

Stability of the $2 + 2$ Fermionic System with Point Interactions

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Received: 26 February 2018 / Accepted: 3 July 2018 / Published online: 23 July 2018
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Abstract We give a lower bound on the ground state energy of a system of two fermions of one species interacting with two fermions of another species via point interactions. We show that there is a critical mass ratio $m_2 \approx 0.58$ such that the system is stable, i.e., the energy is bounded from below, for $m \in [m_2, m_2^{-1}]$. So far it was not known whether this $2 + 2$ system exhibits a stable region at all or whether the formation of four-body bound states causes an unbounded spectrum for all mass ratios, similar to the Thomas effect. Our result gives further evidence for the stability of the more general $N + M$ system.

Keywords Point interactions · Stability · Scattering length · Thomas effect

Mathematics Subject Classification (2010) 81Q10 · 46N50

1 Introduction

Systems of particles interacting via point interactions are frequently used in physics to model short range forces. In these models the shape of the interaction potential

Financial support by the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement No 694227), and by the Austrian Science Fund (FWF), project Nr. P 27533-N27, is gratefully acknowledged.

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enters only via the scattering length. Originally point interactions were introduced in the 1930s to model nuclear interactions [4, 5, 12, 23, 24], and later they were also successfully applied to other areas of physics like polarons (see [15] and references there) or cold atomic gases [25].

Given $N \geq 1$ fermions of one type with mass $1/2$ and $M \geq 1$ fermions of another type with mass $m/2 > 0$, point interaction models give a meaning to the formal expression

$$-\sum_{i=1}^N \Delta_{x_i} - \frac{1}{m} \sum_{j=1}^M \Delta_{y_j} + \gamma \sum_{i=1}^N \sum_{j=1}^M \delta(x_i - y_j) \quad (1.1)$$

for $\gamma \in \mathbb{R}$. Because of the existence of discontinuous functions in $H^1(\mathbb{R}^n)$ for $n \geq 2$, this expression is ill-defined in dimensions larger than one. In the following we restrict our attention to the three-dimensional case but we note that the system also exhibits interesting behavior in two dimensions [9, 10, 14].

A mathematically precise version of (1.1) in three dimensions was constructed in [9, 13] and we will work here with the model introduced there. We note that even though these models are mathematically well-defined it is not established whether they can be obtained as a limit of genuine Schrödinger operators with interaction potentials of shrinking support. (See, however, [1] for the case $N = M = 1$, and [2] for models in one dimension.)

It was already known to Thomas [23] that systems with point interactions are inherently unstable for bosons, in the sense that the energy is not bounded from below, if there are at least three particles involved. It turns out that in the case that the particles are fermions the question of stability is more delicate as it depends on the mass ratio of the two species, in general.

The case $N = M = 1$ is completely understood as it reduces to a one particle problem [1]. In this case there exists a one-parameter family of Hamiltonians describing point interactions parameterized by the inverse scattering length, and they are bounded from below for all masses.

Beside this trivial case also the $2 + 1$ case (i.e., $N = 2$ and $M = 1$), where the two particles of the same species are fermions, is well understood [3, 6–9, 17–20, 22]. There is a critical mass ratio $m^* \approx 0.0735$ such that the system is unstable for $m < m^*$ and stable otherwise. It is remarkable that this critical mass ratio does not depend on the strength of the interaction, i.e., the scattering length. Recently in [3] the spectrum of the $2 + 1$ system was discussed in more detail. Moreover, it was shown in [7, 19] that in a certain mass range other models describing point interactions can be constructed.

For larger systems of fermions even the question of stability is generally open. In [21] the stability result for the $2 + 1$ case was recently extended to the general $N + 1$ problem ($N \geq 2$ and $M = 1$). In particular it was shown that there exists a critical mass $m_1 \approx 0.36$ such that the system's energy is bounded from below, uniformly in N , for $m \geq m_1$. As a consequence of the $2 + 1$ case this $N + 1$ system is unstable for $m < m^*$, but the behavior for $m \in [m^*, m_1)$ is unknown.

By separating particles one can obtain an upper bound on the ground state energy of the general $N + M$ problem using the bounds for the $N + 1$ or the $1 + M$ problem.

We note that the latter is, up to an overall factor, equivalent to the $M + 1$ problem with m replaced by its inverse. Hence the fact that $m_1 < 1$ gives hope that there exists a mass region where the general $N + M$ system is stable for all N and M . The simplest problem of this kind is the $2 + 2$ case. So far there are only numerical results on its stability available [11, 16]. In particular, the analysis in [11] suggests that the critical mass for the $2 + 2$ case should be equal to m^* , i.e., the one for the $2 + 1$ case.

