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David Mitrouskas ; Peter Pickl

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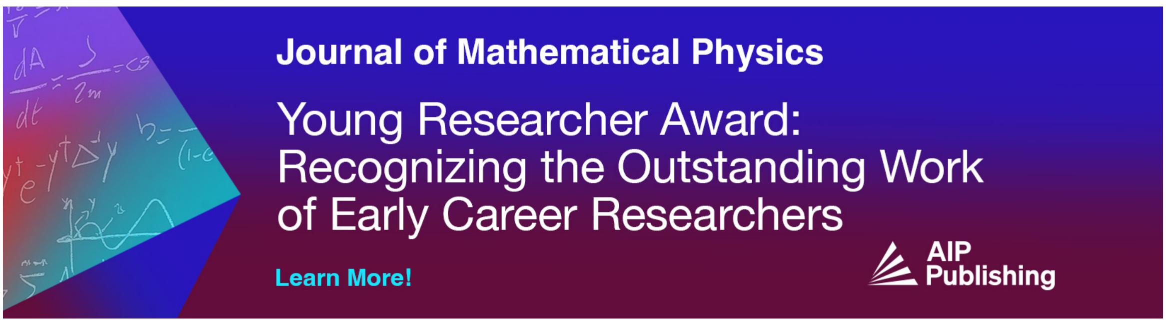

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David Mitrouskas[1,a)] and Peter Pickl[2,b)]

AFFILIATIONS

[1] Institute of Science and Technology Austria (ISTA), Am Campus 1, 3400 Klosterneuburg, Austria
[2] Universität Tübingen, Fachbereich Mathematik, Auf der Morgenstelle 10, 72076 Tübingen, Germany

[a)] Author to whom correspondence should be addressed: mitrouskas@ist.ac.at
[b)] p.pickl@uni-tuebingen.de

ABSTRACT

We consider N trapped bosons in the mean-field limit with coupling constant $\lambda_N = 1/(N-1)$. The ground state of such systems exhibits Bose–Einstein condensation. We prove that the probability of finding ℓ particles outside the condensate wave function decays exponentially in ℓ.

I. INTRODUCTION AND MAIN RESULT

We consider N bosons described by the Hamiltonian

$$H_N = \sum_{i=1}^{N} \left(-\Delta_i + V^{\text{ext}}(x_i)\right) + \frac{1}{N-1} \sum_{1\le i<j\le N} v(x_i - x_j) \tag{1.1}$$

acting on the N-particle Hilbert space $\mathfrak{H}^N = \otimes_{\text{sym}}^N L^2(\mathbb{R}^3)$. We shall work under the following assumptions:

(A1) The external potential $V^{\text{ext}} : \mathbb{R}^3 \to \mathbb{R}$ is measurable, locally bounded and satisfies $V^{\text{ext}}(x) \to \infty$ as $|x| \to \infty$, i.e., it acts as a confining potential.

(A2) The pair potential $v : \mathbb{R}^3 \to \mathbb{R}$ is either (i) a pointwise bounded, even function with non-negative Fourier transform $\hat{v}(k) \ge 0$ or (ii) the repulsive Coulomb potential $v(x) = \lambda |x|^{-1}$ with $\lambda > 0$.

Under these conditions, H_N is essentially self-adjoint and has a unique ground state, which we denote by Ψ_N. It is well understood that the ground state exhibits complete Bose–Einstein condensation (BEC) in the minimizer $\varphi \in L^2(\mathbb{R}^3)$ of the Hartree energy functional $u \mapsto N^{-1}\langle u^{\otimes N}, H_N u^{\otimes N}\rangle$. This means that in the limit $N \to \infty$, most of the particles occupy the same one-particle state $\varphi \in L^2(\mathbb{R}^3)$. To make this statement precise, let $p = |\varphi\rangle\langle\varphi|$ and consider the family of operators $\mathfrak{P}_N(\ell) : \mathfrak{H}^N \to \mathfrak{H}^N$ with $\ell \in \{0, \ldots, N\}$ given by

$$\mathfrak{P}_N(\ell) = \left((1-p)^{\otimes \ell} \otimes p^{\otimes N-\ell}\right)_{\text{sym}}. \tag{1.2}$$

It is straightforward to verify that

$$\mathfrak{P}_N(\ell)\mathfrak{P}_N(k) = \delta_{\ell k} \quad \text{and} \quad \mathbb{1} = \sum_{\ell=0}^{N} \mathfrak{P}_N(\ell). \tag{1.3}$$

The operator $\mathfrak{P}_N(\ell)$ projects onto states that contain $N-\ell$ particles in the condensate wave function φ and ℓ particles in the orthogonal complement $\{\varphi\}^\perp \subseteq L^2(\mathbb{R}^3)$. The number $P_N(\ell) := \|\mathfrak{P}_N(\ell)\Psi_N\|^2$ is thus the probability of finding ℓ particles in the ground state Ψ_N that do not occupy the condensate wave function φ. Complete BEC with optimal rate of condensation can be formulated as $P_N(0) = 1 + O(N^{-1})$ as $N \to \infty$. We refer to Refs. 1, 6, 8, 11, and 22 for results in the mean-field limit and Refs. 2, 3, 5, 9, 12, 13, 15, 18, and 19 for BEC in more singular scaling limits. For finite values of N, there is in general a non-vanishing probability $1 - P_N(0) = \sum_{\ell=1}^{N} P_N(\ell) > 0$ of finding particles outside the condensate. This work aims to establish strong asymptotic bounds on $P_N(\ell)$ for large ℓ and N. Our main result is that $P_N(\ell)$ decays exponentially in ℓ.

Theorem 1.1. *Under Assumptions (A1) and (A2) there exists a constant $\varepsilon > 0$ such that for every $f : \mathbb{N} \to \mathbb{N}$ with $f(n) \xrightarrow{n\to\infty} \infty$*

$$\lim_{N\to\infty} P_N(f(N)) \exp(\varepsilon f(N)) = 0. \tag{1.4}$$

For the homogeneous Bose gas on the torus with bounded pair potential of positive type, the theorem was stated and proved already in 2017 within the Ph.D. thesis (Ref. 14, Theorem 3.1). In this note, we present essentially the same proof, albeit somewhat simplified and with the correction of two minor errors. The generalizations to the trapped Bose gas and the repulsive Coulomb potential require only small modifications. As shown in Ref. 14, the statement can be extended to excited low-energy eigenstates of H_N but for conciseness, we shall address only the ground state here.

In Ref. 4, a related result was obtained under similar assumptions. The authors derived higher-moment bounds for the number of excitations of the form $\sum_{\ell=1}^{N} \ell^n P_N(\ell) \le C_n$ for all $n \in \mathbb{N}$ with unspecified constants C_n [for v bounded, they showed that $C_n \le (C(n+1))^{(n+6)^2}$ for some $C > 0$]. Theorem 1.1 dircetly implies that $C_n \le C^n n!$ for some $C > 0$. In Ref. 4, the higher-moment bounds were employed to obtain an asymptotic series for the ground state energy of H_N in inverse powers of $(N-1)^{-1}$. Our bounds on C_n could thus be potentially relevant for establishing certain resummation properties, such as Borel summability, of this asymptotic series, see Ref. 4 (Remark 3.5).

