Journal of Functional Analysis 285 (2023) 110146

Contents lists available at ScienceDirect

Journal of Functional Analysis

journal homepage: www.elsevier.com/locate/jfa

Full Length Article

Maximal L^p-regularity and H^∞-calculus for block operator matrices and applications ☆

Antonio Agresti [*,1], Amru Hussein

Department of Mathematics, RPTU Kaiserslautern-Landau, Paul-Ehrlich-Straße 31, 67663 Kaiserslautern, Germany

ARTICLE INFO

Article history:
Received 7 October 2021
Accepted 18 August 2023
Available online 29 August 2023
Communicated by P. Auscher

MSC:
primary 47A55
secondary 47D06, 47A60, 35M13, 35Q35

Keywords:
Block operator matrices
Off-diagonal perturbations
Maximal L^p-regularity
Bounded H^∞-calculus

ABSTRACT

Many coupled evolution equations can be described via 2×2-block operator matrices of the form $\mathcal{A} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ in a product space $X = X_1 \times X_2$ with possibly unbounded entries. Here, the case of diagonally dominant block operator matrices is considered, that is, the case where the full operator $\mathcal{A}$ can be seen as a relatively bounded perturbation of its diagonal part with $\mathsf{D}(\mathcal{A}) = \mathsf{D}(A)\times\mathsf{D}(D)$ though with possibly large relative bound. For such operators the properties of sectoriality, $\mathcal{R}$-sectoriality and the boundedness of the H^∞-calculus are studied, and for these properties perturbation results for possibly large but structured perturbations are derived. Thereby, the time dependent parabolic problem associated with $\mathcal{A}$ can be analyzed in maximal L^p_t-regularity spaces, and this is applied to a wide range of problems such as different theories for liquid crystals, an artificial Stokes

☆ The first author has been supported partially by the Nachwuchsring – Network for the promotion of young scientists – at TU Kaiserslautern. Both authors have been supported by MathApp – Mathematics Applied to Real-World Problems - part of the Research Initiative of the Federal State of Rhineland-Palatinate, Germany.

* Corresponding author.

E-mail addresses: antonio.agresti92@gmail.com (A. Agresti), hussein@mathematik.uni-kl.de (A. Hussein).

[1] Current address: Institute of Science and Technology Austria (ISTA), Am Campus 1, 3400 Klosterneuburg, Austria.

https://doi.org/10.1016/j.jfa.2023.110146

system, strongly damped wave and plate equations, and a Keller-Segel model.

Contents

1. Introduction

In this article the abstract Cauchy problem

$$\begin{cases} \partial_t x(t) + \mathcal{A}\, x(t) = f(t), & t > 0, \\ x(0) = x_0, \end{cases} \tag{1.1}$$

is studied for a block operator matrix

$$\mathcal{A} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

in the product space $X = X_1 \times X_2$, where X_1, X_2 are Banach spaces, and

$$\begin{aligned} A &\colon \mathsf{D}(A) \subseteq X_1 \to X_1, &&\text{and} && D\colon \mathsf{D}(D) \subseteq X_2 \to X_2, \\ B &\colon \mathsf{D}(B) \subseteq X_2 \to X_1, &&\text{and} && C\colon \mathsf{D}(C) \subseteq X_1 \to X_2 \end{aligned}$$

are possibly unbounded linear operators with domains $\mathsf{D}(A)$, $\mathsf{D}(D)$, $\mathsf{D}(B)$, and $\mathsf{D}(C)$, respectively. Such block operator matrices arise in a wide range of coupled evolution equations including mixed-order systems. In Section 7 liquid crystal models, an artificial Stokes system, strongly damped wave and plate equations, and a Keller-Segel model are discussed as applications of the general theory presented here. The recurrent theme and main question of this article is to ask under which conditions operator theoretical properties of the diagonal operators can be transferred to the full operator. Thus, the starting point is to assume that the uncoupled problem associated with the diagonal operator is well-understood and to interpret the coupled equations as off-diagonal perturbation, that is,

$$\mathcal{A} = \mathcal{D} + \mathcal{B} \quad \text{with} \quad \mathcal{D} = \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} \quad \text{and} \quad \mathcal{B} = \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}. \tag{1.2}$$

This viewpoint is captured by the assumption of the *diagonal dominance* of $\mathcal{A}$ which requires that $\mathsf{D}(\mathcal{A}) = \mathsf{D}(\mathcal{D})$ and that $\mathcal{B}$ is relatively bounded with respect to $\mathcal{D}$ though with possibly large relative bound, see Section 3 below for the precise setting. For such diagonally dominant block operator matrices $\mathcal{A}$ the properties of sectoriality, $\mathcal{R}$-sectoriality, and the boundedness of the H^∞-calculus are studied assuming that the respective properties hold for the diagonal operator $\mathcal{D}$.

These operator theoretical properties are closely related to the solution theory of the time-dependent problem (1.1). Namely, $\mathcal{R}$-sectoriality implies in UMD-spaces X maximal L^p_t-regularity for $p \in (1, \infty)$, that is, there exists a constant $C > 0$ such that if in (1.1) the right hand side satisfies $f \in L^p(0, \infty; X)$ and $x_0 = 0$, then there is a unique solution x to (1.1) satisfying the maximal regularity estimate

$$\|\partial_t x\|_{L^p(0,\infty;X)} + \|\mathcal{A}\, x\|_{L^p(0,\infty;X)} \leq C\|f\|_{L^p(0,\infty;X)},$$

see e.g. [25,74,96,113] for overviews on this subject. The corresponding notion of stochastic maximal regularity is not equivalent to the deterministic notion, but it is implied for instance by the boundedness of the H^∞-calculus, see e.g. [4,94] and the references given therein for details on this theory. Moreover, sectoriality, $\mathcal{R}$-sectoriality, and the boundedness of the H^∞-calculus give information on the fractional powers of $\mathcal{A}$. Also, the domains and ranges of the fractional powers of $\mathcal{A}$ induce scales of extrapolation spaces and thereon consistent families of operators, compare e.g. [43]. Amongst others, this is helpful in the analysis of many quasi- or semi-linear problems, compare e.g. [5,6,61,80,93,95,96] just to give a small sample of the literature in this direction.

The classical perturbation theorems for sectoriality, $\mathcal{R}$-sectoriality, and the boundedness of the H^∞-calculus deal with smallness conditions often expressed in form of small relative bounds, relative compactness or lower order perturbations, compare e.g. [8,24,25,73]. The aim here is to complement these classical results for general additive perturbations by perturbation theorems for off-diagonal perturbations in the diagonally dominant case. In this situation it turns out that the smallness or lower order perturbation conditions can be imposed on objects describing the coupling rather than on the full additive perturbation. Moreover, assuming that off-diagonal perturbations preserve a certain structure by perturbing “in the right direction”, one can omit even any kind of smallness conditions, see Section 4 below. Also, many assumptions imposed on the general setting can be weakened for the case of off-diagonal perturbations. For instance, for the classical perturbation results for sectoriality and $\mathcal{R}$-sectoriality on relative bounded perturbations with small relative bound, there is no straightforward counterpart for the bounded H^∞-calculus. The available perturbation results for the H^∞-calculus require further assumptions in addition to the smallness conditions. Here we have been able to show a result of this type although only for the case of some off-diagonal perturbations behaving well in some extrapolation scales induced by $\mathcal{D}$, compare Section 5 below.

The approach developed here is based in spirit on a combination of the theory by Kalton, Kunstmann and Weis relating $\mathcal{R}$-sectoriality and the boundedness of the H^∞-calculus, see [57,75], with concepts for diagonally dominant block operator matrices pioneered by Nagel in [90] for C_0-semigroups. This synthesis opens a new perspective on coupled systems in L^p-spaces and is illustrated in the subsequent example and also by a number of applications in Section 7.

1.1. An example

As illustration one can consider the following model case inspired by the linearization of the Beris-Edwards model for liquid crystals. For $p \in (1,\infty)$ let $X = H^{-1,p}(\mathbb{R}^d) \times L^p(\mathbb{R}^d)^d$ and

$$\mathcal{A} \stackrel{\text{def}}{=} \begin{bmatrix} \mathbb{1} - \Delta & \operatorname{div}(\mathbb{1} - \Delta) \\ \nabla & \mathbb{1} - \Delta \end{bmatrix} \quad \text{with} \quad \mathsf{D}(\mathcal{A}) = H^{1,p}(\mathbb{R}^d) \times H^{2,p}(\mathbb{R}^d)^d.$$

The actual linearization of the Beris-Edwards model for liquid crystals – where the first component describes the fluid-like behavior and the second component the orientation of the crystal-like rods – has a similar mixed order structure. However, this simplified Beris-Edwards-type model allows for more direct computations, and therefore it seems better suited as illustrative example, see Subsection 7.6 and 7.7 for a discussion of both models. Here, one directly observes that $\mathcal{A}$ is diagonally dominant, but classical perturbation results are not applicable since the off-diagonal part is neither small nor of lower order. However, this case can be treated within the theory presented here by studying the coupling of the diagonal and the off-diagonal part as will be explained in the subsequent Subsection 1.2.

Special types of diagonally block operator matrices which appear in many applications are of the form

$$\mathcal{A} = \begin{bmatrix} A & B \\ 0 & D \end{bmatrix} \quad \text{or} \quad \mathcal{A} = \begin{bmatrix} 0 & B \\ C & D \end{bmatrix}.$$

In these situations results on $\mathcal{R}$-sectoriality and boundedness of the H^∞-calculus are obtained in Subsection 7.1. Concrete examples such as the simplified Erickson-Leslie model for liquid crystals, the classical Keller-Segel system, the artificial Stokes system and second order problems with strong damping are discussed in Subsections 7.2–7.5.

1.2. Sectoriality and $\mathcal{R}$-sectoriality for block operator matrices

A key tool in our analysis of sectoriality and $\mathcal{R}$-sectoriality is the factorization for diagonally dominant operators

$$\lambda - \mathcal{A} = \mathcal{M}(\lambda)(\lambda - \mathcal{D}) \quad \text{with} \quad \mathcal{M}(\lambda) = \begin{bmatrix} \mathbb{1} & -B(\lambda - D)^{-1} \\ -C(\lambda - A)^{-1} & \mathbb{1} \end{bmatrix},$$

where the inverse of $\mathcal{M}(\lambda)$ can be described by the inverse of one of the operators

$$M_1(\lambda) \stackrel{\text{def}}{=} \mathbb{1} - B(\lambda - D)^{-1}C(\lambda - A)^{-1} \quad \text{for} \quad \lambda \in \rho(A) \cap \rho(D),$$
$$M_2(\lambda) \stackrel{\text{def}}{=} \mathbb{1} - C(\lambda - A)^{-1}B(\lambda - D)^{-1} \quad \text{for} \quad \lambda \in \rho(A) \cap \rho(D),$$

which encode the coupling of the off-diagonal and the diagonal part. This factorization has been studied already in the work [90] by Nagel in the context of C_0-semigroups, and it is discussed in Section 3. Thus sectoriality and $\mathcal{R}$-sectoriality can be traced back to the corresponding estimates on one of the two operator families $M_1(\cdot)$ and $M_2(\cdot)$ – without smallness conditions – assuming that the diagonal operator has these properties, compare Theorem 4.1 and the other statements in Section 4. For the case of the example discussed above in Subsection 1.1 one has for instance

$$\begin{aligned} M_1(\lambda) &= \mathbb{1} - \operatorname{div}(\mathbb{1} - \Delta)(\lambda - \mathbb{1} + \Delta)^{-1}\nabla(\lambda - \mathbb{1} + \Delta)^{-1} \\ &= \mathbb{1} - \Delta(\mathbb{1} - \Delta)(\lambda - \mathbb{1} + \Delta)^{-2} \quad \text{for} \quad \lambda \in \mathbb{C} \setminus [0, \infty) \end{aligned}$$

which can be analyzed explicitly.

Moreover, by this factorization one can derive perturbation results which do not impose the classical smallness conditions on $\mathcal{B}$ as compared to $\mathcal{D}$, see for instance [73] for perturbation results with such smallness conditions. Instead – taking into account the block structure – the smallness is assumed on the coupling expressed by one of the operators $M_j(\lambda)$, $j \in \{1, 2\}$, see Corollaries 4.4 and 4.9, and Proposition 4.8. Here, as in the classical case, behind the smallness assumption lurks the Neumann series.

1.3. Bounded H^∞-calculus for block operator matrices

The boundedness of the H^∞-calculus implies $\mathcal{R}$-sectoriality on Banach space X with the relatively weak property (Δ) (or triangular contraction property), which holds in particular for UMD spaces, cf. [59] or [50, Theorem 10.3.4(2)], and for general X almost $\mathcal{R}$-sectoriality is implied, see [57, Proposition 3.2], but the converse is in general not true. This has been shown for instance by the example constructed by McIntosh and Yagi in [86] where block operator matrices make a *cameo* since the example relies on block triangular operator matrices, the spectrum of which is determined by the diagonal part, however its norm is heavily influenced by the off-diagonal part, cf. also [36] for another counterexample. The question how close $\mathcal{R}$-sectorial operators are to having a bounded H^∞-calculus has been addressed in detail for many situations by Kunstmann, Kalton and Weis in [57,75]. Here, we translate their results to the case of block operator matrices, and we obtain two types of results. First, adding conditions on the orders and on the relations of the blocks A, B, C, D encoded in terms of fractional powers, one can show that $\mathcal{R}$-sectoriality implies boundedness of the H^∞-calculus. This is discussed in Section 5, where also results for small couplings are presented. Second, considering interpolation scales and consistent families of operators thereon, results for certain points in the scale carry over to the full scale or at least to a part of it, compare Section 6. In particular the case of Hilbert spaces is considerably easier to access, and under certain conditions this carries over to scales of Banach spaces. In fact the interpolation results for the H^∞-calculus on different scales from [57] play a key role in the proofs in both Section 5 and 6. Note that in general $\mathcal{R}$-sectoriality or boundedness of the H^∞-calculus does *not* extrapolate as shown by Fackler in [34], however, for the situations considered here it does.

In the literature there is a large body of works on perturbation theory for the boundedness of the H^∞-calculus. Many of these treat additive perturbations $\mathcal{A} = \mathcal{A}_0 + \mathcal{B}$, compare for instance [24,43,74]. A special case is the situation where the perturbative term admits a factorization $\mathcal{B} = TS$ for some operators T and S acting on extrapolation scales induced by $\mathcal{A}_0$, see [9,42]. For the particular situation of block operator matrices (1.2), it seems that there is no application of this except when the blocks B and C already satisfy such factorizations.

1.4. Literature on block operator matrices and coupled parabolic systems

There is a great interest and an extensive literature related to block operator matrices and their applications in the context of spectral and semigroup theory, see e.g. the monographs by Tretter [111] and Jeribi [55]. This subject is also related to the study of mixed order systems, see e.g. the book by Denk and Kaip [27] and the references therein, the elliptic case has been discussed e.g. also in the classical article by Douglis, Agmon and Nirenberg [2]. It seems however, to the best of our knowledge, that so far, there has been no study on $\mathcal{R}$-sectoriality and the boundedness of the H^∞-calculus for a general class of diagonally dominant block operator matrices.

The theory presented here has an overlap with the study of mixed order systems as for instance the example given above in Subsection 1.1 can also be treated as a parabolic mixed order system. The maximal regularity results for mixed order systems as discussed in [27] rely on Fourier multiplier techniques. The approach presented here is however more operator theoretical in spirit. This is of particular advantage when considering scales of spaces and operators thereon including weak settings, and such situations appear in many applications, see e.g. [97] for the scalar case. Also, the approach presented here evades in some situations laborious localization procedures. In comparison, the Newton polygon method presented in [27] allows one to treat not only non-homogeneous orders in space but even in space and time, while the approach given here is restricted to problems of first order in time.

Also, there is an extensive literature on maximal L^p_t-regularity for particular block operators of mixed order arising from parabolic plate and wave equations with damping. In these areas maximal L^p_t-regularity and the boundedness of the H^∞-calculus are usually proven via multiplier and localization methods rather than by general operator theoretical considerations, compare e.g. [26,28,29,35,104,105] and the references therein which is only a small selection of the literature in this direction. These results cover a wide range of dampings and couplings while the diagonally dominant case discussed here occurs only for certain strong dampings. Note that $\mathcal{R}$-sectoriality is discussed also for other very particular block operators as for instance in [12].

In the context of semigroup theory, one of the starting points for the systematic study of diagonally dominant block operator matrices and evolution equations is the work by Nagel [90]. This has been extended by Engel and Nagel in [31,91], respectively, and subsequent works. Early works in this direction are [32,92,115,116] with focus on C_0-semigroups generated by block operator matrices. Along these lines, the case of Hilbert spaces and sesquilinear forms with a block structure have been investigated in [17] and [84], the question of m-sectorial block operator matrices is discussed in [10], second order Cauchy problems for block triangular generators are studied in [89], and special classes of block operator matrices are treated in [1], see also the references in these works.

From the viewpoint of spectral theory different questions have been addressed, see [111] for an overview, and works in this direction deal with the essential spectrum [13,14, 18,51–54], adjoints of block operator matrices [88], closedness and self-adjointness [102], invertibility of block operator matrices [49], the quadratic numerical range [77], or the operator Ricatti equation [70], where the given references are just samples and far from being complete. In most cases 2×2-block operator matrices are discussed where 2×2-matrices serve as an inspiration. For many applications this is sufficient, and some larger block operator matrices can be treated iteratively by different 2×2-block decompositions. However, also larger block structures such as 3×3- or general $n \times n$-block operator matrices have been studied as well, see [15] and also [111, Section 1.11], respectively. Spectral problems for block operator matrices and corresponding sesquilinear forms are discussed in many works, just to mention a few see [40,41,68–70,85,98,101] and also the references therein.

We start by recapitulating some basic notions and facts in Section 2, and in the subsequent Section 3 the setting for diagonally dominant block operator matrices is made precise. The main results are contained in Sections 4–6 and applications are discussed in Section 7.

2. Sectoriality, $\mathcal{R}$-sectoriality and the bounded H^∞-calculus

2.1. Notation

The solution theory of the abstract Cauchy problem

$$\begin{cases} \partial_t x(t) + T\,x(t) = f(t), & t > 0, \\ x(0) = x_0, \end{cases} \tag{2.1}$$

for a linear unbounded operator T in a Banach space X over $\mathbb{C}$ is closely related to the location of the spectrum of T and resolvent estimates in a sector of the complex plane and its complement,

$$\Sigma_\psi \stackrel{\text{def}}{=} \{z \in \mathbb{C} \setminus \{0\} \colon |\arg z| < \psi\}, \quad \text{and} \quad \complement\overline{\Sigma_\psi} \stackrel{\text{def}}{=} \mathbb{C} \setminus \overline{\Sigma_\psi} \quad \text{for } \psi \in (0, \pi),$$

respectively, where we follow the notation in [50]. We will keep this notation throughout this paper and denote by $\sigma(T)$ and $\rho(T)$ the spectrum and the resolvent set of T, respectively, where we always consider spaces over $\mathbb{C}$. If real spaces are needed, these can be obtained for the relevant examples by restriction. By $\mathsf{D}(T)$, $\mathsf{R}(T)$, and $\mathsf{N}(T)$ we denote the domain, the range, and the kernel of an operator T, respectively. We denote by $\mathbb{1}$ the identity map on X and also write to make it short $\lambda - T$ instead of $\lambda\mathbb{1} - T$ for $\lambda \in \mathbb{C}$.

In examples and applications we denote as usual for a domain $\mathcal{O} \subseteq \mathbb{R}^d$ by $L^p(\mathcal{O})$ for $p \in [1, \infty]$ the Lebesgue spaces. For $s > 0$ and $p \in (1, \infty)$ we denote by $H^{s,p}(\mathbb{R}^d)$ the Bessel potential space and set

$$\begin{aligned} &H^{s,p}(\mathcal{O}) \stackrel{\text{def}}{=} \big\{f \in \mathcal{D}'(\mathcal{O}) \colon F|_{\mathcal{O}} = f \text{ for some } F \in H^{s,p}(\mathbb{R}^d)\big\}, \\ &H^{s,p}_0(\mathcal{O}) \stackrel{\text{def}}{=} \overline{C^\infty_0(\mathcal{O})}^{H^{s,p}(\mathcal{O})} \quad \text{and} \quad H^{-s,p}(\mathcal{O}) \stackrel{\text{def}}{=} (H^{s,p'}_0(\mathcal{O}))^* \quad \text{where } \tfrac{1}{p} + \tfrac{1}{p'} = 1, \end{aligned}$$

and where X^* denotes the dual space of a Banach space X. For $s \in \mathbb{N}$ we also use the notation $W^{s,p} = H^{s,p}$, $W^{s,p}_0 = H^{s,p}_0$, and $W^{1,\infty}(\mathcal{O})$ denotes the space of Lipschitz-continuous functions on $\mathcal{O}$. In the case $p = 2$ we simply write $H^s = H^{s,2}$ for $s \in \mathbb{R}$. If not indicated otherwise, all functions in these spaces are complex valued. For vector valued function spaces we denote for a Banach space X by $L^p(\mathcal{O}; X)$ the usual Bochner spaces and by $H^{s,p}(\mathcal{O}; X)$ the corresponding Bessel potential spaces. We identify $L^p(\mathcal{O})^m = L^p(\mathcal{O}; \mathbb{C}^m)$ and $H^{s,p}(\mathcal{O})^m = H^{s,p}(\mathcal{O}; \mathbb{C}^m)$ for $m \in \mathbb{N}$, and correspondingly for other functions spaces.

The real and imaginary part of a complex number $\lambda \in \mathbb{C}$ are denoted by $\Re\lambda$ and $\Im\lambda$, respectively. By $a \lesssim b$ for $a, b \in \mathbb{R}$ we mean that there is a constant $C > 0$ independent of a, b such that $a \leq Cb$, and by $\eqsim$ we mean that $\lesssim$ and $\gtrsim$ hold. Finally, we set $a \vee b = \max\{a, b\}$ and $a \wedge b = \min\{a, b\}$.

2.2. Sectorial operators

We say that the operator

$$T \colon \mathsf{D}(T) \subseteq X \to X$$

with range $\mathsf{R}(T) \subseteq X$ is a *sectorial operator* on the Banach space X provided there exists $\omega \in (0, \pi)$ such that

$$\sigma(T) \subseteq \overline{\Sigma_\omega}, \quad \overline{\mathsf{D}(T)} = \overline{\mathsf{R}(T)} = X, \quad \text{and} \quad \sup_{\lambda \in \complement \overline{\Sigma_\omega}} \|\lambda(\lambda - T)^{-1}\|_{\mathscr{L}(X)} < \infty, \tag{2.2}$$

and the infimum over all such $\omega \in (0, \pi)$ is called the *angle of sectoriality* $\omega(T)$ of T. By [50, Proposition 10.1.7(3)], such T is injective. The operator T is called *pseudo-sectorial* if the assumption $\overline{\mathsf{D}(T)} = \overline{\mathsf{R}(T)} = X$ in (2.2) is dropped, cf. e.g. [96, Definition 3.1.1]. If T is sectorial of angle smaller than $\pi/2$, then $-T$ is the generator of an analytic semigroup.

2.3. Fractional powers and scales of sectorial operators

For a sectorial operator T on a Banach space X fractional powers T^γ with domain $\mathsf{D}(T^\gamma) \subseteq X$ for $\gamma \in \mathbb{R}$ can be defined, see e.g. [25, Section 2.2] and [43, Chapter 3] for a basic construction and also [62–67] for a detailed study of fractional powers. One can show that T^γ is also injective on X, compare e.g. [74, Theorem 15.15]. Then, the corresponding homogeneous space is defined by

$$\dot{\mathsf{D}}(T^\gamma) \stackrel{\text{def}}{=} \big(\mathsf{D}(T^\gamma), \|T^\gamma \cdot\|_X\big)^\sim \quad \text{for } \gamma \in \mathbb{R}, \tag{2.3}$$

where $\sim$ denotes the completion. In [57] these spaces are denoted by $\dot{X}_{\gamma,T} = \dot{X}_\gamma = \dot{\mathsf{D}}(T^\gamma)$. One can check that (see [43, Lemma 6.3.2 a)])

$$\mathsf{D}(T^\gamma) = \dot{\mathsf{D}}(T^\gamma) \cap X \quad \text{for } \gamma \geq 0, \tag{2.4}$$

and in particular if $0 \in \rho(T)$, then $\mathsf{D}(T^\gamma) = \dot{\mathsf{D}}(T^\gamma)$ for $\gamma \geq 0$.

Following [57, Section 2] or [43, Subsection 6.3], let us recall that T uniquely induces an operator on $\dot{\mathsf{D}}(T^\gamma)$ for all $\gamma \in \mathbb{R}$. Indeed, it is easy to check that $T^\gamma \colon \mathsf{D}(T^\gamma) \to \mathsf{R}(T^\gamma)$ extends to an isomorphism $\widetilde{T^\gamma} \colon \dot{\mathsf{D}}(T^\gamma) \to X$ the inverse of which $(\widetilde{T^\gamma})^{-1}$ is an extension of $T^{-\gamma} \colon \mathsf{R}(T^\gamma) \to \mathsf{D}(T^\gamma)$. Note that in general $(\widetilde{T^\gamma})^{-1} \neq (\widetilde{T^{-\gamma}})$ since these operators might act on different spaces. For any $\gamma \in \mathbb{R}$ we define the operator $\dot{T}_\gamma$ on $\dot{\mathsf{D}}(T^\gamma)$ by

$$\begin{array}{ccc} X \supseteq \mathsf{D}(T) & \xrightarrow{\quad T \quad} & X \\ {\scriptstyle \widetilde{T^\gamma}}\big\uparrow & & \big\downarrow{\scriptstyle (\widetilde{T^\gamma})^{-1}} \\ \dot{\mathsf{D}}(T^\gamma) \supseteq \mathsf{D}(\dot{T}_\gamma) & \xrightarrow{\quad \dot{T}_\gamma \quad} & \dot{\mathsf{D}}(T^\gamma) \end{array}$$

that is

$$\dot{T}_\gamma \stackrel{\text{def}}{=} (\widetilde{T^\gamma})^{-1} T \, \widetilde{T^\gamma}, \quad \text{with domain} \quad \mathsf{D}(\dot{T}_\gamma) = \dot{\mathsf{D}}(T^{\gamma+1}) \cap \dot{\mathsf{D}}(T^\gamma). \tag{2.5}$$

In particular, $\dot{T}_\gamma$ is similar to T and therefore it has the same spectral properties as T. For future convenience, let us note that for all $\gamma \in \mathbb{R}$ and $\lambda \in \rho(T) = \rho(\dot{T}_\gamma)$, one has $(\lambda - \dot{T}_\gamma)^{-1} = (\widetilde{T^\gamma})^{-1}(\lambda - T)^{-1}\widetilde{T^\gamma}$ and therefore

$$(\lambda - \dot{T}_\gamma)^{-1}|_{\dot{\mathsf{D}}(T^\gamma) \cap X} = (\lambda - T)^{-1}|_{\dot{\mathsf{D}}(T^\gamma) \cap X}. \tag{2.6}$$

Extrapolation scales play an important role in the proofs in Sections 5 and 6, and they have been used also in perturbation theory presented in [9,42,57,71].

2.4. $\mathcal{R}$-boundedness and maximal L^p-regularity

A family $\mathscr{J} \subseteq \mathscr{L}(E,F)$ is *Rademacher-* or *$\mathcal{R}$-bounded* with *$\mathcal{R}$-bound* $\mathcal{R}(\mathscr{J}) < \infty$ if for any sequence (ε_n) of Rademacher variables, i.e., $\{+1,-1\}$-valued independent random variables on a probability space $(\Omega, \mathscr{A}, \mathbb{P})$ with mean zero, one has for all $T_1, \dots, T_m \subseteq \mathscr{J}$ and $x_1, \dots, x_m \in E$ that

$$\mathbb{E}\Big\| \sum_{n=1}^{m} \varepsilon_n T_n x_n \Big\|_F^2 \leq \mathcal{R}(\mathscr{J})^2 \mathbb{E}\Big\| \sum_{n=1}^{m} \varepsilon_n x_n \Big\|_E^2,$$

where the expectation in $(\Omega, \mathscr{A}, \mathbb{P})$ is denoted by $\mathbb{E}$, cf. e.g. [50, Chapter 8]. In case $\mathscr{J} = \{J_\lambda : \lambda \in \Lambda\}$ for some index set Λ is $\mathcal{R}$-bounded, we write $\mathcal{R}(J_\lambda : \lambda \in \Lambda) < \infty$ instead of $\mathcal{R}(\{J_\lambda : \lambda \in \Lambda\}) < \infty$.

A sectorial operator T in X is called *$\mathcal{R}$-sectorial* if for some $\sigma \in (\omega(T), \pi)$ the family

$$\{\lambda(\lambda - T)^{-1} \colon \lambda \in \complement\overline{\Sigma_\sigma}\}$$

is $\mathcal{R}$-bounded. The *angle of $\mathcal{R}$-sectoriality* of T is

$$\omega_{\mathcal{R}}(T) \stackrel{\text{def}}{=} \inf\Big\{\sigma \in (\omega(T), \pi) \colon \{\lambda(\lambda - T)^{-1} \colon \lambda \in \complement\overline{\Sigma_\sigma}\} \text{ is } \mathcal{R}\text{-bounded}\Big\},$$

compare e.g. [50, Definition 10.3.1].

An operator T in a Banach space X has *maximal L^p-regularity* for $p \in (1,\infty)$ on $[0,\tau)$ with $\tau \in (0,\infty]$ if T is closed and densely defined and for $f \in L^p(0,\tau;X)$, $u_0 = 0$ the solution to the abstract Cauchy problem (2.1) is differentiable almost everywhere and there exists a constant $C_p > 0$ such that for all such f

$$\|\partial_t u\|_{L^p(0,\tau;X)} + \|Tu\|_{L^p(0,\tau;X)} \leq C_p \|f\|_{L^p(0,\tau;X)},$$

cf. e.g. [74, Section 1.3]. Note that if T has maximal L^p-regularity, then $-T$ generates an analytic semigroup, see [22]. In a UMD space X, a closed densely defined operator T has maximal L^p-regularity for $p \in (1,\infty)$ if and only if it is $\mathcal{R}$-sectorial of angle smaller than $\pi/2$, compare [113, Theorem 4.2].

2.5. Bounded H^∞-calculus

For any $\psi \in (0, 2\pi)$, we denote by $H_0^\infty(\Sigma_\psi)$ the set of all holomorphic functions $f : \Sigma_\psi \to \mathbb{C}$ such that $|f(z)| \lesssim |z|^\varepsilon/(1+|z|)^{-2\varepsilon}$ for all $z \in \Sigma_\psi$ and some $\varepsilon > 0$. For a sectorial operator T and all $f \in H_0^\infty(\Sigma_\psi)$ where $\psi > \omega(T)$ the Dunford integral

$$f(T) \stackrel{\text{def}}{=} \frac{1}{2\pi i} \int_{\partial \Sigma_\sigma} f(\lambda)(\lambda - T)^{-1}\,\mathrm{d}\lambda, \tag{2.7}$$

is absolutely convergence and independent of $\sigma \in (\omega(T), \psi)$. We say that T has a bounded $H^\infty(\Sigma_\psi)$-calculus for $\psi \in (\omega(T), \pi)$ if there exists $C > 0$ such that

$$\|f(T)\|_{\mathscr{L}(X)} \leq C\|f\|_{H^\infty(\Sigma_\psi)} \quad \text{for all } f \in H_0^\infty(\Sigma_\psi),$$

where $\|f\|_{H^\infty(\Sigma_\psi)} \stackrel{\text{def}}{=} \sup_{z\in\Sigma_\psi} |f(z)|$, and the H^∞-*angle* of T is

$$\omega_{H^\infty}(T) \stackrel{\text{def}}{=} \inf\{\psi \in (\omega(T), \pi) \colon T \text{ has a bounded } H^\infty(\Sigma_\psi)\text{-calculus}\},$$

compare e.g. [50, Definition 10.2.10]. Finally, we say that T has a bounded H^∞-calculus provided T has a bounded $H^\infty(\Sigma_\sigma)$-calculus for some $\sigma \in (0, \pi)$.

For an UMD space X, one has the inclusions

$$H^\infty(X) \subseteq \mathcal{SMR}(X) \subseteq \mathcal{S}(X) \quad \text{and} \quad H^\infty(X) \subseteq \mathcal{R}(X) \subseteq \mathcal{S}(X),$$

where $H^\infty(X)$, $\mathcal{SMR}(X)$, $\mathcal{R}(X)$, and $\mathcal{S}(X)$ stand for the classes of operators in a UMD space X having a bounded H^∞-calculus, admitting stochastic maximal L^p-regularity, being $\mathcal{R}$-sectorial and sectorial, respectively, compare e.g. [25, Equation (2.15) and Theorem 4.5], [4,94] and [3, Section 6].

By holomorphy, one can check that if T has a bounded H^∞-calculus, then the same holds for $\mu + T$ where $\mu > 0$. The following is a partial converse of the latter observation which follows from [43, Corollary 5.5.5].

