Boundary Superconductivity in BCS Theory

by

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Abstract

Superconductivity has many important applications ranging from levitating trains over qubits to MRI scanners. The phenomenon is successfully modeled by Bardeen-Cooper-Schrieffer (BCS) theory. From a mathematical perspective, BCS theory has been studied extensively for systems without boundary. However, little is known in the presence of boundaries. With the help of numerical methods physicists observed that the critical temperature may increase in the presence of a boundary. The goal of this thesis is to understand the influence of boundaries on the critical temperature in BCS theory and to give a first rigorous justification of these observations. On the way, we also study two-body Schrödinger operators on domains with boundaries and prove additional results for superconductors without boundary.

BCS theory is based on a non-linear functional, where the minimizer indicates whether the system is superconducting or in the normal, non-superconducting state. By considering the Hessian of the BCS functional at the normal state, one can analyze whether the normal state is possibly a minimum of the BCS functional and estimate the critical temperature. The Hessian turns out to be a linear operator resembling a Schrödinger operator for two interacting particles, but with more complicated kinetic energy. As a first step, we study the two-body Schrödinger operator in the presence of boundaries. For Neumann boundary conditions, we prove that the addition of a boundary can create new eigenvalues, which correspond to the two particles forming a bound state close to the boundary.

Second, we need to understand superconductivity in the translation invariant setting. While in three dimensions this has been extensively studied, there is no mathematical literature for the one and two dimensional cases. In dimensions one and two, we compute the weak coupling asymptotics of the critical temperature and the energy gap in the translation invariant setting. We also prove that their ratio is independent of the microscopic details of the model in the weak coupling limit; this property is referred to as universality.

In the third part, we study the critical temperature of superconductors in the presence of boundaries. We start by considering the one-dimensional case of a half-line with contact interaction. Then, we generalize the results to generic interactions and half-spaces in one, two and three dimensions. Finally, we compare the critical temperature of a quarter space in two dimensions to the critical temperatures of a half-space and of the full space.

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About the Author

Barbara Roos completed her BSc and MSc in Mathematics at ETH Zurich before joining ISTA in 2019. Her main research focus lies on quantum many-body systems, although she also enjoys studying classical systems. Barbara did a rotation in the group of Maksym Serbyn, where she worked on the optimal steering of matrix product states, which resulted in a publication in PRX Quantum. For her PhD project advised by Robert Seiringer, she focused on the BCS theory of superconductivity.

List of Collaborators and Publications

The thesis contains the following published papers:

- B. Roos and R. Seiringer. "Two-particle bound states at interfaces and corners". *Journal of functional analysis* **282**.12 (2022), p. 109455. DOI: 10.1016/j.jfa.2022.109455
- C. Hainzl, B. Roos, and R. Seiringer. "Boundary superconductivity in the BCS Model". *Journal of spectral theory* **12**.4 (2022), pp. 1507–1540. DOI: 10.4171/JST/439

It also contains the following preprints:

- J. Henheik, A. B. Lauritsen, and B. Roos. "Universality in low-dimensional BCS theory". Preprint of an article accepted for publication in Reviews in Mathematical Physics. arXiv:2301.05621 [cond-mat, physics:math-ph]. 2023. URL: http://arxiv.org/abs/2301.05621
- B. Roos and R. Seiringer. "BCS Critical Temperature on Half-Spaces". arXiv:2306.05824 [cond-mat, physics:math-ph]. 2023. URL: http://arxiv.org/abs/2306.05824
- B. Roos and R. Seiringer. "Enhanced BCS Superconductivity at a Corner". arXiv:2308.07115 [cond-mat, physics:math-ph]. 2023. URL: http://arxiv.org/abs/2308.07115

The following publication is not included in this thesis, but it is a result of scientific collaborations at ISTA during my PhD:

- M. Ljubotina, B. Roos, D. A. Abanin, and M. Serbyn. "Optimal Steering of Matrix Product States and Quantum Many-Body Scars". *Prx quantum* **3**.3 (2022), p. 030343. DOI: 10.1103/PRXQuantum.3.030343

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CHAPTER 1

Introduction

Superconductivity has many important applications ranging from levitating trains over qubits to MRI scanners. However, there are still gaps in its theoretical understanding. In particular, superconducting properties at the edge of a sample are an active field of research and the focus of this PhD thesis.

For a large class of materials, Bardeen-Cooper-Schrieffer (BCS) theory successfully describes superconductivity [28]. BCS theory is derived from the microscopic laws governing the electrons in a metal. The key observation in BCS theory is that, under suitable conditions, there is an effective attraction between the electrons in the metal. This attraction can arise due to the interaction of the electrons with the vibrations of the lattice of ions in the metal, known as electron-phonon interaction. According to BCS theory, a system is superconducting when the so-called gap function does not vanish. The gap function is related to the energy gap of the system, which is the amount of energy needed to excite the system out of its ground state. Typically, a material becomes superconducting below a certain critical temperature. In the superconducting regime, the resistivity of the material vanishes and there is a jump in the electronic heat capacity at the critical temperature. Superconductivity is sensitive to currents and magnetic fields. For instance, application of strong currents or magnetic fields destroys the superconducting state.

Based on the work of Caroli, de Gennes, and Matricon [12] it was believed for a long time that in BCS theory the gap function can be approximated by a constant function at the edge of a sample, and that the boundary does not affect the critical temperature. There are, however, experimental observations showing that close to the edge of a sample superconductivity persists at higher temperatures than in the bulk [20, 42, 43, 44, 45, 52], see the sketch in Figure 1.1. In [52] the authors measured the resistivity and the heat capacity of a sample and found that the drop in resistivity occurs at a higher temperature than the transition in heat capacity. The heat capacity measurement determines the bulk critical temperature, since the bulk has a much larger volume than the surface. On the other hand, the resistivity measurement gives the critical temperature of surface superconductivity. This observed boundary superconductivity was often attributed to inhomogeneities at the surface. It was recently found by Samoilenka and Babaev [6, 7, 62, 63, 68], however, that already BCS theory predicts boundary superconductivity. They observed localization of the gap function at the boundary and an increased critical temperature in the presence of a boundary. The shape dependence of the critical temperature has also been observed in [64]. Older works [30, 67], which studied the gap function at a

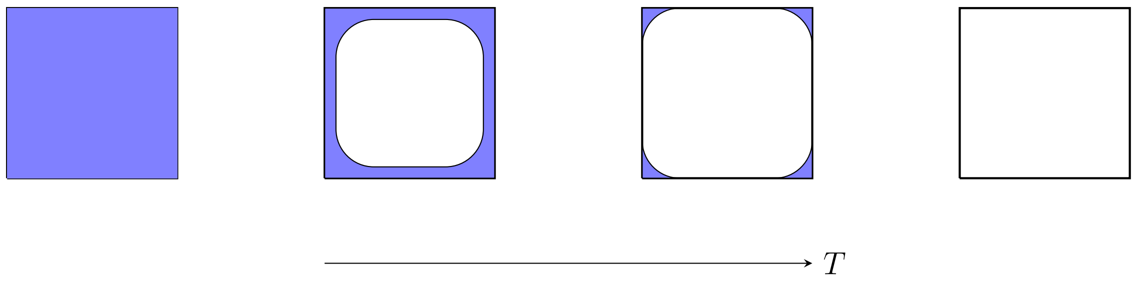

Figure 1.1: Schematic of boundary superconductivity in a two dimensional sample. For a sequence of temperatures increasing from left to right, the shaded regions indicate where the superconducting gap function is large.

boundary, observed only relatively small boundary effects. To our knowledge, there is currently no intuitive explanation for the enhancement of the critical temperature at boundaries.

The effect of the boundary depends on the regime considered. While for thick samples, the critical temperature seems to increase as the thickness is decreased, in very thin samples the critical temperature appears to decrease together with the sample thickness [58, 70].

In this thesis, we prove in a mathematically rigorous way that within the framework of BCS theory, the critical temperature increases in the presence of a boundary, at least in certain regimes. This provides a first step towards justifying the results by Samoilenka and Babaev [62].

There is another manifestation of boundary superconductivity in the context of magnetic fields [28]. Superconductivity may survive at higher magnetic fields close to the surface than in the bulk. This phenomenon is modelled using Ginzburg-Landau (GL) theory, which is a phenomenological theory. In 1959, Gorkov [31] established that close to the critical temperature GL theory can be derived from BCS theory. The first rigorous proof connecting these two theories was given by Frank, Hainzl, Seiringer and Solovej [23] for periodic systems with weak external fields. Their results have been extended in [16, 17] to a larger class of weak external fields. It would be interesting to rigorously derive such an effective GL theory also in the presence of a boundary. Understanding the influence of a boundary on the critical temperature is a crucial first step.

For the mathematical description of the BCS model and to sketch the physical motivation behind it, we follow [32]. We consider a fermionic system confined to a subset $\Omega \subset \mathbb{R}^d$ with one-particle Hilbert-space $L^2(\Omega) \otimes \mathbb{C}^2$. The one particle part of the Hamiltonian is of the form $h = ([-i\nabla + A(x)]^2 + W(x)) \otimes \mathbb{I}$, where A and W denote the external magnetic and electric potential, respectively, and $\mathbb{I}$ is the identity operator on $\mathbb{C}^2$. The fermions interact through a two particle interaction $-2V$. Let $\mu \in \mathbb{R}$ be the chemical potential and T the temperature. For a statistical state with density matrix ρ the pressure functional is given by $\mathcal{F}(\rho) = \mathrm{Tr}\ (\mathbb{H} - \mu\mathbb{N})\rho - TS(\rho)$, where $\mathbb{H}$ and $\mathbb{N}$ denote the Hamiltonian and the number operator on Fock space and $S(\rho) = -\mathrm{Tr}\ \rho \ln \rho$ is the entropy. In statistical equilibrium, the system is in the Gibbs state which minimizes the pressure functional over all states $\rho \geq 0$ with $\mathrm{Tr}\ \rho = 1$. However, it is challenging to compute expectation values of observables in the Gibbs state. This also makes it difficult to study superconductivity since this phenomenon is related to a positive expectation for annihilating a pair of particles in one place and creating a pair in a different spot.

To simplify the problem, one can minimize the pressure functional over a smaller set of particularly nice states, the quasi-free states. Every quasi-free state is determined by two operators $\tilde{\alpha}$ and $\tilde{\gamma}$ on $L^2(\Omega)\otimes\mathbb{C}^2$. The operator $\tilde{\gamma}$ is the one-particle density matrix given by $\langle\phi|\tilde{\gamma}\psi\rangle = \mathrm{Tr}\ a^\dagger(\psi)a(\phi)\rho$, where $a^\dagger, a$ are the creation and annihilation operators on Fock space. The operator $\tilde{\alpha}$ is the pairing expectation $\langle\phi|\tilde{\alpha}\overline{\psi}\rangle = \mathrm{Tr}\ a(\psi)a(\phi)\rho$. For quasi-free states Wick's theorem holds, implying that the expectation of a product of annihilation and creation operators can always be expressed in terms of $\tilde{\gamma}$ and $\tilde{\alpha}$. One further restricts to $SU(2)$-invariant states, for which $\tilde{\gamma}$ and $\tilde{\alpha}$ can be written as $\tilde{\gamma}=\gamma\otimes\mathbb{I}$ and $\tilde{\alpha}=\alpha\otimes\sigma_2$, where σ_2 is the second Pauli matrix and α and γ are operators on $L^2(\Omega)$. This effectively removes the spin degrees of freedom. Since ρ is self-adjoint, it turns out that γ is self-adjoint and α is symmetric.

For $SU(2)$-invariant, quasi-free states ρ it is possible to express the pressure functional $\mathcal{F}(\rho)$ in terms of γ and α. Due to Wick's theorem, the trace over the interaction term in the Hamiltonian splits into three summands. Two of these summands, the so-called direct and exchange terms, are omitted for simplicity. At least for short ranged V, this is expected to be a good approximation. These two terms have little influence apart from an effective change in chemical potential. For translation invariant systems, this was justified in [9]. With the notation

$$\Gamma = \begin{pmatrix} \gamma & \alpha \\ \overline{\alpha} & 1-\overline{\gamma} \end{pmatrix}$$

the remaining terms in the pressure functional give the BCS energy functional

$$\mathcal{F}(\Gamma) = \mathrm{Tr}\ \left([-i\nabla + A(x)]^2 + W(x) - \mu\right)\gamma - TS(\Gamma) - \int\int_{\Omega\times\Omega} |\alpha(x,y)|^2 V(x-y)\mathrm{d}x\mathrm{d}y,$$

up to a factor of 2 coming from the trace over spin. Here, $\alpha(x,y)$ is the integral kernel of α and $S(\Gamma) = -\mathrm{Tr}\ \Gamma\ln\Gamma$ where the trace is over $L^2(\Omega)\oplus L^2(\Omega)$. The BCS energy is obtained by minimizing $\mathcal{F}$ over all admissible pairs γ and α giving

$$F(T,\mu) = \inf_{\Gamma, 0\leq\Gamma\leq 1} \mathcal{F}(\Gamma).$$

The interpretation is that superconductivity occurs if and only if for the minimizer α does not vanish identically. The integral kernel of α can be interpreted as two-particle wave function and is referred to as Cooper pair wave function. A minimizer of the BCS energy functional has to satisfy the Euler-Lagrange equation

$$0 = \begin{pmatrix} h & \Delta \\ \overline{\Delta} & -\overline{h} \end{pmatrix} + T\ln\left(\frac{\Gamma}{1-\Gamma}\right), \tag{1.1}$$

where Δ denotes the operator given by $(\Delta\psi)(y) = -\int_\Omega 2V(x-y)\alpha(x,y)\psi(x)\mathrm{d}x$ for $\psi\in L^2(\Omega)$. The integral kernel of Δ is the gap function. There is one solution of the Euler-Lagrange equation (1.1) with $\alpha=0$, the so-called normal state Γ_0 with $\gamma = (1+\exp((h-\mu)/T))^{-1}$. The normal state minimizes $\mathcal{F}(\Gamma)$ among all states with $\alpha=0$ [24]. To determine whether the system is superconducting or not, one therefore has to check whether $F(T,\mu)$ is smaller or bigger than $\mathcal{F}(\Gamma_0)$.

For translation invariant systems, it turns out that there is a unique critical temperature that separates the superconducting from the normal phase [32, 33]. In general, however, when the temperature is raised it is conceivable that the system could change between superconducting and normal states several times. One can define two critical temperatures, the temperature $\overline{T^{BCS}}$, above which there is no superconductivity, and $\underline{T^{BCS}}$, below which there is always

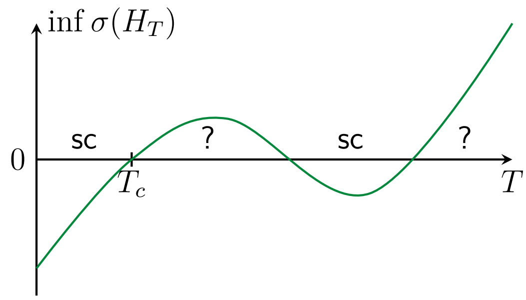

Figure 1.2: Sketch of $\inf \sigma(H_T)$ and the (local) critical temperature T_c for a generic system. The system is superconducting (sc) whenever $\inf \sigma(H_T) < 0$. If $\inf \sigma(H_T) \geq 0$, the state of the system is unknown in general.

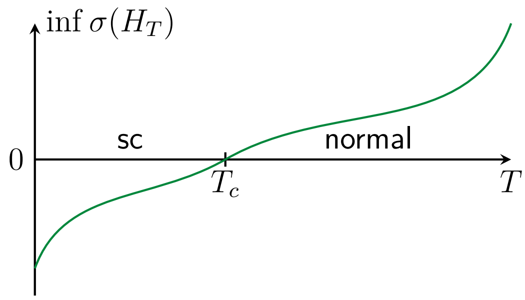

Figure 1.3: Sketch of $\inf \sigma(H_T)$ and the critical temperature T_c for a translation invariant system. For $T < T_c$ the system is superconducting. For $T > T_c$ the system is in the normal state.

superconductivity. For systems without boundary, but where the translation invariance is broken by external fields, these critical temperatures were computed in [16, 17, 24] for weak macroscopic external fields.

In contrast to introducing weak external fields, introducing a boundary is not a small perturbation and computing $\overline{T^{BCS}}$ is currently beyond our reach. Nevertheless, there is a simpler approach which allows to estimate $\underline{T^{BCS}}$. This simpler approach is based on the observation that it is possible to obtain information on the state of the system by analyzing the stability of the normal state. If the normal state is not a (local) minimum, there is a different minimizer with $\alpha \neq 0$, which corresponds to superconductivity. To check the local stability, one can compute the Hessian of $\mathcal{F}$ at the normal state. Positivity of the Hessian indicates that the normal state is a local minimizer, i.e. small perturbations increase $\mathcal{F}$. If the spectrum contains a negative number, the system is superconducting. Since the normal state is optimal in γ, it suffices to consider the Hessian for variations in α only. In [21], this linear operator is computed to be $2(K_T - V)$ acting on $L^2_{symm}(\Omega \times \Omega) = \{\psi \in L^2(\Omega \times \Omega) | \psi(x,y) = \psi(y,x) \text{ for all } x,y \in \Omega\}$ with appropriate boundary conditions, where V acts as $(V\alpha)(x,y) = V(x-y)\alpha(x,y)$ and

$$K_T = \frac{h_x + h_y - 2\mu}{\tanh\left(\frac{h_x-\mu}{2T}\right) + \tanh\left(\frac{h_y-\mu}{2T}\right)}. \tag{1.2}$$

Here, h_x and h_y denote h acting on the x or y variable, respectively. Since h_x and h_y commute, the operator K_T is well-defined through functional calculus. Let

$$H_T = K_T - V.$$

If the infimum of the spectrum $\inf \sigma(H_T) < 0$, the normal state is unstable, and therefore the system is superconducting. For $\inf \sigma(H_T) > 0$, the normal state is a local minimum, but it is unclear whether it is also a global minimum of $\mathcal{F}$. Hence, it remains unclear whether the system is superconducting in this case. The situation in sketched in Figure 1.2.

For translation invariant systems it was shown that the normal state is a global minimum if $\inf \sigma(H_T) > 0$ [32, 33]. The argument involves proving that the minimizer of the BCS functional is translation invariant [32, Section IV.F] and using a monotonicity argument for the gap equation (1.1) [32, Section III.A], which is specific to the translation invariant case. In particular, it suffices to compute $\inf \sigma(H_T)$ to determine whether the system is superconducting or not. Moreover, in the translation invariant case K_T is strictly monotone in T. Hence, there is a unique critical temperature T_c defined by $\inf \sigma(H_{T_c}) = 0$ which separates the normal and the superconducting phase, as visualized in Figure 1.3. For a general system, we define the (local) critical temperature as sketched in Figure 1.2 through

$$T_c := \inf\{T \in (0,\infty) | \inf \sigma(H_T) \geq 0\}.$$

This critical temperature satisfies $T_c \leq \underline{T^{BCS}}$ in general and $T_c = \underline{T^{BCS}} = \overline{T^{BCS}}$ for translation invariant systems.

In this setting, we shall ask two types of questions. Given a system with boundary,

- is the corresponding critical temperature T_c larger than the critical temperature for the system without boundary?
- does the critical temperature T_c increase upon adding another boundary?

In this thesis we focus on domains of the form $\Omega_k = (0,\infty)^k \times \mathbb{R}^{d-k}$, where d is the dimension of the system and k denotes how many times we cut full space in half. We shall assume either Dirichlet or Neumann boundary conditions and that there are no external fields. If T_c^k denotes the critical temperature on Ω_k, we want to find out

- under which conditions $T_c^0 < T_c^k$ for $1 \leq k \leq d$,
- and whether for $d = 2, 3$ the critical temperatures form an increasing sequence $T_c^0 < T_c^1 < T_c^2 (< T_c^3)$.

In terms of the critical temperatures of the BCS functional, since the system on Ω_0 is translation invariant, the first point corresponds to $T_c^0 = \underline{T^{BCS,0}} = \overline{T^{BCS,0}} < T_c^k \leq \underline{T^{BCS,k}}$. This means whenever $T_c^0 < T_c^k$, there is a temperature range, where the system without boundary is in the normal state, whereas the system with boundary is superconducting. The second point corresponds to the surface and corner superconductivity sketched in Figure 1.1 within the framework of local critical temperatures.

1.1 Two-Particle Schrödinger Operator

The critical temperature T_c is determined by the spectrum of $H_T = K_T - V$, where K_T is defined in (1.2). Since the operator K_T is rather complicated, we start by considering a simpler problem. We replace K_T by a Laplacian operator and study the spectrum of the resulting operator.

More precisely, we study the operator

$$H = -\frac{1}{2m_a}\Delta_{x^a} - \frac{1}{2m_b}\Delta_{x^b} + V(x^a - x^b),$$

which is the Schrödinger operator for two interacting particles with masses m_a and m_b. The Hamiltonian H specifies the energy of the system. We assume the interaction to be regular enough for the Hamiltonian to be bounded from below and that it decays at infinity. Furthermore, we assume that the interaction is sufficiently attractive such that in free space the particles form a bound state with energy $\inf \sigma(H) < 0$. We sequentially constrain the particles to the smaller spaces Ω_k. The goal is to understand how adding a boundary affects the spectrum and the ground state. Of course, this depends on the boundary conditions. Dirichlet boundaries tend to repel states and to increase the ground state energy. In our setting all particle domains are infinite and adding a Dirichlet boundary does not affect the spectrum. Hence, we impose Neumann boundary conditions. The general picture is that a Neumann boundary "attracts" the particles. We show that introducing Neumann boundaries creates new bound states with lower energies.

We remove the free center of mass kinetic energy from the Hamiltonian H to work with operators that have eigenvalues; this gives modified Hamiltonians H_k. We show that the ground state energy strictly decreases when cutting space in half, i.e. when going from k to $k+1$. Moreover, the essential spectrum after dividing space starts at the previous ground state energy. Finally, there is only a finite number of eigenvalues. This is the content of our main theorem.

Theorem 1.1.1. *For every $k \in \{1, ..., d\}$, the bottom of the spectrum of the operator H_k is an isolated eigenvalue E^k. The essential spectrum of H_k is $\sigma_{ess}(H_k) = [E^{k-1}, \infty)$. Thus, the ground state energies form a decreasing sequence $E^d < E^{d-1} < ... < E^0 < 0$. Moreover, the operator H_k has only a finite number of eigenvalues below the essential spectrum.*

In one dimension with equal masses $m_a = m_b$, the same properties were already proved in [18]. The proof of Theorem 1.1.1 is provided in Chapter 2 and is the content of the publication

- B. Roos and R. Seiringer. "Two-particle bound states at interfaces and corners". *Journal of functional analysis* **282**.12 (2022), p. 109455. DOI: 10.1016/j.jfa.2022.109455.

Here, we briefly sketch the main methods used in the proof. Assuming that H_{k-1} has an eigenvalue E^{k-1} at the bottom of its spectrum, we first compute that $\sigma_{ess}(H_k) = [E^{k-1}, \infty)$. Second, we show that the operator H_k has an eigenvalue below the essential spectrum. Since the operator H_0 has an eigenvalue at the bottom of its spectrum by assumption, we inductively obtain the decreasing sequence of ground state energies. Third, we show that the number of eigenvalues is finite.

To compute the essential spectrum, we construct Weyl sequences which show $[E^{k-1}, \infty) \subset \sigma_{ess}(H_k)$. For the opposite inclusion, we bound the essential spectrum of H_k from below by introducing additional Neumann boundaries. They split the particle domain into three regions. One of them is bounded, so it does not contribute to the essential spectrum. In the second region, the Hamiltonian approximately becomes $H_{k-1} \otimes \mathbb{I} - \frac{1}{2\mu}\mathbb{I} \otimes \Delta_{z_k}$, where z_k denotes the kth component of the center of mass variable and $\mu = \frac{m_a m_b}{m_a + m_b}$ is the reduced mass. For this operator the essential spectrum starts at E^{k-1}. In the third region, the interaction potential is larger than E^{k-1}. Thus, there is no essential spectrum below E^{k-1}.

To show that there is an eigenvalue below the essential spectrum, we use the variational principle. The operator H_{k-1} has a non-degenerate ground state ψ_{k-1} which can be chosen positive almost everywhere [29]. We choose the trial function $\psi = \psi_{k-1} e^{-\gamma z_k}$, which decays exponentially away from the new boundary. Using the positivity and uniqueness of ψ_{k-1}, we obtain $h_k[\psi] < E^{k-1}\|\psi\|^2$ for $\gamma > 0$ small enough.

To prove the finiteness of the discrete spectrum, we use the standard technique of localization. Using the min-max principle one can bound the number of eigenvalues of H_k below E^k

through the number of eigenvalues below E^k of the localized operators. The localization is conducted in such a way that the localized operators fall into three categories. First, the operator can be compact or second, the potential is larger than E^k. In these cases the number of eigenvalues below E^k is certainly finite or zero. In the third category, the operator is of the form $H_{k-1} \otimes \mathbb{I} - \frac{1}{2\mu}\mathbb{I} \otimes \Delta_{z_k} - G$, where G is a well behaved error term. One estimates this operator by projecting onto $\psi_{k-1} \otimes L^2(\mathbb{R})$ and its orthogonal complement. This reduces the problem to a one dimensional operator. Then the Agmon [3] and Bargman estimate [8] imply that the number of eigenvalues is finite.

1.2 Universality in Low Dimensions

Before being able to investigate boundary superconductivity, one needs to compute the critical temperature for translation invariant systems. We are particularly interested in the weak coupling limit, where we replace V by λV and send $\lambda \to 0$, as this is the regime where boundary superconductivity was observed in [62]. In three dimensions, the asymptotics are well-known and summarized in [32]. We compute the asymptotics for one and two dimensional systems in

- J. Henheik, A. B. Lauritsen, and B. Roos. "Universality in low-dimensional BCS theory". Preprint of an article accepted for publication in Reviews in Mathematical Physics. arXiv:2301.05621 [cond-mat, physics:math-ph]. 2023. URL: `http://arxiv.org/abs/2301.05621`,

which is the content of Chapter 3. Apart from the asymptotics of the critical temperature $T_c^0(\lambda)$, we also compute the asymptotics of the energy gap $\Xi(\lambda)$ at zero temperature. This allows us to prove that the ratio converges to a number independent of V,

$$\lim_{\lambda \to 0} \frac{\Xi(\lambda)}{T_c^0(\lambda)} = \frac{\pi}{e^\gamma} \tag{1.3}$$

in dimensions one and two, where γ is the Euler-Mascheroni constant. The property that the ratio is independent of the microscopic details of the interaction is called universality. The same universal behavior is well known to occur in three dimensions [5, 46], with rigorous proofs in different limits ($\lambda \to 0, \mu \to 0, \mu \to \infty$) in [22, 35, 36, 38, 39, 49].

In the translation invariant setting, the gap function Δ only depends on the relative coordinate. In [33, Appendix A] it is explained that the energy gap is given by

$$\Xi = \inf_{p \in \mathbb{R}^d} \sqrt{(p^2 - \mu)^2 + |\Delta(p)|^2},$$

where $\Delta(p)$ is the Fourier transform of the gap function with p the momentum in the relative coordinate. The asymptotics of T_c^0 and Δ are determined by self-adjoint operators $\mathcal{V}_\mu, \mathcal{W}_\mu : L^2(\mathbb{S}^{d-1}) \to L^2(\mathbb{S}^{d-1})$ to leading and next to leading order, respectively. The integral kernel of $\mathcal{V}_\mu$ is

$$\mathcal{V}_\mu(p, q) = \frac{1}{(2\pi)^{d/2}} \hat{V}(\sqrt{\mu}(p - q)).$$

This operator can be interpreted as restriction of V to the Fermi sphere. The operator $\mathcal{W}_\mu$ is defined via the quadratic form

$$\langle u, \mathcal{W}_\mu u\rangle = \mu^{d/2-1}\Big[\int_{|p|<\sqrt{2}} \frac{1}{|p^2-1|}\left(|\psi(\sqrt{\mu}p)|^2 - |\psi(\sqrt{\mu}p/|p|)|^2\right)\mathrm{d}p + \int_{|p|>\sqrt{2}} \frac{1}{|p^2-1|}|\psi(\sqrt{\mu}p)|^2\mathrm{d}p\Big],$$

where $\psi(p) = \frac{1}{(2\pi)^{d/2}}\int_{\mathbb{S}^{d-1}} \hat{V}(p-\sqrt{\mu}\omega)u(\omega)\mathrm{d}\omega$ and $u \in L^2(\mathbb{S}^{d-1})$. Let $b_\mu(\lambda)$ be given by

$$b_\mu(\lambda) = \sup\sigma(\lambda\mathcal{V}_\mu + \lambda^2\mathcal{W}_\mu).$$

We consider $d \in \{1,2,3\}$, $\mu > 0$ and V regular enough with $e_\mu = \sup\sigma(\mathcal{V}_\mu) > 0$ (the precise assumptions can be found in Theorems 3.2.5 and 3.2.7 for $d = 1, 2$ and [32, Theorem 3.3] for $d = 3$). The critical temperature T_c^0 and the energy gap Ξ satisfy

$$\begin{aligned}\lim_{\lambda\to 0}\left(\ln\left(\frac{\mu}{T_c^0(\lambda)}\right) - \frac{1}{\mu^{d/2-1}b_\mu(\lambda)}\right) &= -\gamma - \ln\left(\frac{2c_d}{\pi}\right), \\ \lim_{\lambda\to 0}\left(\ln\left(\frac{\mu}{\Xi(\lambda)}\right) - \frac{1}{\mu^{d/2-1}b_\mu(\lambda)}\right) &= -\ln(2c_d),\end{aligned} \tag{1.4}$$

where c_d is a dimension-dependent constant. In particular,

$$\begin{aligned}T_c^0(\lambda) &= 2c_d\frac{e^\gamma}{\pi}(1+o(1))\mu e^{1/(\mu^{d/2-1}b_\mu)}, \\ \Xi(\lambda) &= 2c_d(1+o(1))\mu e^{1/(\mu^{d/2-1}b_\mu)},\end{aligned}$$

and (1.3) holds.

The proof in one and two dimensions provided in Chapter 3 is similar to the three dimensional case in [35]. Here, we briefly sketch the main ingredients. For the asymptotics of T_c^0 one uses the Birman Schwinger principle. The Birman-Schwinger operator is defined as

$$A_T = V^{1/2}K_T^{-1}|V|^{1/2}, \tag{1.5}$$

where $V^{1/2}$ denotes multiplication by $\mathrm{sgn}(V(r))|V(r)|^{1/2}$. The condition $\inf\sigma(K_{T_c^0} - \lambda V) = 0$ is equivalent to $1 = \sup\sigma(\lambda A_{T_c^0})$. It turns out that $V^{1/2}K_T^{-1}|V|^{1/2}$ has a logarithmic divergence at the Fermi surface as $T \to 0$ and is bounded otherwise. The divergence is captured by $m_\mu(T)O_\mu$, where $m_\mu(T)$ is a function with asymptotics $m_\mu(T) = \mu^{d/2-1}\ln(\mu/T) + O(1)$ for $T \to 0$ and O_μ is the operator $O_\mu = V^{1/2}\mathcal{F}^\dagger\mathcal{F}|V|^{1/2}$, where $\mathcal{F} : L^1(\mathbb{R}^d) \to L^2(\mathbb{S}^{d-1})$ is the Fourier transform restricted to the Fermi sphere $\mathcal{F}\psi(\omega) = \hat{\psi}(\sqrt{\mu}\omega)$. For suitable radial V, for instance when $\hat{V} \geq 0$, this asymptotic operator O_μ has a non-degenerate eigenvalue at the top of its spectrum. It turns out that O_μ is isospectral to $\mathcal{V}_\mu$ and hence $e_\mu = \sup\sigma(O_\mu)$. The corresponding eigenfunction of O_μ is given by $V^{1/2}j_d$, where

$$j_d(r) = \frac{1}{(2\pi)^{d/2}}\int_{\mathbb{S}^{d-1}} e^{i\sqrt{\mu}\omega\cdot r}\mathrm{d}\omega. \tag{1.6}$$

One obtains that $\lambda\mu^{d/2-1}e_\mu \ln\frac{\mu}{T_c^0(\lambda)} \to 1$ and in particular $T_c^0(\lambda) \to 0$ as $\lambda \to 0$. In order to arrive at the asymptotics of $\ln\frac{\mu}{T_c^0(\lambda)}$ stated in (1.4), one then needs to compute the second order correction.

For the energy gap, the first step is to prove that under suitable conditions on V, involving that V is radial, the gap function Δ is unique, which makes Ξ well-defined. Under these conditions, Δ is also radial and positive. Then one uses the Euler Lagrange equation of the BCS functional to argue that the operator $\sqrt{(p^2-\mu)^2+|\Delta(p)|^2}-\lambda V$ has the Cooper pair wave function α as ground state with energy zero. Similarly as for T_c, one applies the Birman Schwinger principle to compute an asymptotic expression for Δ. It turns out that $\Delta(p)$ vanishes for all p as $\lambda \to 0$. Hence, $\sqrt{(p^2-\mu)^2+|\Delta(p)|^2}$ is minimal close to the Fermi sphere and $\Xi = \Delta(\sqrt{\mu})(1+o(1))$, where we use radiality to write $\Delta(\sqrt{\mu})$ instead of $\Delta(\sqrt{\mu}\hat{p})$ for some $\hat{p} \in \mathbb{S}^{d-1}$. Now $\Delta(\sqrt{\mu})$ takes the role of a vanishing parameter similar to T_c^0 before, and one computes the asymptotics of $\ln \frac{\mu}{\Delta(\sqrt{\mu})}$ to second order in a similar way as for $\ln \frac{\mu}{T_c^0(\lambda)}$.

1.3 Boundary Superconductivity

We are now ready to study superconductors with boundaries. We consider superconductors of the shape $\Omega_k = (0,\infty)^k \times \mathbb{R}^{d-k}$, with either Dirichlet or Neumann boundary conditions and no external fields. In this setting, we want to understand the relationship between the corresponding critical temperatures T_c^k.

In [62] the authors consider the one dimensional case of a half-line versus full line with interaction $\lambda\delta$, where λ is the coupling parameter and δ the Dirac delta. They assume Dirichlet boundary conditions and compute the minimizer of the Hessian H_T on half-space numerically. They report that in the center of mass coordinate it decays exponentially away from the boundary with some oscillation near the boundary. In momentum space, this corresponds to a peak at zero and a dip at $\sqrt{\mu}$. Their results indicate that $T_c^1(\lambda) > T_c^0(\lambda)$ for some values of λ, and that the relative difference $\frac{T_c^1(\lambda)-T_c^0(\lambda)}{T_c^0(\lambda)}$ vanishes as $\lambda \to 0$ and for large enough λ. In Chapter 4, which contains the following publication

- C. Hainzl, B. Roos, and R. Seiringer. "Boundary superconductivity in the BCS Model". *Journal of spectral theory* **12**.4 (2022), pp. 1507–1540. DOI: 10.4171/JST/439

we study this system rigorously and confirm some of the predictions from [62]. Our main result for Dirichlet boundary conditions is

Theorem 1.3.1. *Let $\mu > 0$.*

1. *There is a $\lambda_1 > 0$ such that $T_c^1(\lambda) > T_c^0(\lambda)$ for $0 < \lambda < \lambda_1$.*
2. *In the weak coupling limit*
$$\lim_{\lambda \to 0} \frac{T_c^1(\lambda) - T_c^0(\lambda)}{T_c^0(\lambda)} = 0. \tag{1.7}$$
3. *In the strong coupling limit*
$$\lim_{\lambda \to \infty} \frac{T_c^1(\lambda) - T_c^0(\lambda)}{T_c^0(\lambda)} = 0.$$

This proves that a boundary can increase the critical temperature of a superconductor. Furthermore, it confirms the behavior of the relative temperature difference in the strong and weak coupling limit. However, it is unclear whether the difference vanishes at large enough finite λ already. We extend this result to Neumann boundary conditions, where we obtain that boundary superconductivity exists at all coupling strengths.

Theorem 1.3.2. *Let $\mu > 0$.*

1. *Then $T_c^1(\lambda) > T_c^0(\lambda)$ for all $\lambda > 0$.*
2. *In the weak coupling limit*
$$\lim_{\lambda\to 0} \frac{T_c^1(\lambda) - T_c^0(\lambda)}{T_c^0(\lambda)} = 0. \tag{1.8}$$
3. *In the strong coupling limit*
$$0 < \lim_{\lambda\to\infty} \frac{T_c^1(\lambda) - T_c^0(\lambda)}{T_c^0(\lambda)} < \infty.$$

To look at higher dimensional systems, the numerical works [6, 7, 62, 63, 68] focus on tight binding models on lattices. Our analytic analytic approach allows us to study continuum models in higher dimensions and with generic interactions. In Chapter 5, we consider the case of a half-space in dimensions $d \in \{1, 2, 3\}$. This chapter contains the preprint

- B. Roos and R. Seiringer. "BCS Critical Temperature on Half-Spaces". arXiv:2306.05824 [cond-mat, physics:math-ph]. 2023. URL: http://arxiv.org/abs/2306.05824.

Again, we prove that an increase in critical temperature occurs in the weak coupling limit, meaning we take the interaction λV and look at small λ. As we saw in Section 1.2, the function j_d defined in (1.6) determines the asymptotics of the minimizer of H_T on $\mathbb{R}^d$ in the Birman-Schwinger picture. For $d = 3$, we define

$$\widetilde{m}_3^{D/N}(r;\mu) := \int_{\mathbb{R}} \left(j_3(z_1, r_2, r_3)^2 - |j_3(z_1, r_2, r_3) \mp j_3(r)|^2 \chi_{|z_1|<|r_1|} \right) \mathrm{d}z_1 \mp \frac{\pi}{\mu^{1/2}} j_3(r)^2, \tag{1.9}$$

where the indices D and N as well as the upper/lower signs correspond to Dirichlet/Neumann boundary conditions, respectively. For radial and regular enough V with $\hat{V}(0) > 0$ and such that e_μ is a non-degenerate eigenvalue of $\mathcal{V}_\mu$, we obtain the following result (the precise assumptions are stated in 5.1.1).

Theorem 1.3.3. *Let $d \in \{1, 2, 3\}$, $\mu > 0$. Assume either Dirichlet or Neumann boundary conditions. For $d = 3$ additionally assume that*

$$\int_{\mathbb{R}^3} V(r) \widetilde{m}_3^{D/N}(r;\mu) \mathrm{d}r > 0. \tag{1.10}$$

Then, there is a $\lambda_1 > 0$, such that for all $0 < \lambda < \lambda_1$, $T_c^1(\lambda) > T_c^0(\lambda)$.

Note that for general interactions in one dimension with Neumann boundary conditions, we do not necessarily observe $T_c^1(\lambda) > T_c^0(\lambda)$ for all λ as in Theorem 1.3.2. For $d = 3$ we further prove that (1.10) is satisfied for small enough chemical potential. Numerical evaluation of $\widetilde{m}_3^D$ suggests that $\widetilde{m}_3^D \geq 0$ (see Section 5.5, in particular Figure 5.1), whereas $\widetilde{m}_3^N$ changes sign (Figure 5.2). For Dirichlet boundary conditions (1.10) thus seems to hold under the additional assumption that $V \geq 0$. Hence we expect boundary superconductivity to occur for all $\mu > 0$ also in three dimensions. One may wonder why in lower dimensions no condition like (1.10) is needed. For $d \in \{1,2\}$, if one defines $\widetilde{m}_d^{D/N}(r;\mu)$ by replacing j_3 by j_d in (1.9), the first term diverges and $\widetilde{m}_d^{D/N}(r;\mu) = +\infty$. The analogue of (1.10) is then always satisfied if $\widehat{V}(0) > 0$.

The second main result in Chapter 5 is that the relative shift in critical temperature vanishes as $\lambda \to 0$ for both Dirichlet and Neumann boundary conditions. This generalizes the corresponding results (1.7) and (1.8) to dimensions $d \in \{1,2,3\}$ and generic interactions.

In Chapter 6 we focus on two dimensions and compare critical temperatures T_c^1 and T_c^2 of the half-space Ω_1 and the quarter space Ω_2, respectively. This chapter contains the preprint

- B. Roos and R. Seiringer. "Enhanced BCS Superconductivity at a Corner". arXiv:2308.07115 [cond-mat, physics:math-ph]. 2023. URL: http://arxiv.org/abs/2308.07115

For $\mu > 0$ and interactions V satisfying essentially the same assumptions as for Theorem 1.3.3 (see 6.1.2 for the precise statement), we prove that there is a $\lambda_1 > 0$, such that for all $0 < \lambda < \lambda_1$, $T_c^2(\lambda) > T_c^1(\lambda) > T_c^0(\lambda)$ for Dirichlet and Neumann boundary conditions. Furthermore, we prove that the relative difference between $T_c^2(\lambda)$ and $T_c^0(\lambda)$ vanishes in the weak coupling limit.

The proofs of $T_c^1(\lambda) > T_c^0(\lambda)$ and $T_c^2(\lambda) > T_c^1(\lambda)$ follow the same strategy, which we will illustrate here in the example of dimension $d = 1$. Comparing the Birman-Schwinger operators for Ω_0 and Ω_1, it turns out that the boundary induces a compact perturbation. One can conclude that the Hessian on the half-line always contains the spectrum of the Hessian on the full line, and therefore $T_c^1(\lambda) \geq T_c^0(\lambda)$. To show the strict inequality $T_c^1(\lambda) > T_c^0(\lambda)$, the idea is to use the variational principle. We build a trial state which contains the ground state Φ_λ of $H_{T_c^0(\lambda)}$ on Ω_0. Due to the translation invariance of $H_{T_c^0(\lambda)}$ in the center of mass direction $z = x + y$, Φ_λ is a function of $r = x - y$. We want the trial state to look like $\Phi_\lambda(r)e^{-\epsilon|z|}$, but we symmetrize it to meet the boundary conditions, which leads to the choice

$$\psi_\epsilon(r,z) = \Phi_\lambda(r)e^{-\epsilon|z|} \mp \Phi_\lambda(z)e^{-\epsilon|r|}$$

where the signs $-$ and $+$ correspond to Dirichlet and Neumann boundary conditions, respectively. From the asymptotics of the Birman-Schwinger operator (1.5) described in Section 1.2, it follows that $V^{1/2}\Phi_\lambda$ converges to $V^{1/2}j_1$. Hence, we expect Φ_λ to localize on the Fermi sphere. In particular, the two summands in ψ_ϵ localize at momentum zero and momentum $\sqrt{\mu}$ in z-direction. This agrees with the qualitative behavior of the minimizer for the half-line with delta interaction numerically obtained in [62]. The strategy is to show that for H_T on the half-line

$$\lim_{\lambda \to 0} \lim_{\epsilon \to 0} \langle \psi_\epsilon, H_{T_c^0(\lambda)} \psi_\epsilon \rangle < 0. \tag{1.11}$$

By continuity, and since we already know that $\inf \sigma(H_T) < 0$ for $T < T_c^0(\lambda)$ we then have that $\inf \sigma(H_T) < 0$ for all $0 < T < T_c^0 + \delta$ for a small $\delta > 0$, proving that $T_c^1(\lambda) > T_c^0(\lambda)$ for small λ. Taking the limit $\epsilon \to 0$ in (1.11) gives some expression involving V and Φ_λ. To compute the asymptotics of this expression for $\lambda \to 0$, we use the asymptotics in the Birman-Schwinger picture to effectively replace Φ_λ by j_1. It turns out that the leading order term is negative, giving (1.11). In three dimensions, the term that defines the leading order in one and two dimensions has the same order as the other terms. This is where the condition (1.10) originates.

To prove that the relative difference in critical temperature vanishes in the weak coupling limit, one uses the fact that the boundary causes a bounded perturbation in the Birman-Schwinger picture, while the unperturbed part diverges logarithmically as $T \to 0$. This also allows to compute the asymptotics of the Birman-Schwinger maximizer $V^{1/2}\Phi_\lambda$ on half-space, which is used in the trial state to prove that $T_c^2(\lambda) > T_c^1(\lambda)$.

To sum up, the results in this thesis confirm that a boundary can increase the BCS critical temperature. There are many topics left for future research, including

- the extension of our results to domains of other shapes, different boundary conditions and external fields,
- the investigation of the relationship of T_c defined through the linear criterion, and the critical temperatures $\underline{T^{BCS}}, \overline{T^{BCS}}$ of the full BCS functional,
- the study of the gap function Δ, and the proof of its localization at the boundary, justifying Figure 1.1.

The last two points involve studying the non-linear BCS functional, which we expect to be much more difficult than working with the linear criterion that defines the local critical temperature T_c.

CHAPTER 2

Two-particle Bound States at Interfaces and Corners

Abstract We study two interacting quantum particles forming a bound state in d-dimensional free space, and constrain the particles in k directions to $(0,\infty)^k \times \mathbb{R}^{d-k}$, with Neumann boundary conditions. First, we prove that the ground state energy strictly decreases upon going from k to $k+1$. This shows that the particles stick to the corner where all boundary planes intersect. Second, we show that for all k the resulting Hamiltonian, after removing the free part of the kinetic energy, has only finitely many eigenvalues below the essential spectrum. This paper generalizes the work of Egger, Kerner and Pankrashkin (J. Spectr. Theory 10(4):1413–1444, 2020) to dimensions $d > 1$.

2.1 Introduction and Main Results

We consider two interacting quantum particles in d-dimensional space that form a bound state in free space. We constrain the particles in k directions to $(0,\infty)^k \times \mathbb{R}^{d-k}$ for some $k \in \{1,...,d\}$ and impose Neumann boundary conditions. The goal of this paper is to show that at low energy the particles will stick to the boundary of the domain. In fact, the particles want to be close to as many boundary planes as possible. In particular, they stick to the corner where all boundary planes intersect. Neumann boundary conditions can be interpreted as representing perfect mirrors. It is remarkable that while such boundary conditions are not sufficiently attractive to capture single particles, mutually bound pairs are always attracted to the boundary.

In order to justify the picture of particles sticking to the boundary, we show that introducing a boundary plane lowers the ground state energy. Then it is energetically favorable for the particles to localize at a finite distance to the new boundary plane. Moving the particles away from that boundary plane would reduce the boundary effects and raise the energy to reach the previous ground state energy, which is strictly higher. Since moving just one of the particles to infinity would increase the potential energy between them, both particles stick to the boundary.

This problem was already studied (for particles with equal masses) in the case $d = k = 1$. Kerner and Mühlenbruch [41] considered a hard-wall interaction between the particles. (For a higher-dimensional version of this problem, which is different from the one we consider here, however, see [4].) More general interactions were studied by Egger, Kerner and Pankrashkin in

[18]. Additionally, they showed that the Hamiltonian has only finitely many eigenvalues below the essential spectrum. We show here that this also holds true for particles with different masses and all dimensions d and numbers of boundary planes k. The finiteness of the number of bound states is a consequence of the fact that the effective attractive interaction with the boundary decays exponentially with distance, a decay that is inherited from the corresponding one of the ground state wave function in free space.

Let x^a and x^b be the coordinates of the particles. The Hamiltonian of the system is

$$H = -\frac{1}{2m_a}\Delta_{x^a} - \frac{1}{2m_b}\Delta_{x^b} + V(x^a - x^b) \tag{2.1}$$

acting in $L^2\left((0,\infty)^k \times \mathbb{R}^{d-k}\right) \otimes L^2\left((0,\infty)^k \times \mathbb{R}^{d-k}\right)$, where $V : \mathbb{R}^d \to \mathbb{R}$ is the interaction potential. We change to relative and center-of-mass coordinates $y = x^a - x^b$ and $z = \frac{m_a x^a + m_b x^b}{M}$, where $M = m_a + m_b$ is the total mass. The conditions $x^a_j > 0$ and $x^b_j > 0$ for $1 \leq j \leq k$ result in the coordinates $(z_1, ..., z_k, y_1, ..., y_k)$ lying in the domain

$$Q_k = \left\{(z_1, ..., z_k, y_1, ..., y_k) \in \mathbb{R}^{2k} \mid \forall j \in \{1, ..., k\} : z_j > 0 \text{ and } -\frac{M}{m_b}z_j < y_j < \frac{M}{m_a}z_j\right\}, \tag{2.2}$$

while $(z_{k+1}, ..., z_d)$ and $(y_{k+1}, ..., y_d)$ lie in $\mathbb{R}^{d-k}$. In these coordinates, the Hamiltonian becomes $H = -\frac{1}{2\mu}\Delta_y - \frac{1}{2M}\Delta_z + V(y)$, where $\mu = \frac{m_a m_b}{M}$ is the reduced mass. Separating the variables $(z_{k+1}, ..., z_d)$ from the rest, we write the Hamiltonian as $H = H_k \otimes \mathbb{I} + \mathbb{I} \otimes q$, where $q = -\frac{1}{2M}\Delta$ on $H^2(\mathbb{R}^{d-k})$ and

$$H_k = -\frac{1}{2\mu}\Delta_y - \frac{1}{2M}\sum_{j=1}^{k}\frac{\partial^2}{\partial z_j^2} + V(y) \tag{2.3}$$

acting in $L^2(Q_k \times \mathbb{R}^{d-k})$. To be precise, we define the Hamiltonian H_k via the quadratic form

$$h_k[\psi] = \int_{Q_k \times \mathbb{R}^{d-k}} \left(\frac{1}{2\mu}|\nabla_y \psi|^2 + \frac{1}{2M}\sum_{j=1}^{k}\left|\frac{\partial \psi}{\partial z_j}\right|^2 + V(y)|\psi|^2\right) \mathrm{d}z_1 ... \mathrm{d}z_k \mathrm{d}y_1 ... \mathrm{d}y_d \tag{2.4}$$

with domain $D[h_k] = H^1(Q_k \times \mathbb{R}^{d-k})$. Due to the free part of the kinetic energy q, the Hamiltonian H has no discrete spectrum if $k < d$. We remove this free part and work with H_k instead of H.

We impose the following conditions on the interaction potential V.

Assumption 2.1.1. We assume that

1. $V = v + w$ for some $v \in L^r(\mathbb{R}^d)$ and $w \in L^\infty(\mathbb{R}^d)$, where

$$r = 1 \quad \text{if } d = 1, \tag{2.5}$$
$$r > 1 \quad \text{if } d = 2, \tag{2.6}$$
$$r \geq \frac{d}{2} \quad \text{if } d \geq 3, \tag{2.7}$$

2. the operator $H_0 = -\frac{1}{2\mu}\Delta_y + V(y)$ in $L^2(\mathbb{R}^d)$ has a ground state ψ_0 with energy $E^0 < 0$,

3. $\liminf_{|y|\to\infty} V(y) \geq 0$,

4. V is invariant under permutation of the d coordinates $(y_1, ..., y_d) \in \mathbb{R}^d$.

Remark 2.1.2. Condition 1 implies that in the quadratic form h_k the interaction term is infinitesimally form bounded with respect to the kinetic energy, see Proposition 2.A.3 in the Appendix. The KLMN theorem (see e.g. Theorem 6.24 in [69]) then guarantees that there is a unique self-adjoint operator H_k corresponding to h_k, which is bounded from below. Assumption 2 means that the particles form a bound state in free space. Condition 3 is a rather strong form of decay of the negative part at infinity. Presumably some weaker assumptions would be sufficient, but in our proofs this version is convenient. Also the assumptions on the positive part of V can probably be relaxed. Assumption 4 is imposed for convenience as it implies that it is irrelevant which coordinates are restricted, and without loss of generality we pick the first k. However, our methods easily extend to the general case.

Our first result is that the ground state energy strictly decreases upon adding a Neumann boundary that cuts space in half, i.e. when going from $k \to k+1$. Moreover, the essential spectrum after dividing space starts at the previous ground state energy.

Theorem 2.1.3. *Let V satisfy Assumptions 2.1.1. Then for every $k \in \{1, ..., d\}$, the bottom of the spectrum of the operator H_k is an isolated eigenvalue $E^k = \inf \sigma(H_k)$. Moreover, the essential spectrum of H_k is $\sigma_{ess}(H_k) = [E^{k-1}, \infty)$. In particular, the ground state energies form a decreasing sequence $E^d < E^{d-1} < ... < E^0 < 0$.*

Our second result is that the operators H_k have only finitely many bound states.