Very recently, Nam and Rademacher[17] achieved a major advancement by extending the exponential decay of $P_N(\ell)$ to dilute Bose gases. They consider the homogeneous Bose gas on the unit torus with pair potential $v(x) = N^{3\beta} v(N^{\beta} x)$, $\beta \in [0, 1]$, for non-negative compactly supported $v \in L^3([0, 1])$. This includes, in particular, the physically most relevant Gross–Pitaevskii regime with $\beta = 1$. Their result shows that for every low-energy eigenfunction $\psi_N \in \mathfrak{H}^N$, $\langle \psi_N, e^{\kappa \mathcal{N}} \psi_N \rangle = O(1)$ as $N \to \infty$ for some $\kappa > 0$, where $\mathcal{N} = \sum_{i=1}^{N} (1 - p_i)$ is the operator that counts the number of particles outside the condensate. Since $\langle \psi_N, e^{\kappa \mathcal{N}} \psi_N \rangle = \sum_{\ell=1}^{N} P_N(\ell) \exp(\kappa \ell)$, this proves (1.4) for the dilute Bose gas. Higher-moment bounds of the form $\langle \psi_N, \mathcal{N}^n \psi_N \rangle \le C_n$, $n \in \mathbb{N}$, have been obtained in the Gross–Pitaevskii regime in Ref. 2.

Finally, let us mention that exponential bounds for slightly different observables than $\mathcal{N}$ have been recently studied also in the context of large deviations[7,20,21] [see also Ref. 17 (Remark 1.3)].

The key idea of the Proof of Theorem 1.1 is to show that $P_N(\ell)$ satisfies an inequality of the form $P_N(\ell+2) + P_N(\ell-2) - 2P_N(\ell) \ge \sigma P_N(\ell)$ for some $\sigma > 0$. To obtain such a bound, we take the scalar product of the ground state eigenvalue equation with $\mathfrak{P}_N(\ell)\Psi_N$ and utilize the observation that the two-body potential in H_N acts, after subtraction of the mean-field contribution, effectively as a discrete second derivative in ℓ. To illustrate the simplicity of the idea, we provide a sketch of the argument in Sec. II B.

II. PROOF

The remainder of this note is organized as follows. In Sec. II A, we introduce the Fock space excitation formalism,[10,11] which is convenient for our analysis. In Sec. II A, we give a heuristic discussion of the proof and in Sec. II C, we state the main technical lemma and use this lemma to prove our main result. In Secs. II D and II E, we provide the proof of the technical lemma.

A. Excitation Hamiltonian

We define the Hartree energy as $e_{\mathrm{H}} := N^{-1} \inf_u \langle u^{\otimes N}, H_N u^{\otimes N} \rangle$, where the infimum is taken over all L^2-normalized $u \in H^1(\mathbb{R}^3)$. The corresponding unique positive minimizer is denoted by φ. For a proof of existence, uniqueness and positivity of the minimizer, see e.g. Ref. 4 (Lemma 2.2). Given the Hartree minimizer φ, we introduce the unitary excitation map $U_N(\varphi) : \mathfrak{H}^N \to \mathcal{F}_{\perp}^{\le N} := \oplus_{\ell=0}^{N} \otimes_{\mathrm{sym}}^{\ell} \{\varphi\}^{\perp}$ acting as

$$U_N(\varphi)\Phi_N = \bigoplus_{\ell=0}^{N} q^{\otimes \ell} \left(\frac{a(\varphi)^{\otimes N-\ell}}{\sqrt{(N-\ell)!}} \Phi_N \right), \quad \Phi_N \in \mathfrak{H}^N \tag{2.1}$$

where $q = 1 - |\varphi\rangle\langle\varphi|$ and $a(\varphi)$ is the usual bosonic annihilation operator.

To fix some notation and conventions, we call $\mathcal{F}_{\perp} := \oplus_{\ell=0}^{\infty} \otimes_{\mathrm{sym}}^{\ell} \{\varphi\}^{\perp}$ the full excitation Fock space and $\mathcal{F}_{\perp}^{\le N}$ its truncated version. We will denote the bosonic creation and annihilation operators on both these spaces by $a^*(f)$ and $a(f)$ for $f \in \{\varphi\}^{\perp}$. It will be convenient to also use pointwise creation and annihiliation operators, defined through $a(f) = \int dx \overline{f(x)}\, a_x$. Note that these satisfy $[a_x, a_y^*] = \delta(x-y)$ and $[a_x, a_y] = 0$. The number operator on $\mathcal{F}_{\perp}$ is abbreviated by $\mathcal{N} = \int dx\, a_x^* a_x$.

Next, we define the excitation Hamiltonian $\mathbb{H}$ as an operator acting on the truncated excitation Fock space $\mathcal{F}_{\perp}^{\le N}$ by

$$\begin{aligned} \mathbb{H} &:= U_N(\varphi)(H_N - N e_{\mathrm{H}}) U_N(\varphi)^* \\ &= \mathbb{K}_0 + \frac{1}{N-1} \left(\mathbb{K}_1\, \mathfrak{a}(\mathcal{N}) + (\mathbb{K}_2\, \mathfrak{b}(\mathcal{N}) + \text{h.c.}) + (\mathbb{K}_3\, \mathfrak{c}(\mathcal{N}) + \text{h.c.}) + \mathbb{K}_4 \right) \end{aligned} \tag{2.2}$$

with N-dependent functions

$$\mathfrak{a}(\ell) := N-\ell, \quad \mathfrak{b}(\ell) := \sqrt{(N-\ell)(N-\ell-1)}, \quad \mathfrak{c}(\ell) := \sqrt{N-\ell} \tag{2.3}$$

and N-independent operators $\mathbb{K}_0 := \mathrm{d}\Gamma(qhq)$ with $h: L^2(\mathbb{R}^3) \to L^2(\mathbb{R}^3)$ given by

$$h = -\Delta + V^{\text{ext}} + v * \varphi^2 - \langle \varphi, (-\Delta + V^{\text{ext}} + v * \varphi^2)\varphi \rangle \tag{2.4}$$

and

$$\mathbb{K}_1 := \int \mathrm{d}x_1 \mathrm{d}x_2\, K_1(x_1,x_2) a^*_{x_1} a_{x_2} \tag{2.5}$$

$$\mathbb{K}_2 := \frac{1}{2} \int \mathrm{d}x_1 \mathrm{d}x_2 K_2(x_1,x_2) a^*_{x_1} a^*_{x_2} \tag{2.6}$$

$$\mathbb{K}_3 := \int dx_1 \mathrm{d}x_2 \mathrm{d}x_3\, K_3(x_1,x_2,x_3) a^*_{x_1} a^*_{x_2} a_{x_3} \tag{2.7}$$

$$\mathbb{K}_4 := \frac{1}{2} \int \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3 dx_4\, K_4(x_1,x_2,x_3,x_4) a^*_{x_1} a^*_{x_2} a_{x_3} a_{x_4}. \tag{2.8}$$