Proposition 2.1 *(H^∞-calculus for shifted operators). Let T be a linear operator. Assume that $\mu_0 + T$ has a bounded H^∞-calculus for some $\mu_0 > 0$. Suppose that $\rho(T) \supseteq \{0\} \cup \{z \in \mathbb{C} : |\arg z| \geq \sigma\}$ for some $\sigma > \omega_{H^\infty}(\mu_0 + T)$. Then T has a bounded H^∞-calculus of angle $\leq \sigma$.*

Proof. Let us begin by proving that T is sectorial of angle $\omega(T) \leq \sigma$. It is enough to show that T is sectorial of angle $\omega(T) \leq \psi$ where $\psi > \sigma$ is arbitrary. The sectoriality of $\mu_0 + T$ implies that $\rho(T) \supseteq -\mu_0 + \complement\overline{\Sigma}_\phi$ for all $\phi \in [\psi, \omega_{H^\infty}(\mu_0 + T))$. Fix such ϕ and note that $\complement\overline{\Sigma}_\psi \setminus (-\mu_0 + \complement\overline{\Sigma}_\phi)$ is compact and it is contained in $\{z \in \mathbb{C} : |\arg z| \geq \sigma\}$. Therefore $\rho(T) \supseteq \complement\overline{\Sigma}_\psi$ and the estimate

$$\sup_{\lambda \in \complement\overline{\Sigma}_\psi} \|\lambda(\lambda - T)^{-1}\|_{\mathscr{L}(X)} < \infty$$

follows from the one of $\mu_0 + T$ using [50, Lemma 10.2.4]. Now, applying [43, Corollary 5.5.5] with $A = T + \mu_0$ and $B = -\mu_0$, the statement follows. $\square$

3. Block operator matrices

3.1. Diagonal dominance

The standing assumption throughout this note is

Assumption 3.1 *(Diagonal dominance). Let X_1, X_2 be Banach spaces, and*

$$\begin{aligned} A&\colon \mathsf{D}(A) \subseteq X_1 \to X_1, \quad &\textit{and}\quad D&\colon \mathsf{D}(D) \subseteq X_2 \to X_2,\\ B&\colon \mathsf{D}(B) \subseteq X_2 \to X_1, \quad &\textit{and}\quad C&\colon \mathsf{D}(C) \subseteq X_1 \to X_2 \end{aligned}$$

be linear operators with domains $\mathsf{D}(A)$, $\mathsf{D}(D)$, $\mathsf{D}(B)$, and $\mathsf{D}(C)$, respectively, where

(1) *$A\colon \mathsf{D}(A) \subseteq X_1 \to X_1$ and $D\colon \mathsf{D}(D) \subseteq X_2 \to X_2$ are closed linear operators with dense domains;*
(2) *$\mathsf{D}(D) \subseteq \mathsf{D}(B)$, $\mathsf{D}(A) \subseteq \mathsf{D}(C)$, and there exist $c_A, c_D, L \geq 0$ such that*

$$\begin{aligned} \|Cx\|_{X_2} &\leq c_A\|Ax\|_{X_1} + L\|x\|_{X_1} \quad \textit{for all } x \in \mathsf{D}(A),\\ \|By\|_{X_1} &\leq c_D\|Dy\|_{X_2} + L\|y\|_{X_2} \quad \textit{for all } y \in \mathsf{D}(D). \end{aligned}$$

With this assumption one sets

$$X \stackrel{\text{def}}{=} X_1 \times X_2 \quad \text{and} \quad \mathsf{D}(\mathcal{A}) \stackrel{\text{def}}{=} \mathsf{D}(A) \times \mathsf{D}(D),$$

compare [111, Equation (2.2.3)], and

$$\mathcal{A} : \mathsf{D}(\mathcal{A}) \subseteq X \to X \quad \text{with} \quad \mathcal{A}\begin{bmatrix} x \\ y \end{bmatrix} \stackrel{\text{def}}{=} \begin{bmatrix} A & B \\ C & D \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}, \quad \text{for all } \begin{bmatrix} x \\ y \end{bmatrix} \in \mathsf{D}(\mathcal{A}).$$

The operator $\mathcal{A}$ is called *diagonally dominant* if Assumption 3.1 holds. Here, compared to [111, Definition 2.2.1], closedness of A and D is assumed for simplicity instead of closability, see also [90, Assumption 2.2] for a similar definition including closedness of $\mathcal{A}$. For $\mathcal{A}$ diagonally dominant we also write $\mathcal{A} = \mathcal{D} + \mathcal{B}$, where

$$\mathcal{D} \stackrel{\text{def}}{=} \begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix} \quad \text{and} \quad \mathcal{B} \stackrel{\text{def}}{=} \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} \quad \text{with} \quad \mathsf{D}(\mathcal{D}) = \mathsf{D}(\mathcal{B}) = \mathsf{D}(\mathcal{A}). \tag{3.1}$$

A diagonally dominant $\mathcal{A}$ is then obtained from the diagonal part $\mathcal{D}$ by the relatively bounded perturbation $\mathcal{B}$. In particular its domain is already determined by the diagonal part $\mathcal{D}$ which determines the "degree of unboundedness" of $\mathcal{A}$ as formulated in [90, Assumption 2.2 f.].

Remark 3.2 *(Boundedness of $B(\lambda - D)^{-1}$ and $C(\lambda - A)^{-1}$).* By Assumption 3.1

$$B(\lambda - D)^{-1} \colon X_2 \to X_1 \quad \text{if} \quad \lambda \in \rho(D) \neq \varnothing$$

is bounded, since for all $y \in X_2$

$$\begin{aligned} \|B(\lambda - D)^{-1}y\|_{X_1} &\leq c_D \|D(\lambda - D)^{-1}y\|_{X_2} + L\|(\lambda - D)^{-1}y\|_{X_2} \\ &\leq c_D(1 + \|\lambda(\lambda - D)^{-1}\|_{\mathscr{L}(X_2)}\|y\|_{X_2} + L\|(\lambda - D)^{-1}\|_{\mathscr{L}(X_2)}\|y\|_{X_2}, \end{aligned}$$

and analogously, if $\lambda \in \rho(A) \neq \varnothing$, then $C(\lambda - A)^{-1} \colon X_1 \to X_2$, is bounded. If D is injective with dense range, and if Assumption 3.1 holds with $L = 0$, then

$$\|BD^{-1}y\|_{X_2} \leq c_D \|y\|_{X_2} \quad \text{for all } y \in \mathsf{R}(D),$$

and since $\mathsf{R}(D) \subseteq X_2$ is dense, BD^{-1} has a unique continuous extension in $\mathscr{L}(X_2, X_1)$, and similarly for CA^{-1} in $\mathscr{L}(X_1, X_2)$ if A is injective with dense range.

3.2. A factorization of diagonally dominant operators

For $\mathcal{A}$ diagonally dominant and $\lambda \in \rho(A) \cap \rho(D) = \rho(\mathcal{D}) \neq \varnothing$, one sets

$$\mathcal{M}(\lambda) \stackrel{\text{def}}{=} (\lambda - \mathcal{A})(\lambda - \mathcal{D})^{-1}. \tag{3.2}$$

The operators defined by

$$\begin{aligned} S_1(\lambda) &\stackrel{\text{def}}{=} (\lambda - A) - B(\lambda - D)^{-1}C \quad \text{for} \quad \lambda \in \rho(D), \\ S_2(\lambda) &\stackrel{\text{def}}{=} (\lambda - D) - C(\lambda - A)^{-1}B \quad \text{for} \quad \lambda \in \rho(A), \end{aligned}$$

are called *Schur-complements* of $\mathcal{A}$, see e.g. [111, Definition 2.2.12], and they serve in the theory of block operator matrices as a substitute to the determinant of 2×2-matrices. Factoring out $(\lambda - A)$ and $(\lambda - D)$ respectively, one obtains the pair of operators

$$\begin{aligned} M_1(\lambda) &\stackrel{\text{def}}{=} \mathbb{1} - B(\lambda - D)^{-1}C(\lambda - A)^{-1} \quad \text{for} \quad \lambda \in \rho(A) \cap \rho(D), \\ M_2(\lambda) &\stackrel{\text{def}}{=} \mathbb{1} - C(\lambda - A)^{-1}B(\lambda - D)^{-1} \quad \text{for} \quad \lambda \in \rho(A) \cap \rho(D). \end{aligned} \tag{3.3}$$

These are the main building blocks in the following factorization, where $M_1(\lambda)$ and $M_2(\lambda)$ encode the interaction of the different blocks of $\mathcal{A}$. These operators have been introduced in the context of the spectral theory of block operator matrices, see [90, Lemma 2.1] for the case of bounded block operator matrices, [91, Theorem 2.4] for the unbounded diagonally dominant triangular case, and also e.g. [111, Proposition 2.3.4 ff.] for the general diagonally dominant case.

Proposition 3.3 *(Factorization of diagonally dominant $\mathcal{A}$). Let Assumption 3.1 be satisfied, and let $\lambda \in \rho(A) \cap \rho(D) \neq \varnothing$. Then the following hold:*

(a) *One has*

$$\lambda - \mathcal{A} = \mathcal{M}(\lambda)(\lambda - \mathcal{D}) \quad \text{and} \quad \mathcal{M}(\lambda) = \begin{bmatrix} \mathbb{1} & -B(\lambda - D)^{-1} \\ -C(\lambda - A)^{-1} & \mathbb{1} \end{bmatrix}.$$

(b) *The operators $M_1(\lambda)$, $M_2(\lambda)$, and $\mathcal{M}(\lambda)$ are bounded on X_1, X_2, and X, respectively.*

(c) *The following are equivalent:*

   (1) $\lambda \in \rho(\mathcal{A})$;
   (2) $\mathcal{M}(\lambda)$ *is boundedly invertible;*
   (3) $M_1(\lambda)$ *is boundedly invertible;*
   (4) $M_2(\lambda)$ *is boundedly invertible.*

(d) *If one of the previous conditions in* (c) *is satisfied, then*

$$(\lambda - \mathcal{A})^{-1} = (\lambda - \mathcal{D})^{-1}\mathcal{M}(\lambda)^{-1}$$

where $\mathcal{M}(\lambda)^{-1}$ has the block matrix representations

$$\begin{aligned} \mathcal{M}(\lambda)^{-1} &= \begin{bmatrix} M_1(\lambda)^{-1} & M_1(\lambda)^{-1}B(\lambda - D)^{-1} \\ M_2(\lambda)^{-1}C(\lambda - A)^{-1} & M_2(\lambda)^{-1} \end{bmatrix} \\ &= \begin{bmatrix} \mathbb{1} & B(\lambda - D)^{-1}M_2(\lambda)^{-1} \\ 0 & M_2(\lambda)^{-1} \end{bmatrix} \begin{bmatrix} \mathbb{1} & 0 \\ C(\lambda - A)^{-1} & \mathbb{1} \end{bmatrix} \\ &= \begin{bmatrix} M_1(\lambda)^{-1} & 0 \\ C(\lambda - A)^{-1}M_1(\lambda)^{-1} & \mathbb{1} \end{bmatrix} \begin{bmatrix} \mathbb{1} & B(\lambda - D)^{-1} \\ 0 & \mathbb{1} \end{bmatrix}. \end{aligned}$$

Proof of Proposition 3.3. The factorization and the block representation in (a) can be verified in a straightforward way on $\mathsf{D}(\mathcal{A}) = \mathsf{D}(\mathcal{D})$ using the block representations of $\lambda - \mathcal{A}$ and $(\lambda - \mathcal{D})^{-1}$.

To prove (b), note that by Remark 3.2 the operators $M_1(\lambda)$ and $M_2(\lambda)$ are bounded if $\mathcal{A}$ is diagonally dominant. Similarly, using the block matrix representation in (a) of $\mathcal{M}(\lambda)$ it follows that $\mathcal{M}(\lambda)$ is bounded.

The statement of (c) is essentially given in [111, Corollary 2.3.5], see also [90, Lemma 2.1] for the case of bounded operators and [91, Theorem 2.4] for block triangular operator matrices. For the sake of completeness the proof is given here. The implication (c1)⇒(c2) follows since for $\lambda \in \rho(\mathcal{A}) \cap \rho(\mathcal{D})$, (a) implies that $\mathcal{M}(\lambda)$ is invertible and

$$\mathcal{M}(\lambda)^{-1} = (\lambda - \mathcal{D})(\lambda - \mathcal{A})^{-1} \tag{3.4}$$

is bounded. Note that by a row reduction one obtains the following factorizations

$$\begin{aligned}\mathcal{M}(\lambda) &= \begin{bmatrix} \mathbb{1} & 0 \\ -C(\lambda - A)^{-1} & \mathbb{1} \end{bmatrix} \begin{bmatrix} \mathbb{1} & -B(\lambda - D)^{-1} \\ 0 & M_2(\lambda) \end{bmatrix} \\ &= \begin{bmatrix} \mathbb{1} & -B(\lambda - D)^{-1} \\ 0 & \mathbb{1} \end{bmatrix} \begin{bmatrix} M_1(\lambda) & 0 \\ -C(\lambda - A)^{-1} & \mathbb{1} \end{bmatrix}.\end{aligned} \tag{3.5}$$

From (3.5) one sees also that (c2), (c3) and (c4) are equivalent. From (a) it follows that if one of the conditions (c2), (c3) or (c4) holds for $\lambda \in \rho(\mathcal{D})$, then $\lambda - \mathcal{A}$ is boundedly invertible and hence $\lambda \in \rho(\mathcal{A})$.

Part (d) follows from (a) and the representation (3.5). □

3.3. Comparison to further factorizations

In [31] also the operators

$$N_1(\lambda) \stackrel{\text{def}}{=} \mathbb{1} - (\lambda - A)^{-1}B(\lambda - D)^{-1}C \quad \text{for} \quad \lambda \in \rho(A) \cap \rho(D),$$
$$N_2(\lambda) \stackrel{\text{def}}{=} \mathbb{1} - (\lambda - D)^{-1}C(\lambda - A)^{-1}B \quad \text{for} \quad \lambda \in \rho(A) \cap \rho(D)$$

have been introduced by Engel which give rise to a factorization of the type

$$\lambda - \mathcal{A} = (\lambda - \mathcal{D})\mathcal{N}(\lambda), \quad \text{where} \quad \mathcal{N}(\lambda) = (\lambda - \mathcal{D})^{-1}(\lambda - \mathcal{A}),$$

and its inverse can be represented in terms of $N_1(\lambda)$ and $N_2(\lambda)$. Both factorizations are equivalent for the setting discussed here, since for $\lambda \in \rho(A) \cap \rho(D)$

$$M_1(\lambda) = (\lambda - A)N_1(\lambda)(\lambda - A)^{-1} \quad \text{and} \quad M_2(\lambda) = (\lambda - D)N_2(\lambda)(\lambda - D)^{-1}.$$

A technical advantage of the factorization $\lambda - \mathcal{A} = \mathcal{M}(\lambda)(\lambda - \mathcal{D})$ used here is that $\mathcal{M}(\lambda)$ is a bounded operator in X.

Furthermore, the *Frobenius-Schur factorization* is a classical factorization for block operator matrices. Under the general assumptions $\mathsf{D}(A) \subseteq \mathsf{D}(C)$, $\rho(A) \neq \varnothing$, and that, for some (and hence for all) $\lambda \in \rho(A)$, the operator $(\lambda - A)^{-1}B$ is bounded on $\mathsf{D}(B)$, one has – assuming here for simplicity that $\mathcal{A}$ is closed – that

$$\lambda - \mathcal{A} = \begin{bmatrix} \mathbb{1} & 0 \\ C(\lambda - A)^{-1} & \mathbb{1} \end{bmatrix} \begin{bmatrix} \lambda - A & 0 \\ 0 & S_2(\lambda) \end{bmatrix} \begin{bmatrix} \mathbb{1} & \overline{(\lambda - A)^{-1}B} \\ 0 & \mathbb{1} \end{bmatrix},$$

compare [111, Theorem 2.2.14]. If $\mathcal{A}$ is diagonally dominant, then $\mathsf{D}(S_2(\lambda)) = \mathsf{D}(D)$ for $\lambda \in \rho(A)$, cf. [111, Rem. 2.2.13]. Analogous statements hold for $S_1(\lambda)$. For $\mathcal{A}$ satisfying Assumption 3.1, $\lambda \in \rho(\mathcal{D})$, and $S_j(\lambda)$ closed for $j \in \{1, 2\}$, one has that bounded invertibility of $M_j(\lambda)$ and $S_j(\lambda)$ are equivalent. On the one hand, if for instance $M_1(\lambda)$ is boundedly invertible, then $S_1(\lambda)^{-1} = (\lambda - A)^{-1}M_1(\lambda)^{-1}$ is bijective and bounded. On the other hand, if $S_1(\lambda)$ is boundedly invertible, then $M_1(\lambda) = S_1(\lambda)(\lambda - A)^{-1}$ is closed and bijective, and hence boundedly invertible. The analogous argument applies to the case $j = 2$.

The factorization in Subsection 3.2 is valid only for diagonally dominant operators with non-empty resolvent set, whereas the Frobenius-Schur decomposition applies to a general class of block operator matrices. The Frobenius-Schur decomposition needs the assumption that $(\lambda - A)^{-1}B$ is bounded on $\mathsf{D}(B)$, and this is quite restrictive for the purpose of the analysis presented below, and it does not hold automatically for diagonally dominant operators as discussed in the following Example 3.4. The condition that $(\lambda - A)^{-1}B$ is bounded on $\mathsf{D}(B)$ is related to the orders of A, D and B. Heuristically, since $B(\lambda - D)^{-1}$ is bounded for $\mathcal{A}$ diagonally dominant the order of B is at most the order of D, and similarly, that $(\lambda - A)^{-1}B$ is bounded implies that the order of B is at most the order of A.

Example 3.4 *(The condition on $(\lambda - A)^{-1}B$).* In $L^p(\mathbb{R}^d) \times L^p(\mathbb{R}^d)$ for $p \in (1, \infty)$

$$\mathcal{A}_\alpha = \begin{bmatrix} -\Delta & (-\Delta)^\alpha \\ 0 & (-\Delta)^\alpha \end{bmatrix} \qquad \text{with } \mathsf{D}(\mathcal{A}_\alpha) = H^{2,p}(\mathbb{R}^d) \times H^{2\alpha,p}(\mathbb{R}^d), \quad \alpha > 0$$

is diagonally dominant for all $\alpha > 0$, but for $\alpha > 1$

$$(\lambda - A)^{-1}B = (\lambda + \Delta)^{-1}(-\Delta)^\alpha, \quad \lambda \in \rho(-\Delta) = \mathbb{C} \setminus [0, \infty)$$

does not extend to a bounded operator on $L^p(\mathbb{R}^d)$. Nevertheless in this situation the factorization given in Subsection 3.2 applies.

4. Sectoriality and $\mathcal{R}$-sectoriality for block operator matrices

In this section we present our main results concerning the sectoriality and $\mathcal{R}$-sectoriality of block operator matrices. Further perturbation results for block operators with smallness conditions are given in Subsection 4.1. The proofs of Theorem 4.1, Corollary 4.4, and Proposition 4.5 will be given in Subsection 4.2 below.

Theorem 4.1 *(Characterization of sectoriality and $\mathcal{R}$-sectoriality). Suppose that Assumption 3.1 with $L = 0$ holds and that*

$$\|\mathcal{D}x\|_X \lesssim \|\mathcal{A}x\|_X \qquad \text{for all } x \in \mathsf{D}(\mathcal{D}) = \mathsf{D}(\mathcal{A}). \tag{4.1}$$

(a) *If A and D are sectorial operators, then for each $\psi \in [\omega(A) \vee \omega(D), \pi)$ the following are equivalent:*

  (1) *$\mathcal{A}$ is sectorial of angle ψ;*
  (2) *$\overline{\mathsf{R}(\mathcal{A})} = X$, and for all $\phi > \psi$ and for one $j \in \{1, 2\}$*

$$M_j(\lambda)^{-1} \in \mathscr{L}(X_j) \text{ for all } \lambda \in \complement\overline{\Sigma_\phi}, \text{ and } \sup\{\|M_j(\lambda)^{-1}\| \colon \lambda \in \complement\overline{\Sigma_\phi}\} < \infty.$$

(b) *If A and D are $\mathcal{R}$-sectorial operators, then for each $\psi \in [\omega_{\mathcal{R}}(A) \vee \omega_{\mathcal{R}}(D), \pi)$ the following are equivalent:*

(1) *$\mathcal{A}$ is $\mathcal{R}$-sectorial of angle ψ;*
(2) *$\overline{\mathsf{R}(\mathcal{A})} = X$, and for all $\phi > \psi$ and for one $j \in \{1, 2\}$*

$$M_j(\lambda)^{-1} \in \mathscr{L}(X_j) \text{ for all } \lambda \in \complement\overline{\Sigma_\phi}, \text{ and } \mathcal{R}(M_j(\lambda)^{-1} \colon \lambda \in \complement\overline{\Sigma_\phi}) < \infty.$$

(c) *Finally, if X is reflexive, then the condition $\overline{\mathsf{R}(\mathcal{A})} = X$ in* (a2) *and* (b2) *can be removed.*

Remark 4.2 *(Optimality of the angle).* If $C = B = 0$, then $\mathcal{A} = \mathcal{D}$ and therefore (in general) the inequalities $\omega(\mathcal{A}) \geq \omega(A) \vee \omega(D) = \omega(\mathcal{D})$ and $\omega_{\mathcal{R}}(\mathcal{A}) \geq \omega_{\mathcal{R}}(A) \vee \omega_{\mathcal{R}}(D) = \omega_{\mathcal{R}}(\mathcal{D})$ cannot be improved.

Remark 4.3 *(Closedness of $\mathcal{A}$).* The closedness of $\mathcal{A}$ does not follow from Assumption 3.1. In fact, it can be characterized by the Schur complements, cf. [111, Theorem 2.2.14], and there seems to be no such characterization by the operators $M_1(\cdot)$ and $M_2(\cdot)$ besides that $\mathcal{A}$ is closed if one of the conditions in Proposition 3.3 (c) holds. However, (4.1) together with Assumption 3.1 implies that

$$\|\mathcal{A}x\|_X + \|x\|_X \eqsim \|\mathcal{D}x\|_X + \|x\|_X \quad \text{for all } x \in \mathsf{D}(\mathcal{A}) = \mathsf{D}(\mathcal{D}),$$

and hence together with the closedness of $\mathcal{D}$, the closedness of $\mathcal{A}$ follows. In particular Proposition 4.5 below implies already the closedness of $\mathcal{A}$. For further conditions ensuring the closedness of $\mathcal{A}$ see also [111, Theorem 2.2.8].

Corollary 4.4 *(Characterization of sectoriality and $\mathcal{R}$-sectoriality for invertible $\mathcal{D}$). Let Assumption 3.1 be satisfied, and assume that A and D are boundedly invertible.*

(a) *If A and D are sectorial operators, then for each $\psi \in [\omega(A) \vee \omega(D), \pi)$ the following are equivalent:*

(1) *$\mathcal{A}$ is an invertible sectorial operator of angle ψ;*
(2) *For all $\phi > \psi$ and for one $j \in \{1, 2\}$*

$$M_j(\lambda)^{-1} \in \mathscr{L}(X_j) \text{ for all } \lambda \in \complement\overline{\Sigma_\phi} \cup \{0\}, \text{ and } \sup\{\|M_j(\lambda)^{-1}\| \colon \lambda \in \complement\overline{\Sigma_\phi}\} < \infty.$$

(b) *If A and D are $\mathcal{R}$-sectorial operators, then for each $\psi \in [\omega_{\mathcal{R}}(A) \vee \omega_{\mathcal{R}}(D), \pi)$ the following are equivalent:*

(1) *$\mathcal{A}$ is an invertible $\mathcal{R}$-sectorial operator of angle ψ;*

(2) *For all* $\phi > \psi$ *and for one* $j \in \{1,2\}$

$$M_j(\lambda)^{-1} \in \mathscr{L}(X_j) \text{ for all } \lambda \in \complement\overline{\Sigma_\phi} \cup \{0\}, \text{ and } \mathcal{R}(M_j(\lambda)^{-1} : \lambda \in \complement\overline{\Sigma_\phi}) < \infty.$$

Paraphrasing the above results one has that sectoriality and $\mathcal{R}$-sectoriality for angle larger than $\omega(A) \vee \omega(D)$ and $\omega_{\mathcal{R}}(A) \vee \omega_{\mathcal{R}}(D)$, respectively, of a block operator matrix $\mathcal{A}$ is solely determined by one of the *bounded* operators $M_j(\lambda)$ in X_j for $j \in \{1,2\}$ defined in (3.3), and by (3.5) one sees that if the condition holds for one of the operators $M_j(\lambda)$, $j \in \{1,2\}$, then it also holds for the other.

In the study of long-time behavior of solutions to nonlinear partial differential equations, see e.g. [96, Section 5.3], the assumption $0 \in \rho(A) \cap \rho(D)$ in Corollary 4.4 is too restrictive, though it simplifies the formulation of the statement considerably, and in fact it would exclude even cases such as the Laplace operator on $\mathbb{R}^d$. To avoid this limitation, we assumed in Theorem 4.1 instead condition (4.1). Next, we give sufficient conditions for (4.1) to hold.

Proposition 4.5 *(Criteria for condition* (4.1)*). Let Assumption 3.1 be satisfied with* $L = 0$*, and assume that* $\overline{\mathsf{R}(A)} = X_1$ *and* $\overline{\mathsf{R}(D)} = X_2$*. Then set*

$$G \stackrel{\text{def}}{=} \overline{CA^{-1}} \in \mathscr{L}(X_1, X_2) \quad \text{and} \quad H \stackrel{\text{def}}{=} \overline{BD^{-1}} \in \mathscr{L}(X_2, X_1), \tag{4.2}$$

and if one of the operators

$$\mathbb{1} - HG \quad \text{and} \quad \mathbb{1} - GH$$

is boundedly invertible, then (4.1) *holds. In particular,* (4.1) *holds if for* $\varepsilon > 0$

$$\sup\{\|B(t+D)^{-1}C(t+A)^{-1}\| : t \in (0,\varepsilon)\} < 1, \text{ or}$$
$$\sup\{\|C(t+A)^{-1}B(t+D)^{-1}\| : t \in (0,\varepsilon)\} < 1.$$

Remark 4.6 *(Density of* $\mathsf{R}(\mathcal{A})$*).* By (4.2) the operator $\mathcal{M}(0) \stackrel{\text{def}}{=} \mathcal{A}\mathcal{D}^{-1}$ extends to a bounded linear operator on X, and a factorization analogous to the one in Proposition 3.3 (d) holds also for $\lambda = 0$. Then the condition that one of the operators $\mathbb{1} - HG$ and $\mathbb{1} - GH$ is boundedly invertible implies that $\overline{\mathcal{M}(0)}$ is boundedly invertible. Thus $\mathsf{R}(\mathcal{A}) = \mathsf{R}(\mathcal{M}(0)\mathcal{D})$ and $\overline{\mathsf{R}(\mathcal{A})} = X$ in case $\overline{\mathsf{R}(\mathcal{D})} = X$.

The next proposition can be seen as a variation of [96, Theorem 4.4.4] and [113, Theorem 4.2] where $\mathcal{R}$-bounds on the relevant operators appear only on subsets

$$\ell_\theta \stackrel{\text{def}}{=} \{re^{i\theta} : r > 0\} \cup \{re^{-i\theta} : r > 0\} \subseteq \complement\overline{\Sigma_\psi} \subseteq \mathbb{C} \quad \text{for } \theta \in (0,\pi) \text{ and } \psi \in (0,\theta).$$

Proposition 4.7. *Let Assumption 3.1 be satisfied and A, D be $\mathcal{R}$-sectorial. Fix $\theta \in (\omega_{\mathcal{R}}(A) \vee \omega_{\mathcal{R}}(D), \pi)$, and $a \in (1, \infty)$. Assume that for one $j \in \{1, 2\}$ and for each $\lambda \in \complement\overline{\Sigma_\theta}$, $M_j(\lambda)$ is boundedly invertible, and*

$$\sup_{\lambda \in \complement\overline{\Sigma_\theta}} \|M_j(\lambda)^{-1}\|_{\mathscr{L}(X_j)} < \infty. \tag{4.3}$$

Then there exists $\xi \in (\omega_{\mathcal{R}}(A) \vee \omega_{\mathcal{R}}(D), \theta)$ for which the following hold.

(1) *$M_j(\lambda)$ is invertible for all $\lambda \in \complement\overline{\Sigma_\xi}$ and $\sup_{\lambda \in \complement\overline{\Sigma_\xi}} \|M_j(\lambda)^{-1}\|_{\mathscr{L}(X_j)} < \infty$.*
(2) *$\mathcal{R}(M_j(\lambda)^{-1} \colon \lambda \in \complement\overline{\Sigma_\xi}) < \infty$ provided*

$$\sup_{\lambda \in \ell_\theta} \mathcal{R}\big(M_j(a^k\lambda)^{-1} \colon k \in \mathbb{Z}\big) < \infty. \tag{4.4}$$

Note that if (4.4) holds for one $j \in \{1, 2\}$, (4.1) holds, and $\overline{\mathsf{R}(\mathcal{A})} = X$, then Theorem 4.1 and Proposition 4.7 ensure that $\mathcal{A}$ is $\mathcal{R}$-sectorial of angle $< \theta$. Here the condition $\overline{\mathsf{R}(\mathcal{A})} = X$ is redundant if X is reflexive, and if $0 \in \rho(\mathcal{A})$ then one can also remove the condition (4.1). In applications one typically checks the stronger condition

$$\mathcal{R}(M_j(\lambda)^{-1} \colon \lambda \in \ell_\theta) < \infty$$

instead of (4.4). Thus to prove $\mathcal{R}$-sectoriality of $\mathcal{A}$ one needs to show $\mathcal{R}$-bounds on ℓ_θ instead of the much larger set $\complement\overline{\Sigma_\theta}$. In applications to parabolic problems, the choice $\theta = \frac{\pi}{2}$ is particularly handy.

Proof of Proposition 4.7. Fix $j \in \{1, 2\}$. Up to the choice of a smaller ξ, (2) follows from (1) and [50, Proposition 8.5.8(2)] applies (up to a rotation) to the holomorphic function $M_j^{-1} : \complement\overline{\Sigma_\xi} \to \mathscr{L}(X_j)$. Therefore, it remains to prove (1).

Let $\psi \in (\theta, \pi)$. Due to our assumptions and (4.3), it remains to show that there exists $\varepsilon > 0$ independent of ψ such that $M_j(\lambda)$ is invertible for all

$$\lambda \in L_\psi(\varepsilon), \quad \text{where } L_\psi(\varepsilon) \overset{\text{def}}{=} \{z \in \mathbb{C} \setminus \{0\} : |\arg(z) - \psi| < \varepsilon\}.$$

To prove the above claim, we employ a Neumann series argument. To this end, let $\delta \in (0, \psi - \omega_{\mathcal{R}}(A) \vee \omega_{\mathcal{R}}(D))$. Note that the rotation map

$$\Psi_\delta \colon L_\psi(\delta) \to \Sigma_\delta \cup (-\Sigma_\delta), \quad \lambda \mapsto e^{-i\psi}\lambda$$

is a bi-holomorphism. Below we prove the claim for $L_\psi(\delta)$ replaced by $L_\psi^{\pm}(\delta) = \Psi_\delta^{-1}(\pm\Sigma_\delta)$, the general case follows similarly. For notational convenience, we let $M_1(\lambda) = \mathbb{1} - T_{e^{-i\psi}\lambda}$ on $L_\psi^+(\delta)$ where

$$T_\lambda : \Sigma_\delta \to \Sigma_\delta \quad \text{with} \quad T_\lambda \overset{\text{def}}{=} B(\lambda e^{i\psi} - D)^{-1} C(\lambda e^{i\psi} - A)^{-1}.$$

Next we prove that $\mathbb{1} - T_\lambda$ is invertible for all $\lambda \in \Sigma_{2\varepsilon}$ where $\varepsilon > 0$ is independent of ψ. Fix $\lambda \in \Sigma_{2\varepsilon}$ and let $t = \Re\lambda$. Let c_θ be the supremum in (4.3). If

$$\|T_t - T_\lambda\|_{\mathscr{L}(X_1)} < c_\theta^{-1}, \tag{4.5}$$

then we can write

$$\begin{aligned}(\mathbb{1} - T_\lambda)^{-1} &= (\mathbb{1} - T_t)^{-1}(\mathbb{1} + (\mathbb{1} - T_t)^{-1}(T_t - T_\lambda))^{-1} \\ &= (\mathbb{1} - T_t)^{-1} \sum_{k \geq 0} (-1)^k (\mathbb{1} - T_t)^{-k}(T_t - T_\lambda)^k.\end{aligned}$$

To check (4.5), one can argue as follows. By the assumption in (4.3) one has $\|(\mathbb{1} - T_\lambda)^{-1}\|_{\mathscr{L}(X_1)} \leq c_\theta$ and, for each $\lambda, \mu \in \complement\overline{\Sigma_\psi}$,

$$\begin{aligned}&B(\lambda - D)^{-1}C(\lambda - A)^{-1} - B(\mu - D)^{-1}C(\mu - A)^{-1} \\ &= B(\lambda - D)^{-1}C[(\lambda - A)^{-1} - (\mu - A)^{-1}] - B[(\lambda - D)^{-1} - (\mu - D)^{-1}]C(\mu - A)^{-1} \\ &= \frac{\lambda - \mu}{\mu}\Big[B(\lambda - D)^{-1}C(\lambda - A)^{-1}[\mu(\mu - A)^{-1}] \\ &\qquad\qquad - B(\lambda - D)^{-1}[\mu(\mu - D)^{-1}]C(\mu - A)^{-1}\Big],\end{aligned}$$

where we used the resolvent identity. Applying the previous with $\mu = t$ and using sectoriality of A and D as well as Assumption 3.1 for $L = 0$ one gets $\|T_t - T_\lambda\|_{\mathscr{L}(X_1)} \leq K\frac{|t-\lambda|}{t}$ where K depends only on the sectoriality constants of A, D on $\complement\overline{\Sigma_\theta}$. Since $\frac{|t-\lambda|}{t} \leq \tan(2\varepsilon)$, (4.5) follows by choosing $\varepsilon \overset{\text{def}}{=} \frac{1}{2}\arcsin(\frac{c_\theta}{K})$ which is independent of ψ. □

4.1. Perturbative type results for block operators

In this subsection, as a consequence of Theorem 4.1 and Corollary 4.4, employing perturbative arguments we show sectoriality and $\mathcal{R}$-sectoriality of block operator matrices if the coupling is small, this holds in particular if C is small enough, while B can be *large*. Therefore, this goes beyond the standard perturbation theory.

Proposition 4.8 *(Sectoriality and $\mathcal{R}$-sectoriality for small couplings). Let Assumption 3.1 be satisfied with $L = 0$.*

(a) *Let A, D be sectorial operators, $\psi \in (\omega(A) \vee \omega(D), \pi)$, and assume that*

$$\begin{aligned}&\sup\{\|B(\lambda - D)^{-1}C(\lambda - A)^{-1}\| : \lambda \in \mathbb{C} \setminus \Sigma_\psi\} < 1 \ \textit{or} \\ &\sup\{\|C(\lambda - A)^{-1}B(\lambda - D)^{-1}\| : \lambda \in \mathbb{C} \setminus \Sigma_\psi\} < 1.\end{aligned}$$

Then $\mathcal{A}$ is sectorial on X of angle $\omega(\mathcal{A}) \leq \psi$.

(b) *Let A, D be $\mathcal{R}$-sectorial operator, $\psi \in (\omega_{\mathcal{R}}(A) \vee \omega_{\mathcal{R}}(D), \pi)$, and*

$$\mathcal{R}(B(\lambda - D)^{-1}C(\lambda - A)^{-1} \colon \lambda \in \mathbb{C} \setminus \Sigma_\psi) < 1 \text{ or}$$
$$\mathcal{R}(C(\lambda - A)^{-1}B(\lambda - D)^{-1} \colon \lambda \in \mathbb{C} \setminus \Sigma_\psi) < 1$$

Then $\mathcal{A}$ is $\mathcal{R}$-sectorial on X of angle $\omega_{\mathcal{R}}(\mathcal{A}) \le \psi$.

Proof. To prove the claim in (b) we check the condition in Theorem 4.1 (b2). First note that by Proposition 4.5 condition (4.1) holds and by Remark 4.6 $\mathsf{R}(\mathcal{A}) \subseteq X$ is dense.