In this paper we give a rigorous proof of stability for the $2 + 2$ system in a certain window of mass ratios. We find a critical mass $m_2 \approx 0.58$ such that the system is stable if $m \in [m_2, m_2^{-1}] \approx [0.58, 1.73]$. We note that the critical mass m_2 is not optimal and we cannot make any further statements about the mass range $[m^*, m_2] \cup [m_2^{-1}, m^{*-1}]$. The behavior for these masses, and in particular the question whether $m_2 = m^*$, still represents an open problem.

2 The Model

For $p_1, p_2, k_1, k_2 \in \mathbb{R}^3$ and $m > 0$, let

$$h_0(p_1, p_2, k_1, k_2) = p_1^2 + p_2^2 + \frac{1}{m} (k_1^2 + k_2^2). \tag{2.1}$$

We will work with the quadratic form F_α introduced in [13] for $2 + 2$ particles. Its form domain is given by

$$D(F_\alpha) = \{\psi = \varphi + G_\mu \xi \mid \varphi \in H_{\text{as}}^1(\mathbb{R}^6) \otimes H_{\text{as}}^1(\mathbb{R}^6), \xi \in H^{1/2}(\mathbb{R}^9)\} \tag{2.2}$$

where, for some (arbitrary) $\mu > 0$, $G_\mu \xi$ is the function with Fourier transform

$$\widehat{G_\mu \xi}(p_1, p_2, k_1, k_2) = \sum_{i,j \in \{1,2\}} (-1)^{i+j} (h_0(p_1, p_2, k_1, k_2) + \mu)^{-1} \hat{\xi}(p_i + k_j, \hat{p}_i, \hat{k}_j) \tag{2.3}$$

and we used the notation that $\hat{p}_1 = p_2, \hat{p}_2 = p_1$ and analogously for k . The space $H_{\text{as}}^1(\mathbb{R}^6)$ denotes antisymmetric functions in $H^1(\mathbb{R}^3) \otimes H^1(\mathbb{R}^3)$. Note that because of the requirement $\varphi \in H^1(\mathbb{R}^{12})$ the decomposition $\psi = \varphi + G_\mu \xi$ is unique. Note also that the Hilbert space under consideration consists of functions that are antisymmetric in the first two and last two variables, i.e., under both the exchange $p_1 \leftrightarrow p_2$ and $k_1 \leftrightarrow k_2$.

For $\alpha \in \mathbb{R}$, the quadratic form we consider is given by

$$F_\alpha(\psi) = H(\varphi) - \mu \|\psi\|_2^2 + 4T_\mu(\xi) + 4\alpha \|\xi\|_2^2, \tag{2.4}$$

where

$$H(\varphi) = \int_{\mathbb{R}^{12}} (h_0(p_1, p_2, k_1, k_2) + \mu) |\hat{\varphi}(p_1, p_2, k_1, k_2)|^2 dp_1 dp_2 dk_1 dk_2 \tag{2.5}$$

and $T_\mu(\xi) = \sum_{i=0}^3 \phi_i(\xi)$, with the ϕ_i of the form

$$\phi_0(\xi) = 2\pi^2 \left(\frac{m}{m+1}\right)^{3/2} \int |\hat{\xi}(P, p, k)|^2 \sqrt{\frac{P^2}{1+m} + p^2 + \frac{k^2}{m} + \mu} dP dp dk \tag{2.6}$$

$$\phi_1(\xi) = \int \frac{\hat{\xi}^*(p_1 + k_1, p_2, k_2) \hat{\xi}(p_2 + k_1, p_1, k_2)}{h_0(p_1, p_2, k_1, k_2) + \mu} dp_1 dp_2 dk_1 dk_2 \tag{2.7}$$

$$\phi_2(\xi) = \int \frac{\hat{\xi}^*(p_1 + k_1, p_2, k_2) \hat{\xi}(p_1 + k_2, p_2, k_1)}{h_0(p_1, p_2, k_1, k_2) + \mu} dp_1 dp_2 dk_1 dk_2 \tag{2.8}$$

$$\phi_3(\xi) = - \int \frac{\hat{\xi}^*(p_1 + k_1, p_2, k_2) \hat{\xi}(p_2 + k_2, p_1, k_1)}{h_0(p_1, p_2, k_1, k_2) + \mu} dp_1 dp_2 dk_1 dk_2. \tag{2.9}$$

We note that F_α is independent of the choice of $\mu > 0$. The parameter α corresponds to the inverse scattering length; more precisely, $\alpha = -2\pi^2/a$, with $a \in (-\infty, 0) \cup (0, \infty]$ the scattering length.