With $K(x,y) := \varphi(y) v(x-y) \varphi(x)$ and

$$W(x,y) := v(x-y) - v * \varphi^2(x) - v * \varphi^2(y) + \langle \varphi, v * \varphi^2 \varphi \rangle, \tag{2.9}$$

the different kernels are given by

$$K_1(x_1,x_2) := \int \mathrm{d}y_1 \mathrm{d}y_2\, q(x_1,y_1) K(y_1,y_2) q(y_2,x_2), \tag{2.10}$$

$$K_2(x_1,x_2) := \int \mathrm{d}y_1 \mathrm{d}y_2\, q(x_1,y_1) q(x_2,y_2) K(y_1,y_2), \tag{2.11}$$

$$K_3(x_1,x_2,x_3) := \int \mathrm{d}y_1 \mathrm{d}y_2\, q(x_1,y_1) q(x_2,y_2) W(y_1,y_2) \varphi(y_1) q(y_2,x_3), \tag{2.12}$$

$$K_4(x_1,x_2,x_3,x_4) := \int \mathrm{d}y_1 \mathrm{d}y_2\, q(x_1,y_1) q(x_2,y_2) W(y_1,y_2) q(y_1,x_3) q(y_2,x_4), \tag{2.13}$$

where $q(x,y)$ is the integral kernel of $q = 1 - |\varphi\rangle\langle\varphi|$. For the derivation of (2.2), we refer to Refs. 4 and 11. Before we continue, let us note the important fact that the operator qhq with h defined in (2.4) has a spectral gap above zero, that is, $qhq \geq \tau$ for some number $\tau > 0$. This follows easily from $h\varphi = 0$, $\varphi > 0$ and $q\varphi = 0$. Consequently, we have $\mathbb{K}_0 \geq \tau \mathcal{N}$ as inequality on $\mathcal{F}_\perp$.

Denoting by $\Psi_N \in \mathfrak{H}^N$ the unique normalized ground state of H_N with ground state energy $E_N = \inf \sigma(H_N)$, we set $\chi := U_N(\varphi)\Psi_N$. By unitarity of $U_N(\varphi)$, it satisfies the eigenvalue equation $\mathbb{H}\chi = (E_N - N e_{\mathrm{H}})\chi$. In terms of $\chi = (\chi^{(\ell)})_{\ell=0}^N$ with $\chi^{(\ell)} \in \otimes^\ell_{\text{sym}} \{\varphi\}^\perp$, the probability of finding ℓ excitations outside the condensate wave function φ is given by $P_N(\ell) = \|\chi^{(\ell)}\|^2$.

B. Idea of the proof

To illustrate the idea of the proof Theorem 1.1, we demonstrate the argument for the ground state eigenfunction of the quadratic Bogoliubov approximation of $\mathbb{H}$. That is, we consider the eigenvalue equation $\mathbb{H}_0\phi = E_0\phi$ on $\mathcal{F}_\perp$, where

$$\mathbb{H}_0 = \mathbb{K}_0 + \mathbb{K}_1 + \mathbb{K}_2 + \mathbb{K}_2^\dagger \tag{2.14}$$

and $E_0 < 0$ is the lowest possible eigenvalue of $\mathbb{H}_0$. Existence and uniqueness of the ground state $\phi \in \mathcal{F}_\perp$ follow by unitary diagonalization of $\mathbb{H}_0$.[11,16] Similarly as for χ, the number of particles in ϕ correspond to the number of particles in the state $U_N(\varphi)^* \mathbb{1}(\mathcal{N} \leq N)\phi \in \mathfrak{H}^N$ that are not in the condensate wave function. In the following, we show that $\|\phi^{(\ell)}\|^2 \leq C \exp(-\varepsilon \ell)$ for some constants $C, \varepsilon > 0$ and all $\ell \geq 0$, thus proving an analogous statement to Theorem 1.1 for the Bogoliubov ground state. For the purpose of this demonstration, we assume that v is pointwise bounded with $\|v\|_\infty$ sufficiently small and $\hat{v} \geq 0$.

We start by taking the scalar product on both sides of the eigenvalue equation with $\phi^{(\ell)}$. Using $E_0 \leq 0$ and $\mathbb{K}_0 \geq \tau\mathcal{N}$ with $\tau > 0$, this implies

$$\tau \ell \|\phi^{(\ell)}\|^2 \leq -\left\langle \phi^{(\ell)} \mathbb{K}_2 \phi^{(\ell-2)} \right\rangle - \left\langle \phi^{(\ell)} \mathbb{K}_2^\dagger \phi^{(\ell+2)} \right\rangle - \left\langle \phi^{(\ell)} \mathbb{K}_1 \phi^{(\ell)} \right\rangle. \tag{2.15}$$

Since $\mathbb{K}_1 \geq 0$ (this follows from $\hat{v} \geq 0$), we can apply $4|\langle \phi^{(\ell)}, \mathbb{K}_2 \phi^{(\ell-2)} \rangle| \leq \langle \phi^{(\ell)}, \mathbb{K}_1 \phi^{(\ell)} \rangle + \langle \phi^{(\ell-2)}, \mathbb{K}_1 \phi^{(\ell-2)} \rangle + v(0) \|\phi^{(\ell-2)}\|^2$ and a similar bound for $\mathbb{K}_2^\dagger$ (see Lemma 2.3 for details). This leads to

$$\left(4\tau - \frac{v(0)}{\ell}\right)\ell\|\phi^{(\ell)}\|^2 \le \left\langle \phi^{(\ell+2)}\mathbb{K}_1\phi^{(\ell+2)}\right\rangle + \left\langle \phi^{(\ell-2)}\mathbb{K}_1\phi^{(\ell-2)}\right\rangle + v(0)\|\phi^{(\ell-2)}\|^2 - 2\left\langle \phi^{(\ell)}\mathbb{K}_1\phi^{(\ell)}\right\rangle \tag{2.16}$$

Next, we use $\mathbb{K}_1 \ge 0$ to estimate the last term and introduce the abbreviation $f(\ell) := \ell\|\phi^{(\ell)}\|^2$. Since by a short computation (see Lemma 2.2), one finds $|\langle \phi^{(\ell)}, \mathbb{K}_1\phi^{(\ell)}\rangle| \le C\|v\|_\infty f(\ell)$, we can bound the first two terms on the right-hand side, such that

$$\left(4\tau - \frac{v(0)}{\ell}\right)f(\ell) \le C\|v\|_\infty(f(\ell+2) + f(\ell-2)). \tag{2.17}$$