We provide the required estimate for $M_1(\cdot)$, the one for $M_2(\cdot)$ being similar. Having

$$\mathcal{R}(B(\lambda - D)^{-1}C(\lambda - A)^{-1} \colon \lambda \in \mathbb{C} \setminus \Sigma_\psi) < 1, \tag{4.6}$$

by a Neumann series argument one obtains that

$$M_1(\lambda)^{-1} = \sum_{n\ge 0} [B(\lambda - D)^{-1}C(\lambda - A)^{-1}]^n \quad \text{for all } \lambda \in \mathbb{C} \setminus \Sigma_\psi,$$

where the series converges absolutely in $\mathscr{L}(X_1)$. The previous expression implies

$$\mathcal{R}(M_1(\lambda)^{-1} \colon \lambda \in \mathbb{C} \setminus \Sigma_\psi) \le \sum_{n\ge 0} \Big[\mathcal{R}(B(\lambda - D)^{-1}C(\lambda - A)^{-1} \colon \lambda \in \mathbb{C} \setminus \Sigma_\psi)\Big]^n < \infty,$$

where in the last inequality we have used (4.6). The claim in (a) follows analogously replacing $\mathcal{R}$-bounds by norm-bounds. □

As usual, the condition $L = 0$ in Proposition 4.8 can be removed up to a shift. For a sectorial operator T on a Banach space X we set

$$\mathcal{N}^{\mathcal{S}}_\psi(T) \stackrel{\text{def}}{=} \sup\{\|T(\lambda - T)^{-1}\| \colon \lambda \in \complement\overline{\Sigma_\psi}\} \quad \text{for} \quad \psi > \omega(T),$$

and for an $\mathcal{R}$-sectorial operator T

$$\mathcal{N}^{\mathcal{R}}_\psi(T) \stackrel{\text{def}}{=} \mathcal{R}\big(T(\lambda - T)^{-1} \colon \lambda \in \complement\overline{\Sigma_\psi}\,\big) \quad \text{for} \quad \psi > \omega_{\mathcal{R}}(T). \tag{4.7}$$

Corollary 4.9 *(Sectoriality and $\mathcal{R}$-sectoriality for small C). Let Assumption 3.1 be satisfied, and let c_A and c_D be the relative bounds in Assumption 3.1.*

(a) *If A and D are sectorial, then for any $\psi \in (\omega(A) \vee \omega(D), \pi)$ there are*

$$\varepsilon_0(c_D, \mathcal{N}^{\mathcal{S}}_\psi(A), \mathcal{N}^{\mathcal{S}}_\psi(D)) > 0 \quad \text{and} \quad \nu_0(c_D, \mathcal{N}^{\mathcal{S}}_\psi(A), \mathcal{N}^{\mathcal{S}}_\psi(D), L) > 0$$

such that if $\nu > \nu_0$ and $c_A < \varepsilon_0$, then $\nu + \mathcal{A}$ is sectorial on X of angle $\le \psi$.

(b) *If A and D are $\mathcal{R}$-sectorial, then for any $\psi \in (\omega_{\mathcal{R}}(A) \vee \omega_{\mathcal{R}}(D), \pi)$ there are*

$$\varepsilon_0(c_D, \mathcal{N}^{\mathcal{R}}_\psi(A), \mathcal{N}^{\mathcal{R}}_\psi(D)) > 0 \quad \textit{and} \quad \nu_0(c_D, \mathcal{N}^{\mathcal{R}}_\psi(A), \mathcal{N}^{\mathcal{R}}_\psi(D), L) > 0$$

such that if $\nu > \nu_0$ and $c_A < \varepsilon_0$, then $\nu + \mathcal{A}$ is $\mathcal{R}$-sectorial on X of angle $\le \psi$.

In particular, if $C \in \mathscr{L}(\mathsf{D}(A^\gamma), X_2)$ for some $\gamma \in (0,1)$, then there are $\varepsilon_0 > 0$ and $\nu_0 > 0$ such that in the situations of (a) *and* (b) *the conditions on c_A are satisfied.*

Remark 4.10. Let us stress that ε_0 does not depend on $L > 0$ which will become clear from (4.11) below. The conditions in Proposition 4.8 (a) and (b) hold provided that

$$c_A < \frac{1}{c_D \mathcal{N}^{\mathcal{S}}_\psi(A) \mathcal{N}^{\mathcal{S}}_\psi(D)} \quad \text{and} \quad c_A < \frac{1}{c_D \mathcal{N}^{\mathcal{R}}_\psi(A) \mathcal{N}^{\mathcal{R}}_\psi(D)}, \tag{4.8}$$

respectively, that is if $L = 0$, then $\nu = 0$, and the above estimates give lower bounds on ε_0. However, the more general assumptions in Proposition 4.8 as compared to Corollary 4.9 are useful as well. For instance consider a diagonally dominant block operator $\mathcal{A}$ on $X = X_1 \times X_2$ with $X_2 = X_2^1 \times X_2^2$, $B_{12} \colon \mathsf{D}(D_{22}) \subseteq X_2^2 \to X_1$ and $C_{21} \colon \mathsf{D}(A) \subseteq X_1 \to X_2^1$ of the form

$$\mathcal{A} = \left[\begin{array}{c|cc} A & 0 & B_{12} \\ \hline C_{21} & D_{11} & 0 \\ 0 & 0 & D_{22} \end{array}\right] = \begin{bmatrix} A & B \\ C & D \end{bmatrix}.$$

Then c_A can be larger than ε_0, but for $\lambda \in \rho(\mathcal{D})$

$$\begin{aligned} B(\lambda - D)^{-1} C(\lambda - A)^{-1} &= [0 \quad B_{12}] \begin{bmatrix} (\lambda - D_{11})^{-1} & 0 \\ 0 & (\lambda - D_{22})^{-1} \end{bmatrix} \begin{bmatrix} C_{21} \\ 0 \end{bmatrix} (\lambda - A_{11})^{-1} \\ &= 0. \end{aligned}$$

Proof of Corollary 4.9. The idea is to apply Proposition 4.8 by checking the condition (4.8) with A and D replaced by $\nu + A$ and $\nu + D$. Let us begin by noticing that, for all $\nu > 0$,

$$\mathcal{N}^*_\psi(\nu + A) \le \mathcal{N}^*_\psi(A) \quad \text{and} \quad \mathcal{N}^*_\psi(\nu + D) \le \mathcal{N}^*_\psi(D), \quad * \in \{\mathcal{S}, \mathcal{R}\}, \tag{4.9}$$

and by Assumption 3.1 for all $y \in X_2$

$$\|B(\nu + D)^{-1} y\|_{X_1} \le C_B \|D(\nu + D)^{-1} y\|_{X_2} + L \|(\nu + D)^{-1} y\|_{X_1}.$$

In particular, for all $\nu > 0$ and $x \in \mathsf{D}(D)$, there is a $y \in X_2$ with $x = (\nu + D)^{-1} y$ and hence

$$\|Bx\|_{X_1} \le \Big(C_B \mathcal{N}^*_\psi(D) + (1 + \mathcal{N}^*_\psi(D)) \frac{L}{\nu} \Big) \|(\nu + D) x\|_{X_2}. \tag{4.10}$$

Next, set

$$\nu_0' \stackrel{\text{def}}{=} \frac{L}{C_B}\Big(1+\frac{1}{\mathcal{N}_\psi^*(D)}\Big),$$

and note that the constant on the right hand side of (4.10) is less than $2C_B\mathcal{N}_\psi^*(D)$ provided $\nu \geq \nu_0'$. Analogously, for all $\nu > 0$ and $x \in \mathsf{D}(A)$,

$$\|Cx\|_{X_2} \leq \Big(\varepsilon\mathcal{N}_\psi^*(A) + (1+\mathcal{N}_\psi^*(A))\frac{L}{\nu}\Big)\|(\nu+A)x\|_{X_1}.$$

By (4.9), (4.10) and Proposition 4.8, the claim follows provided $\nu_0 \geq \nu_0'$ and $\varepsilon_0 > 0$ satisfy

$$\varepsilon_0\mathcal{N}_\psi^*(A) + (1+\mathcal{N}_\psi^*(A))\frac{L}{\nu_0} \leq \frac{1}{2C_B\mathcal{N}_\psi^*(D)\mathcal{N}_\psi^*(A)\mathcal{N}_\psi^*(D)} \stackrel{\text{def}}{=} R,$$

and a possible choice is given by

$$\varepsilon_0 = \frac{R}{\mathcal{N}_\psi^*(A)}, \quad \text{and} \quad \nu_0 = \nu_0' \vee \Big(\frac{L}{R}(1+\mathcal{N}_\psi^*(A))\Big). \tag{4.11}$$

It remains to prove the last statement. Thus, we assume that $C \in \mathscr{L}(\mathsf{D}(A^\gamma), X_2)$ for some $\gamma \in (0,1)$. Under this assumption by the Young and moment inequality (see e.g. [96, Theorem 3.3.5]) for any $\varepsilon > 0$ there is an $C_{\varepsilon,\gamma} > 0$ such that

$$\|Cx\|_{X_2} \leq \varepsilon\|Ax\|_{X_1} + C_{\varepsilon,\gamma}\|x\|_{X_1} \quad \text{for all } x \in \mathsf{D}(A),$$

and in particular there exists $\varepsilon_0 > 0$ such that $c_D < \varepsilon_0$. □

4.2. Proofs of Theorem 4.1, Corollary 4.4, and Proposition 4.5

Proof of Theorem 4.1. For part (b), let $\psi \in (\omega_{\mathcal{R}}(A) \vee \omega_{\mathcal{R}}(D), \pi)$ be fixed.

(b1)⇒(b2): Fix $\phi > \psi$. Since $\phi > \omega_{\mathcal{R}}(A) \vee \omega_{\mathcal{R}}(D)$, we have $\rho(A) \cap \rho(D) \supseteq \complement\overline{\Sigma_\phi}$. By Proposition 3.3 and our assumptions, we have that the representations in Proposition 3.3(d) hold for all $\lambda \in \complement\overline{\Sigma_\phi}$. We claim that

$$\mathcal{R}(\mathcal{M}(\lambda)^{-1} \colon \lambda \in \complement\overline{\Sigma_\phi}) < \infty \tag{4.12}$$

where $\mathcal{M}(\lambda)$ is defined in (3.2). Note that (4.12) is in fact stronger than (b2).

Next we prove (4.12). Note that by (3.2)

$$\mathcal{M}(\lambda)^{-1} = (\lambda - \mathcal{D})(\lambda - \mathcal{A})^{-1} \quad \text{for all } \lambda \in \complement\overline{\Sigma_\phi}. \tag{4.13}$$

Recall that $\mathcal{D}$ is injective since A, D (and thus $\mathcal{D}$) are sectorial operators. By (4.1) and $\mathsf{D}(\mathcal{D}) = \mathsf{D}(\mathcal{A})$, it follows that $\mathcal{A}$ is injective as well. In particular, $\mathcal{A}^{-1} : \mathsf{R}(A) \to \mathsf{D}(A)$ is well-defined and $\|\mathcal{D}\mathcal{A}^{-1}x\|_X \lesssim \|x\|_X$ for all $x \in \mathsf{R}(\mathcal{A})$. By $\overline{\mathsf{R}(A)} = X$ we infer

$$\overline{\mathcal{D}\mathcal{A}^{-1}} \stackrel{\text{def}}{=} K \in \mathscr{L}(X) \quad \text{and} \quad \mathcal{D} = K\mathcal{A} \text{ on } \mathsf{D}(\mathcal{A}).$$

Combining the latter with (4.13),

$$\mathcal{M}(\lambda)^{-1} = \lambda(\lambda - \mathcal{A})^{-1} - K\mathcal{A}(\lambda - \mathcal{A})^{-1} \text{ for all } \lambda \in \complement\overline{\Sigma_\phi}.$$

By (b1) and the previous identity, we get (4.12), and by Proposition 3.3 (d) the claim follows, where one uses that for $\lambda_0 \in \rho(A) \cap \rho(D)$

$$\mathcal{R}(C(\lambda - A)^{-1} \colon \lambda \in \complement\overline{\Sigma_\phi}) \le \|C(\lambda_0 - A)^{-1}\| \mathcal{R}\big((\lambda_0 - A)(\lambda - A)^{-1} \colon \lambda \in \complement\overline{\Sigma_\phi}\big) < \infty,$$
$$\mathcal{R}(B(\lambda - D)^{-1} \colon \lambda \in \complement\overline{\Sigma_\phi}) \le \|B(\lambda_0 - D)^{-1}\| \mathcal{R}\big((\lambda_0 - D)(\lambda - D)^{-1} \colon \lambda \in \complement\overline{\Sigma_\phi}\big) < \infty.$$

(b2)$\Rightarrow$(b1): Let $\phi > \psi$. Note that $\rho(\mathcal{D}) \supseteq \complement\overline{\Sigma_\phi}$ and one has the representations in Proposition 3.3 (d). Due to the $\mathcal{R}$-sectoriality of A, D and the choice of ϕ, it remains to prove that $\mathcal{R}(\mathcal{M}(\lambda)^{-1} \colon \lambda \in \complement\overline{\Sigma_\phi}) < \infty$. By (b2), the latter holds provided

$$\mathcal{R}(B(\lambda - D)^{-1} \colon \lambda \in \complement\overline{\Sigma_\phi}) < \infty \quad \text{and} \quad \mathcal{R}(C(\lambda - A)^{-1} \colon \lambda \in \complement\overline{\Sigma_\phi}) < \infty.$$

Again, these bounds follow from the $\mathcal{R}$-sectoriality of A and D, the choice of ϕ and Assumption 3.1 with $L = 0$.

Part (a) follows analogously, replacing $\mathcal{R}$-bounds by norm-bounds.

Proof of part (c): If X is reflexive, then $X = \mathsf{N}(T) \oplus \overline{\mathsf{R}(T)}$, for any (pseudo-) sectorial operator T (see [50, Proposition 10.1.9]). Reasoning as in the implication (b1)$\Rightarrow$(b2), (4.1) yields $\mathsf{N}(\mathcal{A}) = \{0\}$ and therefore $\overline{\mathsf{R}(\mathcal{A})} = X$. $\square$

Proof of Corollary 4.4. The claim follows by Theorem 4.1, noticing that if $0 \in \rho(\mathcal{A})$ then (4.1) follows from $\mathsf{D}(\mathcal{A}) = \mathsf{D}(\mathcal{D})$ and that Assumption 3.1 with $L = 0$ holds since $0 \in \rho(A) \cap \rho(D)$. $\square$

Proof of Proposition 4.5. This proof resembles the one of Proposition 3.3. The operators H and G are well-defined and bounded by Remark 3.2. By (4.2), we have

$$\mathcal{A} = \overline{\mathcal{M}(0)}\mathcal{D}, \quad \text{where} \quad \overline{\mathcal{M}(0)} = \begin{bmatrix} \mathbb{1} & H \\ G & \mathbb{1} \end{bmatrix} \tag{4.14}$$

as in Remark 4.6. To fix the idea, we assume that $\mathbb{1} - HG$ is invertible. Reasoning as in the proof of Proposition 3.3, one can check that $\overline{\mathcal{M}(0)}$ is invertible with inverse given by

$$\overline{\mathcal{M}(0)}^{-1} = \begin{bmatrix} (\mathbb{1} - HG)^{-1} & 0 \\ -G(\mathbb{1} - HG)^{-1} & \mathbb{1} \end{bmatrix} \begin{bmatrix} \mathbb{1} & -H \\ 0 & \mathbb{1} \end{bmatrix} \in \mathscr{L}(X).$$

Thus (4.14) gives $\mathcal{D} = \overline{\mathcal{M}(0)}^{-1}\mathcal{A}$ and therefore $\|\mathcal{D}x\|_X \leq \|\overline{\mathcal{M}(0)}^{-1}\|_{\mathscr{L}(X)}\|\mathcal{A}x\|_X$ for $x \in \mathsf{D}(\mathcal{D})$, as desired.

It remains to prove the last assertion, where it suffices to show in the first case that $\sup\{\|B(t+D)^{-1}C(t+A)^{-1}\| : t \in (0,\varepsilon)\} < 1$ implies $\|HG\|_{\mathscr{L}(X_1)} < 1$. This follows since for all $x \in X$

$$\lim_{t\downarrow 0} B(t+D)^{-1}C(t+A)^{-1}x = \lim_{t\downarrow 0} HD(t+D)^{-1}GA(t+A)^{-1}x = HGx$$

by [50, Proposition 10.1.7 (2)], and the other case follows similarly. □

5. H^∞-calculus for block operator matrices

In this section we give some sufficient condition to check the boundedness of the H^∞-calculus for block operator matrices $\mathcal{A}$. These results will be formulated using

Assumption 5.1. *Let Assumption 3.1 be satisfied. Suppose that A and D are sectorial operators.*

(+) *We say that Assumption 5.1(+) holds if there exists $\delta \in (0,1)$ such that*

$$\begin{aligned} &C(\mathsf{D}(A^{1+\delta})) \subseteq \mathsf{D}(D^\delta) \quad \text{and} \quad \|D^\delta Cx\|_{X_2} \lesssim \|A^{1+\delta}x\|_{X_1} \quad \text{for all } x \in \mathsf{D}(A^{1+\delta}), \\ &B(\mathsf{D}(D^{1+\delta})) \subseteq \mathsf{D}(A^\delta) \quad \text{and} \quad \|A^\delta By\|_{X_1} \lesssim \|D^{1+\delta}y\|_{X_2} \quad \text{for all } y \in \mathsf{D}(D^{1+\delta}). \end{aligned}$$

(−) *We say that Assumption 5.1(−) holds if there exists $\delta \in (0,1)$ such that*

$$\begin{aligned} &\mathsf{R}(C) \subseteq \mathsf{R}(D^\delta) \quad \text{and} \quad \|D^{-\delta}Cx\|_{X_2} \lesssim \|A^{1-\delta}x\|_{X_1} \quad \text{for all } x \in \mathsf{D}(A), \\ &\mathsf{R}(B) \subseteq \mathsf{R}(A^\delta) \quad \text{and} \quad \|A^{-\delta}By\|_{X_1} \lesssim \|D^{1-\delta}y\|_{X_2} \quad \text{for all } y \in \mathsf{D}(D). \end{aligned}$$

Remark 5.2. If $0 \in \rho(A)$, then $\mathsf{R}(A^\delta) = X_1$ for $\delta > 0$. Thus the condition $\mathsf{R}(B) \subseteq \mathsf{R}(A^\delta)$ in Assumption 5.1(−) becomes redundant in this case. A similar consideration holds for $\mathsf{R}(C) \subseteq \mathsf{R}(D^\delta)$ if $0 \in \rho(D)$.

Having in mind applications to differential operators, Assumption 3.1 implies a relation between the orders of C and A, and the orders of B and D. Assumptions 5.1(±) now impose additional relations between the orders of C, D and A, and the orders of B, A and D. This is illustrated by the following

Example 5.3. Consider in $L^p(\mathbb{R}^d) \times L^p(\mathbb{R}^d)$ for $p \in (1,\infty)$ the operator

$$\mathcal{A}_\alpha = \begin{bmatrix} -\Delta + \mathbb{1} & -(-\Delta+\mathbb{1})^\alpha \\ -\Delta+\mathbb{1} & (-\Delta+\mathbb{1})^\alpha \end{bmatrix} \quad \text{with } \mathsf{D}(\mathcal{A}_\alpha) = H^{2,p}(\mathbb{R}^d) \times H^{2\alpha,p}(\mathbb{R}^d), \quad \alpha > 0.$$

It is easy to see that $\mathcal{A}_\alpha$ is diagonally dominant and satisfies Assumption 3.1 for all $\alpha > 0$. However, in the case (+), using that $\mathsf{D}(A^\gamma) = H^{2\gamma,p}(\mathbb{R}^d)$ and $\mathsf{D}(D^\gamma) = H^{2\alpha\gamma,p}(\mathbb{R}^d)$ for $\gamma \geq 0$,

$$\begin{aligned} C(\mathsf{D}(A^{1+\delta})) &= H^{2\delta,p}(\mathbb{R}^d) \subseteq \mathsf{D}(D^\delta) = H^{2\alpha\delta,p}(\mathbb{R}^d) &&\text{only if } \alpha \geq 1, \\ B(\mathsf{D}(D^{1+\delta})) &= H^{2\alpha\delta,p}(\mathbb{R}^d) \subseteq \mathsf{D}(A^\delta) = H^{2\delta,p}(\mathbb{R}^d) &&\text{only if } \alpha \leq 1, \end{aligned}$$

and hence the inclusions Assumption 5.1(+) holds only for $\alpha = 1$. The same argument also proves that Assumption 5.1(−) holds only for $\alpha = 1$. Hence, Assumption 5.1 requires that A and D have the same orders.

We begin by providing a sufficient condition for the boundedness of the H^∞-calculus of $\mathcal{A}$. For the notion of the *type* of a space we refer to [50, Chapter 7] recapped here in the Appendix A. In particular L^p-spaces with $p \in (1, \infty)$ and their closed subspaces have non-trivial type. Note that spaces of non-trivial type are exactly the K-convex spaces by Pisier's theorem, see e.g. [50, Theorem 7.4.23] and also [50, Section 7.4] for the definition and properties of K-convex spaces.

Theorem 5.4 *(Boundedness of the H^∞-calculus for $\mathcal{R}$-sectorial $\mathcal{A}$). Suppose that X_1 and X_2 are reflexive Banach spaces with non-trivial type. Let Assumption 3.1 with $L = 0$, estimate (4.1), and Assumption 5.1 (+) and (−) be satisfied. Then the following implication holds:*

If A and D have a bounded H^∞-calculus of angle $\omega_{H^\infty}(A)$ and $\omega_{H^\infty}(D)$, respectively, and $\mathcal{A}$ is $\mathcal{R}$-sectorial on X with angle $\omega_{\mathcal{R}}(\mathcal{A})$, then $\mathcal{A}$ has a bounded H^∞-calculus of angle $\omega_{H^\infty}(\mathcal{A}) = \omega_{\mathcal{R}}(\mathcal{A})$.

Proof. This follows directly from the more general transference result Theorem A.1 – given in the appendix – applied with $T = \mathcal{D}$ and $S = \mathcal{A}$, where the assumption (A.2) translates into the Assumptions 5.1(±). □

Having only one of the two Assumptions 5.1 (+) and (−), one can still prove the boundedness of the H^∞-calculus, where as in Proposition 4.8, we only require the coupling to be small. The proofs of Theorem 5.5 and Theorem 5.6 will be given in Subsection 5.2 below.

Theorem 5.5 *(Boundedness of the H^∞-calculus for small couplings). Let Assumption 3.1 be satisfied with $L = 0$, and assume that Assumption 5.1 (+) or Assumption 5.1 (−) holds. Let A and D have a bounded H^∞-calculus and fix $\psi \in (\omega_{H^\infty}(A) \vee \omega_{H^\infty}(D), \pi)$. Assume that X_1 or X_2 has non-trivial type and*

$$\begin{aligned} c_1^{\mathcal{R}} &\stackrel{\text{def}}{=} \mathcal{R}(B(\lambda - D)^{-1}C(\lambda - A)^{-1} \colon \lambda \in \complement\Sigma_\psi) < 1/K_{X_1} \ \textit{or} \\ c_2^{\mathcal{R}} &\stackrel{\text{def}}{=} \mathcal{R}(C(\lambda - A)^{-1}B(\lambda - D)^{-1} \colon \lambda \in \complement\Sigma_\psi) < 1/K_{X_2}, \end{aligned} \tag{5.1}$$

receptively, where K_{X_j} are the K-convexity constants of X_j for $j \in \{1,2\}$, then $\mathcal{A}$ has a bounded H^∞-calculus on X of angle $\omega_{H^\infty}(\mathcal{A}) < \psi$.

The geometric conditions on X_1, X_2 can be avoided assuming a particular smallness condition on C which allows one to interpolate pairs of operators rather than $\mathcal{R}$-bounded sets. Recall that $\mathcal{N}_\psi^{\mathcal{R}}$ is defined in (4.7).

Theorem 5.6 *(Boundedness of the H^∞-calculus in space of trivial type and small C). Let Assumption 3.1 be satisfied with $L = 0$. Assume that A and D have a bounded H^∞-calculus and that Assumption 5.1(−) or Assumption 5.1(+) holds. Fix $\psi \in (\omega_{H^\infty}(A) \vee \omega_{H^\infty}(D), \pi)$. If for c_A and c_D, the relative bounds in Assumption 3.1, one has*

$$c_A < \frac{1}{c_D \mathcal{N}_\psi^{\mathcal{R}}(A) \mathcal{N}_\psi^{\mathcal{R}}(D)}, \tag{5.2}$$

then $\mathcal{A}$ has a bounded H^∞-calculus on X of angle $\omega_{H^\infty}(\mathcal{A}) < \psi$.

Again, arguing as in the proof of Corollary 4.9, we may allow lower order terms at the expense of a shifting. Note that Corollary 5.7 goes beyond the well-known lower order perturbation theorems for the H^∞-calculus, cf. e.g. [8, Theorem 2.4]. The results presented in this section admit even perturbations of the same order, and even if C is of lower order B can be of the same order as A and D. General perturbations of the same order under additional assumptions on mapping properties in domains of fractional powers of the unperturbed operator are discussed in [24, Theorem 3.2] and [57, Section 5].

Corollary 5.7 *(Boundedness of the H^∞-calculus of $\mathcal{A}$ for small C). Let Assumption 3.1 be satisfied, and assume that Assumption 5.1 (+) or Assumption 5.1 (−) holds. If A and D have a bounded H^∞-calculus, then for any $\psi \in (\omega_{H^\infty}(A) \vee \omega_{H^\infty}(D), \pi)$ there exist*

$$\varepsilon_0 = \varepsilon_0(c_D, \mathcal{N}_\psi(A), \mathcal{N}_\psi(D)) > 0 \quad \text{and} \quad \nu_0 = \nu_0(c_D, \mathcal{N}_\psi(A), \mathcal{N}_\psi(D), L) > 0$$

such that if $c_A < \varepsilon_0$ and $\nu > \nu_0$, then $\nu + \mathcal{A}$ has a bounded H^∞-calculus on X of angle $\omega_{H^\infty}(\mathcal{A}) \leq \psi$. In particular, if $C \in \mathscr{L}(\mathsf{D}(A^\gamma), X_2)$ for some $\gamma \in (0,1)$, then there exists $\varepsilon_0 > 0$ and $\nu_0 > 0$ such that the conditions on c_A are satisfied.

Remark 5.8 *(Assumptions 5.1(±) are sufficient but not necessary).* On the one hand, the Assumptions 5.1(±) cannot be avoided in general even for the triangular case with $C = 0$. A counterexample for the triangular case is constructed by McIntosh and Yagi in the proof of [86, Theorem 3]. On the other hand, there are block operator matrices with bounded H^∞-calculus which violate both Assumptions 5.1(±). Considering for instance a diagonally dominant $\mathcal{A}$ with $A = \mathbb{1}$, then this already implies that C is bounded. Now, in Assumptions 5.1(+) the inclusion $C(\mathsf{D}(A^{1+\delta})) \subseteq \mathsf{D}(D^\delta)$ would be violated for C surjective and D unbounded. In (−) the estimate $\|A^{-\delta} By\|_{X_2} \lesssim \|D^{1-\delta} y\|_{X_2}$ would fail if B is of the order of D. So, considering for instance in $L^p(\mathbb{R}^d) \times L^p(\mathbb{R}^d)$ for $p \in (1,\infty)$ the operator

$$\mathcal{A}_\mu = \begin{bmatrix} \mu + \mathbb{1} & \Delta \\ \mathbb{1} & \mu - \Delta \end{bmatrix} \quad \text{with } \mathsf{D}(\mathcal{A}_\mu) = L^p(\mathbb{R}^d) \times H^{2,p}(\mathbb{R}^d), \quad \mu > 0,$$

then this is diagonally dominant, but it violates both Assumptions 5.1($\pm$). Nevertheless, it is shown in Corollary 7.2 below, that $\mathcal{A}_\mu$ has a bounded H^∞-calculus for $\mu > 0$ sufficiently large.

5.1. Fractional powers

One of the advantages of the H^∞-calculus is that it implies the boundedness of imaginary powers (BIP) and therefore allows for a complete description of $\mathsf{D}(\mathcal{A}^\theta)$ for $\theta \in (0,1)$, cf. e.g. [25, Sections 2.3 and 2.4]. The description of the fractional powers of negative orders is more delicate (although sometimes useful in applications to nonlinear (stochastic) partial differential equations, see e.g. [5]). For further results in this direction we refer to Proposition 5.14 and Theorem 6.9 below.

Proposition 5.9 *(Fractional powers of $\mathcal{A}$ and $\mathcal{D}$). Let Assumption 3.1 be satisfied. Suppose that $\mathcal{A}$ and $\mathcal{D}$ have a bounded H^∞-calculus and that (4.1) holds. Then for all $\theta \in (0,1)$*

$$\mathsf{D}(\mathcal{A}^\theta) = \mathsf{D}(\mathcal{D}^\theta) \quad \textit{and} \quad \|\mathcal{D}^\theta x\|_X \eqsim \|\mathcal{A}^\theta x\|_X \quad \textit{for all } x \in \mathsf{D}(\mathcal{D}^\theta).$$

In particular $\dot{\mathsf{D}}(\mathcal{A}^\theta) = \dot{\mathsf{D}}(\mathcal{D}^\theta)$ for all $\theta \in (0,1)$.

Proof. Since $\mathcal{A}$ has a bounded H^∞-calculus and $\mathsf{D}(\mathcal{D}) = \mathsf{D}(\mathcal{A})$, we have $\mathsf{D}(\mathcal{A}^\theta) = \mathsf{D}(\mathcal{D}^\theta)$ and $\dot{\mathsf{D}}(\mathcal{A}^\theta) = \dot{\mathsf{D}}(\mathcal{D}^\theta)$ for all $\theta \in [0,1]$, compare [57, Proposition 2.2]. Here, in the second identification we have also used that (4.1) holds. The remaining estimate follows from these identifications and the definition of these spaces in (2.3). □

5.2. Proofs of Theorem 5.5 and Theorem 5.6

We begin by proving the following key lemma.

Lemma 5.10. *Assume that operators S and T have a bounded H^∞-calculus on Banach spaces E and F, respectively. Fix $\sigma > \omega_{H^\infty}(S) \vee \omega_{H^\infty}(T)$ and let $\mathcal{J} : \mathbb{C} \setminus \Sigma_\sigma \to \mathscr{J}$ be a strongly continuous map where $\mathscr{J} \subseteq \mathscr{L}(E,F)$ is assumed to be an $\mathcal{R}$-bounded set. Then, for each $f \in H_0^\infty(\Sigma_\sigma)$,*

$$\Big\| \int_\Gamma f(\lambda) T^\delta (\lambda - T)^{-1} \mathcal{J}(\lambda) S^{1-\delta} (\lambda - S)^{-1} \,\mathrm{d}\lambda \Big\|_{\mathscr{L}(E,F)} \lesssim \mathcal{R}(\mathscr{J}) \|f\|_{H^\infty(\Sigma_\sigma)},$$

where $\delta \in (0,1)$ and $\Gamma = \partial \Sigma_\theta$ with $\theta \in (\omega_{H^\infty}(S) \vee \omega_{H^\infty}(T), \sigma)$.

The proof of Lemma 5.10 is based on standard randomization techniques, see e.g. [96, Theorem 4.5.6]. For the reader's convenience, we provide some details.

Proof. Set $\Gamma^{\pm} \stackrel{\text{def}}{=} \{re^{\pm i\theta} : r > 0\}$, then $\Gamma = \Gamma^{+} \cup \Gamma^{-}$, and we prove the claimed estimate for Γ replaced by Γ^{+}, the one for Γ^{-} follows similarly. For notational convenience, we set $\mathcal{J}_f(\lambda) \stackrel{\text{def}}{=} f(\lambda)\mathcal{J}(\lambda)$. Note that

$$
\begin{aligned}
&\int_{\Gamma^+} f(\lambda) T^{\delta}(\lambda - T)^{-1}\mathcal{J}(\lambda) S^{1-\delta}(\lambda - S)^{-1}\,\mathrm{d}\lambda \\
&= \lim_{N\uparrow\infty} \sum_{j=-N}^{N-1} \int_{2^j}^{2^{j+1}} T^{\delta}(re^{i\theta} - T)^{-1}\mathcal{J}_f(re^{i\theta}) S^{1-\delta}(re^{i\theta} - S)^{-1}\,\mathrm{d}r \\
&= \lim_{N\uparrow\infty} \sum_{j=-N}^{N-1} \int_{2^j}^{2^{j+1}} \Big(\frac{T}{r}\Big)^{\delta}\Big(e^{i\theta} - \frac{T}{r}\Big)^{-1}\mathcal{J}_f(e^{i\theta} r)\Big(\frac{S}{r}\Big)^{1-\delta}\Big(e^{i\theta} - \frac{S}{r}\Big)^{-1}\frac{\mathrm{d}r}{r} \\
&= \lim_{N\uparrow\infty} \int_{1}^{2} \sum_{j=-N}^{N-1} \Big(\frac{T}{t2^j}\Big)^{\delta}\Big(e^{i\theta} - \frac{T}{t2^j}\Big)^{-1}\mathcal{J}_f(e^{i\theta} t2^j)\Big(\frac{S}{t2^j}\Big)^{1-\delta}\Big(e^{i\theta} - \frac{S}{t2^j}\Big)^{-1}\frac{\mathrm{d}t}{t} \\
&= \lim_{N\uparrow\infty} \int_{1}^{2} \sum_{j=-N}^{N-1} h_{\delta}(2^j tT)\mathcal{J}_f(e^{i\theta}2^j t) h_{1-\delta}(2^j tS)\,\frac{\mathrm{d}t}{t},
\end{aligned}
\tag{5.3}
$$

where $h_{\rho}(z) = z^{\rho}(e^{i\theta} - z)^{-1}$. We estimate the operators appearing in the above integral by standard randomization techniques. To this end, let us note that, by the contraction principle (see e.g. [50, Theorem 6.1.13 (ii)]),

$$
\mathcal{R}(\mathcal{J}_f(re^{i\theta}) : r > 0) \leq \tfrac{\pi}{2}\|f\|_{H^{\infty}(\Sigma_{\sigma})}\mathcal{R}(\mathscr{J}). \tag{5.4}
$$

Since each S and T has a bounded H^{∞}-calculus and $h_{\rho}(z) = z^{\rho}(e^{i\theta} - z)^{-1}$ defines a function in $H_0^{\infty}(\Sigma_{\sigma'})$ for all $\rho \in (0,1)$ and $\sigma' \in (\omega_{H^{\infty}}(S) \vee \omega_{H^{\infty}}(T), \theta)$, by [50, Proposition 10.2.20 and Lemma 10.3.8(1)] there exists a constant C such that for all finite collections of scalars $(\alpha_k)_{k\in\mathbb{Z}}, (\beta_k)_{k\in\mathbb{Z}}$ such that $|\alpha_k|, |\beta_k| \leq 1$,

$$
\sup_{t>0}\Big(\Big\|\sum_{j=-N}^{N-1}\beta_k h_{\delta}(2^j tT^*)\Big\|_{\mathscr{L}(F^*)} + \Big\|\sum_{j=-N}^{N-1}\alpha_k h_{1-\delta}(2^j tS)\Big\|_{\mathscr{L}(E)}\Big) \leq C. \tag{5.5}
$$

Here C is a constant independent of t and N (see also below Definition 10.2.12 and Proposition H.2.3 in [50]). Let $(\varepsilon_j)_{j\in\mathbb{Z}}$ be a Rademacher sequence, see Subsection 2.4. Then, for any $x \in E$ and $y^* \in F^*$ we have

$$
\begin{aligned}
&\Big|\sum_{j=-N}^{N-1}\langle y^*, h_\delta(2^j tT)\mathcal{J}_f(e^{i\theta}2^j t)h_{1-\delta}(2^j tS)x\rangle\Big| \\
=&\Big|\sum_{j=-N}^{N-1}\langle h_\delta(2^j tT^*)y^*, \mathcal{J}_f(e^{i\theta}2^j t)h_{1-\delta}(2^j tS)x\rangle\Big| \\
\overset{(i)}{=}&\Big|\mathbb{E}\Big\langle\sum_{j=-N}^{N-1}\varepsilon_j h_\delta(2^j tT^*)y^*, \sum_{n=-N}^{N-1}\varepsilon_n\mathcal{J}_f(e^{i\theta}2^n t)h_{1-\delta}(2^n tS)x\Big\rangle\Big| \\
\leq&\Big\|\sum_{j=-N}^{N-1}\varepsilon_j h_\delta(2^j tT^*)y^*\Big\|_{L^2(\Omega;F^*)}\Big\|\sum_{j=-N}^{N-1}\varepsilon_j\mathcal{J}_f(e^{i\theta}2^j t)h_{1-\delta}(2^j tS)x\Big\|_{L^2(\Omega;E)} \\
\overset{(ii)}{\leq}&\tfrac{\pi}{2}C^2\|f\|_{H^\infty(\Sigma_\sigma)}\mathcal{R}(\mathscr{J})\|y^*\|_{F^*}\|x\|_E,
\end{aligned}
$$

where in (i) we used $\mathbb{E}[\varepsilon_j\varepsilon_n]=\delta_{j,n}$, here $\delta_{j,n}$ denotes Kronecker's delta, and in (ii) the Equations (5.4) and (5.5). Hence,

$$
\Big\|\sum_{j=-N}^{N-1}h_\delta(2^j tT)\mathcal{J}_f(e^{i\theta}2^j t)h_{1-\delta}(2^j tS)\Big\|_{\mathscr{L}(E,F)}\leq\tfrac{\pi}{2}C^2\|f\|_{H^\infty(\Sigma_\sigma)}\mathcal{R}(\mathscr{J}).
$$

Combining the latter with (5.3), one gets the desired estimate. □

Proof of Theorem 5.5 in case that Assumption 5.1$(-)$ holds. Let us begin by collecting some useful facts. By Proposition 4.8, $\mathcal{A}$ is $\mathcal{R}$-sectorial of angle $<\psi$. Fix $\phi\in(\omega_{H^\infty}(\mathcal{A}),\psi)$. Then, by Theorem 4.1, for all $|\arg\lambda|>\omega_{\mathcal{R}}(\mathcal{A})$ the resolvent $(\lambda-\mathcal{A})^{-1}$ is given by the factorization in Proposition 3.3 (d) with $M_1(\lambda)$ and $M_2(\lambda)$ as in (3.3).

