^d)^r \setminus \Delta^r_{\mathbb{R}^d}$, since these are invariant under the respective group actions. The action of $\mathfrak{S}_r$ on K^r_Δ is cellular and free, whereas for $r \geq 3$ the action on $(\mathbb{R}^d)^r \setminus \Delta^r_{\mathbb{R}^d}$ is not free. The orthogonal complement $(\Delta^r_{\mathbb{R}^d})^\perp \subset (\mathbb{R}^d)^r$ of the thin diagonal $\Delta^r_{\mathbb{R}^d}$ is also an $\mathfrak{S}_r$-invariant subspace, and the orthogonal projection $p\colon (\mathbb{R}^d)^r \twoheadrightarrow (\Delta^r_{\mathbb{R}^d})^\perp$ is an $\mathfrak{S}_r$-equivariant map. By restriction, we get an equivariant map

$$\rho\colon (\mathbb{R}^d)^r \setminus \Delta^r_{\mathbb{R}^d} \twoheadrightarrow_{\mathfrak{S}_r} (\Delta^r_{\mathbb{R}^d})^\perp \setminus \{0\}.$$

Furthermore, normalization to unit length gives an $\mathfrak{S}_r$-equivariant map $\nu\colon (\Delta^r_{\mathbb{R}^d})^\perp \setminus \{0\} \twoheadrightarrow_{\mathfrak{S}_r} S^{d(r-1)-1}$, where $S^{d(r-1)-1}$ is the unit sphere in $(\Delta^r_{\mathbb{R}^d})^\perp$.

PROOF OF LEMMA 2. A (PL) map $f\colon K \to \mathbb{R}^d$ induces an $\mathfrak{S}_r$-equivariant (PL) map $f^r\colon K^r \to (\mathbb{R}^d)^r$ by applying f componentwise, i.e., $f^r(x_1,\ldots,x_r) = (f(x_1),\ldots,f(x_r))$. By restriction, we can view this as a an equivariant map $f^r : K^r_\Delta \to_{\mathfrak{S}_r} (\mathbb{R}^d)^r$.

Moreover, an r-fold Tverberg point of f corresponds bijectively to an $\mathfrak{S}_r$-*orbit* of points $(x_1,\ldots,x_r) \in K^r_\Delta$ such that $f^r(x_1,\ldots,x_r) = (y,\ldots,y) \in \Delta^r_{\mathbb{R}^d}$. Thus, f has no r-fold Tverberg point iff the image $f^r(K^r_\Delta)$ is disjoint from the thin diagonal $\Delta^r_{\mathbb{R}^d}$, i.e., iff we can view (the restriction of) f^r as an equivariant map $f^r : K^r_\Delta \to_{\mathfrak{S}_r} (\mathbb{R}^d)^r \setminus \Delta^r_{\mathbb{R}^d}$.

In particular, if $f\colon K \to \mathbb{R}^d$ is a map without r-fold Tverberg point then the composition

$$\widetilde{f} := \nu \circ \rho \circ f^r : K^r_\Delta \to_{\mathfrak{S}_r} S^{d(r-1)-1} \tag{5}$$

is an equivariant map. $\square$

It is easy to see that the maps p, ρ, and ν are equivariant homotopy equivalences (in fact, deformation retractions). Thus, the existence of an equivariant map

$$K^r_\Delta \to_{\mathfrak{S}_r} (\mathbb{R}^d)^r \setminus \Delta^r_{\mathbb{R}^d} \tag{6}$$

[5]We recall that the i-skeleton $\mathrm{skel}_i(K)$ of a simplicial complex K consists of all simplices of K of dimension at most i.

[6]Recall that an *action* by a finite group G on a topological space is given by a family of homeomorphisms $\rho_g\colon X \to X$, $g \in G$, such that $\rho_g \circ \rho_h = \rho_{gh}$ and $\rho_e = \mathrm{id}_X$, where $e \in G$ is the identity element. If X is a CW complex then the action is called *cellular* if, for each $g \in G$, ρ_g is a cellular map and the set $X^g = \{x \in X : \rho_g(x) = x\}$ of points fixed by g is a subcomplex of X. An action is *free* if no $g \in G \setminus \{e\}$ has a fixed point. Often, ρ is suppressed from the notation, and one writes simply gx or $g \cdot x$ instead of $\rho_g(x)$. If G acts on two spaces X and Y then a map from X to Y is called *G-equivariant*, denoted $f\colon X \to_G Y$, if $f(g\cdot x) = g \cdot f(x)$ for all $x \in X, g \in G$.

not only implies but is actually equivalent to the existence of an equivariant map $K^r_\Delta \to_{\mathfrak{S}_r} S^{d(r-1)-1}$. We have chosen the latter for the formulation of Lemma 2 to highlight the similarity with the classical Gauss map, but in what follows, it will be more convenient to argue in the former setting of maps into $(\mathbb{R}^d)^r \setminus \Delta^r_{\mathbb{R}^d}$.

2.2 Intersection Numbers

Let $\sigma_1, \ldots, \sigma_r$ be an ordered list of oriented geometric simplices in $\mathbb{R}^d$. We assume that the simplices are pairwise vertex-disjoint and in general position with respect to one another and that the dimensions $k_i := \dim(\sigma_i)$, $1 \le i \le r$, satisfy $\sum_{i=1}^r (d - k_i) = d$.

We define the intersection sign of the simplices as follows (this can be seen as a very simple special case of Lefschetz intersection theory [14]).