Dividing both sides by $C\|v\|_\infty$, the pre-factor on the left side is strictly larger than two if $\|v\|_\infty$ is chosen sufficiently small. Note that we are only interested in $\|v\|_\infty > 0$ since in the absence of interaction, $f(\ell) = 0$ for all ℓ. Also note that the spectral gap $\tau > 0$ of the operator qhq is uniform in $\|v\|_\infty \to 0$. Thus, we arrive at

$$\sigma f(\ell) \le f(\ell+2) + f(\ell-2) \tag{2.18}$$

for some $\sigma > 2$ and all $\ell \ge 2$. Considering $f(\ell)$ separately for ℓ even/odd, the difference inequality states that the second discrete derivative of $f(\ell)$ is bounded from below by $(\sigma-2)f(\ell)$. On the one hand, this shows that $f(\ell)$ is convex, and thus has at most one minimum $f(\ell_0)$. On the other hand, the inequality implies that $f(\ell) \le C(\sigma-2)^{-\ell+2}(f(1)+f(2))$ for $1 \le \ell \le \ell_0$ and $f(\ell) \ge (\sigma-2)^{\ell-\ell_0}f(\ell_0)$ for $\ell \ge \ell_0$. By normalization of ϕ, i.e., $\sum_{\ell=0}^{\infty}\|\phi^{(\ell)}\|^2 = 1$, and since $\sigma > 2$, this implies that $f(\ell)$ has no minimum. Consequently, $f(\ell) \le (\sigma-2)^{-\ell+2}(f(1) + f(2))$ for $\ell \ge 1$, as claimed.

In Sec. II C, we extend the above argument to the ground state $\chi = U_N(\varphi)\Psi_N$ of the excitation Hamiltonian $\mathbb{H}$ and remove the assumption that $\|v\|_\infty$ is small. This requires some technical modifications: Most importantly, in (2.16) we will not estimate the last term by $-\mathbb{K}_1 \le 0$. Instead, we sum both sides over $\ell - L, \ldots, \ell + L$ for some large but fixed integer L. While on the right side, many terms are canceled, the left-hand side is effectively increased by a factor proportional to L. This will help us to remove the smallness assumption on $\|v\|_\infty$. The Coulomb case requires another approximation argument that will be explained in Sec. II E. An obstacle in considering the full Hamiltonian $\mathbb{H}$ compared to $\mathbb{H}_0$ is the presence of $\mathbb{K}_3$ and $\mathbb{K}_4$ (for the Coulomb potential, $\mathbb{K}_4$ is not relevant since it is non-negative). In order to treat these operators as perturbations, we restrict the derivation of the difference inequality to values $\ell \le \delta N$ for some small δ. This helps because $\mathbb{K}_3$ and $\mathbb{K}_4$ have more than two creation and annihilation operators and additional factors of $(N-1)^{-1/2}$. Having established the exponential decay up to $\ell = \delta N$, it will follow as a simple consequence of the eigenvalue equation that $\|\chi^{(\ell)}\|$ is bounded by $\exp(-\varepsilon N)$ for all $\delta N \le \ell \le N$ and some $\varepsilon > 0$.

C. Difference inequality and Proof of theorem 1.1

The following lemma is the main ingredient for the Proof of Theorem 1.1. It provides a generalization of the difference inequality (2.18) to the ground state $\chi \in \mathcal{F}_\perp^{\le N}$ of $\mathbb{H}$.

Lemma 2.1. Under Assumptions (A1) and (A2) there exist constants $L \ge 1$, $\sigma > 2$ and $\kappa \in (0,1)$ such that the discrete function $F_L(\ell) := \sum_{k=\ell-L}^{\ell+L} k\|\chi^{(k)}\|^2$ satisfies

$$\sigma F_L(\ell) \le F_L(\ell+L) + F_L(\ell-L) \tag{2.19}$$

for all $L \le \ell \le \kappa N$ and N large enough.

Before we embark on the proof of the lemma, we deduce its consequences to obtain a Proof of Theorem 1.1.

Proof of Theorem 1.1. We apply (2.19) for $\ell = L, 2L, 3L, \ldots, nL$ with $n \le \kappa N/L$. In other words, we use that the second discrete derivative of the function $G(\ell) := F_L(\ell L)$ is bounded from below by $(\sigma-2)G(\ell)$,

$$(\sigma-2)G(\ell) \le G(\ell+1) + G(\ell-1) - 2G(\ell) \tag{2.20}$$

for all $\ell \in \{1, \ldots, \kappa' N\}$ with $\kappa' = \kappa/L$. This implies that G is convex and thus attains a unique minimum at some value ℓ_0. The inequality implies further that $G(\ell)$ is exponentially decaying for $\ell \le \ell_0$ and exponentially increasing for $\ell \ge \ell_0$,

$$G(\ell) \le \frac{G(1)}{(\sigma-2)^{\ell-1}} \qquad \forall\ 1 \le \ell \le \ell_0, \tag{2.21}$$

$$G(k) \ge (\sigma-2)^{k-\ell}G(\ell) \qquad \forall\ \ell_0 \le \ell \le k \le \kappa' N, \tag{2.22}$$

where the second case is only relevant if $\ell_0 < \kappa' N$. Using the second bound, we obtain a sufficient estimate for $G(\ell)$ for $\ell_0 \le \ell \le \kappa' N/2$: In fact, choosing $\ell = \kappa' N/2$, $k = \kappa' N$ and since $G \le N$ by normalization of χ, Eq. (2.22) implies $G(\kappa' N/2) \le N(\sigma-2)^{-\kappa' N/2}$ and thus $G(\ell) \le N(\sigma-2)^{-\kappa' N/2}$ for $\ell_0 \le \ell \le \kappa' N/2$.

From the above, we conclude that $G(\ell) \le C\exp(-\varepsilon'\ell)$ for some constants $C, \varepsilon' > 0$ and all $1 \le \ell \le \kappa' N/2$. Recalling $G(\ell) = \sum_{k=(\ell-1)L}^{(\ell+1)L} \ell \|\chi^{(\ell)}\|^2$, we obtain

$$\sup\left\{\|\chi^{(k)}\|^2 : (\ell-1)L \le k \le (\ell+1)L\right\} \le C\exp(-\varepsilon'\ell) \tag{2.23}$$

for all $\ell \le \kappa' N/2$, which implies $\|\chi^{(\ell)}\| \le C\exp(-\varepsilon\ell)$ for all $\ell \le \kappa' N/2$ and some $\varepsilon > 0$.