Next we look at the functional calculus. Fix $f\in H^\infty_0(\Sigma_\phi)$. Consider w.l.o.g. the case where X_1 has non-trivial type and $c_1^{\mathcal{R}}<1/K_{X_1}$, then one has to estimate

$$
I_{11}\overset{\text{def}}{=}\Big\|\int_\Gamma f(\lambda)(\lambda-A)^{-1}M_1(\lambda)^{-1}\mathrm{d}\lambda\Big\|_{\mathscr{L}(X_1)}, \tag{5.6}
$$

$$
I_{21}\overset{\text{def}}{=}\Big\|\int_\Gamma f(\lambda)(\lambda-D)^{-1}C(\lambda-A)^{-1}M_1(\lambda)^{-1}\mathrm{d}\lambda\Big\|_{\mathscr{L}(X_1,X_2)}, \tag{5.7}
$$

$$
I_{12}\overset{\text{def}}{=}\Big\|\int_\Gamma f(\lambda)(\lambda-A)^{-1}M_1(\lambda)^{-1}B(\lambda-D)^{-1}\mathrm{d}\lambda\Big\|_{\mathscr{L}(X_2,X_1)}, \tag{5.8}
$$

$$
I_{22}\overset{\text{def}}{=}\Big\|\int_\Gamma f(\lambda)(\lambda-D)^{-1}\left[\mathbb{1}+C(\lambda-A)^{-1}M_1(\lambda)^{-1}B(\lambda-D)^{-1}\right]\mathrm{d}\lambda\Big\|_{\mathscr{L}(X_2)} \tag{5.9}
$$

by $\lesssim\|f\|_{H^\infty(\Sigma_\psi)}$, where the implicit constants depend only on the $\mathcal{R}$-bound $c_1^{\mathcal{R}}$ in (5.1) and the constants of the H^∞-calculus of A, D. We split the proof into three steps.

Step 1: Assumption 5.1(−) implies that C and B extend uniquely to bounded linear operators

$$C_\delta \in \mathscr{L}(\dot{\mathsf{D}}(A^{1-\delta}), \dot{\mathsf{D}}(D^{-\delta})) \quad \text{and} \quad B_\delta \in \mathscr{L}(\dot{\mathsf{D}}(D^{1-\delta}), \dot{\mathsf{D}}(A^{-\delta})).$$

Reasoning as in the proof of Proposition 5.9, by complex interpolation one has for all $\eta \in [0, \delta]$ that C and B induce uniquely the operators

$$C_\eta \in \mathscr{L}(\dot{\mathsf{D}}(A^{1-\eta}), \dot{\mathsf{D}}(D^{-\eta})) \quad \text{and} \quad B_\eta \in \mathscr{L}(\dot{\mathsf{D}}(D^{1-\eta}), \dot{\mathsf{D}}(A^{-\eta})), \tag{5.10}$$

satisfying $C_\eta x = Cx$ and $B_\eta y = By$ for all $(x, y) \in \mathsf{D}(A) \times \mathsf{D}(D)$. Hence, using [74, Theorem 15.14 b)], we have for all $\eta \in [0, \delta]$, $x \in \mathsf{D}(A) \cap \mathsf{R}(A)$ and $y \in \mathsf{D}(D) \cap \mathsf{R}(D)$,

$$\|D^{-\eta} C A^{\eta-1} x\|_{X_2} \lesssim \|x\|_{X_1} \quad \text{and} \quad \|A^{-\eta} B D^{\eta-1} y\|_{X_1} \lesssim \|y\|_{X_2}.$$

Since $\mathsf{D}(A) \cap \mathsf{R}(A) \hookrightarrow X_1$ and $\mathsf{D}(D) \cap \mathsf{R}(D) \hookrightarrow X_2$ are dense (see e.g. [96, Theorem 3.1.2(iv)]), this ensures that $D^{-\eta} C A^{\eta-1}$ and $A^{-\eta} B D^{\eta-1}$ admit a unique extension to bounded operators, namely

$$G_\eta \stackrel{\text{def}}{=} \overline{A^{-\eta} B D^{\eta-1}} \in \mathscr{L}(X_2, X_1) \quad \text{and} \quad H_\eta \stackrel{\text{def}}{=} \overline{D^{-\eta} C A^{\eta-1}} \in \mathscr{L}(X_1, X_2). \tag{5.11}$$

Step 2: The estimate (5.1) ensures that

$$M_1(\lambda)^{-1} = \sum_{n \geq 0} [B(\lambda - D)^{-1} C(\lambda - A)^{-1}]^n$$

convergences absolutely in $\mathscr{L}(X_1)$, because $K_{X_1} \geq 1$ for any K-convex space, and that $\{M_1(\lambda)^{-1} \colon \lambda \in \mathbb{C} \setminus \Sigma_\psi\}$ is $\mathcal{R}$-bounded. Next we rewrite the series conveniently. Note that, for $\eta \in [0, \delta]$ and on $\mathsf{R}(A^\eta)$,

$$\begin{aligned}
\mathcal{T}(\lambda) &\stackrel{\text{def}}{=} B(\lambda - D)^{-1} C(\lambda - A)^{-1} \\
&= A^\eta (A^{-\eta} B D^{\eta-1}) D^{1-\eta} (\lambda - D)^{-1} D^\eta (D^{-\eta} C A^{\eta-1}) A^{1-\eta} (\lambda - A)^{-1} \\
&= A^\eta G_\eta D(\lambda - D)^{-1} H_\eta A(\lambda - A)^{-1} A^{-\eta} \\
&= A^\eta \mathcal{S}_\eta(\lambda) A^{-\eta},
\end{aligned}$$

where $\mathcal{S}_\eta(\lambda) \stackrel{\text{def}}{=} G_\eta D(\lambda - D)^{-1} H_\eta A(\lambda - A)^{-1} \in \mathscr{L}(X_1)$. Hence $\mathcal{T}(\lambda)$ extends to an operator $\mathcal{T}_\eta(\lambda) \in \mathscr{L}(\dot{\mathsf{D}}(A^{-\eta}))$ and since $\{\mathcal{S}_\eta(\lambda) \colon \lambda \in \mathbb{C} \setminus \Sigma_\psi\}$ is $\mathcal{R}$-bounded in $\mathscr{L}(X_1)$, also

$$\{\mathcal{T}_\eta(\lambda) \colon \lambda \in \mathbb{C} \setminus \Sigma_\psi\} \subseteq \mathscr{L}(\dot{\mathsf{D}}(A^{-\eta})) \text{ satisfies } c^{\mathcal{R}}_{\mathcal{T}_\eta} \stackrel{\text{def}}{=} \mathcal{R}\big(\mathcal{T}_\eta(\lambda) \colon \lambda \in \mathbb{C} \setminus \Sigma_\psi\big) < \infty.$$

By [50, Proposition 8.4.4] $\mathcal{R}$-boundedness interpolates assuming K-convexity. Here, by assumption, X_1 is K-convex and since $\dot{\mathsf{D}}(A^{-\eta})$ is isomorphic to X_1 it is also K-convex. Hence by complex interpolation for $\eta = \theta\delta$ and $\theta \in (0,1)$

$$c^{\mathcal{R}}_{\mathcal{T}_{\theta\delta}} \le K^{1-\theta}_{\dot{\mathsf{D}}(A^{-\eta})} K^{1-\theta}_{X_1} (c^{\mathcal{R}}_{\mathcal{T}_0})^{\theta} (c^{\mathcal{R}}_{\mathcal{T}_\delta})^{1-\theta},$$

where $K_{\dot{\mathsf{D}}(A^{-\eta})}$ and K_{X_1} are the K-convexity constants of the spaces $\dot{\mathsf{D}}(A^{-\eta})$ and X_1, respectively. Since $c^{\mathcal{R}}_{\mathcal{T}_0} = c^{\mathcal{R}}_1 < 1/K_{X_1}$, there exists an $\eta \in (0,\delta]$ with $c^{\mathcal{R}}_{\mathcal{T}_\eta} < 1$. In particular for this η the series

$$\mathcal{U}(\lambda) \stackrel{\text{def}}{=} \sum_{n\ge 0} \mathcal{S}_\eta(\lambda)^n = A^{-\eta}\Big(\sum_{n\ge 0} \mathcal{T}_\eta(\lambda)^n\Big)A^{\eta}$$

converges absolutely in $\mathscr{L}(X_1)$ and $\{\mathcal{U}(\lambda)\colon \lambda \in \mathbb{C}\setminus\Sigma_\psi\}$ is $\mathcal{R}$-bounded.

Step 3: Finally we can estimate (5.6)–(5.9). Using the above argument, one can check that

$$(\lambda - A)^{-1}M_1(\lambda)^{-1} = (\lambda - A)^{-1} + A^{\eta}(\lambda - A)^{-1}\mathcal{J}_{11}(\lambda)A^{1-\eta}(\lambda - A)^{-1},$$

where $\{\mathcal{J}_{11}(\lambda)\colon \lambda \in \mathbb{C}\setminus\Sigma_\psi\}$ is $\mathcal{R}$-bounded and

$$\mathcal{J}_{11}(\lambda) \stackrel{\text{def}}{=} \Big(\sum_{n\ge 2} \mathcal{S}_\eta(\lambda)^n\Big)\Big(G_\eta D(\lambda - D)^{-1}H_\eta\Big).$$

Thus, (5.6) follows by Lemma 5.10 applied with $S = T = A$ and the assumption on A.

To show (5.7), we write similar to the above

$$(\lambda - D)^{-1}C(\lambda - A)^{-1}M_1(\lambda)^{-1} = (\lambda - D)^{-1}D^{\eta}\mathcal{J}_{21}(\lambda)A^{1-\eta}(\lambda - A)^{-1},$$

where $\{\mathcal{J}_{21}(\lambda)\colon \lambda \in \mathbb{C}\setminus\Sigma_\psi\}$ is $\mathcal{R}$-bounded for $\mathcal{J}_{21}(\lambda) \stackrel{\text{def}}{=} H_\eta + H_\eta A(\lambda - A)^{-1}\mathcal{J}_{11}(\lambda)$, so that Lemma 5.10 applies with $T = D$ and $S = A$.

For (5.8) we rewrite

$$(\lambda - A)^{-1}M_1(\lambda)^{-1}B(\lambda - D)^{-1} = (\lambda - A)^{-1}A^{\eta}\mathcal{J}_{12}(\lambda)D^{1-\eta}(\lambda - D)^{-1}$$

where $\{\mathcal{J}_{12}(\lambda)\colon \lambda \in \mathbb{C}\setminus\Sigma_\psi\}$ is $\mathcal{R}$-bounded with $\mathcal{J}_{12}(\lambda) \stackrel{\text{def}}{=} G_\eta + \mathcal{J}_{11}(\lambda)A(\lambda - A)^{-1}G_\eta$.

In (5.9) the first addend can be estimated by the assumption on D and for the second we write using the previous computation

$$(\lambda - D)^{-1}C(\lambda - A)^{-1}M_1(\lambda)^{-1}B(\lambda - D)^{-1} = (\lambda - D)^{-1}D^{\eta}\mathcal{J}_{22}(\lambda)D^{1-\eta}(\lambda - D)^{-1},$$

where $\mathcal{J}_{22}(\lambda) \stackrel{\text{def}}{=} H_\eta A(\lambda - A)^{-1}\mathcal{J}_{12}(\lambda)$ and $\{\mathcal{J}_{22}(\lambda)\colon \lambda \in \mathbb{C}\setminus\Sigma_\psi\}$ is $\mathcal{R}$-bounded. $\square$

Proof of Theorem 5.5 for the case of Assumption 5.1(+). Following [57, Corollary 6.5], we prove the claim by employing a shift argument on a scale of spaces and Theorem 5.5 for the already proven case with Assumption 5.1(−).

By Proposition 4.8, $\mathcal{A}$ is $\mathcal{R}$-sectorial with angle $\omega_{\mathcal{R}}(\mathcal{A}) < \psi$. Below we use the notation introduced in Subsection 2.2. In particular $\dot{A}_\delta$ and $\dot{D}_\delta$ are sectorial operators with bounded H^∞-calculus on $\dot{\mathsf{D}}(A^\delta)$ and $\dot{\mathsf{D}}(D^\delta)$, respectively, which follows by similarity from the Definition in (2.5). Moreover, Assumption 5.1(+) ensures that B and C uniquely induce operators

$$\begin{aligned} B_\delta &\colon \dot{\mathsf{D}}(D^{1+\delta}) \to \dot{\mathsf{D}}(A^\delta), &&\text{with} && B_\delta|_{\dot{\mathsf{D}}(D^{1+\delta})\cap \mathsf{D}(D)} = B, \text{ and}\\ C_\delta &\colon \dot{\mathsf{D}}(A^{1+\delta}) \to \dot{\mathsf{D}}(D^\delta), &&\text{with} && C_\delta|_{\dot{\mathsf{D}}(A^{1+\delta})\cap \mathsf{D}(A)} = C. \end{aligned}$$

By density of $\mathsf{D}(A) \subseteq \dot{\mathsf{D}}(A)$ and $\mathsf{D}(D) \subseteq \dot{\mathsf{D}}(D)$, Assumption 3.1 implies that also

$$\|B_0 y\| \le c_D \|y\|_{\dot{\mathsf{D}}(D)} \text{ for all } y \in \dot{\mathsf{D}}(D), \text{ and } \|C_0 x\| \le c_A \|x\|_{\dot{\mathsf{D}}(A)} \text{ for all } x \in \dot{\mathsf{D}}(A)$$

hold. By interpolation for $\eta \in (0, \delta)$

$$\|B_\eta\| \le c_D^{\eta/\delta} \|B_\delta\|^{1-\eta/\delta} \quad \text{and} \quad \|C_\eta\| \le c_A^{\eta/\delta} \|C_\delta\|^{1-\eta/\delta}.$$

So we may assume, at the expense of choosing $\delta \in (0, 1)$ small enough,

$$\|C_\delta x\|_{\dot{\mathsf{D}}(D^\delta)} \le c'_A \|\dot{A}_\delta x\|_{\dot{\mathsf{D}}(A^\delta)}, \quad \text{and} \quad \|B_\delta y\|_{\dot{\mathsf{D}}(A^\delta)} \le c'_D \|\dot{D}_\delta y\|_{\dot{\mathsf{D}}(D^\delta)}, \tag{5.12}$$

for all $x \in \dot{\mathsf{D}}(A^{1+\delta})$ and $y \in \dot{\mathsf{D}}(D^{1+\delta})$, and some $c'_D \ge c_D$ and $c'_A \ge c_A$, respectively.

Consider then the block operator matrix

$$\widehat{\mathcal{A}}_\delta \overset{\text{def}}{=} \begin{bmatrix} \dot{A}_\delta & B_\delta \\ C_\delta & \dot{D}_\delta \end{bmatrix} : \dot{\mathsf{D}}(\mathcal{D}^{1+\delta}) \cap \dot{\mathsf{D}}(\mathcal{D}^\delta) \subseteq \dot{\mathsf{D}}(\mathcal{D}^\delta) \to \dot{\mathsf{D}}(\mathcal{D}^\delta), \quad \dot{\mathsf{D}}(\mathcal{D}^\delta) = \dot{\mathsf{D}}(A^\delta) \times \dot{\mathsf{D}}(D^\delta).$$

By (5.12) $\widehat{\mathcal{A}}_\delta$ satisfies Assumption 3.1 with $L = 0$ for $X_1 = \dot{\mathsf{D}}(A^\delta)$ and $X_2 = \dot{\mathsf{D}}(D^\delta)$. Note that *a priori* $\widehat{\mathcal{A}}_\delta$ is *not* equal to the extrapolated operator $\dot{\mathcal{A}}_\delta$, since the state space for $\widehat{\mathcal{A}}_\delta$ is $\dot{\mathsf{D}}(\mathcal{D}^\delta)$ which may differ from the state space of $\dot{\mathcal{A}}_\delta$, that is $\dot{\mathsf{D}}(\mathcal{A}^\delta)$. The relative bounds in Assumption 3.1 for $\mathcal{A}$ imply for B_δ and C_δ that

$$\begin{aligned} \|(\dot{D}_\delta)^{-\delta} C_\delta x\|_{\dot{\mathsf{D}}(D^\delta)} &\le c_A \|(\dot{A}_\delta)^{1-\delta} x\|_{\dot{\mathsf{D}}(A^\delta)}, \quad x \in \mathsf{D}(A^{1+\delta}),\\ \|(\dot{A}_\delta)^{-\delta} B_\delta y\|_{\dot{\mathsf{D}}(A^\delta)} &\le c_D \|(\dot{D}_\delta)^{1-\delta} y\|_{\dot{\mathsf{D}}(D^\delta)}, \quad y \in \mathsf{D}(D^{1+\delta}), \end{aligned} \tag{5.13}$$

which is the estimate in Assumption 5.1(−) for $\widehat{\mathcal{A}}_\delta$, and the range conditions hold by construction. Consider w.l.o.g. the case where $c_1^{\mathcal{R}} < 1/K_{X_1}$. Then

$$c_{1,\delta}^{\mathcal{R}} \overset{\text{def}}{=} \mathcal{R}(B_\delta(\lambda - \dot{D}_\delta)^{-1} C_\delta (\lambda - \dot{A}_\delta)^{-1} \colon \lambda \in \mathbb{C} \setminus \Sigma_\psi) < \infty \tag{5.14}$$

and by repeating the interpolation argument used in Step 2 in the proof of the case of Assumption 5.1($-$) above, at the expense of choosing $\delta \in (0,1)$ small enough, we may assume that $c^{\mathcal{R}}_{1,\delta} < 1/K_{X_1}$. Hence Theorem 5.5 for the case of Assumption 5.1($-$), which has been proven already, ensures that $\widehat{\mathcal{A}}_\delta$ has a bounded H^∞-calculus of angle $< \psi$.

Next, let us point out the relation between $\widehat{\mathcal{A}}_\delta$ and $\mathcal{A}$. To this end, let us recall that by (5.1), the fact that $K_{X_1} \geq 1$ and the last claim in Lemma 4.5,

$$\mathsf{D}(\mathcal{A}) = \mathsf{D}(\mathcal{D}) \quad \text{and} \quad \dot{\mathsf{D}}(\mathcal{A}) = \dot{\mathsf{D}}(\mathcal{D}). \tag{5.15}$$

By construction we have $\widehat{\mathcal{A}}_\delta|_{\mathsf{D}(\mathcal{D}^{1+\delta})} = \mathcal{A}|_{\mathsf{D}(\mathcal{D}^{1+\delta})}$, and by the Proposition 3.3

$$(\lambda - \widehat{\mathcal{A}}_\delta)^{-1} =$$
$$\begin{bmatrix} (\lambda - \dot{A}_\delta)^{-1} & 0 \\ 0 & (\lambda - \dot{D}_\delta)^{-1} \end{bmatrix} \begin{bmatrix} \widehat{M}_{1,\delta}(\lambda)^{-1} & 0 \\ C_\delta(\lambda - \dot{A}_\delta)^{-1}\widehat{M}_{1,\delta}(\lambda)^{-1} & \mathbb{1} \end{bmatrix} \begin{bmatrix} \mathbb{1} & B_\delta(\lambda - \dot{D}_\delta)^{-1} \\ 0 & \mathbb{1} \end{bmatrix},$$

where $\widehat{M}_{1,\delta}(\lambda) = \mathbb{1} - B_\delta(\lambda - \dot{D}_\delta)^{-1} C_\delta(\lambda - \dot{A}_\delta)^{-1}$ and since $c^{\mathcal{R}}_{1,\delta} < 1/K_{X_1}$ one has

$$\widehat{M}_{1,\delta}(\lambda)^{-1} = \sum_{n \geq 0} [B_\delta(\lambda - \dot{D}_\delta)^{-1} C_\delta(\lambda - \dot{A}_\delta)^{-1}]^n.$$

Using the mapping properties of $\dot{A}_\delta, \dot{D}_\delta, B_\delta, C_\delta$, it follows that $(\lambda - \widehat{\mathcal{A}}_\delta)^{-1}$ restricts to an operator on $\mathsf{D}(\mathcal{D}^\delta) = \dot{\mathsf{D}}(\mathcal{D}^\delta) \cap X$ and $(\lambda - \widehat{\mathcal{A}}_\delta)^{-1}\mathsf{D}(\mathcal{D}^\delta) \subseteq \dot{\mathsf{D}}(\mathcal{D}^{1+\delta}) \cap \mathsf{D}(\mathcal{D}^\delta) = \mathsf{D}(\mathcal{D}^{1+\delta})$ (cf. (2.4)). Hence, for all $x \in \mathsf{D}(\mathcal{D}^\delta)$,

$$\begin{aligned} (\lambda - \mathcal{A})^{-1}x &= (\lambda - \mathcal{A})^{-1}(\lambda - \widehat{\mathcal{A}}_\delta)(\lambda - \widehat{\mathcal{A}}_\delta)^{-1}x \\ &= (\lambda - \mathcal{A})^{-1}(\lambda - \mathcal{A})(\lambda - \widehat{\mathcal{A}}_\delta)^{-1}x = (\lambda - \widehat{\mathcal{A}}_\delta)^{-1}x. \end{aligned}$$

Since $\mathsf{D}(\mathcal{A}) \overset{(5.15)}{=} \mathsf{D}(\mathcal{D}) \hookrightarrow \mathsf{D}(\mathcal{D}^\delta)$, the previous display yields

$$(\lambda - \widehat{\mathcal{A}}_\delta)^{-1}|_{\mathsf{D}(\mathcal{A})} = (\lambda - \mathcal{A})^{-1}|_{\mathsf{D}(\mathcal{A})} \overset{(2.6)}{=} (\lambda - \dot{\mathcal{A}}_1)^{-1}|_{\mathsf{D}(\mathcal{A})} \text{ for all } \lambda \in \complement\overline{\Sigma_\psi}. \tag{5.16}$$

Next, we prove that

$$\dot{\mathsf{D}}(\widehat{\mathcal{A}}_\delta) = \dot{\mathsf{D}}(\mathcal{D}^{1+\delta}), \quad \text{and} \quad \|\widehat{\mathcal{A}}_\delta x\|_{\dot{\mathsf{D}}(\mathcal{D}^\delta)} \eqsim \|x\|_{\dot{\mathsf{D}}(\mathcal{D}^{1+\delta})} \tag{5.17}$$

for all $x \in \dot{\mathsf{D}}(\mathcal{D}^{1+\delta}) \cap \dot{\mathsf{D}}(\mathcal{D}^\delta)$. By Assumption 5.1(+) we have

$$\|\mathcal{D}^\delta \mathcal{A}x\|_X \lesssim \|\mathcal{D}^{1+\delta}x\|_X \quad \text{for all} \quad x \in \mathsf{D}(\mathcal{D}^{1+\delta}).$$

Thus $\dot{\mathsf{D}}(\mathcal{D}^{1+\delta}) \hookrightarrow \dot{\mathsf{D}}(\widehat{\mathcal{A}}_\delta)$. The reverse inclusion follows from the last claim in Lemma 4.5 applied with $\mathcal{A}$ replaced by $\widehat{\mathcal{A}}_\delta$ and $X = \dot{\mathsf{D}}(\mathcal{D}^\delta)$ using that $c^{\mathcal{R}}_{1,\delta} < 1$ by assumption.

Note that $(\widehat{\mathcal{A}}_\delta)^{\boldsymbol{\cdot}}_{1-\delta}$ has a bounded H^∞-calculus on $\dot{\mathsf{D}}((\widehat{\mathcal{A}}_\delta)^{1-\delta})$ since it is similar to $\widehat{\mathcal{A}}_\delta$. By [57, Proposition 2.2] and (5.17),

$$\dot{\mathsf{D}}((\widehat{\mathcal{A}}_\delta)^{1-\delta}) = [\dot{\mathsf{D}}(\mathcal{D}^\delta), \dot{\mathsf{D}}(\mathcal{D}^{1+\delta})]_{1-\delta} = \dot{\mathsf{D}}(\mathcal{D}) \overset{(5.15)}{=} \dot{\mathsf{D}}(\mathcal{A}). \tag{5.18}$$

Since $\mathsf{D}(\mathcal{A}) \supseteq \mathsf{D}(\mathcal{D}^{1+\delta}) \overset{d}{\hookrightarrow} \dot{\mathsf{D}}((\widehat{\mathcal{A}}_\delta)^{1-\delta})$, we also have

$$(\lambda - (\widehat{\mathcal{A}}_\delta)^{\boldsymbol{\cdot}}_{1-\delta})^{-1}|_{\mathsf{D}(\mathcal{A})} \overset{(2.6)}{=} (\lambda - \widehat{\mathcal{A}}_\delta)^{-1}|_{\mathsf{D}(\mathcal{A})}.$$

By $\mathsf{D}(\mathcal{A}) \overset{d}{\hookrightarrow} \dot{\mathsf{D}}(\mathcal{A})$, (5.16) and (5.18), we infer that $\dot{\mathcal{A}}_1$ has a bounded H^∞-calculus. Therefore $\mathcal{A}$ has a bounded H^∞-calculus on X by similarity with $\dot{\mathcal{A}}_1$. $\square$

Proof of Theorem 5.6. The proof is almost analogous to the proof of Theorem 5.5. The necessary modifications are in Step 2 and 3 of the part of the proof with Assumption 5.1(−). There, one has that $C_\delta \in \mathscr{L}(\dot{\mathsf{D}}(A^{1-\delta}), \dot{\mathsf{D}}(D^{-\delta}))$ and $B_\delta \in \mathscr{L}(\dot{\mathsf{D}}(D^{1-\delta}), \dot{\mathsf{D}}(A^{-\delta}))$, and then by interpolation C and B extend for $\eta \in [0, \delta]$ to operators

$$\begin{aligned} C_\eta &\in \mathscr{L}(\dot{\mathsf{D}}(A^{1-\eta}), \dot{\mathsf{D}}(D^{-\eta})) \quad \text{with} \quad \|C_\eta\| \le c_A^{1-\eta/\delta} \|C_\delta\|^{\eta/\delta}, \\ B_\eta &\in \mathscr{L}(\dot{\mathsf{D}}(D^{1-\eta}), \dot{\mathsf{D}}(A^{-\eta})) \quad \text{with} \quad \|B_\eta\| \le c_D^{1-\eta/\delta} \|B_\delta\|^{\eta/\delta}. \end{aligned}$$

By assumption one has for η by choosing $\delta \in (0,1)$ small enough

$$\|D^{-\eta} C_\eta x\|_{X_2} \le c'_A \|A^{1-\eta} x\|_{X_1}, \quad \text{and} \quad \|A^{-\eta} B y\|_{X_1} \le c'_D \|D^{1-\eta} y\|_{X_2}, \tag{5.19}$$

for all $x \in \mathsf{D}(A)$ and $y \in \mathsf{D}(D)$, and some $c'_D \ge c_D$ and $c'_A \ge c_A$, respectively, with

$$c'_A < \frac{1}{c'_D \mathcal{N}^{\mathcal{R}}_\psi(A) \mathcal{N}^{\mathcal{R}}_\psi(D)}.$$

Since $\mathsf{D}(A) \cap \mathsf{R}(A) \hookrightarrow X_1$ and $\mathsf{D}(D) \cap \mathsf{R}(D) \hookrightarrow X_2$ are dense, one has from the above that $D^{-\eta} C A^{\eta-1}$ and $A^{-\eta} B D^{\eta-1}$ admit unique extensions

$$G \overset{\text{def}}{=} \overline{A^{-\eta} B D^{\eta-1}} \in \mathscr{L}(X_2, X_1) \quad \text{and } H \overset{\text{def}}{=} \overline{D^{-\eta} C A^{\eta-1}} \in \mathscr{L}(X_1, X_2),$$

where by (5.19)

$$\|G\|_{\mathscr{L}(X_2, X_1)} \le c'_D \quad \text{and} \quad \|H\|_{\mathscr{L}(X_1, X_2)} \le c'_A. \tag{5.20}$$

By assumption one has absolute convergence in $\mathscr{L}(X_1)$ of

$$M_1(\lambda)^{-1} = \sum_{n \ge 0} [B(\lambda - D)^{-1} C(\lambda - A)^{-1}]^n.$$

Next, we rewrite the series conveniently using that

$$B(\lambda-D)^{-1}C(\lambda-A)^{-1}=A^{\eta}GD(\lambda-D)^{-1}HA^{\eta-1}(\lambda-A)^{-1},$$

and iterating the above argument, one can check that, for any $n\geq 1$,

$$\begin{aligned}&(\lambda-A)^{-1}[B(\lambda-D)^{-1}C(\lambda-A)^{-1}]^{n}\\&=A^{\eta}(\lambda-A)^{-1}[GD(\lambda-D)^{-1}HA(\lambda-A)^{-1}]^{n-1}[GD(\lambda-D)^{-1}H]A^{1-\eta}(\lambda-A)^{-1}.\end{aligned}$$

Therefore

$$(\lambda-A)^{-1}M_1(\lambda)^{-1}=(\lambda-A)^{-1}+A^{\eta}(\lambda-A)^{-1}\mathcal{S}(\lambda)A^{1-\eta}(\lambda-A)^{-1},\tag{5.21}$$

where

$$\mathcal{S}(\lambda)\stackrel{\text{def}}{=}\Big(\sum_{n\geq 0}[GD(\lambda-D)^{-1}HA(\lambda-A)^{-1}]^{n}\Big)[GD(\lambda-D)^{-1}H].$$

Reasoning as in the proof of Proposition 4.8, one can check that (5.20) implies that the above series expansion is absolutely convergent in $\mathscr{L}(X_1)$ and

$$\mathcal{R}(\mathcal{S}(\lambda):\lambda\in\mathbb{C}\setminus\Sigma_{\psi})<\infty.\tag{5.22}$$

Thus, (5.6) follows from (5.21), (5.22) and Lemma 5.10 applied with $S=T=A$. The rest of the proof is as in the one of Theorem 5.5. The modifications above interpolate operators rather than $\mathcal{R}$-bounded sets, and thereby the K-convexity assumption has been avoided. $\square$

5.3. H^{∞}-calculus on Hilbert spaces

In this subsection we investigate the boundedness of the H^{∞}-calculus assuming that X is a Hilbert space, which we emphasize by writing H, H_1 and H_2 instead of X, X_1 and X_2, where the respective scalar products are denoted by $(\cdot|\cdot)_H$, $(\cdot|\cdot)_{H_1}$, and $(\cdot|\cdot)_{H_2}$.

It is known that if $-T$ generates a C_0–semigroup of contractions on a Hilbert space, then T has a bounded H^{∞}-calculus. The latter result is optimal if $-T$ generates an analytic semigroup. Indeed, a sectorial operator T of angle smaller than $\pi/2$ on a Hilbert space has a bounded H^{∞}-calculus if and only if $-T$ generates a contraction semigroup w.r.t. an equivalent Hilbertian norm (see e.g. [50, Theorem 10.4.22]). In light of the Lumer-Phillips Theorem, compare e.g. [50, Corollary G.4.5], we get the following criteria.

Proposition 5.11 *(Generation of C_0–semigroups of contractions). Let Assumption 3.1 be satisfied. Suppose that $-A$ and $-D$ generate C_0–semigroup of contractions. Let $-1\in\rho(\mathcal{A})$. Suppose that there exists $\gamma\in(0,\infty)$ such that*

$$\gamma\,\Re(Bh_2|h_1)_{H_1} + \Re(Ch_1|h_2)_{H_2} \geq -\gamma\,\Re(Ah_1|h_1)_{H_1} - \Re(Dh_2|h_2)_{H_2} \tag{5.23}$$

for all $h = (h_1, h_2) \in \mathsf{D}(\mathcal{A})$. Then $-\mathcal{A}$ generates a C_0–semigroup of contractions on H. In particular, $\mathcal{A}$ has a bounded H^∞-calculus of angle $\omega_{H^\infty}(\mathcal{A}) = \omega(\mathcal{A}) \leq \frac{\pi}{2}$.