If the common intersection $\bigcap_{i=1}^r \sigma_i$ of the simplices is empty, we set $\operatorname{sign}(\sigma_1, \ldots, \sigma_r) = 0$.

Otherwise, by general position, the intersection $\bigcap_{i=1}^r \sigma_i$ consists of a single point p that lies in the relative interior of each σ_i and at which the simplices intersect *transversely*, i.e., the dimension of the intersection $\sigma_{i_1} \cap \ldots \cap \sigma_{i_q}$ of any q of the simplices, $1 \le q \le r$, has dimension $d - (d - k_{i_1}) - \ldots - (d - k_{i_q})$ (see Figure 1 for an illustration in the case $r = 3 = d$, $k_1 = k_2 = k_3 = 2$).

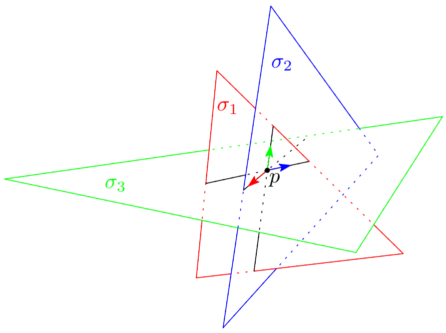

Figure 1: Three triangles intersecting at p.

In particular, for each i, the linear subspaces $L(\sigma_i)$ and $L(\bigcap_{j \ne i} \sigma_j)$ parallel to σ_i and $\bigcap_{j \ne i} \sigma_j$, respectively, are complementary linear subspaces of dimension k_i and $d - k_i$, respectively. Fixing an orientation of σ_i amounts to choosing an ordered basis $B_{\sigma_i} \in \mathbb{R}^{d \times k_i}$ of $L(\sigma_i)$ (i.e., we write the basis elements as the columns of a matrix). For each i, we choose a basis $B_{\overline{\sigma_i}} \in \mathbb{R}^{d \times (d-k_i)}$ of the complementary subspace $L(\bigcap_{j \ne i} \sigma_j)$ such that $[B_{\sigma_i} | B_{\overline{\sigma_i}}] \in \mathbb{R}^{d \times d}$ determines the standard orientation of $\mathbb{R}^d$, i.e., $\det[B_{\sigma_i} | B_{\overline{\sigma_i}}] > 0$. (In Figure 1, each $B_{\overline{\sigma_i}}$ consists of a single vector based at p.)

Then $[B_{\overline{\sigma_1}} | \ldots | B_{\overline{\sigma_r}}] \in \mathbb{R}^{d \times d}$ is another basis of $\mathbb{R}^d$ and we define the r-fold intersection sign of $\sigma_1, \ldots, \sigma_r$ by

$$\operatorname{sign}(\sigma_1, \ldots, \sigma_r) = \operatorname{sign}\left(\det[B_{\overline{\sigma_1}} | \ldots | B_{\overline{\sigma_r}}]\right).$$

It is easy to check that the intersection sign only depends on the orientations of the σ_i's and their order:

Lemma 7. *(a) If we reverse the orientation of any one simplex σ_i then the intersection sign is reversed.*

(b) If we transpose simplies σ_i and σ_j in the list, then the intersection sign changes by $(-1)^{(d-k_i)(d-k_j)}$.

2.3 Equivariant Obstruction Theory

Here, we review the definition of the *generalized r-fold van Kampen obstruction* $\mathfrak{o}_{K^r_\Delta}$, which yields an efficiently computable criterion for the existence of an equivariant map as in (6). This obstruction is a special case of the so-called *primary obstruction* in *equivariant obstruction theory*, and we refer to [34, Sec. 4.1] or [29, Sec. II.3]) for a more general and detailed treatment.

First, consider the $(kr-1)$-skeleton $\operatorname{skel}_{(kr-1)}(K^r_\Delta)$ and an arbitrary equivariant PL map

$$g\colon \operatorname{skel}_{(kr-1)}(K^r_\Delta) \to_{\mathfrak{S}_r} (\mathbb{R}^d)^r \setminus \Delta^r_{\mathbb{R}^d}. \tag{7}$$

It is easy to see that such a map exists: if $f\colon K \to \mathbb{R}^d$ is a PL map in general position then we can take g to be the restriction to $\operatorname{skel}_{(kr-1)}(K^r_\Delta)$ of the map f^r considered in Section 2.1, $f^r(x_1, \ldots, x_r) = (f(x_1), \ldots, f(x_r))$, since the image of the skeleton avoids the thin diagonal, by general position—note that, by the assumptions (4) on the dimensions, $kr = d(r-1)$.

When can we extend the map g to an equivariant map as in (6), defined on all of K^r_Δ? If $C = \sigma_1 \times \ldots \times \sigma_r$ is a kr-cell of K^r_Δ then we can extend g to the interior of C if and only if the restriction

$$g|_{\partial C} \colon \partial C \to (\mathbb{R}^d)^r \setminus \Delta^r_{\mathbb{R}^d}$$

of g to the boundary of C is nullhomotopic. Furthermore, any extension of g to the cell C yields a unique equivariant extension of g to all the whole $\mathfrak{S}_r$-orbit of C, i.e., to all cells of the form $\sigma_{\pi(1)} \times \ldots \times \sigma_{\pi(r)}$, $\pi \in \mathfrak{S}_r$ (here, we use that the action of $\mathfrak{S}_r$ on K^r_Δ is free).