It was shown in [13] that $T_\mu(\xi)$ is well-defined on $H^{1/2}(\mathbb{R}^9)$. To show stability, we need to prove that it is in fact positive. If, on the contrary, there exists a $\mu > 0$ and a $\xi \in H^{1/2}(\mathbb{R}^9)$ such that $T_\mu(\xi) < 0$, a simple scaling argument (choosing $\varphi = 0$ and using the scale invariance of F_0) can be used to deduce that F_α is unbounded from below for all $\alpha \in \mathbb{R}$.

The functionals ϕ_0 and ϕ_1 also appear in a similar form in the discussion of the $2 + 1$ problem, and ϕ_2 can be seen as the analogous $1 + 2$ term. The term ϕ_3 has no analogue in the $2 + 1$ or $1 + 2$ systems. Note that none of the ϕ_i for $1 \leq i \leq 3$ has a sign, and we expect that cancellations occur between them that are important for stability. In our proof below, we will first bound $\phi_0 + \phi_3$ from below by a positive quantity, which we then use to compensate separately the negative parts of ϕ_1 and ϕ_2 . Since we shall neglect some positive terms, we cannot expect to obtain a sharp bound. In particular, whether $m_2 = m^*$, as suggested in [11], cannot be determined using this method.

3 Main Result

For $a \in \mathbb{R}^3, b \geq 0$ and $m > 0$, let $O_{a,b}^m$ be the bounded operator on $L^2(\mathbb{R}^3)$ with integral kernel

$$O_{a,b}^m(p_1, p_2) = \left[(p_1 + a)^2 + b^2 \right]^{-1/4} \left[(p_2 + a)^2 + b^2 \right]^{-1/4} \times \frac{1}{p_1^2 + p_2^2 + \frac{2}{1+m} p_1 \cdot p_2 + \frac{2(2+m)}{(1+m)^2} a^2 + \frac{2m}{(1+m)^2} b^2}. \tag{3.1}$$

Let further

$$\Lambda(m) = -\frac{1}{2\pi^2} \frac{1+m}{\sqrt{m}} \inf_{a \in \mathbb{R}^3, b \geq 0} \inf \text{spec } O_{a,b}^m. \tag{3.2}$$

Theorem 1 For $m > 0$ such that $\Lambda(m) + \Lambda(1/m) \leq 1$, we have

$$T_\mu(\xi) \geq (1 - \Lambda(m) - \Lambda(1/m)) \sqrt{2\mu} \pi^2 \left(\frac{m}{m+1}\right)^{3/2} \|\xi\|_2^2 \tag{3.3}$$

for any $\xi \in H^{1/2}(\mathbb{R}^9)$ and any $\mu > 0$.

This bound readily implies stability for F_α , as the following corollary shows.

Corollary 1 For m such that $\Lambda(m) + \Lambda(1/m) < 1$, we have

$$F_\alpha(\psi) \geq \begin{cases} 0 & \alpha \geq 0 \\ -\alpha^2 \left(\frac{m+1}{m}\right)^3 \frac{1}{2\pi^4(1-\Lambda(m)-\Lambda(1/m))^2} \|\psi\|_2^2 & \alpha < 0 \end{cases} \tag{3.4}$$

for any $\psi \in D(F_\alpha)$.

Proof Without loss of generality we can assume that $\|\psi\|_2 = 1$. Using Theorem 1 and $H(\varphi) \geq 0$, we get

$$\begin{aligned} F_\alpha(\psi) + \mu &\geq 4T_\mu(\xi) + 4\alpha \|\xi\|_2^2 \\ &\geq 4 \left[\alpha + (1 - \Lambda(m) - \Lambda(1/m)) \sqrt{2\mu} \pi^2 \left(\frac{m}{m+1}\right)^{3/2} \right] \|\xi\|_2^2. \end{aligned} \tag{3.5}$$

In case $\alpha \geq 0$ we obtain $F_\alpha(\psi) \geq -\mu$, which shows the result as $\mu > 0$ was arbitrary. If $\alpha < 0$, we choose

$$\mu = \alpha^2 \left(\frac{m+1}{m}\right)^3 \frac{1}{2\pi^4(1-\Lambda(m)-\Lambda(1/m))^2}, \tag{3.6}$$

which yields the desired result. □

We thus proved stability as long as $\Lambda(m) + \Lambda(1/m) < 1$. To investigate the implication on m , let us first check what happens for $a = 0$ and $b = 0$. An explicit calculation following [6] shows that

$$\begin{aligned} \bar{\Lambda}(m) &:= -\frac{1}{2\pi^2} \frac{1+m}{\sqrt{m}} \inf \text{spec } O_{0,0}^m \\ &= \frac{2}{\pi} (1+m)^2 \left(\frac{1}{\sqrt{m}} - \sqrt{2+m} \arcsin \left(\frac{1}{1+m} \right) \right) \end{aligned} \tag{3.7}$$

which satisfies $\bar{\Lambda}(m) + \bar{\Lambda}(1/m) < 1$ for $0.139 \lesssim m \lesssim 7.189$. This range of masses is the largest possible for which our approach can show stability.