It remains to prove the exponential decay for $\kappa' N/2 \le \ell \le N$. To this end, we write $\chi = \chi^{\le} + \chi^{>}$ with $\chi^{\le} := \mathbb{1}(\mathcal{N} \le N\kappa'/2)\chi$. Then, we use the eigenvalue equation $\mathbb{H}\chi = (E_N - Ne_{\mathrm{H}})\chi$ together with $\mathbb{H} \ge E_N - Ne_{\mathrm{H}}$ and the fact that only $\mathbb{K}_2$ and $\mathbb{K}_3$ couple the two different parts of the ground state,

$$0 = \langle \chi, \mathbb{H}\chi\rangle - (E_N - Ne_{\mathrm{H}}) \ge \langle \chi^{>}, \mathbb{H}\chi^{>}\rangle - (E_N - Ne_{\mathrm{H}})\|\chi^{>}\|^2 + 2\mathrm{Re}\langle \chi^{>}, \mathbb{K}_2\, \mathfrak{b}(\mathcal{N})\chi^{\le}\rangle + 2\mathrm{Re}\langle \chi^{>}, \mathbb{K}_3\, \mathfrak{c}(\mathcal{N})\chi^{\le}\rangle. \tag{2.24}$$

The last two terms are bounded by

$$\begin{aligned} &\left|\langle \chi^{>}, \mathbb{K}_2\, \mathfrak{b}(\mathcal{N})\chi^{\le}\rangle\right| + \left|\langle \chi^{>}, \mathbb{K}_3\, \mathfrak{c}(\mathcal{N})\chi^{\le}\rangle\right| \\ &\qquad \le CN\left(\|\chi^{(N\kappa'/2-1)}\| + \|\chi^{(N\kappa'/2)}\|\right) \le \exp(-cN) \end{aligned} \tag{2.25}$$

for some $c > 0$ and large enough N, where we used Lemmas 2.3, 2.4, and 2.6 in the first step and the exponential decay of $\|\chi^{(\ell)}\|$ for $\ell \le \kappa' N/2$ in the second step. Thus,

$$\langle \chi^{>}, \mathbb{H}\chi^{>}\rangle - (E_N - Ne_{\mathrm{H}})\|\chi^{>}\|^2 \le \exp(-cN). \tag{2.26}$$

Since every normalized state ϕ with energy $\langle \phi, \mathbb{H}\phi\rangle \le E_N - Ne_{\mathrm{H}} + o(N)$ as $N \to \infty$ exhibits BEC (Ref. 8, Theorem 3.1), the energy of the state $\chi^{>}/\|\chi^{>}\|$ can not be close to $E_N - Ne_{\mathrm{H}}$, in particular not exponentially close. Hence, (2.26) implies that $\|\chi^{>}\| \le C\exp(-\varepsilon N)$ for some $C, \varepsilon > 0$ and N large enough.

To summarize, we have shown that there exist constants $C, \varepsilon > 0$ such that $\|\chi^{(\ell)}\| \le C\exp(-\varepsilon\ell)$ for all $\ell \in \{1, \ldots, N\}$ and all large N. This implies Theorem 1.1. ∎

D. Proof of the difference inequality

Let us recall Assumption (A2) on the pair potential v: We consider either (i) v even, pointwise bounded and with non-negative Fourier transform or (ii) $v(x) = \lambda|x|^{-1}$ with $\lambda > 0$. For better readability, we first prove Lemma 2.1 in case (i). In Sec. II E, we explain how the proof is adapted to cover case (ii). Note that in both cases, we have $\|v^2 * \varphi^2\|_\infty < \infty$, where φ is the normalized positive Hartree minimizer. For the Coulomb potential, this follows from Hardy's inequality and $\varphi \in H^1(\mathbb{R}^3)$.

Before we come to the Proof of Lemma 2.1, we state and prove some preliminary estimates for the operators appearing in the excitation Hamiltonian. The statements of Lemmas 2.2 and 2.4 hold in both cases, whereas Lemmas 2.3 and 2.5 only hold in case (i).

Lemma 2.2. Under Assumption (A2) we have for all $\xi \in \mathcal{F}_\perp$ that

$$\left|\langle \xi^{(\ell)}, \mathbb{K}_1 \xi^{(\ell)}\rangle\right| \le \|v^2 * \varphi^2\|_\infty^{1/2}\, \ell \|\xi^{(\ell)}\|^2. \tag{2.27}$$

Proof. We apply two times Cauchy–Schwarz to find

$$\begin{aligned} \left|\langle \xi^{(\ell)}, \mathbb{K}_1 \xi^{(\ell)}\rangle\right| &= \left|\int \mathrm{d}x\mathrm{d}y\, \varphi(x)v(x-y)\varphi(y)\langle \xi^{(\ell)}, a_x^* a_y \xi^{(\ell)}\rangle\right| \\ &\le \int dx\, \varphi(x)\left(v^2 * \varphi^2(x)\right)^{1/2} \|a_x \xi^{(\ell)}\|\, \|\mathcal{N}^{1/2}\xi^{(\ell)}\| \\ &\le \|v^2 * \varphi^2\|_\infty^{1/2} \|\varphi\|_2 \|\mathcal{N}^{1/2}\xi^{(\ell)}\|^2. \end{aligned} \tag{2.28}$$

∎

Lemma 2.3. For pointwise bounded v with $\hat{v} \ge 0$, we have

$$4\left|\langle \xi^{(\ell)}, \mathbb{K}_2 \xi^{(\ell-2)}\rangle\right| \le \langle \xi^{(\ell)}, \mathbb{K}_1 \xi^{(\ell)}\rangle + \langle \xi^{(\ell-2)}, \mathbb{K}_1 \xi^{(\ell-2)}\rangle + v(0)\|\xi^{(\ell-2)}\|^2 \tag{2.29}$$

$$4\left|\langle \xi^{(\ell)}, \mathbb{K}_2^\dagger \xi^{(\ell+2)}\rangle\right| \le \langle \xi^{(\ell)}, \mathbb{K}_1 \xi^{(\ell)}\rangle + \langle \xi^{(\ell+2)}, \mathbb{K}_1 \xi^{(\ell+2)}\rangle + v(0)\|\xi^{(\ell)}\|^2 \tag{2.30}$$

for all $\xi \in \mathcal{F}_\perp$.

Proof. We estimate

$$\begin{aligned}\left|\left\langle\xi^{(\ell)},\mathbb{K}_2\xi^{(\ell-2)}\right\rangle\right| &= \frac{1}{2}\left|\int \mathrm{d}x\mathrm{d}y\left\langle\xi^{(\ell)},K(x,y)a_x^*a_y^*\xi^{(\ell-2)}\right\rangle\right|\\ &= \frac{1}{2}\left|\int \mathrm{d}k\,\hat{v}(k)\left\langle\int\mathrm{d}x\,\varphi(x)e^{ikx}a_x\xi^{(\ell)},\int\mathrm{d}y\,\varphi(y)e^{iky}a_y^*\xi^{(\ell-2)}\right\rangle\right|\\ &\le \frac{1}{4}\int\mathrm{d}k\,\hat{v}(k)\left(\left\|\int\mathrm{d}x\,\varphi(x)e^{ikx}a_x\xi^{(\ell)}\right\|^2+\left\|\int\mathrm{d}y\,\varphi(y)e^{iky}a_y^*\xi^{(\ell-2)}\right\|^2\right)\end{aligned}\tag{2.31}$$

and note that

$$\int\mathrm{d}k\,\hat{v}(k)\left\|\int\mathrm{d}x\,\varphi(x)e^{ikx}a_x\xi^{(\ell)}\right\|^2=\left\langle\xi^{(\ell)},\mathbb{K}_1\xi^{(\ell)}\right\rangle\tag{2.32}$$

while

$$\begin{aligned}\int\mathrm{d}k\,\hat{v}(k)\left\|\int\mathrm{d}x\,\varphi(x)e^{ikx}a_x^*\xi^{(\ell-2)}\right\|^2 &= \int dxdy\,K(x,y)\left\langle\xi^{(\ell-2)},a_xa_y^*\xi^{(\ell-2)}\right\rangle\\ &= v(0)\int\mathrm{d}x\,\varphi(x)^2\,\|\xi^{(\ell-2)}\|^2+\left\langle\xi^{(\ell-2)},\mathbb{K}_1\xi^{(\ell-2)}\right\rangle,\end{aligned}\tag{2.33}$$

where we used $a_xa_y^*=\delta(x-y)+a_y^*a_x$ and $K(x,y)=K(y,x)$.