Proof. Proposition 5.11 follows from the Lumer-Phillips Theorem because (5.23) implies

$$\Re(\mathcal{A}h|h)_{H,\gamma} \geq 0 \quad \text{for all } h \in H$$

where $(h|k)_{H,\gamma} \stackrel{\text{def}}{=} \gamma(h_1|k_1)_{H_1} + (h_2|k_2)_{H_2}$ for $h = (h_1, h_2)$, $k = (k_1, k_2) \in H$. The statement on the bounded H^∞-calculus follows by [50, Theorem 10.2.24]. □

Remark 5.12. If A and D are dissipative operators, then (5.23) holds provided

$$\gamma\,\Re(By|x)_{H_1} + \Re(Cx|y)_{H_2} = 0 \quad \text{for all } (x, y) \in \mathsf{D}(\mathcal{A}).$$

The block operator matrix $\mathcal{A}$ is *$\mathcal{J}$-symmetric* if $\mathcal{J}\mathcal{A}$ is symmetric, cf. [111, Section 2.6], where

$$\mathcal{J} \stackrel{\text{def}}{=} \begin{bmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{bmatrix}.$$

For $\mathcal{J}$-symmetric operators $\mathcal{A}$ one has $C \subseteq -B^*$, compare [111, Proposition 2.6.1]. Combining this with Remark 5.12 and Proposition 5.11 one gets the following

Corollary 5.13 *(Bounded H^∞-calculus for $\mathcal{J}$-symmetric operators). If $\mathcal{A}$ satisfies Assumption 3.1, $A = A^*$, $D = D^*$, and it is $\mathcal{J}$-symmetric, then $\mathcal{A}$ has a bounded H^∞-calculus with $\omega_{H^\infty}(\mathcal{A}) = \omega(\mathcal{A}) \leq \frac{\pi}{2}$.*

The following proposition describes negative fractional powers of $\mathcal{A}$ in terms of the ones of $\mathcal{D}$. Below in Section 6, it will also be used to extrapolate the H^∞-calculus in an L^q–setting.

Proposition 5.14 *(Negative fractional powers of $\mathcal{A}$ and $\mathcal{D}$). Let Assumption 3.1 be satisfied. Suppose that (4.1) holds. Assume that $-\mathcal{D}$ and $-\mathcal{A}$ generate C_0–semigroups of contraction. Then*

$$\mathsf{R}(\mathcal{A}^\beta) = \mathsf{R}(\mathcal{D}^\beta) \quad \textit{and} \quad \|\mathcal{A}^{-\beta}h\|_H \eqsim \|\mathcal{D}^{-\beta}h\|_H \textit{ for all } h \in \mathsf{R}(\mathcal{D}^\beta)$$

for all $\beta \in [0, \frac{1}{2})$. In particular $\dot{\mathsf{D}}(\mathcal{D}^\gamma) = \dot{\mathsf{D}}(\mathcal{A}^\gamma)$ for all $\gamma \in (-\frac{1}{2}, 0]$.

Proof. Let $\beta \in [0, \frac{1}{2})$. By [60, Theorem 1.1], $\mathsf{D}((\mathcal{A}^*)^\beta) = \mathsf{D}(\mathcal{A}^\beta)$ and $\mathsf{D}((\mathcal{D}^*)^\beta) = \mathsf{D}(\mathcal{D}^\beta)$ with corresponding homogeneous estimates, namely

$$\|\mathcal{D}^\beta h\|_H \eqsim \|(\mathcal{D}^*)^\beta h\|_H \quad \text{and} \quad \|\mathcal{A}^\beta h\|_H \eqsim \|(\mathcal{A}^*)^\beta h\|_H$$

for all $h \in H$. Combining the latter with Proposition 5.9, one has $\mathsf{D}((\mathcal{A}^*)^\beta) = \mathsf{D}((\mathcal{D}^*)^\beta)$ with a corresponding homogeneous estimate $\|\mathcal{A}^\beta h\|_H \eqsim \|(\mathcal{D}^*)^\beta h\|_H$ for all $h \in \mathsf{D}((\mathcal{A}^*)^\beta)$. The claim now follows from [76, Proposition 11]. □

6. Extrapolation for consistent families of block operators

In applications to (stochastic) partial differential equations many relevant operators can be studied not only in one setting but in a range of spaces. These typically depend on several parameters, e.g. integrability powers and/or Sobolev smoothness. Often there are certain special values of such parameters for which $\mathcal{R}$-sectoriality and the boundedness of the H^∞-calculus are easier to investigate than for the general case. In this section we provide results which allow to extrapolate $\mathcal{R}$-sectoriality and the boundedness of the H^∞-calculus knowing the corresponding property for certain values of the parameters.

6.1. Assumptions and consistency

We begin by introducing the concept of a consistent family of operators. Let S and T be sectorial operators on Y and Z, respectively. The operators S, T are said to be *consistent* if (Y, Z) is an interpolation couple, i.e. $Y \hookrightarrow V$ and $Z \hookrightarrow V$ where V is a topological vector space, cf. e.g. [112], and

$$(\lambda - S)^{-1}|_{Y\cap Z} = (\lambda - T)^{-1}|_{Y\cap Z} \qquad \text{for all } \lambda < 0.$$

A family of sectorial operators $(T_\theta)_{\theta\in I}$ on a family of Banach spaces $(Y_\theta)_{\theta\in I}$, where $I \subseteq \mathbb{R}$ is an interval, is *consistent* if T_θ and T_φ are consistent for all $\theta, \varphi \in I$.

Remark 6.1. In case that $Y \hookrightarrow Z$, then S and T are consistent provided $S = T|_{\mathsf{D}(S)}$. This follows by noticing that by $D(S) = \{x \in D(T) \cap Y : Tx \in Y\}$ one has $D(S) \cap D(T) = D(S)$ and $(\lambda - T)^{-1}|_Y = (\lambda - S)^{-1}$.

The following assumption is in force throughout this section.

Assumption 6.2. *Let $I = [\alpha, \beta]$ for $-\infty < \alpha < \beta < \infty$.*

(1) *Let $(X_{i,\theta})_{\theta\in I}$ for $i \in \{1,2\}$ be a family of Banach spaces such that $X_{i,\alpha}$ have non-trivial type for all $\theta \in I$ and $i \in \{1,2\}$. Moreover, for $i \in \{1,2\}$ assume that for the complex interpolation spaces*

$$[X_{i,\theta}, X_{i,\varphi}]_\gamma = X_{i,(1-\gamma)\theta+\gamma\varphi} \quad \textit{for all } \varphi, \theta \in I \textit{ and } \gamma \in (0,1).$$

(2) *For each $\theta \in I$ the following hold:*

- *A_θ and D_θ are sectorial operators on $X_{1,\theta}$ and $X_{2,\theta}$, respectively;*
- *$C_\theta\colon \mathsf{D}(C_\theta) \subseteq X_{1,\theta} \to X_{2,\theta}$ and $B_\theta : \mathsf{D}(B_\theta) \subseteq X_{2,\theta} \to X_{1,\theta}$ are linear operators with $\mathsf{D}(A_\theta) \subseteq \mathsf{D}(C_\theta)$ and $\mathsf{D}(D_\theta) \subseteq \mathsf{D}(B_\theta)$ and there exist $c_{D,\theta}, c_{A,\theta}, L_\theta \geq 0$ such that*

$$\begin{aligned} \|C_\theta x\|_{X_{2,\theta}} &\leq c_{A,\theta}\|A_\theta x\|_{X_{1,\theta}} + L_\theta\|x\|_{X_{1,\theta}} \quad \text{for all } x \in \mathsf{D}(A_\theta), \\ \|B_\theta y\|_{X_{1,\theta}} &\leq c_{D,\theta}\|D_\theta y\|_{X_{2,\theta}} + L_\theta\|y\|_{X_{2,\theta}} \quad \text{for all } y \in \mathsf{D}(D_\theta). \end{aligned}$$

(3) *$(A_\theta)_{\theta\in I}$, $(D_\theta)_{\theta\in I}$ are consistent families of operators.*
(4) *For all $\theta_1, \theta_2 \in I$, $x \in \mathsf{D}(A_{\theta_1}) \cap \mathsf{D}(A_{\theta_2})$ and $y \in \mathsf{D}(D_{\theta_1}) \cap \mathsf{D}(D_{\theta_2})$,*

$$B_{\theta_1} y = B_{\theta_2} y, \qquad C_{\theta_1} x = C_{\theta_2} x.$$

In this subsection, we extend our standard notation as follows: For each $\theta \in I$ set $X_\theta \overset{\text{def}}{=} X_{1,\theta} \times X_{2,\theta}$, $\mathsf{D}(\mathcal{D}_\theta) = \mathsf{D}(\mathcal{A}_\theta) = \mathsf{D}(A_\theta) \times \mathsf{D}(D_\theta)$,

$$\mathcal{D}_\theta \overset{\text{def}}{=} \begin{bmatrix} A_\theta & 0 \\ 0 & D_\theta \end{bmatrix} \quad \text{and} \quad \mathcal{A}_\theta \overset{\text{def}}{=} \begin{bmatrix} A_\theta & B_\theta \\ C_\theta & D_\theta \end{bmatrix}.$$

Remark 6.3. Assumption 6.2(1) says that the families $(X_\theta)_{\theta\in I}$ and $(X_{i,\theta})_{\theta\in I}$, $i \in \{1,2\}$, are complex interpolation scales, compare e.g. [7, Section V.1]. Assumption 6.2(2) implies that $A_\theta, B_\theta, C_\theta$ and D_θ satisfy Assumption 3.1 for all $\theta \in I$, and Assumption 6.2(3) ensures that $(\mathcal{D}_\theta)_{\theta\in I}$ is a consistent family of operators.

Next we provide a sufficient condition for the consistency of $(\mathcal{A}_\theta)_{\theta\in I}$ in terms of the operators extending (3.3), where we set for all $\theta \in (0,1)$ and $\lambda \in \rho(A_\theta) \cap \rho(D_\theta)$

$$\begin{aligned} M_{1,\theta}(\lambda) &\overset{\text{def}}{=} \mathbb{1} - B_\theta(\lambda - D_\theta)^{-1} C_\theta(\lambda - A_\theta)^{-1} \in \mathscr{L}(X_{1,\theta}), \\ M_{2,\theta}(\lambda) &\overset{\text{def}}{=} \mathbb{1} - C_\theta(\lambda - A_\theta)^{-1} B_\theta(\lambda - D_\theta)^{-1} \in \mathscr{L}(X_{2,\theta}). \end{aligned} \tag{6.1}$$

Lemma 6.4 *(Consistency of $(\mathcal{A}_\theta)_{\theta\in I}$). Let Assumptions 6.2 be satisfied. Assume that $\mathcal{A}_\theta$ is sectorial for all $\theta \in I$, and that*

$$\|\mathcal{D}_\theta x\|_{X_\theta} \lesssim_\theta \|\mathcal{A}_\theta x\|_{X_\theta} \quad \text{for all } x \in \mathsf{D}(\mathcal{D}_\theta) \text{ and } \theta \in I. \tag{6.2}$$

Then, for all $\theta_1, \theta_2 \in I$, $\lambda < 0$ and $j \in \{1,2\}$,

$$M_{j,\theta_1}(\lambda)|_{X_{j,\theta_1} \cap X_{j,\theta_2}} = M_{j,\theta_2}(\lambda)|_{X_{j,\theta_1} \cap X_{j,\theta_2}}. \tag{6.3}$$

Moreover the family $(\mathcal{A}_\theta)_{\theta\in I}$ is consistent if one of the following holds:

(1) *For all* $\lambda < 0$, $j \in \{1,2\}$ *and* $\theta_1, \theta_2 \in I$,

$$M_{j,\theta_1}(\lambda)^{-1}|_{X_{j,\theta_1}\cap X_{j,\theta_2}} = M_{j,\theta_2}(\lambda)^{-1}|_{X_{j,\theta_1}\cap X_{j,\theta_2}}.$$

(2) *For one* $j \in \{1,2\}$ *and for all* $\theta_1, \theta_2 \in I$

$$X_{\theta_1,j} \hookrightarrow X_{\theta_2,j} \quad \text{or} \quad X_{\theta_2,j} \hookrightarrow X_{\theta_1,j}.$$

Recall that $M_{j,\theta}(\lambda)$ is invertible for all $\lambda < 0$ in case that $\mathcal{A}_\theta$ is sectorial by Corollary 4.4. In particular, the inverse in (1) is bounded. Condition (2) can be easily checked in case that $X_{i,\theta}$ is an L^p-space on a finite measure space. Note that (2) does *not* follow from Remark 6.1 since the claimed condition holds only for *one* $j \in \{1,2\}$.

Remark 6.5 *(Optimality of* (1) *in Lemma 6.4).* If $(\mathcal{A}_\theta)_{\theta\in I}$ is consistent and

$$A_{\theta_1}|_{\mathsf{D}(A_{\theta_1})\cap\mathsf{D}(A_{\theta_2})} = A_{\theta_2}|_{\mathsf{D}(A_{\theta_1})\cap\mathsf{D}(A_{\theta_2})}, \quad D_{\theta_1}|_{\mathsf{D}(D_{\theta_1})\cap\mathsf{D}(D_{\theta_2})} = D_{\theta_2}|_{\mathsf{D}(D_{\theta_1})\cap\mathsf{D}(D_{\theta_2})},$$

then (1) in Lemma 6.4 holds. The claim follows from the identity $M_{j,\theta}(\lambda)^{-1} = \mathsf{r}_j(\lambda - \mathcal{D}_\theta)(\lambda - \mathcal{A}_\theta)^{-1}\mathsf{e}_j$, compare Proposition 3.3, where $\mathsf{e}_j\colon X_j \to X$ and $\mathsf{r}_j\colon X \to X_j$ are the extension and restriction operators, respectively.

Remark 6.6 *(Extrapolation of* (6.2)*).* The condition in (6.2) is the analogue of (4.1). If (6.2) holds for some $\theta = \theta^\star \in [\alpha,\beta]$ (e.g. if $0 \in \rho(\mathcal{A}_{\theta^\star})$), then it also holds for all $\theta \in [\alpha,\beta] \cap (\theta^\star - \varepsilon, \theta^\star + \varepsilon)$ where $\varepsilon > 0$ is small. This follows from Assumption 6.2, Lemma 4.5 and Sneiberg's lemma, cf. [103]. For more details we refer to the proof of Theorem 6.7 where a similar situation appears.

Proof of Lemma 6.4. We begin by proving (6.3), where it is enough to show that, for all $\lambda < 0$,

$$C_{\theta_1}(\lambda - A_{\theta_1})^{-1}|_{X_{1,\theta_1}\cap X_{1,\theta_2}} = C_{\theta_2}(\lambda - A_{\theta_2})^{-1}|_{X_{1,\theta_1}\cap X_{1,\theta_2}}, \tag{6.4}$$

$$B_{\theta_1}(\lambda - D_{\theta_1})^{-1}|_{X_{2,\theta_1}\cap X_{2,\theta_2}} = B_{\theta_2}(\lambda - D_{\theta_2})^{-1}|_{X_{2,\theta_1}\cap X_{2,\theta_2}}. \tag{6.5}$$

Note that, for all $\lambda < 0$

$$(\lambda - A_{\theta_1})^{-1}x = (\lambda - A_{\theta_2})^{-1}x \in \mathsf{D}(A_{\theta_1}) \cap \mathsf{D}(A_{\theta_2}) \quad \text{for } x \in X_{1,\theta_1} \cap X_{1,\theta_2},$$

by consistency of $(A_\theta)_{\theta\in I}$. Therefore, by Assumption 6.2(4),

$$C_{\theta_1}(\lambda - A_{\theta_1})^{-1}x = C_{\theta_1}(\lambda - A_{\theta_2})^{-1}x = C_{\theta_2}(\lambda - A_{\theta_2})^{-1}x \quad \text{for } x \in X_{1,\theta_1} \cap X_{1,\theta_2}$$

which proves (6.4), and (6.5) follows similarly.

Statement (1) holds due to Proposition 3.3(d) for $\mathcal{A} = \mathcal{A}_\theta$, the claim follows from the assumption, the consistency of $(A_\theta)_{\theta\in I}$, $(D_\theta)_{\theta\in I}$, (6.4), and (6.5).

To prove (2) we check the condition in (1). Fix $\lambda < 0$ and $j \in \{1,2\}$ and $\theta_1, \theta_2 \in I$ with $\theta_1 < \theta_2$. For simplicity assume the first case, that is, $X_{j,\theta_1} \cap X_{j,\theta_2} = X_{j,\theta_1}$, the second case follows similarly. Thus, by (6.3), we have $M_{j,\theta_1}(\lambda) = M_{j,\theta_2}(\lambda)|_{X_{j,\theta_1}}$. Since $\mathcal{A}_{\theta_i}$ is sectorial, $M_{j,\theta_i}(\lambda)$ is invertible by Theorem 4.1 and therefore

$$M_{j,\theta_2}(\lambda)^{-1}|_{X_{j,\theta_1}} = M_{j,\theta_1}(\lambda)^{-1}. \tag{6.6}$$

Again, using that $X_{j,\theta_1} \cap X_{j,\theta_2} = X_{j,\theta_1}$, the claim follows from (6.6) and (1). □

6.2. Extrapolation results

Here we list the main results of this section, the proof will be given in Subsection 6.3 below. We begin by analyzing $\mathcal{R}$-sectoriality.

Theorem 6.7 *(Extrapolation of $\mathcal{R}$-sectoriality). Let Assumption 6.2 with $L_\theta = 0$ be satisfied for all $\theta \in I$. Suppose that the following are satisfied for some $\psi \in (0, 2\pi)$:*

(a) *$\mathcal{D}_\theta$ is $\mathcal{R}$-sectorial of angle $< \psi$ for all $\theta \in (\alpha, \beta)$;*
(b) *There exists a $\gamma \in (\alpha, \beta)$ such that $\mathcal{A}_\gamma$ is $\mathcal{R}$-sectorial of angle $< \psi$;*
(c) *$\overline{\mathsf{R}(\mathcal{A}_\theta)} = X_\theta$ for all $\theta \in I$.*

Then there exists $\varepsilon > 0$ such that

$$\mathcal{A}_\theta \text{ is } \mathcal{R}\text{-sectorial of angle} \leq \psi \text{ for all } |\theta - \gamma| < \varepsilon.$$

Finally, if $X_{j,\alpha}, X_{j,\beta}$ are reflexive for $j \in \{1,2\}$, then condition (c) *can be omitted.*

Remark 6.8. Fackler showed in [34, Corollary 6.4] that in general $\mathcal{R}$-sectoriality does *not* extrapolate. Therefore Theorem 6.7 is somewhat surprising, and it heavily relies on the block structure of the operator $\mathcal{A}_\theta$ and the assumptions on $\mathcal{D}_\theta$.

Next we turn our attention to the H^∞-calculus. In contrast to Theorem 6.7 the following result requires conditions on the angle. Fortunately, this is always the case in applications with $\omega_{H^\infty}(A) \vee \omega_{H^\infty}(D) < \pi/2$ (see [50, Theorem 10.4.22]).

Theorem 6.9 *(Extrapolation of the H^∞-calculus). Let Assumption 6.2 be satisfied. Assume that $X_{i,\alpha}$ is a Hilbert space for $i \in \{1,2\}$. Suppose that $(\mathcal{A}_\theta)_{\theta\in I}$ is a consistent family of sectorial operators and that*

$$\|\mathcal{D}_\theta x\|_{X_\theta} \lesssim \|\mathcal{A}_\theta x\|_{X_\theta} \quad \text{for all } x \in \mathsf{D}(\mathcal{A}_\theta) \text{ and } \theta \in \{\alpha, \beta\}. \tag{6.7}$$

Let the following be satisfied:

(a) $\mathcal{D}_\alpha$ *and* $\mathcal{D}_\beta$ *have a bounded* H^∞*-calculus.*
(b) $-\mathcal{A}_\alpha$ *generates a* C_0*-semigroup of contractions.*
(c) $\mathcal{A}_\beta$ *is* $\mathcal{R}$*-sectorial.*

Then for all $\theta \in [\alpha, \beta)$ *the following hold.*

(1) $\mathcal{A}_\theta$ *has a bounded* H^∞*-calculus of angle*

$$\omega_{H^\infty}(\mathcal{A}_\theta) \leq \Big(\frac{\beta-\theta}{\beta-\alpha}\Big)\omega(\mathcal{A}_\alpha) + \Big(\frac{\theta-\alpha}{\beta-\alpha}\Big)\omega_{\mathcal{R}}(\mathcal{A}_\beta).$$

(2) *for all* $\delta \in (0, \frac{1}{2}\frac{\beta-\theta}{\beta-\alpha})$*, one has* $\mathsf{R}(\mathcal{A}_\theta^\delta) = \mathsf{R}(\mathcal{D}_\theta^\delta)$ *and*

$$\|\mathcal{A}_\theta^{-\delta} x\|_{X_\theta} \eqsim \|\mathcal{D}_\theta^{-\delta} x\|_{X_\theta} \quad \textit{for all } x \in \mathsf{R}(\mathcal{D}_\theta^\delta).$$

In particular $\dot{\mathsf{D}}(\mathcal{A}_\theta^{-\delta}) = \dot{\mathsf{D}}(\mathcal{D}_\theta^{-\delta})$ *for all* $\delta \in (0, \frac{1}{2}\frac{\beta-\theta}{\beta-\alpha})$.

A condition for a block operator matrix to generate a C_0-semigroup of contractions has been discussed in Subsection 5.3. Conditions for the consistency of the family $(\mathcal{A}_\theta)_{\theta\in I}$ have been given in Lemma 6.4. Note that Theorem 6.9(2) complements Propositions 5.9 and 5.14 since it also holds in a non-Hilbertian setting.

Corollary 6.10 *(Extrapolation of* H^∞*-calculus for small* θ*). Let Assumption 6.2 be satisfied with* $L_\theta = 0$*. Suppose that* $(\mathcal{A}_\theta)_{\theta\in I}$ *is a consistent family of sectorial operators and that (6.7) holds. Let the following be satisfied for some* $\gamma \in I$ *and* $\psi \in (0, \frac{\pi}{2}]$*:*

(a) $\mathcal{D}_\theta$ *has a bounded* H^∞*-calculus of angle* $< \psi$ *for all* $\theta \in I$*;*
(b) $X_{i,\gamma}$ *is a Hilbert space for* $i \in \{1, 2\}$*;*
(c) $-\mathcal{A}_\gamma$ *generates a* C_0*-contraction semigroup;*
(d) $\mathcal{A}_\gamma$ *is sectorial of angle* $< \psi$*.*

Then there exists $\varepsilon > 0$ *such that, for all* $|\theta - \gamma| < \varepsilon$*, the operator* $\mathcal{A}_\theta$ *has a bounded* H^∞*-calculus of angle* $\omega_{H^\infty}(\mathcal{A}_\theta) \leq \psi$*.*

Proof. By Theorem 6.7, $\mathcal{A}_\theta$ is $\mathcal{R}$-sectorial for all $|\gamma - \theta| < \varepsilon$ for some $\varepsilon > 0$. Set $\gamma_\pm \stackrel{\text{def}}{=} \gamma \pm \frac{\varepsilon}{2}$. The claim follows by applying Theorem 6.9 to the families $(\mathcal{A}_\theta)_{\theta\in[\gamma,\gamma_+)}$ and $(\mathcal{A}_{-\theta})_{\theta\in[-\gamma,-\gamma_-)}$. □

6.3. Proof of Theorems 6.7 and 6.9

An important ingredient for our proofs here is Sneiberg's lemma, cf. [103] and [106, Theorem 2.3 and Theorem 3.6]. It has been used already in the context of L^p-theory to extrapolate $\mathcal{R}$-sectoriality, see [72].

Proof of Theorem 6.7. Let us begin by noticing that, up to replacing $(\mathcal{A}_\theta)_{\theta\in I}$ by $(\widehat{A}_\eta)_{\eta\in[0,1]}$ with $\widehat{\mathcal{A}}_\eta = \mathcal{A}_{\alpha+\eta(\beta-\alpha)}$, we may assume $I = [0,1]$. Then, one has for $M_{j,\theta}$ defined in (6.1) that for all $\lambda \in \complement\overline{\Sigma_\psi}$, $\theta_1, \theta_2 \in (0,1)$ and $j \in \{1,2\}$,

$$M_{j,\theta_1}(\lambda)x = M_{j,\theta_2}(\lambda)x \quad \text{for all } \; x \in X_{j,\theta_1} \cap X_{j,\theta_2}. \tag{6.8}$$

To see this, recall that, by Lemma 6.4, (6.8) holds for all $\lambda < 0$. Thus, by the holomorphicity of the maps $\complement\overline{\Sigma_\psi} \ni \lambda \mapsto M_{j,\theta_i}(\lambda)x \in X_{j,\theta_1} + X_{j,\theta_2}$ for all $x \in X_{j,\theta_1} \cap X_{j,\theta_2}$ and $i \in \{1,2\}$, (6.8) holds even for all $\lambda \in \complement\overline{\Sigma_\psi}$.

To prove $\mathcal{R}$-sectoriality by Theorem 4.1 it is enough to show the existence of $\varepsilon \in (0,1)$ such that for $j \in \{1,2\}$

$$\text{for all } |\theta - \gamma| < \varepsilon, \; M_{j,\theta}(\lambda) \text{ is invertible and } \mathcal{R}\big(M_{j,\theta}(\lambda)^{-1} : \lambda \in \complement\overline{\Sigma_\psi}\big) < \infty. \tag{6.9}$$

By assumption, the above statement holds for $\theta = \gamma$. For $\theta \neq \gamma$, we use Sneiberg's lemma, see [103], and here we will employ its quantitative version given in [106, Theorem 2.3 and Theorem 3.6] (see also [30, Subsection 1.3.5]). To this end, fix $N \geq 1$, and for a Banach space Y and a Rademacher sequence $(\varepsilon_j)_{j\geq 1}$ we denote by $\varepsilon_N(Y)$ the space Y^N endowed with the norm

$$\|(x_j)_{j=1}^N\|_{\varepsilon_N(Y)} \stackrel{\text{def}}{=} \mathbb{E}\Big\| \sum_{j=1}^N \varepsilon_j x_j \Big\|_Y. \tag{6.10}$$

By [50, Theorem 7.4.16], $\big(\varepsilon_N(X_\theta)\big)_{\theta\in[0,1]}$ is a complex scale, i.e.

$$[\varepsilon_N(X_{\theta_1}), \varepsilon_N(X_{\theta_2})]_\delta = \varepsilon_N(X_{\theta_1(1-\delta)+\delta\theta_2}) \text{ for all } \theta_1, \theta_2 \in [0,1], \delta \in (0,1). \tag{6.11}$$

By K-convexity of X_0 and X_1, and the fact that $X^{(\theta)} = [X_0, X_1]_\theta$, the constants in the above identification are independent of $N \geq 1$.

Next fix $(\lambda_k)_{k=1}^N \subseteq \complement\overline{\Sigma_\psi}$ and $j \in \{1,2\}$. Let

$$\mathcal{T}_j : \varepsilon_N(X_{j,0}) + \varepsilon_N(X_{j,1}) \to \varepsilon_N(X_{j,0}) + \varepsilon_N(X_{j,1})$$

be given by

$$\mathcal{T}_j \mathbf{x} = \big(M_{j,0}(\lambda_k)x_{k,0} + M_{j,1}(\lambda_k)x_{k,1}\big)_{k=1}^N \tag{6.12}$$

where $\mathbf{x} = (x_k)_{k=1}^N$, $x_k = x_{k,0} + x_{k,1}$ and $x_{k,0} \in X_{j,0}$, $x_{k,1} \in X_{j,1}$. By (6.8), the right hand side in (6.12) does not depend on the decomposition $x_k = x_{k,0} + x_{k,1}$ and therefore $\mathcal{T}_j$ is well defined.

Since $\mathcal{T}_j \mathbf{x} = \big(M_j^{(i)}(\lambda_k)x_k^{(i)}\big)_{k=1}^N$ for all $\mathbf{x} = (x_k)_{k=1}^N \in \varepsilon_N(X_j^{(i)})$ and $\mathcal{D}$ is $\mathcal{R}$-sectorial of angle $\leq \psi$,

$$\|\mathcal{T}_j\|_{\mathscr{L}(\varepsilon_N(X_{j,i}))} \leq 1 + c_A c_D \mathcal{N}_\psi^{\mathcal{R}}(A_i)\mathcal{N}_\psi^{\mathcal{R}}(D_i), \qquad \text{for all } i \in \{0,1\}, \tag{6.13}$$

where we also used Assumption 6.2 with $L_\theta = 0$.

By complex interpolation and (6.11), there exists $C > 0$ independent of $N \geq 1$, the choice of $(\lambda_k)_{k=1}^N \subseteq \complement\overline{\Sigma_\psi}$ and $\theta \in (0,1)$ such that

$$\left\| \mathcal{T}_j|_{\varepsilon_N(X_{j,\theta})} \right\|_{\mathscr{L}(\varepsilon(X_{j,\theta})} \leq C. \tag{6.14}$$

By (6.8) and the density of the embedding $X_{j,0} \cap X_{j,1} \hookrightarrow [X_{j,0}, X_{j,1}]_\theta$,

$$\mathcal{T}_j|_{\varepsilon(X_{j,\theta})}\mathbf{x} = \big(M_j^{(\theta)}(\lambda_k)x_k\big)_{k=1}^N \text{ for all } \mathbf{x} = (x_k) \in \varepsilon_N(X_{j,\theta}) \text{ and } \theta \in (0,1). \tag{6.15}$$

Combining (6.15), Theorem 4.1 and the $\mathcal{R}$-sectoriality of $\mathcal{A}_\gamma$, one can check that $\mathcal{T}_j|_{\varepsilon(X_{j,\gamma})}$ is invertible and the norm of its inverse is independent of $N \geq 1$ and the choice of $(\lambda_k)_{k=1}^N \subseteq \complement\overline{\Sigma_\psi}$. Thus, by Sneiberg's lemma and (6.14), there exist $\varepsilon \in (0,1)$, $C' > 0$ independent of $N \geq 1$ and the choice of $(\lambda_k)_{k=1}^N \subseteq \complement\overline{\Sigma_\psi}$ such that $\mathcal{T}_j|_{\varepsilon_N(X_{j,\theta})}$ is invertible for all $|\theta - \gamma| < \varepsilon$ and the norm of its inverse is $\leq C'$.

By (6.15), the arbitrariness of $N \geq 1$ and $(\lambda_k)_{k=1}^N$, one can check that $M_j(\lambda)$ is invertible for all $\lambda \in \complement\overline{\Sigma_\psi}$ and (6.9) holds. $\square$

Proof of Theorem 6.9. As in the proof of Theorem 6.7, we may assume $I = [0,1]$.

(1): Fix $\theta \in (0,1)$. It is enough to show that $\mathcal{A}_\theta$ has a bounded H^∞-calculus. The bound on the angle follows from [57, Corollary 3.9] and the fact that $\omega_{\mathcal{R}}(\mathcal{A}_0) = \omega(\mathcal{D})$ since X_0 is a Hilbert space. The idea is to apply [76, Theorem 1] to $\mathcal{D}_\theta$ and A_θ extending the argument in [76, Corollary 7]. Note that, by the consistency of $(\mathcal{D}_\theta)_{\theta\in I}$ and the fact that $\mathcal{D}_k$ has a bounded H^∞-calculus for $k \in \{0,1\}$, it follows that $\mathcal{D}_\theta$ has a bounded H^∞-calculus as well (cf. [57, Proposition 4.9]).

Let $\varphi(z) = \psi(z) = z(1+z)^{-2}$ and fix $\gamma \in (0, \frac{1}{2})$. By Propositions 5.9, 5.14 and [76, Proposition 3], for all $\ell \in \mathbb{Z}$,

$$\begin{aligned} \sup_{s,t\in[1,2]} \sup_{j\in\mathbb{Z}} \left\| \varphi(s2^{j+\ell}\mathcal{A}_0)\psi(t2^j\mathcal{D}_0) \right\|_{X_0} &\lesssim 2^{-\gamma|\ell|}, \\ \sup_{s,t\in[1,2]} \sup_{j\in\mathbb{Z}} \left\| \psi(t2^j\mathcal{D}_0)\varphi(s2^{j+\ell}\mathcal{A}_0) \right\|_{X_0} &\lesssim 2^{-\gamma|\ell|}, \end{aligned} \tag{6.16}$$

where we also used that X_0 is a Hilbert space.

By [57, Lemma 3.3] and the $\mathcal{R}$-sectoriality of $\mathcal{D}_1$ and $\mathcal{A}_1$ we get

$$\begin{aligned} \sup_{\ell\in\mathbb{Z}} \sup_{s,t\in[1,2]} \mathcal{R}\Big(\varphi(s2^{j+\ell}\mathcal{A}_1)\psi(t2^j\mathcal{D}_1) \colon j \in \mathbb{Z}\Big) &< \infty, \\ \sup_{\ell\in\mathbb{Z}} \sup_{s,t\in[1,2]} \mathcal{R}\Big(\psi(t2^j\mathcal{D}_1)\varphi(s2^{j+\ell}\mathcal{A}_1) \colon j \in \mathbb{Z}\Big) &< \infty. \end{aligned} \tag{6.17}$$

Reasoning as in the proof of Theorem 6.7, by consistency of $(\mathcal{A}_\theta)_{\theta\in I}$ and the Dunford representation of $\phi(\xi A), \varphi(\eta B)$ for $\eta, \xi > 0$ (see (2.7)), one can check that the operators $\varphi(s2^{j+\ell}\mathcal{A}_k)\psi(t2^j\mathcal{D}_k)$ and $\varphi(s2^{j+\ell}\mathcal{A}_k)\psi(t2^j\mathcal{D}_k)$ for $k \in \{0,1\}$ coincide on $X_0 \cap X_1$, and therefore we can interpolate the bounds (6.16)-(6.17) obtaining

$$\sup_{s,t\in[1,2]} \mathcal{R}\Big(\varphi(s2^{j+\ell}\mathcal{A}_\theta)\psi(t2^j\mathcal{D}_\theta)\colon j\in\mathbb{Z}\Big) \lesssim 2^{-\gamma(1-\theta)|\ell|}, \tag{6.18}$$

$$\sup_{s,t\in[1,2]} \mathcal{R}\Big(\psi(t2^j\mathcal{D}_\theta)\varphi(s2^{j+\ell}\mathcal{A}_\theta)\colon j\in\mathbb{Z}\Big) \lesssim 2^{-\gamma(1-\theta)|\ell|}, \tag{6.19}$$

where we have used [50, Proposition 8.4.4] and the fact that X_0, X_1 have non-trivial type and therefore are K-convex due to Pisier's theorem (see e.g. [50, Theorem 7.4.23]). By K-convexity of X_k for $k \in \{0,1\}$, (6.19) and [50, Proposition 8.4.1] we also get

$$\mathcal{R}\Big(\big(\varphi(s2^{j+\ell}\mathcal{A}_\theta)\big)^*\big(\psi(t2^j\mathcal{D}_\theta)\big)^*\colon j\in\mathbb{Z}\Big) \lesssim 2^{-\gamma(1-\theta)|\ell|}. \tag{6.20}$$

Recall that $\mathcal{D}_\theta$ has a bounded H^∞-calculus. By (6.18) and (6.20), $\mathcal{A}_\theta$ has a bounded H^∞-calculus due to [76, Theorem 1].