Since $\partial C \cong S^{kr-1}$ and $(\mathbb{R}^d)^r \setminus \Delta^r_{\mathbb{R}^d} \simeq S^{d(r-1)-1} = S^{kr-1}$, we can use the classical result of Hopf that the set of homotopy classes of maps from the sphere S^{kr-1} to itself can be identified with the integers $\mathbb{Z}$ via the *degree* of a map; in particular, $g|_{\partial C}$ is nullhomotopic if it has degree zero.

More precisely, defining the degree $\deg(g|_{\partial C}) \in \mathbb{Z}$ involves several choices: First, we choose an orientation of the cell C, which determines an orientation of its boundary ∂C, i.e., a homology class $[\partial C]$ generating $H_{kr-1}(\partial C; \mathbb{Z}) \cong \mathbb{Z}$. Second, we choose a generator of $H_{kr-1}((\mathbb{R}^d)^r \setminus \Delta^r_{\mathbb{R}^d}; \mathbb{Z})$, which amounts to fixing an isomorphism of this homology group with $\mathbb{Z}$; geometrically, such a generator can be represented as the homology class $[\partial \tau_0]$ of the boundary of an oriented linear kr-simplex τ_0 in $(\mathbb{R}^d)^r$ that intersects $\Delta^r_{\mathbb{R}^d}$ precisely once in its interior. For concreteness, we will choose τ_0 so that it intersects $\Delta^r_{\mathbb{R}^d}$ positively.[7] Then the degree $\deg(g|_{\partial C}) \in \mathbb{Z}$ is defined by $g_*([\partial C]) = \deg(g|_{\partial C}) \cdot [\partial \tau]$, where g_* is the induced homomorphism in homology.

We can also compute this degree in terms of intersection numbers as follows:

Lemma 8. *Suppose that the map g is the restriction of an equivariant PL map*[8] *$h\colon K^r_\Delta \to_{\mathfrak{S}_r} (\mathbb{R}^d)^r$ in general position.*

Then for every oriented kr-cell $\sigma_1 \times \ldots \times \sigma_r$, the degree $\deg(g|_{\partial(\sigma_1 \times \ldots \times \sigma_r)})$ equals the sum of intersection signs of all intersection points of $h(\sigma_1 \times \ldots \times \sigma_r)$ with $\Delta^r_{\mathbb{R}^d}$.

[7] To be more precise, we assume that the diagonal $\Delta^r_{\mathbb{R}^r}$ has the orientation given by the basis $(e_1, \ldots, e_1)$, …, $(e_d, \ldots, e_d)$, where $e_1, \ldots, e_d$ is the standard basis of $\mathbb{R}^d$.

[8] Note that for any g, we can choose such an extension h since we do not require it to avoid the diagonal.

PROOF. Note that the boundaries of any two oriented linear kr-simplices that intersect the diagonal positively correspond to the same generator of $H_{kr-1}(\mathbb{R}^{d\times r} \setminus \Delta^r_{\mathbb{R}^d}, \mathbb{Z})$, and if we reverse the orientation of such a simplex τ, so that its intersection sign with $\Delta^r_{\mathbb{R}^d}$ becomes negative, then we also reverse the sign of $[\partial\tau]$ as a generator of the homology group. Furthermore, if τ is disjoint from $\Delta^r_{\mathbb{R}^d}$ then $[\partial\tau] = 0$ in the homology group. The lemma now follows by choosing a sufficiently fine triangulation of the cell $\sigma_1 \times \ldots \times \sigma_r$ on which h is simplexwise linear: Then $g_*([\partial(\sigma_1 \times \ldots \times \sigma_r)]) = \sum_\tau g_*([\partial\tau])$, where τ ranges over all kr-simplices in the triangulation, and $[\partial\tau]$ equals $+[\partial\tau_0]$, $-[\partial\tau_0]$, or zero depending on whether $h(\tau)$ intersects $\Delta^r_{\mathbb{R}^d}$ positively, negatively, or not at all. □

Thus, with each oriented kr-cell $C = \sigma_1 \times \ldots \times \sigma_r$ of K^r_Δ, we have associated an integer

$$\varphi_g(\sigma_1 \times \ldots \times \sigma_r) := \deg(g|_{\partial(\sigma_1\times\ldots\times\sigma_r)}).$$

This defines a kr-dimensional cellular cochain φ_g with integer coefficients on K^r_Δ, and by construction, g can be extended to an equivariant map $K^r_\Delta \to_{\mathfrak{S}_r} (\mathbb{R}^d)^r \setminus \Delta^r_{\mathbb{R}^d}$ if and only if $\varphi_g = 0$. Furthermore φ_g is trivially a cocycle, called the *obstruction cocycle*, since $\dim(K^r_\Delta) = kr$.

Moreover, since the map g is equivariant, the cocycle φ_g is also equivariant, in the following sense: The action of $\mathfrak{S}_r$ on $(\mathbb{R}^d)^r \setminus \Delta^r_{\mathbb{R}^d}$ induces an action of $\mathfrak{S}_r$ on $H_{kr-1}((\mathbb{R}^d)^r \setminus \Delta^r_{\mathbb{R}^d}; \mathbb{Z}) \cong \mathbb{Z}$; let us denote the integers with this $\mathfrak{S}_r$-action by $\mathcal{Z}$ (we will give a concrete formula below). A i-dimensional cochain φ on K^r_Δ, i.e., a labeling of the oriented i-cells of K^r_Δ by integers, is $\mathfrak{S}_r$-equivariant if it commutes with the actions of $\mathfrak{S}_r$ on K^r_Δ and on $\mathcal{Z}$, respectively, i.e., i.e., if $\varphi(\pi \cdot C) = \pi \cdot \varphi(C)$ for all i-cells C and all $\pi \in \mathfrak{S}_r$. The equivariant cochains form a subgroup $C^i_{\mathfrak{S}_r}(K^r_\Delta; \mathbb{Z})$ of the usual (nonequivariant) cochains. It is easy to check that the usual coboundary operator sends equivariant cochains to equivariant cochains, so we get subgroups $B^k_{\mathfrak{S}_r}(K^r_\Delta; \mathcal{Z})$ of equivariant coboundaries (coboundaries of equivariant $(k-1)$-cochains) and $Z^k_{\mathfrak{S}_r}(K^r_\Delta; \mathcal{Z})$ of equivariant cocycles (k-cocycles that are equivariant), and the equivariant cohomology groups are defined by $H^k_{\mathfrak{S}_r}(K^r_\Delta; \mathcal{Z}) = Z^k_{\mathfrak{S}_r}(K^r_\Delta; \mathcal{Z})/B^k_{\mathfrak{S}_r}(K^r_\Delta; \mathcal{Z})$.