Theorem 2.1.4. *Let $1 \leq k \leq d$. Then H_k has a finite number of eigenvalues below the essential spectrum.*

In the one-dimensional case $d = k = 1$ with equal masses $m_a = m_b$, Theorems 2.1.3 and 2.1.4 were proved in [18]. While we follow their main ideas, several new ingredients are needed to extend the results to general d and k. In particular, the localization procedure in the proofs is more complicated and requires several additional steps.

Remark 2.1.5. At various places it will be convenient to switch back to the particle coordinates in the first k components, while keeping the relative coordinate in the last $d-k$ components. We shall from now on use the notation $x^a = (x_1^a, ..., x_k^a)$, $x^b = (x_1^b, ..., x_k^b)$ for the first k components of the particle coordinates and $\tilde{y} = (y_{k+1}, ..., y_d)$ for the remaining components of the relative coordinate. In this notation, $y = (x^a - x^b, \tilde{y})$ and

$$h_k[\psi] = \int_{[0,\infty)^{2k}\times\mathbb{R}^{d-k}} \left(\frac{1}{2m_a}|\nabla_{x^a}\psi|^2 + \frac{1}{2m_b}|\nabla_{x^b}\psi|^2 + \frac{1}{2\mu}|\nabla_{\tilde{y}}\psi|^2 + V(x^a - x^b, \tilde{y})|\psi|^2 \right) \mathrm{d}x^a \mathrm{d}x^b \mathrm{d}\tilde{y} \quad (2.8)$$

with domain $D[h_k] = H^1((0,\infty)^{2k} \times \mathbb{R}^{d-k})$.

Remark 2.1.6. By Corollary 5.1 in [19], if H_k has a ground state, it is non-degenerate and we can choose the corresponding wave function to be positive almost everywhere.

The remainder of this paper is structured as follows. Section 2.2 contains the proof of Theorem 2.1.3. In Section 2.3, we prove Theorem 2.1.4. The Appendix contains an explicit example for $d = 1$ in 2.A.1, the proof of Lemma 2.2.3 in 2.A.2, as well as technical details of the proofs in 2.A.3. The exponential decay of Schrödinger eigenfunctions needed in the proof is discussed in Appendix 2.B by Rupert L. Frank.

2.2 Proof of Theorem 2.1.3

We shall prove the following two statements.

Proposition 2.2.1. *Let $k \in \{1, ..., d\}$. If H_{k-1} has a ground state with energy $E^{k-1} \leq ... \leq E^0$ the essential spectrum of H_k is $[E^{k-1}, \infty)$.*

Proposition 2.2.2. *Let $k \in \{1, ..., d\}$. If H_{k-1} has a ground state ψ_{k-1} with energy E^{k-1} the spectrum of H_k satisfies*

$$E^k = \inf \sigma(H_k) \leq E^{k-1} - \frac{J^2 M}{8\mu^2} \left(1 + 2 \max\left\{\frac{m_a}{m_b}, \frac{m_b}{m_a}\right\}\right)^{-1} < E^{k-1}, \tag{2.9}$$

where $J = \int_{Q_{k-1} \times \mathbb{R}^{d-k+1}} \delta(y_k) |\psi_{k-1}|^2 \mathrm{d}z \mathrm{d}y > 0$ with δ the Dirac delta-function.

The assumption $E^{k-1} \leq ... \leq E^0$ in the first Proposition holds as a consequence of the second Proposition. These two propositions combined yield Theorem 2.1.3.

Proof of Theorem 2.1.3. We proceed by induction. The claim is that H_k has a ground state, and that the ground state energies form a strictly decreasing sequence $E^d < ... < E^0$. For $k = 0$ the former is true by Assumption 2.1.12. For the induction step we apply Propositions 2.2.1 and 2.2.2. Assuming that the claim is true for $k - 1$, Proposition 2.2.2 implies that H_k has spectrum below E^{k-1}. By Proposition 2.2.1 this part of the spectrum must consist of eigenvalues. Since H_k is bounded from below by Proposition 2.A.3, it must have a ground state. The ground state energy E^k is strictly smaller than E^{k-1} by Proposition 2.2.2. □

2.2.1 Proof of Proposition 2.2.1

In order to compute the essential spectrum of H_k, we follow the proof of Proposition 2.1 in [18]. For the inclusion $[E^{k-1}, \infty) \subset \sigma_{\mathsf{ess}}(H_k)$ we use Weyl's criterion (see Section 6.4 in [69]). For the opposite inclusion, we bound the essential spectrum of H_k from below by introducing additional Neumann boundaries. They split the particle domain into several regions. One of them is bounded, so it does not contribute to the essential spectrum. In another, the interaction potential is larger than E^{k-1}, and hence there is no essential spectrum below E^{k-1}. In the remaining regions, the Hamiltonian can be bounded from below by approximately $H_{k-1} \otimes \mathbb{I}$. For this operator the essential spectrum starts at E^{k-1}.

Proof of Proposition 2.2.1. For the inclusion $[E^{k-1}, \infty) \subset \sigma_{\mathsf{ess}}(H_k)$ we construct a Weyl sequence. Remark 2.1.6 allows us to choose the ground state wave function ψ_{k-1} of H_{k-1} to be normalized and positive almost everywhere. Let $l \in [0, \infty)$ and let $\tau : \mathbb{R} \to \mathbb{R}$ be a smooth function satisfying $0 \leq \tau \leq 1$ with $\tau(x) = 0$ for $x \leq 1$ and $\tau(x) = 1$ for $x \geq 2$.

Let us write $\delta = M/\max\{m_a, m_b\}$. For integers $n \geq 5$, choose $\varphi_n(z_1, ..., z_k, y_1, ..., y_d) = f_n(z_1, ..., z_{k-1}, y_1, ..., y_d)g_n(z_k)$ for $(z, y) \in Q_k \times \mathbb{R}^{d-k}$ with

$$f_n(z_1, ..., z_{k-1}, y_1, ..., y_d) = \psi_{k-1}(z_1, ..., z_{k-1}, y_1, ..., y_d)\tau(n - |y_k|/\delta) \tag{2.10}$$

and

$$g_n(z_k) = \cos(lz_k)\tau(z_k - n)\tau(2n - z_k). \tag{2.11}$$

Using the properties of τ, we observe that $g_n(z_k) = \cos(lz_k)$ for $z_k \in [n+2, 2n-2]$. Moreover, for $|y_k| < \delta(n-2)$ we have $f_n = \psi_{k-1}$. Note that for $(z, y) \in Q_k \times \mathbb{R}^{d-k}$ with $z_k \geq n+2$, the variable y_k can take all values satisfying $|y_k| \leq \delta(n+2)$. Therefore,

$$\|\varphi_n\|^2_{L^2(Q_k\times\mathbb{R}^{d-k})} \geq \left(\int_{Q_{k-1}\times[\delta(-n+2),\delta(n-2)]\times\mathbb{R}^{d-k}} \psi_{k-1}^2\right)\left(\int_{n+2}^{2n-2} \cos^2(lz_k)\mathrm{d}z_k\right). \tag{2.12}$$

Since ψ_{k-1} is normalized, the first integral converges to 1 as $n \to \infty$. The second integral is greater than some constant times n. Thus, $\|\varphi_n\|^2_{L^2(Q_k\times\mathbb{R}^{d-k})} \geq C_1 n$ for some constant $C_1 > 0$.

Using the eigenvalue equation for ψ_{k-1}, we have

$$\left(H_k - E^{k-1} - \frac{l^2}{2M}\right)\varphi_n = f_n\Psi_n + \Phi_n g_n \tag{2.13}$$

with

$$\Psi_n(z_k) = \frac{1}{M} l\sin(lz_k)\left[\tau'(z_k - n)\tau(2n - z_k) - \tau(z_k - n)\tau'(2n - z_k)\right]$$
$$-\frac{1}{2M}\cos(lz_k)\left[\tau''(z_k - n)\tau(2n - z_k) - 2\tau'(z_k - n)\tau'(2n - z_k) + \tau(z_k - n)\tau''(2n - z_k)\right] \tag{2.14}$$

and

$$\Phi_n(z_1, ..., z_{k-1}, y_1, ..., y_d) = \frac{1}{\delta\mu}\partial_{y_k}\psi_{k-1}\mathrm{sgn}(y_k)\tau'(n - |y_k|/\delta) - \frac{1}{2\delta^2\mu}\psi_{k-1}\tau''(n - |y_k|/\delta). \tag{2.15}$$

By choice of the function τ, we have $\mathrm{supp}\ \Psi_n \subset [n+1, n+2] \cup [2n-2, 2n-1]$ and $\mathrm{supp}\ \Phi_n \subset Q_{k-1}\times[\delta(-n+1), \delta(-n+2)]\cup[\delta(n-2), \delta(n-1)]\times\mathbb{R}^{d-k}$. Since both τ' and τ'' are bounded, there is a constant $C_2 > 0$ independent of n such that $|\Phi_n| \leq C_2\left(|\partial_{y_k}\psi_{k-1}| + |\psi_{k-1}|\right)$ and $\|\Psi_n\|_\infty \leq C_2$. We the aid of the Schwarz inequality, we therefore have

$$\begin{aligned}
&\left\|\left(H_k - E^{k-1} - \frac{l^2}{2M}\right)\varphi_n\right\|^2_{L^2(Q_k\times\mathbb{R}^{d-k})}\\
&\leq 2\int_{Q_{k-1}\times\mathbb{R}^{d-k+1}} f_n^2 \int_{[n+1,n+2]\cup[2n-2,2n-1]} \Psi_n^2 + 2\int_{Q_{k-1}\times\mathbb{R}^{d-k+1}} \Phi_n^2 \int_{n+1}^{2n-1} g_n^2\\
&\leq 4C_2^2\left(1 + (n-2)\int_{Q_{k-1}\times[\delta(-n+1),\delta(-n+2)]\cup[\delta(n-2),\delta(n-1)]\times\mathbb{R}^{d-k}} \left((\partial_{y_k}\psi_{k-1})^2 + \psi_{k-1}^2\right)\right)
\end{aligned} \tag{2.16}$$

where we used $\|\psi_{k-1}\|_{L^2} = 1$ in the last step. Since $\psi_{k-1} \in H^1(Q_{k-1}\times\mathbb{R}^{d-k+1})$ we obtain

$$\begin{aligned}
&\lim_{n\to\infty} \frac{\|(H_k - E^{k-1} - \frac{l^2}{2M})\varphi_n\|^2_{L^2(Q_k\times\mathbb{R}^{d-k})}}{\|\varphi_n\|^2_{L^2(Q_k\times\mathbb{R}^{d-k})}}\\
&\leq \frac{4C_2^2}{C_1}\lim_{n\to\infty}\int_{Q_{k-1}\times[\delta(-n+1),\delta(-n+2)]\cup[\delta(n-2),\delta(n-1)]\times\mathbb{R}^{d-k}} \left((\partial_{y_k}\psi_{k-1})^2 + \psi_{k-1}^2\right) = 0.
\end{aligned} \tag{2.17}$$

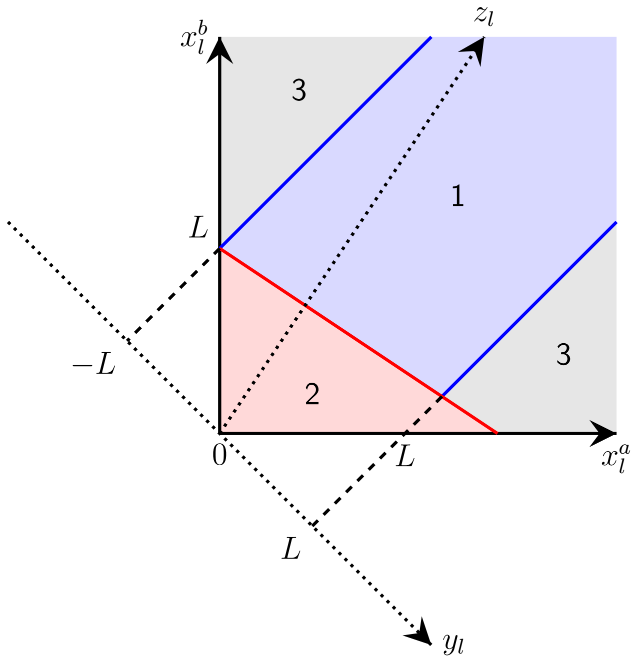

Figure 2.1: In the case $d = k = 1$, the areas labeled 1, 2, and 3 are precisely $\Omega_1, \Omega_2, \Omega_3$, respectively. In higher dimensions, region 1 (blue) is the domain of the lth component of z and y for $(z, y) \in \Omega_l$, $l \le k$. In particular, the domain of y_l is independent of z_l. The (red) triangular area 2 corresponds to the domain of z_j and y_j for $(z, y) \in \Omega_l$ and $j < l \le k+1$.

By Weyl's criterion, we obtain $E^{k-1} + \frac{l^2}{2M} \in \sigma(H_k)$ for all $l \ge 0$. Since the interval $[E^{k-1}, \infty)$ has no isolated points, it belongs to the essential spectrum of H_k.

For the opposite inclusion $\sigma_{\text{ess}}(H_k) \subset [E^{k-1}, \infty)$, we partition the domain $Q_k \times \mathbb{R}^{d-k}$ into $k+2$ subsets. By Assumption 2.1.13 there is a number L_0 such that for all $y \in \mathbb{R}^d$ with $|y| > L_0$ the potential satisfies $V(y) > E^0$. For $L > L_0$ and $1 \le l \le k$ let

$$\Omega_l := \left\{ (z, y) \in Q_k \times \mathbb{R}^{d-k} \;\middle|\; z_l > \frac{L}{\delta}, |y_l| < L, \forall 1 \le j < l : z_j < \frac{L}{\delta} \right\}, \tag{2.18}$$

$$\Omega_{k+1} := \left\{ (z, y) \in Q_k \times \mathbb{R}^{d-k} \;\middle|\; \forall 1 \le j \le k : z_j < \frac{L}{\delta}, \forall j > k : |y_j| < L \right\}, \tag{2.19}$$

$$\Omega_{k+2} := \Omega_0 \setminus \bigcup_{l=1}^{k+1} \overline{\Omega_l}. \tag{2.20}$$

These sets are sketched in Figure 2.1. The set Ω_{k+1} is bounded. For $(z, y) \in \Omega_{k+2}$, we always have $|y| > L$. Moreover, in Ω_l the range of y_l is independent of z_l.

For $1 \le l \le k+2$ we define the quadratic forms $a_l : H^1(\Omega_l) \to \mathbb{R}$ as

$$a_l[\psi] := \int_{\Omega_l} \left(\frac{1}{2M} |\nabla_z \psi|^2 + \frac{1}{2\mu} |\nabla_y \psi|^2 + V(y) |\psi|^2 \right) \mathrm{d}z \mathrm{d}y. \tag{2.21}$$

For $1 \le l \le k+1$, the potential term in a_l is infinitesimally bounded with respect to the kinetic energy term, as will be shown in Lemma 2.A.4. For a_{k+2} the potential is bounded from below. Thus, by the KLMN theorem there is a corresponding self-adjoint operator A_l for all $1 \le l \le k+2$. Let $A = \bigoplus_{l=1}^{k+2} A_l$. There is an isometry $\iota : H^1(\Omega_0) \to \bigoplus_l H^1(\Omega_l)$, $\varphi \mapsto \{\varphi|_{\Omega_l}\}$. Let $\{\varphi_n\}$ be a normalized Weyl sequence such that $\lim_{n\to\infty} \|(H_k - \inf \sigma_{\text{ess}}(H_k))\varphi_n\| = 0$.

Then $\{\iota(\varphi_n)\}$ is an orthonormal sequence with $\lim_{n\to\infty}\langle\iota(\varphi_n)|A\iota(\varphi_n)\rangle = \inf\sigma_{\text{ess}}(H_k)$. By the min-max principle,

$$\inf\sigma_{\text{ess}}(H_k) \geq \inf\sigma_{\text{ess}}(A) = \min_l \inf\sigma_{\text{ess}}(A_l). \tag{2.22}$$

We shall now analyze $\inf\sigma_{\text{ess}}(A_l)$ for all $1 \leq l \leq k+2$. Since Ω_{k+1} is a bounded Lipschitz domain, $H^1(\Omega_{k+1})$ is compactly embedded in $L^2(\Omega_{k+1})$ by the Rellich–Kondrachov theorem [2]. Therefore, A_{k+1} has compact resolvent and the spectrum of A_{k+1} is discrete. In Ω_{k+2}, always at least one of the y_j is larger than L. Therefore, $\inf\sigma(A_{k+2}) \geq \inf_{|y|>L} V(y) \geq E^0$.

Consider now A_l with $l \leq k$. In order to separate the variable z_l from the rest, let q be the quadratic form $q[\varphi] = \frac{1}{2M}\int_{L/\delta}^{\infty} |\varphi'(z_l)|^2\,\mathrm{d}z_l$ with domain $H^1((L/\delta,\infty))$. The remaining variables lie in

$$\Omega_{k-1}^{L,l} := \Big\{(z_1,...,\widehat{z_l},...,z_k,y_1,...,y_d) \in \mathbb{R}^{d+k-1} \,\Big|\, \forall 1 \leq j < l : 0 < z_j < \frac{L}{\delta}, \forall j > l : z_j > 0,$$
$$\forall 1 \leq j \neq l \leq k : -\frac{M}{m_b} z_j < y_j < \frac{M}{m_a} z_j, |y_l| < L\Big\} \tag{2.23}$$

where the hat means that the z_l variable is omitted. Note that for $L \to \infty$ the set $\Omega_{k-1}^{L,l}$ becomes $Q_{k-1} \times \mathbb{R}^{d-k+1}$ with l and k components swapped. Define the quadratic form

$$h_{k-1}^{L,l}[\psi] = \int_{\Omega_{k-1}^{L,l}} \left(\frac{1}{2M}\sum_{\substack{j=1\\ j\neq l}}^{k}\left|\frac{\partial\psi}{\partial z_j}\right|^2 + \frac{1}{2\mu}|\nabla_y\psi|^2 + V(y)|\psi|^2\right)\mathrm{d}z_1...\widehat{\mathrm{d}z_l}...\mathrm{d}z_k\mathrm{d}y \tag{2.24}$$

with domain $D[h_{k-1}^{L,l}] = H^1(\Omega_{k-1}^{L,l})$. In Lemma 2.A.4 we show that there is a self-adjoint operator $H_{k-1}^{L,l}$ corresponding to the quadratic form $h_{k-1}^{L,l}$. By Assumption 2.1.14, the quadratic form $h_{k-1}^{L,l}$ resembles h_{k-1} with l and k components swapped, up to the constraints imposed by the finite number L.

We can decompose

$$a_l = h_{k-1}^{L,l} \otimes \mathbb{I} + \mathbb{I} \otimes q. \tag{2.25}$$

It is well-known that the self-adjoint operator corresponding to q has purely essential spectrum $[0,\infty)$. Therefore, we obtain $\inf\sigma_{\text{ess}}(A_l) = \inf\sigma(H_{k-1}^{L,l})$. Using localization arguments, one can easily prove the following.

Lemma 2.2.3. *Let $1 \leq l \leq k \leq d$ and assume that $E^{k-1} \leq ... \leq E^0$. The self-adjoint operator $H_{k-1}^{L,l}$ defined through the quadratic form (2.24) satisfies $\liminf_{L\to\infty}\inf\sigma(H_{k-1}^{L,l}) \geq E^{k-1}$.*

The proof of Lemma 2.2.3 is rather straightforward and follows similar arguments as in the one-dimensional case in Proposition A.5 in [18]. For completeness, we carry it out in Appendix 2.A.2.

Collecting all estimates and applying (2.22), we see that

$$\inf\sigma_{\text{ess}}(H_k) \geq \min\{E^0, \inf\sigma(H_{k-1}^{L,l})\} \tag{2.26}$$

for all $L > L_0$. With Lemma 2.2.3 and since $E^0 \geq E^{k-1}$, it follows that $\sigma_{\text{ess}}(H_k) \subset [E^{k-1},\infty)$. □

2.2.2 Proof of Proposition 2.2.2

The goal is to find a trial function ψ such that $(\psi, H_k\psi) < E^{k-1}\|\psi\|_2^2$. Then $\inf \sigma(H_k) < E^{k-1}$ by the min-max principle.

We denote the ground state of H_{k-1} by ψ_{k-1} and choose it normalized and positive a.e. (see Remark 2.1.6). Since we expect the ground state of H_k to stick to the boundary, we pick the trial function

$$\psi(z_1, ..., z_k, y_1, ..., y_d) = \psi_{k-1}(z_1, ..., z_{k-1}, y_1, ..., y_d)e^{-\gamma z_k} \tag{2.27}$$

for $\gamma > 0$. We start with a preliminary computation.

Lemma 2.2.4. *Let* $f(y_k) = \chi_{(-\infty,0)}(y_k)e^{-2\gamma m_b|y_k|/M} + \chi_{(0,\infty)}(y_k)e^{-2\gamma m_a|y_k|/M}$, *where* χ *denotes the characteristic function. We have*

$$A := \frac{1}{2}(f\psi_{k-1}, \psi_{k-1}) = \gamma\|\psi\|_2^2. \tag{2.28}$$

Proof. Carrying out the integration over z_k, we have

$$\begin{aligned}
\|\psi\|_2^2 &= \int_{Q_{k-1}\times\mathbb{R}^{d-k+1}} \mathrm{d}z_1...\mathrm{d}z_{k-1}\mathrm{d}y \int_0^\infty \mathrm{d}z_k \ \chi_{\{-\frac{M}{m_b}z_k<y_k<\frac{M}{m_a}z_k\}}\psi_{k-1}^2(z_1, ..., z_{k-1}, y_1, ..., y_d)e^{-2\gamma z_k} \\
&= \frac{1}{2\gamma}\int_{Q_{k-1}\times\mathbb{R}^{d-k+1}} \mathrm{d}z_1...\mathrm{d}z_{k-1}\mathrm{d}y \ \psi_{k-1}^2(z_1, ..., z_{k-1}, y_1, ..., y_d)f(y_k) \\
&= \frac{1}{2\gamma}(f\psi_{k-1}, \psi_{k-1}) = \frac{1}{\gamma}A.
\end{aligned} \tag{2.29}$$

□

Proof of Proposition 2.2.2. We have

$$\begin{aligned}
h_k[\psi] = \int_{Q_k\times\mathbb{R}^{d-k}} \mathrm{d}z_1...\mathrm{d}z_k\mathrm{d}y_1...\mathrm{d}y_d \bigg(&\frac{1}{2M}|\nabla_z\psi_{k-1}|^2 + \frac{1}{2\mu}|\nabla_y\psi_{k-1}|^2 \\
&+\frac{\gamma^2}{2M}\psi_{k-1}^2 + V(y)\psi_{k-1}^2\bigg) e^{-2\gamma z_k}.
\end{aligned} \tag{2.30}$$

We rewrite this as

$$\begin{aligned}
h_k[\psi] &= \frac{\gamma^2\|\psi\|_2^2}{2M} + \int_{Q_{k-1}\times\mathbb{R}^{d-k+1}} \mathrm{d}z_1...\mathrm{d}z_{k-1}\mathrm{d}y_1...\mathrm{d}y_d \\
&\int_0^\infty \mathrm{d}z_k\chi_{\{-\frac{M}{m_b}z_k<y_k<\frac{M}{m_a}z_k\}}\left(\frac{1}{2M}|\nabla_z\psi_{k-1}|^2 + \frac{1}{2\mu}|\nabla_y\psi_{k-1}|^2 + V(y)\psi_{k-1}^2\right) e^{-2\gamma z_k}.
\end{aligned} \tag{2.31}$$

Integrating over z_k as in the proof of Lemma 2.2.4, we obtain

$$\begin{aligned}
h_k[\psi] = \frac{\gamma^2\|\psi\|_2^2}{2M} + \frac{1}{2\gamma}\int_{Q_{k-1}\times\mathbb{R}^{d-k+1}} \mathrm{d}z_1...\mathrm{d}z_{k-1}\mathrm{d}y_1...\mathrm{d}y_d \bigg(&\frac{1}{2M}|\nabla_z\psi_{k-1}|^2 \\
&+\frac{1}{2\mu}|\nabla_y\psi_{k-1}|^2 + V(y)\psi_{k-1}^2\bigg) f(y_k).
\end{aligned} \tag{2.32}$$

We pull the function f into the gradients and write

$$h_k[\psi] = \frac{\gamma^2\|\psi\|_2^2}{2M} + \frac{1}{2\gamma}\int_{Q_{k-1}\times\mathbb{R}^{d-k+1}} \left(\frac{1}{2M}\nabla_z(f\psi_{k-1})\nabla_z\psi_{k-1} + \frac{1}{2\mu}\nabla_y(f\psi_{k-1})\nabla_y\psi_{k-1}\right.$$
$$\left. + \frac{\gamma}{\mu M}\left(-m_b\chi_{(-\infty,0)}e^{-2\gamma\frac{m_b}{M}|y_k|} + m_a\chi_{(0,\infty)}e^{-2\gamma\frac{m_a}{M}|y_k|}\right)\psi_{k-1}\partial_{y_k}\psi_{k-1} + V(y)f\psi_{k-1}^2\right). \quad (2.33)$$

Let us write $h_k[\cdot,\cdot]$ for the sesquilinear form associated to the quadratic form h_k. The previous equation reads

$$h_k[\psi] = \frac{\gamma^2\|\psi\|_2^2}{2M} + \frac{1}{2\gamma}h_{k-1}[f\psi_{k-1},\psi_{k-1}] + B, \quad (2.34)$$

where

$$B = \frac{1}{2\mu M}\int_{Q_{k-1}\times\mathbb{R}^{d-k+1}} \left(-m_b\chi_{(-\infty,0)}e^{-2\gamma\frac{m_b}{M}|y_k|} + m_a\chi_{(0,\infty)}e^{-2\gamma\frac{m_a}{M}|y_k|}\right)\psi_{k-1}\partial_{y_k}\psi_{k-1}. \quad (2.35)$$

Since ψ_{k-1} is the minimizer of the functional $\frac{h_{k-1}[\phi]}{\|\phi\|_2^2}$, for all functions $g \in H^1(Q_{k-1}\times\mathbb{R}^{d-k+1})$ it holds that $h_{k-1}[g,\psi_{k-1}] = E^{k-1}(g,\psi_{k-1})$. With $g = f\psi_{k-1}$ and Lemma 2.2.4, we obtain

$$h_k[\psi] = \left(\frac{\gamma^2}{2M} + E^{k-1}\right)\|\psi\|_2^2 + B. \quad (2.36)$$

We now simplify the integral in B. By the Sobolev embedding theorem (Theorem 4.12 in [2]), the restriction of an H^1-function to a hyperplane is an L^2-function. Therefore, one can restrict the function ψ_{k-1} to $y_k = 0$ and obtain a finite number $J := \int_{Q_{k-1}\times\mathbb{R}^{d-k}} \left(\psi_{k-1}|_{y_k=0}\right)^2$. Integration by parts with respect to y_k gives

$$\begin{aligned}
2\mu M B = &- m_b\int_{Q_{k-1}\times(-\infty,0)\times\mathbb{R}^{d-k}} e^{-2\gamma\frac{m_b}{M}|y_k|}\psi_{k-1}\partial_{y_k}\psi_{k-1} \\
&+ m_a\int_{Q_{k-1}\times(0,\infty)\times\mathbb{R}^{d-k}} e^{-2\gamma\frac{m_a}{M}|y_k|}\psi_{k-1}\partial_{y_k}\psi_{k-1} \\
=& -\frac{m_b}{2}\int_{Q_{k-1}\times\mathbb{R}^{d-k}}\left(\psi_{k-1}|_{y_k=0}\right)^2 + \gamma\frac{m_b^2}{M}\int_{Q_{k-1}\times(-\infty,0)\times\mathbb{R}^{d-k}} e^{-2\gamma\frac{m_b}{M}|y_k|}\psi_{k-1}^2 \\
&- \frac{m_a}{2}\int_{Q_{k-1}\times\mathbb{R}^{d-k}}\left(\psi_{k-1}|_{y_k=0}\right)^2 + \gamma\frac{m_a^2}{M}\int_{Q_{k-1}\times(0,\infty)\times\mathbb{R}^{d-k}} e^{-2\gamma\frac{m_a}{M}|y_k|}\psi_{k-1}^2 \\
=& -\frac{M}{2}J \\
&+ \frac{\gamma}{M}\int_{Q_{k-1}\times\mathbb{R}^{d-k+1}}\left(m_b^2\chi_{(-\infty,0)}(y_k)e^{-2\gamma\frac{m_b}{M}|y_k|} + m_a^2\chi_{(0,\infty)}(y_k)e^{-2\gamma\frac{m_a}{M}|y_k|}\right)\psi_{k-1}^2.
\end{aligned} \quad (2.37)$$

The last integral is bounded from above by $2\max\{m_a^2, m_b^2\}A$. With (2.36), Lemma 2.2.4 and the min-max principle we obtain

$$\inf\sigma(H_k) \le \frac{h_k[\psi]}{\|\psi\|_2^2} \le E^{k-1} + \frac{\gamma}{A}\left(\left(\frac{1}{2} + \max\left\{\frac{m_a}{m_b},\frac{m_b}{m_a}\right\}\right)\frac{\gamma A}{M} - \frac{J}{4\mu}\right). \quad (2.38)$$

This holds for all $\gamma > 0$. Minimizing with respect to γ yields

$$\inf\sigma(H_k) \le E^{k-1} - \frac{J^2 M}{32\mu^2 A^2}\left(1 + 2\max\left\{\frac{m_a}{m_b},\frac{m_b}{m_a}\right\}\right)^{-1}. \quad (2.39)$$

Moreover, since ψ_{k-1} is normalized we have

$$A = \frac{1}{2}\int_{Q_{k-1}\times\mathbb{R}^{d-k+1}} f\psi_{k-1}^2 \leq \frac{1}{2}\int_{Q_{k-1}\times\mathbb{R}^{d-k+1}} \psi_{k-1}^2 = \frac{1}{2}. \tag{2.40}$$

This yields (2.9).

We are left with showing that $J > 0$. Suppose that $J = 0$. Define a new function $\widetilde{\psi}_{k-1} = \psi_{k-1}\left(\chi_{y_k<0} - \chi_{y_k>0}\right)$. Since $J = 0$, the function $\widetilde{\psi}_{k-1} \in H^1(Q_{k-1}\times\mathbb{R}^{d-k+1})$. Moreover, $\widetilde{\psi}_{k-1}$ is a ground state of H_{k-1} because $\frac{h_{k-1}[\widetilde{\psi}_{k-1}]}{\|\widetilde{\psi}_{k-1}\|_2^2} = \frac{h_{k-1}[\psi_{k-1}]}{\|\psi_{k-1}\|_2^2}$. Since ψ_{k-1} and $\widetilde{\psi}_{k-1}$ are linearly independent, this contradicts the uniqueness of the ground state (Remark 2.1.6). Hence, $J > 0$ and $\inf\sigma(H_k) < E^{k-1}$. □

2.3 Finiteness of the Discrete Spectrum

In this section we shall give the proof of Theorem 2.1.4. An important ingredient will be the exponential decay of the ground state wave function ψ_k of H_k. In fact, the Agmon estimate (Corollary 4.2. in [3]) implies that for any $a < \sqrt{\inf\sigma_{\text{ess}}(H_k) - E^k}$ we have

$$\int_{Q_k\times\mathbb{R}^{d-k}} |\psi_k|^2 e^{2a\sqrt{2M|z|^2+2\mu|y|^2}}\mathrm{d}z\mathrm{d}y < \infty. \tag{2.41}$$

Strictly speaking, the assumptions on the interaction potential stated in [3] are slightly stronger than ours. However, the Agmon estimate only requires V to be form-bounded with respect to the kinetic energy with form bound less than 1, as shown in Theorem 2.B.1 in Appendix 2.B by Rupert Frank. As we argue in Proposition 2.A.3, this is the case given Assumptions 2.1.1.

In order to derive (2.41) from Theorem 2.B.1, we remove the boundaries in the particle domain via mirroring and consider the operator $\widetilde{H}_k$ acting on $H^1(\mathbb{R}^{d+k})$ (see Proposition 2.A.1). It suffices to prove the exponential decay for the ground state $\widetilde{\psi}_k$ of $\widetilde{H}_k$. We rescale the variables to remove the masses in front of the Laplacians using the unitary transform $U\varphi(z,y) = \sqrt{2M}^k\sqrt{2\mu}^d\varphi(\sqrt{2M}z,\sqrt{2\mu}y)$ on $H^1(\mathbb{R}^{d+k})$. Switching to relative and center of mass coordinates and writing $\widetilde{V}(z,y) = V((|x_j^a| - |x_j^b|)_{j=1}^k, \tilde{y})$ and $\widetilde{V}_U(z,y) = \widetilde{V}(z/\sqrt{2M}, y/\sqrt{2\mu})$ we have

$$\widetilde{H}_k = -\frac{1}{2M}\Delta_z - \frac{1}{2\mu}\Delta_y + \widetilde{V} = U\left(-\Delta_z - \Delta_y + \widetilde{V}_U\right)U^\dagger. \tag{2.42}$$

The ground state φ_k of $-\Delta_z - \Delta_y + \widetilde{V}_U$ satisfies $\widetilde{\psi}_k = U\varphi_k$. For any $a < \sqrt{\inf\sigma_{\text{ess}}(H_k) - E^k} = \sqrt{\inf\sigma_{\text{ess}}(\widetilde{H}_k) - E^k}$ we thus have

$$\int_{\mathbb{R}^{d+k}} |\widetilde{\psi}_k|^2 e^{2a\sqrt{2M|z|^2+2\mu|y|^2}}\mathrm{d}z\mathrm{d}y = \int_{\mathbb{R}^{d+k}} |\varphi_k|^2 e^{2a\sqrt{|z|^2+|y|^2}}\mathrm{d}z\mathrm{d}y < \infty \tag{2.43}$$

by Theorem 2.B.1. Hence (2.41) holds.

Definition 2.3.1. Let $n \in \mathbb{Z}^{\geq 0}$ and A be a self-adjoint operator with corresponding quadratic form a. We define

$$E_n(A) := \inf_{\substack{V\subset D[a]\\ \dim V = n+1}} \sup_{\substack{\varphi\in V\\ \varphi\neq 0}} \frac{a[\varphi]}{\|\varphi\|^2}. \tag{2.44}$$

By the min-max principle, if n is larger than the number of eigenvalues below the essential spectrum, we have $E_n(A) = \inf\sigma_{\text{ess}}(A)$. Otherwise, E_{n-1} is the n-th eigenvalue of A below the essential spectrum counted with multiplicities.

Definition 2.3.2. For a self-adjoint operator A and a number $\lambda \in \mathbb{R}$, let $N(A, \lambda)$ denote the number of eigenvalues in $(-\infty, \lambda)$ if $\sigma_{\text{ess}}(A) \cap (-\infty, \lambda) = \emptyset$. Otherwise, set $N(A, \lambda) = \infty$. When $N(A, \lambda) \neq 0$, one can write

$$N(A, \lambda) = \sup \left\{ n \in \mathbb{Z}^{\geq 1} | E_{n-1}(A) < \lambda \right\}. \tag{2.45}$$

In the case $k = d = 1$, Theorem 2.1.4 was already shown in [18]. We generalize the proof using similar ideas. The overall strategy is to construct localized operators A and bound $N(H_k, E^{k-1})$ using $N(A, E^{k-1})$. The localized operators fall into three categories. First, they can have compact resolvent or second, the corresponding potential is larger than E^{k-1}. In these cases, the number of eigenvalues below E^{k-1} is certainly finite (or even zero). In the third category, the operator is of the form $\mathbb{I} \otimes H_{k-1} - \frac{1}{2M} \Delta_{z_j} \otimes \mathbb{I} - K$, where K is a well behaved error term. One estimates this operator by projecting onto $L^2(\mathbb{R}) \otimes \psi_{k-1}$ and its orthogonal complement. This reduces the problem to a one-dimensional operator. Then, (2.41) and the Bargmann estimate [8] imply that the number of eigenvalues is finite.

Proof of Theorem 2.1.4. Let $\chi_1, \chi_2 : \mathbb{R} \to [0, 1]$ and $\chi_3 : \mathbb{R}^2 \to [0, 1]$ be continuously differentiable functions satisfying $\chi_1(t) = 0$ for $t \geq 2$, $\chi_1(t) = 1$ for $t \leq 1$, $\chi_1(t)^2 + \chi_2(t)^2 = 1$ for all t and $\chi_3(s, t)^2 + \chi_2(s)^2 \chi_2(t)^2 = 1$ for all t and s. Note that for $j = 1, 2, 3$ we have $\|(\nabla \chi_j)^2\|_\infty < \infty$.

Let $\Omega_0 = (0, \infty)^{2k} \times \mathbb{R}^{d-k}$. The boundary of the particle domain consists of k orthogonal $d-1$-dimensional hyperplanes. We start by localizing into two separate regions, distinguishing whether there is a particle close to all the hyperplanes, or whether both particles are far from some hyperplane. For $R > 0$, let

$$\begin{aligned}
\Omega_1 &= \left\{ (x^a, x^b, \tilde{y}) \in \Omega_0 | x^a \in (0, 2R)^k \text{ or } x^b \in (0, 2R)^k \right\} \\
&= \left\{ (x^a, x^b, \tilde{y}) \in \Omega_0 | \max\{x_1^a, ..., x_k^a\} < 2R \text{ or } \max\{x_1^b, ..., x_k^b\} < 2R \right\}, \qquad (2.46) \\
\Omega_2 &= \left\{ (x^a, x^b, \tilde{y}) \in \Omega_0 | x^a \notin [0, R]^k \text{ and } x^b \notin [0, R]^k \right\} \\
&= \left\{ (x^a, x^b, \tilde{y}) \in \Omega_0 | \max\{x_1^a, ..., x_k^a\} > R \text{ and } \max\{x_1^b, ..., x_k^b\} > R \right\}. \qquad (2.47)
\end{aligned}$$

We define the functions

$$\begin{aligned}
f_1^R(x^a, x^b) &= \chi_3 \left(\frac{\max\{x_1^a, ..., x_k^a\}}{R}, \frac{\max\{x_1^b, ..., x_k^b\}}{R} \right), \qquad (2.48) \\
f_2^R(x^a, x^b) &= \chi_2 \left(\frac{\max\{x_1^a, ..., x_k^a\}}{R} \right) \chi_2 \left(\frac{\max\{x_1^b, ..., x_k^b\}}{R} \right). \qquad (2.49)
\end{aligned}$$

Note that for all functions $\varphi \in L^2(\Omega_0)$ we have support $\operatorname{supp} f_j^R \varphi \subset \Omega_j$. By the IMS localization formula we have for all $\varphi \in H^1(\Omega_0)$ that

$$h_k[f_1^R \varphi] + h_k[f_2^R \varphi] = h_k[\varphi] + \int_{(0,\infty)^{2k} \times \mathbb{R}^{d-k}} W_R |\varphi|^2 \, \mathrm{d}x^a \mathrm{d}x^b \mathrm{d}\tilde{y}, \tag{2.50}$$

where

$$W_R(x^a, x^b, \tilde{y}) = \frac{1}{R^2}\left[\frac{1}{2m_a}(\nabla_{x^a}\chi_3)\left(\frac{\max\{x_1^a, ..., x_k^a\}}{R}, \frac{\max\{x_1^b, ..., x_k^b\}}{R}\right)^2 + \frac{1}{2m_b}(\nabla_{x^b}\chi_3)\left(\frac{\max\{x_1^a, ..., x_k^a\}}{R}, \frac{\max\{x_1^b, ..., x_k^b\}}{R}\right)^2 + \frac{1}{2m_a}\chi_2'\left(\frac{\max\{x_1^a, ..., x_k^a\}}{R}\right)^2 \chi_2\left(\frac{\max\{x_1^b, ..., x_k^b\}}{R}\right)^2 + \frac{1}{2m_b}\chi_2\left(\frac{\max\{x_1^a, ..., x_k^a\}}{R}\right)^2 \chi_2'\left(\frac{\max\{x_1^b, ..., x_k^b\}}{R}\right)^2\right]. \quad (2.51)$$

Note that there is a constant $c_1 > 0$ such that $\|W_R\|_\infty \leq \frac{c_1}{R^2}$. For $j = 1, 2$, define the quadratic forms

$$a_j[\varphi] = \int_{\Omega_j}\left(\frac{1}{2m_a}|\nabla_{x^a}\varphi|^2 + \frac{1}{2m_b}|\nabla_{x^b}\varphi|^2 + \frac{1}{2\mu}|\nabla_{\tilde{y}}\varphi|^2 + \left(V(x^a - x^b, \tilde{y}) - W_R(x^a, x^b, \tilde{y})\right)|\varphi|^2\right)\mathrm{d}x^a\mathrm{d}x^b\mathrm{d}\tilde{y} \quad (2.52)$$

with domains

$$D[a_1] = \left\{\varphi \in H^1(\Omega_0) | \varphi(x^a, x^b, \tilde{y}) = 0 \text{ if } \max\{x_1^a, ..., x_k^a\} \geq 2R \text{ and } \max\{x_1^b, ..., x_k^b\} \geq 2R\right\}, \quad (2.53)$$

$$D[a_2] = \left\{\varphi \in H^1(\Omega_0) | \varphi(x^a, x^b, \tilde{y}) = 0 \text{ if } \max\{x_1^a, ..., x_k^a\} \leq R \text{ or } \max\{x_1^b, ..., x_k^b\} \leq R\right\}. \quad (2.54)$$

For all quadratic forms a_j in this proof, let A_j denote the corresponding self-adjoint operator. In Lemma 2.A.5, we verify that these operators exist. For $\varphi \in D[h_k]$, the restriction of the function $f_j^R\varphi$ to Ω_j belongs to $D[a_j]$. With $(f_1^R)^2 + (f_2^R)^2 = 1$, it follows that $h_k[\varphi] = a_1[f_1^R\varphi] + a_2[f_2^R\varphi]$. Let $\hat{A}$ denote the operator $\hat{A} = A_1 \oplus A_2$. The map $J : H^1(\Omega_0) \to H^1(\Omega_0) \oplus H^1(\Omega_0), \varphi \mapsto (f_1^R\varphi, f_2^R\varphi)$ is an L^2-isometry and thus injective. By the min-max principle, we have

$$E_n(H_k) = \inf_{\substack{V \subset D[h_k] \\ \dim V = n+1}} \sup_{\substack{\varphi \in V \\ \varphi \neq 0}} \frac{h_k[\varphi]}{\|\varphi\|^2_{L^2(\Omega_0)}} = \inf_{\substack{V \subset D[h_k] \\ \dim V = n+1}} \sup_{\substack{\varphi \in V \\ \varphi \neq 0}} \frac{\hat{a}[J\varphi]}{\|J\varphi\|^2_{L^2(\Omega_0) \oplus L^2(\Omega_0)}} = \inf_{\substack{V \subset JD[h_k] \\ \dim V = n+1}} \sup_{\substack{\varphi \in V \\ \varphi \neq 0}} \frac{\hat{a}[\varphi]}{\|\varphi\|^2_{L^2(\Omega_0) \oplus L^2(\Omega_0)}} \geq \inf_{\substack{V \subset D[\hat{a}] \\ \dim V = n+1}} \sup_{\substack{\varphi \in V \\ \varphi \neq 0}} \frac{\hat{a}[\varphi]}{\|\varphi\|^2_{L^2(\Omega_0) \oplus L^2(\Omega_0)}} = E_n(\hat{A}) \quad (2.55)$$

for all $n \in \mathbb{Z}^{\geq 0}$. Thus, $N(H_k, E^{k-1}) \leq N(\hat{A}, E^{k-1}) = N(A_1, E^{k-1}) + N(A_2, E^{k-1})$.

Let

$$\tilde{\Omega}_{1,\text{int}} = \left\{(x^a, x^b, \tilde{y}) \in \Omega_0 | (x^a - x^b, \tilde{y}) \in (-R, R)^d\right\} \quad \text{and} \quad (2.56)$$

$$\tilde{\Omega}_{1,\text{ext}} = \left\{(x^a, x^b, \tilde{y}) \in \Omega_0 | (x^a - x^b, \tilde{y}) \notin [-R, R]^d\right\}. \quad (2.57)$$

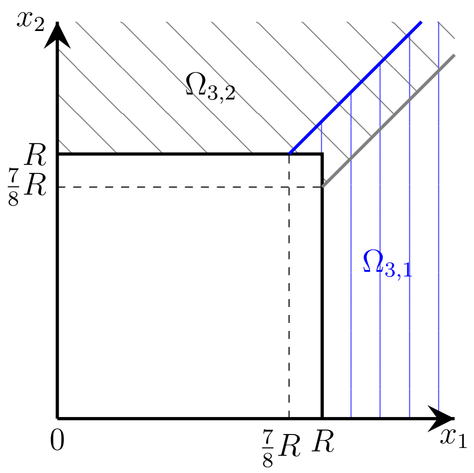

Figure 2.2: Let $k = 2$. In Ω_2 both x^a and x^b lie outside the square $(0, R)^2$. If x^a lies below the upper diagonal, the configuration belongs to $\Omega_{3,1}$. If x^a lies above the lower diagonal, the configuration belongs to $\Omega_{3,2}$.

Moreover, let $\Omega_{1,\bullet} = \tilde{\Omega}_{1,\bullet} \cap \Omega_1$ for $\bullet \in \{\text{int}, \text{ext}\}$. Define quadratic forms $a_{1,\text{int}}, a_{1,\text{ext}}$ through expression (2.52) with domain

$$D[a_{1,\bullet}] = \left\{\varphi \in H^1(\tilde{\Omega}_{1,\bullet}) | \varphi(x^a, x^b, \tilde{y}) = 0 \text{ if } \max\{x^a_1, ..., x^a_k\} \geq 2R \text{ and } \max\{x^b_1, ..., x^b_k\} \geq 2R\right\}, \tag{2.58}$$

for $\bullet \in \{\text{int}, \text{ext}\}$. Again, there is an isometry

$$D[a_1] \to D[a_{1,\text{int}}] \oplus D[a_{1,\text{ext}}], \varphi \mapsto (\varphi|_{\tilde{\Omega}_{1,\text{int}}}, \varphi|_{\tilde{\Omega}_{1,\text{ext}}}), \tag{2.59}$$

and therefore, $N(A_1, E^{k-1}) \leq N(A_{1,\text{int}}, E^{k-1}) + N(A_{1,\text{ext}}, E^{k-1})$. Since the negative part of V vanishes at infinity by Assumption 2.1.13 and since $\|W_R\|_\infty \leq \frac{c_1}{R^2}$, there is a $R_0 > 0$ such that for $R \geq R_0$ and $|(x^a - x^b, \tilde{y})| \geq R_0$ we have $V(x^a - x^b, \tilde{y}) - W_R(x^a, x^b, \tilde{y}) > E^{k-1}$. Choosing $R \geq R_0$, we have $N(A_{1,\text{ext}}, E^{k-1}) = 0$. Since $\Omega_{1,\text{int}}$ is a bounded Lipschitz domain, $A_{1,\text{int}}$ has purely discrete spectrum. As $A_{1,\text{int}}$ is bounded from below, we have $N(A_{1,\text{int}}, E^{k-1}) < \infty$.

We are left with showing that $N(A_2, E^{k-1}) < \infty$. For $k = 1$, wave functions in the support of A_2 are localized away from the boundary. Effectively, the boundary has thus disappeared and one can directly make a comparison with $H_{k-1} = H_0$. For $k > 1$, the domain Ω_2 is more complicated and we need to continue localizing in order to effectively eliminate one of the boundary planes. For now, assume $k > 1$ and let $r = R/8$. We localize x^a in the k sectors

$$\Omega_{3,j} = \{(x^a, x^b, \tilde{y}) \in \Omega_2 | x^a_j > \max\{x^a_1, ..., x^a_k\} - r\} \quad \text{for } 1 \leq j \leq k. \tag{2.60}$$

In the sector $\Omega_{3,j}$, the largest component of x^a is x^a_j up to the constant r. The domains are sketched in Figure 2.2 for the case $k = 2$. For the localization, we need functions $f^r_{3,j}$ on Ω_2 which are supported in $\Omega_{3,j}$, satisfy $\sum_{j=1}^k (f^r_{3,j})^2 = 1$, and their derivatives scale as $1/r$. We construct auxiliary functions $f_{3,j}$ corresponding to the case $r = 1$ and set

$$f^r_{3,j}(x^a, x^b, \tilde{y}) = f_{3,j}(x^a/r). \tag{2.61}$$

The idea behind the construction of the auxiliary functions is as follows. We want that $f_{3,1}$ equals 1 on $\Omega_{3,1}$ apart from the boundary region which overlaps with other $\Omega_{3,j}$. The expression

$\max\{x_2^a, ..., x_k^a\} - x_1^a$ measures the distance to the boundary of $\Omega_{3,1}$ and is large outside $\Omega_{3,1}$. Hence, to define $f_{3,1}$, we apply χ_1 to this expression (up to some constants). For the sum condition to hold, the remaining $f_{3,j}$ will contain the corresponding factor χ_2. This χ_2 factor takes care of the behavior at the boundary towards large x_1^a. For the next function $f_{3,2}$, we proceed analogously to before, but ignoring the x_1^a direction. Inductively, for $x^a \in (0, \infty)^k$ and $1 \le j \le k-1$ we define

$$f_{3,j}(x^a) = \chi_1\left(\frac{k}{2}\left(\max\{x_{j+1}^a, ..., x_k^a\} - x_j^a\right) + \frac{3}{2}\right)\prod_{l=1}^{j-1}\chi_2\left(\frac{k}{2}\left(\max\{x_{l+1}^a, ..., x_k^a\} - x_l^a\right) + \frac{3}{2}\right),$$
$$f_{3,k}(x^a) = \prod_{l=1}^{k-1}\chi_2\left(\frac{k}{2}\left(\max\{x_{l+1}^a, ..., x_k^a\} - x_l^a\right) + \frac{3}{2}\right), \tag{2.62}$$

where the product in the first line has to be understood as 1 for $j = 1$. Note that for all $1 \le j \le k$ the derivatives are bounded, i.e. $\|(\nabla f_{3,j})^2\|_\infty < \infty$. By construction, we have $\sum_{j=1}^k (f_{3,j})^2 = 1$. That the functions $f_{3,j}^r$ indeed have the correct support is the content of the following Lemma, which is proved at the end of this section.

Lemma 2.3.3. *For $1 \le j \le k$, the functions $f_{3,j}^r$ defined through (2.61) and (2.62) satisfy*

$$\operatorname{supp} f_{3,j}^r \cap \Omega_2 \subset \overline{\Omega_{3,j}}. \tag{2.63}$$

Moreover,

$$\operatorname{supp} \nabla f_{3,j}^r \cap \Omega_2 \subset \{(x^a, x^b, \tilde{y}) \in \Omega_2 | \max\{x_1^a, ..., \widehat{x_j^a}, ..., x_k^a\} - r \le x_j^a \le \max\{x_1^a, ..., \widehat{x_j^a}, ..., x_k^a\} + r\}, \tag{2.64}$$

where $\widehat{x_j^a}$ means that this variable is omitted.