While we do not know whether $\Lambda(m) = \bar{\Lambda}(m)$, we shall give in Section 5 a rough upper bound on $\Lambda(m)$ which shows that $\Lambda(m) + \Lambda(1/m) < 1$ for $0.58 \lesssim m \lesssim 1.73$.

4 Proof of Theorem 1

We shall split the proof into several steps.

4.1 Bound on ϕ_3

We shall rewrite ϕ_3 in (2.9) using center-of-mass and relative coordinates for each of the pairs (p_1, k_1) and (p_2, k_2) . With $P_1 = p_1 + k_1$, $q_1 = \frac{m}{1+m}p_1 - \frac{1}{1+m}k_1$, $P_2 = p_2 + k_2$ and $q_2 = \frac{m}{1+m}p_2 - \frac{1}{1+m}k_2$, we have

$$\phi_3(\xi) = - \int dP_1 dP_2 dq_1 dq_2 \times \frac{\hat{\xi}^*(P_1, \frac{P_2}{1+m} + q_2, \frac{mP_2}{1+m} - q_2)\hat{\xi}(P_2, \frac{P_1}{1+m} + q_1, \frac{mP_1}{1+m} - q_1)}{\frac{1}{1+m}(P_1^2 + P_2^2) + \frac{1+m}{m}(q_1^2 + q_2^2) + \mu}. \tag{4.1}$$

By completing the square, we can write, for any positive function w ,

$$\phi_3(\xi) = \int \frac{dP_1 dP_2 dq_1 dq_2}{w(q_2, P_1, P_2)w(q_1, P_2, P_1)} \times \frac{\frac{1}{2}|\chi_w(q_2, P_1, P_2) - \chi_w(q_1, P_2, P_1)|^2 - |\chi_w(q_2, P_1, P_2)|^2}{\frac{1}{1+m}(P_1^2 + P_2^2) + \frac{1+m}{m}(q_1^2 + q_2^2) + \mu} \tag{4.2}$$

where we denote $\chi_w(q, P_1, P_2) = \hat{\xi}(P_1, \frac{P_2}{1+m} + q, \frac{mP_2}{1+m} - q)w(q, P_1, P_2)$. We shall choose

$$w(q, P_1, P_2) = q^2 + \lambda^2 \left(\frac{m}{(1+m)^2} (P_1^2 + P_2^2) + \frac{m}{1+m} \mu \right) \tag{4.3}$$

for some constant $\lambda \geq 0$. The first term in the numerator on the right side of (4.2) is manifestly positive. Performing the integration over q_1 , the integral over the second term equals

$$\int dP_1 dP_2 dq_2 \left(-\frac{2\pi^2 m}{1+m} \right) \left| \hat{\xi}(P_1, \frac{1}{1+m}P_2 + q_2, \frac{m}{1+m}P_2 - q_2) \right|^2 \times \frac{q_2^2 + \lambda^2 \left(\frac{m}{(1+m)^2} (P_1^2 + P_2^2) + \frac{m}{1+m} \mu \right)}{\lambda \sqrt{\frac{m}{(1+m)^2} (P_1^2 + P_2^2) + \frac{m}{1+m} \mu} + \sqrt{q_2^2 + \frac{m}{(1+m)^2} (P_1^2 + P_2^2) + \frac{m}{1+m} \mu}}. \tag{4.4}$$

Let us compare this latter expression with ϕ_0 in (2.6), which can be rewritten as

$$\phi_0(\xi) = \frac{2\pi^2 m}{m+1} \int |\hat{\xi}(P_1, \frac{1}{1+m}P_2 + q_2, \frac{m}{1+m}P_2 - q_2)|^2 \times \sqrt{q_2^2 + \frac{m}{(1+m)^2} (P_1^2 + P_2^2) + \frac{m}{1+m} \mu} dP_1 dP_2 dq_2. \tag{4.5}$$

