Vertical Visibility among Parallel Polygons in Three Dimensions*

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Abstract. Let $\mathcal{C} = \{C_1, \ldots, C_n\}$ denote a collection of translates of a regular convex k-gon in the plane with the stacking order. The collection $\mathcal{C}$ forms a *visibility clique* if for every $i < j$ the intersection C_i and C_j is not covered by the elements that are stacked between them, i.e., $(C_i \cap C_j) \setminus \bigcup_{i<l<j} C_l \neq \emptyset$.
We show that if $\mathcal{C}$ forms a visibility clique its size is bounded from above by $O(k^4)$ thereby improving the upper bound of 2^{2^k} from the aforementioned paper. We also obtain an upper bound of $2^{2\binom{k}{2}+2}$ on the size of a visibility clique for homothetes of a convex (not necessarily regular) k-gon.

1 Introduction

In a visibility representation of a graph $G = (V, E)$ we identify the vertices of V with sets in the Euclidean space, and the edge set E is defined according to some visibility rule. Investigation of visibility graphs, driven mainly by applications to VLSI wire routing and computer graphics, goes back to the 1980s [12,14]. This also includes a significant interest in three-dimensional visualizations of graphs [3,4,8,10].

Babilon et al. [1] studied the following three-dimensional visibility representations of complete graphs. The vertices are represented by translates of a regular convex polygon lying in distinct planes parallel to the xy-plane and two translates are joined by an edge if they can *see* each other, which happens if it is possible to connect them by a line segment orthogonal to the xy-plane avoiding all the other translates. They showed that the maximal size $f(k)$ of a clique represented by regular k-gons satisfies $\left\lfloor \frac{k+1}{2} \right\rfloor + 2 \leq f(k) \leq 2^{2^k}$ and that $f(3) \geq 14$. Hence, $\lim_{k\to\infty} f(k) = \infty$. Fekete et al. [8] proved that $f(4) = 7$ thereby showing that $f(k)$ is not monotone in k. Nevertheless, it is plausible that $f(k+2) \geq f(k)$ for every k, and surprisingly enough this is stated as an open problem in [1]. Another interesting open problem from the same paper is to decide if the limit $\lim_{k\to\infty} \frac{f(k)}{k}$ exists. In the present note we improve the above upper bound on $f(k)$ to $O(k^4)$ [3] and we extend our investigation to families of homothetes of

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general convex polygons. The main tool to obtain the result is Dilworth Theorem [6], which was also used by Babilon et al. to obtain the doubly exponential bound in [1]. Roughly speaking, our improvement is achieved by applying Dilworth Theorem only once whereas Babilon et al. used its k successive applications.

Fekete et al. [8] observed that a clique of arbitrary size can be represented by translates of a disc. Their construction can be adapted to translates of any convex set whose boundary is partially smooth, or to translates of possibly rotated copies of a convex polygon. The same is true for non-convex shapes, see Fig. 1.

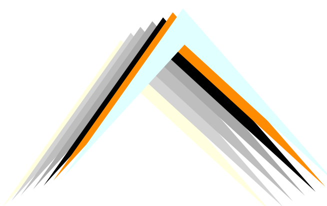


Fig. 1. A visibility clique formed by translates of a non-convex 4-gon.

An analogous question was extensively studied for arbitrary, i.e. not necessarily translates or homothetes of, axis parallel rectangles [3,8], see also [11]. Bose et al. [3] showed that in this case a clique on 22 vertices can be represented. On the other hand, they showed that a clique of size 57 cannot be represented by rectangles.

For convenience, we restate the problem of Babilon et al. as follows. Let $\mathcal{C} = \{C_1, \ldots, C_n\}$ denote a collection of sets in the plane with the *stacking order* given by the indices of the elements in the collection. By a standard perturbation argument, we assume that the boundaries of no three sets in $\mathcal{C}$ pass through a common point. The collection $\mathcal{C}$ forms a *visibility clique* if for every i and j, $i < j$, the intersection C_i and C_j is not covered by the elements that are stacked between them, i.e., $(C_i \cap C_j) \setminus \bigcup_{i<k<j} C_k \neq \emptyset$. Note that reversing the stacking order of $\mathcal{C}$ does not change the property of $\mathcal{C}$ forming a visibility clique. We are interested in the maximum size of $\mathcal{C}$, if $\mathcal{C}$ is a collection of translates and homothetes, resp., of a convex k-gon. We prove the following.