The second bound of the lemma follows from $\ell\mapsto\ell+2$ and $(\mathbb{K}_2^\dagger)^\dagger=\mathbb{K}_2$. ■

Lemma 2.4. Under Assumption (A2) there is a constant $C>0$ so that for all $\ell\le\delta N$, $\delta\in(0,1)$, we have

$$\left|\left\langle\xi^{(\ell)},\mathbb{K}_3\xi^{(\ell-1)}\right\rangle\right|\le(\delta N)^{1/2}\left(C\ell\|\xi^{(\ell)}\|^2+(\ell-1)\|\xi^{(\ell-1)}\|^2\right)\tag{2.34}$$

$$\left|\left\langle\xi^{(\ell)},\mathbb{K}_3^\dagger\xi^{(\ell+1)}\right\rangle\right|\le(\delta N)^{1/2}\left(C\ell\|\xi^{(\ell)}\|^2+(\ell+1)\|\xi^{(\ell+1)}\|^2\right)\tag{2.35}$$

for every $\xi\in\mathcal{F}_\perp$.

Proof. Using $\|W^2*\varphi^2\|_\infty\le C$, it follows again by Cauchy–Schwarz that

$$\begin{aligned}\left|\left\langle\xi^{(\ell)},\mathbb{K}_3\xi^{(\ell-1)}\right\rangle\right| &= \left|\int\mathrm{d}x_2\,W(x_1,x_2)\varphi(x_1)\left\langle\xi^{(\ell)},a_{x_1}^*a_{x_2}^*a_{x_2}\xi^{(\ell-1)}\right\rangle\right|\\ &\le C\int dx_2\left(W^2*\varphi^2(x)\right)^{1/2}\|a_{x_2}\mathcal{N}^{1/2}\xi^{(\ell)}\|\,\|a_{x_2}\xi^{(\ell-1)}\|\\ &\le C\ell^{3/2}\|\xi^{(\ell-1)}\|\,\|\xi^{(\ell)}\|\end{aligned}\tag{2.36}$$

and similarly, for the bound involving $\mathbb{K}_3^\dagger$. ■

Lemma 2.5. For $\|v\|_\infty\le C$, we have for $\ell\le\delta N$, $\delta\in(0,1)$,

$$\left|\left\langle\xi^{(\ell)},\mathbb{K}_4\xi^{(\ell)}\right\rangle\right|\le C\delta N\ell\|\xi^{(\ell)}\|^2\tag{2.37}$$

for every $\xi\in\mathcal{F}_\perp$.

The proof is straightforward and thus omitted.

Proof of Lemma 2.1. We shall prove the lemma in two steps.

Step 1. We take the scalar product with the state $\chi^{(\ell)}$ on both sides of the eigenvalue equation $\mathbb{H}\chi=(E_N-Ne_{\mathrm{H}})\chi$. Using $E_N-Ne_{\mathrm{H}}\le0$, $\mathcal{N}\chi^{(\ell)}=\ell\chi^{(\ell)}$ and multiplying both sides by $N-1$, we obtain

$$\begin{aligned}0\ge&(N-1)\left\langle\chi^{(\ell)},\mathbb{K}_0\chi^{(\ell)}\right\rangle+\mathfrak{a}(\ell)\left\langle\chi^{(\ell)},\mathbb{K}_1\chi^{(\ell)}\right\rangle\\ &+\mathfrak{b}(\ell-2)\left\langle\chi^{(\ell)},\mathbb{K}_2\chi^{(\ell-2)}\right\rangle+\mathfrak{b}(\ell)\left\langle\chi^{(\ell)},\mathbb{K}_2^\dagger\chi^{(\ell+2)}\right\rangle\\ &+\mathfrak{c}(\ell-1)\left\langle\chi^{(\ell)},\mathbb{K}_3\chi^{(\ell-1)}\right\rangle+\mathfrak{c}(\ell)\left\langle\chi^{(\ell)},\mathbb{K}_3^\dagger\chi^{(\ell+1)}\right\rangle+\left\langle\chi^{(\ell)}\mathbb{K}_4\chi^{(\ell)}\right\rangle\end{aligned}\tag{2.38}$$

and invoking $\mathbb{K}_0 \geq \tau\mathcal{N}$, we find

$$
\begin{aligned}
(N-1)\tau\ell\|\chi^{(\ell)}\|^2 + \mathfrak{a}(\ell)\Big\langle\chi^{(\ell)}, \mathbb{K}_1\chi^{(\ell)}\Big\rangle \\
&\leq -\mathfrak{b}(\ell-2)\Big\langle\chi^{(\ell)}, \mathbb{K}_2\chi^{(\ell-2)}\Big\rangle - \mathfrak{b}(\ell)\Big\langle\chi^{(\ell)}, \mathbb{K}_2^\dagger\chi^{(\ell+2)}\Big\rangle \\
&\quad - \mathfrak{c}(\ell-1)\Big\langle\chi^{(\ell)}, \mathbb{K}_3\chi^{(\ell-1)}\Big\rangle - \mathfrak{c}(\ell)\Big\langle\chi^{(\ell)}, \mathbb{K}_3^\dagger\chi^{(\ell+1)}\Big\rangle \\
&\quad - \Big\langle\chi^{(\ell)}, \mathbb{K}_4\chi^{(\ell)}\Big\rangle.
\end{aligned}
\tag{2.39}
$$

To facilitate the reading, let us abbreviate $f(\ell) := \ell\|\chi^{(\ell)}\|^2$, $g(\ell) := \Big\langle\chi^{(\ell)}, \mathbb{K}_1\chi^{(\ell)}\Big\rangle$ and

$$R_2(\ell) := -\mathfrak{b}(\ell-2)\Big\langle\chi^{(\ell)}, \mathbb{K}_2\chi^{(\ell-2)}\Big\rangle - \mathfrak{b}(\ell)\Big\langle\chi^{(\ell)}, \mathbb{K}_2^\dagger\chi^{(\ell+2)}\Big\rangle \tag{2.40}$$

$$R_3(\ell) := -\mathfrak{c}(\ell-1)\Big\langle\chi^{(\ell)}, \mathbb{K}_3\chi^{(\ell-1)}\Big\rangle - \mathfrak{c}(\ell)\Big\langle\chi^{(\ell)}, \mathbb{K}_3^\dagger\chi^{(\ell+1)}\Big\rangle \tag{2.41}$$

$$R_4(\ell) := -\Big\langle\chi^{(\ell)}, \mathbb{K}_4\chi^{(\ell)}\Big\rangle. \tag{2.42}$$

such that Inequality (2.39) reads

$$(N-1)\tau f(\ell) + \mathfrak{a}(\ell)g(\ell) \leq R_2(\ell) + R_3(\ell) + R_4(\ell). \tag{2.43}$$