By the IMS formula, we have for all $\varphi \in D[a_2]$

$$\sum_{j=1}^k a_2[f_{3,j}^r \varphi] = a_2[\varphi] + \int_{\Omega_2} F_r(x^a, x^b, \tilde{y})|\varphi|^2 \mathrm{d}x^a \mathrm{d}x^b \mathrm{d}\tilde{y}, \tag{2.65}$$

where

$$F_r(x^a, x^b, \tilde{y}) = \frac{1}{r^2}\sum_{j=1}^k \frac{1}{2m_a}(\nabla f_{3,j})^2 (x^a/r). \tag{2.66}$$

For $1 \le j \le k$, define the quadratic forms

$$a_{3,j}[\varphi] = \int_{\Omega_{3,j}}\left(\frac{1}{2m_a}|\nabla_{x^a}\varphi|^2 + \frac{1}{2m_b}|\nabla_{x^b}\varphi|^2 + \frac{1}{2\mu}|\nabla_{\tilde{y}}\varphi|^2 + \left(V(x^a - x^b, \tilde{y}) - W_R(x^a, x^b, \tilde{y}) - F_r(x^a, x^b, \tilde{y})\right)|\varphi|^2\right)\mathrm{d}x^a\mathrm{d}x^b\mathrm{d}\tilde{y} \tag{2.67}$$

with domains

$$D[a_{3,j}] = \Big\{\varphi \in H^1(\Omega_0) | \varphi(x^a, x^b, \tilde{y}) = 0 \quad \text{if} \quad \max\{x_1^a, ..., x_k^a\} \le R \,\text{or}\, \max\{x_1^b, ..., x_k^b\} \le R \\ \text{or}\; x_j^a \le \max\{x_1^a, ..., x_k^a\} - r\Big\}. \tag{2.68}$$

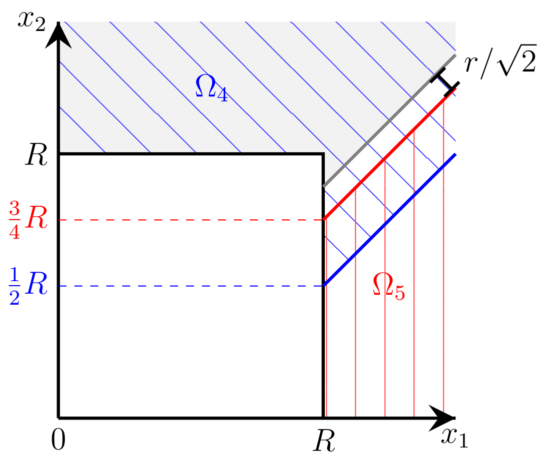

Figure 2.3: In $\Omega_{3,2}$, the first particle's coordinate x^a lies in the shaded area, while the second particle at x^b lies outside the square $(0,R)^2$. If x^b lies above the lowest diagonal (blue), the configuration belongs to Ω_4. If x^b lies below the middle diagonal (red), the configuration belongs to Ω_5. Note that for any configuration in Ω_5, the particles are separated by at least distance $r/\sqrt{2}$.

Again we have $N(A_2, E^{k-1}) \leq \sum_{j=1}^{k} N(A_{3,j}, E^{k-1})$. We will show that $N(A_{3,k}, E^{k-1}) < \infty$. For $1 \leq j < k$, by Assumption 2.1.14 the same argument with vector components $k \leftrightarrow j$ swapped gives $N(A_{3,j}, E^{k-1}) < \infty$.

We localize x^b close and far from the domain of x^a. Define the sets

$$\Omega_4 = \left\{(x^a, x^b, \tilde{y}) \in \Omega_{3,k} | x_k^b > \max\{x_1^b, ..., x_{k-1}^b\} - 4r\right\} \quad \text{and} \tag{2.69}$$

$$\Omega_5 = \left\{(x^a, x^b, \tilde{y}) \in \Omega_{3,k} | x_k^b < \max\{x_1^b, ..., x_{k-1}^b\} - 2r\right\}. \tag{2.70}$$

For $k = 2$, they are sketched in Figure 2.3. Let $f_4^r(x^b) = \chi_1\left(\frac{\max\{x_1^b,...,x_{k-1}^b\} - x_k^b}{2r}\right)$ and $f_5^r(x^b) = \chi_2\left(\frac{\max\{x_1^b,...,x_{k-1}^b\} - x_k^b}{2r}\right)$. By the IMS formula, we have for all $\varphi \in D[a_{3,k}]$

$$a_{3,k}[f_4^r\varphi] + a_{3,k}[f_5^r\varphi] = a_{3,k}[\varphi] + \int_{\Omega_{3,k}} G_r(x^a, x^b, \tilde{y})|\varphi|^2 \mathrm{d}x^a \mathrm{d}x^b \mathrm{d}\tilde{y}, \tag{2.71}$$

where

$$G_r(x^a, x^b, \tilde{y}) = \frac{1}{4r^2 m_b}\left[\chi_1'\left(\frac{\max\{x_1^b, ..., x_{k-1}^b\} - x_k^b}{2r}\right)^2 + \chi_2'\left(\frac{\max\{x_1^b, ..., x_{k-1}^b\} - x_k^b}{2r}\right)^2\right]. \tag{2.72}$$

For $j = 4, 5$, define the quadratic forms

$$a_j[\varphi] = \int_{\Omega_j} \left(\frac{1}{2m_a}|\nabla_{x^a}\varphi|^2 + \frac{1}{2m_b}|\nabla_{x^b}\varphi|^2 + \frac{1}{2\mu}|\nabla_{\tilde{y}}\varphi|^2\right.$$
$$\left. + \left(V(x^a - x^b, \tilde{y}) - W_R(x^a, x^b, \tilde{y}) - F_r(x^a, x^b, \tilde{y}) - G_r(x^a, x^b, \tilde{y})\right)|\varphi|^2\right) \mathrm{d}x^a \mathrm{d}x^b \mathrm{d}\tilde{y} \tag{2.73}$$

with domains

$$D[a_4] = \left\{\varphi \in H^1(\Omega_0) | \varphi(x^a, x^b, \tilde{y}) = 0 \quad \text{if} \quad \max\{x_1^a, ..., x_k^a\} \leq R \text{ or } \max\{x_1^b, ..., x_k^b\} \leq R\right.$$
$$\left. \text{or } x_k^a \leq \max\{x_1^a, ..., x_{k-1}^a\} - r \text{ or } x_k^b \leq \max\{x_1^b, ..., x_{k-1}^b\} - 4r\right\}, \tag{2.74}$$

$$D[a_5] = \Big\{\varphi \in H^1(\Omega_0) | \varphi(x^a, x^b, \tilde{y}) = 0 \quad \text{if} \quad \max\{x_1^a, ..., x_k^a\} \leq R \text{ or } \max\{x_1^b, ..., x_k^b\} \leq R$$
$$\text{or } x_k^a \leq \max\{x_1^a, ..., x_{k-1}^a\} - r \text{ or } x_k^b \geq \max\{x_1^b, ..., x_{k-1}^b\} - 2r\Big\}. \quad (2.75)$$

Again, we have $N(A_{3,k}, E^{k-1}) \leq N(A_4, E^{k-1}) + N(A_5, E^{k-1})$.

For $(x^a, x^b, \tilde{y}) \in \Omega_5$, we claim that

$$|(x^a - x^b, \tilde{y})| \geq r/\sqrt{2} = R/(8\sqrt{2}). \quad (2.76)$$

Let l be the index such that $x_l^b = \max\{x_1^b, ..., x_{k-1}^b\}$. We estimate

$$|(x^a - x^b, \tilde{y})|^2 \geq (x_l^a - x_l^b)^2 + (x_k^a - x_k^b)^2 \geq \frac{1}{2}\left(x_l^a - x_k^a - x_l^b + x_k^b\right)^2. \quad (2.77)$$

Since $\max\{x_1^a, ..., x_{k-1}^a\} \geq x_l^a$ we have in the set Ω_5 (see (2.70) and (2.60))

$$x_k^a > x_l^a - r \quad \text{and} \quad x_k^b < x_l^b - 2r \qquad \Leftrightarrow \qquad x_l^a - x_k^a < r \quad \text{and} \quad x_l^b - x_k^b > 2r. \quad (2.78)$$

Combining this with (2.77) yields (2.76). Moreover, we have $\|W_R\|_\infty + \|F_r\|_\infty + \|G_r\|_\infty \leq \frac{c_2}{R^2}$. By Assumption 2.1.13, there is $R_1 > 0$ such that for $R > R_1$ we have $a_5 > E^{k-1}$. Choosing R large enough, we thus have $N(A_5, E^{k-1}) = 0$.

For $k = 1$, we set $F_r = G_r = 0$ and $a_4 = a_2$. For any choice of $k \geq 1$, we now just need to show $N(A_4, E^{k-1}) < \infty$. At the boundaries which constrain the kth component of x^a and x^b, the operator A_4 has Dirichlet boundary conditions. The idea is to extend the domain of x_k^a and x_k^b to $\mathbb{R}$, which leads to the new operator $\hat{A}_4$ defined below. In $\hat{A}_4$, the boundary hyperplane in the kth direction has disappeared. This makes it possible to compare the operator $\hat{A}_4$ to the Hamiltonian H_{k-1} of the problem with $k-1$ boundary hyperplanes. Let us write $K_R = (W_R + F_r + G_r)\chi_{(0,\infty)^{2k} \times \mathbb{R}^{d-k}}$. Let $\hat{\Omega}_4 = \left((0,\infty)^{k-1} \times \mathbb{R}\right)^2 \times \mathbb{R}^{d-k}$ and define the quadratic form

$$\hat{a}_4[\varphi] = \int_{\hat{\Omega}_4} \Bigg(\frac{1}{2m_a}|\nabla_{x^a}\varphi|^2 + \frac{1}{2m_b}|\nabla_{x^b}\varphi|^2 + \frac{1}{2\mu}|\nabla_{\tilde{y}}\varphi|^2$$
$$+ \Big(V(x^a - x^b, \tilde{y}) - K_R(x^a, x^b, \tilde{y})\Big)|\varphi|^2\Bigg)\mathrm{d}x^a\mathrm{d}x^b\mathrm{d}\tilde{y} \quad (2.79)$$

with domain $D[\hat{a}_4] = H^1(\hat{\Omega}_4)$. We have $N(A_4, E^{k-1}) \leq N(\hat{A}_4, E^{k-1})$.

Let us change to relative and center-of-mass coordinates $y = (x^a - x^b, \tilde{y})$ and $z = \frac{m_a x^a + m_b x^b}{M}$. Then

$$\hat{a}_4[\varphi] = \int_{\mathbb{R}} \mathrm{d}z_k \int_{Q_{k-1} \times \mathbb{R}^{d-k+1}} \mathrm{d}z_1..\mathrm{d}z_{k-1}\mathrm{d}y \Bigg(\frac{1}{2\mu}|\nabla_y\varphi|^2 + \frac{1}{2M}|\nabla_z\varphi|^2$$
$$+ \left[V(y) - K_R\left(z + \frac{m_b}{M}(y_1, ..., y_k), z - \frac{m_a}{M}(y_1, ..., y_k), \tilde{y}\right)\right]|\varphi|^2\Bigg) \quad (2.80)$$

with $D[\hat{a}_4] = H^1(\mathbb{R} \times Q_{k-1} \times \mathbb{R}^{d-k+1})$. Note that we can separate z_k from the other variables and write the corresponding operator as $\hat{A}_4 = \mathbb{I} \otimes H_{k-1} - \frac{1}{2M}\Delta_{z_k} \otimes \mathbb{I} - K_R$. Recall that H_{k-1} has the ground state ψ_{k-1} with energy E^{k-1}. Let Π denote the orthogonal projection onto $L^2(\mathbb{R}) \otimes \psi_{k-1}$ in $L^2(\mathbb{R} \times Q_{k-1} \times \mathbb{R}^{d-k+1})$, and $\Pi^\perp := \mathbb{I} - \Pi$. For $\varphi \in H^1(\mathbb{R} \times Q_{k-1} \times \mathbb{R}^{d-k+1})$ both $\Pi\varphi$ and $\Pi^\perp\varphi$ belong to $H^1(\mathbb{R} \times Q_{k-1} \times \mathbb{R}^{d-k+1})$. We have

$$\hat{a}_4[\varphi] = \hat{a}_4[\Pi\varphi] + \hat{a}_4[\Pi^\perp\varphi] - 2K_R[\Pi^\perp\varphi, \Pi\varphi], \quad (2.81)$$

where

$$K_R[\varphi,\psi] = \int_{\mathbb{R}\times Q_{k-1}\times\mathbb{R}^{d-k+1}} \overline{\varphi(z,y)} K_R\left(z+\frac{m_b}{M}(y_1,...,y_k), z-\frac{m_a}{M}(y_1,...,y_k),\tilde{y}\right)\psi(z,y) \, \mathrm{d}z_k\mathrm{d}z_1...\mathrm{d}z_{k-1}\mathrm{d}y. \quad (2.82)$$

Using the Schwarz inequality, we estimate

$$|2K_R[\Pi^\perp\varphi,\Pi\varphi]| \le R\|K_R\Pi\varphi\|^2_{L^2(\mathbb{R}\times Q_{k-1}\times\mathbb{R}^{d-k+1})} + \frac{1}{R}\|\Pi^\perp\varphi\|^2_{L^2(\mathbb{R}\times Q_{k-1}\times\mathbb{R}^{d-k+1})}. \quad (2.83)$$

Since E^{k-1} is a discrete and non-degenerate eigenvalue of H_{k-1}, we have $E_1^{k-1} = \inf(\sigma(H_{k-1})\setminus\{E^{k-1}\}) > E^{k-1}$, and $(\mathbb{I}\otimes h_{k-1})[\Pi^\perp\varphi] \ge E_1^{k-1}\|\Pi^\perp\varphi\|^2_{L^2(\mathbb{R}\times Q_{k-1}\times\mathbb{R}^{d-k+1})}$. Together with the positivity of $-\Delta_{z_k}\otimes\mathbb{I}$ and $\|K_R\|_\infty \le \frac{c_2}{R^2}$ it follows that

$$\hat{a}_4[\Pi^\perp\varphi] \ge \left(E_1^{k-1} - \frac{c_2}{R^2}\right)\|\Pi^\perp\varphi\|^2_{L^2(\mathbb{R}\times Q_{k-1}\times\mathbb{R}^{d-k+1})}. \quad (2.84)$$

In total, we have

$$\hat{a}_4[\varphi] \ge \hat{a}_4[\Pi\varphi] - R\|K_R\Pi\varphi\|^2_{L^2(\mathbb{R}\times Q_{k-1}\times\mathbb{R}^{d-k+1})} + \left(E_1^{k-1} - \frac{1}{R} - \frac{c_2}{R^2}\right)\|\Pi^\perp\varphi\|^2_{L^2(\mathbb{R}\times Q_{k-1}\times\mathbb{R}^{d-k+1})}. \quad (2.85)$$

We choose R large enough such that $E_1^{k-1} - E^{k-1} > \frac{1}{R} + \frac{c_2}{R^2}$. Let B_1 be the self-adjoint operator corresponding to

$$b_1[\varphi] = \hat{a}_4[\varphi] - R\|K_R\varphi\|^2_{L^2(\mathbb{R}\times Q_{k-1}\times\mathbb{R}^{d-k+1})} \quad (2.86)$$

in $\operatorname{ran}\Pi$. Then $N(\hat{A}_4, E^{k-1}) \le N(B_1, E^{k-1})$ by the min-max principle.

We can write any function $\varphi\in\operatorname{ran}\Pi$ as $\varphi(z,y) = f(z_k)\psi_{k-1}(z_1,...,z_{k-1},y)$ for some $f\in H^1(\mathbb{R})$. Integrating over $z_1,...,z_{k-1},y$, we have

$$\hat{a}_4[f\otimes\psi_{k-1}] = \int_{\mathbb{R}}\left(\frac{1}{2M}|f'(z_k)|^2 + (E^{k-1} - U_R(z_k))f(z_k)^2\right)\mathrm{d}z_k, \quad (2.87)$$

where

$$U_R(z_k) = \int_{Q_{k-1}\times\mathbb{R}^{d-k+1}} K_R\left(z+\frac{m_b}{M}(y_1,...,y_k), z-\frac{m_a}{M}(y_1,...,y_k),\tilde{y}\right)\psi_{k-1}(z_1,...z_{k-1},y)^2 \, \mathrm{d}z_1...\mathrm{d}z_{k-1}\mathrm{d}y. \quad (2.88)$$

Moreover,

$$\|K_R(f\otimes\psi_{k-1})\|^2_{L^2(\mathbb{R}\times Q_{k-1}\times\mathbb{R}^{d-k+1})} = \int_{\mathbb{R}} V_R(z_k)f(z_k)^2\mathrm{d}z_k \quad (2.89)$$

with

$$V_R(z_k) = \int_{Q_{k-1}\times\mathbb{R}^{d-k+1}} K_R\left(z+\frac{m_b}{M}(y_1,...,y_k), z-\frac{m_a}{M}(y_1,...,y_k),\tilde{y}\right)^2\psi_{k-1}(z_1,...,z_{k-1},y)^2 \, \mathrm{d}z_1...\mathrm{d}z_{k-1}\mathrm{d}y. \quad (2.90)$$

Let $Z_R = U_R + RV_R$. With

$$b_2[f] = \int_{\mathbb{R}}\left(\frac{1}{2M}|f'(z)|^2 - Z_R(z)f(z)^2\right)\mathrm{d}z, \quad (2.91)$$

we can write $b_1[f \otimes \psi_{k-1}] = E^{k-1}\|f\|^2_{L^2(\mathbb{R})} + b_2[f]$. Therefore, $N(B_1, E^{k-1}) = N(B_2, 0)$.

In the following, we bound the function Z_R from above by an exponentially decaying function. With this bound it is easy to see that $N(B_2, 0) < \infty$ using e.g. the Bargmann estimate (see Chapter 2, Theorem 5.3 in [8]). This concludes the proof of $N(H_k, E^{k-1}) < \infty$.

To bound Z_R, first use that K_R is bounded to obtain

$$Z_R(z_k) \leq \left(\|K\|_\infty + R\|K\|^2_\infty\right) I(z_k), \tag{2.92}$$

where

$$I(z_k) = \int_{Q_{k-1}\times\mathbb{R}^{d-k+1}} \chi_{\text{supp } K_R}(z,y)\psi^2_{k-1}\mathrm{d}z_1...\mathrm{d}z_{k-1}\mathrm{d}y. \tag{2.93}$$

By construction, $I(z_k) = 0$ for $z_k < 0$. We shall show that $I(z_k)$ decays exponentially for $z_k \geq 0$. In fact, if z_k is large and $K_R(z,y) \neq 0$, then necessarily one of the remaining coordinates $z_1, ..., z_{k-1}, y_1, ..., y_d$ has to be large as well. This is essentially the content of the following Lemma.

Lemma 2.3.4. *Let $a > 0$. For $z_k \geq 2R$ the function*

$$\alpha(z,y) = e^{a\sqrt{2M|z_1|^2+..+2M|z_{k-1}|^2+2\mu|y|^2}}\chi_{\text{supp } K_R}(z,y) \tag{2.94}$$

satisfies $\alpha(z,y) \geq e^{ac(z_k-2R)}\chi_{\text{supp } K_R}(z,y)$ with $c = \sqrt{2M}(1 + 2\max\{\frac{m_a}{m_b}, \frac{m_b}{m_a}\})^{-1/2}$.

The Agmon estimate (2.41) tells us that there is a constant $a > 0$ such that

$$c_3 := \int_{Q_{k-1}\times\mathbb{R}^{d-k+1}} \psi^2_{k-1}e^{a\sqrt{2M|z_1|^2+...+2M|z_{k-1}|^2+2\mu|y|^2}}\mathrm{d}z_1...\mathrm{d}z_{k-1}\mathrm{d}y < \infty. \tag{2.95}$$

We apply Lemma 2.3.4 with this constant a and conclude that

$$\chi_{\text{supp } K_R}(z,y) \leq e^{-c_4(z_k-2R)}\alpha(z,y) \tag{2.96}$$

for $z_k \geq 2R$ and suitable constant $c_4 > 0$. In particular,

$$\begin{aligned} I(z_k) &\leq e^{-c_4(z_k-2R)}\int_{Q_{k-1}\times\mathbb{R}^{d-k+1}} \alpha(z,y)\psi_{k-1}(z_1, ..., z_{k-1}, y)^2\mathrm{d}z_1...\mathrm{d}z_{k-1}\mathrm{d}y \\ &\leq c_3 e^{-c_4(z_k-2R)}. \end{aligned} \tag{2.97}$$

for $z_k \geq 2R$. Recall that Z_R vanishes on $(-\infty, 0)$ and $\|Z_R\|_\infty < \infty$. With (2.92) we thus conclude the desired exponentially decaying bound. □

It remains to give the proof of Lemmas 2.3.3 and 2.3.4.

Proof of Lemma 2.3.4. Recall the definitions of W_R, F_r and G_r in (2.51), (2.66) and (2.72), respectively. Since $\text{supp } K_R \subset \text{ supp } W_R \cup \text{ supp } F_r \cup \text{ supp } G_r$, we estimate α on each of these three sets. In $\text{supp } W_R$, at least one particle is close to the corner, i.e. in the hypercube $(0, 2R)^k$. If z_k is large, this means that the two particles are far apart and y_k is large. To be precise, using $x^a_j = z_j + \frac{m_b}{M}y_j$ and $x^b_j = z_j - \frac{m_a}{M}y_j$ we have

$$\begin{aligned} \text{supp } W_R &\subset \left\{(z,y) \in Q_k \times \mathbb{R}^{d-k} | 0 \leq \frac{z_k + \frac{m_b}{M}y_k}{R} \leq 2 \text{ or } 0 \leq \frac{z_k - \frac{m_a}{M}y_k}{R} \leq 2\right\} \\ &\subset \left\{(z,y) \in Q_k \times \mathbb{R}^{d-k} | z_k - 2R \leq \frac{\max\{m_a, m_b\}}{M}|y_k|\right\}. \end{aligned} \tag{2.98}$$

For $(z, y) \in \operatorname{supp} W_R$ with $z_k \geq 2R$, we therefore have

$$M \sum_{j=1}^{k-1} |z_j|^2 + \mu \sum_{j=1}^{k} |y_k|^2 \geq \frac{\mu M^2 (z_k - 2R)^2}{\max\{m_a^2, m_b^2\}} = \frac{M(z_k - 2R)^2}{\max\{\frac{m_a}{m_b}, \frac{m_b}{m_a}\}}, \tag{2.99}$$

which implies the desired bound on α.

For $k = 1$, both F_r and G_r are identically zero, hence to estimate α on their support we can restrict our attention to the case $k > 1$. Observe that in $\operatorname{supp} F_r$ every coordinate x_j^a for $1 \leq j \leq k$ is smaller than or similar in magnitude to the largest of the other coordinates x_i^a, $i \neq j$; in particular, this applies to $j = k$. Intuitively, for large z_k either x_k^a or $|y_k|$ needs to be large. If x_k^a is large, also some other x_j^a with $j < k$ has to be large. Phrased precisely, by Lemma 2.3.3 we have

$$\begin{aligned}
&\operatorname{supp} F_r \subset \\
&\bigcup_{j=1}^{k} \left\{ (z, y) \in Q_k \times \mathbb{R}^{d-k} | \max_{\substack{1 \leq l \leq k, \\ l \neq j}} \{z_l + \frac{m_b}{M} y_l\} - r \leq z_j + \frac{m_b}{M} y_j \leq \max_{\substack{1 \leq l \leq k, \\ l \neq j}} \{z_l + \frac{m_b}{M} y_l\} + r \right\} \\
&\qquad \subset \left\{ (z, y) \in Q_k \times \mathbb{R}^{d-k} | z_k - r \leq -\frac{m_b}{M} y_k + \max_{1 \leq j \leq k-1} \{\frac{m_b}{M} y_j + z_j\} \right\} =: S_F.
\end{aligned} \tag{2.100}$$

The constraint in S_F can be written as $z_k - r \leq (\sqrt{M} z, \sqrt{\mu} y) \cdot e$ for a vector $e \in \mathbb{R}^{k+d}$. A simple Schwarz inequality therefore shows that on the set S_F we have

$$M \sum_{j=1}^{k-1} |z_j|^2 + \mu \sum_{j=1}^{k} |y_k|^2 \geq \frac{(z_k - r)^2}{\|e\|^2} = \frac{M(z_k - r)^2}{1 + 2\frac{m_b}{m_a}} \tag{2.101}$$

as long as $z_k \geq r$, which yields the desired bound on α.

Similarly to the previous case, in $\operatorname{supp} G_r$ the coordinate x_k^b is of similar magnitude as the largest of the other coordinates x_j^b. We have

$$\begin{aligned}
\operatorname{supp} G_r &\subset \left\{ (z, y) \in Q_k \times \mathbb{R}^{d-k} | 2r \leq \max_{1 \leq j \leq k-1} \{z_j - \frac{m_a}{M} y_j\} + \frac{m_a}{M} y_k - z_k \leq 4r \right\} \\
&\subset \left\{ (z, y) \in Q_k \times \mathbb{R}^{d-k} | z_k + 2r \leq \max_{1 \leq j \leq k-1} \{z_j - \frac{m_a}{M} y_j\} + \frac{m_a}{M} y_k \right\} =: S_G.
\end{aligned} \tag{2.102}$$

Analogously to before, on the set S_G we have

$$M \sum_{j=1}^{k-1} |z_j|^2 + \mu \sum_{j=1}^{k} |y_k|^2 \geq \frac{M(z_k + 2r)^2}{1 + 2\frac{m_a}{m_b}}. \tag{2.103}$$

This concludes the proof. □

Proof of Lemma 2.3.3. Suppose $(x^a, x^b, \tilde{y}) \in \operatorname{supp} f_{3,j}^r$. If $j < k$, we need

$$k \frac{\max\{x_{j+1}^a, ..., x_k^a\} - x_j^a}{2r} + \frac{3}{2} \leq 2 \tag{2.104}$$

for the factor χ_1 to be non-zero. This is equivalent to $\max\{x_{j+1}^a, ..., x_k^a\} \leq x_j^a + \frac{r}{k}$. Thus, for any $1 \leq j \leq k$ we have $\max\{x_j^a, ..., x_k^a\} \leq x_j^a + \frac{r}{k}$ on the support of $f_{3,j}^r$. Let us argue inductively why $\max\{x_1^a, ..., x_k^a\} \leq x_j^a + r$. Suppose we know for some $1 < l \leq j$

that $\max\{x_l^a, ..., x_k^a\} \leq x_j^a + (j+1-l)\frac{r}{k}$. If $x_{l-1}^a \leq \max\{x_l^a, ..., x_k^a\}$, we trivially have $\max\{x_{l-1}^a, ..., x_k^a\} \leq x_j^a + (j+1-(l-1))\frac{r}{k}$. If $x_{l-1}^a > \max\{x_l^a, ..., x_k^a\}$, for the factor $\chi_2\left(k\frac{\max\{x_l^a,...,x_k^a\}-x_{l-1}^a}{2r} + \frac{3}{2}\right)$ not to vanish we have $\max\{x_l^a, ..., x_k^a\} + \frac{r}{k} \geq x_{l-1}^a$. Thus,

$$\max\{x_{l-1}^a, ..., x_k^a\} = x_{l-1}^a \leq \max\{x_l^a, ..., x_k^a\} + \frac{r}{k} \leq x_j^a + (j+1-(l-1))\frac{r}{k}. \quad (2.105)$$

Inductively, we see that for every j we have $\max\{x_1^a, ..., x_k^a\} \leq x_j^a + j\frac{r}{k} \leq x_j^a + r$. Thus, $\operatorname{supp} f_{3,j} \cap \Omega_2 \subset \Omega_{3,j}$.

For the support of $\nabla f_{3,j}$, we have

$$\operatorname{supp} \nabla f_{3,j}^r \cap \Omega_2 \subset \operatorname{supp} f_{3,j}^r \cap \Omega_2 \subset \Omega_{3,j} = \{(x^a, x^b, \tilde{y}) \in \Omega_2 | x_j^a \geq \max\{x_1^a, ..., \widehat{x_j^a}, ..., x_k^a\} - r\}. \quad (2.106)$$

Now, suppose $x_j^a > \max\{x_1^a, ..., \widehat{x_j^a}, ..., x_k^a\} + r$. It is sufficient to show that $f_{3,j}^r \equiv 1$ in this region. For $j < k$, we have

$$k\frac{\max\{x_{j+1}^a, ..., x_k^a\} - x_j^a}{2r} + \frac{3}{2} \leq k\frac{\max\{x_1^a, ..., \widehat{x_j^a}, ..., x_k^a\} - x_j^a}{2r} + \frac{3}{2} < -\frac{k}{2} + \frac{3}{2} \leq 1. \quad (2.107)$$

Thus, $\chi_1\left(k\frac{\max\{x_{j+1}^a,...,x_k^a\}-x_j^a}{2r} + \frac{3}{2}\right) = 1$. For $l < j \leq k$, we have

$$k\frac{\max\{x_{l+1}^a, ..., x_k^a\} - x_l^a}{2r} + \frac{3}{2} = k\frac{x_j^a - x_l^a}{2r} + \frac{3}{2} \geq k\frac{x_j^a - \max\{x_1^a, ..., \widehat{x_j^a}, ..., x_k^a\}}{2r} + \frac{3}{2} > \frac{k}{2} + \frac{3}{2} \geq 2. \quad (2.108)$$

Thus, $\chi_2\left(k\frac{\max\{x_{l+1}^a,...,x_k^a\}-x_l^a}{2r} + \frac{3}{2}\right) = 1$. In total, $f_{3,j} \equiv 1$ for $x_j^a > \max\{x_1^a, ..., \widehat{x_j^a}, ..., x_k^a\} + r$. $\square$

Appendix

2.A Appendix

2.A.1 Explicit example in one dimension

To illustrate the effect of a boundary on two-particle bound states, we present an explicit example in one dimension. We consider particles with equal masses $m_a = m_b = \frac{1}{2}$ and with delta-interaction $V(y) = -\alpha\delta(y)$ for $\alpha > 0$. The full Hamiltonian is

$$H = -\left(\frac{\partial}{\partial x^a}\right)^2 - \left(\frac{\partial}{\partial x^b}\right)^2 - \alpha\delta(x^a - x^b), \tag{2.109}$$

either on $L^2(\mathbb{R}^2)$ or on $L^2((0,\infty)^2)$ with Neumann boundary conditions. In the first case, corresponding to $k = 0$, we look at the operator $H_0 = -2\frac{\partial^2}{\partial y^2} - \alpha\delta(y)$ on $L^2(\mathbb{R})$. It has the ground state $\psi_0(y) = e^{-\frac{\alpha}{4}|y|}$ with corresponding energy $E^0 = -\frac{\alpha^2}{8}$.

The second case corresponds to $k = 1$. To compute the ground state of $H = H_1$ on $L^2((0,\infty)^2)$, we mirror the problem along the $x^a = 0$ and $x^b = 0$ boundaries, and look for the ground state of the modified Hamiltonian

$$\widetilde{H}_1 = -\left(\frac{\partial}{\partial x^a}\right)^2 - \left(\frac{\partial}{\partial x^b}\right)^2 - \alpha\delta(x^a - x^b) - \alpha\delta(x^a + x^b) \tag{2.110}$$

on $L^2(\mathbb{R}^2)$. This is exactly the operator considered in Proposition 2.A.1. Switching to relative and center of mass coordinates $y = x^a - x^b$ and $z = \frac{x^a + x^b}{2}$, we obtain

$$\widetilde{H}_1 = \left(-2\frac{\partial^2}{\partial y^2} - \alpha\delta(y)\right) + \frac{1}{2}\left(-\frac{\partial^2}{\partial z^2} - \alpha\delta(z)\right). \tag{2.111}$$

The ground state of $\tilde{H}_1$ is $\widetilde{\psi}_1(y,z) = \psi_0(y)e^{-\frac{\alpha}{2}|z|}$, which decays exponentially away from the Neumann boundary. The ground state energy $E^1 = -\frac{\alpha^2}{4}$ is strictly lower than E^0.

2.A.2 Proof of Lemma 2.2.3

Let $1 \leq k \leq d$. First, we shall prove that the claim is true for $l = 1$, i.e.

$$\lim_{L\to\infty} \inf \sigma(H_{k-1}^{L,1}) \geq E^{k-1}. \tag{2.112}$$

In $\Omega_{k-1}^{L,1}$, the first component of y is constrained to $|y_1| < L$. Apart from that, $\Omega_{k-1}^{L,1}$ is the same as $Q_{k-1} \times \mathbb{R}^{d-k+1}$ with components 1 and k swapped. We localize in the y_1 direction, analogously to the one-dimensional case in Proposition A.5 in [18]. For this, let $\chi_1, \chi_2 : \mathbb{R} \to$

$[0,1]$ be continuously differentiable functions satisfying $\chi_1(t) = 0$ for $t \geq 1$, $\chi_1(t) = 1$ for $t \leq \frac{1}{2}$, and $\chi_1(t)^2 + \chi_2(t)^2 = 1$ for all t. Note that $c := \max\{\|(\chi_1')^2\|_\infty, \|(\chi_2')^2\|_\infty\} < \infty$. We choose the localizing functions f_j on $\Omega_{k-1}^{L,1}$ as $f_j(z_2, \dots z_k, y) = \chi_j(|y_1|/L)$. By the IMS localization formula, we have for all $\psi \in H^1(\Omega_{k-1}^{L,1})$

$$h_{k-1}^{L,1}[\psi] = h_{k-1}^{L,1}[f_1\psi] + h_{k-1}^{L,1}[f_2\psi] - \frac{1}{2\mu}\int_{\Omega_{k-1}^{L,1}} \left((\nabla f_1)^2 + (\nabla f_2)^2\right)|\psi|^2. \tag{2.113}$$

Note that $(\nabla f_j)^2 = \frac{1}{L^2}(\chi_j'(|y_1|/L))^2 \leq \frac{c}{L^2}$. Since $f_2\psi$ is nonzero only for $|y_1| > L/2$, for large enough L, we have $h_{k-1}^{L,1}[f_2\psi] \geq E^{k-1}\|f_2\psi\|_2^2$ by Assumption 2.1.13. Furthermore, since $f_1\psi$ satisfies Dirichlet boundary conditions at $|y_1| = L$, we can extend the function by zero to $y_1 \in \mathbb{R}$. Additionally, let us swap the first and the kth components and call the function obtained this way $\iota(f_1\psi)$. Note that $\iota(f_1\psi) \in H^1(Q_{k-1} \times \mathbb{R}^{d-k+1})$ and $\|\iota(f_1\psi)\|_2^2 = \|f_1\psi\|_2^2$. Therefore,

$$\frac{h_{k-1}^{L,1}[f_1\psi]}{\|f_1\psi\|_2^2} = \frac{h_{k-1}[\iota(f_1\psi)]}{\|\iota(f_1\psi)\|_2^2} \geq E^{k-1}. \tag{2.114}$$

Combining the estimates, we obtain for large L that

$$\frac{h_{k-1}^{L,1}[\psi]}{\|\psi\|^2} \geq E^{k-1}\frac{\|f_1\psi\|^2 + \|f_2\psi\|^2}{\|\psi\|_2^2} - \frac{c}{\mu L^2} = E^{k-1} - \frac{c}{\mu L^2}. \tag{2.115}$$

Hence, $\inf\sigma(H_{k-1}^{L,1}) \geq E^{k-1} - \frac{c}{\mu L^2}$ and the claim follows.

Note that for $k = 1$, $l = 1$ was the only possible case. Consider $k \geq 2$. We proceed by induction. For $l \geq 2$, assume the claim holds for $l-1$. The strategy is to bound $h_{k-1}^{L,l}$ using $h_{k-1}^{L,l-1}$ and $h_{k-2}^{L,l-1}$. In $\Omega_{k-1}^{L,l}$, each of the first $l-1$ components are restricted to the (red) triangular domain 2 in Figure 2.1. Furthermore, $y_l \in (-L, L)$ while in the z-coordinate the lth component is omitted. In the components $l+1$ to k we have the full quadrant. Recall that $\delta = M/\max\{m_a, m_b\}$. In the $(l-1)$th component, we localize such that one function has Dirichlet boundary conditions along the (red) line $z_{l-1} = L/\delta$ in Figure 2.1 and the other is localized at $L/2\delta < z_{l-1} < L/\delta$, with a Dirichlet boundary at $z_{l-1} = L/2\delta$. For this, we use the functions $f_j(z_1, \dots, \hat{z}_l, \dots, z_k, y) = \chi_j(\delta z_{l-1}/L)$. By the IMS localization formula, we have for all $\psi \in H^1(\Omega_{k-1}^{L,l})$

$$h_{k-1}^{L,l}[\psi] = h_{k-1}^{L,l}[f_1\psi] + h_{k-1}^{L,l}[f_2\psi] - \frac{1}{2M}\int_{\Omega_{k-1}^{L,l}} \left((\nabla f_1)^2 + (\nabla f_2)^2\right)|\psi|^2. \tag{2.116}$$

Note that $(\nabla f_j)^2 = \frac{\delta^2}{L^2}(\chi_j'(\delta z_{l-1}/L))^2 \leq \frac{\delta^2 c}{L^2}$. Since $f_1\psi$ satisfies Dirichlet boundary conditions along $z_{l-1} = L/\delta$, one can extend the function by zero to the quadrant Q_1 in the $(l-1)$th component. Additionally swap y_{l-1} and y_l to define $\iota_1(f_1\psi) \in H^1(\Omega_{k-1}^{L,l-1})$. Then $\|\iota_1(f_1\psi)\|_2^2 = \|f_1\psi\|_2^2$ and hence

$$\frac{h_{k-1}^{L,l}[f_1\psi]}{\|f_1\psi\|_2^2} = \frac{h_{k-1}^{L,l-1}[\iota_1(f_1\psi)]}{\|\iota_1(f_1\psi)\|_2^2} \geq \inf\sigma(H_{k-1}^{L,l-1}). \tag{2.117}$$

To estimate $h_{k-1}^{L,l}[f_2\psi]$, we localize in the y_{l-1}-direction, such that the first function satisfies Dirichlet boundary conditions at $y_{l-1} = L/2$ and the second function is nonzero only for $y_{l-1} > L/4$. For this, we use the functions $g_j(z_1, \dots, \hat{z}_l, \dots, z_k, y) = \chi_j(2y_{l-1}/L)$. The IMS localization formula gives

$$h_{k-1}^{L,l}[f_2\psi] = h_{k-1}^{L,l}[g_1 f_2\psi] + h_{k-1}^{L,l}[g_2 f_2\psi] - \frac{1}{2\mu}\int_{\Omega_{k-1}^{L,l}} \left((\nabla g_1)^2 + (\nabla g_2)^2\right)|f_1\psi|^2, \tag{2.118}$$

where $(\nabla g_j)^2 = \frac{4}{L^2}(\chi_j'(2|y_{l-1}|/L))^2 \le \frac{4c}{L^2}$. For L large enough, by Assumption 2.1.13, we have $h_{k-1}^{L,l}[g_2 f_2 \psi] \ge E^{k-1}\|g_2 f_2 \psi\|_2^2$. In the $(l-1)$th component, the function $g_1 f_2 \psi$ is supported in the parallelogram $(z_{l-1}, y_{l-1}) \in (L/2\delta, L/\delta) \times (-L/2, L/2)$ and satisfies Dirichlet boundary conditions at $|y_{l-1}| = L/2$ and $z_{l-1} = L/2\delta$. We extend the function $g_1 f_2 \psi$ by zero to $y_{l-1} \in \mathbb{R}$. Then we define $\iota_2(g_1 f_2 \psi)$ on $\Omega_{k-2}^{L,l-1} \times (L/2\delta, L/\delta)$ as

$$\begin{aligned}&\iota_2(g_1 f_2 \psi)(z_1 \dots, \hat{z}_{l-1}, \dots z_{k-1}, y, x)\\ &\quad = g_1 f_2 \psi(z_1, \dots, z_{l-2}, x, z_l, \dots, z_{k-1}, y_1, \dots, y_{l-2}, y_k, y_{l-1}, \dots, y_{k-1}, y_{k+1}, \dots y_d). \end{aligned} \tag{2.119}$$

Observe that $h_{k-1}^{L,l}$ now can effectively be decomposed into $h_{k-2}^{L,l-1}$ plus a Laplacian in the x-direction

$$\frac{h_{k-1}^{L,l}[g_1 f_2 \psi]}{\|g_1 f_2 \psi\|_2^2} = \frac{(h_{k-2}^{L,l-1} \otimes \mathbb{I} + \mathbb{I} \otimes q)[\iota_2(g_1 f_2 \psi)]}{\|\iota_2(g_1 f_2 \psi)\|_2^2}, \tag{2.120}$$

where q is defined on $H^1((L/2\delta, L/\delta))$ through

$$q[\varphi] = \int_{L/2\delta}^{L/\delta} \frac{1}{2M}|\varphi'(x)|^2 \mathrm{d}x. \tag{2.121}$$

Since $\inf \sigma(H_{k-2}^{L,l-1} \otimes \mathbb{I} - \frac{1}{2M}\mathbb{I} \otimes \Delta_x) \ge \inf \sigma(H_{k-2}^{L,l-1})$, we obtain

$$\frac{h_{k-1}^{L,l}[g_1 f_2 \psi]}{\|g_1 f_2 \psi\|_2^2} \ge \inf \sigma(H_{k-2}^{L,l-1}). \tag{2.122}$$

Combining all the estimates, we obtain that for large L and all $\psi \in H^1(\Omega_{k-1}^{L,l})$

$$\frac{h_{k-1}^{L,l}[\psi]}{\|\psi\|^2} \ge \min\{\inf \sigma(H_{k-1}^{L,l-1}), \inf \sigma(H_{k-2}^{L,l-1}), E^{k-1}\} - \frac{\delta^2 c}{ML^2} - \frac{4c}{\mu L^2}. \tag{2.123}$$

Taking $L \to \infty$ the claim now follows from the induction hypothesis.

2.A.3 Technical details

By mirroring along the $x_j^a = 0$ and $x_j^b = 0$ hyperplanes, we can relate H_k to an operator $\widetilde{H}_k$ defined in $L^2(\mathbb{R}^{d+k})$.

Proposition 2.A.1. *Let $\widetilde{H}_k$ be the operator defined by the quadratic form*

$$\begin{aligned}\widetilde{h}_k[\psi] = \int_{\mathbb{R}^{d+k}} \Bigg(&\frac{1}{2m_a}|\nabla_{x^a}\psi|^2 + \frac{1}{2m_b}|\nabla_{x^b}\psi|^2 + \frac{1}{2\mu}|\nabla_{\tilde{y}}\psi|^2 \\ &+ V((|x_j^a| - |x_j^b|)_{j=1}^k, \tilde{y})|\psi|^2 \Bigg) \mathrm{d}x^a \mathrm{d}x^b \mathrm{d}\tilde{y} \end{aligned} \tag{2.124}$$

with domain $D[\widetilde{h}_k] = H^1(\mathbb{R}^{d+k})$. Then $\inf \sigma(H_k) = \inf \sigma(\widetilde{H}_k)$ and $\inf \sigma_{ess}(H_k) = \inf \sigma_{ess}(\widetilde{H}_k)$. Moreover, the function ψ_k is a ground state of H_k if and only if the function

$$\widetilde{\psi}_k(x^a, x^b, \tilde{y}) = \psi_k((|x_j^a|)_{j=1}^k, (|x_j^b|)_{j=1}^k, \tilde{y}) \tag{2.125}$$

is a ground state of $\widetilde{H}_k$.

Proof. The operator $\widetilde{H}_k$ commutes with all reflections along the $x_j^a = 0$ or $x_j^b = 0$ hyperplanes. Reflections along different hyperplanes commute as well. Therefore, the Hilbert space $\mathbb{H} = L^2(\mathbb{R}^{d+k})$ splits into subspaces $\mathbb{H} = \bigoplus_r \mathbb{H}_r$ characterized by the eigenvalues ± 1 of these reflections. We can write $\widetilde{H}_k = \bigoplus_r \widetilde{H}_k^r$, where $\widetilde{H}_k^r$ is the restriction of $\widetilde{H}_k$ to $\mathbb{H}_r$. For the spectrum, we obtain $\inf \sigma(\widetilde{H}_k) = \min_r \inf \sigma(\widetilde{H}_k^r)$ and $\inf \sigma_{\text{ess}}(\widetilde{H}_k) = \min_r \inf \sigma_{\text{ess}}(\widetilde{H}_k^r)$.

The subspace that is symmetric under all reflections corresponds to Neumann boundary conditions on $[0,\infty)^{2k} \times \mathbb{R}^{d-k}$. The other subspaces $\mathbb{H}_r$ are antisymmetric under at least one reflection, so they have Dirichlet boundary conditions along the corresponding hyperplane. Thus, the domains of the quadratic forms for $\widetilde{H}_k^r$ satisfy $D[\widetilde{h_k^r}] \subset D[\widetilde{h_k^{\text{sym}}}]$. By the min-max principle, $E_n(\widetilde{H}_k^r) \geq E_n(\widetilde{H}_k^{\text{sym}})$. Therefore, both $\inf \sigma(\widetilde{H}_k) = \inf \sigma(\widetilde{H}_k^{\text{sym}})$ and $\inf \sigma_{\text{ess}}(\widetilde{H}_k) = \inf \sigma_{\text{ess}}(H_k^{\text{sym}})$.

Note that the map $U : L^2([0,\infty)^{2k} \times \mathbb{R}^{d-k}) \to L^2_{\text{sym}}(\mathbb{R}^{d+k})$ that maps ψ to $\widetilde{\psi}(x^a, x^b, \tilde{y}) = \frac{1}{2^k}\psi((|x_j^a|)_j, (|x_j^b|)_j, \tilde{y})$ is unitary. Since $\widetilde{H}_k^{\text{sym}} = U H_k U^{-1}$, the operators are unitarily equivalent and $\sigma(\widetilde{H}_k^{\text{sym}}) = \sigma(H_k)$. □

The next lemma follows from the Sobolev inequality, see e.g. Sections 8.8 and 11.3 in [50].

Lemma 2.A.2. *Let $\Omega \subset \mathbb{R}^d$ be a domain satisfying the cone property (as defined in [50]) with radius R and opening angle θ. Let V satisfy Assumption 2.1.11. Then, for any $0 < a < 1$ there is a constant $b \in \mathbb{R}$ (depending only on d, R, θ, V and a) such that*

$$\int_\Omega |V||f|^2 \leq a\|\nabla f\|^2_{L^2(\Omega)} + b\|f\|^2_{L^2(\Omega)}, \tag{2.126}$$

for all $f \in H^1(\Omega)$.

Proposition 2.A.3. *Let $0 \leq k \leq d$. Assumption 2.1.11 implies that in the quadratic form h_k in* (2.4) *the interaction term is infinitesimally form bounded with respect to the kinetic energy. By the KLMN theorem, there is a unique self-adjoint operator H_k corresponding to h_k, and both h_k and H_k are bounded from below.*

Proof. The quadratic form $q_k : H^1([0,\infty)^{2k} \times \mathbb{R}^{d-k}) \to \mathbb{R}$ given by

$$q_k[\psi] = \int_{[0,\infty)^{2k}\times\mathbb{R}^{d-k}} \left(\frac{1}{2m_a}|\nabla_{x^a}\psi|^2 + \frac{1}{2m_b}|\nabla_{x^b}\psi|^2 + \frac{1}{2\mu}|\nabla_y \psi|^2 \right) \mathrm{d}x^a \mathrm{d}x^b \mathrm{d}\tilde{y} \tag{2.127}$$

is closed and bounded from below. In order to apply the KLMN theorem, we need to show that there are constants $a < 1, b \in \mathbb{R}$ such that for all $\psi \in H^1([0,\infty)^{2k} \times \mathbb{R}^{d-k})$

$$K[\psi] := \left| \int_{[0,\infty)^{2k}\times\mathbb{R}^{d-k}} V(x^a - x^b, \tilde{y})|\psi|^2 \mathrm{d}x^a \mathrm{d}x^b \mathrm{d}\tilde{y} \right| \leq a q_k[\psi] + b\|\psi\|_2^2. \tag{2.128}$$

Let $\psi \in H^1([0,\infty)^{2k} \times \mathbb{R}^{d-k})$ and define $\widetilde{\psi}(x^a, x^b, \tilde{y}) = \frac{1}{2^k}\psi((|x_j^a|)_j, (|x_j^b|)_j, \tilde{y})$ for $(x^a, x^b, \tilde{y}) \in \mathbb{R}^k \times \mathbb{R}^k \times \mathbb{R}^{d-k}$. We have $\|\widetilde{\psi}\|_2^2 = \|\psi\|_2^2$ and $\|\nabla\widetilde{\psi}\|_2^2 = \|\nabla\psi\|_2^2$. Moreover, ψ and $2^k\widetilde{\psi}$ agree on $[0,\infty)^{2k} \times \mathbb{R}^{d-k}$. Hence,

$$K[\psi] \leq 4^k \int_{[0,\infty)^{2k}\times\mathbb{R}^{d-k}} |V(x^a - x^b, \tilde{y})||\widetilde{\psi}(x^a, x^b, \tilde{y})|^2 \mathrm{d}x^a \mathrm{d}x^b \mathrm{d}\tilde{y}. \tag{2.129}$$

Since the integrand is nonnegative, extending the domain of integration from $[0,\infty)^{2k}\times\mathbb{R}^{d-k}$ to $\mathbb{R}^{2k}\times\mathbb{R}^{d-k}$ gives the upper bound

$$
\begin{aligned}
K[\psi] &\le 4^k \int_{\mathbb{R}^{2k}\times\mathbb{R}^{d-k}} |V(x^a-x^b,\tilde{y})||\widetilde{\psi}(x^a,x^b,\tilde{y})|^2 \mathrm{d}x^a\mathrm{d}x^b\mathrm{d}\tilde{y} \\
&= 4^k \int_{\mathbb{R}^k\times\mathbb{R}^d} |V(y)||\widetilde{\psi}(w+(y_1,...,y_k)/2, w-(y_1,...,y_k)/2,\tilde{y})|^2 \mathrm{d}w\mathrm{d}y, \quad (2.130)
\end{aligned}
$$

where we changed to coordinates $w=\frac{x^a+x^b}{2}$ and y. For almost every $w\in\mathbb{R}^k$, the function $f(y)=\widetilde{\psi}(w+(y_1,...,y_k)/2, w-(y_1,...,y_k)/2,\tilde{y})$ lies in $H^1(\mathbb{R}^d)$ by Fubini's theorem. By Lemma 2.A.2, for any $0<\tilde{a}$ there is a constant b independent of f such that $\int_{\mathbb{R}^d}|V||f|^2\le \tilde{a}\|\nabla f\|_2^2+b\|f\|_2^2$. Integrating over w then gives

$$
K[\psi]\le 4^k\left(\tilde{a}\int_{\mathbb{R}^k\times\mathbb{R}^d}\left|\nabla_y\widetilde{\psi}(w+(y_1,...,y_k)/2, w-(y_1,...,y_k)/2,\tilde{y})\right|^2\mathrm{d}w\mathrm{d}y + b\|\widetilde{\psi}\|_2^2\right). \quad (2.131)
$$

For $1\le j\le k$,

$$
\left|\partial_{y_j}\widetilde{\psi}(w+(y_1,...,y_k)/2, w-(y_1,...,y_k)/2,\tilde{y})\right|^2 = \frac{1}{4}\left|\partial_{x_j^a}\widetilde{\psi}-\partial_{x_j^b}\widetilde{\psi}\right|^2 \le \frac{1}{2}\left(\left|\partial_{x_j^a}\widetilde{\psi}\right|^2+\left|\partial_{x_j^b}\widetilde{\psi}\right|^2\right). \quad (2.132)
$$

Therefore,

$$
K[\psi]\le 4^k\left(\tilde{a}\|\nabla\widetilde{\psi}\|_2^2+b\|\widetilde{\psi}\|_2^2\right)=4^k\tilde{a}\|\nabla\psi\|_2^2+4^kb\|\psi\|_2^2. \quad (2.133)
$$

For any $0<a<1$, pick $\tilde{a}=2^{-2k-1}\min(m_a^{-1},m_b^{-1})a$ to obtain $K[\psi]\le aq_k[\psi]+4^kb\|\psi\|_2^2$. □

Lemma 2.A.4. *The quadratic forms defined in the proof of Proposition 2.2.1 in Eqs.* (2.21) *and* (2.24) *correspond to unique self-adjoint operators.*

Proof. In all cases we prove that the potential term in the quadratic form is infinitesimally bounded with respect to the kinetic energy term. The claim then follows from the KLMN theorem.