For $0 \leq \lambda \leq 1$, one readily checks that

$$\begin{aligned}
 &L_\lambda(P_1, P_2, q) \\
 &:= \sqrt{q^2 + \frac{m}{(1+m)^2} (P_1^2 + P_2^2) + \frac{m}{1+m} \mu} \\
 &\quad - \frac{q^2 + \lambda^2 \left(\frac{m}{(1+m)^2} (P_1^2 + P_2^2) + \frac{m}{1+m} \mu \right)}{\lambda \sqrt{\frac{m}{(1+m)^2} (P_1^2 + P_2^2) + \frac{m}{1+m} \mu} + \sqrt{q^2 + \frac{m}{(1+m)^2} (P_1^2 + P_2^2) + \frac{m}{1+m} \mu}}
 \end{aligned} \tag{4.6}$$

is non-negative. What we have shown here is that

$$\begin{aligned}
 &\phi_0(\xi) + \phi_3(\xi) \\
 &\geq \frac{2\pi^2 m}{m+1} \int |\hat{\xi}(P_1, \frac{1}{1+m} P_2 + q, \frac{m}{1+m} P_2 - q)|^2 L_\lambda(P_1, P_2, q) dP_1 dP_2 dq
 \end{aligned} \tag{4.7}$$

for any $\lambda \geq 0$.

Note that for $\lambda^2 = 1/2$, L_λ takes the simple form

$$L_{1/\sqrt{2}}(P_1, P_2, q) = \frac{1}{\sqrt{2}} \sqrt{\frac{m}{(1+m)^2} (P_1^2 + P_2^2) + \frac{m}{1+m} \mu} \tag{4.8}$$

and is, in particular, independent of q .

4.2 Bound on ϕ_1

For the term ϕ_1 in (2.7), we shall switch to center-of-mass and relative coordinates for the particles (p_1, p_2, k_1) . With $P = p_1 + p_2 + k_1$, $q_1 = \frac{1+m}{2+m} p_1 - \frac{1}{2+m} (p_2 + k_1)$ and $q_2 = \frac{1+m}{2+m} p_2 - \frac{1}{2+m} (p_1 + k_1)$, as well as $k = k_2$ for short, we have

$$\begin{aligned}
 \phi_1(\xi) &= \frac{m}{1+m} \int dP dq_1 dq_2 dk \\
 &\quad \times \frac{\hat{\xi}^* \left(\frac{1+m}{2+m} P - q_2, \frac{P}{2+m} + q_2, k \right) \hat{\xi} \left(\frac{1+m}{2+m} P - q_1, \frac{P}{2+m} + q_1, k \right)}{q_1^2 + q_2^2 + \frac{2}{1+m} q_1 \cdot q_2 + \frac{m}{(1+m)(2+m)} P^2 + \frac{1}{1+m} k^2 + \frac{m}{1+m} \mu}.
 \end{aligned} \tag{4.9}$$

Defining

$$\ell_\lambda(q, P, k) = L_\lambda \left(\frac{1+m}{2+m} P - q, \frac{P}{2+m} + q + k, \frac{mq}{1+m} + \frac{mP}{(1+m)(2+m)} - \frac{k}{1+m} \right) \tag{4.10}$$

our aim is to obtain a lower bound on the operator on $L^2(\mathbb{R}^3)$ with integral kernel

$$\frac{\ell_\lambda(q_1, P, k)^{-1/2} \ell_\lambda(q_2, P, k)^{-1/2}}{q_1^2 + q_2^2 + \frac{2}{1+m} q_1 \cdot q_2 + \frac{m}{(1+m)(2+m)} P^2 + \frac{1}{1+m} k^2 + \frac{m}{1+m} \mu} \tag{4.11}$$

for suitable λ , uniformly in the fixed parameters P and k .

Let us take $\lambda^2 = 1/2$ for simplicity, in which case we have

$$\ell_{1/\sqrt{2}}(q, P, k) = \frac{\sqrt{m}}{1+m} \sqrt{\left(q + \frac{1}{2}k - \frac{m}{2(2+m)}P\right)^2 + \frac{1}{4}(P+k)^2 + \frac{1+m}{2}\mu}. \tag{4.12}$$

Note also that

$$\begin{aligned} & \frac{m}{(1+m)(2+m)}P^2 + \frac{1}{1+m}k^2 \\ &= \frac{2m}{(1+m)^2} \left[\frac{2+m}{m} \left(\frac{1}{2}k - \frac{m}{2(2+m)}P\right)^2 + \frac{1}{4}(P+k)^2 \right]. \end{aligned} \tag{4.13}$$

With

$$a = \frac{1}{2}k - \frac{m}{2(2+m)}P, \quad b^2 = \frac{1}{4}(P+k)^2 + \frac{1+m}{2}\mu \tag{4.14}$$

our task is thus to find a lower bound on the operator with integral kernel $\frac{1+m}{\sqrt{m}}O_{a,b}^m(q_1, q_2)$, defined in (3.1). The best lower bound equals $-2\pi^2\Lambda(m)$, by definition.