The obstruction cocycle φ_g depends on the map g that we start out with (and it is zero if and only if g can be extended to an equivariant map $K^r_\Delta \to_{\mathfrak{S}_r} (\mathbb{R}^d)^r \setminus \Delta^r_{\mathbb{R}^d}$). However, it turns out that the equivariant cohomology class $[\varphi_g]$ is independent of g. This is the following special case of equivariant obstruction theory (see [34, Cor. 4.2], [29, Sec. II.3]).

PROPOSITION 9. *The equivariant cohomology class* $\mathfrak{o}_{K^r_\Delta} = [\varphi_g] \in H^{kr}_{\mathfrak{S}_r}(K^r_\Delta; \mathcal{Z})$ *is independent of the choice of the equivariant map* $g\colon \mathrm{skel}_{(kr-1)}(K^r_\Delta) \to_{\mathfrak{S}_r} (\mathbb{R}^d)^r \setminus \Delta^r_{\mathbb{R}^d}$ *used to define the representing obstruction cocycle.*

Moreover, there exists an equivariant map $K^r_\Delta \to_{\mathfrak{S}_r} (\mathbb{R}^d)^r \setminus \Delta^r_{\mathbb{R}^d}$ *if and only if the* $\mathfrak{o}_{K^r_\Delta} = 0$.

We call the cohomology class $\mathfrak{o}_{K^r_\Delta}$ the generalized r-fold van Kampen obstruction.

In the special case that $g = f^r|_{\mathrm{skel}_{(kr-1)}(K^r_\Delta)}$ for some PL map $f\colon K \to \mathbb{R}^d$ in general position, we will slightly abuse notation and denote the obstruction cocycle by φ_f instead of φ_{f^r}. Thus, for any oriented kr-cell $\sigma_1 \times \ldots \times \sigma_r$ of K^r_Δ,

$$\varphi_f(\sigma_1 \times \ldots \times \sigma_r) = \deg\left(f^r|_{\partial(\sigma_1\times\ldots\times\sigma_r)}\cdot\right), \qquad (8)$$

As mentioned above, in this case, we can further reformulate Lemma 8 in terms of r-fold intersection numbers.

LEMMA 10. *Suppose that* $f\colon K \to \mathbb{R}^d$ *is a PL map in general position. Consider an ordered list* $\sigma_1, \ldots, \sigma_r$ *of pairwise oriented* k*-simplices of* K*. The orientations of the* σ_i *yield an orientation of the cell* $\sigma_1 \times \ldots \times \sigma_r$ *of* K^r_Δ*, and* $\deg(f^r|_{\partial(\sigma_1\times\ldots\times\sigma_r)})$ *is equal to the sum of signs of* r*-fold intersection points of* $f(\sigma_1), \ldots, f(\sigma_r)$*, up to a sign* ε *that depends only on* k *and* r*. Specifically,* ε *is* -1 *if* k *is odd and* r *is* 2 *mod* 4*, and* 1 *otherwise.*

The proof of this lemma is an exercise in linear algebra, which we omit from this extended abstract.

Combining Lemmas 10 and 7, we get that transposing two oriented k-simplices σ_i and σ_j in a cell $\sigma_1 \times \cdots \times \sigma_r$ corresponds to changing the sign of $\varphi_f(\sigma_1 \times \cdots \times \sigma_r)$ by a factor $(-1)^{d-k}$. Thus, in concrete terms, $\mathcal{Z}$ denotes the group of integers with the $\mathfrak{S}_r$-action given by letting each transposition act by multiplication by $(-1)^{d-k}$.

2.4 Elementary coboundaries and r-fold van Kampen Finger Moves

For any dimension ℓ, we get a basis of the ℓ-dimensional equivariant cochains $C^\ell_{\mathfrak{S}_r}(K^r_\Delta; \mathcal{Z})$ as follows: Choose an ℓ-dimensional oriented cell $\eta_1 \times \cdots \times \eta_r$ of K^r_Δ (i.e., the product of pairwise disjoint simplices of K with $\sum_{i=1}^r \dim(\eta_i) = \ell$). We define the cochain $\mathbf{1}_{\mathfrak{S}_r\cdot(\eta_1\times\cdots\times\eta_r)}$ to take value 1 on $\eta_1 \times \cdots \times \eta_r$, extend equivariantly over the $\mathfrak{S}_r$-orbit of the cell, and set the value to 0 on all other cells.