Let us begin with the quadratic form $h_{k-1}^{l,L}$ in (2.24). The idea is to use the same mirroring argument as in Prop. 2.A.3 for the coordinate components from $l+1$ to k. In the first $l-1$ components, we extend the triangular domain in Figure 2.1 via a suitable mirroring, in order to be able to apply Lemma 2.A.2. To be precise, we define the map ϕ taking $(0,L/\delta)\times\left(-\frac{ML}{m_b\delta},\frac{ML}{m_a\delta}\right)$ to the triangular domain $\{(z,y)\in(0,L/\delta)\times\mathbb{R}|-\frac{M}{m_b}z<y<\frac{M}{m_a}z\}$ as

$$
\phi(z,y)=(z,y) \qquad \text{if } x^a=z+\frac{m_b}{M}y\ge 0 \text{ and } x^b=z-\frac{m_a}{M}y\ge 0 \quad (2.134)
$$

$$
\phi(z,y)=\left(\frac{m_a}{M}y,\frac{M}{m_a}z\right) \qquad \text{if } x^b\le 0 \quad (2.135)
$$

$$
\phi(z,y)=\left(\frac{m_b}{M}y,\frac{M}{m_b}z\right) \qquad \text{if } x^a\le 0 \quad (2.136)
$$

Let us use the notation $\phi=(\phi_1,\phi_2)$. Note that for a function f defined on the triangular domain, we have

$$
\|f\circ\phi\|_2^2=2\|f\|_2^2, \quad (2.137)
$$

where one contribution of $\|f\|_2^2$ comes from the triangular domain, and the second $\|f\|_2^2$ is the sum of the contributions with $x^b<0$ and $x^a<0$. In the region with $x^b<0$ we have

$$\int_0^{\frac{ML}{m_a\delta}}\mathrm{d}y\int_0^{\frac{m_a y}{M}}\mathrm{d}z|f(\phi(z,y))|^2=\int_0^{\frac{ML}{m_a\delta}}\mathrm{d}y\int_0^{\frac{m_a y}{M}}\mathrm{d}z|f(m_a y/M,Mz/m_a)|^2$$
$$=\int_0^{L/\delta}\mathrm{d}\tilde z\int_0^{\frac{M\tilde z}{m_a}}\mathrm{d}\tilde y|f(\tilde z,\tilde y)|^2, \quad (2.138)$$

where we substituted $\tilde z=m_a y/M$ and $\tilde y=Mz/m_a$. Similarly, for $x^a<0$

$$\int_{-\frac{ML}{m_b\delta}}^0\mathrm{d}y\int_0^{\frac{m_b y}{M}}\mathrm{d}z|f(\phi(z,y))|^2=\int_0^{L/\delta}\mathrm{d}\tilde z\int_{-\frac{M\tilde z}{m_b}}^0\mathrm{d}\tilde y|f(\tilde z,\tilde y)|^2. \quad (2.139)$$

Moreover, if $f\in H^1$, then $f\circ\phi\in H^1$ by the Lipschitz continuity of ϕ.

Let us work in center of mass and relative coordinates in the first l components, and with the x^a and x^b coordinates in components $l+1$ to k. The kinetic part of $h_{k-1}^{l,L}$ is then

$$q_{k-1}^{l,L}[\psi]:=\int_{\Omega_{k-1}^{l,L}}\Bigg[\sum_{j=1}^{l-1}\left(\frac{1}{2M}|\nabla_{z_j}\psi|^2+\frac{1}{2\mu}|\nabla_{y_j}\psi|^2\right)+\frac{1}{2\mu}|\nabla_{y_l}\psi|^2+\sum_{j=l+1}^{k}\Bigg(\frac{1}{2m_a}|\nabla_{x_j^a}\psi|^2$$
$$+\frac{1}{2m_b}|\nabla_{x_j^b}\psi|^2\Bigg)+\frac{1}{2\mu}|\nabla_{\tilde y}\psi|^2\Bigg]\mathrm{d}z_1\dots\mathrm{d}z_{l-1}\mathrm{d}x_{l+1}^a\dots\mathrm{d}x_k^a\mathrm{d}y_1\dots\mathrm{d}y_l\mathrm{d}x_{l+1}^b\dots\mathrm{d}x_k^b\mathrm{d}\tilde y. \quad (2.140)$$

For $\psi\in H^1(\Omega_{k-1}^{l,L})$ define $\widetilde\psi$ on

$$\widetilde\Omega_{k-1}^{l,L}:=\Big\{(z_1,\dots z_{l-1},x_{l+1}^a,\dots x_k^a,y_1,\dots,y_l,x_{l+1}^b,\dots x_k^b,\tilde y)|\forall j<l: z_j\in(0,L/\delta),$$
$$y_j\in\left(-\frac{ML}{m_b\delta},\frac{ML}{m_a\delta}\right),y_l\in(-L,L),\forall l<j\le k: x_j^a\in\mathbb{R},x_j^b\in\mathbb{R},\tilde y\in\mathbb{R}^{d-k}\Big\} \quad (2.141)$$

as

$$\widetilde\psi(z,y)=\frac{1}{2^{(l-1)/2}}\frac{1}{2^{k-l}}\psi\left((\phi_1(z_j,y_j))_{j=1}^{l-1},(|x_j^a|)_{j=l+1}^k,(\phi_2(z_j,y_j))_{j=1}^{l-1},y_l,(|x_j^b|)_{j=l+1}^k,\tilde y\right). \quad (2.142)$$

By (2.137) we have $\|\widetilde\psi\|_2^2=\|\psi\|_2^2$. Furthermore, $\|\nabla\widetilde\psi\|_2^2\le\left(\frac{M^2}{\min\{m_a,m_b\}^2}+1\right)^{l-1}\|\nabla\psi\|_2^2$.

Analogously to (2.129)-(2.130) we obtain

$$K[\psi]:=\Bigg|\int_{\Omega_{k-1}^{l,L}}V(y_1,\dots y_l,x_{l+1}^a-x_{l+1}^b,\dots x_k^a-x_k^b,\tilde y)|\psi|^2$$
$$\mathrm{d}z_1\dots\mathrm{d}z_{l-1}\mathrm{d}x_{l+1}^a\dots\mathrm{d}x_k^a\mathrm{d}y_1\dots\mathrm{d}y_l\mathrm{d}x_{l+1}^b\dots\mathrm{d}x_k^b\mathrm{d}\tilde y\Bigg|$$
$$\le 2^{l-1}4^{k-l}\int_{\widetilde\Omega_{k-1}^{l,L}}|V(y)||\widetilde\psi(z_1,\dots z_{l-1},(w_j+\frac{y_j}{2})_{j=l+1}^k,y_1,\dots,y_l,(w_j-\frac{y_j}{2})_{j=l+1}^k,\tilde y)|^2$$
$$\mathrm{d}z_1\dots\mathrm{d}z_{l-1}\mathrm{d}w_{l+1}\dots\mathrm{d}w_k\mathrm{d}y, \quad (2.143)$$

where we changed the coordinates x_j^a,x_j^b to $w_j=\frac{x^a+x^b}{2}$ and y_j. Let $D_y=\left(-\frac{ML}{m_b\delta},\frac{ML}{m_a\delta}\right)^{l-1}\times(-L,L)\times\mathbb{R}^{d-k}$. For almost every $(z_1,\dots z_{l-1},w_{l+1},\dots w_k)\in(0,L/\delta)^{l-1}\times\mathbb{R}^{k-l}$, the function

$f(y) = \widetilde{\psi}(z_1, ...z_{l-1}, (w_j + \frac{y_j}{2})_{j=l+1}^k, y_1, ..., y_l, (w_j - \frac{y_j}{2})_{j=l+1}^k, \tilde{y})$ lies in $H^1(D_y)$ by Fubini's theorem. Applying Lemma 2.A.2 with $\Omega = D_y$ and integrating over z and w one obtains

$$K[\psi] \leq 2^{l-1}4^{k-l}a \int_{\tilde{\Omega}_{k-1}^{l,L}} \left|\nabla_y \widetilde{\psi}(z_1, ...z_{l-1}, (w_j + \frac{y_j}{2})_{j=l+1}^k, y_1, ..., y_l, (w_j - \frac{y_j}{2})_{j=l+1}^k, \tilde{y})\right|^2 \mathrm{d}z_1 \dots \mathrm{d}z_{l-1}\mathrm{d}w_{l+1} \dots \mathrm{d}w_k \mathrm{d}y + 2^{l-1}4^{k-l}b\|\widetilde{\psi}\|_2^2 \quad (2.144)$$

for any $a > 0$ and a suitable constant b. As in (2.132) we have

$$K[\psi] \leq 2^{l-1}4^{k-l}\left(a\|\nabla\widetilde{\psi}\|_2^2 + b\|\widetilde{\psi}\|_2^2\right) \leq 2^{l-1}4^{k-l}\left(\frac{M^2}{\min\{m_a, m_b\}^2} + 1\right)^{l-1} a\|\nabla\psi\|_2^2 + 2^{l-1}4^{k-l}b\|\psi\|_2^2. \quad (2.145)$$

Since a can be arbitrarily small, the interaction term is infinitesimally bounded w.r.t. $q_{k-1}^{l,L}$.

Let us now consider the quadratic form a_l in (2.21). For $l = k+2$, the potential term is bounded from below since $|y| > L$, and is hence infinitesimally bounded w.r.t the kinetic energy.

The kinetic part of a_l is

$$q_l[\psi] := \int_{\Omega_l} \left[\sum_{j=1}^{l} \left(\frac{1}{2M}|\nabla_{z_j}\psi|^2 + \frac{1}{2\mu}|\nabla_{y_j}\psi|^2\right) + \sum_{j=l+1}^{k} \left(\frac{1}{2m_a}|\nabla_{x_j^a}\psi|^2 + \frac{1}{2m_b}|\nabla_{x_j^b}\psi|^2\right) + \frac{1}{2\mu}|\nabla_{\tilde{y}}\psi|^2 \right] \mathrm{d}z_1 \dots \mathrm{d}z_l \mathrm{d}x_{l+1}^a \dots \mathrm{d}x_k^a \mathrm{d}y_1 \dots \mathrm{d}y_l \mathrm{d}x_{l+1}^b \dots \mathrm{d}x_k^b \mathrm{d}\tilde{y}. \quad (2.146)$$

First, we consider $1 \leq l \leq k$. Then, a_l is closely related to $h_{k-1}^{l,L}$ through (2.25). Let $\psi \in H^1(\Omega_l)$. For every $z_l \in (L/\delta, \infty)$, the function $\psi(\cdot, \dots, z_l, \dots, \cdot)$ belongs to $H^1(\Omega_{k-1}^{l,L})$. In (2.143)-(2.145), we saw that for any $a > 0$ there is a constant b such that

$$\int_{\Omega_{k-1}^{l,L}} |V(y)|\,|\psi(z,y)|^2 \mathrm{d}y\mathrm{d}z_1 \dots \widehat{\mathrm{d}z_l} \dots \mathrm{d}z_k \leq a q_{k-1}^{l,L}[\psi(\cdot, z_l, \cdot)] + b\int |\psi(z,y)|^2 \mathrm{d}y\mathrm{d}z_1 \dots \widehat{\mathrm{d}z_l} \dots \mathrm{d}z_k. \quad (2.147)$$

Integrating the inequality over z_l, we obtain

$$\int_{\Omega_l} |V(y)||\psi(z,y)|^2 \mathrm{d}y\mathrm{d}z \leq a\int_{L/\delta}^{\infty} q_{k-1}^{l,L}[\psi(\cdot, z_l, \cdot)]\mathrm{d}z_l + b\|\psi\|_2^2 \leq a q_l[\psi] + b\|\psi\|_2^2. \quad (2.148)$$

Hence, the potential term is infinitesimally bounded w.r.t q_l.

For $l = k+1$, we use the map ϕ in the first k components. For $\psi \in H^1(\Omega_{k+1})$ define $\widetilde{\psi}$ on

$$\widetilde{\Omega}_{k+1} := (0, L/\delta)^k \times \left(-\frac{ML}{m_b\delta}, \frac{ML}{m_a\delta}\right)^k \times (-L, L)^{d-k} \quad (2.149)$$

as

$$\widetilde{\psi}(z,y) = \frac{1}{2^{k/2}}\psi\left((\phi_1(z_j, y_j))_{j=1}^k, (\phi_2(z_j, y_j))_{j=1}^k, \tilde{y}\right). \quad (2.150)$$

By (2.137) we have $\|\widetilde{\psi}\|_2^2 = \|\psi\|_2^2$. Furthermore, $\|\nabla\widetilde{\psi}\|_2^2 \leq \left(\frac{M^2}{\min\{m_a,m_b\}^2}+1\right)^k \|\nabla\psi\|_2^2$. Analogously to (2.129)-(2.130) we obtain

$$K[\psi] := \left|\int_{\Omega_{k+1}} V(y)|\psi(z,y)|^2 \mathrm{d}z\mathrm{d}y\right| \leq 2^k \int_{\widetilde{\Omega}_{k+1}} |V(y)||\widetilde{\psi}(z,y)|^2 \mathrm{d}z\mathrm{d}y. \tag{2.151}$$

Let $D_y = \left(-\frac{ML}{m_b\delta}, \frac{ML}{m_a\delta}\right)^k \times (-L,L)^{d-k}$. For almost every $z \in (0, L/\delta)^k$, the function $f(y) = \widetilde{\psi}(z,y)$ lies in $H^1(D_y)$ by Fubini's theorem. Applying Lemma 2.A.2 with $\Omega = D_y$ and integrating over z gives

$$K[\psi] \leq 2^k a \int_{\widetilde{\Omega}_{k+1}} \left|\nabla_y \widetilde{\psi}(z,y)\right|^2 \mathrm{d}z\mathrm{d}y + 2^k b\|\widetilde{\psi}\|_2^2 \leq 2^k a\|\nabla\widetilde{\psi}\|_2^2 + 2^k b\|\widetilde{\psi}\|_2^2 \tag{2.152}$$

for any $a > 0$ and a suitable constant b. Hence,

$$K[\psi] \leq 2^k \left(\frac{M^2}{\min\{m_a,m_b\}^2}+1\right)^k a\|\nabla\psi\|_2^2 + 2^k b\|\psi\|_2^2. \tag{2.153}$$

Since a can be arbitrarily close to zero, the interaction term is infinitesimally bounded w.r.t. q_{k+1}. □

Lemma 2.A.5. *The quadratic forms defined in the proof of Theorem 2.1.4 in Eqs. (2.52), (2.58), (2.67), (2.73), (2.79), (2.86) and (2.91) correspond to unique self-adjoint operators.*

Proof. The quadratic forms a_j with $j \in \{1,2,4,5\}$ in Eqs. (2.52) and (2.73) and the forms $a_{3,j}$ for $1 \leq j \leq k$ in (2.67) have the form

$$a_j[\varphi] = \int_{\Omega_j} \left(\frac{1}{2m_a}|\nabla_{x^a}\varphi|^2 + \frac{1}{2m_b}|\nabla_{x^b}\varphi|^2 + \frac{1}{2\mu}|\nabla_{\tilde{y}}\varphi|^2 \right.$$
$$\left. + \left(V(x^a - x^b, \tilde{y}) + V_\infty(x^a, x^b, \tilde{y})\right)|\varphi|^2\right) \mathrm{d}x^a \mathrm{d}x^b \mathrm{d}\tilde{y} \tag{2.154}$$

for some bounded potential V_∞. The quadratic form $q_j : H^1(\Omega_j) \to \mathbb{R}$ given by

$$q_j[\varphi] = \int_{\Omega_j} \left(\frac{1}{2m_a}|\nabla_{x^a}\varphi|^2 + \frac{1}{2m_b}|\nabla_{x^b}\varphi|^2 + \frac{1}{2\mu}|\nabla_{\tilde{y}}\varphi|^2\right) \mathrm{d}x^a \mathrm{d}x^b \mathrm{d}\tilde{y} \tag{2.155}$$

is closed and bounded from below. Using that $\varphi \in D[a_j]$ vanishes outside $\overline{\Omega_j}$ and applying Proposition 2.A.3, we obtain

$$\left|\int_{\Omega_j} V(x^a - x^b, \tilde{y}) + V_\infty(x^a, x^b, \tilde{y})|\varphi|^2\right| \leq \left|\int_{Q_k \times \mathbb{R}^{d-k}} V(y)|\varphi|^2\right| + \|V_\infty\|_\infty \|\varphi\|_2^2$$
$$\leq a q_j[\varphi] + (b + \|V_\infty\|_\infty)\|\varphi\|_2^2 \tag{2.156}$$

for some $a < 1$ and $b \in \mathbb{R}$. By the KLMN theorem, there is a unique self-adjoint operator A_j corresponding to a_j.

For $\hat{a}_4$ in (2.79), note that K_R is bounded. Adapting the argument in Proposition 2.A.3, we show that the interaction term is infinitesimally bounded with respect to the kinetic part $\hat{q} : H^1(((0,\infty)^{k-1} \times \mathbb{R})^2 \times \mathbb{R}^{d-k}) \to \mathbb{R}$ given by

$$\hat{q}[\varphi] = \int_{\hat{\Omega}_4} \left(\frac{1}{2m_a}|\nabla_{x^a}\varphi|^2 + \frac{1}{2m_b}|\nabla_{x^b}\varphi|^2 + \frac{1}{2\mu}|\nabla_{\tilde{y}}\varphi|^2\right) \mathrm{d}x^a \mathrm{d}x^b \mathrm{d}\tilde{y}. \tag{2.157}$$

For $\psi \in H^1(\hat{\Omega}_4)$, define $\widetilde{\psi}(x^a, x^b, \tilde{y}) = \frac{1}{2^{k-1}}\psi((|x^a_j|)_{j=1}^{k-1}, x^a_k, (|x^b_j|)_{j=1}^{k-1}, x^b_k, \tilde{y})$ for $(x^a, x^b, \tilde{y}) \in \mathbb{R}^k \times \mathbb{R}^k \times \mathbb{R}^{d-k}$. We have $\|\widetilde{\psi}\|_2^2 = \|\psi\|_2^2$ and $\|\nabla\widetilde{\psi}\|_2^2 = \|\nabla\psi\|_2^2$. Following the same steps as in Proposition 2.A.3 from (2.129)-(2.133) with this adapted choice of $\widetilde{\psi}$, we obtain that for any $0 < a$ there is a b such that

$$K[\psi] := \left|\int_{\hat{\Omega}_4} V(x^a - x^b, \tilde{y})|\psi|^2 \mathrm{d}x^a \mathrm{d}x^b \mathrm{d}\tilde{y}\right| \leq 4^{k-1}\left(a\|\nabla\widetilde{\psi}\|_2^2 + b\|\widetilde{\psi}\|_2^2\right)$$
$$= 4^{k-1}a\|\nabla\psi\|_2^2 + 4^k b\|\psi\|_2^2. \quad (2.158)$$

By the KLMN theorem, $\hat{a}_4$ corresponds to a self-adjoint operator. Since b_1 in (2.86) differs from $\hat{a}_4$ by a bounded term, it also corresponds to a self-adjoint operator. For b_2 in (2.91) and $a_{1,\text{ext}}$ in (2.58), the potential is bounded. Thus, these forms also correspond to self-adjoint operators.

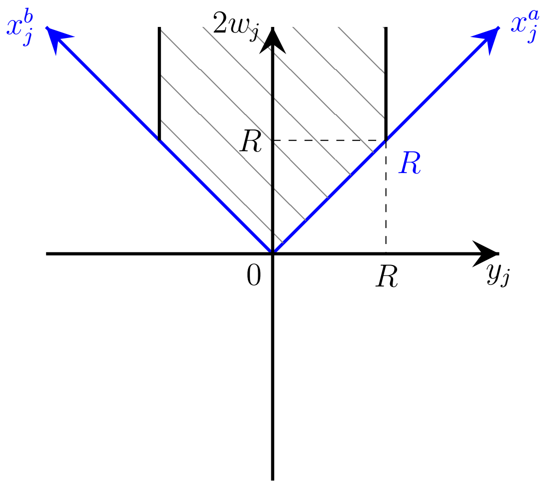

Figure 2.4: In the domain of ψ for $1 \leq j \leq k$, the coordinates (x^a_j, x^b_j) lie in the hatched set. We have $y_j = x^a_j - x^b_j$ and $w_j = \frac{x^a_j + x^b_j}{2}$.

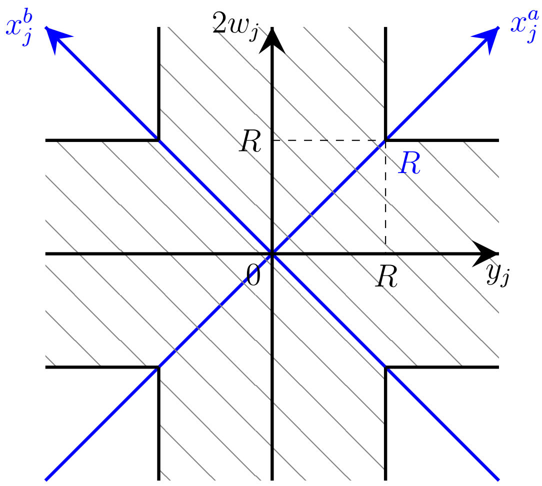

Figure 2.5: Mirroring ψ along $x^a_j = 0$ and $x^b_j = 0$ defines $\tilde{\psi}$. For $1 \leq j \leq k$, the coordinates (x^a_j, x^b_j) or equivalently (w_j, y_j) lie in the hatched set.

For $a_{1,\text{int}}$ in (2.58), we proceed similarly to Proposition 2.A.3. Let $\psi \in D[a_{1,\text{int}}]$. The domain of ψ is sketched in Figure 2.4. Mirroring the domain along the $x^a_j = 0$ and $x^b_j = 0$ hyperplanes, we obtain the set $\tilde{\Omega}$ sketched in Figure 2.5. For $(x^a, x^b, \tilde{y}) \in \tilde{\Omega}$ define $\tilde{\psi}(x^a, x^b, \tilde{y}) = \frac{1}{2^k}\psi((|x^a_j|)_j, (|x^b_j|)_j, \tilde{y})$. We have $\|\tilde{\psi}\|_2^2 = \|\psi\|_2^2$ and $\|\nabla\tilde{\psi}\|_2^2 = \|\nabla\psi\|_2^2$. Using the triangle inequality and enlarging the domain of integration to $\tilde{\Omega}$, we have

$$K[\psi] := \left|\int_{\Omega_{1,\text{int}}} V(x^a - x^b, \tilde{y})|\psi(x^a, x^b, \tilde{y})|^2 \mathrm{d}x^a \mathrm{d}x^b \mathrm{d}\tilde{y}\right|$$
$$\leq 4^k \int_{\tilde{\Omega}} |V(x^a - x^b, \tilde{y})||\tilde{\psi}(x^a, x^b, \tilde{y})|^2 \mathrm{d}x^a \mathrm{d}x^b \mathrm{d}\tilde{y}. \quad (2.159)$$

We change to coordinates $w = \frac{x^a + x^b}{2}$ and y. For every $w \in \mathbb{R}^k$, the set

$$\Omega_w = \{\ y \in \mathbb{R}^d | (w + (y_1, ..., y_k)/2, w - (y_1, ..., y_k)/2, \tilde{y}) \in \tilde{\Omega}\} \quad (2.160)$$

is equal to $I_1 \times ... \times I_k \times \mathbb{R}^{d-k}$, where each $I_j \in \{\mathbb{R}, (-R, R)\}$ (Figure 2.5). Thus, there is an angle θ and radius r such that all the sets Ω_w satisfy the cone property with parameters

θ, r. For almost every $w \in \mathbb{R}^k$, the function $f(y) = \widetilde{\psi}(w + (y_1, ..., y_k)/2, w - (y_1, ..., y_k)/2, \tilde{y})$ lies in $H^1(\Omega_w)$. By Lemma 2.A.2, for any $0 < \tilde{a}$ there is a constant b independent of f_w and w such that

$$\int_{\Omega_w} |V(y)||f(y)|^2 \mathrm{d}y \leq \tilde{a}\|\nabla f\|_2^2 + b\|f\|_2^2. \tag{2.161}$$

Integrating inequality (2.161) over w and using (2.132) gives

$$\int_{\tilde{\Omega}} |V(x^a - x^b, \tilde{y})||\widetilde{\psi}(x^a, x^b, \tilde{y})|^2 \mathrm{d}x^a \mathrm{d}x^b \mathrm{d}\tilde{y} \leq \tilde{a}\|\nabla_y \widetilde{\psi}\|^2 + b\|\widetilde{\psi}\|_2^2 \leq \tilde{a}\|\nabla\widetilde{\psi}\|^2 + b\|\widetilde{\psi}\|_2^2. \tag{2.162}$$

In total, we thus have

$$K[\psi] \leq 4^k \tilde{a}\|\nabla\psi\|_2^2 + 4^k b\|\psi\|_2^2. \tag{2.163}$$

For any $0 < a < 1$, pick $\tilde{a} = 2^{-2k-1}\min(m_a^{-1}, m_b^{-1})a$ to obtain $K[\psi] \leq a q_{1,\mathsf{int}}[\psi] + 4^k b\|\psi\|_2^2$. The KLMN theorem thus implies that there is a self-adjoint $A_{1,\mathsf{int}}$, which is bounded from below. □

2.B Exponential decay of Schrödinger eigenfunctions (by Rupert L. Frank[1])

It is a folklore theorem that eigenfunctions of Schrödinger operators corresponding to eigenvalues below the bottom of their essential spectrum decay exponentially. This was raised to high art by Agmon [3] and others; see, for instance, the review [66]. It may be of interest to note that the most basic one of these bounds holds under rather minimal assumptions of the potential. This is what we record here.

Let $V \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ be real and set $V_\pm := \max\{\pm V, 0\}$. Given $\alpha \in [0,1]$, we say that V_- is $-\Delta$-form bounded with form bound α if there is a $C_\alpha < \infty$ such that

$$\int_{\mathbb{R}^d} V_-|\psi|^2\,dx \leq \alpha \int_{\mathbb{R}^d} |\nabla\psi|^2\,dx + C_\alpha \int_{\mathbb{R}^d} |\psi|^2\,dx \qquad \text{for all } \psi \in H^1(\mathbb{R}^d)\,.$$

In this case, we define a quadratic form h by

$$D[h] := \left\{\psi \in H^1(\mathbb{R}^d) : \int_{\mathbb{R}^d} V_+|\psi|^2\,dx < \infty\right\},$$
$$h[\psi] := \int_{\mathbb{R}^d} \left(|\nabla\psi|^2 + V|\psi|^2\right)dx \qquad \text{for } \psi \in D[h]\,.$$

This quadratic form is lower semibounded in $L^2(\mathbb{R}^d)$ and, if $\alpha < 1$, closed. Thus, it corresponds to a selfadjoint, lower semibounded operator, which we denote by $-\Delta + V$. We abbreviate

$$E_\infty := \inf \sigma_{\mathrm{ess}}(-\Delta + V) \in \mathbb{R} \cup \{+\infty\}.$$

Theorem 2.B.1. *Assume that $V_+ \in L^1_{loc}(\mathbb{R}^d)$ and that V_- is $-\Delta$-form bounded with bound < 1. For every $E' < E_\infty$ there is a constant $C_{E'} < \infty$ such that if $E \leq E'$ and if $\psi \in D(-\Delta + V)$ satisfies $(-\Delta + V)\psi = E\psi$, then*

$$\int_{\mathbb{R}^d} e^{2\sqrt{E'-E}\,|x|}\left(|\nabla\psi|^2 + V_+|\psi|^2 + (E' - E)|\psi|^2\right)dx \leq C_{E'}\,\|\psi\|^2\,. \tag{2.164}$$

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We emphasize that E_∞ may be equal to $+\infty$, in which case E' may be taken arbitrarily large. If $E_\infty < \infty$, the decay exponent $\sqrt{E'-E}$ can be any number $< \sqrt{E_\infty - E}$.

Note that under the assumptions of the theorem, ψ is not necessarily bounded, so one cannot expect pointwise exponential decay bounds. The bounds in the theorem control the quantities that are natural from the definition of the operator in the form sense.

In order to prove Theorem 2.B.1, we use a geometric characterization of the bottom of the essential spectrum due to Persson [57]. Let $K \subset \mathbb{R}^d$ be a compact set and define

$$E_1(-\Delta + V|_{\mathbb{R}^d\setminus K}) = \inf \left\{ \frac{h[\psi]}{\|\psi\|^2} :\ \psi \in D[h],\, \psi \equiv 0 \text{ on } K \right\}.$$

Clearly, $E_1(-\Delta + V|_{\mathbb{R}^d\setminus K})$ is nondecreasing in K and therefore its supremum over all compact $K \subset \mathbb{R}^d$ exists in $\mathbb{R} \cup \{+\infty\}$.

Theorem 2.B.2. *Assume that $V_+ \in L^1_{loc}(\mathbb{R}^d)$ and that V_- is $-\Delta$-form bounded with bound < 1. Then*

$$E_\infty = \sup_{K\subset\mathbb{R}^d \text{ compact}} E_1(-\Delta + V|_{\mathbb{R}^d\setminus K}) .$$

We first assume Theorem 2.B.2 and show how it implies Theorem 2.B.1. Then we will provide a proof of Theorem 2.B.2 under our assumptions on V.

Proof of Theorem 2.B.1. Fix $E_\infty > E'' > E'$. By Theorem 2.B.2, there is an $R' > 0$ such that

$$h[u] \geq E''\|u\|^2$$

for all $u \in D[h]$ with $u \equiv 0$ in $\overline{B_{R'/2}}$. Next, for an $R > 0$ to be specified, we choose two smooth, real-valued functions $\chi_<$ and $\chi_>$ on $\mathbb{R}^d$ such that

$$\operatorname{supp} \chi_< \subset \overline{B_{2R}} \quad \text{and} \quad \operatorname{supp} \chi_> \subset \mathbb{R}^d \setminus B_R \tag{2.165}$$

and such that $\chi_<^2 + \chi_>^2 \equiv 1$ on $\mathbb{R}^d$. By scaling an R-independent quadratic partition of unity, we may assume that

$$|\nabla\chi_<|^2 + |\nabla\chi_>|^2 \leq CR^{-2} \tag{2.166}$$

with a constant C independent of R. By increasing R' if necessary, we can make sure that $C(R')^{-2} \leq (E''-E')/2 =: \epsilon$ with C from (2.166). Let $f : \mathbb{R}^d \to \mathbb{R}$ be a bounded Lipschitz function and take $\varphi = e^{2f}\psi \in D[h]$ as a trial function in the quadratic form version of the equation $(-\Delta + V)\psi = E\psi$ to obtain, after an integration by parts,

$$E \int_{\mathbb{R}^d} e^{2f}|\psi|^2\,dx = \int_{\mathbb{R}^d} \left(|\nabla(e^f\psi)|^2 + (V - |\nabla f|^2)|e^f\psi|^2 \right) dx\,. \tag{2.167}$$

Thus, in view of the IMS formula (see, e.g., [14, Theorem 3.2]),

$$\begin{aligned} E \int_{\mathbb{R}^d} |e^f\chi_<\psi|^2\,dx + E \int_{\mathbb{R}^d} |e^f\chi_>\psi|^2\,dx &= \int_{\mathbb{R}^d} \left(|\nabla(e^f\chi_<\psi)|^2 + \tilde V|e^f\chi_<\psi|^2 \right) dx \\ &\quad + \int_{\mathbb{R}^d} \left(|\nabla(e^f\chi_>\psi)|^2 + \tilde V|e^f\chi_>\psi|^2 \right) dx \end{aligned}$$

with $\tilde V := V - |\nabla f|^2 - |\nabla\chi_<|^2 - |\nabla\chi_>|^2$. For $R \geq R'$ we bound the terms on the right side from below by

$$\int_{\mathbb{R}^d} \left(|\nabla(e^f\chi_<\psi)|^2 + \tilde V|e^f\chi_<\psi|^2 \right) dx \geq \left(E_1 - \|\nabla f\|_\infty^2 - \epsilon \right) \int_{\mathbb{R}^d} |e^f\chi_<\psi|^2\,dx$$

with $E_1 := \inf \sigma(-\Delta + V)$, and

$$\int_{\mathbb{R}^d} \left(|\nabla(e^f \chi_> \psi)|^2 + \tilde{V} |e^f \chi_> \psi|^2 \right) dx \geq \left(E'' - \|\nabla f\|_\infty^2 - \epsilon \right) \int_{\mathbb{R}^d} |e^f \chi_> \psi|^2 \, dx \, .$$

Thus,

$$\left(E'' - E - \|\nabla f\|_\infty^2 - \epsilon \right) \int_{\mathbb{R}^d} |e^f \chi_> \psi|^2 \, dx \leq \left(E - E_1 + \|\nabla f\|_\infty^2 + \epsilon \right) \int_{\mathbb{R}^d} |e^f \chi_< \psi|^2 \, dx \, ,$$

and therefore

$$\begin{aligned} \left(E'' - E - \|\nabla f\|_\infty^2 - \epsilon \right) \int_{\mathbb{R}^d} |e^f \psi|^2 \, dx &\leq (E'' - E_1) \int_{\mathbb{R}^d} |e^f \chi_< \psi|^2 \, dx \\ &\leq (E'' - E_1) \, \|\psi\|^2 \, \sup_{B_R} e^{2f} \, . \end{aligned}$$

Ideally, we would want to choose $f(x) = \kappa |x|$ with κ as large as possible. The wish to have a positive constant (ϵ, say) in front of the integral on the left side then dictates our choice $\kappa = \sqrt{E'' - E - 2\epsilon} = \sqrt{E' - E}$. The problem with this 'ideal' choice of f is that the function $|x|$ is Lipschitz, but not bounded. We remedy this by taking $|x|/(1 + \delta|x|)$ instead and proving bounds which are uniform in the parameter $\delta > 0$, which we will let tend to zero at the end. Thus, let us choose

$$f(x) := \sqrt{E' - E} \, \frac{|x|}{1 + \delta |x|}$$

with a (small) parameter $\delta > 0$. This is a Lipschitz function satisfying $\|\nabla f\|_\infty = \sqrt{E' - E}$. Thus, the previous inequality with $R = R'$ becomes

$$\epsilon \int_{\mathbb{R}^d} |e^f \psi|^2 \, dx \leq (E'' - E_1) \, \|\psi\|^2 \, e^{2R' \sqrt{E' - E}} \, .$$

Since the right side is independent of δ, we can take the limit $\delta \to 0$ and obtain by monotone convergence

$$\epsilon \int_{\mathbb{R}^d} |e^{\sqrt{E' - E}|x|} \psi|^2 \, dx \leq (E'' - E_1) \, \|\psi\|^2 \, e^{2R' \sqrt{E' - E}} \, .$$

This is already one of the inequalities claimed in the theorem.

To prove boundedness of the terms involving the gradient term and V_+ we recall that, by form boundedness,

$$h[e^f \psi] \geq (1 - \alpha) \int_{\mathbb{R}^d} |\nabla(e^f \psi)|^2 \, dx + \int_{\mathbb{R}^d} V_+ |e^f \psi|^2 \, dx - C_\alpha \int_{\mathbb{R}^d} |e^f \psi|^2 \, dx \, .$$

This, together with identity (2.167), implies

$$\left(E + \|\nabla f\|_\infty^2 + C_\alpha \right) \int_{\mathbb{R}^d} |e^f \psi|^2 \, dx \geq (1 - \alpha) \int_{\mathbb{R}^d} |\nabla(e^f \psi)|^2 \, dx + \int_{\mathbb{R}^d} V_+ |e^f \psi|^2 \, dx \, .$$

Using

$$\begin{aligned} |\nabla(e^f \psi)|^2 = e^{2f} |\nabla \psi + \psi \nabla f|^2 &= e^{2f} \left(|\nabla \psi|^2 + 2 \operatorname{Re} \overline{\psi} \nabla \psi \cdot \nabla f + |\psi|^2 |\nabla f|^2 \right) \\ &\geq e^{2f} \left(\frac{1}{2} |\nabla \psi|^2 - |\psi|^2 |\nabla f|^2 \right) , \end{aligned}$$

we obtain

$$\left(E + (2 - \alpha) \|\nabla f\|_\infty^2 + C_\alpha \right) \int_{\mathbb{R}^d} |e^f \psi|^2 \, dx \geq \frac{1 - \alpha}{2} \int_{\mathbb{R}^d} |e^f \nabla \psi|^2 \, dx + \int_{\mathbb{R}^d} V_+ |e^f \psi|^2 \, dx \, .$$

Since we have already shown an upper bound on the left side, this completes the proof of the theorem. □

Thus, we are left with proving Theorem 2.B.2. We use the following abstract characterization of the essential spectrum.

Lemma 2.B.3. *Let a be a lower semibounded, closed quadratic form in a Hilbert space and A the corresponding self-adjoint operator. Then*

$$\inf \sigma_{ess}(A) = \inf \left\{ \liminf_{j\to\infty} a[\xi_j] : \ \xi_j \rightharpoonup 0 \, , \ \|\psi_j\| = 1 \right\}$$

(with the convention that $\inf \emptyset = +\infty$). Moreover, if both sides are finite, then there is a sequence (ξ_j) with $\|\xi_j\| = 1$, $a[\xi_j] \to \inf \sigma_{ess}(A)$ and $\xi_j \rightharpoonup 0$ in $D[a]$.

This lemma is classical. The proof in [25, Lemma 1.20] shows the first assertion and, in the case of finiteness, the existence of a normalized sequence with $a[\xi_j] \to \inf \sigma_{\mathrm{ess}}(A)$ and $\xi_j \rightharpoonup 0$. Since this sequence is bounded in $D[a]$, a subsequence converges weakly in $D[a]$ and, since $D[a]$ is continuously embedded into the Hilbert space, the weak limit is necessarily zero, as claimed.

Proof of Theorem 2.B.2. We abbreviate $E'_\infty := \sup_{K \text{ compact}} E_1(-\Delta + V|_{\mathbb{R}^d \setminus K})$.

We begin by proving $E_\infty \geq E'_\infty$. We may assume that $E_\infty < \infty$ and we shall show that for all $R > 0$,

$$E_1(-\Delta + V|_{B_R^c}) \leq E_\infty \, , \tag{2.168}$$

for then the claimed inequality follows as $R \to \infty$. Fix $R > 0$ and let $\chi_<$ and $\chi_>$ be as in the proof of Theorem 2.B.1. By Lemma 2.B.3, there is a sequence $(\xi_j) \subset D[h]$ with $\|\xi_j\| = 1$ such that $\xi_j \rightharpoonup 0$ in $D[h]$ and $h[\xi_j] \to E_\infty$. Then

$$E_1(-\Delta + V|_{B_R^c}) \leq h\left[\frac{\chi_> \xi_j}{\|\chi_> \xi_j\|} \right] \tag{2.169}$$

and our goal is to estimate the right side as $j \to \infty$.

By Rellich's compactness theorem, $\xi_j \to 0$ in $L^2_{\mathrm{loc}}(\mathbb{R}^d)$, so $\chi_< \xi_j \to 0$ in $L^2(\mathbb{R}^d)$ and

$$\|\chi_> \xi_j\|^2 = \|\xi_j\|^2 - \|\chi_< \xi_j\|^2 \to 1 \qquad \text{as } j \to \infty \, . \tag{2.170}$$

Moreover, by the IMS formula,

$$h\left[\chi_> \xi_j\right] = h\left[\xi_j\right] - h\left[\chi_< \xi_j\right] + \left\| \left(|\nabla \chi_<|^2 + |\nabla \chi_>|^2 \right)^{1/2} \xi_j \right\|^2 . \tag{2.171}$$

The last term vanishes as $j \to \infty$ again by Rellich's theorem. Moreover,

$$h\left[\chi_< \xi_j\right] \geq E_1 \|\chi_< \xi_j\|^2$$

and therefore

$$\liminf_{j\to\infty} h\left[\chi_< \xi_j\right] \geq \liminf_{j\to\infty} E_1 \|\chi_< \xi_j\|^2 = 0 \, .$$

Putting this into (2.171), we learn that

$$\limsup_{j\to\infty} h\left[\chi_> \xi_j\right] \leq \limsup_{j\to\infty} h\left[\xi_j\right] = E_\infty \, .$$

This, together with (2.169) and (2.170), yields (2.168).

We now prove the converse inequality $E_\infty \leq E'_\infty$. Let $(R_j) \subset (0,\infty)$ be a sequence with $R_j \to \infty$ and let $(\psi_j) \subset D[h]$ be a sequence with $\|\psi_j\| = 1$, $\psi_j \equiv 0$ in $\{|x| < R_j\}$ and $h[\psi_j] - E_1(-\Delta + V|_{B^c_{R_j}}) \to 0$. The support condition implies that $\psi_j \rightharpoonup 0$ in $L^2(\mathbb{R}^d)$ and therefore, by Lemma 2.B.3,

$$E_\infty \leq \liminf_{j\to\infty} h[\psi_j] = \liminf_{j\to\infty} E_1(-\Delta + V|_{B^c_{R_j}}) \leq E'_\infty \,,$$

which proves the theorem. □

CHAPTER 3

Universality in low-dimensional BCS theory

Abstract It is a remarkable property of BCS theory that the ratio of the energy gap at zero temperature Ξ and the critical temperature T_c is (approximately) given by a universal constant, independent of the microscopic details of the fermionic interaction. This universality has rigorously been proven quite recently in three spatial dimensions and three different limiting regimes: weak coupling, low density, and high density. The goal of this short note is to extend the universal behavior to lower dimensions $d = 1, 2$ and give an exemplary proof in the weak coupling limit.

3.1 Introduction

The Bardeen–Cooper–Schrieffer (BCS) theory of superconductivity [5] is governed by the *BCS gap equation*. For translation invariant systems without external fields the BCS gap equation is

$$\Delta(p) = -\frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \hat{V}(p-q) \frac{\Delta(q)}{E_\Delta(p)} \tanh\left(\frac{E_\Delta(p)}{2T}\right) \mathrm{d}q \tag{3.1}$$

with dispersion relation $E_\Delta(p) = \sqrt{(p^2-\mu)^2 + |\Delta(p)|^2}$. Here, $T \geq 0$ denotes the temperature and $\mu > 0$ the chemical potential. We consider dimensions $d \in \{1, 2, 3\}$. The Fourier transform of the potential $V \in L^1(\mathbb{R}^d) \cap L^{p_V}(\mathbb{R}^d)$ (with a d-dependent $p_V \geq 1$ to be specified below), modeling their effective interaction, is denoted by $\hat{V}(p) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} V(x) \mathrm{e}^{-\mathrm{i}p \cdot x} \mathrm{d}x$.

According to BCS theory, a system is in a superconducting state, if there exists a non-zero solution Δ to the gap equation (3.1). The question of existence of such a non-trivial solution Δ hinges, in particular, on the temperature T. It turns out, there exists a critical temperature $T_c \geq 0$ such that for $T < T_c$ there exists a non-trivial solution, and for $T \geq T_c$ it does not [33, Theorem 3.1.3 and Definition 3.1.4]. This critical temperature is one of the key (physically measurable) quantities of the theory and its asymptotic behavior, in three spatial dimensions, has been studied in three physically rather different limiting regimes: In a weak-coupling limit (i.e. replacing $V \to \lambda V$ and taking $\lambda \to 0$) [22, 35], in a low-density limit (i.e. $\mu \to 0$) [36], and in a high-density limit (i.e. $\mu \to \infty$) [38].

As already indicated above, at zero temperature, the function E_Δ may be interpreted as the dispersion relation of a certain 'approximate' Hamiltonian of the quantum many-body system, see [33, Appendix A]. In particular

$$\Xi := \inf_{p \in \mathbb{R}^d} E_\Delta(p) \tag{3.2}$$

has the interpretation of an energy gap associated with the approximate BCS Hamiltonian and as such represents a second key quantity of the theory. Analogously to the critical temperature, the asymptotic behavior of this energy gap, again in three spatial dimensions, has been studied in the same three different limiting regimes: In a weak coupling limit [35], in a low density limit [49], and in a high density limit [39].

In this paper, we focus on a remarkable feature of BCS theory, which is well known in the physics literature [5, 47, 55]: The ratio of the energy gap Ξ and critical temperature T_c tends to a *universal constant, independent of the microscopic details* of the interaction between the fermions, i.e. the potential V. More precisely, in three spatial dimension, it holds that

$$\frac{\Xi}{T_c} \approx \frac{\pi}{e^\gamma} \approx 1.76\,, \tag{3.3}$$

where $\gamma \approx 0.577$ is the Euler-Mascheroni constant, in each of the three physically very different limits mentioned above. This result follows as a limiting equality by combining asymptotic formulas for the critical temperature T_c (see [22, 35, 36, 38]) and the energy gap Ξ (see [35, 39, 49]) in the three different regimes. Although these scenarios (weak coupling, low density, and high density) are physically rather different, they all have in common that 'superconductivity is weak' and one can hence derive an asymptotic formula for T_c and Ξ as they depart from being zero (in the extreme cases $\lambda = 0$, $\mu = 0$, $\mu = \infty$, respectively). However, all the asymptotic expressions are *not perturbative*, as they depend exponentially on the natural dimensionless small parameter in the respective limit. We refer to the above mentioned original works for details.

The goal of this note is to prove the *same* universal behavior (3.3), which has already been established in three spatial dimension, also in dimensions $d = 1, 2$ in the weak coupling limit (i.e. replacing $V \to \lambda V$ and taking $\lambda \to 0$). This situation serves as a showcase for the methods involved in the proofs of the various limits in three dimensions (see Remark 3.3.5 and Remark 3.3.8 below). Apart from the mathematical curiosity in $d = 1, 2$, there have been recent studies in lower-dimensional superconductors in the physics literature, out of which we mention one-dimensional superconducting nanowires [54] and two-dimensional 'magic angle' graphene [11].

In the remainder of this introduction, we briefly recall the mathematical formulation of BCS theory, which has been developed mostly by Hainzl and Seiringer, but also other co-authors [22, 32, 33]. Apart from the universality discussed here, also many other properties of BCS theory have been shown using this formulation: Most prominently, Ginzburg-Landau theory, as an effective theory describing superconductors close to the critical temperature, has been derived from BCS theory [16, 17, 23, 26]. More recently, it has been shown that the effect of boundary superconductivity occurs in the BCS model [34]. We refer to [32] for a more comprehensive review of developments in the mathematical formulation of BCS theory. The universal behavior in the weak coupling limit for lower dimensions $d = 1, 2$ is presented in Section 3.2. Finally, in Section 3.3, we provide the proofs of the statements from Section 3.2.

3.1.1 Mathematical formulation of BCS theory

We will now briefly recall the mathematical formulation [32, 33] of BCS theory [5], which is an effective theory developed for describing superconductivity of a fermionic gas. In the following, we consider these fermions in $\mathbb{R}^d$, $d = 1, 2$, at temperature $T \geq 0$ and chemical potential $\mu \in \mathbb{R}$, interacting via a two-body potential V, for which we assume the following.

Assumption 3.1.1. We have that V is real-valued, reflection symmetric, i.e. $V(x) = V(-x)$ for all $x \in \mathbb{R}^d$, and it satisfies $V \in L^{p_V}(\mathbb{R}^d)$, where $p_V = 1$ if $d = 1$, $p_V \in (1, \infty)$ if $d = 2$.

Moreover, we neglect external fields, in which case the system is translation invariant.

The central object in the mathematical formulation of the theory is the BCS functional, which can naturally be viewed as a function of BCS states Γ. These states are given by a pair of functions (γ, α) and can be conveniently represented as a 2×2 matrix valued Fourier multiplier on $L^2(\mathbb{R}^d) \oplus L^2(\mathbb{R}^d)$ of the form

$$\hat{\Gamma}(p) = \begin{pmatrix} \hat{\gamma}(p) & \hat{\alpha}(p) \\ \overline{\hat{\alpha}(p)} & 1 - \hat{\gamma}(p) \end{pmatrix} \tag{3.4}$$

for all $p \in \mathbb{R}^d$. In (3.4), $\hat{\gamma}(p)$ denotes the Fourier transform of the one particle density matrix and $\hat{\alpha}(p)$ is the Fourier transform of the Cooper pair wave function. We require reflection symmetry of $\hat{\alpha}$, i.e. $\hat{\alpha}(-p) = \hat{\alpha}(p)$, as well as $0 \leq \hat{\Gamma}(p) \leq 1$ as a matrix.

The *BCS free energy functional* takes the form

$$\mathcal{F}_T[\Gamma] := \int_{\mathbb{R}^d} (p^2 - \mu)\hat{\gamma}(p)\mathrm{d}p - TS[\Gamma] + \int_{\mathbb{R}^d} V(x)|\alpha(x)|^2 \mathrm{d}x\,, \qquad \Gamma \in \mathcal{D}, \tag{3.5}$$

$$\mathcal{D} := \left\{ \hat{\Gamma}(p) = \begin{pmatrix} \hat{\gamma}(p) & \hat{\alpha}(p) \\ \overline{\hat{\alpha}(p)} & 1 - \hat{\gamma}(p) \end{pmatrix} : 0 \leq \hat{\Gamma} \leq 1\,,\ \hat{\gamma} \in L^1(\mathbb{R}^d, (1+p^2)\mathrm{d}p)\,,\ \alpha \in H^1_{\mathrm{sym}}(\mathbb{R}^d) \right\},$$

where the *entropy density* is defined as

$$S[\Gamma] = -\int_{\mathbb{R}^d} \mathrm{Tr}_{\mathbb{C}^2}\left[\hat{\Gamma}(p) \log \hat{\Gamma}(p)\right] \mathrm{d}p\,.$$

The minimization problem associated with (3.5) is well defined. In fact, the following result has only been proven for $d = 3$ and $V \in L^{3/2}(\mathbb{R}^3)$, but its extension to $d = 1, 2$ is straightforward.

Proposition 3.1.2 ([33], see also [32]). *Under Assumption 3.1.1 on V, the BCS free energy is bounded below on $\mathcal{D}$ and attains its minimum.*

The BCS gap equation (3.1) arises as the Euler–Lagrange equations of this functional [33]. Namely by defining $\Delta = -2\widehat{V\alpha}$, the Euler–Lagrange equation for α takes the form of the BCS gap equation (3.1). Additionally, one has the following linear criterion for the BCS gap equation to have non-trivial solutions. Again, so far, a proof has only been given in spatial dimension $d = 3$ and for $V \in L^{3/2}(\mathbb{R}^3)$, but its extension to $d = 1, 2$ is straightforward.

Theorem 3.1.3 ([33, Thm. 1]). *Let V satisfy Assumption 3.1.1 and let $\mu \in \mathbb{R}$ as well as $T \geq 0$. Then, writing $\mathcal{F}_T[\Gamma] \equiv \mathcal{F}_T(\gamma, \alpha)$, the following are equivalent.*

1. *The minimizer of $\mathcal{F}_T$ is not attained with $\alpha = 0$, i.e.*

$$\inf_{(\gamma,\alpha)\in\mathcal{D}} \mathcal{F}_T(\gamma, \alpha) < \inf_{(\gamma,0)\in\mathcal{D}} \mathcal{F}_T(\gamma, 0),$$

2. *There exists a pair $(\gamma, \alpha) \in \mathcal{D}$ with $\alpha \neq 0$ such that $\Delta = -2\widehat{V\alpha}$ satisfies the BCS gap equation* (3.1),

3. *The linear operator* $K_T + V$*, where* $K_T(p) = \frac{p^2-\mu}{\tanh((p^2-\mu)/(2T))}$ *has at least one negative eigenvalue.*

The third item immediately leads to the following definition of the *critical temperature* T_c for the existence of non-trivial solutions of the BCS gap equation (3.1).

Definition 3.1.4 (Critical temperature, see [22, Def. 1])**.** For V satisfying Assumption 3.1.1, we define the critical temperature $T_c \geq 0$ as

$$T_c := \inf\{T > 0 : K_T + V \geq 0\}. \tag{3.6}$$

By $K_T(p) \geq 2T$ and the asymptotic behavior $K_T(p) \sim p^2$ for $|p| \to \infty$, Sobolev's inequality [50, Thm. 8.3] implies that the critical temperature is well defined.

The other object we study is the energy gap Ξ defined in (3.2). The energy gap depends on the solution Δ of the gap equation (3.1) at $T = 0$. A priori, Δ may not be unique. However, for potentials with non-positive Fourier transform, this possibility can be ruled out.

Proposition 3.1.5 (see [35, (21)-(22) and Lemma 2])**.** *Let* V *satisfy Assumption 3.1.1 (and additionally* $V \in L^1(\mathbb{R}^2)$ *in case that* $d = 2$*). Moreover, we assume that* $\hat{V} \leq 0$ *and* $\hat{V}(0) < 0$*. Then, there exists a unique minimizer* Γ *of* $\mathcal{F}_0$ *(up to a constant phase in* α*). One can choose the phase such that* α *has strictly positive Fourier transform* $\hat{\alpha} > 0$*.*

In particular, we conclude that Δ is strictly positive. Moreover, by means of the gap equation (3.1), Δ is continuous and thus $\Xi > 0$.

3.2 Main Results

As explained in the introduction, our main result in this short note is the extension of the universality (3.3) from $d = 3$ to lower spatial dimensions $d = 1, 2$ in the limit of weak coupling (i.e., replacing $V \to \lambda V$ and taking $\lambda \to 0$). We assume the following properties for the interaction potential V.