To summarize, what we have shown here is that

$$\phi_1(\xi) \geq -\Lambda(m) \frac{2\pi^2m}{m+1} \int \left| \hat{\xi} \left(\frac{1+m}{2+m}P - q, \frac{P}{2+m} + q, k \right) \right|^2 \ell_{1/\sqrt{2}}(q, P, k) dP dq dk. \tag{4.15}$$

Using (4.10), a simple change of variables shows that this is equivalent to

$$\begin{aligned} & \phi_1(\xi) \\ & \geq -\Lambda(m) \frac{2\pi^2m}{m+1} \int \left| \hat{\xi} \left(P_1, \frac{P_2}{1+m} + q, \frac{mP_2}{1+m} - q \right) \right|^2 L_{1/\sqrt{2}}(P_1, P_2, q) dP_1 dP_2 dq. \end{aligned} \tag{4.16}$$

4.3 Bound on ϕ_2

In exactly the same way we proceed with ϕ_2 in (2.8), which we rewrite as

$$\begin{aligned} & \phi_2(\xi) \\ &= \frac{m}{1+m} \int dP dq_1 dq_2 dp \\ & \quad \times \frac{\hat{\xi}^* \left(\frac{1+m}{1+2m}P - q_2, p, q_2 + \frac{mP}{1+2m} \right) \hat{\xi} \left(\frac{1+m}{1+2m}P - q_1, p, q_1 + \frac{mP}{1+2m} \right)}{q_1^2 + q_2^2 + \frac{2m}{1+m}q_1 \cdot q_2 + \frac{m}{(1+m)(1+2m)}P^2 + \frac{m}{1+m}P^2 + \frac{m}{1+m}\mu}. \end{aligned} \tag{4.17}$$

If we now define

$$\tilde{\ell}_\lambda(q, P, p) = L_\lambda \left(\frac{1+m}{1+2m}P - q, p + q + \frac{mP}{1+2m}, \frac{mp}{1+m} - \frac{q}{1+m} - \frac{mP}{(1+m)(1+2m)} \right) \tag{4.18}$$

we need a lower bound on the operator on $L^2(\mathbb{R}^3)$ with integral kernel

$$\frac{\tilde{\ell}_\lambda(q_1, P, p)^{-1/2} \tilde{\ell}_\lambda(q_2, P, p)^{-1/2}}{q_1^2 + q_2^2 + \frac{2m}{1+m}q_1 \cdot q_2 + \frac{m}{(1+m)(1+2m)}P^2 + \frac{m}{1+m}P^2 + \frac{m}{1+m}\mu} \tag{4.19}$$

for fixed P and p . By proceeding as in the previous subsection, one readily checks that, for $\lambda^2 = 1/2$, its best lower bound is $-2\pi^2\Lambda(1/m)$, with Λ defined in (3.2). In particular, we have

$$\begin{aligned} &\phi_2(\xi) \\ &\geq -\Lambda(1/m) \frac{2\pi^2 m}{m+1} \int |\hat{\xi}(P_1, \frac{P_2}{1+m} + q, \frac{mP_2}{1+m} - q)|^2 L_{1/\sqrt{2}}(P_1, P_2, q) dP_1 dP_2 dq. \end{aligned} \tag{4.20}$$

4.4 Combining above bounds

By combining the bounds (4.7), (4.16) and (4.20) from the previous three subsections, we obtain

$$\begin{aligned} T_\mu(\xi) &= \sum_{j=0}^3 \phi_j(\xi) \\ &\geq (1 - \Lambda(m) - \Lambda(1/m)) \frac{2\pi^2 m}{m+1} \\ &\quad \times \int |\hat{\xi}(P_1, \frac{1}{1+m} P_2 + q, \frac{m}{1+m} P_2 - q)|^2 L_{1/\sqrt{2}}(P_1, P_2, q) dP_1 dP_2 dq \end{aligned} \tag{4.21}$$

with $L_{1/\sqrt{2}}$ defined in (4.8). In the case $\Lambda(m) + \Lambda(1/m) \leq 1$, we can further use $L_{1/\sqrt{2}}(P_1, P_2, q) \geq \sqrt{m\mu/(2(1+m))}$ for a lower bound. This completes the proof of Theorem 1.