In particular, the equivariant coboundaries $B^{\ell+1}_{\mathfrak{S}_r}(K^r_\Delta; \mathcal{Z})$ are generated by *elementary equivariant coboundaries* of the form $\delta\mathbf{1}_{\mathfrak{S}_r\cdot(\eta_1\times\cdots\times\eta_r)}$. In particular, if $f\colon K \to \mathbb{R}^d$ is a PL map in general position then $\mathfrak{o}_{K^r_\Delta} = [\varphi_f] = 0$ if and only if φ_f can be written as a sum of elementary coboundaries.

This observation easily leads to a proof that the obstruction $\mathfrak{o}_{K^r_\Delta}$ is computable (Lemma 4), and it also yields a proof of Lemma 5, by a repeated application of the following:

LEMMA 11. *If* $f\colon K \to \mathbb{R}^d$ *is a PL map in general position and if* $\delta\mathbf{1}_{\mathfrak{S}_r\cdot(\eta_1\times\cdots\times\cdots\eta_r)}$ *is an elementary equivariant* kr*-dimensional coboundary then there exists a map* $f'\colon K \to \mathbb{R}^d$ *such that* $\varphi_{f'} = \varphi_f - \delta\mathbf{1}_{\mathfrak{S}_r\cdot(\eta_1\times\cdots\times\cdots\eta_r)}$.

The proof is a generalization of the classical *van Kampen finger moves.* It is omitted here for reasons of space and will be presented in the forthcoming full version of the paper.

3. THE r-FOLD WHITNEY TRICK

In this section, we prove Theorem 6 and show how it implies Theorem 3. Throughout, we work under the assumptions (4). The proof uses a number of standard notions and techniques from PL topology, for which we refer to [24].

PROOF OF THEOREM 3 USING THEOREM 6. By Lemma 2 and Prop. 9, it is enough to prove the "if" direction of Theorem 3. That is, we assume that $\mathfrak{o}_{K^r_\Delta} = 0$ and we want to show that there exists a map $K \to \mathbb{R}^d$ without r-Tverberg point. By Lemma 5, there exists a map $f\colon K \to \mathbb{R}^d$ such that for any r pairwise disjoint k-simplices $\sigma_1, \ldots, \sigma_r$, the sum of intersection signs $\sum_y \mathrm{sign}_y(f(\sigma_1), \ldots, f(\sigma_r))$ over all r-intersection points y is zero. Thus, we can group these r-intersection points into pairs $\{y_+, y_-\}$ of opposite signs. Using Theorem 6, we eliminate these pairs one by one. To do this, we pick, for $1 \le i \le r$, a path λ_i in the relative interiors of $f(\sigma_i)$ that avoids all of the following "obstacles": all other images $f(\sigma_j)$, $j \neq i$ outside small neighborhoods of y_+ and y_- (in particular, λ_i avoids all other r-intersection points), all images $f(\tau)$ of simplices τ of K other than $\sigma_1, \ldots, \sigma_r$, and also all double points of $f(\sigma_i)$; we can do this since the codimension is sufficiently large (for this, $d-k \ge 2$ would be enough). Thus, by repeatedly applying Theorem 6, we can successively remove all pairs $\{y_+, y_-\}$ without introducing any new r-Tverberg points. □

The proof of Theorem 6 is an induction on r. The base case is $r = 2$ is the PL version of the Whitney trick due to Weber [35]. Thus, we work under the assumptions

$$r \ge 3 \quad \text{and} \quad \text{Theorem 6 holds for smaller values of } r. \qquad (9)$$

3.1 Reduction to a local situation

The first step of the proof is to reduce the problem to a "local situation", i.e., to restrict ourselves to the "small" ball mentioned in the statement of Theorem 6. That is we reduce Theorem 6 to

Proposition 12. *Assume $\sigma_1, \ldots, \sigma_r$ are k-simplices properly embedded in general position in a ball B^d (i.e., $\sigma_i \cap \partial B = \partial\sigma_i$). Furthermore, assume that $\sigma_1 \cap \sigma_2 \cap \cdots \cap \sigma_r = \{x, y\}$, where x and y have opposite intersection signs (this is independent of the choice of orientations of the σ_i). Also, suppose that for $i \neq j$ the intersection $\sigma_i \cap \sigma_j$ is the disjoint union of two "flat" $(2k-d) = (r-2)m$-balls, say B_1^{ij} and B_2^{ij}, each properly contained in B^d, such that $x \in B_1^{ij}$ and $y \in B_2^{ij}$, and that the triple of spheres $(\partial\sigma_i, \partial B_1^{ij}, \partial B_2^{ij})$ is unlinked.*[9]

Then there exist $r-1$ ambient isotopies $H_t^2, \ldots, H_t^r : B \to B$ of B, constant on ∂B and such that

$$\sigma_1 \cap H_1^2(\sigma_2) \cap \cdots \cap H_t^r(\sigma_r) = \emptyset.$$

Proof of Theorem 6 using Prop. 12. Let $\{x, y\}$ be the pair of r-intersection points of opposite signs that we want to remove. Using general position and the assumptions on the dimensions, we choose, for $1 \le i \le r$, a path λ_i in the relative interior of $f(\sigma_i)$ connecting x and y (Figure 2) and avoiding all the following obstacles: outside of small ε-neighborhoods of x and y, λ_i avoids (and hence has at least some fixed positive distance from) all images $f(\sigma_j)$, $j \neq i$, and λ_i completely avoids all images $f(\tau)$ of other simplices τ of K as well as the double image points of $f|_{\sigma_i}$.