Assumption 3.2.1. Let $d \in \{1, 2\}$ and assume that V satisfies Assumption 3.1.1 as well as $\hat{V} \leq 0$, $\hat{V}(0) < 0$. Moreover, for $d = 1$ we assume that $(1 + |\cdot|^{\epsilon})V \in L^1(\mathbb{R}^1)$ for some $\epsilon > 0$. Finally, in case that $d = 2$, we suppose that $V \in L^1(\mathbb{R}^2)$ is radial.

By Proposition 3.1.5, this means that, in particular, the minimizer of $\mathcal{F}_0$ is unique (up to a phase) and the associated energy gap at zero temperature (3.2) is strictly positive, $\Xi > 0$. We are now ready to state our main result.

Theorem 3.2.2 (BCS Universality in one and two dimensions)**.** *Let* V *be as in Assumption 3.2.1. Then the critical temperature* $T_c(\lambda)$ *(defined in* (3.6)*) and the energy gap* $\Xi(\lambda)$ *(defined in* (3.2)*) are strictly positive for all* $\lambda > 0$ *and it holds that*

$$\lim_{\lambda \to 0} \frac{\Xi(\lambda)}{T_c(\lambda)} = \frac{\pi}{e^{\gamma}},$$

where $\gamma \approx 0.577$ *is the Euler-Mascheroni constant.*

To prove the universality, we separately establish asymptotic formulas for T_c (see Theorem 3.2.5) and Ξ (see Theorem 3.2.7), valid to second order, and compare them by taking their ratio. The asymptotic formula for T_c is valid under weaker conditions on V than Assumption 3.2.1, because we do not need uniqueness of Δ. To obtain the asymptotic formulas, we first introduce two self-adjoint operators $\mathcal{V}_\mu^{(d)}$ and $\mathcal{W}_\mu^{(d)}$ mapping $L^2(\mathbb{S}^{d-1}) \to L^2(\mathbb{S}^{d-1})$ and as such measuring the strength of the interaction $\hat{V}$ on the (rescaled) Fermi surface (see [35, 38, 39]). To assure that $\mathcal{V}_\mu^{(d)}$ and $\mathcal{W}_\mu^{(d)}$ will be well-defined and compact, we assume the following.

Assumption 3.2.3. Let V satisfy Assumption 3.1.1. Additionally, assume that for $d = 1$, $\Big(1 + (\ln(1 + |\cdot|))^2\Big)V \in L^1(\mathbb{R}^1)$ and for $d = 2$, $V \in L^1(\mathbb{R}^2)$.

First, in order to capture the strength to leading order, we define $\mathcal{V}_\mu^{(d)}$ via

$$(\mathcal{V}_\mu^{(d)} u)(p) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{S}^{d-1}} \hat{V}(\sqrt{\mu}(p-q))u(q)\mathrm{d}\omega(q)\,,$$

where $\mathrm{d}\omega$ is the Lebesgue measure on $\mathbb{S}^{d-1}$. Since $V \in L^1(\mathbb{R}^d)$, we have that $\hat{V}$ is a bounded continuous function and hence $\mathcal{V}_\mu^{(d)}$ is a Hilbert-Schmidt operator (in fact, trace class with trace being equal to $(2\pi)^{-d}|\mathbb{S}^{d-1}| \int_{\mathbb{R}^d} V(x)\mathrm{d}x$). Therefore, its lowest eigenvalue $e_\mu^{(d)} := \inf \operatorname{spec} \mathcal{V}_\mu^{(d)}$ satisfies $e_\mu^{(d)} \leq 0$ and it is strictly negative if e.g. $\int V < 0$ as in Assumption 3.2.1.

Second, in order to capture the strength of $\hat{V}$ to next to leading order, we define the operator $\mathcal{W}_\mu^{(d)}$ via its quadratic form

$$\begin{aligned}
&\left\langle u \middle| \mathcal{W}_\mu^{(d)} \middle| u \right\rangle \\
&\quad = \mu^{d/2-1}\left[\int_{|p|<\sqrt{2}} \frac{1}{|p^2-1|}\left(|\psi(\sqrt{\mu}p)|^2 - |\psi(\sqrt{\mu}p/|p|)|^2\right)\mathrm{d}p + \int_{|p|>\sqrt{2}} \frac{1}{|p^2-1|}|\psi(\sqrt{\mu}p)|^2\mathrm{d}p\right],
\end{aligned}$$

where $\psi(p) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{S}^{d-1}} \hat{V}(p - \sqrt{\mu}q)u(q)\mathrm{d}\omega(q)$ and $u \in L^2(\mathbb{S}^{d-1})$. The proof of the following proposition shall be given in Section 3.3.3.

Proposition 3.2.4. *Let $d \in \{1, 2\}$ and let V satisfy Assumption 3.2.3. The operator $\mathcal{W}_\mu^{(d)}$ is well-defined and Hilbert-Schmidt.*

Next, we define the self-adjoint Hilbert-Schmidt operator

$$\mathcal{B}_\mu^{(d)}(\lambda) := \frac{\pi}{2}\left(\lambda\mathcal{V}_\mu^{(d)} - \lambda^2\mathcal{W}_\mu^{(d)}\right)$$

on $L^2(\mathbb{S}^{d-1})$ and its ground state energy

$$b_\mu^{(d)}(\lambda) := \inf \operatorname{spec}\left(\mathcal{B}_\mu^{(d)}(\lambda)\right). \tag{3.7}$$

Note that if $e_\mu^{(d)} < 0$, then also $b_\mu^{(d)}(\lambda) < 0$ for small enough λ. After these preparatory definitions, we are ready to state the separate asymptotic formulas for the critical temperature and the energy gap in one and two dimensions, which immediately imply Theorem 3.2.2.

Theorem 3.2.5 (Critical Temperature for $d = 1, 2$)**.** *Let $\mu > 0$. Let V satisfy Assumption 3.2.3 and additionally $e_\mu^{(d)} < 0$. Then the critical temperature T_c, given in Definition 3.1.4, is strictly positive and satisfies*

$$\lim_{\lambda \to 0}\left(\ln\left(\frac{\mu}{T_c(\lambda)}\right) + \frac{\pi}{2\,\mu^{d/2-1}\,b_\mu^{(d)}(\lambda)}\right) = -\gamma - \ln\left(\frac{2c_d}{\pi}\right),$$

where γ denotes the Euler-Mascheroni constant and $c_1 = \frac{4}{1+\sqrt{2}}$ and $c_2 = 1$.

Here, the Assumptions on V are weaker than Assumption 3.2.1, since $\hat{V}(0) < 0$ implies that $e_\mu^{(d)} < 0$. We thus have the asymptotic behavior

$$T_c(\lambda) = 2c_d \, \frac{\mathrm{e}^\gamma}{\pi} \Big(1 + o(1)\Big) \mu \, \mathrm{e}^{\pi/(2\mu^{d/2-1} b_\mu^{(d)}(\lambda))}$$

in the limit of small λ.

Remark 3.2.6. Theorem 3.2.5 is essentially a special case of [37, Theorem 2]. We give the proof here for two main reasons.

1. There is still some work required to translate the statement of [37, Theorem 2] into a form in which it is comparable to that of Theorem 3.2.7 (in order to prove Theorem 3.2.2). The main difficulty is that the operator $\mathcal{W}_\mu^{(d)}$ in [37] is only defined via a limit, [37, Equation (2.10)].
2. The goal of this paper is to give an exemplary proof of Theorem 3.2.5 in order to compare it to the proofs of the similar statements in the literature concerning the asymptotic behavior of the critical temperature in various limits [35, 36, 38].

Theorem 3.2.5 is complemented by the following asymptotics for the energy gap.

Theorem 3.2.7 (Energy Gap for $d = 1, 2$)**.** *Let V satisfy Assumption 3.2.1 and let $\mu > 0$. Then there exists a unique radially symmetric minimizer (up to a constant phase) of the BCS functional* (3.5) *at temperature $T = 0$. The associated energy gap Ξ, given in* (3.2)*, is strictly positive and satisfies*

$$\lim_{\lambda \to 0} \left(\ln\left(\frac{\mu}{\Xi}\right) + \frac{\pi}{2\,\mu^{d/2-1}\, b_\mu^{(d)}(\lambda)} \right) = -\ln(2c_d)\,,$$

where $b_\mu^{(d)}$ is defined in (3.7) *and $c_1 = \frac{4}{1+\sqrt{2}}$ and $c_2 = 1$.*

In other words, we have the asymptotic behavior

$$\Xi(\lambda) = 2c_d \Big(1 + o(1)\Big) \mu \, \mathrm{e}^{\pi/(2\mu^{d/2-1} b_\mu^{(d)}(\lambda))}$$

in the limit of small λ. Now, Theorem 3.2.2 follows immediately from Theorems 3.2.5 and 3.2.7.

Remark 3.2.8 (Other limits in dimensions $d = 1, 2$)**.** Similarly to the presented results, one could also consider the limits of low and high density, i.e. $\mu \to 0$ and $\mu \to \infty$, respectively. We expect that also here the universality $\frac{\Xi}{T_c} \approx \frac{\pi}{\mathrm{e}^\gamma}$ holds. Indeed, one would expect that the proofs of BCS universality in dimension $d = 3$ should carry over to one and two dimensions with some minor technical modifications. Note that, even for the (technically less demanding) case of a weak coupling limit, which we consider here, there are still some technical details that are different in dimensions $d = 1, 2$ compared to dimension $d = 3$. Hence, it is not a trivial matter to generalize the arguments of [36, 38, 39, 49] to one and two dimensions. Moreover, for the case of low density, there is even an issue of what exactly low density *means* in dimensions one and two: In three spatial dimensions [36, 49], the asymptotic formulas for T_c and Ξ were obtained for potentials V with *negative* scattering length but not creating bound states for the Laplacian. This latter condition ensures that $\mu \to 0$ actually corresponds to the limit of low density. However, in spatial dimensions one and two, attractive potentials, no matter how weak, always give rise to bound states of $-\nabla^2 + V$, see [65]. Thus for $\mu = 0$ the particle density is non-zero. We will not deal with the low- and high-density limits here.

The rest of the paper is devoted to proving Theorem 3.2.5 and Theorem 3.2.7.

3.3 Proofs

The overall structure of our proofs is as follows: The principal idea is to derive two different formulas for each of the two integrals

$$m_\mu^{(d)}(T) := \frac{1}{|\mathbb{S}^{d-1}|}\int_{|p|<\sqrt{2\mu}} \frac{1}{K_T(p)}\mathrm{d}p \tag{3.8}$$

and

$$m_\mu^{(d)}(\Delta) := \frac{1}{|\mathbb{S}^{d-1}|}\int_{|p|<\sqrt{2\mu}} \frac{1}{E_\Delta(p)}\mathrm{d}p. \tag{3.9}$$

The first set of formulas is derived by studying the Birman-Schwinger operators

$$B_T^{(d)} := \lambda V^{1/2} K_T^{-1} |V|^{1/2} \quad \text{and} \quad B_\Delta^{(d)} := \lambda V^{1/2} E_\Delta^{-1} |V|^{1/2}\,,$$

associated to the Schrödinger type operators $K_T + \lambda V$ and $E_\Delta + \lambda V$, respectively. In particular, spectral properties of these unbounded Schrödinger type operators naturally translate to their associated Birman-Schwinger operators, which are compact and as such much simpler to analyze. The second set of formulas is obtained by just calculating the integrals $m_\mu^{(d)}$ directly.

Indeed, for the critical temperature we obtain the following asymptotics, which, by combining them, immediately prove Theorem 3.2.5.

Proposition 3.3.1. *Let $\mu > 0$. Let V satisfy Assumption 3.2.3 and additionally $e_\mu^{(d)} < 0$. Then, the critical temperature T_c is positive and, as $\lambda \to 0$, we have that*

$$\begin{aligned} m_\mu^{(d)}(T_c) &= -\frac{\pi}{2b_\mu^{(d)}(\lambda)} + o(1)\,, \\ m_\mu^{(d)}(T_c) &= \mu^{d/2-1}\left(\ln\left(\frac{\mu}{T_c}\right) + \gamma + \ln\left(\frac{2c_d}{\pi}\right) + o(1)\right)\,. \end{aligned}$$

For the energy gap we obtain the following asymptotics, which, again by combining them, immediately prove Theorem 3.2.7.

Proposition 3.3.2. *Let V satisfy Assumption 3.2.1 and let $\mu > 0$. Then (by Proposition 3.1.5) we have a strictly positive radially symmetric gap function Δ and associated energy gap Ξ, which, as $\lambda \to 0$, satisfy the asymptotics*

$$\begin{aligned} \Xi &= \Delta(\sqrt{\mu})\big(1 + o(1)\big) \\ m_\mu^{(d)}(\Delta) &= -\frac{\pi}{2b_\mu^{(d)}(\lambda)} + o(1) \\ m_\mu^{(d)}(\Delta) &= \mu^{d/2-1}\left(\ln\left(\frac{\mu}{\Delta(\sqrt{\mu})}\right) + \ln(2c_d) + o(1)\right) \end{aligned}$$

With a slight abuse of notation, using radiality of Δ, we wrote $\Delta(\sqrt{\mu})$ instead of $\Delta(\sqrt{\mu}\hat{p})$ for some $\hat{p} \in \mathbb{S}^{d-1}$.

In the remainder of this paper, where we give the proofs of Propositions 3.3.1 and 3.3.2, we shall frequently use the notation $\mathfrak{F}_\mu^{(d)} : L^1(\mathbb{R}^d) \to L^2(\mathbb{S}^{d-1})$ for the (scaled) Fourier transform restricted to the (rescaled) Fermi sphere,

$$\left(\mathfrak{F}_\mu^{(d)}\psi\right)(p) := \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \psi(x) e^{-i\sqrt{\mu}p\cdot x} \mathrm{d}x\,.$$

Note that for an L^1-function, pointwise values of its Fourier transform are well-defined by the Riemann–Lebesgue lemma. (In particular the restriction to a co–dimension 1 manifold of a sphere is well-defined.)

Remark 3.3.3. In [13], Cuenin and Merz use the Tomas-Stein theorem to define $\mathfrak{F}_\mu^{(d)}$ on a larger space than $L^1(\mathbb{R}^d)$. With this they are able to prove a general version of Theorem 3.2.5 under slightly weaker conditions on V. However, we do not pursue this here, see Remark 3.2.6.

3.3.1 Proof of Proposition 3.3.1

Proof of Proposition 3.3.1. The argument is divided into several steps.

1. A priori spectral information on $K_{T_c} + \lambda V$. First note that, due to Theorem 3.1.3 and Definition 3.1.4, the critical temperature T_c is determined by the lowest eigenvalue of $K_T + \lambda V$ being 0 exactly for $T = T_c$.

2. Birman-Schwinger principle. Next, we employ the Birman-Schwinger principle, which says that the compact Birman-Schwinger operator $B_T^{(d)} = \lambda V^{1/2} K_T^{-1} |V|^{1/2}$ (denoting $V(x)^{1/2} = \mathrm{sgn}(V(x))|V(x)|^{1/2}$) has -1 as its lowest eigenvalue exactly for $T = T_c$, see [22, 35].

Using the notation for the Fourier transform restricted to the rescaled Fermi sphere introduced above, we now decompose the Birman-Schwinger operator as

$$B_T^{(d)} = \lambda m_\mu^{(d)}(T) V^{1/2} (\mathfrak{F}_\mu^{(d)})^\dagger \mathfrak{F}_\mu^{(d)} |V|^{1/2} + \lambda V^{1/2} M_T^{(d)} |V|^{1/2},$$

where $M_T^{(d)}$ is defined through the integral kernel

$$M_T^{(d)}(x,y) = \frac{1}{(2\pi)^d} \left[\int_{|p|<\sqrt{2\mu}} \frac{1}{K_T(p)} \left(e^{ip\cdot(x-y)} - e^{i\sqrt{\mu}p/|p|\cdot(x-y)} \right) \mathrm{d}p + \int_{|p|>\sqrt{2\mu}} \frac{1}{K_T} e^{ip\cdot(x-y)} \mathrm{d}p \right]. \tag{3.10}$$

We claim that $V^{1/2} M_T^{(d)} |V|^{1/2}$ is uniformly bounded.

Lemma 3.3.4. *Let $\mu > 0$. Let V satisfy Assumption 3.2.3. Then we have for all $T \geq 0$*

$$\left\| V^{1/2} M_T^{(d)} |V|^{1/2} \right\|_{\mathsf{HS}} \leq C\,,$$

where $C > 0$ denotes some positive constant and $\|\cdot\|_{HS}$ is the Hilbert-Schmidt norm.

Armed with this bound, we have that for sufficiently small λ that $1 + \lambda V^{1/2} M_T^{(d)} |V|^{1/2}$ is invertible, and hence

$$1 + B_T^{(d)} = (1 + \lambda V^{1/2} M_T^{(d)} |V|^{1/2}) \left(1 + \frac{\lambda m_\mu^{(d)}(T)}{1 + \lambda V^{1/2} M_T^{(d)} |V|^{1/2}} V^{1/2} (\mathfrak{F}_\mu^{(d)})^\dagger \mathfrak{F}_\mu^{(d)} |V|^{1/2} \right).$$

Thus, the fact that $B_T^{(d)}$ has lowest eigenvalue -1 at $T = T_c$ is equivalent to

$$\lambda m_\mu^{(d)}(T)\mathfrak{F}_\mu^{(d)}|V|^{1/2}\frac{1}{1+\lambda V^{1/2}M_T^{(d)}|V|^{1/2}}V^{1/2}(\mathfrak{F}_\mu^{(d)})^\dagger \tag{3.11}$$

having lowest eigenvalue -1, again at $T = T_c$, as it is isospectral to the rightmost operator on the right-hand-side above. (Recall that for bounded operators A, B, the operators AB and BA have the same spectrum apart from possibly at 0. However, in our case, both operators are compact on an infinite dimensional space and hence 0 is in both spectra.)

We now prove Lemma 3.3.4.

Proof of Lemma 3.3.4. We want to bound the integral kernel (3.10) of $M_T^{(d)}$ uniformly in T. Hence, we will bound $K_T \geq |p^2-\mu|$. The computation is slightly different in $d = 1$ and $d = 2$, so we do them separately.

<u>$d = 1$.</u> The second integral in (3.10) is bounded by

$$2\int_{|p|>\sqrt{2\mu}}\frac{1}{|p^2-\mu|}\mathrm{d}p = \frac{2\,\mathrm{arcoth}\sqrt{2}}{\sqrt{\mu}}.$$

For the first integral, we use that $|e^{ix}-e^{iy}| \leq \min\{|x-y|,2\}$, $|p^2-\mu| \geq \sqrt{\mu}||p|-\sqrt{\mu}|$, and increase the domain of integration to obtain the bound

$$\begin{aligned}\frac{2}{\sqrt{\mu}}\int_0^{2\sqrt{\mu}}\frac{\min\left\{||p-\sqrt{\mu}||x-y|,2\right\}}{|p-\sqrt{\mu}|}\mathrm{d}p &= \frac{8}{\sqrt{\mu}}\left[1+\ln\left(\max\left\{\frac{|x-y|\sqrt{\mu}}{2},1\right\}\right)\right]\\ &\leq \frac{8}{\sqrt{\mu}}(1+\ln(1+\sqrt{\mu}\max\{|x|,|y|\}).\end{aligned}$$

We conclude that $|M_T^{(1)}(x,y)| \lesssim \frac{1}{\sqrt{\mu}}(1+\ln(1+\sqrt{\mu}\max\{|x|,|y|\}))$. Hence,

$$\left\|V^{1/2}M_T^{(1)}|V|^{1/2}\right\|_{\mathsf{HS}}^2 \lesssim \frac{1}{\mu}\left(\|V\|_{L^1(\mathbb{R})}^2+\|V\|_{L^1(\mathbb{R})}\int_{\mathbb{R}}|V(x)|(1+\ln(1+\sqrt{\mu}|x|))^2\mathrm{d}x\right).$$

<u>$d = 2$</u>. We first compute the angular integral. Note that $\int_{\mathbb{S}^1} e^{ipx}\mathrm{d}\omega(p) = 2\pi J_0(|x|)$, where J_0 is the zeroth order Bessel function. For the second integral in (3.10) we may bound $|p^2-\mu| \geq cp^2$. Up to some finite factor, the second integral is hence bounded by

$$\int_{\sqrt{2\mu}}^\infty \frac{1}{p}|J_0(p|x-y|)|\mathrm{d}p \leq C\int_{\sqrt{2\mu}}^\infty \frac{1}{p^{1+\lambda}}|x-y|^{-\lambda}\mathrm{d}p \leq C_\lambda|x-y|^{-\lambda},$$

for any $0 < \lambda \leq 1/2$ since $|J_0(x)| \leq C$ and $\sqrt{x}J_0(x) \leq C$, see e.g. [10, (9.55f), (9.57a)]. For the first integral we get the bound

$$\int_0^{\sqrt{2\mu}}\frac{p}{|p^2-\mu|}\left|J_0(p|x-y|)-J_0(\sqrt{\mu}|x-y|)\right|\mathrm{d}p.$$

Here we use that J_0 is Lipschitz, since its derivative J_{-1} is bounded (see e.g. [10, (9.55a), (9.55f)]), so that

$$|J_0(x)-J_0(y)| \leq C|x-y|^{1/3}(|J_0(x)|+|J_0(y)|)^{2/3} \leq C|x-y|^{1/3}\left(x^{-1/3}+y^{-1/3}\right).$$

That is

$$|J_0(p|x-y|) - J_0(\sqrt{\mu}|x-y|)| \leq C\frac{|p-\sqrt{\mu}|^{1/3}}{p^{1/3}+\sqrt{\mu}^{1/3}}.$$

This shows that the first integral is bounded. We conclude that $|M_T^{(2)}(x,y)| \lesssim 1 + \frac{1}{|x-y|^\lambda}$ for any $0 < \lambda \leq 1/2$. Then, by the Hardy–Littlewood–Sobolev inequality [50, Theorem 4.3] we have that

$$\left\|V^{1/2}M_T^{(2)}|V|^{1/2}\right\|_{\mathrm{HS}}^2 = \iint |V(x)||M_T^{(2)}(x,y)|^2|V(y)|\mathrm{d}x\mathrm{d}y \lesssim \|V\|_{L^1(\mathbb{R}^2)}^2 + \|V\|_{L^p(\mathbb{R}^2)}^2$$

for any $1 < p \leq 4/3$. □

3. First order. Evaluating (3.11) at $T = T_c$ and expanding the geometric series to first order we get

$$\begin{aligned}-1 &= \lambda m_\mu^{(d)}(T_c)\inf\operatorname{spec}\left(\mathfrak{F}_\mu^{(d)}|V|^{1/2}\frac{1}{1+\lambda V^{1/2}M_{T_c}^{(d)}|V|^{1/2}}V^{1/2}(\mathfrak{F}_\mu^{(d)})^\dagger\right)\\ &= \lambda\, m_\mu^{(d)}(T_c)\,\inf\operatorname{spec}\mathcal{V}_\mu^{(d)}(1+O(\lambda)) = \lambda\, m_\mu^{(d)}(T_c)\, e_\mu^{(d)}(1+O(\lambda))\end{aligned}$$

where we used $\mathcal{V}_\mu^{(d)} = \mathfrak{F}_\mu^{(d)}V(\mathfrak{F}_\mu^{(d)})^\dagger$. Since by assumption $e_\mu^{(d)} < 0$, this shows that $m_\mu^{(d)}(T_c) \to \infty$ as $\lambda \to 0$.

4. A priori bounds on T_c. By (3.8), the divergence of $m_\mu^{(d)}$ as $\lambda \to 0$ in particular shows that $T_c/\mu \to 0$ in the limit $\lambda \to 0$.

5. Calculation of the integral $m_\mu^{(d)}(T_c)$. This step is very similar to [35, Lemma 1] and [34, Lemma 3.5], where the asymptotics have been computed for slightly different definitions of $m_\mu^{(d)}$ in three and one spatial dimension, respectively. Integrating over the angular variable and substituting $s = \left|\frac{|p|^2}{\mu} - 1\right|$, we get

$$m_\mu^{(d)}(T_c) = \mu^{d/2-1}\int_0^1 \tanh\left(\frac{s}{2(T_c/\mu)}\right)\frac{(1+s)^{d/2-1}+(1-s)^{d/2-1}}{2s}\mathrm{d}s.$$

According to [35, Lemma 1],

$$\lim_{T_c\downarrow 0}\left(\int_0^1\frac{\tanh\left(\frac{s}{2(T_c/\mu)}\right)}{s}\mathrm{d}s - \ln\frac{\mu}{T_c}\right) = \gamma - \ln\frac{\pi}{2}.$$

By monotone convergence, it follows that

$$m_\mu^{(d)}(T_c) = \mu^{d/2-1}\left[\ln\frac{\mu}{T_c} + \gamma - \ln\frac{\pi}{2} + \int_0^1\frac{(1-s)^{d/2-1}+(1+s)^{d/2-1}-2}{2s}\mathrm{d}s + o(1)\right].$$

The remaining integral equals $\ln c_d$ and we have thus proven the second item in Proposition 3.3.1.

Combining this with the third step, one immediately sees that the critical temperature vanishes exponentially fast, $T_c \sim e^{1/\lambda e_\mu}$, as $\lambda \to 0$, recalling that $e_\mu^{(d)} < 0$ by assumption.

6. Second order. Now, to show the universality, we need to compute the next order correction.

To do so, we expand the geometric series in (3.11) and employ first order perturbation theory, yielding that

$$m_\mu^{(d)}(T_c) = \frac{-1}{\lambda \left\langle u \middle| \mathfrak{F}_\mu^{(d)} V (\mathfrak{F}_\mu^{(d)})^\dagger \middle| u \right\rangle - \lambda^2 \left\langle u \middle| \mathfrak{F}_\mu^{(d)} V M_{T_c}^{(d)} V (\mathfrak{F}_\mu^{(d)})^\dagger \middle| u \right\rangle + O(\lambda^3)}, \tag{3.12}$$

where u is the (normalized) ground state (eigenstate of lowest eigenvalue) of $\mathfrak{F}_\mu^{(d)} V (\mathfrak{F}_\mu^{(d)})^\dagger$. (In case of a degenerate ground state, u is the ground state minimizing the second order term.)

This second order term in the denominator of (3.12) is close to $\mathcal{W}_\mu^{(d)}$. More precisely, it holds that

$$\lim_{\lambda \to 0} \left\langle u \middle| \mathfrak{F}_\mu^{(d)} V M_{T_c}^{(d)} V (\mathfrak{F}_\mu^{(d)})^\dagger \middle| u \right\rangle = \left\langle u \middle| \mathcal{W}_\mu^{(d)} \middle| u \right\rangle , \tag{3.13}$$

which easily follows from dominated convergence, noting that $\frac{1}{K_T}$ increases to $\frac{1}{|p^2-\mu|}$ as $T \to 0$. We then conclude that

$$\lim_{\lambda \to 0} \left(m_\mu^{(d)}(T_c) + \frac{\pi}{2 b_\mu^{(d)}(\lambda)} \right) = 0 \,,$$

since $\left\langle u \middle| \lambda \mathcal{V}_\mu^{(d)} - \lambda^2 \mathcal{W}_\mu^{(d)} \middle| u \right\rangle = \inf \operatorname{spec}(\lambda \mathcal{V}_\mu^{(d)} - \lambda^2 \mathcal{W}_\mu^{(d)}) + O(\lambda^3) = \frac{\pi}{2} b_\mu^{(d)}(\lambda) + O(\lambda^3)$, again by first-order perturbation theory. This concludes the proof of Proposition 3.3.1. □

We conclude this subsection with several remarks, comparing our proof with those of similar results from the literature.

Remark 3.3.5 (Structure here vs. in earlier papers on T_c)**.** We compare the structure of our proof to that of the different limits in three dimensions [35, 36, 38]:

- **Weak coupling:** The structure of the proof we gave here is quite similar to that of [35], only they do Steps 5 and 6 in the opposite order. Also the leading term for T_c was shown already in [22], where a computation somewhat similar to Steps 1–4 is given.
- **High denisty:** For $\mu \to \infty$, the structure of the proof in [38] is slightly different compared to the one given here. This is basically due to the facts that (i) the necessary a priori bound $T_c = o(\mu)$ already requires the Birman-Schwinger decomposition and (ii) the second order requires strengthened assumptions compared to the first order. To conclude, the order of steps in [38] can be thought of as: 1, 5, 4 (establishing $T_c = O(\mu)$), 2, 3, 4 (establishing $T_c = o(\mu)$), 2 (again), 6. Here the final step is much more involved than in the other limits considered.
- **Low density:** As above, for the proof of the low density limit in [36] the structure is slightly different. One first needs the a priori bound $T_c = o(\mu)$ on the critical temperature before one uses the Birman-Schwinger principle and decomposes the Birman-Schwinger operator.[1] Also, the decomposition of the Birman-Schwinger operator is again different. For the full decomposition and analysis of the Birman-Schwinger operator one needs also the first-order analysis, that is Step 2, which is done in two parts. The order of the steps in [36] can then mostly be though of as: 1, 4, 5, 2, 3, 2 (again), 6.

[1] Strictly speaking, in [36], it is only proven that $T_c = O(\mu)$ (which is sufficient for applying the Birman-Schwinger principle), while the full $T_c = o(\mu)$ itself requires the Birman-Schwinger decomposition (see [48, Remark 4.12] for details).

3.3.2 Proof of Proposition 3.3.2

Proof of Proposition 3.3.2. The structure of the proof is parallel to that of Proposition 3.3.1 for the critical temperature.

1. A priori spectral information on $E_\Delta + \lambda V$. First, it is proven in [35, Lemma 2] that $\mathcal{F}_0$ has a unique minimizer α which has strictly positive Fourier transform. Using radiality of V, it immediately follows that this minimizer is rotationally symmetric (since otherwise rotating α would give a different minimizer) and hence also $\Delta = -2\lambda \hat{V} \star \hat{\alpha}$ is rotation invariant. It directly follows from [35, (43) and Lemma 3] that that $E_\Delta + \lambda V$ has lowest eigenvalue 0, and that the minimizer α is the corresponding eigenfunction.

2. Birman-Schwinger principle. This implies, by means of the Birman-Schwinger principle, that the Birman-Schwinger operator $B_\Delta^{(d)} = \lambda V^{1/2} E_\Delta^{-1} |V|^{1/2}$ has -1 as its lowest eigenvalue. As in the proof of Proposition 3.3.1, we decompose it as

$$B_\Delta^{(d)} = \lambda m_\mu^{(d)}(\Delta) V^{1/2} (\mathfrak{F}_\mu^{(d)})^\dagger \mathfrak{F}_\mu^{(d)} |V|^{1/2} + \lambda V^{1/2} M_\Delta^{(d)} |V|^{1/2}$$

and prove the second summand to be uniformly bounded.

Lemma 3.3.6. *Let $\mu > 0$. Let V satisfy Assumption 3.2.3. Then, uniformly in small λ, we have*

$$\left\| V^{1/2} M_\Delta^{(d)} |V|^{1/2} \right\|_{\mathsf{HS}} \leq C\,.$$

With this one may similarly factor

$$1+B_\Delta^{(d)} = (1+\lambda V^{1/2} M_\Delta^{(d)} |V|^{1/2}) \left(1 + \frac{\lambda m_\mu^{(d)}(\Delta)}{1+\lambda V^{1/2} M_\Delta^{(d)} |V|^{1/2}} V^{1/2} (\mathfrak{F}_\mu^{(d)})^\dagger \mathfrak{F}_\mu^{(d)} |V|^{1/2} \right) \quad (3.14)$$

and conclude that

$$T_\Delta^{(d)} := \lambda m_\mu^{(d)}(\Delta) \mathfrak{F}_\mu^{(d)} |V|^{1/2} \frac{1}{1+\lambda V^{1/2} M_\Delta^{(d)} |V|^{1/2}} V^{1/2} (\mathfrak{F}_\mu^{(d)})^\dagger \quad (3.15)$$

has lowest eigenvalue -1.

Proof of Lemma 3.3.6. Note that M_Δ has kernel

$$M_\Delta(x,y) = \frac{1}{(2\pi)^d} \left[\int_{|p|<\sqrt{2\mu}} \frac{1}{E_\Delta(p)} \left(e^{ip\cdot(x-y)} - e^{i\sqrt{\mu}p/|p|\cdot(x-y)} \right) \mathrm{d}p + \int_{|p|>\sqrt{2\mu}} \frac{1}{E_\Delta(p)} e^{ip\cdot(x-y)} \mathrm{d}p \right].$$

We may bound this exactly as in the proof of Lemma 3.3.4 using that $E_\Delta(p) \geq |p^2 - \mu|$. □

3. First order. Expanding the geometric series in (3.15) to first order, we see that

$$\begin{aligned} -1 &= \lambda m_\mu^{(d)}(\Delta) \inf \operatorname{spec} \left(\mathfrak{F}_\mu^{(d)} |V|^{1/2} \frac{1}{1+\lambda V^{1/2} M_\Delta^{(d)} |V|^{1/2}} V^{1/2} (\mathfrak{F}_\mu^{(d)})^\dagger \right) \\ &= \lambda m_\mu^{(d)}(\Delta) \inf \operatorname{spec} \mathcal{V}_\mu^{(d)} (1+O(\lambda)) = \lambda e_\mu^{(d)} m_\mu^{(d)}(\Delta)(1+O(\lambda)). \end{aligned}$$

Hence, in particular, $m_\mu^{(d)}(\Delta) \sim -\frac{1}{\lambda e_\mu^{(d)}} \to \infty$ as $\lambda \to 0$.

4. A priori bounds on Δ. We now prepare for the computation of the integral $m_\mu^{(d)}(\Delta)$

in terms of $\Delta(\sqrt{\mu})$. This requires two types of bounds on Δ: One bound estimating the gap function $\Delta(p)$ at general momentum $p \in \mathbb{R}^d$ in terms of $\Delta(\sqrt{\mu})$ (see (3.16)), and one bound controlling the difference $|\Delta(p) - \Delta(q)|$ in some kind of Hölder-continuity estimate (see (3.17)).

Lemma 3.3.7. *Suppose that V is as in Assumption 3.2.1. Then for λ small enough*

$$\Delta(p) = f(\lambda) \left(\int_{\mathbb{S}^{d-1}} \hat{V}(p - \sqrt{\mu} q) \mathrm{d}\omega(q) + \lambda \eta_\lambda(p) \right),$$

where f is some function of λ and $\|\eta_\lambda\|_{L^\infty(\mathbb{R}^d)}$ is bounded uniformly in λ.

Proof. Recall that α is the eigenfunction of $E_\Delta + \lambda V$ with lowest eigenvalue 0. Then, by the Birman-Schwinger principle, $\phi = V^{1/2}\alpha$ satisfies

$$B_\Delta \phi = \lambda V^{1/2} \frac{1}{E_\Delta} |V|^{1/2} V^{1/2} \alpha = -\phi.$$

With the decomposition (3.14) then ϕ is an eigenfunction of

$$\frac{\lambda m_\mu^{(d)}(\Delta)}{1 + \lambda V^{1/2} M_\Delta^{(d)} |V|^{1/2}} V^{1/2} (\mathfrak{F}_\mu^{(d)})^\dagger \mathfrak{F}_\mu^{(d)} |V|^{1/2}$$

of eigenvalue -1. Thus, $\mathfrak{F}_\mu^{(d)} |V|^{1/2} \phi$ is an eigenfunction of $T_\Delta^{(d)}$ of (lowest) eigenvalue -1. Now $u = |\mathbb{S}^{d-1}|^{-1/2}$ is the unique eigenfunction corresponding to the lowest eigenvalue of $\mathcal{V}_\mu^{(d)}$ by radiality of V and the assumption $\hat{V} \leq 0$ (see e.g. [22]). Hence, for λ small enough, u is the unique eigenfunction of $T_\Delta^{(d)}$ of smallest eigenvalue. Thus,

$$\phi = f(\lambda) \frac{1}{1 + \lambda V^{1/2} M_\Delta^{(d)} |V|^{1/2}} V^{1/2} (\mathfrak{F}_\mu^{(d)})^\dagger u = f(\lambda) \left(V^{1/2} (\mathfrak{F}_\mu^{(d)})^\dagger u + \lambda \xi_\lambda \right)$$

for some number $f(\lambda)$. The function ξ_λ satisfies $\|\xi_\lambda\|_{L^2(\mathbb{R}^d)} \leq C$ by Lemma 3.3.6. Noting that $\Delta = -2 \widehat{|V|^{1/2}\phi}$ and bounding $\left\| \widehat{|V|^{1/2}\xi_\lambda} \right\|_{L^\infty} \leq \|V\|_{L^1}^{1/2} \|\xi_\lambda\|_{L^2}$ we get the desired. □

Evaluating the formula in Lemma 3.3.7 at $p = \sqrt{\mu}$ we get $|f(\lambda)| \leq C\Delta(\sqrt{\mu})$ for λ small enough. This in turn implies that

$$\Delta(p) \leq C \Delta(\sqrt{\mu}). \tag{3.16}$$

For the Hölder-continuity, we have by rotation invariance

$$\begin{aligned}
&\left| \int \hat{V}(p - \sqrt{\mu} r) - \hat{V}(q - \sqrt{\mu} r) \mathrm{d}\omega(r) \right| = \left| \int \hat{V}(|p| e_1 - \sqrt{\mu} r) - \hat{V}(|q| e_1 - \sqrt{\mu} r) \mathrm{d}\omega(r) \right| \\
&= \left| \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \mathrm{d}x \left(V(x) \left(e^{i|p|x_1} - e^{i|q|x_1} \right) \int_{\mathbb{S}^{d-1}} e^{-i\sqrt{\mu} x \cdot r} \mathrm{d}\omega(r) \right) \right| \\
&\leq C_\epsilon \mu^{-\epsilon/2} ||p| - |q||^\epsilon \int \mathrm{d}x \left(|V(x)| (\sqrt{\mu}|x|)^\epsilon \left| \int_{\mathbb{S}^{d-1}} e^{-i\sqrt{\mu} x \cdot r} \mathrm{d}\omega(r) \right| \right),
\end{aligned}$$

for any $0 < \epsilon \leq 1$. For $d = 2$ we have $V \in L^1(\mathbb{R}^2)$ and

$$\left| \int_{\mathbb{S}^{d-1}} e^{-i\sqrt{\mu} x r} \mathrm{d}\omega(r) \right| = |J_0(\sqrt{\mu}|x|)| \leq (\sqrt{\mu}|x|)^{-1/2}.$$

For $d = 1$ we have $|x|^\epsilon V \in L^1(\mathbb{R})$ for some $\epsilon > 0$ and

$$\left|\int_{\mathbb{S}^{d-1}} e^{-i\sqrt{\mu}x\cdot r}\mathrm{d}\omega(r)\right| = 2|\cos(\sqrt{\mu}|x|)| \leq 2.$$

We conclude that with $\epsilon = 1/2$ for $d = 2$ and small enough $\epsilon > 0$ for $d = 1$

$$|\Delta(p) - \Delta(q)| \leq C|f(\lambda)| \left(\mu^{-\epsilon/2}||p| - |q||^\epsilon + \lambda\right) \leq C|\Delta(\sqrt{\mu})| \left(\mu^{-\epsilon/2}||p| - |q||^\epsilon + \lambda\right). \tag{3.17}$$

Additionally, since $m_\mu^{(d)}(\Delta) \to \infty$ we have that $\Delta(p) \to 0$ at least for some $p \in \mathbb{R}^d$ by (3.9). Then it follows from Lemma 3.3.7 that $f(\lambda) \to 0$, i.e. that $\Delta(p) \to 0$ for all p.

5. Calculation of the integral $m_\mu^{(d)}(\Delta)$. Armed with the apriori bounds (3.16) and (3.17), we can now compute the integral $m_\mu^{(d)}(\Delta)$. Carrying out the angular integration and substituting $s = \left|\frac{|p|^2-\mu}{\mu}\right|$ we have

$$\begin{aligned} m_\mu^{(d)}(\Delta) = \frac{\mu^{d/2-1}}{2} \Bigg[&\int_0^1 \left(\frac{(1-s)^{d/2-1}-1}{\sqrt{s^2+x_-(s)^2}} + \frac{(1+s)^{d/2-1}-1}{\sqrt{s^2+x_+(s)^2}}\right)\mathrm{d}s \\ &+ \int_0^1 \left(\frac{1}{\sqrt{s^2+x_-(s)^2}} + \frac{1}{\sqrt{s^2+x_+(s)^2}}\right)\mathrm{d}s\Bigg], \end{aligned}$$

where $x_\pm(s) = \frac{\Delta(\sqrt{\mu}\sqrt{1\pm s})}{\mu}$. By dominated convergence, using that $x_\pm(s) \to 0$, the first integral is easily seen to converge to

$$\int_0^1 \left(\frac{(1-s)^{d/2-1}-1}{s} + \frac{(1+s)^{d/2-1}-1}{s}\right)\mathrm{d}s = 2\ln c_d$$

for $\lambda \to 0$. For the second integral, we will now show that

$$\int_0^1 \left(\frac{1}{\sqrt{s^2+x_\pm(s)^2}} - \frac{1}{\sqrt{s^2+x_\pm(0)^2}}\right)\mathrm{d}s \to 0\,.$$

In fact, the integrand is bounded by

$$\begin{aligned} &\left|\frac{1}{\sqrt{s^2+x_\pm(s)^2}} - \frac{1}{\sqrt{s^2+x_\pm(0)^2}}\right| \\ &= \frac{|x_\pm(0)^2 - x_\pm(s)^2|}{\sqrt{s^2+x_\pm(s)^2}\sqrt{s^2+x_\pm(0)^2}(\sqrt{s^2+x_\pm(s)^2}+\sqrt{s^2+x_\pm(0)^2})} \\ &\leq \frac{Cx_\pm(0)(s^\epsilon+\lambda)}{\sqrt{s^2+x_\pm(s)^2}\sqrt{s^2+x_\pm(0)^2}}, \end{aligned}$$

using the Hölder continuity from (3.17). By continuity of $\hat{V}$ there exists some s_0 (independent of λ) such that for $s < s_0$ we have $x_\pm(s) \geq cx_\pm(0)$. We now split the integration into $\int_0^{s_0}$ and $\int_{s_0}^1$. For the first we have

$$\int_0^{s_0}\left|\frac{1}{\sqrt{s^2+x_\pm(s)^2}} - \frac{1}{\sqrt{s^2+x_\pm(0)^2}}\right|\mathrm{d}s \leq C\int_0^{s_0}\frac{x_\pm(0)}{s^2+x_\pm(0)^2}(s^\epsilon+\lambda)\mathrm{d}s = O(x_\pm(0)^\epsilon+\lambda)\,.$$

For the second we have

$$\int_{s_0}^{1}\left|\frac{1}{\sqrt{s^2+x_\pm(s)^2}}-\frac{1}{\sqrt{s^2+x_\pm(0)^2}}\right|\mathrm{d}s \le C\int_{s_0}^{1} x_\pm(0)\frac{s^\epsilon+\lambda}{s^2}\mathrm{d}s = O(x_\pm(0))\,.$$

Collecting all the estimates, we have thus shown that $m_\mu^{(d)}(\Delta)$ equals

$$\begin{aligned}&\mu^{d/2-1}\left(\ln c_d+\int_0^1\frac{1}{\sqrt{s^2+\Delta(\sqrt{\mu})^2/\mu^2}}\mathrm{d}s+o(1)\right)\\ &=\mu^{d/2-1}\left(\ln c_d+\ln\left(\frac{\mu+\sqrt{\mu^2+\Delta(\sqrt{\mu})^2}}{|\Delta(\sqrt{\mu})|}\right)+o(1)\right)=\mu^{d/2-1}\ln\left(\frac{2\mu c_d}{|\Delta(\sqrt{\mu})|}+o(1)\right)\,.\end{aligned}$$

This proves the third inequality in Proposition 3.3.2.

Combining this with the third step, one immediately sees that the gap function evaluated on the Fermi sphere vanishes exponentially fast, $\Delta(\sqrt{\mu})\sim e^{1/\lambda e_\mu}$, as $\lambda\to 0$, recalling that $e_\mu^{(d)}<0$ by assumption.

6. Second order. To obtain the next order, we recall that $T_\Delta^{(d)}$ has lowest eigenvalue -1 (see (3.15)), and hence, by first-order perturbation theory,

$$m_\mu^{(d)}(\Delta)=\frac{-1}{\lambda\left\langle u\middle|\mathfrak{F}_\mu^{(d)}V(\mathfrak{F}_\mu^{(d)})^\dagger\middle|u\right\rangle-\lambda^2\left\langle u\middle|\mathfrak{F}_\mu^{(d)}VM_\Delta^{(d)}V(\mathfrak{F}_\mu^{(d)})^\dagger\middle|u\right\rangle+O(\lambda^3)},\tag{3.18}$$

where $u(p)=|\mathbb{S}^{d-1}|^{-1/2}$ is the constant function on the sphere. Recall that u is the unique ground state of $\mathcal{V}_\mu^{(d)}$.

In the second order term we have that

$$\lim_{\lambda\to 0}\left\langle u\middle|\mathfrak{F}_\mu^{(d)}VM_\Delta^{(d)}V(\mathfrak{F}_\mu^{(d)})^\dagger\middle|u\right\rangle=\left\langle u\middle|\mathcal{W}_\mu^{(d)}\middle|u\right\rangle\,,$$

which follows from a simple dominated convergence argument as for T_c, noting that $\Delta(p)\to 0$ pointwise.

By again employing first–order perturbation theory, similarly to the last step in the proof of Proposition 3.3.1, we conclude the second equality in Proposition 3.3.2.

7. Comparing $\Delta(\sqrt{\mu})$ to Ξ. To prove the first equality in Proposition 3.3.2 we separately prove upper and lower bounds. The upper bound is immediate from

$$\Xi=\inf_{p\in\mathbb{R}^d}E_\Delta(p)=\inf_{p\in\mathbb{R}^d}\sqrt{|p^2-\mu|+\Delta(p)^2}\le\Delta(\sqrt{\mu})\,.$$

Hence, for the lower bound, take $p\in\mathbb{R}^d$ with $\sqrt{|p^2-\mu|}\le\Xi\le\Delta(\sqrt{\mu})$. Then by (3.17)

$$\Delta(p)\ge\Delta(\sqrt{\mu})-|\Delta(p)-\Delta(\sqrt{\mu})|\ge\Delta(\sqrt{\mu})-C\Delta(\sqrt{\mu})\left(||p|-\sqrt{\mu}|^\epsilon+\lambda\right)\ge\Delta(\sqrt{\mu})(1+o(1)).$$

In combination with the upper bound, we have thus shown that $\Xi=\Delta(\sqrt{\mu})(1+o(1))$ as desired. This concludes the proof of Proposition 3.3.2. □

We conclude this subsection with several remarks, comparing our proof with those of similar results from the literature.

Remark 3.3.8 (Structure here vs. in earlier papers on Ξ)**.** We now compare the proof above to the proofs of the three different limits in 3 dimensions [35, 39, 49]:

- **Weak coupling:** The structure of our proof here is very similar to that of [35]. Essentially, only the technical details in Lemma 3.3.6 and the calculation of $m_\mu^{(d)}(\Delta)$ in Step 5 are different.
- **High density:** For the high-density limit in [39], we needed some additional a priori bounds on Δ before we could employ the Birman-Schwinger argument. Apart from that, in [39] the comparison of $\Delta(\sqrt{\mu})$ and Ξ are done right after these a priori bounds. Additionally, since one starts with finding a priori bounds on Δ, one does not need the first-order analysis in Step 3. One may think of the structure in [39] as being ordered in the above steps as follows: 4, 7, 1, 2, 4 (again), 5, 6.
- **Low density:** For the low-density limit in [49] the structure is quite different. Again, one first needs some a priori bounds on Δ before one can use the Birman-Schwinger argument. One then improves these bounds on Δ using the Birman-Schwinger argument, which in turn can be used to get better bounds on the error term in the decomposition of the Birman–Schwinger operator. In this sense, the Steps 2–4 are too interwoven to be meaningfully separated. Also, Step 5 is done in two parts.

3.3.3 Proof of Proposition 3.2.4

Note that $\mathcal{W}_\mu^{(d)} = \mathfrak{F}_\mu^{(d)} V M_0^{(d)} V (\mathfrak{F}_\mu^{(d)})^\dagger$, where $M_0^{(d)}$ is defined in (3.10). By Lemma 3.3.4, $V^{1/2} M_0^{(d)} V^{1/2}$ is Hilbert-Schmidt. The integral kernel of $\mathcal{W}_\mu^{(d)}$ is bounded by

$$|\mathcal{W}_\mu^{(d)}(p,q)| \leq \frac{1}{(2\pi)^d} \int_{\mathbb{R}^{2d}} |V(x)||M_0^{(d)}(x,y)||V(y)| \mathrm{d}x \mathrm{d}y \leq \frac{1}{(2\pi)^d} \|V\|_1 \|V^{1/2} M_0^{(d)} V^{1/2}\|_{\mathrm{HS}}. \tag{3.19}$$

It follows that $\|\mathcal{W}_\mu^{(d)}\|_{\mathrm{HS}} \leq \frac{|\mathbb{S}^{d-1}|}{(2\pi)^d} \|V\|_1 \|V^{1/2} M_0^{(d)} V^{1/2}\|_{\mathrm{HS}}$. □

CHAPTER 4

Boundary Superconductivity in the BCS Model

Abstract We consider the linear BCS equation, determining the BCS critical temperature, in the presence of a boundary, where Dirichlet boundary conditions are imposed. In the one-dimensional case with point interactions, we prove that the critical temperature is strictly larger than the bulk value, at least at weak coupling. In particular, the Cooper-pair wave function localizes near the boundary, an effect that cannot be modeled by effective Neumann boundary conditions on the order parameter as often imposed in Ginzburg–Landau theory. We also show that the relative shift in critical temperature vanishes if the coupling constant either goes to zero or to infinity.

4.1 Introduction and Main Result

We study how a boundary influences the critical temperature of a superconductor in the Bardeen–Cooper–Schrieffer (BCS) model. At superconductor–insulator (or superconductor–vacuum) boundaries, it is natural to impose Dirichlet boundary conditions on the Cooper-pair wave function. In several works [1, 12, 15] it was concluded that the presence of the boundary only affects the Cooper-pair wave function on microscopic scales; in particular, on larger scales described by Ginzburg–Landau theory (GL), the effect of the Dirichlet boundary conditions disappears and consequently the GL order parameter should satisfy *Neumann* boundary conditions [28, Ch. 7.3], [56, Ch. 6]. This seems to implicitly assume that the effect of the boundary on the critical temperature is negligible. Recent computations [6, 7, 62] indicate, however, that the Cooper-pair wave function can localize near the boundary, leading to an increase in the critical temperature compared to its bulk value. In this paper, we shall give a rigorous proof of the occurrence of this phenomenon in the simplest setting of one dimension, with δ-interactions among the particles. We consider a system on the half-line, where the boundary is then just a point.

The increase of the critical temperature in the presence of a boundary has some far-reaching implications. First of all, it implies that boundary superconductivity in the BCS model sets in already above the bulk value of the critical temperature. Second, it questions the validity of the often employed phenomenological GL theory in the presence of boundaries, as detailed in [63]. Note that GL theory has so far only been rigorously derived from the BCS model for periodic systems without boundaries [23]. (In the low-density BEC limit at zero temperature it was shown in [27] that the effective Gross–Pitaevskii theory inherits the microscopic Dirichlet boundary conditions.)

In mathematical terms, the presence of a boundary manifests itself in a compact perturbation of a translation-invariant operator, and we shall show that at weak coupling this leads to the appearance of discrete eigenvalues outside the continuous spectrum. In particular, there is an effective attraction to the boundary, which is strong enough to create bound states.

In the following, we shall consider a superconductor on a domain Ω, with either $\Omega = \mathbb{R}$ or $\Omega = \mathbb{R}_+ = (0, \infty)$. The main quantity of interest is the linear two-particle operator

$$H_T^\Omega = \frac{-\Delta_x - \Delta_y - 2\mu}{\tanh\left(\frac{-\Delta_x - \mu}{2T}\right) + \tanh\left(\frac{-\Delta_y - \mu}{2T}\right)} - v\delta(x-y) \tag{4.1}$$

acting in $L^2_{\text{symm}}(\Omega^2) = \{\psi \in L^2(\Omega^2) | \psi(x,y) = \psi(y,x) \text{ for all } x, y \in \Omega\}$, where Δ denotes the *Dirichlet* Laplacian on Ω, and the subscripts x and y, respectively, indicate the variable on which Δ acts. The first term is defined through functional calculus. In the second term, δ is the Dirac delta distribution, and $v > 0$ is a coupling constant. Moreover, $T > 0$ denotes the temperature, and $\mu \in \mathbb{R}$ is the chemical potential.