Theorem 1. *If $\mathcal{C}$ is a collection of translates of a regular convex k-gon forming a visibility clique, the size of $\mathcal{C}$ is bounded from above by $O(k^4)$.*

Theorem 2. *If $\mathcal{C}$ is a collection of homothetes of a convex k-gon forming a visibility clique, the size of $\mathcal{C}$ is bounded from above by $2^{2\binom{k}{2}+2}$.*

The paper is organized as follows. In Section 2 we give a proof of Theorem 1. In Section 3 we give a proof of Theorem 2. We conclude with open problems in Section 4.

2 Proof of Theorem 1

We let $\mathcal{C} = \{C_1, \ldots, C_n\}$ denote a collection of translates of a regular convex k-gon C in the plane with the stacking order given by the indices of the elements in the collection.

Let $\mathbf{c_i}$ denote the center of gravity of C_i. We assume that $\mathcal{C}$ forms a visibility clique. We label the vertices of C by natural numbers starting in the clockwise fashion from the topmost vertex, which gets label 1. We label in the same way the vertices in the copies of C. The proof is carried out by successively selecting a large and in some sense regular subset of $\mathcal{C}$. Let W_i be the convex wedge with the apex $\mathbf{c_1}$ bounded by the rays orthogonal to the sides of C_1 incident to the vertex with label i. The set $\mathcal{C}$ is *homogenous* if for every $1 \leq i \leq k$ all the vertices of C_j's with label i are contained in W_i. We remark that already in the proof of the following lemma our proof falls apart if C can be arbitrary or only centrally symmetric convex k-gon.

Lemma 1. *If C is a regular k-gon then $\mathcal{C}$ contains a homogenous subset of size at least $\Omega\left(\frac{n}{k^2}\right)$.*

Let $(C_{i_1}, \ldots, C_{i_n})$ be the order in which the ray bounding W_i orthogonal to the segment $i[(i-1) \bmod k]$ of C_1 intersects the boundaries of C_j's. The set $\mathcal{C}$ forms an *i-staircase* if the order $(C_{i_1}, \ldots, C_{i_n})$ is the stacking order. As a direct consequence of Dilworth Theorem or Erdős–Szekeres Lemma [6,7] we obtain that if $\mathcal{C}$ is homogenous, it contains a subset of size at least $\sqrt{|\mathcal{C}|}$ forming an i-staircase.

A graph $G = (\{1, \ldots, n\}, E)$ is a *permutation graph* if there exists a permutation π such that $ij \in E$, where $i < j$, iff $\pi(i) > \pi(j)$. Let $G_i = (\mathcal{C}', E)$ denote a graph such that $\mathcal{C}'$ is a homogenous subset of $\mathcal{C}$, and two vertices C'_j and C'_k of G_i are joined by an edge if and only if the orders in which the rays bounding W_i intersect the boundaries of C'_j and C'_k are reverse of each other. In other words, the boundaries of C'_j and C'_k intersect inside W_i, see Fig. 2(a). Thus, G_i's form a family of permutation graphs sharing the vertex set. Note that every pair of boundaries of elements in $\mathcal{C}'$ cross exactly twice.

Since for an even k a regular k-gon is centrally symmetric the graphs G_i and $G_{i+k/2 \bmod k}$ are identical. For an odd k, we only have $G_i \subseteq G_{i+\lceil k/2 \rceil \bmod k} \cup G_{i+\lfloor k/2 \rfloor \bmod k}$. The notion of the i-staircase and homogenous set is motivated by the following simple observation illustrated by Fig. 2(b).

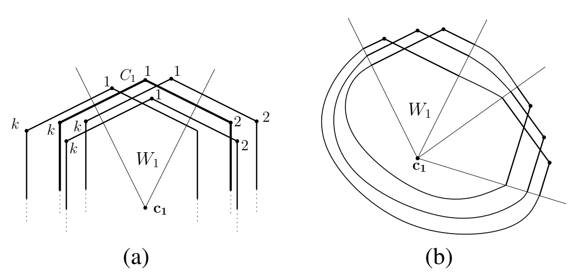


Fig. 2. (a) The wedge W_1 containing all the copies of vertex 1. (b) The 1-staircase giving rise to a clique of size three in G_1 and G_j for some j that cannot appear in a visibility clique.