Next, consider the embedded circle $\lambda_1 \cup \lambda_2$. Using general position and the fact that $d-k \ge 3$, we choose an embedded 2-dimensional PL *filling disk* D_{12} in $\mathbb{R}^d$, i.e., whose boundary is $\lambda_1 \cup \lambda_2$, such that D_{12} intersects $f(\sigma_1)$ and $f(\sigma_2)$ precisely in λ_1 and λ_2, respectively, and completely avoids the other images $f(\sigma_i)$, $i \neq 1, 2$ (except at x and y) as well as all images $f(\tau)$ of other simplices, see Figure 3).

Repeating the same construction on each successive circle $\lambda_i \cup \lambda_{i+1}$ up to $i = r-1$, we get a sequence of filling disks $D_{12}, D_{23}, \ldots, D_{(r-1)r}$, which we can choose to be pairwise internally disjoint and whose union then forms an embedded PL 2-disk D with boundary $\lambda_1 \cup \lambda_r$.

We take a sufficiently small *regular neighborhood* of D [24, Ch. 3]. This neighborhood is a PL d-ball B that can be assumed to sufficiently small, so that B intersects each $f(\sigma_i)$

[9]See [24, p. 69].

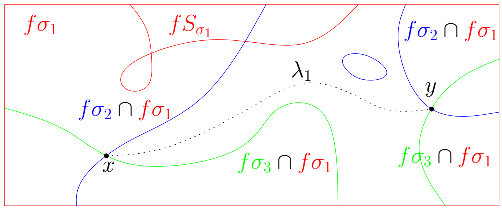

Figure 2: On $f\sigma_1$, the path λ_1 joins x and y.

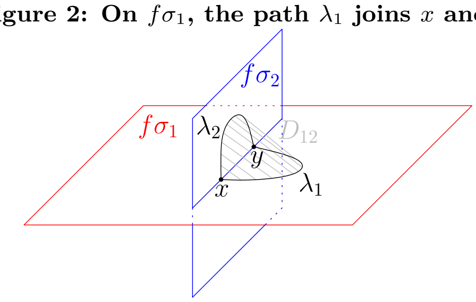

Figure 3: The disk D_{12} fills the circle $\lambda_1 \cup \lambda_2$.

in a regular ε-neighborhood τ_i of λ_i. This neighborhood τ_i is a k-ball properly contained in D, and if we choose ε sufficiently small, then any two such neighborhoods τ_i and τ_j intersect only in two $(2k-d)$-balls with the required properties. □

3.2 Piping and Connectivity

To prove Proposition 12 by induction, the general idea is to restrict ourselves to σ_1, to solve the situation inside this k-dimensional ball inductively, and then to extend the resulting isotopy to the whole B^d. For this strategy to work, one needs to check that for any given point $y \in \sigma_1 \cap \cdots \cap \sigma_r$, the sign of y as an r-intersection point of k-surfaces (PL k-manifolds with boundary) $\sigma_1, \ldots, \sigma_r$ in $\mathbb{R}^d$ is the same as the sign of y as an $(r-1)$-intersection point of the $(k-m)$-surfaces $(\sigma_1 \cap \sigma_2), \ldots, (\sigma_1 \cap \sigma_r)$ in σ_1. This is an exercise in linear algebra that, due to space constrains, we do not present here.

Note that here the intersections $\sigma_1 \cap \sigma_i$ are given suitable orientations compatible with orientations of σ_1 and σ_i and the orientation of the 'ambient space', i.e., in our case the standard orientation of $\mathbb{R}^d$, see [14] (our earlier definition of pairwise intersection signs is a special case of this).

Thus, x and y also have opposite signs as $(r-1)$-intersection points of the $(\sigma_1 \cap \sigma_i)$ inside σ_i. We restrict our-

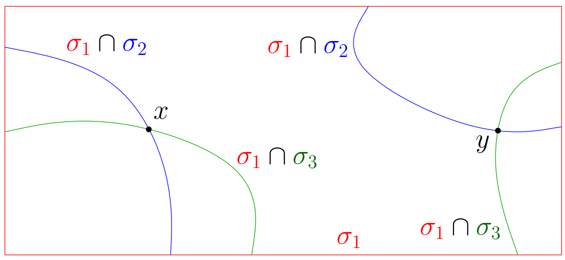

Figure 4: On σ_1, the two intersection points x and y are not "connected".

selves to σ_1 and look at the intersection pattern of the $\sigma_i \cap \sigma_1$

inside $\sigma_1 \cong B^k \subset B^d$ (see Figure 4). Since $\sigma_1 \cap \sigma_i$, $2 \leq i \leq r$ is not connected but the disjoint union two $(2k-d)$-balls, we cannot immediately apply induction to eliminate x and y by suitable isotopies defined inside σ_1. First, we need to 'force connectivity'. That is, for each $\sigma_1 \cap \sigma_i \cong B_1 \sqcup B_2$, we pick $b_1 \in B_1$ and $b_2 \in B_2$, and we connect b_1 and b_2 on σ_1 by a path λ_i that avoids the other $\sigma_1 \cap \sigma_j$ for $j \neq i$ (by general position). We run a pipe from σ_i to itself along λ_i [24, p. 67]. Such a piping can be made compatible with the orientation of σ_i. We call the resulting "piped" k-surface σ_i^* (see Figure 6). The intersection of σ_i^* and σ_1 is a piping of B_1 and B_2 along the path λ_i (see Figure 7). Since orientations are preserved (see the reference), x and y have opposite signs as intersections points of $\sigma_1 \cap \sigma_2^*, \ldots, \sigma_1 \cap \sigma_r^*$.

Eventually, these pipes that we have just added will have to be removed again, which is the purpose of the next lemma.