As explained in [21], H_T^Ω characterizes the local stability of the normal state in BCS theory. If H_T^Ω has spectrum below zero, i.e. $\inf \sigma(H_T^\Omega) < 0$, the normal state is unstable and the system in Ω is superconducting. If $\inf \sigma(H_T^\Omega) \geq 0$, the normal state is locally stable. We define the critical temperatures T_c^Ω as

$$T_c^\Omega(v) := \inf\left\{T \in (0,\infty) | \inf \sigma(H_T^\Omega) \geq 0\right\}. \tag{4.2}$$

The sample is thus superconducting for $T < T_c^\Omega$. In the translation-invariant case, i.e. $\Omega = \mathbb{R}$, it is also known that local stability of the normal state implies global stability [33]; in particular, the sample is always in a normal state for $T \geq T_c^{\mathbb{R}}$ in this case, i.e. $T_c^{\mathbb{R}}$ separates the superconducting and the normal phases. For the point interactions considered in (4.1), one can derive the explicit relation

$$\frac{1}{2\pi} \int_{\mathbb{R}} \frac{\tanh\left(\frac{q^2-\mu}{2T_c^{\mathbb{R}}(v)}\right)}{q^2 - \mu} \, \mathrm{d}q = \frac{1}{v}. \tag{4.3}$$

Because of translation invariance, $H_T^{\mathbb{R}}$ has purely essential spectrum. Moreover, $H_T^{\mathbb{R}_+}$ has the same essential spectrum and possibly additional eigenvalues below it. In particular, for all $v > 0$ the critical temperatures satisfy

$$T_c^{\mathbb{R}_+}(v) \geq T_c^{\mathbb{R}}(v). \tag{4.4}$$

Our main result states that this inequality is actually strict, at least for small v, proving that the boundary increases the critical temperature. Moreover, the relative difference between the two critical temperatures vanishes both in the weak and in the strong coupling limit.

Theorem 4.1.1. *Let* $\mu > 0$.

1. *There is a* $\tilde{v} > 0$ *such that*

$$T_c^{\mathbb{R}_+}(v) > T_c^{\mathbb{R}}(v) \tag{4.5}$$

for $0 < v < \tilde{v}$.

2. *In the weak coupling limit*

$$\lim_{v\to 0} \frac{T_c^{\mathbb{R}_+}(v) - T_c^{\mathbb{R}}(v)}{T_c^{\mathbb{R}}(v)} = 0 \tag{4.6}$$

3. *In the strong coupling limit*

$$\lim_{v\to \infty} \frac{T_c^{\mathbb{R}_+}(v) - T_c^{\mathbb{R}}(v)}{T_c^{\mathbb{R}}(v)} = 0 \tag{4.7}$$

This result can be viewed as a rigorous justification of the observations in [62]. Numerics shows that the ratio $T_c^{\mathbb{R}_+}(v)/T_c^{\mathbb{R}}(v)$ can be as large as 1.06, see [62, Fig. 2]. Moreover, numerics also suggests that $T_c^{\mathbb{R}_+}(v)$ and $T_c^{\mathbb{R}}(v)$ actually agree for v large enough, but it remains an open problem to show this.

Part 1 of Theorem 4.1.1 follows from the existence of an eigenvalue of $H_T^{\mathbb{R}_+}$ below the spectrum of $H_T^{\mathbb{R}}$. It is quite remarkable that a Dirichlet boundary can decrease the ground state energy and create bound states. In contrast, for two-particle Schrödinger operators of the form $-\Delta_x - \Delta_y + V(x-y)$, only Neumann boundaries can bind states [18, 59].

While we restrict our attention in this article to the one-dimensional setting with point interactions, we expect that our methods can be generalized to a larger class of interaction potentials, as well as to higher dimensions and the corresponding more complicated geometries possible. We shall leave these generalizations for future investigations, however.

Remark 4.1.2. Our techniques can also be applied in case of Neumann boundary conditions for Δ on $\mathbb{R}_+$. In this case one obtains the following results instead.

1. For all $v > 0$

$$T_c^{\mathbb{R}_+}(v) > T_c^{\mathbb{R}}(v) \tag{4.8}$$

2. In the weak coupling limit

$$\lim_{v\to 0} \frac{T_c^{\mathbb{R}_+}(v) - T_c^{\mathbb{R}}(v)}{T_c^{\mathbb{R}}(v)} = 0 \tag{4.9}$$

3. In the strong coupling limit

$$0 < \lim_{v\to \infty} \frac{T_c^{\mathbb{R}_+}(v) - T_c^{\mathbb{R}}(v)}{T_c^{\mathbb{R}}(v)} < \infty \tag{4.10}$$

In the remainder of this article we shall give the proof of Theorem 4.1.1. In the next Section 4.2, we shall use the Birman–Schwinger principle to conveniently reformulate the problem in terms of bounded operators and compact perturbations. Section 4.3 contains the proof of part 1, the existence of boundary superconductivity. The analysis of the weak and strong coupling limits in parts 2 and 3 is the content of Sections 4.4 and 4.5, respectively. Finally, Section 4.6 contains the proofs of some auxiliary Lemmas.

4.2 Preliminaries

Let us fix the notation

$$L_{T,\mu}(p,q) := \frac{\tanh\left(\frac{p^2-\mu}{2T}\right) + \tanh\left(\frac{q^2-\mu}{2T}\right)}{p^2+q^2-2\mu}. \tag{4.11}$$

Using the partial fraction expansion for $\tanh$ (Mittag-Leffler series), one can obtain the series representation [21]

$$L_{T,\mu}(p,q) = 2T \sum_{n\in\mathbb{Z}} \frac{1}{p^2-\mu-iw_n} \frac{1}{q^2-\mu+iw_n} \tag{4.12}$$

for $w_n = \pi(2n+1)T$. Moreover, let

$$F_{T,\mu}(p) := L_{T,\mu}(p,p) = \frac{\tanh\left(\frac{p^2-\mu}{2T}\right)}{p^2-\mu} \tag{4.13}$$

and

$$B_{T,\mu}(p,q) := L_{T,\mu}\left(\frac{p+q}{2}, \frac{p-q}{2}\right) \tag{4.14}$$

In order to control the kinetic energy in H_T^Ω the following bounds turn out to be useful. We shall prove them in Section 4.6.1.

Lemma 4.2.1. *Let $T>0$. There are constants $C_1(T,\mu), C_2(T,\mu) > 0$ such that for all $p,q\in\mathbb{R}$*

$$C_1(T,\mu)(1+p^2+q^2) \le L_{T,\mu}(p,q)^{-1} \le C_2(T,\mu)(1+p^2+q^2) \tag{4.15}$$

Moreover, for $T_0>0$ there is a $C_3(T_0,\mu)>0$ such that

$$C_3(T_0,\mu)(T+p^2+q^2) \le L_{T,\mu}(p,q)^{-1} \tag{4.16}$$

for all $T>T_0$ and $p,q\in\mathbb{R}$.

Since $v\delta(x-y)$ is infinitesimally form bounded with respect to $-\Delta_x - \Delta_y$, it follows that the H_T^Ω are self-adjoint operators defined via the KLMN theorem. Moreover, the operators H_T^Ω become positive for T large enough. In particular, the critical temperatures defined in (4.2) are finite in both cases $\Omega=\mathbb{R}$ and $\Omega=\mathbb{R}_+$.

Let $L_{T,\mu}^\Omega$ denote the operator $L_{T,\mu}(-i\nabla_x, -i\nabla_y)$ defined through functional calculus. Of course, $L_{T,\mu}^\Omega$ depends on the domain Ω and on the boundary conditions imposed on Δ. Its integral kernel is given by

$$L_{T,\mu}^\Omega(x,y;x',y') = \int_{\mathbb{R}^2} \mathrm{d}p\,\mathrm{d}q\, \overline{t_\Omega(xp)t_\Omega(yq)} L_{T,\mu}(p,q) t_\Omega(x'p) t_\Omega(y'q)\,, \tag{4.17}$$

where for the problem on the full real line $t_{\mathbb{R}}(x) = \frac{1}{\sqrt{2\pi}}e^{-ix}$ and on the half-line with Dirichlet boundary condition $t_{\mathbb{R}_+}(x) = \frac{1}{\sqrt{\pi}}\sin(x)$. For Neumann boundary conditions, one would have $t_{\mathbb{R}_+}(x) = \frac{1}{\sqrt{\pi}}\cos(x)$ instead.

It is convenient to switch to the Birman–Schwinger formulation of the problem. For a more regular interaction V instead of δ, the Birman-Schwinger operator would be $V^{1/2}L_{T,\mu}^\Omega V^{1/2}$. For the δ-case, it turns out that $V^{1/2}$ has to be understood as restriction of a two-body wave function to its diagonal. Hence, the Birman-Schwinger operator has kernel $L^{\Omega}_{T,\mu}(x,x;x',x')$ and acts on functions of one variable only. For the two domains under consideration, the Birman-Schwinger operators $A^{\mathbb{R}_+}_{T,\mu} : L^2((0,\infty)) \to L^2((0,\infty))$ and $A^{\mathbb{R}}_{T,\mu} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ are explicitly given by

$$(A^{\mathbb{R}_+}_{T,\mu}\alpha)(x) = \frac{1}{\pi^2}\int_{\mathbb{R}} \mathrm{d}p \int_{\mathbb{R}} \mathrm{d}q \int_0^\infty \mathrm{d}y \sin(px)\sin(qx) L_{T,\mu}(p,q)\sin(py)\sin(qy)\alpha(y) \quad (4.18)$$

and

$$(A^{\mathbb{R}}_{T,\mu}\beta)(x) = \frac{1}{4\pi^2}\int_{\mathbb{R}} \mathrm{d}p \int_{\mathbb{R}} \mathrm{d}q \int_{\mathbb{R}} \mathrm{d}y\, e^{i(p+q)(x-y)} L_{T,\mu}(p,q)\beta(y) \quad (4.19)$$

Lemma 4.2.2. *The condition* $\inf\sigma(H^{\Omega}_T) < 0$ *is equivalent to*

$$\sup\sigma(A^{\Omega}_{T,\mu}) > \frac{1}{v} \quad (4.20)$$

for either $\Omega = \mathbb{R}$ *or* $\Omega = \mathbb{R}_+$.

Proof. The quadratic form corresponding to H^{Ω}_T is defined on the Sobolev space $D_\Omega = H^1_0(\Omega^2)$. Since the operator $L^{\Omega}_{T,\mu}$ is positive definite, one can write

$$H^{\Omega}_T = \left(L^{\Omega}_{T,\mu}\right)^{-1} - v\delta = \frac{1}{\sqrt{L^{\Omega}_{T,\mu}}}\left(\mathbb{I} - v\sqrt{L^{\Omega}_{T,\mu}}\delta\sqrt{L^{\Omega}_{T,\mu}}\right)\frac{1}{\sqrt{L^{\Omega}_{T,\mu}}}\,. \quad (4.21)$$

Hence, $\inf\sigma(H^{\Omega}_T) < 0$ is equivalent to

$$\sup_{\psi\in(L^{\Omega}_{T,\mu})^{-1/2}D_\Omega, \|\psi\|_2=1} \left\langle \psi \middle| \sqrt{L^{\Omega}_{T,\mu}}\delta\sqrt{L^{\Omega}_{T,\mu}} \middle| \psi \right\rangle > \frac{1}{v}\,. \quad (4.22)$$

By Lemma 4.2.1, $\sqrt{L^{\Omega}_{T,\mu}} : L^2(\Omega^2) \to D_\Omega$ and its inverse are bounded. Hence, $(L^{\Omega}_{T,\mu})^{-1/2}D_\Omega = L^2(\Omega^2)$. The projection onto the diagonal $H^1(\Omega^2) \to L^2(\Omega)$, $\psi(x,y) \mapsto \psi(x,x)$ defines a bounded operator [2, Thm 4.12]. Let $M_\Omega : L^2(\Omega^2) \to L^2(\Omega)$ be the composition of $\sqrt{L^{\Omega}_{T,\mu}}$ with the projection $H^1(\Omega^2) \to L^2(\Omega)$. Explicitly, M_Ω is given by

$$M_\Omega\psi(x) = \int_{\mathbb{R}^2} \mathrm{d}p\,\mathrm{d}q \int_{\Omega^2} \mathrm{d}x'\mathrm{d}y'\, \overline{t_\Omega(xp)t_\Omega(xq)}\sqrt{L_{T,\mu}(p,q)}\,t_\Omega(x'p)t_\Omega(y'q)\psi(x',y') \quad (4.23)$$

where $t_{\mathbb{R}}(x) = \frac{1}{\sqrt{2\pi}}e^{-ix}$ and $t_{\mathbb{R}_+}(x) = \frac{1}{\sqrt{\pi}}\sin(x)$. Note that $\sqrt{L^{\Omega}_{T,\mu}}\delta\sqrt{L^{\Omega}_{T,\mu}} = M^\dagger_\Omega M_\Omega$ and $A^{\Omega}_{T,\mu} = M_\Omega M^\dagger_\Omega$. Hence, $\sigma(A^{\Omega}_{T,\mu}) \setminus \{0\} = \sigma(\sqrt{L^{\Omega}_{T,\mu}}\delta\sqrt{L^{\Omega}_{T,\mu}}) \setminus \{0\}$ and the claim follows. □

From now on we will work with the operators $A^{\Omega}_{T,\mu}$ rather than H^{Ω}_T. In momentum space, the operator $A^{\mathbb{R}}_{T,\mu}$ is multiplication by the function

$$A_{T,\mu}(p) = \frac{1}{4\pi}\int_{\mathbb{R}} B_{T,\mu}(p,q)\mathrm{d}q\,, \quad (4.24)$$

where B is defined in (4.14).

Lemma 4.2.3 (Momentum representation of $A^{\mathbb{R}}_{T,\mu}$)**.** *With* $\widehat{\beta}(p) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\beta(x)e^{ipx}\mathrm{d}x$ *we have for all* $\beta_1, \beta_2 \in D(A^{\mathbb{R}}_{T,\mu})$

$$\langle\beta_1|A^{\mathbb{R}}_{T,\mu}|\beta_2\rangle = \int_{\mathbb{R}} \overline{\widehat{\beta}_1}(p)A_{T,\mu}(p)\widehat{\beta}_2(p)\mathrm{d}p. \quad (4.25)$$

The following Lemma shows that adding the boundary to the system effectively introduces the perturbation $\frac{1}{4\pi}B_{T,\mu}$, where $B_{T,\mu}$ is short for the operator with integral kernel $B_{T,\mu}(p,q)$.

Lemma 4.2.4 (Momentum representation of $A^{\mathbb{R}_+}_{T,\mu}$)**.** *With* $\widehat{\alpha}(p) = \int_0^\infty \alpha(x)\frac{1}{\sqrt{\pi}}\cos(px)\mathrm{d}x$ *we have for all* $\alpha_1, \alpha_2 \in D(A^{\mathbb{R}_+}_{T,\mu})$

$$\langle \alpha_1 | A^{\mathbb{R}_+}_{T,\mu} | \alpha_2 \rangle = \int_{\mathbb{R}} \overline{\widehat{\alpha}_1}(p) A_{T,\mu}(p) \widehat{\alpha}_2(p) \mathrm{d}p - \frac{1}{4\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} \overline{\widehat{\alpha}_1}(p) B_{T,\mu}(p,q)\,\widehat{\alpha}_2(q)\mathrm{d}p\mathrm{d}q. \tag{4.26}$$

Note that here we work with the cosine transform and not the sine transform as might be expected from (4.18). This is because α is the diagonal of a function which is antisymmetric under both $x \to -x$ and $y \to -y$ and hence symmetric under $(x,y) \to (-x,-y)$.

Proof or Lemma 4.2.4. Using that $\sin(px)\sin(qx) = \frac{1}{2}[\cos((p-q)x) - \cos((p+q)x)]$ and substituting $p' = p - q$ and $q' = p + q$ gives

$$\begin{aligned}
\langle \alpha_1 | A^{\mathbb{R}_+}_{T,\mu} | \alpha_2 \rangle &= \frac{1}{\pi^2}\int_{\mathbb{R}^2} \mathrm{d}p\,\mathrm{d}q \int_0^\infty \mathrm{d}x \int_0^\infty \mathrm{d}y\, \overline{\alpha_1(x)} \sin(px)\sin(qx) L_{T,\mu}(p,q)\sin(py)\sin(qy)\alpha_2(y) \\
&= \frac{1}{4\pi^2}\int_{\mathbb{R}^2} \frac{\mathrm{d}p'\mathrm{d}q'}{2} \int_0^\infty \mathrm{d}x \int_0^\infty \mathrm{d}y \Bigg[\overline{\alpha_1(x)}[\cos(p'x) - \cos(q'x)] L_{T,\mu}\left(\frac{p'+q'}{2}, \frac{p'-q'}{2}\right) \\
&\qquad \times [\cos(p'y) - \cos(q'y)]\alpha_2(y) \Bigg] \\
&= \int_{\mathbb{R}^2} \frac{\mathrm{d}p'\mathrm{d}q'}{8\pi} \left[\overline{\widehat{\alpha}_1(p')} - \overline{\widehat{\alpha}_1(q')}\right] B_{T,\mu}(p',q')\,[\widehat{\alpha}_2(p') - \widehat{\alpha}_2(q')]\,.
\end{aligned} \tag{4.27}$$

Since $B(p',q') = B(q',p')$, this reduces to

$$\langle \alpha_1 | A^{\mathbb{R}_+}_{T,\mu} | \alpha_2 \rangle = \int_{\mathbb{R}^2} \frac{\mathrm{d}p'\mathrm{d}q'}{4\pi} \overline{\widehat{\alpha}_1(p')} B_{T,\mu}(p',q')\,[\widehat{\alpha}_2(p') - \widehat{\alpha}_2(q')]\,. \tag{4.28}$$

□

Lemma 4.2.3 follows from an analogous computation.

Since the operator $A^{\mathbb{R}}_{T,\mu}$ is multiplication by the function (4.24), it has purely essential spectrum. The perturbation $B_{T,\mu}$ in $A^{\mathbb{R}_+}_{T,\mu}$ is Hilbert–Schmidt and thus compact. Hence, $\sigma(A^{\mathbb{R}}_{T,\mu}) = \sigma_{\text{ess}}(A^{\mathbb{R}_+}_{T,\mu})$. It follows that for all $T < T_c^{\mathbb{R}}(v)$ we have $\sup\sigma(A^{\mathbb{R}_+}_{T,\mu}) \geq \sup\sigma(A^{\mathbb{R}}_{T,\mu}) > 1/v$, which implies (4.4).

Remark 4.2.5. Choosing Neumann instead of Dirichlet boundary conditions amounts to changing the minus sign in (4.26) into a plus sign.

It is possible to give a more explicit expression for $\sup\sigma(A^{\mathbb{R}}_{T,\mu})$. The following is proved in Section 4.6.1.

Lemma 4.2.6. *For all* $p \in \mathbb{R}$

$$\int_{\mathbb{R}} B_{T,\mu}(p,q)\mathrm{d}q \leq \int_{\mathbb{R}} B_{T,\mu}(0,q)\mathrm{d}q\,. \tag{4.29}$$

Consequently,

$$a_{T,\mu} := \sup \sigma(A^{\mathbb{R}}_{T,\mu}) = \frac{1}{4\pi}\int_{\mathbb{R}} B_{T,\mu}(0,q)\mathrm{d}q\,. \tag{4.30}$$

Hence, in the translation invariant case superconductivity is equivalent to $a_{T,\mu} > \frac{1}{v}$ and the critical temperature is determined by (4.3). Note that $a_{T,\mu}$ is decreasing in T. Therefore, $T_c^{\mathbb{R}}(v)$ is a monotonically increasing function of v.

4.3 Existence of Boundary Superconductivity

From now on we assume that $\mu > 0$. In this Section, we show that for weak coupling the half-line critical temperature is higher than the bulk critical temperature. The idea is to prove that for T below a threshold $T_0 > 0$ we have

$$\sup \sigma(A^{\mathbb{R}_+}_{T,\mu}) > a_{T,\mu}\,. \tag{4.31}$$

Then consider $v < \tilde{v} := a^{-1}_{T_0,\mu}$. We must have $T_c^{\mathbb{R}}(v) < T_0$ by the monotonicity of $T_c^{\mathbb{R}}(v)$. By definition and continuity of $\inf \sigma(H_T^{\mathbb{R}^+})$ in T, $\sup \sigma\left(A^{\mathbb{R}_+}_{T_c^{\mathbb{R}_+}(v),\mu}\right) = \frac{1}{v} = a_{T_c^{\mathbb{R}}(v),\mu}$. If $T_c^{\mathbb{R}}(v) = T_c^{\mathbb{R}_+}(v)$, we would get a contradiction to (4.31). Thus, $T_c^{\mathbb{R}}(v) \neq T_c^{\mathbb{R}_+}(v)$ and, together with (4.4), part 1 of Theorem 4.1.1 follows.

To prove (4.31), we use the variational principle with a trial function mimicking the ground state found in [62]. We choose $\psi_\epsilon^\lambda(x) = e^{-\epsilon|x|} + \lambda g(x)$, where $\lambda \in \mathbb{R}$ and the cosine Fourier transform $\widehat{g}(p) = \frac{1}{\sqrt{\pi}}\int_0^\infty g(x)\cos(px)\mathrm{d}x$ is real, continuous and centered at $2\sqrt{\mu}$.

Proposition 4.3.1. *Let $\widehat{g}(p) = e^{-(|p|-2\sqrt{\mu})^2/b}$ for some constant $b > 0$. For $\mu > 0$ there exists $T_0 > 0$ such that for $T < T_0$*

$$\max_{\lambda} \lim_{\epsilon\to 0} \langle \psi_\epsilon^\lambda | A^{\mathbb{R}_+}_{T,\mu} - a_{T,\mu}\mathbb{I} | \psi_\epsilon^\lambda\rangle > 0.$$

As discussed above, Theorem 4.1.1 1 follows directly from Prop. 4.3.1.

Proof. Let $h_\epsilon(x) = e^{-\epsilon|x|}$. The cosine Fourier transform of the trial state is $\widehat{\psi}_\epsilon^\lambda(p) = \widehat{h}_\epsilon(p) + \lambda\widehat{g}(p)$, where $\widehat{h}_\epsilon(p) = \frac{1}{\sqrt{\pi}}\frac{\epsilon}{\epsilon^2+p^2}$. We have $\lim_{\epsilon\to 0}\langle \psi_\epsilon^\lambda | A^{\mathbb{R}_+}_{T,\mu} - a_{T,\mu}\mathbb{I} | \psi_\epsilon^\lambda\rangle = \lim_{\epsilon\to 0}\langle h_\epsilon | A^{\mathbb{R}_+}_{T,\mu} - a_{T,\mu}\mathbb{I} | h_\epsilon\rangle + 2\lambda \lim_{\epsilon\to 0}\langle g | A^{\mathbb{R}_+}_{T,\mu} - a_{T,\mu}\mathbb{I} | h_\epsilon\rangle + \lambda^2 \langle g | A^{\mathbb{R}_+}_{T,\mu} - a_{T,\mu}\mathbb{I} | g\rangle$. In Lemma 4.3.3 we show $\langle g | A^{\mathbb{R}_+}_{T,\mu} - a_{T,\mu}\mathbb{I} | g\rangle < 0$. Maximizing over λ thus yields

$$\max_{\lambda} \lim_{\epsilon\to 0} \langle \psi_\epsilon^\lambda | A^{\mathbb{R}_+}_{T,\mu} - a_{T,\mu}\mathbb{I} | \psi_\epsilon^\lambda\rangle = \lim_{\epsilon\to 0}\langle h_\epsilon | A^{\mathbb{R}_+}_{T,\mu} - a_{T,\mu}\mathbb{I} | h_\epsilon\rangle - \frac{\lim_{\epsilon\to 0}\langle g | A^{\mathbb{R}_+}_{T,\mu} - a_{T,\mu}\mathbb{I} | h_\epsilon\rangle^2}{\langle g | A^{\mathbb{R}_+}_{T,\mu} - a_{T,\mu}\mathbb{I} | g\rangle} \tag{4.32}$$

We now compute the two limits. Note that for bounded continuous functions f, we have $\lim_{\epsilon\to 0}\int_{\mathbb{R}} \frac{1}{\sqrt{\pi}}\frac{\epsilon}{\epsilon^2+p^2} f(p)\mathrm{d}p = \sqrt{\pi} f(0)$. Moreover, for bounded functions f such that $\lim_{p\to 0}\frac{f(p)}{p}$ exists, $\lim_{\epsilon\to 0}\int_{\mathbb{R}} \frac{1}{\pi}\frac{\epsilon^2}{(\epsilon^2+p^2)^2} f(p)\mathrm{d}p = \frac{1}{\pi}\lim_{p\to 0}\frac{f(p)}{p}$. With the momentum space representation of $A^{\mathbb{R}_+}_{T,\mu}$ in Lemma 4.2.4 we thus obtain

$$\begin{aligned}\lim_{\epsilon\to 0}\langle h_\epsilon | A^{\mathbb{R}_+}_{T,\mu} - a_{T,\mu}\mathbb{I} | g\rangle &= \lim_{\epsilon\to 0}\int_{\mathbb{R}} \mathrm{d}p\ \widehat{h}_\epsilon(p)\widehat{g}(p)\left(A_{T,\mu}(p) - A_{T,\mu}(0)\right) \\ &\quad - \lim_{\epsilon\to 0}\int_{\mathbb{R}} \mathrm{d}p\, \widehat{h}_\epsilon(p)\int_{\mathbb{R}} \mathrm{d}q\ \frac{1}{4\pi} B_{T,\mu}(p,q)\widehat{g}(q) = -\frac{1}{4\sqrt{\pi}}\int_{\mathbb{R}} \mathrm{d}q\ B_{T,\mu}(0,q)\widehat{g}(q).\end{aligned} \tag{4.33}$$

Moreover,

$$\begin{aligned}
&\lim_{\epsilon\to 0}\langle h_\epsilon|A^{\mathbb{R}_+}_{T,\mu} - a_{T,\mu}\mathbb{I}|h_\epsilon\rangle \\
&= \lim_{\epsilon\to 0}\int_{\mathbb{R}} \mathrm{d}p\, \widehat{h}^2_\epsilon(p)\int_{\mathbb{R}}\mathrm{d}q\,\frac{1}{4\pi}(B_{T,\mu}(p,q)-B_{T,\mu}(0,q)) - \lim_{\epsilon\to 0}\int_{\mathbb{R}}\mathrm{d}p\,\widehat{h}_\epsilon(p)\int_{\mathbb{R}}\mathrm{d}q\,\frac{1}{4\pi}B_{T,\mu}(p,q)\widehat{h}_\epsilon(q) \\
&= \frac{1}{\pi}\lim_{p\to 0}\frac{1}{p}\int_{\mathbb{R}}\mathrm{d}q\,\frac{1}{4\pi}(B_{T,\mu}(p,q)-B_{T,\mu}(0,q)) - \frac{1}{4}B_{T,\mu}(0,0). \qquad (4.34)
\end{aligned}$$

In the first summand, we want to interchange limit and integration using dominated convergence. The following Lemma is proved below.

Lemma 4.3.2. *The function* $f(p,q) = \frac{1}{p}(B_{T,\mu}(p,q) - B_{T,\mu}(0,q))$

1. *is continuous at* $p = 0$ *and satisfies* $f(0,q) = 0$ *for all* q.
2. *There is a* $g \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ *such that* $|f(p,q)| \le g(q)$ *for all* p *and* q.

By dominated convergence the first term on the right hand side of (4.34) vanishes and thus $\lim_{\epsilon\to 0}\langle h_\epsilon|A^{\mathbb{R}_+}_{T,\mu} - a_{T,\mu}\mathbb{I}|h_\epsilon\rangle = -\frac{1}{4}B_{T,\mu}(0,0)$. Combining this with (4.32) and (4.33) yields

$$\max_\lambda \lim_{\epsilon\to 0}\langle \psi^\lambda_\epsilon|A^{\mathbb{R}_+}_{T,\mu} - a_{T,\mu}\mathbb{I}|\psi^\lambda_\epsilon\rangle = -\frac{1}{4}B_{T,\mu}(0,0) - \frac{1}{16\pi}\frac{\left(\int_{\mathbb{R}} B_{T,\mu}(0,q)\widehat{g}(q)\mathrm{d}q\right)^2}{\langle g|A^{\mathbb{R}_+}_{T,\mu} - a_{T,\mu}|g\rangle} \qquad (4.35)$$

For $T \to 0$ the term $B_{T,\mu}(0,0)$ is bounded while the second summand diverges logarithmically, which is content of the following Lemma.

Lemma 4.3.3. *Let* $\widehat{g}(p) = e^{-\frac{(|p|-2\sqrt{\mu})^2}{b}}$ *for some* $b > 0$. *Then,*

1. $\frac{4}{\sqrt{\mu}}e^{-\frac{4\mu}{b}} < \lim_{T\to 0}\left(\ln\frac{\mu}{T}\right)^{-1}\int_{\mathbb{R}} B_{T,\mu}(0,q)\widehat{g}(q)\mathrm{d}q < \frac{4}{\sqrt{\mu}}$,
2. $0 \ge \lim_{T\to 0}\left(\ln\frac{\mu}{T}\right)^{-1}\langle g|A^{\mathbb{R}_+}_{T,\mu} - a_{T,\mu}|g\rangle > -\infty$.

Therefore, the last term in (4.35) dominates for small T and makes the right hand side positive. This completes the proof of Prop. 4.3.1. □

Remark 4.3.4. For Neumann boundary conditions, one obtains $\lim_{\epsilon\to 0}\langle h_\epsilon|A^{\mathbb{R}_+}_{T,\mu} - a_{T,\mu}\mathbb{I}|h_\epsilon\rangle = \frac{1}{4}L_{T,\mu}(0,0) > 0$. Hence, the trial state h_ϵ suffices to prove $\sup\sigma(A^{\mathbb{R}_+}_{T,\mu}) > a_{T,\mu}$ for all $T > 0$.

Proof of Lemma 4.3.2. Using (4.12) one obtains the series representation

$$f(p,q) = \frac{T}{8}\sum_{n\in\mathbb{Z}}\frac{8\mu p - p^3 + 2pq^2 - 16iqw_n}{\left(\left(\frac{p+q}{2}\right)^2 - \mu - iw_n\right)\left(\left(\frac{p-q}{2}\right)^2 - \mu + iw_n\right)\left(\left(\frac{q}{2}\right)^2 - \mu - iw_n\right)\left(\left(\frac{q}{2}\right)^2 - \mu + iw_n\right)} \qquad (4.36)$$

where $w_n = (2n+1)\pi T$. From this, claim 1 is easy to see. For part 2, note that by Lemma 4.2.1, $|f(p,q)| < \frac{C}{1+q^2} =: g_1(q)$ for $|p| > \sqrt{\mu}$. For $|p| < \sqrt{\mu}$,

$$\begin{aligned}
&\sup_{(p,q)\in\mathbb{R}^2, |p|<\sqrt{\mu}} \frac{|8\mu p - p^3 + 2pq^2|}{\left|\left(\left(\frac{p+q}{2}\right)^2 - \mu - iw_n\right)\left(\left(\frac{p-q}{2}\right)^2 - \mu + iw_n\right)\right|} \\
&\le \sup_{(p,q)\in\mathbb{R}^2, |p|<\sqrt{\mu}} \frac{8\mu|p| + |p|^3 + 2|p|q^2}{\sqrt{\left[\left(\frac{p+q}{2}\right)^2 - \mu\right]^2 + w_0^2}\sqrt{\left[\left(\frac{p-q}{2}\right)^2 - \mu\right]^2 + w_0^2}} =: c_1 < \infty \qquad (4.37)
\end{aligned}$$

and

$$\sup_{(p,q)\in\mathbb{R}^2} \frac{16|qw_n|}{\left|\left(\left(\frac{p+q}{2}\right)^2-\mu-iw_n\right)\left(\left(\frac{p-q}{2}\right)^2-\mu+iw_n\right)\right|}$$
$$\leq \sup_{(p,q)\in\mathbb{R}^2} \frac{16|q|}{\sqrt{\left[\left(\frac{|p|+|q|}{2}\right)^2-\mu\right]^2+w_0^2}} =: c_2 < \infty \quad (4.38)$$

With these estimates, one obtains for $|p| < \sqrt{\mu}$

$$|f(p,q)| \leq \frac{T(c_1+c_2)}{8}\sum_{n\in\mathbb{Z}} \frac{1}{\left(\frac{q^2}{4}-\mu\right)^2+w_n^2} \quad (4.39)$$

Using that the summands are decreasing in n, we can estimate the sum by an integral

$$|f(p,q)| \leq \frac{T(c_1+c_2)}{4}\left[\frac{1}{\left(\frac{q^2}{4}-\mu\right)^2+w_0^2}+\int_{1/2}^{\infty}\frac{1}{\left(\frac{q^2}{4}-\mu\right)^2+4\pi^2T^2x^2}\mathrm{d}x\right]$$
$$= \frac{T(c_1+c_2)}{4}\left[\frac{1}{\left(\frac{q^2}{4}-\mu\right)^2+w_0^2}+\frac{\arctan\left(\frac{|\frac{q^2}{4}-\mu|}{\pi T}\right)}{2\pi T|\frac{q^2}{4}-\mu|}\right] =: g_2(q) \quad (4.40)$$

Clearly, $g=\max\{g_1,g_2\}\in L^1(\mathbb{R})\cap L^\infty(\mathbb{R})$. □

The logarithmic divergence in Lemma 4.3.3 originates from the following asymptotics proved in Section 4.6.2.

Lemma 4.3.5. *Let $\mu>0$. As $T\to 0$*

$$\int_{\mathbb{R}} F_{T,\mu}(p)\mathrm{d}p = \frac{2}{\sqrt{\mu}}\left(\ln\frac{\mu}{T}+\gamma+\ln\frac{8}{\pi}\right)+o(1) = \int_{-\sqrt{2\mu}}^{\sqrt{2\mu}} F_{T,\mu}(p)\mathrm{d}p + O(1), \quad (4.41)$$

where γ denotes the Euler–Mascheroni constant.

Proof of Lemma 4.3.3. Part 1. On the interval $[-2\sqrt{2\mu}, 2\sqrt{2\mu}]$ the minimum of $\widehat{g}$ is $e^{-\frac{4\mu}{b}}$. We estimate

$$\int_{-2\sqrt{2\mu}}^{2\sqrt{2\mu}} B_{T,\mu}(0,p)e^{-\frac{4\mu}{b}}\mathrm{d}p \leq \int_{\mathbb{R}} B_{T,\mu}(0,p)\widehat{g}(p)\mathrm{d}p$$
$$\leq \int_{-2\sqrt{2\mu}}^{2\sqrt{2\mu}} B_{T,\mu}(0,p)\mathrm{d}p + \int_{\mathbb{R}} \chi_{|p|>2\sqrt{2\mu}}\frac{e^{-\frac{(|p|-2\sqrt{\mu})^2}{b}}}{(p/2)^2-\mu}\mathrm{d}p, \quad (4.42)$$

where we used $\widehat{g}(k)\leq 1$ and $\tanh(x)\leq 1$. The last summand is some constant independent of T. Using that $B_{T,\mu}(0,p)=F_{T,\mu}(p/2)$ and Lemma 4.3.5 the asymptotic behavior for $T\to 0$ is

$$\int_{-2\sqrt{2\mu}}^{2\sqrt{2\mu}} B_{T,\mu}(0,p)\mathrm{d}p = \int_{-2\sqrt{2\mu}}^{2\sqrt{2\mu}} F_{T,\mu}(p/2)\mathrm{d}p = 2\int_{-\sqrt{2\mu}}^{\sqrt{2\mu}} F_{T,\mu}(p)\mathrm{d}p = \frac{4}{\sqrt{\mu}}\ln\frac{\mu}{T}+O(1) \quad (4.43)$$

and the claim follows.

Part 2. Recall that

$$\langle g|A_{T,\mu}^{\mathbb{R}_+}-a_{T,\mu}|g\rangle = \int_{\mathbb{R}} \mathrm{d}p\, \widehat{g}(p)^2\,(A_{T,\mu}(p)-a_{T,\mu}) - \int_{\mathbb{R}} \mathrm{d}p\, \widehat{g}(p)\int_{\mathbb{R}} \mathrm{d}q\, \frac{1}{4\pi}B_{T,\mu}(p,q)\widehat{g}(q). \tag{4.44}$$

By Lemma 4.2.6, the first summand is negative and thus also $\langle g|A_{T,\mu}^{\mathbb{R}_+}-a_{T,\mu}|g\rangle<0$. Moreover, using Lemma 4.2.6 and $0<\widehat{g}(p)\leq 1$ we have

$$|\langle g|A_{T,\mu}^{\mathbb{R}_+}-a_{T,\mu}|g\rangle| \leq \int_{\mathbb{R}} \mathrm{d}p\, \widehat{g}(p)^2 a_{T,\mu} + \int_{\mathbb{R}} \mathrm{d}p\, \widehat{g}(p)\int_{\mathbb{R}} \mathrm{d}q\, \frac{1}{4\pi}B_{T,\mu}(0,q). \tag{4.45}$$

In both terms, the integral over p gives a finite constant independent of T. The claim follows from the asymptotics in Lemma 4.3.5. □

4.4 Weak Coupling Limit

In [62] it was observed by numerical and non-rigorous analytical computations that the effect of boundary superconductivity disappears in the weak coupling limit, in the sense that $\frac{T_c^{\mathbb{R}_+}(v)-T_c^{\mathbb{R}}(v)}{T_c^{\mathbb{R}}(v)}\to 0$ for $v\to 0$. In this section we shall verify this claim.

Recall that the bulk critical temperature $T_c^{\mathbb{R}}(v)$ is the unique $T>0$ such that $a_{T,\mu}=\frac{1}{v}$. For the system on the half-line, we have by continuity of $\inf\sigma(H_T^{\mathbb{R}^+})$ in T

$$T_c^{\mathbb{R}_+}(v)=\min\{T\in[0,\infty)|\sup\sigma(A_{T,\mu}^{\mathbb{R}_+})=v^{-1}\}. \tag{4.46}$$

We want to invert this function and view v as function of $T_c^{\mathbb{R}_+}$. We define $\mathfrak{v}(T):=(\sup\sigma(A_{T,\mu}^{\mathbb{R}_+}))^{-1}$. Note that $\mathfrak{v}\circ T_c^{\mathbb{R}_+}=\mathrm{id}$ and for all $T>0$ we have $T_c^{\mathbb{R}_+}(\mathfrak{v}(T))\leq T$.

The claim can be reformulated in terms of the operator $A_{T,\mu}^{\mathbb{R}_+}$ and $a_{T,\mu}$ in the following way.

Lemma 4.4.1. $\lim_{v\to 0}\frac{T_c^{\mathbb{R}_+}(v)-T_c^{\mathbb{R}}(v)}{T_c^{\mathbb{R}}(v)}=0 \Leftrightarrow \lim_{T\to 0}\inf\sigma(a_{T,\mu}\mathbb{I}-A_{T,\mu}^{\mathbb{R}_+})=0.$

Proof. By definition, we have $\sup\sigma(A_{T,\mu}^{\mathbb{R}_+})=\frac{1}{\mathfrak{v}(T)}=a_{T_c^{\mathbb{R}}(\mathfrak{v}(T)),\mu}$. Hence,

$$\lim_{T\to 0}\inf\sigma(a_{T,\mu}\mathbb{I}-A_{T,\mu}^{\mathbb{R}_+})=\lim_{T\to 0}(a_{T,\mu}-a_{T_c^{\mathbb{R}}(\mathfrak{v}(T)),\mu})=\frac{1}{\pi\sqrt{\mu}}\lim_{T\to 0}\ln\left(\frac{T_c^{\mathbb{R}}(\mathfrak{v}(T))}{T}\right) \tag{4.47}$$

where in the last equality we used Lemma 4.3.5 and that $T\geq T_c^{\mathbb{R}_+}(\mathfrak{v}(T))\geq T_c^{\mathbb{R}}(\mathfrak{v}(T))\geq 0$ and thus $\lim_{T\to 0}T_c^{\mathbb{R}}(\mathfrak{v}(T))=0$. Therefore,

$$\lim_{T\to 0}\inf\sigma(a_{T,\mu}\mathbb{I}-A_{T,\mu}^{\mathbb{R}_+})=0\Leftrightarrow\lim_{T\to 0}\frac{T-T_c^{\mathbb{R}}(\mathfrak{v}(T))}{T_c^{\mathbb{R}}(\mathfrak{v}(T))}=0\,. \tag{4.48}$$

There exists a sequence (T_n) such that $T_n\to 0$ as $n\to\infty$ and $T_c^{\mathbb{R}_+}(\mathfrak{v}(T_n))=T_n$ for all n. Therefore,

$$\lim_{T\to 0}\frac{T-T_c^{\mathbb{R}}(\mathfrak{v}(T))}{T_c^{\mathbb{R}}(\mathfrak{v}(T))}=\lim_{T\to 0}\frac{T_c^{\mathbb{R}_+}(\mathfrak{v}(T))-T_c^{\mathbb{R}}(\mathfrak{v}(T))}{T_c^{\mathbb{R}}(\mathfrak{v}(T))}\,. \tag{4.49}$$

Since $\lim_{T\to 0}T_c^{\mathbb{R}}(\mathfrak{v}(T))=0$, also $\lim_{T\to 0}\mathfrak{v}(T)=0$. Thus,

$$\lim_{T\to 0}\frac{T_c^{\mathbb{R}_+}(\mathfrak{v}(T))-T_c^{\mathbb{R}}(\mathfrak{v}(T))}{T_c^{\mathbb{R}}(\mathfrak{v}(T))}=\lim_{v\to 0}\frac{T_c^{\mathbb{R}_+}(v)-T_c^{\mathbb{R}}(v)}{T_c^{\mathbb{R}}(v)} \tag{4.50}$$

and the claim follows. □

Recall the definition of $A_{T,\mu}$ in (4.24). With the notation

$$E_{T,\mu}(p) = 4\pi\left(a_{T,\mu} - A_{T,\mu}(p)\right) \tag{4.51}$$

we have for all $\psi \in L^2((0,\infty))$

$$4\pi(a_{T,\mu}\mathbb{I} - A_{T,\mu}^{\mathbb{R}_+})\psi(p) = E_{T,\mu}(p)\psi(p) + \int_{\mathbb{R}} B_{T,\mu}(p,q)\psi(q)\mathrm{d}q. \tag{4.52}$$

For the proof of Theorem 4.1.1 2, we need the following intermediate results which are proved in Section 4.4.1.

Lemma 4.4.2. *Let $\mu > 0$. Then*

$$\sup_{T>0}\|B_{T,\mu}\| < \infty \tag{4.53}$$

Lemma 4.4.3. *Let $\mathbb{I}_{\leq\epsilon}$ denote multiplication with the characteristic function of the interval $[-\epsilon,\epsilon]$ in momentum space. Let $\mu > 0$. Then*

$$\lim_{\epsilon\to 0}\sup_T\|\mathbb{I}_{\leq\epsilon}B_{T,\mu}\mathbb{I}_{\leq\epsilon}\| \leq \lim_{\epsilon\to 0}\sup_T\|\mathbb{I}_{\leq\epsilon}B_{T,\mu}\mathbb{I}_{\leq\epsilon}\|_{HS} = 0, \tag{4.54}$$

where $\|\cdot\|_{HS}$ denotes the Hilbert–Schmidt norm.

Lemma 4.4.4. *Let $0 < \epsilon < 2\sqrt{\mu}$. For $|p| > \epsilon$ we have*

$$E_{T,\mu}(p) \geq c_1 \ln\left(\frac{c_2}{T}\right) \tag{4.55}$$

for constants $c_1, c_2 > 0$ and T small enough.

Proof of Theorem 4.1.1 2. By Lemma 4.4.1 it suffices to prove $0 = \lim_{T\to 0}\inf\sigma(a_{T,\mu}\mathbb{I} - A_{T,\mu}^{\mathbb{R}_+}) = \lim_{T\to 0}\frac{1}{4\pi}\inf\sigma(E_{T,\mu}+B_{T,\mu})$. By (4.4), we only need to show that $\lim_{T\to 0}\inf\sigma(E_{T,\mu}+B_{T,\mu}) \geq 0$. For $\delta > 0$ we can write

$$E_{T,\mu} + B_{T,\mu} + \delta = \sqrt{E_{T,\mu}+\delta}\left(\mathbb{I} + \frac{1}{\sqrt{E_{T,\mu}+\delta}}B_{T,\mu}\frac{1}{\sqrt{E_{T,\mu}+\delta}}\right)\sqrt{E_{T,\mu}+\delta} \tag{4.56}$$

since $E_{T,\mu}(p) \geq 0$ by Lemma 4.2.6. We shall show that for all $\delta > 0$

$$\lim_{T\to 0}\left\|\frac{1}{\sqrt{E_{T,\mu}+\delta}}B_{T,\mu}\frac{1}{\sqrt{E_{T,\mu}+\delta}}\right\| = 0\,. \tag{4.57}$$

Hence, the operator in the bracket in (4.56) is positive for small T. This implies that for all $\delta > 0$ for T small enough we have $\inf\sigma(E_{T,\mu} + B_{T,\mu} + \delta) > 0$. Since δ can be arbitrarily small, the theorem follows.

To prove (4.57), we use the notation of Lemma 4.4.3 and estimate for an arbitrary $0 < \epsilon < 2\sqrt{\mu}$

$$\begin{aligned}\left\|\frac{1}{\sqrt{E_{T,\mu}+\delta}}B_{T,\mu}\frac{1}{\sqrt{E_{T,\mu}+\delta}}\right\| &\leq \left\|\mathbb{I}_{\leq\epsilon}\frac{1}{\sqrt{E_{T,\mu}+\delta}}B_{T,\mu}\frac{1}{\sqrt{E_{T,\mu}+\delta}}\mathbb{I}_{\leq\epsilon}\right\| \\ &+ \left\|\mathbb{I}_{\leq\epsilon}\frac{1}{\sqrt{E_{T,\mu}+\delta}}B_{T,\mu}\frac{1}{\sqrt{E_{T,\mu}+\delta}}\mathbb{I}_{>\epsilon}\right\| + \left\|\mathbb{I}_{>\epsilon}\frac{1}{\sqrt{E_{T,\mu}+\delta}}B_{T,\mu}\frac{1}{\sqrt{E_{T,\mu}+\delta}}\right\|\,. \end{aligned} \tag{4.58}$$

Now we use that $E_{T,\mu} \geq 0$ and Lemma 4.4.4 to obtain

$$\lim_{T\to 0}\left\|\frac{1}{\sqrt{E_{T,\mu}+\delta}}B_{T,\mu}\frac{1}{\sqrt{E_{T,\mu}+\delta}}\right\| \leq \lim_{T\to 0}\frac{1}{\delta}\|\mathbb{I}_{\leq\epsilon}B_{T,\mu}\mathbb{I}_{\leq\epsilon}\| + \lim_{T\to 0}\frac{2c_1^{1/2}}{(\delta|\ln(c_2/T)|)^{1/2}}\|B_{T,\mu}\|. \tag{4.59}$$

With Lemma 4.4.2 it follows that the second term vanishes and

$$\lim_{T\to 0}\left\|\frac{1}{\sqrt{E_{T,\mu}+\delta}}B_{T,\mu}\frac{1}{\sqrt{E_{T,\mu}+\delta}}\right\| \leq \sup_T\frac{1}{\delta}\|\mathbb{I}_{\leq\epsilon}B_{T,\mu}\mathbb{I}_{\leq\epsilon}\|. \tag{4.60}$$

Since $\epsilon > 0$ was arbitrary, (4.57) follows from Lemma 4.4.3. □

Remark 4.4.5. In the case of Neumann boundary conditions, the same argument proves (4.9).

4.4.1 Proofs of Intermediate Results

Proof of Lemma 4.4.2. In order to bound $B_{T,\mu}(p,q)$ we apply the following inequality proved in Section 4.6.3.

Lemma 4.4.6. *For all $x,y\in\mathbb{R}$ and $T>0$ it holds that*

$$\frac{\tanh(x/T)+\tanh(y/T)}{x+y} < \frac{2}{|x|+|y|}. \tag{4.61}$$

Hence, $B_{T,\mu}(p,q)$ is bounded above by

$$f(p,q) = \frac{2}{|\left(\frac{p+q}{2}\right)^2-\mu| + |\left(\frac{p-q}{2}\right)^2-\mu|}. \tag{4.62}$$

The function f has singularities at the four points where $\{|p|,|q|\} = \{0, 2\sqrt{\mu}\}$. Since f diverges linearly at those points, the idea is to do a Schur test with a test function of the form $d(p)^\alpha$, where $d(p)$ is the distance from the singularities in variable p and $\alpha\in(0,1)$. We choose the function $h(p) = \min\{|p|, |2\sqrt{\mu}-|p||\}^{1/2}$. The Schur test gives

$$\sup_T\|B_{T,\mu}\| \leq \sup_T\sup_p h(p)\int_{\mathbb{R}}\frac{B_{T,\mu}(p,q)}{h(q)}\mathrm{d}q \leq \sup_p h(p)\int_{\mathbb{R}}\frac{f(p,q)}{h(q)}\mathrm{d}q = 2\sup_{p>0}h(p)\int_0^\infty\frac{f(p,q)}{h(q)}\mathrm{d}q, \tag{4.63}$$

where we used that $\frac{h(p)f(p,q)}{h(q)} = \frac{h(|p|)f(|p|,|q|)}{h(|q|)}$ for the last equality.

In order to estimate $h(p)\int_0^\infty\frac{f(p,q)}{h(q)}\mathrm{d}q$, we split the domain into nine regions as indicated in Figure 4.1. The finiteness of the right hand side of (4.63) follows from the bounds listed in Table 4.1. In the following, we prove the bounds in Table 4.1.

In region 1, we have

$$\begin{aligned}\int_{2\sqrt{\mu}+p}^\infty\frac{f(p,q)}{h(q)}\mathrm{d}q &= \int_{2\sqrt{\mu}+p}^\infty\frac{4}{p^2+q^2-4\mu}\frac{1}{(q-2\sqrt{\mu})^{1/2}}\mathrm{d}q \leq \int_{2\sqrt{\mu}+p}^\infty\frac{4}{(q+2\sqrt{\mu})(q-2\sqrt{\mu})^{3/2}}\mathrm{d}q\\ &\leq \frac{1}{\sqrt{\mu}}\int_{2\sqrt{\mu}+p}^\infty\frac{1}{(q-2\sqrt{\mu})^{3/2}}\mathrm{d}q = \frac{2}{\sqrt{\mu}p^{1/2}}.\end{aligned} \tag{4.64}$$

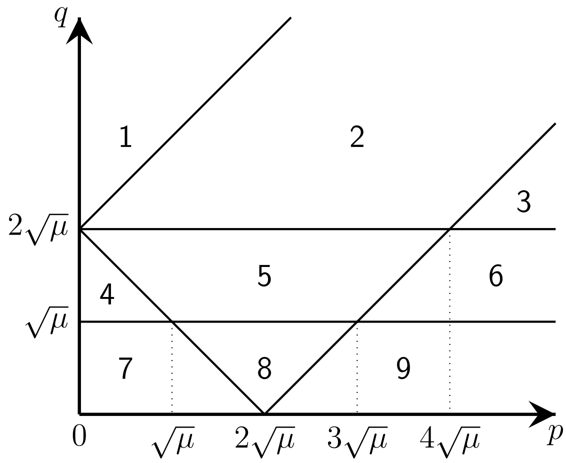

Figure 4.1: The nine regions of the domain of p, q in the proof of Lemma 4.4.2.