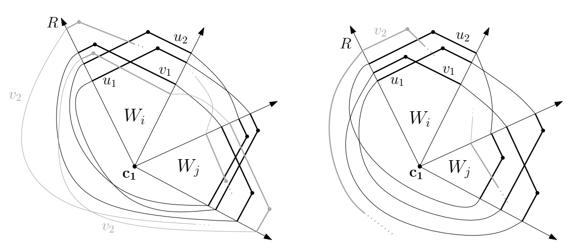
Observation 1 *If $\mathcal{C}'$ forms an i-staircase then there do not exist two indices i and j, $i \neq j$, such that both G_i and G_j contain the same clique of size three.*

The following lemma lies at the heart of the proof of Theorem 1.

Lemma 2. *Suppose that $\mathcal{C}'$ forms an i-staircase, and that there exists a pair of identical induced subgraphs $G_i' \subseteq G_i$ and $G_j' \subseteq G_j$, where $i \neq j$, containing a matching of size two. Then $\mathcal{C}'$ does not form a visibility clique.*

Proof. The lemma can be proved by a simple case analysis as follows. There are basically two cases to consider depending on the stacking order of the elements of $\mathcal{C}'$ supporting the matching M of size two in G_i'. Let u_1, v_1 and u_2, v_2, respectively, denote the vertices (or elements of $\mathcal{C}'$) of the first and the second edge in M, such that u_1 is the first one in the stacking order. By symmetry and without loss of generality we assume that the ray R bounding W_i orthogonal to the segment $i[(i-1) \mod k]$ of C_1 intersects the boundary of u_1 before intersecting the boundaries of u_2, v_1 and v_2, and the boundary of u_2 before v_2.

First, we assume that R intersects the boundary of u_2 before the boundary of v_1. In the light of Observation 1, u_1, v_1 and u_2 look combinatorially like in the Fig. 3(a). Then all the possibilities for the position of v_2 cause that the first and last element in the stacking order do not see each other. Otherwise, R intersects the boundary of v_1 before the boundary of u_2. In the light of Observation 1, u_1, v_1 and u_2 look combinatorially like in the Fig. 3(b), but then v_2 cannot see u_1. ■



(a) (b)

Fig. 3. The case analysis of possible combinatorial configurations of the boundaries of u_1, v_1, u_2 and v_2, after the first three boundaries were fixed. (a) If R intersects the boundary of u_2 before v_1 the first and the last element in the stacking order cannot see each other. (b) If R intersects the boundary of v_1 before u_2 then u_1 cannot see v_2.

Finally, we are in a position to prove Theorem 1. We consider two cases depending on whether k is even or odd. First, we treat the case when k is even which is easier.

Thus, let C be a regular convex k-gon for an even k. By Lemma 1 and Dilworth Theorem we obtain a homogenous subset $\mathcal{C}'$ of $\mathcal{C}$ of size at least $\Omega(\sqrt{\frac{n}{k^2}})$ forming a 1-staircase. Note that for $\mathcal{C}'$ the hypothesis of Lemma 2 is satisfied with $i = 1$ and $j = 1 + k/2$. Since $\mathcal{C}'$ forms a visibility clique, the graph G_1 does not contain a matching of size two. Hence, $G_1 = (\mathcal{C}' = \mathcal{C}_1, E)$ contains a dominating set of vertices $\mathcal{C}_1'$ of size at most two. Let $\mathcal{C}_2 = \mathcal{C}_1 \setminus \mathcal{C}_1'$. Note that $\mathcal{C}_2$ forms a 2-staircase and that the hypothesis of Lemma 2 is satisfied with $\mathcal{C}' = \mathcal{C}_2, i = 2$ and $j = 2 + k/2 \mod k$.

Thus, $G_2 = (\mathcal{C}_2, E)$ contains a dominating set of vertices $\mathcal{C}_2'$ of size at most two. Hence, $\mathcal{C}_3 = \mathcal{C}_2 \setminus \mathcal{C}_2'$ forms a 3-staircase. In general, $\mathcal{C}_i = \mathcal{C}_{i-1} \setminus \mathcal{C}_{i-1}'$ forms an i-staircase and the hypothesis of Lemma 2 is satisfied with $\mathcal{C}' = \mathcal{C}_i, i = i$ and $j = i + k/2 \mod k$. Note that $|\mathcal{C}_{k/2+1}| \leq 1$. Thus, $|\mathcal{C}'| \leq k + 1$. Consequently, $n = O(k^4)$.