LEMMA 13 (UNPIPING LEMMA). *The space $B^d \setminus (\sigma_1 \setminus \sigma_i^*)$ is homotopically equivalent to a sphere S^{k-1}. In particular, it is $(k-2)$-connected. Thus, by Hurewicz's Theorem, the* homologically trivial *$(k-1)$-sphere $\partial\sigma_i = \partial\sigma_i^*$ is also homotopically trivial in $B^d \setminus (\sigma_1 \setminus \sigma_i^*)$.*

PROOF. *Step I: Reduction to a standard situation.* By codimension, we can assume that (B^d, σ_1) is unlinked [24, p. 91], i.e., B^d is the join of the k-ball σ_1 and a $(d-k-1)$-sphere,

$$B^d = \sigma_1 * S^{d-k-1}.$$

Let us write $\partial\sigma_1 \cap \sigma_i^*$ as the disjoint union of two $(2k-d-1)$-spheres

$$\partial\sigma_1 \cap \sigma_i = \partial\sigma_1 \cap \sigma_i^* := S_1^{2k-d-1} \sqcup S_2^{2k-d-1}.$$

The link of spheres $(\partial\sigma_1, S_1^{2k-d-1}, S_2^{2k-d-1})$ is homologically trivial, and hence homeomorphic to any other homologically trivial link [24, p. 70] (also by codimension ≥ 3, the pairs $(\partial\sigma_1, S_i^{2k-d-1})$ are unknotted [24, p. 69]). Therefore, we can envisage S_1^{2k-d-1} and S_2^{2k-d-1} as opposite to each other on $\partial\sigma_1$, after an homeomorphism of $\partial\sigma_1$. Such an homeomorphism extend to σ_1 [24, p. 8], and to $B^d = \sigma_1 * S^{d-k-1}$.

Furthermore, once S_1^{2k-d-1} and S_2^{2k-d-1} are opposite to each other, we can build an homotopy from $\sigma_1 \cap \sigma_i^* \cong S^{2k-d-1} \times I$ to an 'obvious' embedding of $S^{2k-d-1} \times I$ with the same boundary (that is, simply a straight tube $S^{2k-d-1} \times I$ in σ_1), here $I = [-1, 1]$.

By general theory [38, Ch X, p 198, Thm 10.1], there exists an homeomorphism of σ_1 that throws $\sigma_1 \cap \sigma_i^*$ to this 'straight' embedding of $S^{2k-d-1} \times I$, and this homeomorphism extend to B^d.

Step II: Sequence of retractions. See Figure 5. Recall that $B^d = \sigma_1 * S^{d-k-1}$, and σ_1 contains a 'straight line' embedding of $S^{2k-d-1} \times I$, we want to elucidate the homotopy type of

$$(\sigma_1 * S^{d-k-1}) \setminus (\sigma_1 \setminus (S^{2k-d-1} \times I)). \qquad (10)$$

First, we retract along the I direction, that is we retract

$$\sigma_1 \setminus (S^{2k-d-1} \times I) \quad \text{to} \quad B^{k-1} \setminus S^{2k-d-1}.$$

which leads to a retraction of (10) to

$$(B^{k-1} * S^{d-k-1}) \setminus (B^{k-1} \setminus S^{2k-d-1}). \qquad (11)$$

We retract B^{k-1} to $S^{2k-d-1} * B^0$, hence we can retract (11) to

$$((S^{2k-d-1} * B^0) * S^{d-k-1}) \setminus ((S^{2k-d-1} * B^0) \setminus S^{2k-d-1}).$$

Finally, we retract in the last expression from B^0 to the boundary $S^{2k-d-1} * S^{d-k-1} \cong S^{k-1}$. □

3.3 Induction

PROOF OF PROPOSITION 12. Here, we carry out the inductive strategy. First, we pipe all the k-simplices $\sigma_2, \ldots, \sigma_r$ to form $\sigma_2^*, \ldots, \sigma_r^*$, as described in Section 3.2. The piped k-surfaces intersect on σ_1 in two points of opposite signs. We would like to apply the induction hypothesis to the $(k-m)$-surfaces $\sigma_i^* \cap \sigma_1$ inside σ_1. A minor technical issue is that these surfaces are not PL balls but PL cylinders, $\sigma_i^* \cap \sigma_1 \cong S^{k-m-1} \times [0, 1]$. To get around this issue, we use the same localization trick as in Section 3.1, using just that the surfaces $\sigma_i^* \cap \sigma_1$ are path connected, so we can restrict to a small ball $B'^d \subset B^d$ that contains the two points x and y and that intersects each $\sigma_i^* \cap \sigma_1$ in a ball of dimension $k-m$, and then we can find suitable isotopies of $\sigma_1 \cap B'^d$ that are the identity outside of B'^d (which we can also interpret as isotopies of σ_1). Thus, inductively, there exist isotopies $H_t^3, \ldots, H_t^r$ of σ_1 (one for each $\sigma_3^*, \ldots, \sigma_r^*$), such that

$$(\sigma_1 \cap \sigma_2^*) \cap H_1^3(\sigma_1 \cap \sigma_3^*) \cap \cdots \cap H_1^r(\sigma_1 \cap \sigma_r^*) = \emptyset.$$

By the Hudson Isotopy Extension Theorem [12, Thm 1], we can extend these isotopies to B^d to get isotopies $H_t^i : B^d \to B^d$ fixed on the boundary, $3 \leq i \leq r$.