Since the left-hand side of (2.43) is non-negative, we can apply Lemma 2.6 to estimate

$$|R_2(\ell)| \leq \frac{1}{4}\big(\mathfrak{b}(\ell-2)g(\ell) + \mathfrak{b}(\ell-2)g(\ell-2) + \mathfrak{b}(\ell)g(\ell) + \mathfrak{b}(\ell)g(\ell+2)\big) + CN\ell^{-1}f(\ell) + C\|\chi^{(\ell-2)}\|^2 \tag{2.44}$$

where we used that $\mathfrak{b}(\ell-2), \mathfrak{b}(\ell) \leq N$ and $v(0) \leq C$. Note that for $\ell - 2 \geq \delta^{-1/2}$ for some $\delta \in (0,1)$, we can bound the last term by $\|\chi^{(\ell-2)}\|^2 \leq \delta^{1/2}f(\ell-2)$. If we further restrict the values of ℓ to $\ell \leq \delta N$, we have by Lemma 2.4 that

$$|R_3(\ell)| \leq \sqrt{N}\Big(\Big|\Big\langle\chi^{(\ell)}, \mathbb{K}_3\chi^{(\ell-1)}\Big\rangle\Big| + \Big|\Big\langle\chi^{(\ell)}, \mathbb{K}_3\chi^{(\ell+1)}\Big\rangle\Big|\Big) \leq CN\delta^{1/2}f(\ell) + N\delta^{1/2}f(\ell-1) + N\delta^{1/2}f(\ell+1) \tag{2.45}$$

where we used $\mathfrak{c}(\ell-1) \leq \mathfrak{c}(\ell) \leq \sqrt{N}$. Moreover, by Lemma 2.5, $|R_4(\ell)| \leq CN\delta f(\ell)$. Thus, we arrive at

$$
\begin{aligned}
4(N-1)\Big(\tau - C\ell^{-1} - C\delta^{1/2}\Big)f(\ell) &\leq \mathfrak{b}(\ell-2)g(\ell) + \mathfrak{b}(\ell-2)g(\ell-2) - 2\mathfrak{a}(\ell)g(\ell) \\
&\quad + \mathfrak{b}(\ell)g(\ell) + \mathfrak{b}(\ell)g(\ell+2) - 2\mathfrak{a}(\ell)g(\ell) \\
&\quad + CN\delta^{1/2}\big(f(\ell+1) + f(\ell-1) + f(\ell-2)\big).
\end{aligned}
\tag{2.46}
$$

We now choose $\ell \geq c$ large enough and δ sufficiently small so that the left-hand side is bounded from below by $2N\tau f(\ell)$. Moreover, we write the first two lines of the right-hand side as

$$\mathfrak{b}(\ell-2)g(\ell-2) - \mathfrak{b}(\ell)g(\ell) + \mathfrak{b}(\ell)g(\ell+2) - \mathfrak{b}(\ell-2)g(\ell) + 2(\mathfrak{b}(\ell) - \mathfrak{a}(\ell))g(\ell) + 2(\mathfrak{b}(\ell-2) - \mathfrak{a}(\ell))g(\ell) \tag{2.47}$$

and use that

$$\mathfrak{b}(\ell) - \mathfrak{a}(\ell) \leq 0, \quad \mathfrak{b}(\ell-2) - \mathfrak{a}(\ell) \leq C, \quad g(\ell) \leq Cf(\ell), \tag{2.48}$$

where the last bound follows from Lemma 2.2. Thus, we obtain the inequality

$$
\begin{aligned}
N\tau f(\ell) &\leq \mathfrak{b}(\ell-2)g(\ell-2) - \mathfrak{b}(\ell)g(\ell) + \mathfrak{b}(\ell)g(\ell+2) - \mathfrak{b}(\ell-2)g(\ell) \\
&\quad + CN\delta^{1/2}\big(f(\ell+1) + f(\ell-1) + f(\ell-2)\big).
\end{aligned}
\tag{2.49}
$$

Now, we sum both sides over $\{L-\ell, \ldots, L+\ell\}$. On the left-hand side, this gives

$$N\tau F_L(\ell) := N\tau\sum_{k=\ell-L}^{\ell+L} k\|\chi^{(k)}\|^2, \tag{2.50}$$

whereas the terms on the right-hand side are bounded by

$$\sum_{k=\ell-L}^{\ell+L}\big(\mathfrak{b}(k-2)g(k-2)-\mathfrak{b}(k)g(k)\big)\le \mathfrak{b}(\ell-L-2)g(\ell-L-2)\le CNf(\ell-L-2) \tag{2.51}$$

and

$$\sum_{k=\ell-L}^{L+\ell}\big(\mathfrak{b}(k)g(k+2)-\mathfrak{b}(k-2)g(k)\big)\le \mathfrak{b}(\ell+L)g(\ell+L+2)\le CNf(\ell+L+2) \tag{2.52}$$

and

$$\sum_{k=\ell-L}^{\ell+L} N\delta^{1/2}\big(f(\ell+1)+f(\ell-1)+f(\ell-2)\big)$$
$$\le 3N\delta^{1/2}F_L(\ell)+N\big(f(\ell-L-1)+f(\ell+L+1)+f(\ell-L-2)\big), \tag{2.53}$$

where we used $\delta^{1/2}\le 1/2$.

Putting everything together, we arrive at the conclusion that there is a constant $0<\mu\le C(\tau-C\delta^{1/2})$ such that for all allowed values of ℓ, that is, for $2+\delta^{-1/2}\le\ell\le\delta N$ for sufficiently small δ and all large N, we have

$$\mu F_L(\ell)\le f(\ell+L+2)+f(\ell+L+1)+f(\ell-L-1)+f(\ell-L-2). \tag{2.54}$$

Step 2. We proceed by estimating

$$F_L(\ell+L)+F_L(\ell-L)\ge\sum_{k=\ell+L+1}^{\ell+2L}f(k)+\sum_{k=\ell-2L}^{\ell-L-1}f(k)$$
$$\ge\mu\big(F_L(\ell)+F_{L+2}(\ell)+F_{L+4}(\ell)+\cdots+F_{2L-2}(\ell)\big) \tag{2.55}$$

where we used the definition of $F_L(\ell)$ in the first step and applied Inequality (2.54) with $L\to L+j$ in the second step, that is,

$$f(\ell+L+j+2)+f(\ell+L+j+1)+f(\ell-L-j-2)+f(\ell-L-j-1)\ge\mu F_{L+j}(\ell) \tag{2.56}$$

for $j=0,\dots,L-2$.