5 Bound on $\Lambda(m)$

Note that $\Lambda(m) \geq \bar{\Lambda}(m)$. To obtain an upper bound, we use the Schur test. We first drop the positive part of the operator with integral kernel

$$k(p_1, p_2) = \left[p_1^2 + p_2^2 + \frac{2}{1+m} p_1 \cdot p_2 + \frac{2(2+m)}{(1+m)^2} a^2 + \frac{2m}{(1+m)^2} b^2 \right]^{-1}. \tag{5.1}$$

It follows from [21, Lemma 3] that the negative part of this operator has the integral kernel

$$\begin{aligned} k_-(p_1, p_2) &= \frac{-k(p_1, p_2) + k(p_1, -p_2)}{2} \\ &= \frac{2}{1+m} \frac{p_1 \cdot p_2}{\left[p_1^2 + p_2^2 + \frac{2(2+m)}{(1+m)^2} a^2 + \frac{2m}{(1+m)^2} b^2 \right]^2 - \frac{4(p_1 \cdot p_2)^2}{(1+m)^2}}. \end{aligned}$$

By applying the Cauchy-Schwarz inequality, we obtain, for any positive function h on \mathbb{R}^3 (possibly depending on a and b)

$$\Lambda(m) \leq \frac{1}{\pi^2 \sqrt{m}} \sup_{p_1, a, b} \int_{\mathbb{R}^3} \frac{h(p_1)}{h(p_2)} \frac{|p_1 \cdot p_2|}{\left[p_1^2 + p_2^2 + \frac{2(2+m)}{(1+m)^2} a^2 + \frac{2m}{(1+m)^2} b^2 \right]^2 - \frac{4(p_1 \cdot p_2)^2}{(1+m)^2}} \times \left[(p_2 + a)^2 + b^2 \right]^{-1/2} dp_2. \tag{5.2}$$

By monotonicity, we can set $b = 0$, i.e.,

$$\Lambda(m) \leq \frac{1}{\pi^2 \sqrt{m}} \sup_{p_1, a} \int_{\mathbb{R}^3} \frac{h(p_1)}{h(p_2)} \frac{|p_1 \cdot p_2|}{\left[p_1^2 + p_2^2 + \frac{2(2+m)}{(1+m)^2} a^2 \right]^2 - \frac{4(p_1 \cdot p_2)^2}{(1+m)^2}} |p_2 + a|^{-1} dp_2. \tag{5.3}$$

We shall choose h to be even, i.e., $h(p) = h(-p)$, in which case we can symmetrize to get

$$\begin{aligned} \Lambda(m) &\leq \frac{1}{\pi^2 \sqrt{m}} \sup_{p_1, a} \int_{\mathbb{R}^3} \frac{h(p_1)}{h(p_2)} \frac{|p_1 \cdot p_2|}{\left[p_1^2 + p_2^2 + \frac{2(2+m)}{(1+m)^2} a^2 \right]^2 - \frac{4(p_1 \cdot p_2)^2}{(1+m)^2}} \\ &\quad \times \frac{1}{2} \left(\frac{1}{|p_2 + a|} + \frac{1}{|p_2 - a|} \right) dp_2 \\ &\leq \frac{1}{\pi^2 \sqrt{m}} \sup_{p_1, a} \int_{\mathbb{R}^3} \frac{h(p_1)}{h(p_2)} \frac{|p_1 \cdot p_2|}{\left[p_1^2 + p_2^2 + \frac{2(2+m)}{(1+m)^2} a^2 \right]^2 - \frac{4(p_1 \cdot p_2)^2}{(1+m)^2}} \\ &\quad \times \sqrt{\frac{p_2^2 + a^2}{(p_2^2 + a^2)^2 - 4(p_2 \cdot a)^2}} dp_2. \end{aligned} \tag{5.4}$$

To maximize the right side, a wants to be parallel to p_1 , i.e., $a = \kappa p_1$ for $\kappa \in \mathbb{R}$. This is a direct consequence of [21, Lemma 5]. We shall choose $h(p) = |p|$. By scale invariance we can set $|p_1| = 1$. We then obtain

$$\begin{aligned} \Lambda(m) &\leq \frac{4}{\pi \sqrt{m}} \sup_{\kappa \in \mathbb{R}} \int_0^1 dt \int_0^\infty dr \frac{r^2 t}{\left[1 + r^2 + \frac{2(2+m)}{(1+m)^2} \kappa^2 \right]^2 - \frac{4r^2 t^2}{(1+m)^2}} \\ &\quad \times \sqrt{\frac{r^2 + \kappa^2}{(r^2 + \kappa^2)^2 - 4\kappa^2 r^2 t^2}}. \end{aligned} \tag{5.5}$$