(2): [76, Theorem 1] and (6.18), (6.20) also yield, for all $\delta \in (0, \gamma(1-\theta))$,

$$\mathsf{R}(\mathcal{A}_\theta^\delta) = \mathsf{R}(\mathcal{D}_\theta^\delta) \quad \text{and} \quad \|\mathcal{A}_\theta^{-\delta}x\|_{X_\theta} \eqsim \|\mathcal{D}_\theta^{-\delta}x\|_{X_\theta} \quad \text{for all } x \in \mathsf{R}(\mathcal{D}_\theta^\delta).$$

The conclusion follows by letting $\gamma \uparrow \frac{1}{2}$ and recalling that $\alpha = 0, \beta = 1$. □

7. Applications

The analysis of quasi- or semi-linear problems in maximal L^p_t-regularity spaces typically comes in two steps: First a linearization is considered for which one has to prove maximal L^p_t-regularity. Then, as a second step, Lipschitz estimates on the non-linearities are needed in certain interpolation spaces to apply Banach's fixed point theorem, see e.g. [5,6,61,80,95,96]. In this section we focus on properties of the linearized problems which are relevant to characterize the relevant interpolation spaces, and which are also helpful for the stability analysis for the non-linear problem.

We begin by deriving some consequences of our results for triangular matrices. These results are used in Subsections 7.2–7.5.

7.1. The block triangular case

In this subsection we consider a block triangular diagonally dominant operator matrix $\mathcal{A}$, this means that $C = 0$, then the statements in Sections 4 and 5 simplify considerably. Using classical results for bounded perturbations, one can include the case $C \in \mathscr{L}(X_1, X_2)$ as well.

Corollary 7.1. *Let*

$$\mathcal{A} = \begin{bmatrix} A & B \\ 0 & D \end{bmatrix}, \quad \textit{with} \quad B \colon \mathsf{D}(D) \subseteq X_2 \to X_1, \quad \textit{and} \quad \|By\|_{X_1} \lesssim \|Dy\|_{X_2} \quad y \in \mathsf{D}(D).$$

(1) *If A and D are sectorial with angles $\omega(A)$ and $\omega(D)$, respectively, then $\mathcal{A}$ is sectorial with angle $\omega(\mathcal{A}) \leq \omega(A) \vee \omega(D)$.*
(2) *If A and D are $\mathcal{R}$-sectorial with angles $\omega_{\mathcal{R}}(A)$ and $\omega_{\mathcal{R}}(D)$, respectively, then $\mathcal{A}$ is $\mathcal{R}$-sectorial with $\mathcal{R}$-sectoriality angle $\omega_{\mathcal{R}}(\mathcal{A}) \leq \omega_{\mathcal{R}}(A) \vee \omega_{\mathcal{R}}(D)$.*
(3) *If A and D have a bounded H^∞-calculus of angle $\omega_{H^\infty}(A)$ and $\omega_{H^\infty}(D)$, respectively, and there exist $\delta > 0$, such that*

$$\mathsf{R}(B) \subseteq \mathsf{R}(A^\delta) \textit{ and } \|A^{-\delta}By\|_{X_1} \lesssim \|D^{1-\delta}y\|_{X_2} \quad \textit{for all } x \in \mathsf{D}(D), \quad \textit{or}$$
$$B(\mathsf{D}(D^{1+\delta})) \subseteq \mathsf{D}(A^\delta) \textit{ and } \|A^\delta By\|_{X_1} \lesssim \|D^{1+\delta}y\|_{X_2} \quad \textit{for all } y \in \mathsf{D}(D^{1+\delta}).$$

Then $\mathcal{A}$ has a bounded H^∞-calculus on X of angle $\leq \omega_{H^\infty}(A) \vee \omega_{H^\infty}(D)$.

Proof. In the situation of Corollary 7.1 Assumption 3.1 holds with $L = 0$. Statements (1) and (2) follow from Proposition 4.8, and statement (3) follows from Theorem 5.6, where one has in each case $\varepsilon = 0$. $\square$

The diagonally dominant case with $A = 0$ has also a particular structure, that is,

$$\mathcal{A} = \begin{bmatrix} 0 & B \\ C & D \end{bmatrix},$$

where by a bounded perturbation argument one can include also $A \in \mathscr{L}(X_1)$. Applications for this case are the artificial Stokes system in Subsection 7.4 and second order Cauchy problems with strong damping in Subsection 7.5.

Corollary 7.2. *Let A be bounded, D be a sectorial operator, and $\mathcal{A}$ satisfy Assumption 3.1. Then the following hold.*

(1) *For all $\psi \in (\omega(D), \pi)$, there exists $\nu \geq 0$ such that $\nu + \mathcal{A}$ is sectorial with angle $\omega(\nu + \mathcal{A}) \leq \psi$.*
(2) *If D is $\mathcal{R}$-sectorial (resp. has a bounded H^∞-calculus), then* (1) *holds with $\omega(\nu + \mathcal{A})$ replaced by $\omega_{\mathcal{R}}(\nu + \mathcal{A})$ (resp. $\omega_{H^\infty}(\nu + \mathcal{A})$).*

Remark 7.3. The condition $A \in \mathscr{L}(X_1)$ together with $\mathcal{A}$ diagonally dominant implies already that $C \colon \mathsf{D}(A) = X_1 \to X_2$ is bounded, while B can be unbounded. Note that, comparing Corollary 7.2(2) with ω replaced by ω_{H^∞} and the results in Section 5, then one observes that for $\mathcal{A}'$ below Assumptions 5.1(+) trivially holds while $\mathcal{A}$ can violate both Assumptions 5.1($\pm$), compare Remark 5.8.

Proof. By assumption $A \in \mathscr{L}(X_1)$, and by Remark 7.3 we infer $C \in \mathscr{L}(X_1, X_2)$. Considering

$$\mathcal{A}' \stackrel{\text{def}}{=} \begin{bmatrix} 0 & B \\ 0 & D \end{bmatrix},$$

by Remark 7.3 one can apply Corollary 7.1. The statement follows since $\mathcal{A}$ is a bounded perturbation of $\mathcal{A}'$. $\square$

7.2. Simplified Ericksen–Leslie model for nematic liquid crystals

The continuum theory of liquid crystals was developed by Ericksen [33] and Leslie [81]. A simplified model has been introduced by Lin and Liu [82], and here we consider the following simplified model normalizing all constants to one

$$\begin{cases} \partial_t u + (u \cdot \nabla)u - \Delta u + \nabla \pi = -\operatorname{div}([\nabla d]^\top \nabla d) & \text{in } \mathbb{R}_+ \times \mathcal{O}, \\ \partial_t d + (u \cdot \nabla) d = \Delta d + |\nabla d|^2 d & \text{in } \mathbb{R}_+ \times \mathcal{O}, \\ \operatorname{div} u = 0 & \text{in } \mathbb{R}_+ \times \mathcal{O}, \\ u = 0 \quad \text{and} \quad \partial_n d = 0 & \text{in } \mathbb{R}_+ \times \partial\mathcal{O}, \end{cases} \tag{7.1}$$

with initial data $u(0) = u_0$ and $d(0) = d_0$. Here $u : \mathbb{R}_+ \times \mathcal{O} \to \mathbb{R}^3$ denotes the velocity field of the fluid, $\pi : \mathbb{R}_+ \times \mathcal{O} \to \mathbb{R}$ the pressure, and $d : \mathbb{R}_+ \times \mathcal{O} \to \mathbb{R}^3$ denotes the molecular orientation of the liquid crystal at the macroscopic level referred to as the director field. This physical interpretation of d imposes the condition $|d| = 1$ in $\mathbb{R}_+ \times \mathcal{O}$. Recent developments and the literature on this subject are discussed by Hieber and Prüss in the survey [45].

The simplified Ericksen-Leslie model (7.1) has been investigated by Hieber, Nesensohn, Prüss, and Schade in [44] as a quasi-linear evolution equation in maximal L^q_t-L^p_x-regularity spaces for a smooth bounded domain $\mathcal{O} \subseteq \mathbb{R}^3$, see also e.g. [21] for a semilinear approach where the term $\operatorname{div}([\nabla d]^\top \nabla d)$ is estimated as non-linear right hand side in a negative Sobolev space. To define the relevant operator introduced in [44] to linearize (7.1) we need some preparations and we assume that $\mathcal{O} \subseteq \mathbb{R}^3$ is a bounded C^2-domain. Let $p \in (1, \infty)$, and set

$$L^p_\sigma(\mathcal{O}) \stackrel{\text{def}}{=} \{u \in L^p(\mathcal{O})^3 \colon \operatorname{div} u = 0 \text{ in } \mathcal{D}'(\mathcal{O}) \text{ and } n \cdot u|_{\partial\mathcal{O}} = 0\},$$

where n denotes the exterior normal vector field on $\partial\mathcal{O}$. Recall also that $n \cdot u \in \mathcal{D}'(\partial\mathcal{O})$ as $\operatorname{div} u = 0$, see e.g. [38, Theorem III.2.2]. Then we denote by $\mathbb{P}_p : L^p(\mathcal{O})^3 \to L^p_\sigma(\mathcal{O})$ the Helmholtz projection, and by the $\mathbb{P}_p\Delta_p$ the Stokes operator, i.e.

$$\begin{aligned} \mathsf{D}(\mathbb{P}_p\Delta_p) &\stackrel{\text{def}}{=} \{u \in H^{2,p}(\mathcal{O})^3 \cap L^p_\sigma(\mathcal{O})^3 \colon u = 0 \text{ on } \partial\mathcal{O}\}, \\ \mathbb{P}_p\Delta_p &\colon \mathsf{D}(\mathbb{P}_p\Delta_p) \subseteq L^p_\sigma(\mathcal{O}) \to L^p_\sigma(\mathcal{O}), \quad u \mapsto \mathbb{P}_p(\Delta u), \end{aligned}$$

see e.g. [96, Section 7.3]. Also, we define the Neumann Laplacian $\Delta_{N,p}$ on $L^p(\mathcal{O})^3$ as the operator $u \mapsto \Delta u$ with domain

$$\mathsf{D}(\Delta_{N,p}) \stackrel{\text{def}}{=} \{u \in H^{2,p}(\mathcal{O})^3 \colon \partial_n u|_{\partial\mathcal{O}} = 0\},$$

compare e.g. [96, Section 7.4]. Noticing that $[\operatorname{div}([\nabla d]^\top \nabla d)]_i = (\partial_i d_\ell)\Delta d_\ell + (\partial_k d_\ell)(\partial^2_{i,k} d_\ell)$ (here, the summation over repeated indexes is employed), the linearization of (7.1) for fixed d is given by

$$\mathcal{A}_p^{\mathsf{EL}}(d) = \begin{bmatrix} -\mathbb{P}_p\Delta_p & \mathbb{P}_p\mathscr{B}(d) \\ 0 & -\Delta_{N,p} \end{bmatrix}, \quad [\mathscr{B}(d)u]_i = (\partial_i d_\ell)\Delta u_\ell + (\partial_k d_\ell)(\partial^2_{i,k} u_\ell), \tag{7.2}$$

for $i \in \{1,2,3\}$ on

$$X = L^p_\sigma(\mathcal{O}) \times L^p(\mathcal{O})^3 \quad \text{with domain} \quad \mathsf{D}(\mathcal{A}_p^{\mathsf{EL}}(d)) = \mathsf{D}(\mathbb{P}_p\Delta_p) \times \mathsf{D}(\Delta_{N,p}).$$

The main result of this subsection reads as follows.

Proposition 7.4. *Let $\mathcal{O} \subseteq \mathbb{R}^3$ be a bounded C^2-domain. Let $p \in (1,\infty)$ and $\mathcal{A}_p^{\mathsf{EL}}(d)$ be as in (7.2). Then for all $\nu > 0$ the following hold.*

(1) *If $d \in W^{1,\infty}(\overline{\mathcal{O}})^3$, then $\nu + \mathcal{A}_p^{\mathsf{EL}}(d)$ is $\mathcal{R}$-sectorial of angle 0.*
(2) *If $d \in C^{1,\alpha}(\mathcal{O})^3$ for some $\alpha > 0$, then $\nu + \mathcal{A}_p^{\mathsf{EL}}(d)$ has a bounded H^∞-calculus of angle 0.*

The constants in (1) (resp. (2)) depend on d only through $\|d\|_{W^{1,\infty}}$ (resp. $\|d\|_{C^{1,\alpha}}$).

Remark 7.5. For the deterministic setting in [44] it has been sufficient to prove maximal L^q_t-L^p_x-regularity for $\mathcal{A}_p^{\mathsf{EL}}(d)$ to solve the non-linear problem. Proposition 7.4(2) also implies stochastic maximal regularity, and therefore the quasilinear approach to (7.1) developed in [44] – with the regularity assumptions as in [44, Remark 4.2] – can be extended to the stochastic setting using the results by Veraar and the first author in [5,6] to solve non-linear SPDEs.

Proof of Proposition 7.4. To prove Proposition 7.4(1) we apply Corollary 7.1(2). Note that, for all $u \in \mathsf{D}(\Delta_{N,p})$,

$$\begin{aligned}\|\mathbb{P}_p\mathscr{B}(d)u\|_{L^p_\sigma(\mathcal{O})} \lesssim \|\mathscr{B}(d)u\|_{L^p(\mathcal{O})^3} &\lesssim \|d\|_{W^{1,\infty}(\mathcal{O})^3}\Big(\sup_{i,j\in\{1,2,3\}} \|\partial^2_{i,j}u\|_{L^p(\mathcal{O})^3}\Big) \\ &\lesssim \|d\|_{W^{1,\infty}(\mathcal{O})^3}\|(\nu - \Delta_{N,p})u\|_{L^p(\mathcal{O})^3},\end{aligned}$$

where in the last inequality we have used that $\nu - \Delta_{N,p}$ is invertible for all $\nu > 0$. Thus, $\nu + \mathcal{A}^{\mathsf{EL}}(d)$ is $\mathcal{R}$-sectorial of angle 0 by Corollary 7.1(2).

To prove Proposition 7.4(2), we apply Corollary 7.1(3) in the $(-)$-case. Recall that $\mathbb{P}_p\Delta_p$ is invertible and therefore $\mathsf{R}((-\mathbb{P}_p\Delta_p)^\gamma) = L^p_\sigma(\mathcal{O})$ for all $\gamma \in (0,1)$. Moreover, since $\mathcal{O}$ is a C^2-domain, $(\mathbb{P}_p\Delta_p)^* = \mathbb{P}_{p'}\Delta_{p'}$, where $\frac{1}{p}+\frac{1}{p'}=1$, and

$$\begin{aligned}\dot{\mathsf{D}}((-\mathbb{P}_p\Delta_p)^{-\gamma}) &= \big(\dot{\mathsf{D}}((-\mathbb{P}_{p'}\Delta_{p'})^{\gamma})\big)^* \\ &= \big(H^{2\gamma,p'}(\mathcal{O})^3 \cap L^{p'}_\sigma(\mathcal{O})\big)^* \quad \text{for all } \gamma \in \Big(0,\frac{1}{2p}\Big).\end{aligned} \tag{7.3}$$

By [57, Proposition 9.14] and interpolation, $\mathbb{P}_p$ extends uniquely to a map

$$\mathbb{P}_p : H^{-2\gamma,p}(\mathcal{O})^3 \to \dot{\mathsf{D}}((-\mathbb{P}_p\Delta_p)^{-\gamma}) \quad \text{for all } \gamma \in \Big(0,\frac{1}{2p}\Big) \tag{7.4}$$

where we used that $\dot{\mathsf{D}}((-\Delta_{D,p})^{-\gamma}) = H^{-2\gamma,p}(\mathcal{O})^3$ for all $\gamma \in (0,\frac{1}{2p})$ by [100]. Here, as in [57, Proposition 9.14], $\Delta_{D,p}$ denotes the Dirichlet Laplacian.

Since $\mathsf{D}(\nu-\Delta_{N,p}) \hookrightarrow H^{2,p}(\mathcal{O})^3$ for all $\nu>0$,

$$\mathsf{D}((\nu-\Delta_{N,p})^\gamma) \hookrightarrow H^{2\gamma,p}(\mathcal{O})^3 \quad \text{for all } \gamma \in (0,1). \tag{7.5}$$

As above, without loss of generality we assume $\alpha \in (0,\frac{1}{p})$. With this preparation, for all $\nu>0$ and $u \in \mathsf{D}((\nu-\Delta_{N,p})^{1-\beta})$ with $\beta<\frac{\alpha}{2}$, we can estimate

$$\begin{aligned}&\|(\nu-\mathbb{P}_p\Delta_p)^{-\beta}[\mathbb{P}_p\mathscr{B}(d)u]\|_{L^p_\sigma(\mathcal{O})} \\ &\qquad \eqsim \|(-\mathbb{P}_p\Delta_p)^{-\beta}[\mathbb{P}_p\mathscr{B}(d)u]\|_{L^p_\sigma(\mathcal{O})} \\ &\qquad \eqsim \|\mathbb{P}_p\mathscr{B}(d)u\|_{\dot{\mathsf{D}}((-\mathbb{P}_p\Delta_p)^{-\beta})} \\ &\qquad \overset{(7.4)}{\lesssim} \|\mathscr{B}(d)u\|_{H^{-2\beta,p}(\mathcal{O})^3} \\ &\qquad \overset{(7.5)}{\lesssim} \|d\|_{C^{1,\alpha}(\mathcal{O})^3}\|(\nu-\Delta_{N,p})^{1-\beta}u\|_{L^p(\mathcal{O})^3},\end{aligned} \tag{7.6}$$

where we also used that C^α maps are pointwise multipliers on $H^{-2\beta,p}$. Now, Corollary 7.1(3) in the $(-)$-case ensures that $\nu+\mathcal{A}^{\mathsf{EL}}_p(d)$ has a bounded H^∞-calculus for all $\nu>0$ with a corresponding estimate in terms of $\|d\|_{C^{1,\alpha}(\mathcal{O})^3}$. $\square$

Remark 7.6. To prove Proposition 7.4(2) one can also employ Corollary 7.1 in the $(+)$-case. However, one has then to assume that $\mathcal{O}$ is a $C^{2,\alpha}$-domain for some $\alpha>0$. Then, by elliptic regularity, one can check that

$$\mathbb{P}_p : H^{s,p}(\mathcal{O})^3 \to H^{s,p}(\mathcal{O})^3 \cap L^p_\sigma(\mathcal{O}) \quad \text{for all } s \in \big[0,\tfrac{1}{p}\big). \tag{7.7}$$

Without loss of generality, we assume $\alpha \in (0,\frac{1}{p})$. By Proposition 5.9, for all $\beta \in (0,\alpha/2)$ and $\nu>0$,

$$\mathsf{D}((\nu - \mathbb{P}_p\Delta_p)^\beta) = \mathsf{D}((-\mathbb{P}_p\Delta_p)^\beta) = H^{2\beta,p}(\mathcal{O})^3 \cap L^p_\sigma(\mathcal{O}), \tag{7.8}$$

$$\mathsf{D}((\nu - \Delta_{N,p})^\beta) = H^{2\beta,p}(\mathcal{O})^3. \tag{7.9}$$

The former, the fact that $\mathcal{O} \in C^{2,\alpha}$ and elliptic regularity yield for all $\beta \in (0, \frac{\alpha}{2})$,

$$\mathsf{D}((\nu - \Delta_{N,p})^{1+\beta}) = \Big\{u \in H^{2+2\beta,p}(\mathcal{O})^3 \colon \partial_n u|_{\partial\mathcal{O}} = 0\Big\}. \tag{7.10}$$

Thus, for all $\beta \in (0, \frac{\alpha}{2})$ and $u \in \mathsf{D}((\nu - \Delta_{N,p})^{1+\beta})$,

$$\begin{aligned} \|(\nu - \mathbb{P}_p\Delta_p)^\beta[\mathbb{P}_p\mathscr{B}(d)u]\|_{L^p_\sigma(\mathcal{O})} &\overset{(7.8)}{\eqsim} \|\mathbb{P}_p\mathscr{B}(d)u\|_{H^{2\beta,p}(\mathcal{O})^3\cap L^p_\sigma(\mathcal{O})} \\ &\overset{(7.7)}{\lesssim} \|\mathscr{B}(d)u\|_{H^{2\beta,p}(\mathcal{O})^3} \\ &\overset{(i)}{\lesssim} \|d\|_{C^{1,\alpha}(\mathcal{O})^3}\|u\|_{H^{2+2\beta,p}(\mathcal{O})^3} \\ &\overset{(7.10)}{\eqsim} \|d\|_{C^{1,\alpha}(\mathcal{O})^3}\|(\nu - \Delta_{N,p})^{1+\beta}u\|_{L^p(\mathcal{O})^3} \end{aligned} \tag{7.11}$$

where in (i) we use as before that C^α functions are pointwise multipliers on $H^{2\beta,p}$ ones. The estimate (7.11) ensures that Corollary 7.1 in the $(+)$-case can be applied.

The argument in the above proof can be extended to prove the boundedness of the H^∞-calculus for $\mu + \mathcal{A}^{\mathsf{EL}}_p(d)$ for some $\mu \geq 0$ in case $-\mathbb{P}_p\Delta_p$ is replaced by $-\mathbb{P}_p\mathscr{A}_p$ where $\mathscr{A}_p(x)u = \sum_{i,j=1}^3 a^{i,j}(x)\partial^2_{i,j}u$, $a^{i,j} \in C^\alpha(\mathcal{O})^{3\times 3}$ and $a^{i,j}$ are uniformly elliptic. To see this, note that in the above proof the Stokes operator plays a role only through (7.8). The latter also holds if $-\mathbb{P}_p\Delta_p$ is replaced by $\mu - \mathbb{P}_p\mathscr{A}_p$ provided the latter operator has a bounded H^∞-calculus for μ large enough. To prove the latter, recall that $\mu - \mathbb{P}_p\mathscr{A}_p$ is $\mathcal{R}$-sectorial on $L^p_\sigma(\mathcal{O})$ of angle $< \pi/2$ for μ large enough by [96, Chapter 7]. Thus it also has a bounded H^∞-calculus by Theorem A.1 applied with $A = -\mathbb{P}_p\Delta_p$ and $B = \mu - \mathbb{P}_p\mathscr{A}_p$ (cf. (7.8), (7.9) and (7.3)).

7.3. The weak Keller-Segel operator

Keller-Segel equations arise in the mathematical modeling of chemotaxis, see e.g. [46–48,78] for surveys and further literature. Here, we consider the classical Keller-Segel system given by

$$\begin{cases} \partial_t u - \Delta u + \nabla\cdot(u\nabla v) = 0, & \text{in } \mathbb{R}_+ \times \mathcal{O}, \\ \partial_t v - \Delta v + v - u = 0, & \text{in } \mathbb{R}_+ \times \mathcal{O}, \\ u(0) = u_0, \quad v(0) = v_0, & \text{in } \mathbb{R}_+ \times \mathcal{O}, \end{cases} \tag{7.12}$$

where $u\colon \mathbb{R}_+ \times \mathcal{O} \to \mathbb{R}$ represents the density of a cell population and $v\colon \mathbb{R}_+ \times \mathcal{O} \to \mathbb{R}$ the concentration of a chemoattractant. We complement the above system with non-linear boundary conditions

$$\partial_n u - u\,\partial_n v = 0, \quad \text{and} \quad v = 0, \quad \text{in } \mathbb{R}_+ \times \partial\mathcal{O}, \tag{7.13}$$

where ∂_n denotes the outer normal derivative at the boundary.

In applications to (stochastic) partial differential equations, the weak setting has two advantages. Firstly, it (typically) requires less regularity assumption on $\partial\mathcal{O}$ compared to the strong setting, and secondly, in the stochastic framework it also requires minimal compatibility conditions for the noise. To obtain the weak formulation of (7.12)-(7.13), we multiple the first equation in (7.12) by $\varphi \in C^\infty(\overline{\mathcal{O}})$. Using that

$$\int_{\mathcal{O}} [-\Delta u + \nabla\cdot(u\cdot\nabla v)]\varphi\,\mathrm{d}x \overset{(7.13)}{=} \int_{\mathcal{O}} [\nabla u\cdot\nabla\varphi - u\nabla v\cdot\nabla\varphi]\,\mathrm{d}x,$$

one can linearize (7.12)-(7.13) in the weak setting by writing for $U = (u,v)^T$

$$U' + \mathcal{A}_p^{\mathsf{KS}}(u)U = 0, \quad U(0) = U_0, \quad \text{where} \quad \mathcal{A}_p^{\mathsf{KS}}(z) \overset{\text{def}}{=} \begin{bmatrix} -\Delta^{\mathsf{w}}_{N,p} & \mathscr{B}_p(z) \\ -\mathbb{1} & \mathbb{1} - \Delta^{\mathsf{w}}_{D,p} \end{bmatrix}$$

on $X_0 = (W^{1,p'}(\mathcal{O}))^* \times (W^{1,p'}_0(\mathcal{O}))^*$ with $\mathsf{D}(\mathcal{A}_p^{\mathsf{KS}}(z)) \overset{\text{def}}{=} W^{1,p}(\mathcal{O}) \times W^{1,p}_0(\mathcal{O})$. Here, for all $(u,v),(u',v') \in W^{1,p}(\mathcal{O}) \times W^{1,p}_0(\mathcal{O})$,

$$\langle u', \Delta^{\mathsf{w}}_{N,p} u\rangle \overset{\text{def}}{=} -\int_{\mathcal{O}} \nabla u\cdot\nabla u'\,\mathrm{d}x, \qquad \langle v', \Delta^{\mathsf{w}}_{D,p} v\rangle \overset{\text{def}}{=} -\int_{\mathcal{O}} \nabla v\cdot\nabla v'\,\mathrm{d}x,$$
$$\langle u', \mathscr{B}_p(z)v\rangle \overset{\text{def}}{=} -\int_{\mathcal{O}} z\,\nabla v\cdot\nabla u'\,\mathrm{d}x \quad \text{where } z \in L^\infty(\mathcal{O}).$$

Proposition 7.7. *For all $p \in (1,\infty)$ the following hold.*

1. *If $\mathcal{O} \subseteq \mathbb{R}^d$ is a bounded C^1-domain and $z \in L^\infty(\mathcal{O})$, then there exists $\nu \geq 0$ such that $\nu + \mathcal{A}_p^{\mathsf{KS}}(z)$ is $\mathcal{R}$-sectorial of angle 0.*
2. *If $\mathcal{O}$ is a $C^{1,\alpha}$-domain and $z \in C^\alpha(\mathcal{O})$ for some $\alpha > 0$, then there exists $\nu \geq 0$ such that $\nu + \mathcal{A}_p^{\mathsf{KS}}(z)$ has a bounded H^∞-calculus of angle 0.*

The constants in (1) (resp. (2)) depend on z only through $\|z\|_{L^\infty(\mathcal{O})}$ (resp. $\|z\|_{C^\alpha(\mathcal{O})}$).

Proof. By a standard result for bounded perturbations (see e.g. [96, Corollary 3.3.15 and Proposition 4.4.3]), it suffices to show (1)-(2) for $\mathcal{A}_p^{\mathsf{KS}}(z)$ replaced by

$$\widehat{\mathcal{A}}_p(z) \overset{\text{def}}{=} \begin{bmatrix} \mathbb{1} - \Delta^{\mathsf{w}}_{N,p} & \mathscr{B}_p(z) \\ 0 & 2 - \Delta^{\mathsf{w}}_{D,p} \end{bmatrix} \quad \text{with} \quad \mathsf{D}(\widehat{\mathcal{A}}_p(z)) = \mathsf{D}(\mathcal{A}_p^{\mathsf{KS}}(z)).$$

Let us recall that by [11, Theorem 11.5] $\mathbb{1} - \Delta^{\mathsf{w}}_{N,p}$ and $2 - \Delta^{\mathsf{w}}_{D,p}$ have a bounded H^∞-calculus of angle 0. In particular, they are also $\mathcal{R}$-sectorial of angle 0. So, (1) follows immediately from Corollary 7.1(2).

The claim (2) follows from Theorem 6.9 for $p \neq 2$ if Corollary 7.1(3) applies to the case $p = 2$. Theorem 6.9 is then applicable. Its assumptions are indeed satisfied: First, as $\widehat{\mathcal{A}}_p(z)$ for $p \in (1, \infty)$ is a consistent family of operators, (6.7) holds since $\widehat{\mathcal{D}}_p = \operatorname{diag}\{\mathbb{1} - \Delta^{\mathsf{w}}_{N,p}, 2 - \Delta^{\mathsf{w}}_{D,p}\}$ and $\widehat{\mathcal{A}}_p(z)$ are boundedly invertible. Second, by the previous argument $\widehat{\mathcal{A}}_p(z)$ is $\mathcal{R}$-sectorial for $p \in (1, \infty)$. Third, Corollary 7.1(3) and $\omega(\widehat{\mathcal{A}}_2(z)) = 0$ imply by e.g. [50, Corollary 10.4.10 and Theorem 10.4.22] that $\widehat{\mathcal{A}}_2(z)$ generates a contraction semigroup w.r.t. an equivalent Hilbertian norm.

To apply Corollary 7.1(3) for $p = 2$, we check the condition for the (+)-case of (3), that is,

$$\mathscr{B}_2(z) \colon \mathsf{D}((2 - \Delta^{\mathsf{w}}_{D,2})^{1+\beta}) \to \mathsf{D}((\mathbb{1} - \Delta^{\mathsf{w}}_{N,2})^{\beta}) \text{ is bounded for some } \beta > 0. \tag{7.14}$$

Since $\mathbb{1} - \Delta^{\mathsf{w}}_{N,2}$ has a bounded H^∞-calculus and by [16, Theorem 4.6.1 and Corollary 4.5.2], we have for all $\gamma \in (0, 1/2)$ that

$$\begin{aligned} \mathsf{D}((1 - \Delta^{\mathsf{w}}_{N,2})^{\gamma}) &= [(H^1(\mathcal{O}))^*, H^1(\mathcal{O})]_\gamma \\ &= [(H^1(\mathcal{O}))^*, L^2(\mathcal{O})]_{2\gamma} \\ &= \big([H^1(\mathcal{O}), L^2(\mathcal{O})]_{2\gamma}\big)^* = \big(H^{1-2\gamma}(\mathcal{O})\big)^*. \end{aligned}$$

Since $2 - \Delta^{\mathsf{w}}_{D,2}$ has a bounded H^∞-calculus as well, $\mathsf{D}((2 - \Delta^{\mathsf{w}}_{D,2})^{\gamma}) = H^{-1+2\gamma}(\mathcal{O})$ for all $\gamma \in (0, 1/2)$. By elliptic regularity and the fact that $\mathcal{O}$ is a $C^{1,\alpha}$-domain we have, for all $2\beta \in \big(0, \alpha \wedge \frac{1}{2}\big)$,

$$\mathsf{D}((2 - \Delta^{\mathsf{w}}_{D,2})^{1+\beta}) = \big\{v \in H^1_0(\mathcal{O}) \,:\, \Delta^{\mathsf{w}}_D v \in H^{-1+2\beta}(\mathcal{O})\big\} = H^{1+2\beta}_0(\mathcal{O}).$$

With this at hand, we prove (7.14). Fix $0 < \beta < \eta < \frac{\alpha}{2} \wedge \frac{1}{4}$. For any $u \in H^{1-2\beta}(\mathcal{O})$ there exists an extension $U \in H^{1-2\beta}(\mathbb{R}^d)$ such that $U|_{\mathcal{O}} = u$ and $\|U\|_{H^{1-2\beta}(\mathbb{R}^d)} \lesssim \|u\|_{H^{1-2\beta}(\mathcal{O})}$ (with implicit constant independent of u). Similarly, choose $Z \in C^\eta(\mathbb{R}^d)$ such that $Z|_{\mathcal{O}} = z$ and $\|Z\|_{C^\eta(\mathbb{R}^d)} \lesssim \|z\|_{C^\eta(\mathcal{O})}$. Let E_0 be the extension by 0 outside $\mathcal{O}$. Then, for all $v \in C^\infty_0(\mathcal{O})$ and v as above,

$$\begin{aligned} \langle u, \mathscr{B}_2(z) v\rangle = \Big| \int_{\mathcal{O}} z \, \nabla u \cdot \nabla v \, \mathrm{d}x \Big| &= \Big| \int_{\mathbb{R}^d} Z \, \nabla U \cdot \nabla(\mathrm{E}_0 v) \, \mathrm{d}x \Big| \\ &\lesssim \|Z \, \nabla(\mathrm{E}_0 v)\|_{H^{2\beta}(\mathbb{R}^d)} \|U\|_{H^{1-2\beta}(\mathbb{R}^d)} \\ &\lesssim \|z\|_{C^\eta(\mathcal{O})} \|v\|_{H^{1+2\beta}(\mathcal{O})} \|u\|_{H^{1-2\beta}(\mathcal{O})}. \end{aligned}$$

By density of $C^\infty_0(\mathcal{O})$ in $H^{1+\beta}_0(\mathcal{O})$ and taking the supremum over all $\|u\|_{H^{1-2\beta}(\mathcal{O})} \le 1$, (7.14) follows. □

Remark 7.8. The boundary conditions considered here have been discussed recently in [37]. A variety of zero-flux boundary conditions boundary is discussed in [46, Section 2], and there are also Keller-Segel models with pure Dirichlet or Neumann boundary conditions and combinations of the various cases. It seems that the above proof can be adapted to these situations by using different extension operators.

7.4. Artificial compressible Stokes system

The artificial compressible Stokes system has been introduced in the context of steady state solutions to the Navier-Stokes equations, see [19,20,107–109]. It is formally given by

$$\mathcal{A}^{\mathsf{AS}} = \begin{bmatrix} 0 & \frac{1}{\epsilon^2}\mathrm{div} \\ \nabla & -\Delta + v_s \cdot \nabla + (\nabla v_s)^T \end{bmatrix}, \quad \varepsilon > 0, \tag{7.15}$$

with a given real valued vector field v_s. The spectral properties of this operator have been investigated recently in detail for the Hilbert space case in [56,110]. However, the L^p-theory for this operator has not been studied so far. Therefore, we consider here $\mathcal{A}_p^{\mathsf{AS}} = \mathcal{A}^{\mathsf{AS}}$ in the space $X = X_1 \times X_2$ with

$$X_1 = H^{1,p}(\mathcal{O}) \quad \text{and} \quad X_2 = L^p(\mathcal{O})^3$$

for $p \in (1, \infty)$ with domain

$$\mathsf{D}(\mathcal{A}_p^{\mathsf{AS}}) = H^{1,p}(\mathcal{O}) \times \big(H^{2,p}(\mathcal{O})^3 \cap H_0^{1,p}(\mathcal{O})^3\big).$$

It turns out that Corollary 7.2 is applicable here, and it guarantees some basic operator theoretical properties by purely perturbative methods and properties of the Laplacian in L^p-spaces. Using the particular structure of the off-diagonal perturbation more detailed properties can be derived as in [56,110] for $p = 2$.