Region	Expression	Upper bound	Proof
1	$h(p)\int_{2\sqrt{\mu}+p}^{\infty}\frac{f(p,q)}{h(q)}\mathrm{d}q$	$\frac{2}{\sqrt{\mu}}$	(4.64)
2	$h(p)\int_{\max\{2\sqrt{\mu},p-2\sqrt{\mu}\}}^{2\sqrt{\mu}+p}\frac{f(p,q)}{h(q)}\mathrm{d}q$	$\frac{2}{\sqrt{\mu}}$	(4.65)
3	$h(p)\int_{2\sqrt{\mu}}^{p-2\sqrt{\mu}}\frac{f(p,q)}{h(q)}\mathrm{d}q$	$\frac{2}{\sqrt{\mu}}$	(4.66)
4	$h(p)\int_{\sqrt{\mu}}^{2\sqrt{\mu}-p}\frac{f(p,q)}{h(q)}\mathrm{d}q$	$\frac{8\cdot 2^{1/2}}{\sqrt{\mu}}$	(4.67)
5	$h(p)\int_{\max\{2\sqrt{\mu}-p,\sqrt{\mu},p-2\sqrt{\mu}\}}^{2\sqrt{\mu}}\frac{f(p,q)}{h(q)}\mathrm{d}q$	$\frac{4}{\sqrt{\mu}}$	(4.68)
6	$h(p)\int_{\sqrt{\mu}}^{\min\{2\sqrt{\mu},p-2\sqrt{\mu}\}}\frac{f(p,q)}{h(q)}\mathrm{d}q$	$\frac{2^{1/2}8}{\sqrt{\mu}(3-3^{3/4})}$	(4.69)
7	$h(p)\int_{0}^{\min\{\sqrt{\mu},2\sqrt{\mu}-p\}}\frac{f(p,q)}{h(q)}\mathrm{d}q$	$\frac{4}{\sqrt{\mu}}$	(4.70)
8	$h(p)\int_{\lvert 2\sqrt{\mu}-p\rvert}^{\sqrt{\mu}}\frac{f(p,q)}{h(q)}\mathrm{d}q$	$\frac{4}{\sqrt{\mu}}$	(4.71)
9	$h(p)\int_{0}^{\min\{\sqrt{\mu},p-2\sqrt{\mu}\}}\frac{f(p,q)}{h(q)}\mathrm{d}q$	$\frac{2}{\sqrt{\mu}}$	(4.72)

Table 4.1: Overview of the estimates used in the proof of Lemma 4.4.2.

In region 2, we have

$$\int_{\max\{2\sqrt{\mu},p-2\sqrt{\mu}\}}^{2\sqrt{\mu}+p}\frac{f(p,q)}{h(q)}\mathrm{d}q=\int_{\max\{2\sqrt{\mu},p-2\sqrt{\mu}\}}^{2\sqrt{\mu}+p}\frac{2}{pq(q-2\sqrt{\mu})^{1/2}}\mathrm{d}q\le\frac{2}{p}\int_{2\sqrt{\mu}}^{2\sqrt{\mu}+p}\frac{1}{q(q-2\sqrt{\mu})^{1/2}}\mathrm{d}q$$
$$\le\frac{1}{p\sqrt{\mu}}\int_{2\sqrt{\mu}}^{2\sqrt{\mu}+p}\frac{1}{(q-2\sqrt{\mu})^{1/2}}\mathrm{d}q=\frac{2}{\sqrt{\mu}p^{1/2}}. \quad (4.65)$$

In region 3, we have $p>4\sqrt{\mu}$ and

$$\int_{2\sqrt{\mu}}^{p-2\sqrt{\mu}}\frac{f(p,q)}{h(q)}\mathrm{d}q=\int_{2\sqrt{\mu}}^{p-2\sqrt{\mu}}\frac{4}{p^2+q^2-4\mu}\frac{1}{(q-2\sqrt{\mu})^{1/2}}\mathrm{d}q$$
$$\le\frac{4}{p^2}\int_{2\sqrt{\mu}}^{p-2\sqrt{\mu}}\frac{1}{(q-2\sqrt{\mu})^{1/2}}\mathrm{d}q=\frac{8}{p^2}(p-4\sqrt{\mu})^{1/2}\le\frac{8}{p^{3/2}}\le\frac{2}{\sqrt{\mu}p^{1/2}} \quad (4.66)$$

where we used $p>4\sqrt{\mu}$ in the last inequality.

In region 4, we have $p < \sqrt{\mu}$ and

$$
\begin{aligned}
\int_{\sqrt{\mu}}^{2\sqrt{\mu}-p} \frac{f(p,q)}{h(q)}\mathrm{d}q &= \int_{\sqrt{\mu}}^{2\sqrt{\mu}-p} \frac{4}{4\mu-p^2-q^2}\frac{1}{(2\sqrt{\mu}-q)^{1/2}}\mathrm{d}q \\
&= \int_{\sqrt{\mu}}^{2\sqrt{\mu}-p} \frac{4}{(\sqrt{4\mu-p^2}+q)(\sqrt{4\mu-p^2}-q)}\frac{1}{(2\sqrt{\mu}-q)^{1/2}}\mathrm{d}q \\
\leq \frac{1}{\sqrt{\mu}}\int_{\sqrt{\mu}}^{2\sqrt{\mu}-p} &\frac{4}{(\sqrt{4\mu-p^2}-q)}\frac{1}{(2\sqrt{\mu}-q)^{1/2}}\mathrm{d}q \leq \frac{1}{\sqrt{\mu}}\int_{-\infty}^{2\sqrt{\mu}-p}\frac{4}{(\sqrt{4\mu-p^2}-q)^{3/2}}\mathrm{d}q \\
= \frac{8}{\sqrt{\mu}(\sqrt{4\mu-p^2}-2\sqrt{\mu}+p)^{1/2}} &= \frac{8}{\sqrt{\mu}(2\sqrt{\mu}-p)^{1/4}\left[(2\sqrt{\mu}+p)^{1/2}-(2\sqrt{\mu}-p)^{1/2}\right]^{1/2}} \\
&= \frac{8\left[(2\sqrt{\mu}+p)^{1/2}+(2\sqrt{\mu}-p)^{1/2}\right]^{1/2}}{\sqrt{\mu}(2\sqrt{\mu}-p)^{1/4}\,(2p)^{1/2}} \leq \frac{8\left(4\mu^{1/4}\right)^{1/2}}{\sqrt{\mu}\mu^{1/8}\,(2p)^{1/2}} = \frac{8\cdot 2^{1/2}}{\sqrt{\mu}p^{1/2}},
\end{aligned}
\tag{4.67}
$$

where we used $p < \sqrt{\mu}$ in the last inequality.

In region 5, we have

$$
\begin{aligned}
\int_{\max\{2\sqrt{\mu}-p,\sqrt{\mu},p-2\sqrt{\mu}\}}^{2\sqrt{\mu}} \frac{f(p,q)}{h(q)}\mathrm{d}q &= \int_{\max\{2\sqrt{\mu}-p,\sqrt{\mu},p-2\sqrt{\mu}\}}^{2\sqrt{\mu}} \frac{2}{pq}\frac{1}{(2\sqrt{\mu}-q)^{1/2}}\mathrm{d}q \\
\leq \frac{2}{p\sqrt{\mu}}\int_{\max\{2\sqrt{\mu}-p,\sqrt{\mu},p-2\sqrt{\mu}\}}^{2\sqrt{\mu}} \frac{1}{(2\sqrt{\mu}-q)^{1/2}}\mathrm{d}q &= \frac{4}{p\sqrt{\mu}}\min\{p,\sqrt{\mu},4\sqrt{\mu}-p\}^{1/2} \leq \frac{4}{\sqrt{\mu}p^{1/2}}.
\end{aligned}
\tag{4.68}
$$

In region 6, we have $p > 3\sqrt{\mu}$ and

$$
\begin{aligned}
\int_{\sqrt{\mu}}^{\min\{2\sqrt{\mu},p-2\sqrt{\mu}\}} \frac{f(p,q)}{h(q)}\mathrm{d}q &= \int_{\sqrt{\mu}}^{\min\{2\sqrt{\mu},p-2\sqrt{\mu}\}} \frac{4}{p^2+q^2-4\mu}\frac{1}{(2\sqrt{\mu}-q)^{1/2}}\mathrm{d}q \\
\leq \frac{4}{p^2-3\mu}\int_0^{2\sqrt{\mu}}\frac{1}{(2\sqrt{\mu}-q)^{1/2}}\mathrm{d}q &= \frac{8}{p^2-3\mu}(2\sqrt{\mu})^{1/2} = \frac{8}{(p+\sqrt{3\mu})(p-\sqrt{3\mu})}(2\sqrt{\mu})^{1/2} \\
&\leq \frac{8}{\sqrt{3\mu}(p^{1/2}-(3\mu)^{1/4})(p^{1/2}+(3\mu)^{1/4})}(2\sqrt{\mu})^{1/2} \leq \frac{2^{1/2}8}{\sqrt{\mu}(3-3^{3/4})p^{1/2}}.
\end{aligned}
\tag{4.69}
$$

In region 7, we have

$$
\begin{aligned}
\int_0^{\min\{\sqrt{\mu},2\sqrt{\mu}-p\}} \frac{f(p,q)}{h(q)}\mathrm{d}q &= \int_0^{\min\{\sqrt{\mu},2\sqrt{\mu}-p\}} \frac{4}{4\mu-p^2-q^2}\frac{1}{q^{1/2}}\mathrm{d}q \\
\leq \frac{4}{4\mu-p^2-\min\{\sqrt{\mu},2\sqrt{\mu}-p\}^2}&\int_0^{\min\{\sqrt{\mu},2\sqrt{\mu}-p\}}\frac{1}{q^{1/2}}\mathrm{d}q = \frac{8\min\{\sqrt{\mu},2\sqrt{\mu}-p\}^{1/2}}{4\mu-p^2-\min\{\sqrt{\mu},2\sqrt{\mu}-p\}^2} \\
&= \begin{cases} \frac{8\mu^{1/4}}{3\mu-p^2} & \text{if } p<\sqrt{\mu} \\ \frac{4}{p(2\sqrt{\mu}-p)^{1/2}} & \text{if } p>\sqrt{\mu}\end{cases} \leq \begin{cases} \frac{4\mu^{1/4}}{\mu} & \text{if } p<\sqrt{\mu} \\ \frac{4}{\sqrt{\mu}(2\sqrt{\mu}-p)^{1/2}} & \text{if } p>\sqrt{\mu}\end{cases}
\end{aligned}
\tag{4.70}
$$

In region 8, we have $p > \sqrt{\mu}$ and

$$
\int_{|2\sqrt{\mu}-p|}^{\sqrt{\mu}} \frac{f(p,q)}{h(q)}\mathrm{d}q = \int_{|2\sqrt{\mu}-p|}^{\sqrt{\mu}} \frac{2}{pq}\frac{1}{q^{1/2}}\mathrm{d}q \leq \frac{2}{\sqrt{\mu}}\int_{|2\sqrt{\mu}-p|}^{\infty}\frac{1}{q^{3/2}}\mathrm{d}q = \frac{4}{\sqrt{\mu}}|2\sqrt{\mu}-p|^{-1/2}. \tag{4.71}
$$

In region 9, we have $p > 2\sqrt{\mu}$ and

$$\int_0^{\min\{\sqrt{\mu},p-2\sqrt{\mu}\}} \frac{f(p,q)}{h(q)}\mathrm{d}q = \int_0^{\min\{\sqrt{\mu},p-2\sqrt{\mu}\}} \frac{4}{p^2+q^2-4\mu}\frac{1}{q^{1/2}}\mathrm{d}q$$
$$\leq \frac{4}{p^2-4\mu}\int_0^{\min\{\sqrt{\mu},p-2\sqrt{\mu}\}} \frac{1}{q^{1/2}}\mathrm{d}q = \frac{8}{(p+2\sqrt{\mu})(p-2\sqrt{\mu})}\min\{\mu^{1/4},(p-2\sqrt{\mu})^{1/2}\}$$
$$\leq \frac{2}{\sqrt{\mu}(p-2\sqrt{\mu})^{1/2}} \quad (4.72)$$

□

Proof of Lemma 4.4.3. Let $0 < \epsilon < \sqrt{\mu}$. For $0 \leq |p|,|q| \leq \epsilon$ we have $2\mu - \left(\frac{p+q}{2}\right)^2 - \left(\frac{p-q}{2}\right)^2 \geq 2\mu - 2\epsilon^2$. Together with $0 \leq \tanh(x) \leq 1$ for $x \geq 0$ we obtain

$$0 \leq B_{T,\mu}(p,q) \leq \frac{1}{\mu-\epsilon^2}. \quad (4.73)$$

Using this estimate, we bound the Hilbert–Schmidt norm as

$$\|\mathbb{I}_{\leq\epsilon}B_{T,\mu}\mathbb{I}_{\leq\epsilon}\|_{\mathsf{HS}}^2 = \int_{-\epsilon}^{\epsilon}\int_{-\epsilon}^{\epsilon} B_{T,\mu}(p,q)^2\mathrm{d}p\mathrm{d}q \leq \frac{4\epsilon^2}{(\mu-\epsilon^2)^2}. \quad (4.74)$$

□

Proof of Lemma 4.4.4. Recall that $E_{T,\mu}(p) = 4\pi a_{T,\mu} - \int_{\mathbb{R}} B_{T,\mu}(p,q)\mathrm{d}q$. The idea is to show that the supremum $\sup_{p>\epsilon,T>0}\int_{\mathbb{R}} B_{T,\mu}(p,q)\mathrm{d}q < \infty$. Then, for $T \to 0$ we have $\inf_{|p|>\epsilon} E_{T,\mu}(p) \sim 4\pi a_{T,\mu} \sim \frac{4}{\sqrt{\mu}}\ln\frac{\mu}{T}$.

We shall prove that the following four expressions are finite.

$$I_1 := \sup_{p>\epsilon,T>0}\int_{p+2\sqrt{\mu}}^{\infty} B_{T,\mu}(p,q)\mathrm{d}q \quad (4.75)$$

$$I_2 := \sup_{2\sqrt{\mu}>p>\epsilon,T>0}\int_0^{2\sqrt{\mu}-p} B_{T,\mu}(p,q)\mathrm{d}q \quad (4.76)$$

$$I_3 := \sup_{p>2\sqrt{\mu},T>0}\int_0^{p-2\sqrt{\mu}} B_{T,\mu}(p,q)\mathrm{d}q \quad (4.77)$$

$$I_4 := \sup_{p>\epsilon,T>0}\int_{|p-2\sqrt{\mu}|}^{p+2\sqrt{\mu}} B_{T,\mu}(p,q)\mathrm{d}q \quad (4.78)$$

From this, together with $B_{T,\mu}(p,q) = B_{T,\mu}(|p|,|q|)$ it follows that

$$\sup_{|p|>\epsilon,T>0}\int_{\mathbb{R}} B_{T,\mu}(p,q)\mathrm{d}q \leq 2\max\left\{\sup_{2\sqrt{\mu}>p>\epsilon,T>0}\int_0^{\infty} B_{T,\mu}(p,q)\mathrm{d}q, \sup_{p>2\sqrt{\mu},T>0}\int_0^{\infty} B_{T,\mu}(p,q)\mathrm{d}q\right\}$$
$$\leq 2\max\{I_2+I_4+I_1, I_3+I_4+I_1\} < \infty. \quad (4.79)$$

The following inequality is proved in Section 4.6.3.

Lemma 4.4.7. *For $x,y > 0$*

$$\frac{\tanh(x)-\tanh(y)}{x-y} \leq 4e^{-2\min\{x,y\}} \quad (4.80)$$

Applying Lemmas 4.4.6 and 4.4.7 we estimate

$$B_{T,\mu}(p,q) \leq \begin{cases} \frac{2}{T}\exp\left(-\min\{(p+q)^2-4\mu, 4\mu-(p-q)^2\}/4T\right) & \text{for } |p-2\sqrt{\mu}| < q < p+2\sqrt{\mu} \\ \frac{8}{|(p+q)^2-4\mu|+|(p-q)^2-4\mu|} & \text{otherwise.} \end{cases} \tag{4.81}$$

With (4.81) we have

$$I_1 \leq \sup_{p>\epsilon} \int_{p+2\sqrt{\mu}}^{\infty} \frac{4}{p^2+q^2-4\mu} \mathrm{d}q \leq \int_{\epsilon+2\sqrt{\mu}}^{\infty} \frac{4}{q^2-4\mu} \mathrm{d}q < \infty \,. \tag{4.82}$$

Furthermore,

$$\begin{aligned} I_2 \leq \sup_{2\sqrt{\mu}>p>\epsilon} \int_0^{2\sqrt{\mu}-p} \frac{4}{4\mu-p^2-q^2} \mathrm{d}q &\leq \sup_{2\sqrt{\mu}>p>\epsilon} (2\sqrt{\mu}-p) \frac{4}{4\mu-p^2-(2\sqrt{\mu}-p)^2} \\ &= \sup_{2\sqrt{\mu}>p>\epsilon} \frac{2}{p} = \frac{2}{\epsilon} \end{aligned} \tag{4.83}$$

Moreover,

$$I_3 \leq \sup_{p>2\sqrt{\mu}} \int_0^{p-2\sqrt{\mu}} \frac{4}{p^2+q^2-4\mu} \mathrm{d}q = \sup_{p>2\sqrt{\mu}} \frac{4\arctan\left(\sqrt{\frac{p-2\sqrt{\mu}}{p+2\sqrt{\mu}}}\right)}{\sqrt{p^2-4\mu}} \leq \sup_{0<x<1} \frac{\arctan(x)}{\sqrt{\mu}x} \leq \frac{1}{\sqrt{\mu}} \tag{4.84}$$

In order to estimate I_4, note that $(p+q)^2-4\mu < 4\mu-(p-q)^2 \Leftrightarrow q < \sqrt{4\mu-p^2}$. Let

$$I_5 := \sup_{\epsilon<p<2\sqrt{\mu},T>0} \frac{2}{T} \int_{2\sqrt{\mu}-p}^{\sqrt{4\mu-p^2}} e^{\mu/T-(p+q)^2/4T} \mathrm{d}q, \tag{4.85}$$

$$I_6 := \sup_{\epsilon<p<2\sqrt{\mu},T>0} \frac{2}{T} \int_{\sqrt{4\mu-p^2}}^{2\sqrt{\mu}+p} e^{(q-p)^2/4T-\mu/T} \mathrm{d}q, \tag{4.86}$$

and

$$I_7 := \sup_{p>2\sqrt{\mu},T>0} \frac{2}{T} \int_{p-2\sqrt{\mu}}^{p+2\sqrt{\mu}} e^{(q-p)^2/4T-\mu/T} \mathrm{d}q. \tag{4.87}$$

Then we have $I_4 \leq \max\{I_5+I_6, I_7\}$. We can bound both I_6 and I_7 using

$$\begin{aligned} I_6, I_7 \leq \sup_{p>\epsilon,T>0} \frac{2}{T} \int_{p-2\sqrt{\mu}}^{p+2\sqrt{\mu}} e^{(q-p)^2/4T-\mu/T} \mathrm{d}q &= \sup_{T>0} \frac{2}{T} \int_{-2\sqrt{\mu}}^{2\sqrt{\mu}} e^{q^2/4T-\mu/T} \mathrm{d}q \\ &= \sup_{T>0} \frac{4\sqrt{\pi}e^{-\mu/T}}{\sqrt{T}} \operatorname{erfi}\left(\sqrt{\frac{\mu}{T}}\right) = \frac{4\sqrt{\pi}}{\sqrt{\mu}} \sup_{x>0} xe^{-x^2}\operatorname{erfi}(x). \end{aligned} \tag{4.88}$$

Since $\sqrt{\pi}\lim_{x\to\infty} xe^{-x^2}\operatorname{erfi}(x) = 1$, it follows that $I_6, I_7 < \infty$.

Finally,

$$\begin{aligned} I_5 = \sup_{\epsilon<p<2\sqrt{\mu},T>0} \frac{2e^{\mu/T}}{T} \int_{2\sqrt{\mu}}^{\sqrt{4\mu-p^2}+p} e^{-q^2/4T} \mathrm{d}q &\leq \sup_{T>0} \frac{2e^{\mu/T}}{T} \int_{2\sqrt{\mu}}^{\infty} e^{-q^2/4T} \mathrm{d}q \\ &= \sup_{T>0} \frac{2\sqrt{\pi}e^{\mu/T}}{\sqrt{T}} \operatorname{erfc}\left(\sqrt{\frac{\mu}{T}}\right) = \frac{2\sqrt{\pi}}{\sqrt{\mu}} \sup_{x>0} xe^{x^2}\operatorname{erfc}(x) \end{aligned} \tag{4.89}$$

Since $0 \leq \operatorname{erfc}(x) \leq 1$ and for $x \to \infty$ asymptotically $\operatorname{erfc}(x) \sim \frac{e^{-x^2}}{x\sqrt{\pi}} + o(e^{-x^2}/x)$, we have $\sup_{x>0} xe^{x^2}\operatorname{erfc}(x) < \infty$ and obtain $I_5 < \infty$. □

4.5 Strong Coupling Limit

The goal of this section is to prove part 3 of Theorem 4.1.1. As for the weak coupling limit, we first translate the question about the relative temperature difference into a condition on $A^{\mathbb{R}_+}_{T,\mu}$ and $a_{T,\mu}$. While the weak coupling limit turned out to be equivalent to a low temperature limit, the strong coupling limit corresponds to a high temperature limit. In this limit, the relevant quantities behave as follows.

Lemma 4.5.1. *Let* $\mu > 0$. *Then*

1. $\lim_{v\to\infty} T_c^{\mathbb{R}_+}(v) = \infty$
2. $\lim_{T\to\infty} T_c^{\mathbb{R}}(\mathfrak{v}(T)) = \infty$
3. $\lim_{T\to\infty} T^{1/2} a_{T,\mu} = a_{1,0}$
4. $\lim_{T\to\infty} T^{1/2} \sup\sigma(A^{\mathbb{R}_+}_{T,\mu}) = \sup\sigma(A^{\mathbb{R}_+}_{1,0})$

The proof is provided in Section 4.5.1. We can reformulate Theorem 4.1.13 as follows.

Lemma 4.5.2.

$$\lim_{v\to\infty} \frac{T_c^{\mathbb{R}_+}(v) - T_c^{\mathbb{R}}(v)}{T_c^{\mathbb{R}}(v)} = 0 \Leftrightarrow \sup\sigma(A^{\mathbb{R}_+}_{1,0}) = a_{1,0} \tag{4.90}$$

Proof. By Lemma 4.5.14 and the definition of $\mathfrak{v}(T)$ we have

$$\sup\sigma(A^{\mathbb{R}_+}_{1,0}) = \lim_{T\to\infty} T^{1/2}\sup\sigma(A^{\mathbb{R}_+}_{T,\mu}) = \lim_{T\to\infty} T^{1/2} a_{T_c^{\mathbb{R}}(\mathfrak{v}(T)),\mu} \tag{4.91}$$

By Lemma 4.5.12 and 3 we get

$$\lim_{T\to\infty} T^{1/2} a_{T_c^{\mathbb{R}}(\mathfrak{v}(T)),\mu} = a_{1,0}\lim_{T\to\infty}\left(\frac{T}{T_c^{\mathbb{R}}(\mathfrak{v}(T))}\right)^{1/2} = a_{1,0}\lim_{v\to\infty}\left(\frac{T_c^{\mathbb{R}_+}(v)}{T_c^{\mathbb{R}}(v)}\right)^{1/2} \tag{4.92}$$

where we used Lemma 4.5.11 and $\mathfrak{v}(T_c^{\mathbb{R}_+}(v)) = v$ for the second equality. Since $a_{1,0} > 0$, the claim follows. □

Remark 4.5.3. In the case of Neumann boundary conditions, $d := \sup\sigma(A^{\mathbb{R}_+}_{1,0}) - a_{1,0} > 0$. With the argument in Lemma 4.5.2, we have

$$\lim_{v\to\infty} \frac{T_c^{\mathbb{R}_+}(v) - T_c^{\mathbb{R}}(v)}{T_c^{\mathbb{R}}(v)} = \left(\frac{d}{a_{1,0}} + 1\right)^2 - 1 > 0\,. \tag{4.93}$$

We are thus left with showing that $\sup\sigma(A^{\mathbb{R}_+}_{1,0}) = a_{1,0}$. Recall that $\sup\sigma_{\mathsf{ess}}(A^{\mathbb{R}_+}_{1,0}) = a_{1,0}$. Hence it suffices to prove that for all $\psi \in L^2((0,\infty))$

$$\langle\psi|A^{\mathbb{R}_+}_{1,0}|\psi\rangle = \frac{1}{8\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} B_{1,0}(p,q)|\psi(p)-\psi(q)|^2\mathrm{d}p\mathrm{d}q \leq \frac{1}{4\pi}\int_{\mathbb{R}}|\psi(p)|^2\mathrm{d}p\int_{\mathbb{R}} B_{1,0}(0,q)\mathrm{d}q = \|\psi\|_2^2 a_{1,0}. \tag{4.94}$$

In order to show this, we shall bound $B_{1,0}$ by a positive definite kernel K, in such a way that the right hand side of (4.94) does not change.

Lemma 4.5.4. *Let K be the operator on $L^2(\mathbb{R}^2)$ with integral kernel*

$$K(p,q) = \min\{B_{1,0}(p,0), B_{1,0}(q,0)\} \tag{4.95}$$

Then K satisfies

1. $B_{1,0}(p,q) \leq K(p,q)$ *for all* $p,q \in \mathbb{R}$
2. $K(p,q) = K(q,p)$ *for all* $p,q \in \mathbb{R}$
3. K *is positive definite*
4. $\int_{\mathbb{R}} K(p,q)\mathrm{d}q \leq \int_{\mathbb{R}} K(0,q)\mathrm{d}q$ *for all* $p \in \mathbb{R}$
5. $B_{1,0}(p,0) = K(p,0)$ *for all* $p \in \mathbb{R}$

This implies (4.94) and hence part 3 of Theorem 4.1.1 since

$$\begin{aligned}\frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} B_{1,0}(p,q)|\psi(p)-\psi(q)|^2\mathrm{d}p\mathrm{d}q &\leq \frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} K(p,q)|\psi(p)-\psi(q)|^2\mathrm{d}p\mathrm{d}q \\ &= \int_{\mathbb{R}}|\psi(p)|^2\int_{\mathbb{R}}K(p,q)\mathrm{d}q\mathrm{d}p - \langle\psi|K|\psi\rangle \\ &\leq \int_{\mathbb{R}}|\psi(p)|^2\int_{\mathbb{R}}K(p,q)\mathrm{d}q\mathrm{d}p \\ &\leq \|\psi\|_2^2\int_{\mathbb{R}}K(0,q)\mathrm{d}q = \|\psi\|_2^2\int_{\mathbb{R}}B_{1,0}(0,q)\mathrm{d}q\,. \end{aligned} \tag{4.96}$$

Proof of Lemma 4.5.4. Property 2 is obvious. Properties 4 and 5 follow from the fact that

$$K(p,q) = \min\{F_{1,0}(p/2), F_{1,0}(q/2)\} = F_{1,0}(\max\{|p|,|q|\}/2), \tag{4.97}$$

where $F_{1,0}(p) = \frac{\tanh(p^2/2)}{p^2}$ has a maximum at $p=0$ and is monotonously decreasing for $p>0$. For 1 consider the following inequality, which is proved in Section 4.6.4.

Lemma 4.5.5. *For all* $p,q \in \mathbb{R}$

$$B_{1,0}(p,q) \leq \frac{\tanh\left(\frac{p^2+q^2}{8}\right)}{\frac{p^2+q^2}{4}} \tag{4.98}$$

Together with the monotonicity of $\tanh(p)/p$ for $p \geq 0$, it implies 1. For property 3 it suffices to show that there is a real-valued function g such that

$$K(p,q) = \int_{\mathbb{R}} g(r,p)g(r,q)\mathrm{d}r. \tag{4.99}$$

In fact, let $g(r,p) = \sqrt{h(r)}\chi_{r>p^2}$ with

$$h(r) = \frac{\mathrm{d}}{\mathrm{d}x}\frac{\tanh(x/2)}{x}\bigg|_{x=-r} \geq 0. \tag{4.100}$$

With this choice, (4.99) holds since

$$\begin{aligned}\int_{\mathbb{R}} g(r,p)g(r,q)\mathrm{d}r = \int_{\max\{p^2,q^2\}}^{\infty} h(r)\mathrm{d}r = \int_{-\infty}^{-\max\{p^2,q^2\}} \frac{\mathrm{d}}{\mathrm{d}x}\frac{\tanh(x/2)}{x}\bigg|_{x=r}\mathrm{d}r \\ = \frac{\tanh(\max\{p^2,q^2\}/2)}{\max\{p^2,q^2\}} = K(p,q)\end{aligned} \tag{4.101}$$

□

4.5.1 Proof of Lemma 4.5.1

Proof of Lemma 4.5.1. For part 1 we have $\lim_{v\to\infty} T_c^{\mathbb{R}_+}(v) \geq \lim_{v\to\infty} T_c^{\mathbb{R}}(v) = \infty$ by (4.4).

Part 2 follows easily from part 4: Clearly 4 implies that

$$\lim_{T\to\infty} \sup \sigma(A_{T,\mu}^{\mathbb{R}_+}) = 0. \tag{4.102}$$

Since $a_{T_c^{\mathbb{R}}(\mathfrak{v}(T)),\mu} = \sup \sigma(A_{T,\mu}^{\mathbb{R}_+})$ this is equivalent to

$$\lim_{T\to\infty} a_{T_c^{\mathbb{R}}(\mathfrak{v}(T)),\mu} = 0. \tag{4.103}$$

Using that $a_{T,\mu}$ is strictly decreasing in T with $\lim_{T\to\infty} a_{T,\mu} = 0$, this in turn is equivalent to

$$\lim_{T\to\infty} T_c^{\mathbb{R}}(\mathfrak{v}(T)) = \infty. \tag{4.104}$$

For part 3 we have after substituting $q/2T^{1/2} \to q$

$$\lim_{T\to\infty} T^{1/2} a_{T,\mu} = \frac{1}{2\pi} \lim_{T\to\infty} \int_{\mathbb{R}} \frac{\tanh\left(\frac{q^2-\mu/T}{2}\right)}{q^2-\mu/T} \mathrm{d}q. \tag{4.105}$$

Fix some $T_0 > 0$. Since $\tanh(x)/x$ is decreasing for $x \geq 0$ and bounded by 1, the integrand is bounded by $\frac{1}{2}\chi_{|q|<2\sqrt{\mu/T_0}} + \frac{1}{q^2-\mu/T_0}\chi_{|q|>2\sqrt{\mu/T_0}}$ for $T > T_0$. This is an L^1 function, so by dominated convergence we can pull the limit into the integral and arrive at the claim.

Part 4: Let U_T denote the unitary transformation $U_T\psi(p) = T^{1/4}\psi(T^{1/2}p)$ on $L^2(\mathbb{R}^2)$. We shall prove that $\lim_{T\to\infty} \| U_T T^{1/2} A_{T,\mu}^{\mathbb{R}_+} U_T^\dagger - A_{1,0}^{\mathbb{R}_+} \| = 0$, which implies the claim. Note that

$$U_T T^{1/2} A_{T,\mu}^{\mathbb{R}_+} U_T^\dagger = A_{1,\mu/T}^{\mathbb{R}_+} \tag{4.106}$$

Therefore, we have

$$\begin{aligned} \lim_{T\to\infty} \| U_T T^{1/2} A_{T,\mu}^{\mathbb{R}_+} U_T^\dagger - A_{1,0}^{\mathbb{R}_+} \| &= \lim_{\mu\to 0} \| A_{1,\mu}^{\mathbb{R}_+} - A_{1,0}^{\mathbb{R}_+} \| \\ &\leq \frac{1}{4\pi} \lim_{\mu\to 0} \sup_p \left| \int_{\mathbb{R}} (B_{1,\mu}(p,q) - B_{1,0}(p,q)) \mathrm{d}q \right| + \frac{1}{4\pi} \lim_{\mu\to 0} \| B_{1,\mu} - B_{1,0} \| \end{aligned} \tag{4.107}$$

For the second term on the second line of (4.107) we bound the operator norm by the Hilbert–Schmidt norm

$$\| B_{1,\mu} - B_{1,0} \|^2 \leq \| B_{1,\mu} - B_{1,0} \|_{\mathsf{HS}}^2 = \int_{\mathbb{R}} \mathrm{d}p \int_{\mathbb{R}} \mathrm{d}q \, (B_{1,\mu}(p,q) - B_{1,0}(p,q))^2 \tag{4.108}$$

Using that $B_{T,\mu}(p,q) \leq 1/2T$ and $|\tanh(x)| \leq 1$ one can bound

$$B_{1,\mu}(p,q)^2 \leq \frac{1}{4}\chi_{p^2+q^2\leq 4\mu}(p,q) + \chi_{p^2+q^2>4\mu}(p,q) \min\left\{ \frac{1}{4}, \frac{16}{(p^2+q^2-4\mu)^2} \right\} =: f_\mu(p,q). \tag{4.109}$$

By the monotonicity of f_μ in μ, we have for all $\nu \leq \mu$ that $(B_{1,\nu}(p,q) - B_{1,0}(p,q))^2 \leq 2f_\mu(p,q)$. Since f_μ is an L^1 function, dominated convergence implies $\lim_{\mu\to 0} \| B_{1,\mu} - B_{1,0} \| = 0$.

For the first term in the second line of (4.107) we estimate

$$\lim_{\mu\to 0}\sup_p \left|\int_{\mathbb{R}}(B_{1,\mu}(p,q)-B_{1,0}(p,q))\mathrm{d}q\right| = \lim_{\mu\to 0}\sup_p \left|\int_{\mathbb{R}}\int_0^{\mu}\frac{\partial}{\partial\nu}B_{1,\nu}(p,q)\mathrm{d}\nu\mathrm{d}q\right|$$
$$\leq \lim_{\mu\to 0}\mu\sup_p\sup_{\nu\in[0,\mu]}\int_{\mathbb{R}}\left|\frac{\partial}{\partial\nu}B_{1,\nu}(p,q)\right|\mathrm{d}q, \quad (4.110)$$

where we used the triangle inequality and Fubini's theorem in the last step. By (4.12) we may write

$$\frac{\partial}{\partial\mu}B_{1,\mu}(p,q) = 2\sum_{n\in\mathbb{Z}}\frac{1}{\left(\left(\frac{p+q}{2}\right)^2-\mu-iw_n\right)^2}\frac{1}{\left(\frac{p-q}{2}\right)^2-\mu+iw_n}$$
$$+\frac{1}{\left(\frac{p+q}{2}\right)^2-\mu-iw_n}\frac{1}{\left(\left(\frac{p-q}{2}\right)^2-\mu+iw_n\right)^2}, \quad (4.111)$$

where $w_n=\pi(2n+1)$. Observe that

$$\left|\left(\frac{p+q}{2}\right)^2-\mu-iw_n\right| \geq w_n\chi_{|q|<2\sqrt{\mu}}+\sqrt{\left(q^2/4-\mu\right)^2+w_n^2}\chi_{|q|>2\sqrt{\mu}} \quad (4.112)$$

and

$$\left|\left(\frac{p-q}{2}\right)^2-\mu+iw_n\right|\geq w_n. \quad (4.113)$$

Applying Fubini's theorem to swap integration and summation, we have for all p and μ

$$\int_{\mathbb{R}}\left|\frac{\partial}{\partial\mu}B_{1,\mu}(p,q)\right|\mathrm{d}q \leq 2\sum_{n\in\mathbb{Z}}\int_{\mathbb{R}}\mathrm{d}q\left(\frac{2}{w_n^3}\chi_{|q|\leq 2\sqrt{\mu}}+\frac{\chi_{|q|>2\sqrt{\mu}}}{w_n\left((q^2/4-\mu)^2+w_n^2\right)}\right.$$
$$\left.+\frac{\chi_{|q|>2\sqrt{\mu}}}{w_n^2\sqrt{(q^2/4-\mu)^2+w_n^2}}\right)$$
$$=2\sum_{n\in\mathbb{Z}}\left[\frac{8\sqrt{\mu}}{w_n^3}+\frac{2}{w_n^{5/2}}\int_0^{\infty}\mathrm{d}s\left(\frac{1}{(s^2+1)\sqrt{s+\mu/w_n}}+\frac{1}{\sqrt{(s^2+1)(s+\mu/w_n)}}\right)\right], \quad (4.114)$$

where we substituted $s=w_n^{-1}(q^2/4-\mu)$. For $\mu<1$ we therefore obtain a μ-independent bound

$$\sup_p\sup_{\nu\in[0,\mu]}\int_{\mathbb{R}}\left|\frac{\partial}{\partial\nu}B_{1,\nu}(p,q)\right|\mathrm{d}q$$
$$\leq 2\sum_{n\in\mathbb{Z}}\left[\frac{8}{w_n^3}+\frac{2}{w_n^{5/2}}\int_0^{\infty}\mathrm{d}s\left(\frac{1}{(s^2+1)\sqrt{s}}+\frac{1}{\sqrt{(s^2+1)s}}\right)\right]<\infty. \quad (4.115)$$

Thus, the last expression in (4.110) vanishes and the claim follows. □

4.6 Proofs of Auxiliary Results

4.6.1 From Section 4.2

Proof of Lemma 4.2.1. Note that for all $p, q \in \mathbb{R}$

$$L_{T,\mu}(p,q) \leq \min\left\{\frac{1}{2T}, \frac{2}{|p^2+q^2-2\mu|}\right\} \tag{4.116}$$

Hence, $L_{T,\mu}(p,q)(1+p^2+q^2) \leq \frac{1+4T+2\mu}{2T}$ and $L_{T,\mu}(p,q)(T+p^2+q^2) \leq \frac{5T+2\mu}{2T}$. So with $C_1(T,\mu) = \frac{2T}{1+4T+2\mu}$ and $C_3(T_0,\mu) = \frac{2T_0}{5T_0+2\mu}$ the respective inequalities hold.

For the remaining inequality, note that $L_{T,\mu}$ vanishes only at infinity. Let $\epsilon > 0$. There is a constant c_1 such that $L_{T,\mu}(p,q) > c_1$ for all $|p|, |q| \leq \sqrt{\max\{2\mu, 0\} + \epsilon}$. Moreover, if $|p|$ or $|q| > \sqrt{\max\{2\mu, 0\} + \epsilon}$, we have

$$L_{T,\mu}(p,q) \geq \frac{\tanh((|\mu|+\epsilon)/2T) - \tanh(\mu/2T)}{p^2+q^2-2\mu} \geq \frac{c_2}{p^2+q^2+\max\{-2\mu,0\}} \tag{4.117}$$

In particular, $L_{T,\mu}(p,q)(1+p^2+q^2) \geq \min\{c_1, c_2, c_2/\max\{-2\mu, 0\}\}$. □

Proof of Lemma 4.2.6. First, we show that for every $x, y \in \mathbb{R}$

$$\frac{\tanh(x)+\tanh(y)}{x+y} \leq \frac{1}{2}\left(\frac{\tanh(x)}{x} + \frac{\tanh(y)}{y}\right) \tag{4.118}$$

Since changing $x \to -x, y \to -y$ does not change the expressions, we may assume without loss of generality that $x \geq |y|$. Note that

$$\frac{\tanh(x)+\tanh(y)}{x+y} = \frac{1}{2(x+y)}\left[(x+y)\left(\frac{\tanh(x)}{x} + \frac{\tanh(y)}{y}\right) + (x-y)\left(\frac{\tanh(x)}{x} - \frac{\tanh(y)}{y}\right)\right] \tag{4.119}$$

Since $\tanh(x)/x \leq \tanh(y)/y$, the last term is not positive and the inequality (4.118) follows.

For $p \in \mathbb{R}$ we therefore have

$$\int_{\mathbb{R}} B_{T,\mu}(p,q)\mathrm{d}q \leq \frac{1}{2}\int_{\mathbb{R}}\left[F_{T,\mu}\left(\frac{p+q}{2}\right) + F_{T,\mu}\left(\frac{p-q}{2}\right)\right]\mathrm{d}q = \int_{\mathbb{R}} F_{T,\mu}(q/2)\mathrm{d}q. \tag{4.120}$$

Since $F_{T,\mu}(q/2) = B(0,q)$, the claim follows. □

4.6.2 From Section 4.3

Proof of Lemma 4.3.5. Substituting by $p^2 - \mu = t$ for $p^2 > \mu$ and $\mu - p^2 = t$ for $p^2 < \mu$ we get

$$\int_{\mathbb{R}} F_{T,\mu}(p)\mathrm{d}p = 2\int_0^\infty \frac{\tanh\left(\frac{p^2-\mu}{2T}\right)}{p^2-\mu}\mathrm{d}p = 2\int_0^\infty \frac{\tanh(t/2T)}{2t\sqrt{\mu+t}}\mathrm{d}t + 2\int_0^\mu \frac{\tanh(t/2T)}{2t\sqrt{\mu-t}}\mathrm{d}t. \tag{4.121}$$

It was shown in [35, Lemma 1] that

$$\lim_{T\to 0}\left(\int_0^\mu \frac{\tanh(t/2T)}{t}\mathrm{d}t - \ln\frac{\mu}{T}\right) = \gamma - \ln\frac{\pi}{2}. \tag{4.122}$$

By monotone convergence, we observe that

$$\lim_{T\to 0}\int_0^{\mu}\frac{\tanh(t/2T)}{2t}\left(\frac{1}{\sqrt{\mu-t}}-\frac{1}{\sqrt{\mu}}\right)\mathrm{d}t=\int_0^{\mu}\frac{1}{2t}\left(\frac{1}{\sqrt{\mu-t}}-\frac{1}{\sqrt{\mu}}\right)\mathrm{d}t=\frac{\ln 4}{2\sqrt{\mu}} \quad (4.123)$$

as well as

$$\lim_{T\to 0}\int_0^{\mu}\frac{\tanh(t/2T)}{2t}\left(\frac{1}{\sqrt{\mu+t}}-\frac{1}{\sqrt{\mu}}\right)\mathrm{d}t=\int_0^{\mu}\frac{1}{2t}\left(\frac{1}{\sqrt{\mu+t}}-\frac{1}{\sqrt{\mu}}\right)\mathrm{d}t=\frac{\ln\left(2(\sqrt{2}-1)\right)}{\sqrt{\mu}}. \quad (4.124)$$

Using monotone convergence once more, we obtain

$$\lim_{T\to 0}\int_{\mu}^{\infty}\frac{\tanh(t/2T)}{2t\sqrt{\mu+t}}\mathrm{d}t=\int_{\mu}^{\infty}\frac{1}{2t\sqrt{\mu+t}}\mathrm{d}t=\frac{\ln(\sqrt{2}+1)}{\sqrt{\mu}}. \quad (4.125)$$

Combining all the terms we arrive at the first equality in (4.41). Observe that

$$0<\int_{\mathbb{R}}\chi_{|p|>\sqrt{2\mu}}F_{T,\mu}(p)\mathrm{d}p\leq 2\int_{\sqrt{2\mu}}^{\infty}\frac{1}{p^2-\mu}\mathrm{d}p<\infty. \quad (4.126)$$

Therefore, this term is of order one for $T\to 0$ and $\int_{\mathbb{R}}F_{T,\mu}(p)\mathrm{d}p=\int_{-\sqrt{2\mu}}^{\sqrt{2\mu}}F_{T,\mu}(p)\mathrm{d}p+O(1)$. □

4.6.3 From Section 4.4

Proof of Lemma 4.4.6. In the case $xy>0$, the inequality follows immediately from the fact that $|\tanh(z)|<1$ for all $z\in\mathbb{R}$. In the case $xy<0$, let us replace $y\to -y$ and assume without loss of generality that $x>y>0$. Since the function $s\mapsto e^{-2s}$ is convex, we have

$$\frac{e^{-2y}-e^{-2x}}{x-y}\leq -\frac{\mathrm{d}}{\mathrm{d}s}e^{-2s}\Big|_{s=y}=2e^{-2y} \quad (4.127)$$

We estimate

$$\begin{aligned}\frac{x+y}{x-y}(\tanh(x)-\tanh(y))=\frac{2(x+y)}{1+e^{-2y}}\frac{e^{-2y}-e^{-2x}}{(x-y)(1+e^{-2x})}&\leq\frac{2(x+y)e^{-2y}}{1+e^{-2y}}\min\left\{2,\frac{1}{x-y}\right\}\\&\leq\frac{4(2y+1/2)e^{-2y}}{1+e^{-2y}},\end{aligned} \quad (4.128)$$

where we maximized over x in the last step. The maximum of the last expression over y is attained at the value $y=\tilde{y}$ satisfying $e^{-2\tilde{y}}=2\tilde{y}-1/2$. Therefore, we get

$$\frac{x+y}{x-y}(\tanh(x)-\tanh(y))\leq 4(2\tilde{y}-1/2). \quad (4.129)$$

The function e^{-2y} is decreasing in y and $2y-1/2$ is increasing. For $y=1/2$ we have $e^{-1}<1/2$, hence the intersection point $\tilde{y}$ satisfies $0<\tilde{y}<1/2$. Thus, $\frac{x+y}{x-y}(\tanh(x)-\tanh(y))<2$, which proves the claim. □

Proof of Lemma 4.4.7. Without loss of generality, we may assume that $y<x$. We have

$$\begin{aligned}\tanh(x)-\tanh(y)=\frac{e^x-e^{-x}}{e^x+e^{-x}}-\frac{e^y-e^{-y}}{e^y+e^{-y}}&=2\frac{e^{x-y}-e^{y-x}}{(e^x+e^{-x})(e^y+e^{-y})}\\&\leq 2\frac{e^{x-y}-e^{y-x}}{e^{x+y}}=2(e^{-2y}-e^{-2x})\end{aligned}$$

Applying (4.127) the claim follows. □

4.6.4 From Section 4.5

Proof of Lemma 4.5.5. By concavity of $\tanh(x)$ for $x \geq 0$ for $x, y \geq 0$ it holds that

$$\frac{\tanh(x) + \tanh(y)}{2} \leq \tanh\left(\frac{x+y}{2}\right) \Leftrightarrow \frac{\tanh(x) + \tanh(y)}{2(x+y)} \leq \frac{\tanh\left(\frac{x+y}{2}\right)}{x+y} \tag{4.130}$$

Choosing $x = (p+q)^2/8$ and $y = (p-q)^2/8$ gives the desired inequality. □

CHAPTER 5

BCS Critical Temperature on Half-Spaces

Abstract We study the BCS critical temperature on half-spaces in dimensions $d = 1, 2, 3$ with Dirichlet or Neumann boundary conditions. We prove that the critical temperature on a half-space is strictly higher than on $\mathbb{R}^d$, at least at weak coupling in $d = 1, 2$ and weak coupling and small chemical potential in $d = 3$. Furthermore, we show that the relative shift in critical temperature vanishes in the weak coupling limit.

5.1 Introduction and Results

We study the effect of a boundary on the critical temperature of a superconductor in the Bardeen-Cooper-Schrieffer model. It was recently observed [6, 7, 62, 63, 68] that the presence of a boundary may increase the critical temperature. For a one-dimensional system with δ-interaction, a rigorous mathematical justification was given in [34]. Here, we generalize this result to generic interactions and higher dimensions. While in dimensions $d = 2, 3$ the existing numerical works only consider lattice models, our analytic approach allows us to study continuum models. We compare the half infinite superconductor with shape $\Omega_1 = (0, \infty) \times \mathbb{R}^{d-1}$ to the superconductor on $\Omega_0 = \mathbb{R}^d$ in dimensions $d = 1, 2, 3$. We impose either Dirichlet or Neumann boundary conditions, and prove that in the presence of a boundary the critical temperature can increase. The critical temperature can be determined from the spectrum of the two-body operator

$$H_T^\Omega = \frac{-\Delta_x - \Delta_y - 2\mu}{\tanh\left(\frac{-\Delta_x - \mu}{2T}\right) + \tanh\left(\frac{-\Delta_y - \mu}{2T}\right)} - \lambda V(x - y) \tag{5.1}$$

acting in $L^2_{\text{sym}}(\Omega \times \Omega) = \{\psi \in L^2(\Omega \times \Omega) | \psi(x, y) = \psi(y, x) \text{ for all } x, y \in \Omega\}$ with appropriate boundary conditions [21]. Here, Δ denotes the Dirichlet or Neumann Laplacian on Ω and the subscript indicates on which variable it acts. Furthermore, T denotes the temperature, μ is the chemical potential, V is the interaction and λ is the coupling constant. The first term in H_T^Ω is defined through functional calculus.

Importantly, the system is superconducting if $\inf \sigma(H_T^\Omega) < 0$. For translation invariant systems, i.e. $\Omega = \mathbb{R}^d$, it was shown in [33] that superconductivity is equivalent to $\inf \sigma(H_T^\Omega) < 0$. In this case, there is a unique critical temperature T_c determined by $\inf \sigma(H_{T_c}^\Omega) = 0$ which separates the superconducting and the normal phase. The critical temperatures T_c^0 and T_c^1 for $\Omega = \mathbb{R}^d$ and $\Omega = \Omega_1$, respectively, are defined as

$$T_c^j(\lambda) := \inf\{T \in (0,\infty) | \inf \sigma(H_T^{\Omega_j}) \geq 0\}. \tag{5.2}$$

In Lemma 5.2.3 we prove that the $\inf \sigma(H_T^{\Omega_1}) \leq \inf \sigma(H_T^{\Omega_0})$. Therefore, $T_c^1(\lambda) \geq T_c^0(\lambda)$. The main part is to show that the inequality is strict.

Our strategy involves proving $\inf \sigma(H_{T_c^0(\lambda)}^{\Omega_1}) < 0$ using the variational principle. The idea is to construct a trial state involving the ground state of $H_{T_c^0(\lambda)}^{\Omega_0}$. However, $H_T^{\Omega_0}$ is translation invariant in the center of mass coordinate and thus has purely essential spectrum. To obtain a ground state eigenfunction, we remove the translation invariant directions, and instead consider the reduced operator

$$H_T^0 = \frac{-\Delta - \mu}{\tanh\left(\frac{-\Delta-\mu}{2T}\right)} - \lambda V(r) \tag{5.3}$$

acting in $L_{\mathrm{s}}^2(\mathbb{R}^d)$, where $L_{\mathrm{s}}^2(\Omega) = \{\psi \in L^2(\Omega) | \psi(r) = \psi(-r)\}$ (c.f. Lemma 5.2.4). Our trial state hence involves the ground state of $H_{T_c^0(\lambda)}^0$. In the weak coupling limit, $\lambda \to 0$, we can compute the asymptotic form of this ground state provided that $\mu > 0$ and the operator $\mathcal{V}_\mu : L_{\mathrm{s}}^2(\mathbb{S}^{d-1}) \to L_{\mathrm{s}}^2(\mathbb{S}^{d-1})$ with integral kernel

$$\mathcal{V}_\mu(p,q) = \frac{1}{(2\pi)^{d/2}} \widehat{V}(\sqrt{\mu}(p-q)) \tag{5.4}$$

has a non-degenerate eigenvalue $e_\mu = \sup \sigma(\mathcal{V}_\mu) > 0$ at the top of its spectrum [32, 40]. Here, $\widehat{V}(p) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} V(r) e^{-ip\cdot r} \mathrm{d}r$ denotes the Fourier transform of V. For $d = 1$, since $L_{\mathrm{s}}^2(\mathbb{S}^0)$ is a one-dimensional vector space, $\mathcal{V}_\mu$ is just multiplication by the number $e_\mu = \frac{\widehat{V}(0)+\widehat{V}(\mu)}{2(2\pi)^{1/2}}$.

We make the following assumptions on the interaction potential.

Assumption 5.1.1. Let $d \in \{1,2,3\}$ and $\mu > 0$. Assume that

1. $V \in L^1(\mathbb{R}^d) \cap L^{p_d}(\mathbb{R}^d)$, where $p_d = 1$ for $d = 1$, and $p_d > d/2$ for $d \in \{2,3\}$,
2. V is radial, $V \not\equiv 0$,
3. $|\cdot| V \in L^1(\mathbb{R}^d)$,
4. $\widehat{V}(0) > 0$,
5. $e_\mu = \sup \sigma(\mathcal{V}_\mu)$ is a non-degenerate eigenvalue.

Remark 5.1.2. The assumption $V \in L^1(\mathbb{R}^d)$ implies that $\widehat{V}$ is continuous and bounded. The operator $\mathcal{V}_\mu$ is thus Hilbert-Schmidt and in particular compact. Due to Assumption 5 we have $e_\mu > 0$. This in turn implies that the critical temperature $T_c^0(\lambda)$ for the system on $\mathbb{R}^d$ is positive for all $\lambda > 0$ ([32, Theorem 3.2] for $d = 3$, and [40, Theorem 2.5] for $d = 1, 2$). Furthermore, radiality of V and Assumption 5 imply that the eigenfunction corresponding to e_μ must be rotation invariant, i.e. the constant function. Assumption 5 is satisfied if $\widehat{V} \geq 0$ [32].