We further bound $t \leq 1$ in the denominator of the first term in the integrand in (5.5), and use that

$$\left[1 + r^2 + \frac{2(2+m)}{(1+m)^2} \kappa^2 \right]^2 - \frac{4r^2}{(1+m)^2} \geq \frac{m(m+2)}{(1+m)^2} \left[1 + r^2 + \frac{2\sqrt{2+m}}{(1+m)\sqrt{m}} \kappa^2 \right]^2. \tag{5.6}$$

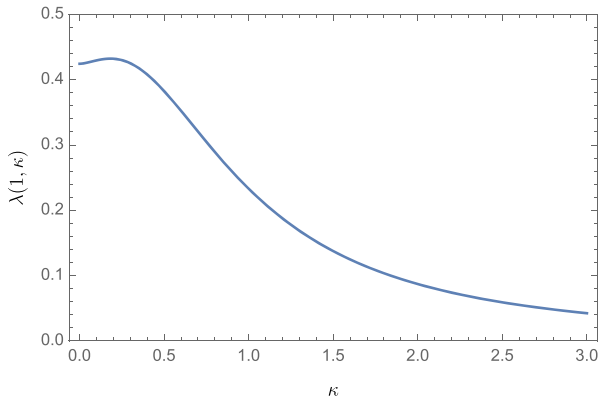


Fig. 1 The function $\lambda(1, \kappa)$, with $\lambda(1) = \sup_{\kappa} \lambda(1, \kappa) \approx 0.427$

Since

$$\int_0^1 dt t \sqrt{\frac{r^2 + \kappa^2}{(r^2 + \kappa^2)^2 - 4\kappa^2 r^2 t^2}} = \frac{1}{2r^2} \sqrt{r^2 + \kappa^2} \min\{1, r^2/\kappa^2\} \tag{5.7}$$

we therefore get

$$\Lambda(m) \leq \frac{2}{\pi} \frac{(1+m)^2}{m^{3/2}(m+2)} \sup_{\kappa \in \mathbb{R}} \int_0^\infty dr \frac{\sqrt{r^2 + \kappa^2}}{\left[1 + r^2 + \frac{2\sqrt{2+m}}{(1+m)\sqrt{m}} \kappa^2\right]^2} . \tag{5.8}$$

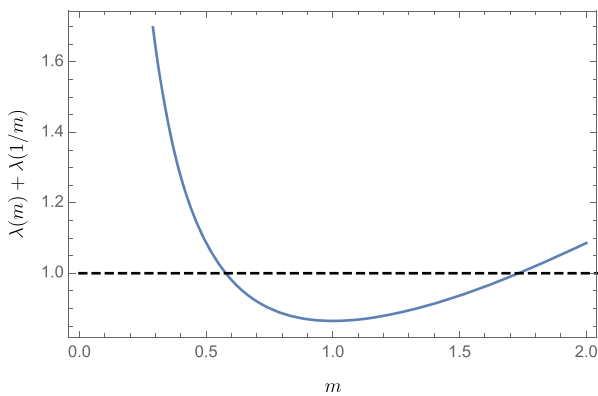


Fig. 2 Our upper bound on $\Lambda(m) + \Lambda(1/m)$, given by $\lambda(m) + \lambda(1/m)$

We define $c_m = 2\sqrt{2+m}/((1+m)\sqrt{m})$. After explicitly doing the integral, the bound (5.8) reads $\Lambda(m) \leq \lambda(m) := \sup_{\kappa>0} \lambda(m, \kappa)$ with

$$\lambda(m, \kappa) := \frac{1}{\pi} \frac{(1+m)^2}{m^{3/2}(m+2)} \frac{1}{1+c_m\kappa^2} \left(1 + \frac{\kappa^2}{\sqrt{1+c_m\kappa^2}\sqrt{1+\kappa^2(c_m-1)}} \right) \times \ln \left(\frac{\sqrt{1+c_m\kappa^2} + \sqrt{1+\kappa^2(c_m-1)}}{\kappa} \right). \quad (5.9)$$

For our purpose it is important that $\lambda(1) \approx 0.427 < 1/2$ (see Fig. 1). By continuity, this implies that $\Lambda(m) + \Lambda(1/m) < 1$ for a window of mass ratios around 1. In fact, a numerical optimization over κ leads to the conclusion that $\Lambda(m) + \Lambda(1/m) < 1$ whenever $0.58 \approx m_2 < m < m_2^{-1} \approx 1.73$ (see Fig. 2).

Acknowledgments Open access funding provided by Austrian Science Fund (FWF).

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