Proposition 7.9. *Let $\mathcal{O}$ be a bounded C^2 domain in $\mathbb{R}^3$. If $v_s \in H^{1,q}(\mathcal{O})^3$ for $p, q \in (1, \infty)$ with $q > 3/2$ and $q \geq p$, then for each $\psi > 0$ there exists $\mu \geq 0$ such that the shifted artificial Stokes system $\mu + \mathcal{A}_p^{\mathsf{AS}}$ has a bounded H^∞-calculus of angle $\leq \psi$.*

Remark 7.10. For the case $v_s \equiv 0$ one can consider the operator

$$M_2(\lambda) = \mathbb{1} - \tfrac{\varepsilon^2}{\lambda} \nabla \mathrm{div}(\lambda - \Delta)^{-1}, \quad \lambda \in \mathbb{C} \setminus (\sigma(-\Delta) \cup \{0\}).$$

This is related to the second Schur complement of $\mathcal{A}_p^{\mathsf{AS}}$

$$S_2(\lambda) = (\lambda - \Delta) - \tfrac{\varepsilon^2}{\lambda} \nabla \mathrm{div}, \quad \lambda \in \mathbb{C} \setminus (\sigma(-\Delta) \cup \{0\}), \quad \mathsf{D}(S_2(\lambda)) = \mathsf{D}(D)$$

which is in fact – up to a shift – a Lamé operator studied for instance in [23,71,87]. Following the proof of [87, Theorem 4.1], $S_2(\lambda)$ is boundedly invertible on $L^p(\mathcal{O})$ for $\lambda > 0$. Hence $M_2(\lambda) = S_2(\lambda)(\lambda - D)^{-1}$ is a closed bijective operator and hence boundedly invertible. In particular, one can choose in this situation any $\mu > 0$. The case with $v_s \equiv 0$ is comparable to the situation analyzed in [39].

Proof of Proposition 7.9. For $v_s \in H^{1,q}(\mathcal{O})^3$ with $q > 3/2$ and $q \geq p$ one has

$$\|v_s \cdot \nabla v\|_{L^p} \leq \|v_s\|_{L^{sp}} \|v\|_{H^{1,rp}} \lesssim \|v_s\|_{H^{1,q}} \|v\|_{H^{2-\delta,p}}$$

using Hölder's inequality and Sobolev embeddings with

$$s = \tfrac{3q}{p(3-q)}, \quad \tfrac{1}{s} + \tfrac{1}{r} = 1, \quad \delta = \tfrac{3}{q},$$

and

$$\|(\nabla v_s)^T v\|_{L^p} \leq \|v_s\|_{H^{1,sp}} \|v\|_{L^{rp}} \lesssim \|v_s\|_{H^{1,q}} \|v\|_{H^{2-\delta,p}},$$

here using Hölder's inequality and Sobolev embeddings with

$$s = \tfrac{p}{q}, \quad \tfrac{1}{s} + \tfrac{1}{r} = 1, \quad \delta = 2 - \tfrac{3}{q}.$$

Since the operator $D_0 = -\Delta$ with $\mathsf{D}(D_0) = H^{2,p}(\mathcal{O})^3 \cap H^{1,p}_0(\mathcal{O})^3$ has a bounded H^∞-calculus of angle zero, and as shown by the estimate above $v_s \cdot \nabla + (\nabla v_s)^T$ is a lower order perturbation, there exists a $\mu_0 \geq 0$ such that $\mu_0 + \mathcal{D}$ has a bounded H^∞-calculus of angle zero. Therefore, the operator $\mathcal{A}_p^{\mathsf{AS}}$ is diagonally dominant as

$$\|\nabla p\|_{L^p(\mathcal{O})^3} \leq \|p\|_{H^{1,p}(\mathcal{O})^3} \quad \text{and} \quad \|\tfrac{1}{\epsilon^2} \operatorname{div} v\|_{H^{1,p}(\mathcal{O})} \leq \tfrac{1}{\epsilon^2} \|v\|_{H^{2,p}(\mathcal{O})^3},$$

and by Corollary 7.2 it follows that there is a $\mu \geq \mu_0$ such that $\mu + \mathcal{A}$ has a bounded H^∞-calculus of angle zero. $\square$

7.5. Second order Cauchy problems with strong damping

Second order Cauchy problems of the form

$$\partial_t^2 u + T\partial_t u + Su = f, \quad u(0) = u_0, \quad \partial_t u(0) = v_0 \tag{7.16}$$

for operators S and T in Y can be re-written as a first order Cauchy problem by formally setting $v = \partial_t u$ to obtain

$$\mathcal{A}^{\mathsf{SD}} = \begin{bmatrix} 0 & -\mathbb{1} \\ S & T \end{bmatrix}. \tag{7.17}$$

Classical examples are d'Alembert's wave equation, where $T = 0$ and $S = -\Delta$, and the beam equation where $T = 0$ and $S = \Delta^2$. These operators are not diagonally dominant and therefore cannot be treated by the methods presented here, and in fact these equations are not parabolic. In [99] for S with $\mathsf{D}(S) \subseteq Y$ the case where $T = S^{1/2}$, and $X_1 = \mathsf{D}(S^{1/2})$ while $X_2 = Y$ is considered. This is parabolic, but not diagonally dominant, and most of the damped wave and plate equations are rather lower dominant, that is, $\mathsf{D}(\mathcal{A}) = \mathsf{D}(C) \times \mathsf{D}(D)$, compare [111, Definition 2.2.1], than diagonally dominant. Only when one adds a relatively strong damping such as a *Kelvin-Voigt-type damping*, then one has that T and S are of the same order. The following consequence of Corollary 7.2 captures such situations. In the literature there are many works on the regularity properties of solutions to plate or wave equations, but there seems to be little known about the H^∞-calculus of such operators.

Corollary 7.11. *Let T and S be operators in a Banach space Y. Assume that T is closed and densely defined and let S be T-bounded, i.e. there exist $c_0, L_0 > 0$ such that*

$$\|Sy\|_Y \le c_0 \|Ty\|_Y + L_0 \|y\|_Y \quad \textit{for all } y \in \mathsf{D}(T).$$

Then the operator $\mathcal{A}^{\mathsf{SD}}$ with strong damping as in (7.17) on the space

$$X = \mathsf{D}(T) \times Y \quad \textit{with} \quad \mathsf{D}(\mathcal{A}^{\mathsf{SD}}) = \mathsf{D}(T) \times \mathsf{D}(T)$$

is diagonally dominant. Moreover, if T is sectorial, then for all $\psi \in (\omega(T), \pi)$ there exists $\mu \ge 0$ such that $\mathcal{A}^{\mathsf{SD}} + \mu$ is sectorial of angle $\le \psi$. The respective statement holds for $\mathcal{R}$-sectoriality and for the boundedness of the H^∞-calculus.

Example 7.12. An example is the Kelvin-Voigt plate-like equation

$$\partial_t^2 u + \varepsilon \Delta^2 \partial_t u + \Delta^2 u = f, \quad u(0) = u_0, \quad \partial_t u(0) = v_0, \quad \varepsilon > 0,$$

discussed in [79, Section 5.5]. Setting $\partial_t u = v$, it translates to

$$\mathcal{A}^{\mathsf{SD}} = \begin{bmatrix} 0 & -\mathbb{1} \\ \Delta^2 & \varepsilon \Delta^2 \end{bmatrix}$$

which following Corollary 7.11 can be considered in

$$H^{4,p}(\mathbb{R}^d) \times L^p(\mathbb{R}^d) \quad \text{with} \quad \mathsf{D}(\mathcal{A}^{\mathsf{SD}}) = H^{4,p}(\mathbb{R}^d) \times H^{4,p}(\mathbb{R}^d).$$

Moreover, by Corollary 7.11, for each $\psi > 0$ there exists $\mu \ge 0$ such that $\mu + \mathcal{A}^{\mathsf{SD}}$ has a bounded H^∞-calculus of angle $\le \psi$.

Considering a smooth bounded domain, one can still apply Corollary 7.11 as long as the boundary conditions assure that the bi-Laplacian Δ^2 is sectorial, $\mathcal{R}$-sectorial, or has a bounded H^∞-calculus, respectively.

Example 7.13. The strongly damped wave equation given in [79, Section 3.8] is

$$\partial_t^2 u - \varepsilon\Delta\partial_t u - \Delta u = f, \quad u(0) = u_0, \quad \partial_t u(0) = v_0, \quad \varepsilon > 0.$$

This yields to

$$\mathcal{A}^{\mathsf{SD}} = \begin{bmatrix} 0 & -\mathbb{1} \\ -\Delta & -\varepsilon\Delta \end{bmatrix}$$

which by Corollary 7.11 can be considered in

$$H^{2,p}(\mathbb{R}^d) \times L^p(\mathbb{R}^d) \quad \text{with} \quad \mathsf{D}(\mathcal{A}^{\mathsf{SD}}) = H^{2,p}(\mathbb{R}^d) \times H^{2,p}(\mathbb{R}^d).$$

Corollary 7.11 also ensures that – up to a shift – $\mathcal{A}^{\mathsf{SD}}$ has a bounded H^∞–calculus of arbitrary small angle.

As in Example 7.12, we may also consider smooth domains with suitable boundary conditions as long as the corresponding Laplacian $-\Delta$ is sectorial, $\mathcal{R}$-sectorial or has bounded H^∞–calculus with such boundary conditions, respectively.

Example 7.14. Consider a damped thermoelastic plate equation of the type

$$\begin{aligned} \partial_t^2 u + \varepsilon\Delta^2\partial_t u + \Delta^2 u + \Delta\theta &= f, \\ \partial_t\theta - \Delta\theta - \Delta\partial_t u &= g, \\ u(0) = u_0, \quad \partial_t u(0) &= v_0, \\ \theta(0) &= \theta_0, \end{aligned}$$

where one couples to the Kelvin-Voigt plate-like equation above the temperature similar to [79, 3.11.1]. Then one obtains the first order system with

$$\mathcal{A}^{\mathsf{SD}} = \left[\begin{array}{cc|c} 0 & -\mathbb{1} & 0 \\ \Delta^2 & \varepsilon\Delta^2 & \Delta \\ \hline 0 & -\Delta & -\Delta \end{array}\right] = \begin{bmatrix} A & B \\ C & D \end{bmatrix}.$$

Here, one considers the operators A and D on the diagonal in $X_1 = H^{4,p}(\mathbb{R}^d) \times L^p(\mathbb{R}^d)$ and $X_2 = L^p(\mathbb{R}^d)$ with $\mathsf{D}(A) = H^{4,p}(\mathbb{R}^d) \times H^{4,p}(\mathbb{R}^d)$ and $\mathsf{D}(D) = H^{2,p}(\mathbb{R}^d)$, respectively. Therefore, we consider the above operator $\mathcal{A}^{\mathsf{SD}}$ on the space $X = X_1 \times X_2$ and $\mathsf{D}(\mathcal{A}^{\mathsf{SD}}) = \mathsf{D}(A) \times \mathsf{D}(D)$. Note that A was studied in Example 7.12.

The block B is D-bounded (more precisely, we have $\|Bv\|_{X_1} \lesssim \|v\|_{\mathsf{D}(D)}$ for all $v \in \mathsf{D}(D)$), and the block C given by

$$\left\| \begin{bmatrix} 0 & -\Delta \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} \right\|_{L^p(\mathbb{R}^d)} = \|\Delta v\|_{L^p(\mathbb{R}^d)}$$

is a lower order term compared to A, and therefore by Corollary 5.7 – up to a shift – the operator $\mathcal{A}$ has a bounded H^∞-calculus of arbitrarily small angle. Here again these operators on domains can also be considered along the same lines provided that the boundary conditions guarantee that the operators $\varepsilon\Delta^2$ and $-\Delta$ have a bounded H^∞-calculus.

7.6. Beris-Edwards Q-tensor model for liquid crystals

In the Beris-Edwards model of nematic liquid crystal, the molecular orientation is described by a so-called Q-tensor, a function

$$Q\colon \mathbb{R}_+ \times \mathcal{O} \to S^d_{0,\mathbb{C}} \stackrel{\text{def}}{=} \{Q \in \mathbb{C}^{d\times d} : \, Q = Q^T \text{ and } \operatorname{tr} Q = 0\},$$

and the fluid properties by the velocity field $u\colon \mathbb{R}_+ \times \mathcal{O} \to \mathbb{R}^3$. The Beris-Edwards model has been investigated recently by Wrona in [114] in maximal L^p_t-L^2_x-spaces. There, a linearization of the full quasi-linear model in the strong setting for fixed $Q_0\colon \mathcal{O} \to S^d_{0,\mathbb{C}}$ in the space

$$X = L^q_\sigma(\mathcal{O}) \times H^{1,q}(\mathcal{O}; S^d_{0,\mathbb{C}})$$

is given by the block operator matrix

$$\mathcal{A}^{\mathsf{EB}}_q(Q_0) = \begin{bmatrix} -\mathbb{P}_q\Delta & -\mathbb{P}\operatorname{div} S_\xi(Q_0)(\mathbb{1} - \Delta_{N,q}) \\ -\tilde{S}_\xi(Q_0)\nabla & \mathbb{1} - \Delta_{N,q} \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

with domain

$$\mathsf{D}(\mathcal{A}^{\mathsf{EB}}_q(Q_0)) = \big(H^{1,q}_0(\mathcal{O})^3 \cap H^{2,q}_\sigma(\mathcal{O})^3\big) \times \{Q \in H^{3,q}(\mathcal{O}; S^d_{0,\mathbb{C}})\colon \partial_n Q = 0\}.$$

Here $\partial_n Q$ denotes the outer normal derivative of Q, $\mathbb{P}_q$ denotes the Helmholtz projection in $L^q(\mathcal{O}; \mathbb{R}^3)$, and

$$\begin{aligned} S_\xi(Q)P &= [Q, P] - \tfrac{2\xi}{d} P - \xi\{Q, P\} + 2\xi(Q + \mathbb{1}/d) \operatorname{tr}(QP), \\ \tilde{S}_\xi(Q)P &= [\overline{Q}, P^W] - \tfrac{2\xi}{d} P^D - \xi\{\overline{Q}, P^D\} + 2\xi(\overline{Q} + \mathbb{1}/d) \operatorname{tr}(\overline{Q}P^D), \end{aligned}$$

with $[Q, P] = QP - PQ$ and $\{Q, P\} = QP + PQ$ denoting the commutator and the anti-commutator, respectively, $\overline{Q}$ the complex conjugate of Q, and

$$P^D = \tfrac{1}{2}(P + P^T) \quad \text{and} \quad P^W = \tfrac{1}{2}(P - P^T)$$

the symmetric and anti-symmetric part of $P \in S^d_{0,\mathbb{C}}$ respectively. The parameter $\xi \in \mathbb{R}$ describes the ration of tumbling and aligning effects. First, we verify that these operators fit into the theory presented here.

Lemma 7.15. *Let $q \in (1,\infty)$, $\mathcal{O} \subseteq \mathbb{R}^3$ be a bounded domain with $\partial\mathcal{O} \in C^3$ and $Q_0 \in W^{1,\infty}(\mathcal{O}; S^d_{0,\mathbb{C}})$, then $\mathcal{A}^{\mathsf{EB}}_q(Q_0)$ satisfies Assumption 3.1 with $L = 0$, and it satisfies the estimates in Assumption 5.1* (−).

Proof. Diagonal dominance follows from the estimates

$$\begin{aligned}\|BQ\|_{L^q} &\lesssim \|(\mathrm{div} S_\xi(Q_0))(-\Delta+\mathbb{1})Q\|_{L^q} + \|(S_\xi(Q_0))\mathrm{div}(-\Delta+\mathbb{1})Q\|_{L^q}\\ &\lesssim \|S_\xi(Q_0)\|_{W^{1,\infty}}\|Q\|_{H^{3,q}} \lesssim \|Q_0\|_{W^{1,\infty}}\|DQ\|_{H^{1,q}},\\ \|Cu\|_{H^{1,q}} &= \|\tilde{S}_\xi(Q_0)\nabla u\|_{H^{1,q}} \le \|\tilde{S}_\xi(Q_0)\|_{W^{1,\infty}}\|u\|_{H^{2,q}} \lesssim \|Q_0\|_{W^{1,\infty}}\|Au\|_{L^q}\end{aligned}$$

for $u \in \mathsf{D}(A)$ and $Q \in \mathsf{D}(D)$, where one uses that $A = -\mathbb{P}_q\Delta$ and $D = \mathbb{1} - \Delta_{N,q}$ are boundedly invertible. $\square$

In [114, Corollary 3.2.7 and Theorem 3.2.16] it is shown that in the Hilbert space case $q = 2$ the operator $\mathcal{A}^s_2(Q_0)$ is $\mathcal{J}$-symmetric, sectorial, and it generates a contraction semigroup. We summarize this result without proof here.

Proposition 7.16 *(The case $q = 2$). Let $\mathcal{O} \subseteq \mathbb{R}^d$ be a bounded domain with $\partial\mathcal{O} \in C^3$ and $Q_0 \in W^{1,\infty}(\mathcal{O}; S^d_{0,\mathbb{C}})$, then $\mathcal{A}^{\mathsf{EB}}_2(Q_0)$ is an invertible $\mathcal{J}$-symmetric and sectorial of angle $< \pi/2$.*

Corollary 6.10 yields the boundedness of the H^∞-calculus for q near 2.

Proposition 7.17 *($\mathcal{R}$-sectoriality and bounded H^∞-calculus near $q = 2$). Let $\mathcal{O} \subseteq \mathbb{R}^d$ be a bounded domain with $\partial\mathcal{O} \in C^3$ and $Q_0 \in W^{1,\infty}(\mathcal{O}; S^d_{0,\mathbb{C}})$, then there exists $\delta > 0$ such that for all $q \in (2-\delta, 2+\delta)$ the operators $\mathcal{A}^{\mathsf{EB}}_q(Q_0)$ are invertible and have a bounded H^∞-calculus of angle less than $\pi/2$. In particular $\mathcal{A}^{\mathsf{EB}}_q(Q_0)$ are $\mathcal{R}$-sectorial of angle less than $\pi/2$.*

Proof. The operators $\mathcal{A}^{\mathsf{EB}}_q(Q_0)$ are a consistent family of operators for $q \in (1,\infty)$ by Lemma 6.4(2) (note that (6.2) holds for q close to 2 by Remark 6.6). For $q = 2$ the statement follows by Corollary 5.13 and Proposition 7.16. Hence, using Lemma 7.15, one can now apply Corollary 6.10, and the statement follows. $\square$

Remark 7.18. One can extend the above argument to prove the boundedness of the H^∞-calculus for the Beris-Edwards operator on

$$H^{s,q}_\sigma(\mathcal{O}) \times H^{1+s,q}(\mathcal{O}; S^d_{0,\mathbb{C}}) \quad \text{for all } q \in (2-\delta, 2+\delta) \text{ and } s \in (-\delta, \delta),$$

for some $\delta > 0$ assuming $Q_0 \in C^{1,\alpha}(\mathcal{O})$ with $\alpha > 0$.

7.7. A differential operator of Beris-Edwards type

In this subsection we study the following differential operator on $X = H^{-1,p}(\mathbb{R}^d) \times L^p(\mathbb{R}^d)^d$

$$\mathcal{A}_p^{\Delta} \overset{\text{def}}{=} \begin{bmatrix} \mathbb{1}-\Delta & \operatorname{div}(\mathbb{1}-\Delta) \\ \nabla & \mathbb{1}-\Delta \end{bmatrix} \quad \text{with} \quad \mathsf{D}(\mathcal{A}_p^{\Delta}) = H^{1,p}(\mathbb{R}^d) \times H^{2,p}(\mathbb{R}^d)^d \tag{7.18}$$

where $p \in (1,\infty)$. This differential operator has a structure similar to the one studied in Subsection 7.6 which arises in the study of Beris-Edwards model for liquid crystals. In contrast to the latter, $\mathcal{A}_p^{\Delta}$ allows us to give more direct computations and therefore is more suited for illustrative purposes.

Let us denote by Δ_p^{s} and Δ_p^{w} the realization of the Laplace operator on $L^p(\mathbb{R}^d)$ and $H^{-1,p}(\mathbb{R}^d)$, respectively. Thus $\mathcal{A}_p^{\Delta}$ is a block operator matrix with the choice

$$A = \mathbb{1} - \Delta_p^{\mathsf{w}}, \qquad B = \operatorname{div}(\mathbb{1} - \Delta_p^{\mathsf{s}}), \qquad C = \nabla, \qquad D = \mathbb{1} - \Delta_p^{\mathsf{s}},$$

where Δ_p^{s} in the definition of B and D acts component-wise on $H^{2,p}(\mathbb{R}^d)^d$.

Proposition 7.19. *Let $p \in (1,\infty)$. Then $\mathcal{A}_p^{\Delta}$ has a bounded H^∞-calculus of angle* 0*. Moreover, for all $\beta \in \big(-\big((1-\frac{1}{p}) \vee \frac{1}{p}\big), 1\big]$,*

$$\dot{\mathsf{D}}((\mathcal{A}_p^{\Delta})^{\beta}) = H^{-1+2\beta,p}(\mathbb{R}^d) \times H^{2\beta,p}(\mathbb{R}^d)^d. \tag{7.19}$$

Proof. It is easy to check that $\mathcal{A}_p^{\Delta}$ is diagonally dominant, and it does not fit into the special cases analyzed in Subsection 7.1. For the reader's convenience we split the proof into several steps.

Step 1: $\mathcal{A}_p^{\Delta}$ is $\mathcal{R}$-sectorial of angle 0. Fix $\psi \in (0,\pi)$. By Theorem 4.1 and the fact that $(1-\Delta)^{s/2} : H^{\beta+s,p}(\mathbb{R}^d) \to H^{\beta,p}(\mathbb{R}^d)$ is an isomorphism for all $s, \beta \in \mathbb{R}$, it is enough to show that

$$\begin{aligned} M_1(\lambda) &\overset{\text{def}}{=} \mathbb{1} - \operatorname{div}(\mathbb{1}-\Delta)((\lambda-1)+\Delta)^{-1}\nabla((\lambda-1)+\Delta)^{-1} \\ &= \mathbb{1} - \Delta(\mathbb{1}-\Delta)((\lambda-1)+\Delta)^{-2} \in \mathscr{L}(L^p(\mathbb{R}^d)), \end{aligned}$$

is invertible for all $\lambda \in \complement\overline{\Sigma_\psi}$ and

$$\mathcal{R}(M_1(\lambda)^{-1} \colon \lambda \in \complement\overline{\Sigma_\psi}) < \infty. \tag{7.20}$$

To check (7.20), it is enough to check that the symbol of $(M_1(\lambda))^{-1}$ satisfies the Lizorkin condition (see e.g. [96, Theorem 4.3.9]). Note that $(M_1(\lambda))^{-1}$ has symbol

$$\widetilde{m}_\lambda(\xi) \overset{\text{def}}{=} \Big(1 + \frac{|\xi|^2(1+|\xi|^2)}{(\lambda-1-|\xi|^2)^2}\Big)^{-1}$$

$$= \frac{(\lambda - 1 - |\xi|^2)^2}{(\lambda - 1 - |\xi|^2)^2 + |\xi|^2(1 + |\xi|^2)} \qquad \text{for all } \xi \in \mathbb{R}^d.$$

By standard computations, one can check that there exists $C > 0$ independent of λ, ξ such that

$$\sup \left\{ |\xi^\alpha \partial_\xi^\alpha \widetilde{m}_\lambda(\xi)| \, : \, \alpha = (\alpha_1, \dots, \alpha_d) \text{ such that } \alpha_k \in \{0, 1\} \right\} \leq C.$$

Thus (7.20) follows from [96, Theorem 4.3.9].

Step 2: Boundedness of the H^∞-calculus. The claim of this step can be proven also by employing Theorem 6.9 and Proposition 5.11. Indeed, by Theorem 6.9 and Step 1, it is enough to show that $\mathcal{A}_2^\Delta$ has a bounded H^∞-calculus. Note that the consistency follows from Lemma 6.4 and Step 1. Since Δ_2^{w} and Δ_2^{s} generate C_0–semigroup of contractions, by Theorem 6.9 and Proposition 5.11, it is enough to check skew-symmetry as in Remark 5.12 with $\gamma = (4\pi^2)^{-1}$, that is for all $f \in H^2(\mathbb{R}^d)^d$ and $g \in H^1(\mathbb{R}^d)$,

$$\gamma \Re(\operatorname{div}(\mathbb{1} - \Delta) f | g)_{H^{-1}(\mathbb{R}^d)} + \Re(\nabla g | f)_{L^2(\mathbb{R}^d;\mathbb{R}^d)} = 0. \tag{7.21}$$

To prove (7.21), we equip the space $H^{-1}(\mathbb{R}^d)$ with the inner product

$$(f|g)_{H^{-1}(\mathbb{R}^d)} = \int_{\mathbb{R}^d} (1 + |\xi|^2)^{-1} \widehat{f}(\xi) \cdot \overline{\widehat{g}(\xi)} \, \mathrm{d}\xi, \tag{7.22}$$

where $\widehat{h}(\xi) \stackrel{\text{def}}{=} \int_{\mathbb{R}^d} h(x) e^{-2\pi i x \cdot \xi} \, \mathrm{d}x$ denotes the Fourier transform of h at $\xi \in \mathbb{R}^d$. Since the Fourier transform of $\operatorname{div}(\mathbb{1} - \Delta) f$ is given by $-4\pi^2(1 + |\xi|^2)(2\pi i \xi \cdot \widehat{f})$, we have

$$\begin{aligned}
\big(\operatorname{div}(\mathbb{1} - \Delta) f | g\big)_{H^{-1}(\mathbb{R}^d)} &= -4\pi^2 \int_{\mathbb{R}^d} (2\pi i \xi \cdot \widehat{f}(\xi)) \overline{\widehat{g}(\xi)} \, \mathrm{d}\xi \\
&= -4\pi^2 \int_{\mathbb{R}^d} \widehat{f}(\xi) \cdot \overline{\widehat{\nabla g}(\xi)} \, \mathrm{d}\xi \\
&= -4\pi^2 (f | \nabla g)_{L^2(\mathbb{R}^d)^d} = -4\pi^2 \overline{(\nabla g | f)_{L^2(\mathbb{R}^d)^d}}.
\end{aligned}$$

This yields (7.21) and therefore $\mathcal{A}_2^\Delta$ has a bounded H^∞-calculus by Proposition 5.11 and Remark 5.12.

Step 3: Proof of (7.19). The case $\beta \in [0, 1]$ follows from Proposition 5.9 recalling that, for all $\beta \in \mathbb{R}$,

$$\dot{\mathsf{D}}((\mathbb{1} - \Delta_p^{\mathsf{s}})^\beta) = H^{2\beta,p}(\mathbb{R}^d) \quad \text{ and } \quad \dot{\mathsf{D}}((\mathbb{1} - \Delta_p^{\mathsf{w}})^\beta) = \dot{\mathsf{D}}((-\Delta_p^{\mathsf{s}})^{\beta - 1}). \tag{7.23}$$

The case $p = 2$ follows from Step 2, Proposition 5.14 and (7.23). It remains to study the case $\beta < 0$ and $p \neq 2$. To this end, we apply Theorem 6.9(2).

Case $p \in (2,\infty)$: Let $r \in (p,\infty)$ be arbitrary. For all $\theta \in [0,1]$ we set $\mathcal{A}_\theta = \mathcal{A}_q^{\Delta}$ where $\frac{1}{q} = \frac{1-\theta}{2} + \frac{\theta}{r}$. By Step 1 and Lemma 6.4, $(\mathcal{A}_\theta)_{\theta\in[0,1]}$ is a consistent family of sectorial operators. By Step 2, (7.23) and Theorem 6.9(2),

$$\dot{\mathsf{D}}((\mathcal{A}_p^{\Delta})^{-\beta}) = \dot{\mathsf{D}}((\Delta_p^{\mathsf{w}})^{-\beta}) \times \dot{\mathsf{D}}((\Delta_p^{\mathsf{s}})^{-\beta}) = \dot{H}^{-1+2\beta,p}(\mathbb{R}^d) \times \dot{H}^{2\beta,p}(\mathbb{R}^d)^d \tag{7.24}$$

for all $\beta \in (0, \frac{1}{2}(1-\varphi))$ where $\varphi \in (0,1)$ satisfy $\frac{1}{q} = \frac{1-\varphi}{2} + \frac{\varphi}{r}$. Since $1-\varphi = \frac{2}{q}\frac{r-q}{r-2}$, (7.24) holds for all $\beta \in (0, \frac{1}{q}\frac{r-q}{r-2})$. Thus (7.19) in this case follows by letting $r \to \infty$.

Case $p \in (1,2)$: The argument is similar to the previous case considering $\mathcal{A}_\theta = \mathcal{A}_q^{\Delta}$ where $\frac{1}{q} = \frac{1-\theta}{r} + \frac{\theta}{2}$ where $r \in (1,p)$ is arbitrary and letting $r \to 1$. $\square$

Data availability

No data was used for the research described in the article.

Acknowledgment

We would like to thank Tim Binz, Emiel Lorist and Mark Veraar for valuable discussions. We also thank the anonymous referees for their helpful comments and suggestions, and for the very accurate reading of the manuscript.

Appendix A. A transference result for the H^∞-calculus

Let X be a Banach space and $p \in [1,2]$, then the space X has *type* p if there exists a constant $C_p \geq 0$ such that for all finite sequences $x_1,\dots,x_N$ in X and $\varepsilon_1,\dots,\varepsilon_N$ being Rademacher sequences on a probability space $(\Omega,\mathscr{A},\mathbb{P})$ one has

$$\mathbb{E}\Big\| \sum_{n=1}^N \varepsilon_n x_n \Big\|_X^p \leq C_p \sum_{n=1}^N \|x\|_X^p,$$

and X has *non-trivial type* if it has type p for some $p \in (1,2]$, compare e.g. [50, Definition 7.1.1 f.].

Theorem A.1. *Assume that X reflexive Banach space with non-trivial type. Let T be a linear operator on X with a bounded H^∞-calculus and let S be an $\mathcal{R}$-sectorial operator on X such that*

$$\mathsf{D}(T) = \mathsf{D}(S) \quad \textit{and} \quad \|Tx\|_X \eqsim \|Sx\|_X \ \textit{ for all } x \in \mathsf{D}(T). \tag{A.1}$$

Suppose that for some $\delta \in (0,1)$ one has

$$\begin{aligned} S(\mathsf{D}(T^{1+\delta})) \subseteq \mathsf{D}(T^\delta) \ \textit{and} \quad & \|T^\delta S x\|_X \lesssim \|T^{1+\delta}x\|_X \ \textit{for all } x \in \mathsf{D}(T^{1+\delta}), \\ \mathsf{R}(S) \subseteq \mathsf{R}(T^\delta) \ \textit{and} \quad & \|T^{-\delta}Sx\|_X \lesssim \|T^{1-\delta}x\|_X \ \textit{for all } x \in \mathsf{D}(T). \end{aligned} \tag{A.2}$$

Then S has a bounded H^∞-calculus and $\omega_{\mathcal{R}}(S) = \omega_{H^\infty}(S)$.

The above result is folklore known to experts. For the reader's convenience we provide the proof extending the arguments in [57, Section 9]. It seems that the geometric assumptions on X can be removed by using in the proof below the results in [58, Subsection 5.3], instead of [57, Corollary 7.8].

Proof. Let us begin by collecting some useful facts. By [50, Corollary 10.4.10], it is enough to show that S has a bounded H^∞-calculus. Recall that $(\dot{\mathsf{D}}(T^\beta))_{\beta\in\mathbb{R}}$ is a complex interpolation scale w.r.t. β, cf. [57, Proposition 2.2], that is

$$[\dot{\mathsf{D}}(T^{\beta_1}), \dot{\mathsf{D}}(T^{\beta_2})]_\theta = \dot{\mathsf{D}}(T^{\beta_1(1-\theta)+\beta_2\theta}), \qquad \text{for all } \theta \in (0,1), \tag{A.3}$$

since T has a bounded H^∞-calculus and hence bounded imaginary powers. Recall that, by the definition of $\dot{\mathsf{D}}(T^\beta)$ in Subsection 2.3,

$$\mathsf{D}(T^\alpha) \hookrightarrow \dot{\mathsf{D}}(T^\beta) \text{ is dense for all } 0 \le \beta \le \alpha < \infty. \tag{A.4}$$

Moreover, the operator T induces operators

$$\dot{T}_\theta \colon \dot{\mathsf{D}}(T^{1+\theta}) \cap \dot{\mathsf{D}}(T^\theta) \subseteq \dot{\mathsf{D}}(T^\theta) \to \dot{\mathsf{D}}(T^\theta) \quad \text{ for all } \theta \in \mathbb{R}, \tag{A.5}$$

on the $\dot{\mathsf{D}}(T^\theta)$-scale of spaces, compare [57, Proposition 2.1] or Subsection 2.3.

By (A.2), (A.4), $\delta < 1$ and the definition of the homogeneous scale (2.3), the operator S extends uniquely to bounded linear operators $\widetilde{S}_{\pm\delta} \in \mathscr{L}(\dot{\mathsf{D}}(T^{1\pm\delta}), \dot{\mathsf{D}}(T^{\pm\delta}))$ satisfying $\widetilde{S}_{\pm\delta}x = Sx$ for all $x \in \dot{\mathsf{D}}(T^{1\pm\delta}) \cap \mathsf{D}(T)$. By restriction, we obtain the following unbounded linear operator

$$\begin{gathered} S_{\pm\delta} \colon \dot{\mathsf{D}}(T^{1\pm\delta}) \cap \dot{\mathsf{D}}(T^{\pm\delta}) \subseteq \dot{\mathsf{D}}(T^{\pm\delta}) \to \dot{\mathsf{D}}(T^{\pm\delta}), \\ \text{satisfying} \quad S_{\pm\delta}x = Sx \ \text{ for all } x \in \dot{\mathsf{D}}(T^{1\pm\delta}) \cap \mathsf{D}(T). \end{gathered}$$

Similarly, by restriction, interpolation and (A.3), S induced uniquely linear operators

$$S_\theta \colon \dot{\mathsf{D}}(T^{\theta+1}) \cap \dot{\mathsf{D}}(T^\theta) \subseteq \dot{\mathsf{D}}(T^\theta) \to \dot{\mathsf{D}}(T^\theta) \quad \text{ for all } \theta \in [-\delta, \delta]. \tag{A.6}$$

Next we will need the following lemma which is proven below.