In the case when k is odd we proceed analogously as in the case when k was even except that for $\mathcal{C}'$ as defined above the hypothesis of Lemma 2 might not be satisfied, since we cannot guarantee that G_i and G_j are identical for some $i \neq j$. Nevertheless, since the two tangents between a pair of intersecting translates of a convex k-gon in the plane are parallel we still have $G_i \subseteq G_{i+\lceil \frac{k}{2} \rceil \mod k} \cup G_{i+\lfloor \frac{k}{2} \rfloor \mod k}$ The previous property will help us to find a pair of identical induced subgraphs in G_i, and $G_{i+\lceil \frac{k}{2} \rceil \mod k}$ or $G_{i+\lfloor \frac{k}{2} \rfloor \mod k}$ to which Lemma 2 can be applied, if G_i contains a matching M of size c, where c is a sufficiently big constant determined later. It will follow that G_i does not contain a matching of size c, and thus, the inductive argument as in the case when k was even applies. (Details will appear in the full version.)

3 Homothetes

The aim of this section is to prove Theorem 2. Let C denote a convex polygon in the plane. Let $\mathcal{C} = \{C_1, C_2, \ldots, C_n\}$ denote a finite set of homothetes of C with the stacking order. Unlike as in previous sections, this time we assume that the indices correspond to the order of the centers of gravity of C_i's from left to right. Let $\mathbf{c_i}$ denote the center of gravity of C_i. Let $x(\mathbf{p})$ and $y(\mathbf{p})$, resp., denote x and y-coordinate of $\mathbf{p}$. Thus, we assume that $x(\mathbf{c_1}) < x(\mathbf{c_2}) < \ldots < x(\mathbf{c_n})$

Suppose that $\mathcal{C}$ forms a visibility clique. Similarly as in the previous sections we label the vertices of C by natural numbers starting in the clockwise fashion from the topmost vertex, which gets label 1. We label in the same way the vertices in the copies of C. Consider the poset $(\mathcal{C}, \subset)$ and note that it contains no chain of size five. By Dilworth theorem it contains an anti-chain of size at least $\frac{1}{4}|\mathcal{C}|$. Since we are interested only in the order of magnitude of the size of the biggest visibility clique, from now on we assume that no pair of elements in $\mathcal{C}$ is contained one in another.

Every pair of elements in $\mathcal{C}$ has exactly two common tangents, since every pair intersect and no two elements are contained one in another. We color the edges of the clique $G = (\mathcal{C}, \binom{\mathcal{C}}{2})$ as follows. Each edge C_iC_j, $i < j$, is colored by an ordered pair, in which the first component is an unordered pair of vertices of G supporting the common tangents of C_i and C_j, and the second pair is an indicator equal to one if C_i is below C_j in the stacking order, and zero otherwise.

Lemma 3. *The visibility clique G does not contain a monochromatic path of length two of the form $C_iC_jC_k$, $i < j < k$.*

We say that a path $P = C_1C_2 \ldots C_k$ in G is monotone if $x(\mathbf{c_1}) < x(\mathbf{c_2}) < \ldots < x(\mathbf{c_k})$. It was recently shown [9, Theorem 2.1] that if we color the edges of an ordered complete graph on $2^c + 1$ vertices with c colors we obtain a monochromatic monotone path of length two. We remark that this result is tight and generalizes Erdős–Szekeres Lemma [7]. Thus, if G contains more than $2^{2\binom{k}{2}+2}$ vertices it contains a monochromatic path of length two which is a contradiction by Lemma 3.

4 Open problems

Since we could not improve the lower bound from [1] even in the case of homothetes, we conjecture that the polynomial upper bound in k on the size of the visibility clique holds also for any family of homothetes of an arbitrary convex k-gon. To prove Theorem 2 we used a Ramsey-type theorem [9, Theorem 2.1] for ordered graphs. We wonder if the recent developments in the Ramsey theory for ordered graphs [2,5] could shed more light on our problem.

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