Finally, we invoke $F_{L+j}(\ell)\ge F_L(\ell)$ to arrive at the desired inequality

$$F_L(\ell+L)+F_L(\ell-L)\ge\frac{\mu}{2}(L-3)F_L(\ell)=:\sigma F_L(\ell). \tag{2.57}$$

By choosing L large enough, we have $\sigma>2$, which completes the Proof of Lemma 2.1. ∎

E. Extension to the repulsive Coulomb potential

We briefly explain how the Proof of Lemma 2.1 presented in Sec. II D needs to be adapted to cover the Coulomb potential $v(x)=\lambda|x|^{-1}$ with $\lambda>0$. Since $\|v^2*\varphi^2\|_\infty<\infty$, Lemmas 2.2 and 2.4 still apply. Lemma 2.5, on the other hand, is not needed, since we can use $\mathbb{K}_4\ge 0$ in (2.38). (Note that we have not assumed positivity of v in the bounded case). The only obstacle comes from the use of Lemma 2.3, which requires $v(0)<\infty$. For the repulsive Coulomb potential, we replace Lemma 2.3 by the following statement.

Lemma 2.6. Let $v(x)=\lambda|x|^{-1}$ with $\lambda>0$. There is a constant $C>0$ such that for every $\varepsilon>0$ there exists a constant $\nu(\varepsilon)>0$, such that

$$4\left|\left\langle\xi^{(\ell)},\mathbb{K}_2\xi^{(\ell-2)}\right\rangle\right|\le g(\ell)+g(\ell-2)+\nu(\varepsilon)\|\xi^{(\ell-2)}\|^2+\varepsilon\big(f(\ell-2)+f(\ell)\big) \tag{2.58}$$

for all $\xi\in\mathcal{F}_\perp$, where $g(\ell)=\langle\xi^{(\ell)},\mathbb{K}_1\xi^{(\ell)}\rangle$ and $f(\ell)=\ell\|\xi^{(\ell)}\|^2$.

The bound for $|\langle\xi^{(\ell)},\mathbb{K}_2^\dagger\xi^{(\ell+2)}\rangle|$ is obtained by $\ell\mapsto\ell+2$ and $(\mathbb{K}_2^\dagger)^\dagger=\mathbb{K}_2$. After invoking Lemma 2.6 to bound $|R_2(\ell)|$ in (2.44), the crucial point is that we can choose ε as small as we want (but always fixed w.r.t. N), say $\varepsilon\le\delta^{1/2}$. This comes at the cost of a large factor $\nu(\varepsilon)$, which can be compensated by restricting the values of ℓ to $\ell\ge 2+\delta^{-1/2}\nu(\varepsilon)$. This way, we can bound the third term in (2.58) by $\nu(\varepsilon)\|\xi^{(\ell-2)}\|^2=\frac{\nu(\varepsilon)}{\ell-2}f(\ell-2)\le\delta^{1/2}f(\ell-2)$. Hence, the ε-dependent terms in (2.58) are bounded by $\delta^{1/2}(2f(\ell-2)+f(\ell))$ and with this at hand, the remaining steps of the proof are completely analogous to the bounded case.

Proof of Lemma 2.6. We write $v=v_\kappa+v_\kappa^\perp$ with $v_\kappa(x)=v(x)\,(1-\exp(-|x|/\kappa))$, $\kappa>0$, and observe that $v_\kappa(0)=\lambda\kappa^{-1}$ and $\widehat{v_\kappa}\ge 0$. The non-negativity of the Fourier transform follows from the fact that the Fourier transform of the Yukawa potential $x\mapsto v(x)\exp(-|x|/\kappa)$ is smaller than the Fourier transform of the Coulomb potential. Moreover, we have $\|(v_\kappa^\perp)^2 * \varphi^2\|_\infty \to 0$ as $\kappa \to 0$, which is a consequence of $f_\kappa(x) := (v_\kappa^\perp)^2 * \varphi^2(x)$ being strictly monotone decreasing as $\kappa \to 0$, i.e., $f_\kappa(x) - f_\eta(x) = \lambda \int dy |x-y|^{-2}(e^{-2|x-y|/\kappa} - e^{-2|x-y|/\eta})\varphi(y)^2 > 0$ for all $x \in \mathbb{R}^3$ and $\kappa > \eta$, where we used positivity of φ. In analogy to the definitions in Sec. II A, we introduce $K_\kappa(x,y) = \varphi(x) v_\kappa(x-y)\varphi(y)$ as well as $\mathbb{K}_{2,\kappa}$ and $\mathbb{K}_{1,\kappa}$. For the part involving v_κ, we can proceed as in the Proof of Lemma 2.3, which gives

$$4\left|\left\langle \xi^{(\ell)}, \mathbb{K}_{2,\kappa}\xi^{(\ell-2)}\right\rangle\right| \le \left\langle \xi^{(\ell)}, \mathbb{K}_{1,\kappa}\xi^{(\ell)}\right\rangle + \left\langle \xi^{(\ell-2)}, \mathbb{K}_{1,\kappa}\xi^{(\ell-2)}\right\rangle + \lambda\kappa^{-1}\|\xi^{(\ell-2)}\|^2. \tag{2.59}$$

Applying Lemnma 2.2 for $v_\kappa^\perp = v - v_\kappa$ and using $\|(v_\kappa^\perp)^2 * \varphi^2\|_\infty \to 0$ as $\kappa \to 0$, we further have

$$\left|\left\langle \xi^{(\ell)}, (\mathbb{K}_{1,\kappa} - \mathbb{K}_1)\xi^{(\ell)}\right\rangle\right| \le \|(v_\kappa^\perp)^2 * \varphi^2\|_\infty^{1/2} \ell \|\xi^{(\ell)}\|^2 \le \delta(\kappa)\ell\|\xi^{(\ell)}\|^2 \tag{2.60}$$

for some sequence $\delta(\kappa) \to 0$ as $\kappa \to 0$. To estimate the remainder term, we apply two times Cauchy–Schwarz (similarly as in the Proof of Lemma 2.2) to obtain

$$\begin{aligned}\left|\left\langle \xi^{(\ell)}, (\mathbb{K}_2 - \mathbb{K}_{2,\kappa})\xi^{(\ell-2)}\right\rangle\right| &= \left|\int dxdy\, \varphi(x) v_\kappa^\perp(x-y)\varphi(y)\left\langle \chi^{(\ell)}, a_x^* a_y^* \chi^{(\ell-2)}\right\rangle\right| \\ &\le C\|(v_\kappa^\perp)^2 * \varphi^2\|_\infty^{1/2}\, \ell\|\xi^{(\ell)}\|\,\|\xi^{(\ell-2)}\| \le \delta(\kappa)\ell\|\xi^{(\ell)}\|\,\|\xi^{(\ell-2)}\|.\end{aligned} \tag{2.61}$$

This implies the statement of the lemma. ■

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AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

Author Contributions

David Mitrouskas: Conceptualization (equal). **Peter Pickl**: Conceptualization (equal).

DATA AVAILABILITY

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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