Lemma A.2. *Under the assumptions of Theorem A.1 there exists a $\delta_0 \in (0,1)$ such that $S_{-\delta_0}$ is $\mathcal{R}$-sectorial on $\dot{\mathsf{D}}(T^{-\delta_0})$ with angle $\le \omega_{\mathcal{R}}(S) \vee \omega_{H^\infty}(T)$, and moreover $\dot{\mathsf{D}}(S_{-\delta_0}) = \dot{\mathsf{D}}(T^{1-\delta_0})$.*

To verify that [57, Corollary 7.8] is applicable to $S_{-\delta_0}$ one needs that by Lemma A.2 $S_{-\delta_0}$ is $\mathcal{R}$-sectorial on $\dot{\mathsf{D}}(T^{-\delta_0})$ and that the spaces $\dot{\mathsf{D}}(T^\theta)$ have non-trivial type, which is equivalent to B-convexity, see [50, Proposition 7.6.8]. This follows since $\dot{\mathsf{D}}(T^\theta)$ are isomorphic to X, compare [57, Section 2]. Next, let $\langle\cdot,\cdot\rangle_\theta$ be the Rademacher interpolation functor introduced in [57, Section 7] (see also [58,83]). By [57, Corollary 7.8], for all $\theta\in(0,1)$, the operator $S_{\theta-\delta_0}|_{\langle\dot{\mathsf{D}}((S_{-\delta_0})^0),\dot{\mathsf{D}}(S_{-\delta_0})\rangle_\theta}$ induced by $S_{-\delta_0}$ on $\langle\dot{\mathsf{D}}((S_{-\delta_0})^0),\dot{\mathsf{D}}(S_{-\delta_0})\rangle_\theta$ has a bounded H^∞-calculus (cf. [74, Proposition 15.24] and [83, Proposition 5.3.5] for similar situations). Note that, by using the last assertion of Lemma A.2,

$$\begin{aligned}\langle\dot{\mathsf{D}}((S_{-\delta_0})^0),\dot{\mathsf{D}}(S_{-\delta_0})\rangle_\theta &= \langle\dot{\mathsf{D}}(T^{-\delta_0}),\dot{\mathsf{D}}(T^{1-\delta_0})\rangle_\theta\\ &\overset{(i)}{=} [\dot{\mathsf{D}}(T^{-\delta_0}),\dot{\mathsf{D}}(T^{1-\delta_0})]_\theta = \dot{\mathsf{D}}(T^{\theta-\delta_0}),\end{aligned}$$

where in (i) we applied [57, Theorem 7.4] to T which guarantees that here the complex and the Rademacher interpolation scale agree for T. Collecting the previous facts, we have

$$S_{\theta-\delta_0}|_{\dot{\mathsf{D}}(T^{\theta-\delta_0})} \text{ has a bounded } H^\infty\text{-calculus for all } \theta\in(0,1) \text{ on } \dot{\mathsf{D}}(T^{\theta-\delta_0}).$$

Choosing $\theta=\delta_0<1$ and using that $\dot{\mathsf{D}}(T^0)=X$, we obtain that $S_0=S$ has a bounded H^∞-calculus on X. This completes the proof of Theorem A.1. □

Proof of Lemma A.2. We split the proof into two steps.

Step 1: There exists $\delta_1\in(0,\delta)$ such that $\dot{\mathsf{D}}(S_\theta)=\dot{\mathsf{D}}(T^{1+\theta})$ for all $\theta\in(-\delta_1,\delta_1)$. Firstly, let us note that the sectoriality of S and (A.1) imply that S extends to an isomorphism $\widetilde{S}$ between $\dot{\mathsf{D}}(T)$ and X. Secondly, recall that S_θ are defined as explained before (A.6) via complex interpolation and restriction. Hence, by (A.3) and the Sneiberg lemma (see e.g. [106, Theorem 2.3 and 3.6]) there exists a $\delta_1\in(0,\delta)$ such that $\widetilde{S}_\theta$ is an isomorphism for all $\theta\in[-\delta_1,\delta_1]$. In particular

$$\|\widetilde{S}_\theta x\|_{\dot{\mathsf{D}}(T^\theta)} \eqsim \|x\|_{\dot{\mathsf{D}}(T^{1+\theta})} \quad \text{for all } \theta\in[-\delta_1,\delta_1].$$

Now the conclusion follows from the definition of S_θ as the restriction of $\widetilde{S}_\theta$ on $\dot{\mathsf{D}}(T^\theta)$ and a density argument (see e.g. [58, Proposition 5.3.1]).

Step 2: There exists $\delta_0\in(0,1)$ such that $S_{-\delta_0}$ is $\mathcal{R}$-sectorial on $\dot{\mathsf{D}}(T^{-\delta_0})$ with angle $\le \omega_{\mathcal{R}}(S)\vee\omega_{H^\infty}(T)$. Let $\delta_1\in(0,\delta)$ be as in Step 1. We begin by introducing suitable spaces of sequences. To this end, let $(\lambda_j)_{j\ge1}\subseteq \complement\Sigma_\phi$ be a dense subset with $\phi\in(\omega_{\mathcal{R}}(S)\vee\omega_{H^\infty}(T),\pi)$, and let $(\varepsilon_j)_{j\ge1}$ be a Rademacher sequence over a probability space $(\Omega,\mathscr{A},\mathbb{P})$. Then, for $\theta\in[-\delta_1,\delta_1]$, we set

$$\begin{aligned}\mathcal{X}_\theta &\overset{\text{def}}{=} \Big\{(x_j)_{j\ge1}\subseteq\dot{\mathsf{D}}(T^\theta)\colon \mathbb{E}\Big\|\sum_{j\ge1}\varepsilon_j x_j\Big\|_{\dot{\mathsf{D}}(T^\theta)}<\infty\Big\},\\ \mathcal{Y}_\theta &\overset{\text{def}}{=} \Big\{(x_j)_{j\ge1}\subseteq\dot{\mathsf{D}}(T^{\theta+1})\cap\dot{\mathsf{D}}(T^\theta)\colon \mathbb{E}\Big\|\sum_{j\ge1}(\lambda_j-\dot{T}_\theta)\varepsilon_j x_j\Big\|_{\dot{\mathsf{D}}(T^\theta)}<\infty\Big\},\end{aligned} \tag{A.7}$$

endowed with the natural norms.

Since X has non-trivial type and $\dot{\mathsf{D}}(T^\theta)$ is isomorphic to X, compare [57, Section 2], $\dot{\mathsf{D}}(T^\theta)$ has non-trivial type as well and therefore, it is K-convex due to [50, Theorem 7.4.23]. In particular, $(\mathcal{X}_\theta)_{\theta\in[-\delta_1,\delta_1]}$ is a complex interpolation scale by [50, Theorem 7.4.16(1)] and (A.3). By $\mathcal{R}$-sectoriality of T, the assignment $(x_j)_{j\geq 1} \mapsto \big((\lambda_j - \dot{T}_\theta)^{-1}x_j\big)_{j\geq 1}$ induces an isomorphism between $\mathcal{X}_\theta$ and $\mathcal{Y}_\theta$, cf. (2.5). By compatibility (2.6) and [57, Proposition 2.1], $(\mathcal{Y}_\theta)_{\theta\in[-\delta_1,\delta_1]}$ is also a complex scale.

For any sequence $\mathbf{x} = (x_j)_{j\geq 1} \in \mathcal{Y}_\theta$ we set

$$\mathcal{T}_\theta(\mathbf{x}) = ((\lambda_j - S_\theta)x_j)_{j\geq 1}.$$

By (A.5)–(A.6) and $\phi > \omega_{\mathcal{R}}(S) \vee \omega_{H^\infty}(T)$, we get that

$$\mathcal{T}_\theta\colon \mathcal{Y}_\theta \to \mathcal{X}_\theta \text{ is bounded for } \theta \in [-\delta_1, \delta_1].$$

Next we prove that $\mathcal{T}_0 \in \mathscr{L}(\mathcal{Y}_0, \mathcal{X}_0)$ is an isomorphism with inverse

$$\mathcal{S}_0(\mathbf{x}) = ((\lambda_j - S)^{-1}x_j)_{j\geq 1},$$

where $S = S_0$. Clearly, $\mathcal{T}_0\mathcal{S}_0$ and $\mathcal{S}_0\mathcal{T}_0$ are equal to the identity on the subset of finite sequences in $\mathcal{Y}_0$ and $\mathcal{X}_0$, respectively. By density it remains to prove that $\mathcal{S}_0 \in \mathscr{L}(\mathcal{X}_0, \mathcal{Y}_0)$. To this end, recall that $\mathsf{D}(T) = \mathsf{D}(S)$ and thus $\|Tx\|_X \eqsim \|Sx\|_X$ for all $x \in \mathsf{D}(T)$. The latter implies

$$\begin{aligned}
&\|\mathcal{S}_0(\mathbf{x})\|_{\mathcal{Y}_0}\\
&\leq \mathbb{E}\Big\|\sum_{j\geq 1}\varepsilon_j\lambda_j(\lambda_j - S)^{-1}x_j\Big\|_X + \mathbb{E}\Big\|T\sum_{j\geq 1}\varepsilon_j(\lambda_j - S)^{-1}x_j\Big\|_X\\
&\eqsim \mathbb{E}\Big\|\sum_{j\geq 1}\varepsilon_j\lambda_j(\lambda_j - S)^{-1}x_j\Big\|_X + \mathbb{E}\Big\|\sum_{j\geq 1}\varepsilon_j S(\lambda_j - S)^{-1}x_j\Big\|_X\\
&\overset{(i)}{\leq} \sup_{J\geq 1}\Big(\mathbb{E}\Big\|\sum_{1\leq j\leq J}\varepsilon_j\lambda_j(\lambda_j - S)^{-1}x_j\Big\|_X + \mathbb{E}\Big\|\sum_{1\leq j\leq J}\varepsilon_j S(\lambda_j - S)^{-1}x_j\Big\|_X\Big)\\
&\overset{(ii)}{\lesssim} \mathcal{R}(S(\lambda - S)^{-1} : \lambda \in \Sigma_\phi)\,\mathbb{E}\Big\|\sum_{j\geq 1}\varepsilon_j x_j\Big\|_X \lesssim \|\mathbf{x}\|_{\mathcal{X}_0},
\end{aligned} \tag{A.8}$$

where in (i) we used Fatou's lemma and in (ii) the $\mathcal{R}$-sectoriality of S, $(\lambda_j)_{j\geq 1} \subseteq \complement\overline{\Sigma_\phi}$ and $\phi > \omega_{\mathcal{R}}(S) \vee \omega_{H^\infty}(T)$. Note that the constants (A.8) are independent of the choice of the sequence $(\lambda_j)_{j\geq 1} \subseteq \complement\Sigma_\phi$.

Similar to Step 1, Sneiberg's lemma [106, Theorem 2.3 and 3.6] is applicable to $\mathcal{T}_\theta$. Thus there exists $\delta_0 \in (0, \delta_1)$, independent of $(\lambda_j)_{j\geq 1} \subseteq \complement\overline{\Sigma_\phi}$, such that

$$\mathcal{T}_\theta : \mathcal{Y}_\theta \to \mathcal{X}_\theta \text{ is boundedly invertible for all } \theta \in [-\delta_0, \delta_0] \tag{A.9}$$

and

$$\|\mathcal{T}_\theta^{-1}\|_{\mathscr{L}(\mathcal{X}_\theta,\mathcal{Y}_\theta)} \leq C_\theta \mathcal{R}(S(\lambda-S)^{-1} : \lambda \in \Sigma_\phi) \stackrel{\text{def}}{=} C_{S,\theta},$$

where $C_\theta > 0$ is independent of the sequence $(\lambda_j)_{j\geq 1}$.

Since $\delta_0 < \delta < 1$, by [58, Proposition 5.3.1] the embedding $\mathsf{D}(T)\cap\mathsf{R}(T) \hookrightarrow \dot{\mathsf{D}}(T^\theta)$ is dense for all $\theta \in [-\delta_0,\delta_0]$. Hence, one can check that $\mathcal{T}_\theta^{-1} : \mathcal{X}_\theta \to \mathcal{Y}_\theta$ for all $\theta \in [-\delta_0,\delta_0]$ is given by

$$\mathcal{T}_\theta^{-1}(\mathbf{x}) \stackrel{\text{def}}{=} ((\lambda_j - S_\theta)^{-1}x_j)_{j\geq 1}.$$

Since $\mathcal{T}_{\delta_0}^{-1} \in \mathscr{L}(\mathcal{X}_{-\delta_0},\mathcal{Y}_{-\delta_0})$, for all finite set $J \subseteq \mathbb{Z}$ and finite sequence $(x_j)_{j=1}^{\# J} \subseteq \dot{\mathsf{D}}(T^{-\delta_0})$ we have

$$\mathbb{E}\Big\|\sum_{j\in J}\varepsilon_j\lambda_j(\lambda_j - S_{-\delta_0})^{-1}x_j\Big\|_{\dot{\mathsf{D}}(T^{-\delta_0})} \leq C_{S,\theta}\,\mathbb{E}\Big\|\sum_{j\in J}\varepsilon_j x_j\Big\|_{\dot{\mathsf{D}}(T^{-\delta_0})}. \tag{A.10}$$

Recall that $(\lambda_j)_{j\geq 1} \supseteq \complement\overline{\Sigma_\phi}$ is dense. Equation (A.10) and the continuity of the resolvent map $\lambda \mapsto (\lambda - S_{-\delta_0})^{-1}$ ensure that $\rho(S_{-\delta_0}) \supseteq \complement\overline{\Sigma_\phi}$ and (A.10) holds for all finite set $(\lambda_j)_{j\in J} \subseteq \mathbb{C}\setminus\Sigma_\phi$. In particular, $S_{-\delta_0}$ is $\mathcal{R}$-sectorial with angle $< \phi$. The arbitrariness of ϕ yields $\omega_{\mathcal{R}}(S_{-\delta_0}) \leq \omega_{\mathcal{R}}(S) \vee \omega_{H^\infty}(T)$. $\square$

References

[1] M. Adler, K.-J. Engel, Spectral theory for structured perturbations of linear operators, J. Spectr. Theory 8 (4) (2018) 1393–1442.

[2] S. Agmon, A. Douglis, L. Nirenberg, Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions, II, Commun. Pure Appl. Math. 17 (1964) 35–92.

[3] A. Agresti, N. Lindemulder, M.C. Veraar, On the trace embedding and its applications to evolution equations, Math. Nachr. 296 (4) (2023) 1319–1350.

[4] A. Agresti, M.C. Veraar, Stability properties of stochastic maximal L^p-regularity, J. Math. Anal. Appl. 482 (2) (2020) 123553.

[5] A. Agresti, M.C. Veraar, Nonlinear parabolic stochastic evolution equations in critical spaces part I. Stochastic maximal regularity and local existence, Nonlinearity 35 (8) (2022) 4100–4210.

[6] A. Agresti, M.C. Veraar, Nonlinear parabolic stochastic evolution equations in critical spaces part II, J. Evol. Equ. 22 (2) (2022) 56.

[7] H. Amann, Linear and Quasilinear Parabolic Problems. Vol. I, Abstract Linear Theory, Monographs in Mathematics, vol. 89, Birkhäuser Boston Inc., Boston, MA, 1995.

[8] H. Amann, M. Hieber, G. Simonett, Bounded H_∞-calculus for elliptic operators, Differ. Integral Equ. 7 (3–4) (1994) 613–653.

[9] A. Amansag, H. Bounit, A. Driouich, S. Hadd, Staffans-Weiss perturbations for maximal L^p-regularity in Banach spaces, J. Evol. Equ. 22 (1) (2022) 15.

[10] Y. Arlinskiĭ, On sectorial block operator matrices, Mat. Fiz. Anal. Geom. 9 (4) (2002) 533–571.

[11] P. Auscher, N. Badr, R. Haller-Dintelmann, J. Rehberg, The square root problem for second-order, divergence form operators with mixed boundary conditions on L^p, J. Evol. Equ. 15 (1) (2015) 165–208.

[12] T. Bárta, On R-sectorial derivatives on Bergman spaces, Bull. Aust. Math. Soc. 77 (2) (2008) 305–313.

[13] A. Bátkai, P. Binding, A. Dijksma, R. Hryniv, H. Langer, Spectral problems for operator matrices, Math. Nachr. 278 (12–13) (2005) 1408–1429.

[14] A. Bátkai, P. Csomós, K.-J. Engel, B. Farkas, Stability and convergence of product formulas for operator matrices, Integral Equ. Oper. Theory 74 (2) (2012) 281–299.
[15] A. Ben Amar, A. Jeribi, B. Krichen, Essential spectra of a 3×3 operator matrix and an application to three-group transport equations, Integral Equ. Oper. Theory 68 (1) (2010) 1–21.
[16] J. Bergh, J. Löfström, Interpolation Spaces. An Introduction, Grundlehren der Mathematischen Wissenschaften, vol. 223, Springer-Verlag, Berlin, 1976.
[17] S. Cardanobile, D. Mugnolo, Qualitative properties of coupled parabolic systems of evolution equations, Ann. Sc. Norm. Super. Pisa, Cl. Sci. (5) 7 (2) (2008) 287–312.
[18] S. Charfi, J. Aref, R. Moalla, Essential spectra of operator matrices and applications, Math. Methods Appl. Sci. 37 (4) (2014) 597–608.
[19] A.J. Chorin, The numerical solution of the Navier-Stokes equations for an incompressible fluid, Bull. Am. Math. Soc. 73 (1967) 928–931.
[20] A.J. Chorin, Numerical solution of the Navier-Stokes equations, Math. Comput. 22 (1968) 745–762.
[21] A.P. Choudhury, A. Hussein, P. Tolksdorf, Nematic liquid crystals in Lipschitz domains, SIAM J. Math. Anal. 50 (4) (2018) 4282–4310.
[22] Thierry Coulhon, Damien Lamberton, Régularité L^p pour les équations d'évolution, in: Séminaire d'Analyse Fonctionelle 1984/1985, in: Publ. Math. Univ. Paris VII, vol. 26, Univ. Paris VII, Paris, 1986, pp. 155–165.
[23] R. Danchin, On the solvability of the compressible Navier-Stokes system in bounded domains, Nonlinearity 23 (2) (2010) 383–407.
[24] R. Denk, G. Dore, M. Hieber, J. Prüss, A. Venni, New thoughts on old results of R.T. Seeley, Math. Ann. 328 (4) (2004) 545–583.
[25] R. Denk, M. Hieber, J. Prüss, R-boundedness, Fourier multipliers and problems of elliptic and parabolic type, Mem. Am. Math. Soc. 166 (788) (2003).
[26] R. Denk, F. Hummel, Dispersive mixed-order systems in L^p-Sobolev spaces and application to the thermoelastic plate equation, Adv. Differ. Equ. 24 (7–8) (2019) 377–406.
[27] R. Denk, M. Kaip, General Parabolic Mixed Order Systems in L_p and Applications, Operator Theory: Advances and Applications, vol. 239, Birkhäuser/Springer, Cham, 2013.
[28] R. Denk, R. Racke, Y. Shibata, L_p theory for the linear thermoelastic plate equations in bounded and exterior domains, Adv. Differ. Equ. 14 (7–8) (2009) 685–715.
[29] R. Denk, Y. Shibata, Maximal regularity for the thermoelastic plate equations with free boundary conditions, J. Evol. Equ. 17 (1) (2017) 215–261.
[30] M. Egert, On Kato's conjecture and mixed boundary conditions, PhD thesis, TU Darmstadt, Sierke, 2015.
[31] K.-J. Engel, Operator matrices and systems of evolution equations, Habilitationsschrift, Universität Tübingen, 1995.
[32] K.-J. Engel, R. Nagel, On the spectrum of certain systems of linear evolution equations, in: Differential Equations in Banach Spaces, Bologna, 1985, in: Lecture Notes in Math., vol. 1223, Springer, Berlin, 1986, pp. 102–109.
[33] J.L. Ericksen, Hydrostatic theory of liquid crystals, Arch. Ration. Mech. Anal. 9 (1962) 371–378.
[34] S. Fackler, The Kalton-Lancien theorem revisited: maximal regularity does not extrapolate, J. Funct. Anal. 266 (1) (2014) 121–138.
[35] S. Fackler, T. Nau, Local strong solutions for the non-linear thermoelastic plate equation on rectangular domains in L^p-spaces, Nonlinear Differ. Equ. Appl. 21 (6) (2014) 775–794.
[36] Stephan Fackler, Regularity properties of sectorial operators: counterexamples and open problems, in: Operator Semigroups Meet Complex Analysis, Harmonic Analysis and Mathematical Physics, in: Oper. Theory Adv. Appl., vol. 250, Birkhäuser/Springer, Cham, 2015, pp. 171–197.
[37] J. Fuhrmann, J. Lankeit, M. Winkler, A double critical mass phenomenon in ano-flux-Dirichlet Keller-Segel system, arXiv preprint, arXiv:2101.06748, 2021.
[38] G. Galdi, An Introduction to the Mathematical Theory of the Navier-Stokes Equations: Steady-State Problems, Springer Science & Business Media, 2011.
[39] L. Grubišić, V. Kostrykin, K.A. Makarov, S. Schmitz, K. Veselić, The Tan 2Θ theorem in fluid dynamics, J. Spectr. Theory 9 (4) (2019) 1431–1457.
[40] L. Grubišić, V. Kostrykin, K.A. Makarov, K. Veselić, Representation theorems for indefinite quadratic forms revisited, Mathematika 59 (1) (2013) 169–189.
[41] L. Grubišić, V. Kostrykin, K.A. Makarov, K. Veselić, The Tan 2Θ theorem for indefinite quadratic forms, J. Spectr. Theory 3 (1) (2013) 83–100.
[42] B.H. Haak, M. Haase, P.C. Kunstmann, Perturbation, interpolation, and maximal regularity, Adv. Differ. Equ. 11 (2) (2006) 201–240.

[43] M. Haase, The Functional Calculus for Sectorial Operators, Operator Theory: Advances and Applications, vol. 169, Birkhäuser Verlag, Basel, 2006.
[44] M. Hieber, M. Nesensohn, J. Prüss, K. Schade, Dynamics of nematic liquid crystal flows: the quasilinear approach, Ann. Inst. Henri Poincaré, Anal. Non Linéaire 33 (2) (2016) 397–408.
[45] M. Hieber, J. Prüss, Modeling and analysis of the Ericksen-Leslie equations for nematic liquid crystal flows, in: Handbook of Mathematical Analysis in Mechanics of Viscous Fluids, Springer, Cham, 2018, pp. 1075–1134.
[46] T. Hillen, K.J. Painter, A user's guide to PDE models for chemotaxis, J. Math. Biol. 58 (1–2) (2009) 183–217.
[47] D. Horstmann, From 1970 until present: the Keller-Segel model in chemotaxis and its consequences, I, Jahresber. Dtsch. Math.-Ver. 105 (3) (2003) 103–165.
[48] D. Horstmann, From 1970 until present: the Keller-Segel model in chemotaxis and its consequences, II, Jahresber. Dtsch. Math.-Ver. 106 (2) (2004) 51–69.
[49] J. Huang, J. Sun, A. Chen, C. Trunk, Invertibity of 2×2 operator matrices, Math. Nachr. 292 (11) (2019) 2411–2426.
[50] T.P. Hytönen, J.M.A.M. van Neerven, M.C. Veraar, L. Weis, Analysis in Banach Spaces. Vol. II. Probabilistic Methods and Operator Theory, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge, vol. 67, Springer, 2017.
[51] O.O. Ibrogimov, Essential spectrum of non-self-adjoint singular matrix differential operators, J. Math. Anal. Appl. 451 (1) (2017) 473–496.
[52] O.O. Ibrogimov, H. Langer, M. Langer, C. Tretter, Essential spectrum of systems of singular differential equations, Acta Sci. Math. (Szeged) 79 (3–4) (2013) 423–465.
[53] O.O. Ibrogimov, P. Siegl, C. Tretter, Analysis of the essential spectrum of singular matrix differential operators, J. Differ. Equ. 260 (4) (2016) 3881–3926.
[54] O.O. Ibrogimov, C. Tretter, Essential spectrum of elliptic systems of pseudo-differential operators on $L^2(\mathbb{R}^N) \oplus L^2(\mathbb{R}^N)$, J. Pseudo-Differ. Oper. Appl. 8 (2) (2017) 147–166.
[55] A. Jeribi, Spectral theory and applications of linear operators and block operator matrices, Springer, Cham, 2015.
[56] Y. Kagei, T. Nishida, Y. Teramoto, On the spectrum for the artificial compressible system, J. Differ. Equ. 264 (2) (2018) 897–928.
[57] N.J. Kalton, P.C. Kunstmann, L. Weis, Perturbation and interpolation theorems for the H^∞-calculus with applications to differential operators, Math. Ann. 336 (4) (2006) 747–801.
[58] N.J. Kalton, E. Lorist, L. Weis, Euclidean Structures and Operator Theory in Banach Spaces, 2019.
[59] N.J. Kalton, L. Weis, The H^∞-calculus and sums of closed operators, Math. Ann. 321 (2) (2001) 319–345.
[60] T. Kato, Fractional powers of dissipative operators, J. Math. Soc. Jpn. 13 (1961) 246–274.
[61] M. Köhne, J. Prüss, M. Wilke, On quasilinear parabolic evolution equations in weighted L_p-spaces, J. Evol. Equ. 10 (2) (2010) 443–463.
[62] H. Komatsu, Fractional powers of operators, Pac. J. Math. 19 (1966) 285–346.
[63] H. Komatsu, Fractional powers of operators. II. Interpolation spaces, Pac. J. Math. 21 (1967) 89–111.
[64] H. Komatsu, Fractional powers of operators. III. Negative powers, J. Math. Soc. Jpn. 21 (1969) 205–220.
[65] H. Komatsu, Fractional powers of operators. IV. Potential operators, J. Math. Soc. Jpn. 21 (1969) 221–228.
[66] H. Komatsu, Fractional powers of operators. V. Dual operators, J. Fac. Sci. Univ. Tokyo, Sect. I 17 (1970) 373–396.
[67] H. Komatsu, Fractional powers of operators. VI. Interpolation of non-negative operators and imbedding theorems, J. Fac. Sci. Univ. Tokyo, Sect. IA, Math. 19 (1972) 1–63.
[68] V. Kostrykin, K.A. Makarov, A.K. Motovilov, Perturbation of spectra and spectral subspaces, Trans. Am. Math. Soc. 359 (1) (2007) 77–89.
[69] V. Kostrykin, K.A. Makarov, A.K. Motovilov, A generalization of the $\tan 2\Theta$ theorem, in: Current Trends in Operator Theory and Its Applications, in: Oper. Theory Adv. Appl., vol. 149, Birkhäuser, Basel, 2004, pp. 349–372.
[70] V. Kostrykin, K.A. Makarov, A.K. Motovilov, On the existence of solutions to the operator Riccati equation and the $\tan\Theta$ theorem, Integral Equ. Oper. Theory 51 (1) (2005) 121–140.
[71] P.C. Kunstmann, M. Uhl, L^p-spectral multipliers for some elliptic systems, Proc. Edinb. Math. Soc. (2) 58 (1) (2015) 231–253.

[72] P.C. Kunstmann, On elliptic non-divergence operators with measurable coefficients, in: Nonlinear Elliptic and Parabolic Problems, in: Progr. Nonlinear Differential Equations Appl., vol. 64, Birkhäuser, Basel, 2005, pp. 265–272.
[73] P.C. Kunstmann, L. Weis, Perturbation theorems for maximal L_p-regularity, Ann. Sc. Norm. Super. Pisa, Cl. Sci. (4) 30 (2) (2001) 415–435.
[74] P.C. Kunstmann, L. Weis, Maximal L_p-regularity for parabolic equations, Fourier multiplier theorems and H^∞-functional calculus, in: Functional Analytic Methods for Evolution Equations, in: Lecture Notes in Math., vol. 1855, Springer, Berlin, 2004, pp. 65–311.
[75] P.C. Kunstmann, L. Weis, Erratum to: Perturbation and interpolation theorems for the H^∞-calculus with applications to differential operators, Math. Ann. 357 (2) (2013).
[76] P.C. Kunstmann, L. Weis, New criteria for the H^∞-calculus and the Stokes operator on bounded Lipschitz domains, J. Evol. Equ. 17 (1) (2017) 387–409.
[77] H. Langer, A. Markus, V. Matsaev, C. Tretter, A new concept for block operator matrices: the quadratic numerical range, Linear Algebra Appl. 330 (1–3) (2001) 89–112.
[78] J. Lankeit, M. Winkler, Facing low regularity in chemotaxis systems, Jahresber. Dtsch. Math.-Ver. 122 (1) (2020) 35–64.
[79] I. Lasiecka, R. Triggiani, Control Theory for Partial Differential Equations: Continuous and Approximation Theories. I. Abstract Parabolic Systems, Encyclopedia of Mathematics and Its Applications, vol. 74, Cambridge University Press, Cambridge, 2000.
[80] J. LeCrone, J. Prüss, M. Wilke, On quasilinear parabolic evolution equations in weighted L_p-spaces II, J. Evol. Equ. 14 (3) (2014) 509–533.
[81] F.M. Leslie, Some constitutive equations for liquid crystals, Arch. Ration. Mech. Anal. 28 (4) (1968) 265–283.
[82] F.-H. Lin, C. Liu, Nonparabolic dissipative systems modeling the flow of liquid crystals, Commun. Pure Appl. Math. 48 (5) (1995) 501–537.
[83] N. Lindemulder, E. Lorist, A discrete framework for the interpolation of Banach spaces, arXiv preprint, arXiv:2105.08373, 2021.
[84] J. Liu, J. Huang, A. Chen, Semigroup generations of unbounded block operator matrices based on the space decomposition, Oper. Matrices 14 (2) (2020) 295–304.
[85] K.A. Makarov, S. Schmitz, A. Seelmann, On invariant graph subspaces, Integral Equ. Oper. Theory 85 (3) (2016) 399–425.
[86] A. McIntosh, A. Yagi, Operators of type ω without a bounded H_∞ functional calculus, in: Miniconference on Operators in Analysis, Sydney, 1989, in: Proc. Centre Math. Anal. Austral. Nat. Univ., vol. 24, Austral. Nat. Univ., Canberra, 1990, pp. 159–172.
[87] M. Mitrea, S. Monniaux, Maximal regularity for the Lamé system in certain classes of non-smooth domains, J. Evol. Equ. 10 (4) (2010) 811–833.
[88] M. Möller, F.H. Szafraniec, Adjoints and formal adjoints of matrices of unbounded operators, Proc. Am. Math. Soc. 136 (6) (2008) 2165–2176.
[89] D. Mugnolo, Matrix methods for wave equations, Math. Z. 253 (4) (2006) 667–680.
[90] R. Nagel, Towards a "matrix theory" for unbounded operator matrices, Math. Z. 201 (1) (1989) 57–68.
[91] R. Nagel, Characteristic equations for the spectrum of generators, Ann. Sc. Norm. Super. Pisa, Cl. Sci. (4) 24 (4) (1998) 703–717, 1997.
[92] Rainer Nagel, Well-posedness and positivity for systems of linear evolution equations, Conf. Semin. Mat. Univ. Bari 203 (1985) 29.
[93] J.M.A.M. van Neerven, M.C. Veraar, L. Weis, Maximal L^p-regularity for stochastic evolution equations, SIAM J. Math. Anal. 44 (3) (2012) 1372–1414.
[94] J.M.A.M. van Neerven, M.C. Veraar, L. Weis, Stochastic maximal L^p-regularity, Ann. Probab. 40 (2) (2012) 788–812.
[95] J. Prüss, G. Simonett, Maximal regularity for evolution equations in weighted L_p-spaces, Arch. Math. 82 (5) (2004) 415–431.
[96] J. Prüss, G. Simonett, Moving Interfaces and Quasilinear Parabolic Evolution Equations, Monographs in Mathematics, vol. 105, Birkhäuser/Springer, 2016.
[97] J. Prüss, G. Simonett, M. Wilke, Critical spaces for quasilinear parabolic evolution equations and applications, J. Differ. Equ. 264 (3) (2018) 2028–2074.
[98] S. Schmitz, Representation theorems for indefinite quadratic forms without spectral gap, Integral Equ. Oper. Theory 83 (1) (2015) 73–94.
[99] R. Schnaubelt, M. Veraar, Structurally damped plate and wave equations with random point force in arbitrary space dimensions, Differ. Integral Equ. 23 (9–10) (2010) 957–988.

[100] R. Seeley, Interpolation in L^p with boundary conditions, Stud. Math. 44 (1972) 47–60.

[101] A. Seelmann, Notes on the subspace perturbation problem for off-diagonal perturbations, Proc. Am. Math. Soc. 144 (9) (2016) 3825–3832.

[102] A.A. Shkalikov, C. Trunk, On stability of closedness and self-adjointness for 2×2 operator matrices, Mat. Zametki 100 (6) (2016) 932–938.

[103] I.Ja. Šneĭberg, Spectral properties of linear operators in interpolation families of Banach spaces, Mat. Issled. 9 (2(32)) (1974) 214–229, 254–255.

[104] Suma'inna, The existence of $\mathcal{R}$-bounded solution operators of the thermoelastic plate equation with Dirichlet boundary conditions, Math. Methods Appl. Sci. 41 (4) (2018) 1578–1599.

[105] Suma'inna, H. Saito, Y. Shibata, On some nonlinear problem for the thermoplate equations, Evol. Equ. Control Theory 8 (4) (2019) 755–784.

[106] A. Tabacco Vignati, M. Vignati, Spectral theory and complex interpolation, J. Funct. Anal. 80 (2) (1988) 383–397.

[107] R. Temam, Sur l'approximation de la solution des équations de Navier-Stokes par la méthode des pas fractionnaires, I, Arch. Ration. Mech. Anal. 32 (1969) 135–153.

[108] R. Temam, Sur l'approximation de la solution des équations de Navier-Stokes par la méthode des pas fractionnaires, II, Arch. Ration. Mech. Anal. 33 (1969) 377–385.

[109] R. Temam, Navier-Stokes Equations: Theory and Numerical Analysis, AMS Chelsea Publishing, Providence, RI, 2001, Reprint of the 1984 edition.

[110] Y. Teramoto, Stability of bifurcating stationary solutions of the artificial compressible system, J. Math. Fluid Mech. 20 (3) (2018) 1213–1228.

[111] C. Tretter, Spectral Theory of Block Operator Matrices and Applications, Imperial College Press, London, 2008.

[112] H. Triebel, Interpolation Theory, Function Spaces, Differential Operators, second edition, Johann Ambrosius Barth, Heidelberg, 1995.

[113] L. Weis, Operator-valued Fourier multiplier theorems and maximal L_p-regularity, Math. Ann. 319 (4) (2001) 735–758.

[114] M. Wrona, Liquid Crystals and the Primitive Equations: An Approach by Maximal Regularity, PhD thesis, Technische Universität, Darmstadt, 2020.

[115] J. Zabczyk, On decomposition of generators, SIAM J. Control Optim. 16 (4) (1978) 523–534.

[116] J. Zabczyk, Erratum: "On decomposition of generators" [SIAM J. Control Optim. 16 (1978), no. 4, 523–534; MR 58 #23757], SIAM J. Control Optim. 18 (3) (1980) 325.