Using the Unpiping Lemma 13, we can find, for each $\sigma_2^*, \ldots, \sigma_r^*$, a (non-PL, non-embedded) ball $\tau_i \in B^d \setminus (\sigma_1 \setminus \sigma_i^*)$ with $\partial\tau_i = \partial\sigma_i$, and $\sigma_1 \cap \tau_2 \cap H_1^3(\tau_2) \cap \cdots \cap H_1^r(\tau_r) = \emptyset$.

Finally, because $d-k \geq 3$ and hence $2k-d+1 \leq k-2$, we can apply Irwin's Theorem [38, Ch. VIII, p. 4, Thm. 23], hence we can assume that the τ_i are embedded. To apply Irwin's Theorem, we need to 'stretch' $B^d \setminus (\sigma_1 \setminus \sigma_i^*)$ to turn it into a manifold. Using the same trick as in Lemma 13 we reduce the situation to a 'standard' unknotted one. It is straightforward to check that the homotopy type of the space remains unchanged.

Then by [38, Ch IV, Cor 1, p. 16] we can find ambient isotopies of B^d fixed on its boundary and throwing σ_i to τ_i fo $i = 2, \ldots, r$. □

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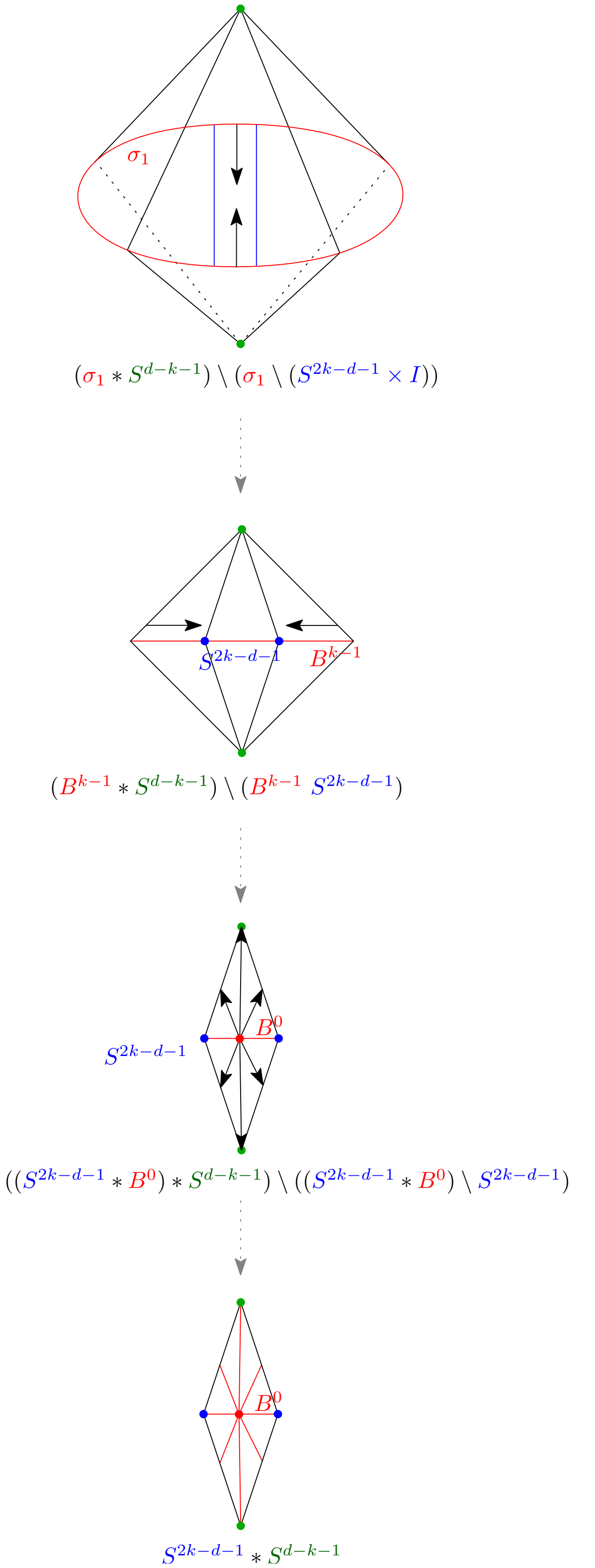

Figure 5: The retraction sequence in the proof of Lemma 13.

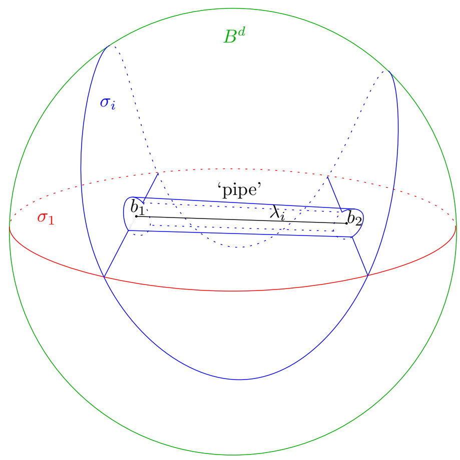

Figure 6: σ_i is piped along $\lambda_i \in \sigma_1$, forming σ_i^*.

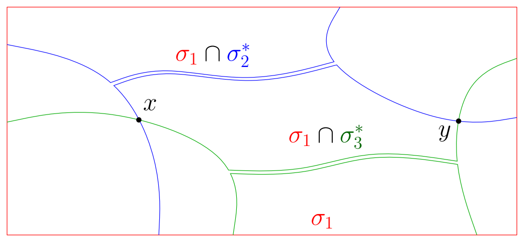

Figure 7: The "piped" surfaces σ_2^* and σ_3^* intersected with σ_1.

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