These assumptions suffice to observe boundary superconductivity in $d = 1, 2$. For $d = 3$, we need one additional condition. Let

$$j_d(r;\mu) := \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{S}^{d-1}} e^{i\omega\cdot r\sqrt{\mu}} \mathrm{d}\omega. \tag{5.5}$$

Define

$$\widetilde{m}_3^{D/N}(r;\mu) := \int_{\mathbb{R}} \left(j_3(z_1,r_2,r_3;\mu)^2 - |j_3(z_1,r_2,r_3;\mu) \mp j_3(r;\mu)|^2 \chi_{|z_1|<|r_1|} \right) \mathrm{d}z_1 \mp \frac{\pi}{\mu^{1/2}} j_3(r;\mu)^2, \tag{5.6}$$

where the indices D and N as well as the upper/lower signs correspond to Dirichlet/Neumann boundary conditions, respectively. Our main result is as follows:

Theorem 5.1.3. *Let $d \in \{1,2,3\}$, $\mu > 0$ and let V satisfy 5.1.1. Assume either Dirichlet or Neumann boundary conditions. For $d = 3$ additionally assume that*

$$\int_{\mathbb{R}^3} V(r)\widetilde{m}_3^{D/N}(r;\mu)\mathrm{d}r > 0. \tag{5.7}$$

Then there is a $\lambda_1 > 0$, such that for all $0 < \lambda < \lambda_1$, $T_c^1(\lambda) > T_c^0(\lambda)$.

For $d = 3$ we prove that (5.7) is satisfied for small enough chemical potential.

Theorem 5.1.4. *Let $d = 3$ and let V satisfy 5.1.11-4. For Dirichlet boundary conditions, additionally assume that $|\cdot|^2 V \in L^1(\mathbb{R}^3)$ and $\int_{\mathbb{R}^3} V(r)r^2\mathrm{d}r > 0$. Then there is a $\mu_0 > 0$ such that for all $0 < \mu < \mu_0$, $\int_{\mathbb{R}^3} V(r)\widetilde{m}_3^{D/N}(r;\mu)\mathrm{d}r > 0$. In particular, if V additionally satisfies 5.1.15 for small μ (e.g. if $\widehat{V} \geq 0$), then for small μ there is a $\lambda_1(\mu) > 0$ such that $T_c^1(\lambda) > T_c^0(\lambda)$ for $0 < \lambda < \lambda_1(\mu)$.*

Remark 5.1.5. Numerical evaluation of $\widetilde{m}_3^D$ suggests that $\widetilde{m}_3^D \geq 0$ (see Section 5.5, in particular Figure 5.1). Hence, for Dirichlet boundary conditions (5.7) appears to hold under the additional assumption that $V \geq 0$. We therefore expect that for Dirichlet boundary conditions also in 3 dimensions boundary superconductivity occurs for all values of μ. There is no proof so far, however.

Remark 5.1.6. One may wonder why in $d = 1, 2$ no condition like (5.7) is needed. Actually, in $d = 1, 2$ the analogous condition is always satisfied if $\widehat{V}(0) > 0$. The reason is that if one defines $\widetilde{m}_d^{D/N}(r;\mu)$ by replacing j_3 by j_d in (5.6), the first term diverges and $\widetilde{m}_d^{D/N}(r;\mu) = +\infty$.

Our second main result is that the relative shift in critical temperature vanishes as $\lambda \to 0$. This generalizes the corresponding result for $d = 1$ with contact interaction in [34].

Theorem 5.1.7. *Let $d \in \{1,2,3\}$, $\mu > 0$ and let V satisfy 5.1.1 and $V \geq 0$. Then*

$$\lim_{\lambda\to 0} \frac{T_c^1(\lambda) - T_c^0(\lambda)}{T_c^0(\lambda)} = 0. \tag{5.8}$$

We expect that the additional assumption $V \geq 0$ in Theorem 5.1.7 is not necessary; it is required in our proof, however.

The rest of the paper is organized as follows. In Section 5.2 we prove the Lemmas mentioned in the introduction. In Section 5.3 we use the Birman-Schwinger principle to study the ground state of $H^0_{T_c^0(\lambda)}$. Section 5.4 contains the proof of Theorem 5.1.3. Section 5.5 discusses the conditions under which (5.7) holds and in particular contains the proof Theorem 5.1.4. In Section 5.6 we study the relative temperature shift and prove Theorem 5.1.7. Section 5.7 contains the proof of auxiliary Lemmas from Section 5.6.

5.2 Preliminaries

The following functions will occur frequently

$$K_{T,\mu}(p,q) := \frac{p^2+q^2-2\mu}{\tanh\left(\frac{p^2-\mu}{2T}\right)+\tanh\left(\frac{q^2-\mu}{2T}\right)} \tag{5.9}$$

and

$$B_{T,\mu}(p,q) := \frac{1}{K_{T,\mu}(p+q,p-q)}. \tag{5.10}$$

We will suppress the subscript μ and write K_T, B_T when the μ-dependence is not relevant. The following estimate [34, Lemma 2.1] will prove useful.

Lemma 5.2.1. *For every $T_0 > 0$ there is a constant $C_1(T_0,\mu) > 0$ such that for $T > T_0$, $C_1(T+p^2+q^2) \le K_T(p,q)$. For every $T > 0$ there is a constant $C_2(T,\mu) > 0$ such that $K_T(p,q) \le C_2(p^2+q^2+1)$.*

The minimal value of K_T is $2T$. Since $|\tanh(x)| < 1$, we have for all $p,q \in \mathbb{R}^d$ and $T \ge 0$

$$B_T(p,q) \le \frac{1}{\max\{|p^2+q^2-\mu|, 2T\}} \quad \text{and} \quad B_T(p,q)\chi_{p^2+q^2>2\mu>0} \le \frac{C(\mu)}{1+p^2+q^2}, \tag{5.11}$$

where $C(\mu)$ depends only on μ.

Remark 5.2.2. Assumption 5.1.11 guarantees that V is infinitesimally form bounded with respect to $-\Delta_x - \Delta_y$ [50, 59]. By Lemma 5.2.1, H_T^Ω defines a self-adjoint operator via the KLMN theorem. Furthermore, H_T^Ω becomes positive for T large enough and hence the critical temperatures are finite.

Let K_T^Ω be the kinetic term in H_T^Ω. The corresponding quadratic form acts as $\langle \psi, K_T^\Omega \psi\rangle = \int_{\Omega^4} \overline{\psi(x,y)} K_T^\Omega(x,y;x',y')\psi(x',y')\mathrm{d}x\mathrm{d}y\mathrm{d}x'\mathrm{d}y'$ where $K_T^\Omega(x,y;x',y')$ is the distribution

$$K_T^\Omega(x,y;x',y') = \int_{\mathbb{R}^{2d}} \overline{F_\Omega(x,p)F_\Omega(y,q)} K_T(p,q) F_\Omega(x',p)F_\Omega(y',q)\mathrm{d}p\mathrm{d}q, \tag{5.12}$$

with

$$F_{\mathbb{R}^d}(x,p) = \frac{e^{-ip\cdot x}}{(2\pi)^{d/2}} \quad \text{and} \quad F_{\Omega_1}(x,p) = \frac{(e^{-ip_1x_1} \mp e^{ip_1x_1})e^{-i\tilde{p}\cdot\tilde{x}}}{2^{1/2}(2\pi)^{d/2}}, \tag{5.13}$$

where the $-/+$ sign corresponds to Dirichlet and Neumann boundary conditions, respectively. Here, $\tilde{x}$ denotes the vector containing all but the first component of x. (In the case $d=1$, $\tilde{x}$ is empty and can be omitted.)

Lemma 5.2.3. *Let $T,\lambda > 0$, $d \in \{1,2,3\}$, and let V satisfy 5.1.11. Then $\inf\sigma(H_T^{\Omega_1}) \le \inf\sigma(H_T^{\Omega_0})$.*

The following Lemma shows that we may use H_T^0 instead of $H_T^{\Omega_0}$ to compute $T_c^0(\lambda)$.

Lemma 5.2.4. *Let $T,\lambda > 0$, $d \in \{1,2,3\}$, and let V satisfy 5.1.11. Then $\inf\sigma(H_T^{\Omega_0}) = \inf\sigma(H_T^0)$.*

Remark 5.2.5. The essential spectrum of H_T^0 satisfies $\inf\sigma_{\text{ess}}(H_T^0) = 2T$ (see e.g. [48, Proof of Thm 3.7]). Due to continuity of $\inf\sigma(H_T^0)$ in T (see Lemma 5.4.1), $\inf\sigma(H^0_{T_c^0(\lambda)}) = 0$. In particular, zero is an eigenvalue of $H^0_{T_c^0(\lambda)}$.

5.2.1 Proof of Lemma 5.2.3

Proof of Lemma 5.2.3. Let S_l be the shift to the right by l in the first component, i.e. $S_l\psi(x,y) = \psi(((x_1-l),\tilde{x}),(y_1-l,\tilde{y}))$. Let ψ be a compactly supported function in $H^1_{\mathrm{sym}}(\mathbb{R}^{2d})$, the Sobolev space restricted to functions satisfying $\psi(x,y)=\psi(y,x)$. For l big enough, $S_l\psi$ is supported on half-space and satisfies both Dirichlet and Neumann boundary conditions. The goal is to prove that $\lim_{l\to\infty}\langle S_l\psi, H_T^{\Omega_1}S_l\psi\rangle = \langle\psi, H_T^{\Omega_0}\psi\rangle$. Then, since compactly supported functions are dense in $H^1_{\mathrm{sym}}(\mathbb{R}^{2d})$, the claim follows.

Note that $\langle S_l\psi, VS_l\psi\rangle = \langle\psi, V\psi\rangle$. Furthermore, using symmetry of K_T in p_1 and q_1 one obtains

$$\langle S_l\psi, K_T^{\Omega_1}S_l\psi\rangle = \int_{\mathbb{R}^{2d}} \overline{\widehat{\psi}(p,q)}K_T(p,q)\Big[\widehat{\psi}(p,q) \mp \widehat{\psi}((-p_1,\tilde{p}),q)e^{i2lp_1} \mp \widehat{\psi}(p,(-q_1,\tilde{q}))e^{i2lq_1} + \widehat{\psi}((-p_1,\tilde{p}),(-q_1,\tilde{q}))e^{i2l(p_1+q_1)}\Big]\mathrm{d}p\mathrm{d}q \quad (5.14)$$

for l big enough such that ψ is supported on the half-space. The first term is exactly $\langle\psi, K_T^{\Omega_0}\psi\rangle$. Note that by the Schwarz inequality and Lemma 5.2.1, the function

$$(p,q)\mapsto \overline{\widehat{\psi}(p,q)}K_T(p,q)\widehat{\psi}((-p_1,\tilde{p}),q) \quad (5.15)$$

is in $L^1(\mathbb{R}^{2d})$ since $\psi\in H^1(\mathbb{R}^{2d})$. By the Riemann-Lebesgue Lemma, the second term in (5.14) vanishes for $l\to\infty$. By the same argument, also the remaining terms vanish in the limit. □

5.2.2 Proof of Lemma 5.2.4

First, we prove the following inequality.

Lemma 5.2.6. *For all $x,y\in\mathbb{R}$ we have*

$$\frac{x+y}{\tanh(x)+\tanh(y)} \geq \frac{1}{2}\left(\frac{x}{\tanh(x)}+\frac{y}{\tanh(y)}\right) \quad (5.16)$$

Proof of Lemma 5.2.6. Suppose $|x|\neq|y|$. Without loss of generality we may assume $x>|y|$. Since $\frac{x}{\tanh x}\geq\frac{y}{\tanh y}$,

$$\frac{x}{2\tanh x}\frac{\tanh x-\tanh y}{\tanh x+\tanh y} \geq \frac{y}{2\tanh y}\frac{\tanh x-\tanh y}{\tanh x+\tanh y} \quad (5.17)$$

This inequality is equivalent to (5.16), as can be seen using $\frac{\tanh x-\tanh y}{\tanh x+\tanh y} = \frac{2\tanh x}{\tanh x+\tanh y}-1 = 1-\frac{2\tanh y}{\tanh x+\tanh y}$ on the left and right side, respectively. By continuity, (5.16) also holds in the case $|x|=|y|$. □

Proof of Lemma 5.2.4. Let U denote the unitary transform $U\psi(r,z)=\frac{1}{2^{d/2}}\psi((r+z)/2,(z-r)/2)$ for $\psi\in L^2(\mathbb{R}^{2d})$. By Lemma 5.2.6 we have

$$\begin{aligned} UH_T^{\Omega_0}U^\dagger &= \frac{-(\nabla_r+\nabla_z)^2-(\nabla_r-\nabla_z)^2-2\mu}{\tanh\left(\frac{-(\nabla_r+\nabla_z)^2-\mu}{2T}\right)+\tanh\left(\frac{-(\nabla_r-\nabla_z)^2-\mu}{2T}\right)}+V(r) \\ &\geq \frac{1}{2}\left(\frac{-(\nabla_r+\nabla_z)^2-\mu}{\tanh\left(\frac{-(\nabla_r+\nabla_z)^2-\mu}{2T}\right)}+V(r)\right)+\frac{1}{2}\left(\frac{-(\nabla_r-\nabla_z)^2-\mu}{\tanh\left(\frac{-(\nabla_r-\nabla_z)^2-\mu}{2T}\right)}+V(r)\right) \end{aligned} \quad (5.18)$$

Both summands are unitarily equivalent to $\frac{1}{2}H_T^0 \otimes \mathbb{I}$, where $\mathbb{I}$ acts on $L^2(\mathbb{R}^d)$. Therefore, $\inf \sigma(H_T^{\Omega_0}) \geq \inf \sigma(H_T^0)$.

For the opposite inequality let $f \in H^1(\mathbb{R}^d)$ with $f(r) = f(-r)$ and $\psi_\epsilon(r,z) = e^{-\epsilon \sum_{j=1}^d |z_j|} f(r)$. Note that $\|\psi_\epsilon\|_2^2 = \frac{1}{\epsilon^d}\|f\|_2^2$. Since the Fourier transform of $e^{-\epsilon|r_1|}$ in $L^2(\mathbb{R})$ is $\sqrt{\frac{2}{\pi}}\frac{\epsilon}{\epsilon^2+p_1^2}$, we have $\widehat{\psi_\epsilon}(p,q) = \frac{2^{d/2}}{\pi^{d/2}} \prod_{j=1}^d \frac{\epsilon}{(\epsilon^2+p_j^2)} \widehat{f}(q)$. Therefore,

$$
\begin{aligned}
\frac{\langle \psi_\epsilon | U H_T^{\Omega_0} U^\dagger \psi_\epsilon \rangle}{\|\psi_\epsilon\|^2} &= \frac{2^d}{\pi^d \|f\|^2} \int_{\mathbb{R}^{2d}} K_T(p+q, p-q) \prod_{j=1}^d \frac{\epsilon^3}{(\epsilon^2+p_j^2)^2} |\widehat{f}(q)|^2 \mathrm{d}p\mathrm{d}q \\
&= \frac{2^d}{\pi^d \|f\|^2} \int_{\mathbb{R}^{2d}} K_T(\epsilon p+q, \epsilon p-q) \left(\prod_{j=1}^d \frac{1}{(1+p_j^2)^2} \right) |\widehat{f}(q)|^2 \mathrm{d}p\mathrm{d}q, \quad (5.19)
\end{aligned}
$$

where we substituted $p \to \epsilon p$ in the second step. By Lemma 5.2.1,

$$
K_T(\epsilon p+q, \epsilon p-q) \left(\prod_{j=1}^d \frac{1}{(1+p_j^2)^2} \right) |\widehat{f}(q)|^2 \leq C(1+d\epsilon^2+q^2) \left(\prod_{j=1}^d \frac{1}{1+p_j^2} \right) |\widehat{f}(q)|^2, \quad (5.20)
$$

which is integrable. With $\int_{\mathbb{R}} \frac{1}{(1+p_j^2)^2} \mathrm{d}p_j = \pi/2$ it follows by dominated convergence that

$$
\lim_{\epsilon \to 0} \frac{\langle \psi_\epsilon | U H_T^{\Omega_0} U^\dagger \psi_\epsilon \rangle}{\|\psi_\epsilon\|^2} = \frac{\langle f | H_T^0 f \rangle}{\|f\|^2}. \quad (5.21)
$$

□

5.3 Ground State of $H^0_{T_c^0(\lambda)}$

To study the ground state of $H^0_{T_c^0(\lambda)}$, it is convenient to apply the Birman-Schwinger principle. For $q \in \mathbb{R}^d$ let $B_T(\cdot, q)$ denote the operator on $L^2(\mathbb{R}^d)$ which acts as multiplication by $B_T(p,q)$ (defined in (5.10)) in momentum space. The Birman-Schwinger operator corresponding to H_T^0 acts on $L^2_{\mathsf{s}}(\mathbb{R}^d)$ and is given by

$$
A_T^0 = V^{1/2} B_T(\cdot, 0) |V|^{1/2}, \quad (5.22)
$$

where we use the notation $V^{1/2}(x) = \mathrm{sgn}(V(x))|V|^{1/2}(x)$. This operator is compact [32, 40]. It follows from the Birman-Schwinger principle that $\sup \sigma(A_T^0) = 1/\lambda$ exactly for $T = T_c^0(\lambda)$ and that the eigenvalue 0 of $H^0_{T_c^0(\lambda)}$ has the same multiplicity as the largest eigenvalue $A^0_{T_c^0(\lambda)}$.

Let $\mathcal{F} : L^1(\mathbb{R}^d) \to L^2(\mathbb{S}^{d-1})$ act as $\mathcal{F}\psi(\omega) = \widehat{\psi}(\sqrt{\mu}\omega)$ and define $O_\mu = V^{1/2}\mathcal{F}^\dagger\mathcal{F}|V|^{1/2}$ on $L^2_{\mathsf{s}}(\mathbb{R}^d)$. Furthermore, let

$$
m_\mu(T) = \int_0^{\sqrt{2\mu}} B_T(t,0) t^{d-1} \mathrm{d}t. \quad (5.23)
$$

Note that $m_\mu(T) = \mu^{d/2-1}(\ln(\mu/T) + c_d) + o(1)$ for $T \to 0$, where c_d is a number depending only on d [40, Prop 3.1].

The operator O_μ captures the singularity of A_T^0 as $T \to 0$. The following has been proved in [22, Lemma 2] for $d = 3$ and in [40, Lemma 3.4] for $d = 1, 2$.

Lemma 5.3.1. *Let $d \in \{1,2,3\}$ and $\mu > 0$ and assume 5.1.1. Then,*

$$\sup_{T\in(0,\infty)} \left\| A_T^0 - m_\mu(T) O_\mu \right\|_{HS} < \infty, \tag{5.24}$$

where $\|\cdot\|_{HS}$ denotes the Hilbert-Schmidt norm.

Thus, the asymptotic behavior of $\sup \sigma(A_T^0)$ depends on the largest eigenvalue of O_μ. Note that O_μ is isospectral to $\mathcal{V}_\mu = \mathcal{F} V \mathcal{F}^\dagger$, since both operators are compact. The eigenfunction of O_μ corresponding to the eigenvalue e_μ is

$$\Psi(r) := V^{1/2}(r) j_d(r;\mu), \tag{5.25}$$

where j_d was defined in (5.5). Note that

$$j_1(r;\mu) = \sqrt{\frac{2}{\pi}} \cos(\sqrt{\mu} r), \quad j_2(r;\mu) = J_0(\sqrt{\mu}|r|), \quad j_3(r;\mu) = \frac{2}{(2\pi)^{1/2}} \frac{\sin\sqrt{\mu}|r|}{\sqrt{\mu}|r|}, \tag{5.26}$$

where J_0 is the Bessel function of order 0. Furthermore

$$e_\mu = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{S}^{d-1}} \widehat{V}(\sqrt{\mu}((1,0,...,0)-p)) \mathrm{d}p = \frac{1}{|\mathbb{S}^{d-1}|} \int_{\mathbb{R}^d} V(r) j_d(r;\mu)^2 \mathrm{d}r \tag{5.27}$$

The following asymptotics of $T_c^0(\lambda)$ for $\lambda \to 0$ was computed in [32, Theorem 3.3] and [40, Theorem 2.5].

Lemma 5.3.2. *Let $\mu > 0$, $d \in \{1,2,3\}$ and assume 5.1.1. Then*

$$\lim_{\lambda\to 0} \left| e_\mu m_\mu(T_c^0(\lambda)) - \frac{1}{\lambda} \right| = \lim_{\lambda\to 0} \left| e_\mu \mu^{d/2-1} \ln\left(\frac{\mu}{T_c^0(\lambda)}\right) - \frac{1}{\lambda} \right| < \infty. \tag{5.28}$$

Lemma 5.3.1 does not only contain information about eigenvalues, but also about the corresponding eigenfunctions. In the following we prove that the eigenstate corresponding to the maximal eigenvalue of A_T^0 converges to Ψ.

Lemma 5.3.3. *Let $\mu > 0$, $d \in \{1,2,3\}$ and assume 5.1.1.*

1. *There is a $\lambda_0 > 0$ such that for $\lambda \le \lambda_0$, the largest eigenvalue of $A^0_{T_c^0(\lambda)}$ is non-degenerate.*

2. *Let $\lambda \le \lambda_0$ and let $\Psi_{T_c^0(\lambda)}$ be the eigenvector of $A^0_{T_c^0(\lambda)}$ corresponding to the largest eigenvalue, normalized such that $\|\Psi_{T_c^0(\lambda)}\|_2 = \|\Psi\|_2$. Pick the phase of $\Psi_{T_c^0(\lambda)}$ such that $\langle \Psi_{T_c^0(\lambda)}, \Psi\rangle \ge 0$. Then*
$$\lim_{\lambda\to 0} \frac{1}{\lambda} \|\Psi - \Psi_{T_c^0(\lambda)}\|_2^2 < \infty \tag{5.29}$$

Remark 5.3.4. Let λ_0 be as in Lemma 5.3.3. By the Birman-Schwinger principle, the multiplicity of the largest eigenvalue of $A^0_{T_c^0(\lambda)}$ equals the multiplicity of the ground state of $H^0_{T_c^0(\lambda)}$. Hence, $H^0_{T_c^0(\lambda)}$ has a unique ground state for $\lambda \le \lambda_0$. For $d \ge 2$, since $H^0_{T_c^0(\lambda)}$ is rotation invariant, uniqueness of the ground state implies that the ground state is radial.

For values of λ such that the operator $H^0_{T^0_c(\lambda)}$ has a non-degenerate eigenvalue at the bottom of its spectrum let Φ_λ be the corresponding eigenfunction, with normalization and phase chosen such that $\Psi_{T^0_c(\lambda)} = V^{1/2}\Phi_\lambda$. The following Lemma with regularity and convergence properties of Φ_λ will be useful.

Lemma 5.3.5. *Let $d \in \{1,2,3\}$, $\mu > 0$ and assume 5.1.1. For all $0 < \lambda < \infty$ such that $H^0_{T^0_c(\lambda)}$ has a non-degenerate ground state Φ_λ, we have*

1. $|\widehat{\Phi_\lambda}(p)| \leq \frac{C(\lambda)}{1+p^2}|\widehat{V\Phi_\lambda}(p)| \leq \frac{C(\lambda)\|V\|_1^{1/2}\|\Psi\|_2}{1+p^2}$ *for some number $C(\lambda)$ depending on λ,*
2. $p \mapsto \widehat{\Phi_\lambda}(p)$ *is continuous,*
3. $\|\widehat{\Phi}_\lambda\|_1 < \infty$ *and* $\|\Phi_\lambda\|_\infty < \infty$.

Furthermore, in the limit $\lambda \to 0$

4. $\|\widehat{\Phi}_\lambda \chi_{p^2>2\mu}\|_1 = O(\lambda)$,
5. $\|\widehat{\Phi}_\lambda\|_1 = O(1)$,
6. *and in particular* $\|\Phi_\lambda\|_\infty = O(1)$.

In three dimensions, because of the additional condition (5.7), we need to compute the limit of Φ_λ.

Lemma 5.3.6. *Let $d = 3$, $\mu > 0$ and assume 5.1.1. Then $\|\Phi_\lambda - j_3\|_\infty = O(\lambda^{1/2})$ as $\lambda \to 0$.*

5.3.1 Proof of Lemma 5.3.3

Proof of Lemma 5.3.3. Part 1: The proof uses ideas from [35, Proof of Thm 1]. Let $M_T = B_T(\cdot,0) - m_\mu(T)\mathcal{F}^\dagger\mathcal{F}$. By Lemma 5.3.1, for λ small enough the operator $1-\lambda V^{1/2}M_T|V|^{1/2}$ is invertible for all T. Then we can write

$$1-\lambda A^0_T = (1-\lambda V^{1/2}M_T|V|^{1/2})\left(1-\frac{\lambda m_\mu(T)}{1-\lambda V^{1/2}M_T|V|^{1/2}}V^{1/2}\mathcal{F}^\dagger\mathcal{F}|V|^{1/2}\right) \tag{5.30}$$

Recall that the largest eigenvalue of $A^0_{T^0_c(\lambda)}$ equals $1/\lambda$. Hence, 1 is an eigenvalue of

$$\frac{\lambda m_\mu(T^0_c(\lambda))}{1-\lambda V^{1/2}M_{T^0_c(\lambda)}|V|^{1/2}}V^{1/2}\mathcal{F}^\dagger\mathcal{F}|V|^{1/2} \tag{5.31}$$

and it has the same multiplicity as the eigenvalue $1/\lambda$ of $A^0_{T^0_c(\lambda)}$. This operator is isospectral to the self-adjoint operator

$$\mathcal{F}|V|^{1/2}\frac{\lambda m_\mu(T^0_c(\lambda))}{1-\lambda V^{1/2}M_{T^0_c(\lambda)}|V|^{1/2}}V^{1/2}\mathcal{F}^\dagger. \tag{5.32}$$

Note that the operator difference

$$\mathcal{F}|V|^{1/2}\frac{1}{1-\lambda V^{1/2}M_{T^0_c(\lambda)}|V|^{1/2}}V^{1/2}\mathcal{F}^\dagger - \mathcal{V}_\mu = \lambda\mathcal{F}|V|^{1/2}\frac{V^{1/2}M_{T^0_c(\lambda)}|V|^{1/2}}{1-\lambda V^{1/2}M_{T^0_c(\lambda)}|V|^{1/2}}V^{1/2}\mathcal{F}^\dagger \tag{5.33}$$

has operator norm of order $O(\lambda)$ according to Lemma 5.3.1. By assumption, the largest eigenvalue of $\mathcal{V}_\mu$ has multiplicity one, and $\lambda m_\mu(T_c^0(\lambda))e_\mu = 1 + O(\lambda)$ by Lemma 5.3.2. Let $\alpha < 1$ be the ratio between the second largest and the largest eigenvalue of $\mathcal{V}_\mu$. The second largest eigenvalue of $\lambda m_\mu(T_c^0(\lambda))\mathcal{V}_\mu$ is of order $\alpha + O(\lambda)$. Therefore, the largest eigenvalue of (5.32) must have multiplicity 1 for small enough λ, and it is of order $1 + O(\lambda)$, whereas the rest of the spectrum lies below $\alpha + O(\lambda)$. Hence, 1 is the maximal eigenvalue of (5.32) and it has multiplicity 1 for small enough λ.

Part 2: Note that $\Psi_{T_c^0(\lambda)}$ is an eigenvector of (5.31) with eigenvalue 1. Furthermore, let ψ_λ be a normalized eigenvector of (5.32) with eigenvalue 1. Then

$$\tilde{\Psi}_{T_c^0(\lambda)} = \frac{\|\Psi\|_2}{\|\frac{1}{(1-\lambda V^{1/2}M_{T_c^0(\lambda)}|V|^{1/2})}V^{1/2}\mathcal{F}^\dagger\psi_\lambda\|_2}\frac{1}{1-\lambda V^{1/2}M_{T_c^0(\lambda)}|V|^{1/2}}V^{1/2}\mathcal{F}^\dagger\psi_\lambda \tag{5.34}$$

agrees with $\Psi_{T_c^0(\lambda)}$ up to a constant phase. Since $\|\Psi_{T_c^0(\lambda)} - \Psi\|^2 \leq \|\tilde{\Psi}_{T_c^0(\lambda)} - \Psi\|^2$, it suffices to prove that the latter is of order $O(\lambda)$ for a suitable choice of phase for ψ_λ.

Let $\psi(p) = \frac{1}{|\mathbb{S}^{d-1}|^{1/2}}$. This is the eigenfunction of $\mathcal{V}_\mu$ corresponding to the maximal eigenvalue, and $\Psi = V^{1/2}\mathcal{F}^\dagger\psi$. In particular, for all $\phi \in L^2(\mathbb{S}^{d-1})$,

$$\langle\phi, \mathcal{V}_\mu\phi\rangle \leq e_\mu|\langle\phi,\psi\rangle|^2 + \alpha e_\mu(\|\phi\|_2^2 - |\langle\phi,\psi\rangle|^2) \tag{5.35}$$

We choose the phase of ψ_λ such that $\langle\psi_\lambda,\psi\rangle \geq 0$. We shall prove that $\|\psi_\lambda - \psi\|_2^2 = O(\lambda)$. We have by (5.33) and (5.35)

$$\begin{aligned} O(\lambda) &= \langle\psi_\lambda, (1-\lambda m_\mu(T_c^0(\lambda))\mathcal{V}_\mu)\psi_\lambda\rangle \\ &\geq 1 - \lambda m_\mu(T_c^0(\lambda))e_\mu|\langle\psi_\lambda,\psi\rangle|^2 - \lambda m_\mu(T_c^0(\lambda))\alpha e_\mu(1-|\langle\psi_\lambda,\psi\rangle|^2) \\ &= O(\lambda) + (1-\alpha)(1-|\langle\psi_\lambda,\psi\rangle|^2) \end{aligned} \tag{5.36}$$

where we used Lemma 5.3.2 for the last equality. In particular, $1 - |\langle\psi_\lambda,\psi\rangle|^2 = O(\lambda)$. Hence,

$$\|\psi - \psi_\lambda\|_2^2 = 2(1 - \langle\psi_\lambda,\psi\rangle) = 2\frac{1-\langle\psi_\lambda,\psi\rangle^2}{1+\langle\psi_\lambda,\psi\rangle} = O(\lambda). \tag{5.37}$$

Using Lemma 5.3.1 and that $V^{1/2}\mathcal{F}^\dagger : L^2(\mathbb{S}^{d-1}) \to L^2(\mathbb{R}^d)$ is a bounded operator, and then (5.37) we obtain

$$\frac{1}{1-\lambda V^{1/2}M_{T_c^0(\lambda)}|V|^{1/2}}V^{1/2}\mathcal{F}^\dagger\psi_\lambda = V^{1/2}\mathcal{F}^\dagger\psi_\lambda + O(\lambda) = V^{1/2}\mathcal{F}^\dagger\psi + O(\lambda^{1/2}), \tag{5.38}$$

where $O(\lambda)$ here denotes a vector with L^2-norm of order $O(\lambda)$. Furthermore,

$$\begin{aligned} &\Big|\|(1-\lambda V^{1/2}M_{T_c^0(\lambda)}|V|^{1/2})^{-1}V^{1/2}\mathcal{F}^\dagger\psi_\lambda\|_2 - \|V^{1/2}\mathcal{F}^\dagger\psi\|_2\Big| \\ &\qquad \leq \|(1-\lambda V^{1/2}M_{T_c^0(\lambda)}|V|^{1/2})^{-1}V^{1/2}\mathcal{F}^\dagger\psi_\lambda - V^{1/2}\mathcal{F}^\dagger\psi\|_2 = O(\lambda^{1/2}). \end{aligned} \tag{5.39}$$

In total, we have

$$\begin{aligned} \tilde{\Psi}_{T_c^0(\lambda)} &= \frac{\|\Psi\|_2}{\|V^{1/2}\mathcal{F}^\dagger\psi\|_2 + O(\lambda^{1/2})}(V^{1/2}\mathcal{F}^\dagger\psi + O(\lambda^{1/2})) = \frac{\|\Psi\|_2}{\|V^{1/2}\mathcal{F}^\dagger\psi\|_2}V^{1/2}\mathcal{F}^\dagger\psi + O(\lambda^{1/2}) \\ &= \Psi + O(\lambda^{1/2}) \end{aligned} \tag{5.40}$$

□

5.3.2 Regularity and convergence of Φ_λ

In this section, we prove Lemma 5.3.5 and Lemma 5.3.6. The following standard results (see e.g. [50, Sections 11.3, 5.1]) will be helpful.

Lemma 5.3.7. *1. Let $V \in L^p(\mathbb{R}^d)$, where $p = 1$ for $d = 1$, $p > 1$ for $d = 2$ and $p = 3/2$ for $d = 3$. Let $\psi \in H^1(\mathbb{R}^d)$. Then $V^{1/2}\psi \in L^2$.*

2. If $V \in L^1(\mathbb{R}^d)$, and $\psi \in L^2(\mathbb{R}^d)$, then $V^{1/2}\psi \in L^1$ and hence $\widehat{V^{1/2}\psi}$ is continuous and bounded.

3. For $1 \leq t$, $\|\widehat{V^{1/2}\psi}\|_s \leq C\|V\|_t^{1/2}\|\psi\|_2$, where $s = 2t/(t-1)$ and C is some constant independent of ψ and V.

4. Let f be a radial, measurable function on $\mathbb{R}^3$ and $p \geq 1$. Then there is a constant C independent of f such that $\sup_{p_1\in\mathbb{R}}\|f(p_1,\cdot)\|_{L^p(\mathbb{R}^2)} = \|f(0,\cdot)\|_{L^p(\mathbb{R}^2)} \leq C(\|f\|^p_{L^p(\mathbb{R}^3)} + \|f\|^p_{L^\infty(\mathbb{R}^3)})^{1/p}$.

Proof. For parts 1 and 2 see e.g. [50, Sections 11.3, 5.1]. For part 3 let $s \geq 2$. Applying the Hausdorff-Young and Hölder inequality gives

$$\|\widehat{V^{1/2}\psi}\|_s \leq C\|V^{1/2}\psi\|_p \leq C\|V\|_t^{1/2}\|\psi\|_2, \tag{5.41}$$

where $1 = 1/p + 1/s$ and $1 = p/2t + p/2$. Hence, $s = 2t/(t-1)$.

For part 4 we write

$$\begin{aligned}\|f(p_1,\cdot)\|^p_{L^p(\mathbb{R}^2)} &= 2\pi\int_0^\infty |f(\sqrt{p_1^2+t^2})|^p t\mathrm{d}t = 2\pi\int_{|p_1|}^\infty |f(s)|^p s\mathrm{d}s \leq \|f(0,\cdot)\|^p_{L^p(\mathbb{R}^2)} \\ &\leq 2\pi\int_0^1 |f(s)|^p\mathrm{d}s + 2\pi\int_0^\infty |f(s)|^p s^2\mathrm{d}s \leq 2\pi\|f\|^p_\infty + \frac{1}{2}\|f\|^p_p,\end{aligned} \tag{5.42}$$

where in the second step we substituted $s = \sqrt{p_1^2 + t^2}$ and in the third step we used $s \leq \max\{1, s^2\}$. □

Proof of Lemma 5.3.5. The eigenvalue equation $H^0_{T_c^0(\lambda)}\Phi_\lambda = 0$ implies that

$$\widehat{\Phi}_\lambda(p) = \lambda B_{T_c^0(\lambda)}(p,0)\widehat{V\Phi}_\lambda(p). \tag{5.43}$$

Part (1) follows with Lemma 5.2.1 and 5.3.73 and the normalization $\|V^{1/2}\Phi_\lambda\|_2 = \|\Psi\|_2$. For part (2), note that $p \mapsto B_T(p,0)$ is continuous for $T > 0$. Since $\Phi_\lambda \in H^1(\mathbb{R}^d)$, continuity of $\widehat{V\Phi}_\lambda$ follows by Lemma 5.3.71 and 2.

Note that $\|\Phi_\lambda\|_\infty \leq (2\pi)^{-d/2}\|\widehat{\Phi_\lambda}\|_1 = (2\pi)^{-d/2}(\|\widehat{\Phi_\lambda}\chi_{p^2<2\mu}\|_1 + \|\widehat{\Phi_\lambda}\chi_{p^2>2\mu}\|_1)$. In particular, the second part of (3) and (6) follow from the first part of (3) and (5), respectively. Using (5.43) and $\|\Psi_{T_c^0(\lambda)}\|_2 = \|\Psi\|_2$ we obtain

$$\|\widehat{\Phi_\lambda}\chi_{p^2<2\mu}\|_1 \leq \lambda m_\mu(T_c^0(\lambda))|\mathbb{S}^{d-1}|\|\widehat{V^{1/2}\Psi_{T_c^0(\lambda)}}\|_\infty \leq \lambda m_\mu(T_c^0(\lambda))|\mathbb{S}^{d-1}|\|V\|_1^{1/2}\|\Psi\|_2, \tag{5.44}$$

where m_μ was defined in (5.23). In particular, for fixed λ, $\|\widehat{\Phi_\lambda}\chi_{p^2<2\mu}\|_1 < \infty$ and from Lemma 5.3.2 it follows that $\|\widehat{\Phi_\lambda}\chi_{p^2<2\mu}\|_1$ is bounded for $\lambda \to 0$.

It only remains to prove that $\|\widehat{\Phi}_\lambda \chi_{p^2>2\mu}\|_1$ is bounded for fixed λ and is $O(\lambda)$ for $\lambda \to 0$. By (5.11) $B_T(p,0)\chi_{p^2>2\mu} \leq C/(1+p^2)$ for some C independent of T. Using (5.43) and applying Hölder's inequality and Lemma 5.3.73,

$$\|\widehat{\Phi}_\lambda \chi_{p^2>2\mu}\|_s \leq C\lambda \left\| \frac{1}{1+|\cdot|^2} \right\|_p \|\widehat{V^{1/2}\Psi_{T_c^0(\lambda)}}\|_q \leq C\lambda \left\| \frac{1}{1+|\cdot|^2} \right\|_p \|V\|_t^{1/2} \|\Psi\|_2 \tag{5.45}$$

where $1/s = 1/p + 1/q$ and $q = 2t/(t-1)$. For $d = 1$ the claim follows with the choice $t = p = 1$. For $d = 2$, $V \in L^{1+\epsilon}$ for some $0 < \epsilon \leq 1$. With the choice $t = 1+\epsilon, p = 2t/(t+1) > 1$ the claim follows.

For $d = 3$, we may choose $1 \leq t \leq 3/2$ and $3/2 < p \leq \infty$ which gives

$$\|\widehat{\Phi}_\lambda \chi_{p^2>2\mu}\|_s = O(\lambda) \tag{5.46}$$

for all $6/5 < s \leq \infty$. We use a bootstrap argument to decrease s to one. Let us use the short notation B for multiplication with $B_T(p,0)$ in momentum space and $F : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ the Fourier transform. Using (5.43) one can find by induction that

$$\widehat{\Phi}_\lambda \chi_{p^2>2\mu} = \lambda^n (\chi_{p^2>2\mu} BFVF^\dagger)^n \widehat{\Phi}_\lambda \chi_{p^2>2\mu} + \sum_{j=1}^{n} \lambda^j (\chi_{p^2>2\mu} BFVF^\dagger)^j \widehat{\Phi}_\lambda \chi_{p^2<2\mu} \tag{5.47}$$

for any $n \in \mathbb{Z}_{\geq 1}$. The strategy is to prove that applying $\chi_{p^2>2\mu} BFVF^\dagger$ to an L^r function will give a function in $L^s \cap L^\infty$, where $s/r < c < 1$ for some fixed constant c. For n large enough, the first term will be in L^1, while the second term is in L^1 for all n since $\widehat{\Phi}_\lambda \chi_{p^2<2\mu}$ is L^1.

Lemma 5.3.8. *Let $V \in L^1 \cap L^{3/2+\epsilon}(\mathbb{R}^3)$ for some $0 < \epsilon \leq 1/2$ and let $1 \leq r \leq 3/2$ and $f \in L^r(\mathbb{R}^3)$. Let $2 \geq q \geq r$ and $3/2 < t \leq \infty$.*

1. *Then,*
$$\|\chi_{p^2>2\mu} BFVF^\dagger f\|_s \leq C(r,q) \left\| \frac{1}{1+|\cdot|^2} \right\|_t \|V\|_q \|f\|_r \tag{5.48}$$
where $1/s = 1/t + 1/r - 1/q$ and $C(r,q)$ is a finite number. (For $s < 1$, $\|\cdot\|_s$ has to be interpreted as $\|f\|_s = (\int_{\mathbb{R}^3} |f(p)|^s \mathrm{d}p)^{1/s}$.)

2. *Let $c = \frac{\epsilon}{(3+\epsilon)(3+2\epsilon)} > 0$ and let $r/(1+c) \leq s \leq \infty$. Then $\|\chi_{p^2>2\mu} BFVF^\dagger f\|_s \leq C(r,s)\|f\|_r$, where $C(r,s)$ is a finite number.*

Proof of Lemma 5.3.8. Part 1: Using (5.11) we have $|\chi_{p^2>2\mu} BFVF^\dagger f(p)| \leq \frac{C}{1+p^2} |\widehat{V} * f(p)|$. By the Young and Hausdorff-Young inequalities, the convolution satisfies

$$\|\widehat{V} * f\|_p \leq C(q,r) \|V\|_q \|f\|_r \tag{5.49}$$

for some finite constant $C(q,r)$ where $1/p = 1/r - 1/q$. The claim follows from Hölder's inequality.

Part 2: For fixed r and choosing q,t in the range $r \leq q \leq 3/2 + \epsilon$ and $3/2 + \epsilon/2 \leq t \leq \infty$, $s = (1/t + 1/r - 1/q)^{-1}$ can take all values in $[r/(1+c), \infty]$. The claim follows immediately from part 1. □

Let n be the smallest integer such that $\frac{7}{5}\frac{1}{(1+c)^n} \leq 1$. To bound the first term in (5.47), recall from (5.46) that $\|\widehat{\Phi}_\lambda \chi_{p^2>2\mu}\|_s = O(\lambda)$ for $s = 7/5$. We apply the second part of Lemma 5.3.8 n times. After the jth step, we have $\|(\chi_{p^2>2\mu} BFVF^\dagger)^j \widehat{\Phi_\lambda \chi_{p^2>2\mu}}\|_s = O(\lambda)$ for $s = \frac{7}{5}\frac{1}{(1+c)^j}$. In the nth step we pick $s=1$ and obtain $\|(\chi_{p^2>2\mu} BFVF^\dagger)^n \widehat{\Phi_\lambda \chi_{p^2>2\mu}}\|_1 = O(\lambda)$. To bound the second term in (5.47) recall that $\|\widehat{\Phi_\lambda \chi_{p^2<2\mu}}\|_1 = O(1)$. Applying the first part of Lemma 5.3.8 with $r=1, t=q=3/2+\epsilon$ implies that

$$\left\| \sum_{j=1}^n \lambda^j (\chi_{p^2>2\mu} BFVF^\dagger)^j \widehat{\Phi_\lambda \chi_{p^2<2\mu}} \right\|_1 \leq \sum_{j=1}^n \lambda^j \left(C(1,3/2+\epsilon) \left\| \frac{1}{1+|\cdot|^2} \right\|_{3/2+\epsilon} \|V\|_{3/2+\epsilon} \right)^j \|\widehat{\Phi_\lambda \chi_{p^2<2\mu}}\|_1 = O(\lambda). \quad (5.50)$$

It follows that $\|\widehat{\Phi}_\lambda \chi_{p^2>2\mu}\|_1$ is finite and $O(\lambda)$ for $d=3$. □

Proof of Lemma 5.3.6. Using the eigenvalue equation (5.43), we write

$$\begin{aligned} \Phi_\lambda(r) = &\int_{|p|>\sqrt{2\mu}} \frac{e^{ip\cdot r}}{(2\pi)^{3/2}} \widehat{\Phi}_\lambda(p) \mathrm{d}p \\ &+ \lambda \int_{|p|<\sqrt{2\mu}} \frac{e^{ip\cdot(r-r')} - e^{i\sqrt{\mu}\frac{p}{|p|}\cdot(r-r')}}{(2\pi)^3} B_{T_c^0(\lambda)}(p,0)|V|^{1/2}(r')\Psi_{T_c^0(\lambda)}(r')\mathrm{d}p\mathrm{d}r' \\ &+ \lambda \int_{|p|<\sqrt{2\mu}} \frac{e^{i\sqrt{\mu}\frac{p}{|p|}\cdot(r-r')}}{(2\pi)^3} B_{T_c^0(\lambda)}(p,0)|V|^{1/2}(r')(\Psi_{T_c^0(\lambda)}(r') - V^{1/2}(r')j_3(r'))\mathrm{d}p\mathrm{d}r' \\ &+ \lambda \int_{|p|<\sqrt{2\mu}} \frac{e^{i\sqrt{\mu}\frac{p}{|p|}\cdot(r-r')}}{(2\pi)^3} B_{T_c^0(\lambda)}(p,0)V(r')j_3(r')\mathrm{d}p\mathrm{d}r' \end{aligned} \quad (5.51)$$

We prove that the first three terms have L^∞-norm of order $O(\lambda^{1/2})$. For the first term this follows from Lemma 5.3.5. For the second term in (5.51), we proceed as in the proof of [32, Lemma 3.1]. First, integrate over the angular variables

$$\begin{aligned} &\int_{|p|<\sqrt{2\mu}} \left[e^{ip\cdot(r-r')} - e^{i\sqrt{\mu}\frac{p}{|p|}\cdot(r-r')} \right] B_{T_c^0(\lambda)}(p,0)\mathrm{d}p \\ &\qquad = \int_{|p|<\sqrt{2\mu}} \left[\frac{\sin|p||r-r'|}{|p||r-r'|} - \frac{\sin\sqrt{\mu}|r-r'|}{\sqrt{\mu}|r-r'|} \right] B_{T_c^0(\lambda)}(|p|,0)|p|^2 \mathrm{d}|p|, \end{aligned} \quad (5.52)$$

where we slightly abuse notation writing $B_T(|p|,0)$ for the radial function $B_T(p,0)$. Bounding the absolute value of this using $|\sin x/x - \sin y/y| < C|x-y|/|x+y|$ and $B_T(p,0) \leq 1/|p^2-\mu|$ gives

$$(5.52) \leq C \int_{|p|<\sqrt{2\mu}} \frac{|p|^2}{(|p|+\sqrt{\mu})^2} \mathrm{d}|p| =: \tilde{C} < \infty. \quad (5.53)$$

In particular, the second term in (5.51) is bounded uniformly in r by

$$\lambda \frac{\tilde{C}}{(2\pi)^3} \|V\|_1^{1/2} \|\Psi_{T_c^0(\lambda)}\|_2 \quad (5.54)$$

which is of order $O(\lambda)$.

To bound the absolute value of the third term in (5.51), we pull the absolute value into the integral, carry out the integration over p and use the Schwarz inequality in r'. This results in the bound

$$\lambda \frac{|\mathbb{S}^2|}{(2\pi)^3} m_\mu(T_c^0(\lambda)) \|V\|_1^{1/2} \|\Psi_{T_c^0(\lambda)} - \Psi\|_2. \tag{5.55}$$

By Lemma 5.3.2, $\lambda m_\mu(T_c^0(\lambda))$ is bounded and by Lemma 5.3.3, $\|\Psi_{T_c^0(\lambda)} - \Psi\|_2$ decays like $\lambda^{1/2}$ for small λ.

The fourth term in (5.51) equals $\lambda m_\mu(T_c^0(\lambda))\mathcal{F}^\dagger \mathcal{F} V j_3$, where we carried out the radial part of the p integration. Recall that $j_3 = \mathcal{F}^\dagger 1_{\mathbb{S}^2}$ and $\mathcal{V}_\mu 1_{\mathbb{S}^2} = e_\mu 1_{\mathbb{S}^2}$, where $1_{\mathbb{S}^2}$ is the constant function with value 1 on $\mathbb{S}^2$. Hence, $\mathcal{F}^\dagger \mathcal{F} V j_3 = \mathcal{F}^\dagger \mathcal{V}_\mu 1_{\mathbb{S}^2} = e_\mu j_3$ and the fourth term in (5.51) equals $\lambda m_\mu(T_c^0(\lambda)) e_\mu j_3$. By Lemma 5.3.2, $\lambda m_\mu(T_c^0(\lambda)) e_\mu = 1 + O(\lambda)$ as $\lambda \to 0$. Thus, $\|\Phi_\lambda - j_3\|_\infty = |\lambda m_\mu(T_c^0(\lambda)) e_\mu - 1| \|j_3\|_\infty + O(\lambda) = O(\lambda)$. □

5.4 Proof of Theorem 5.1.3

Instead of directly looking at $H_T^{\Omega_1}$, we extend the domain to $L^2(\mathbb{R}^{2d})$ by extending the wave-functions (anti)symmetrically across the boundary. Recall that $\tilde{x}$ denotes the vector containing all but the first component of x. The half-space operator $H_T^{\Omega_1}$ with Dirichlet/Neumann boundary conditions is unitarily equivalent to

$$H_T^{\text{ext}} = K_T^{\mathbb{R}^d} - \lambda V(x-y)\chi_{|x_1-y_1|<|x_1+y_1|} - \lambda V(x_1+y_1, \tilde{x}-\tilde{y})\chi_{|x_1+y_1|<|x_1-y_1|} \tag{5.56}$$

on $L^2(\mathbb{R}^d \times \mathbb{R}^d)$ restricted to functions antisymmetric/symmetric under swapping $x_1 \leftrightarrow -x_1$ and symmetric under exchange of $x \leftrightarrow y$. Next, we express H_T^{ext} in relative and center of mass coordinates $r = x - y$ and $z = x + y$. Let U be the unitary on $L^2(\mathbb{R}^{2d})$ given by $U\psi(r,z) = 2^{-d/2}\psi((r+z)/2, (z-r)/2)$. Then

$$H_T^1 := U H_T^{\text{ext}} U^\dagger = U K_T^{\mathbb{R}^d} U^\dagger - \lambda V(r)\chi_{|r_1|<|z_1|} - \lambda V(z_1, \tilde{r})\chi_{|z_1|<|r_1|} \tag{5.57}$$

on $L^2(\mathbb{R}^{2d})$ restricted to functions antisymmetric/symmetric under swapping $r_1 \leftrightarrow z_1$ and symmetric in r. The spectra of H_T^1 and $H_T^{\Omega_1}$ agree.

For an upper bound on $\inf \sigma(H_T^1)$, we restrict H_T^1 to zero momentum in the translation invariant center of mass directions and call the resulting operator $\tilde{H}_T^1$. The operator $\tilde{H}_T^1$ acts on $\{\psi \in L^2(\mathbb{R}^d \times \mathbb{R}) | \psi(r, z_1) = \psi(-r, z_1) = \mp\psi((z_1, \tilde{r}), r_1)\}$. The kinetic part of $\tilde{H}_T^1$ reads

$$\tilde{K}_T(r, z_1; r', z_1') = \int_{\mathbb{R}^{d+1}} \frac{e^{ip(r-r')+iq_1(z_1-z_1')}}{(2\pi)^{d+1}} B_T^{-1}(p, (q_1, \tilde{0})) \mathrm{d}p \mathrm{d}q_1. \tag{5.58}$$

An important property is the continuity of $\inf \sigma(H_T^1)$, proved in Section 5.4.1.

Lemma 5.4.1. *Let $d \in \{1,2,3\}$ and let V satisfy 5.1.1. Then $\inf \sigma(H_T^0)$ and $\inf \sigma(H_T^1)$ depend continuously on T for $T > 0$.*

To prove Theorem 5.1.3 we show that there is a $\lambda_1 > 0$ such that for $\lambda \leq \lambda_1$, $\inf \sigma(H^1_{T_c^0(\lambda)}) \leq \inf \sigma(\tilde{H}^1_{T_c^0(\lambda)}) < 0$. For all $T < T_c^0(\lambda)$ we have by Lemma 5.2.3 that $\inf \sigma(H_T^1) \leq \inf \sigma(H_T^{\Omega_0}) < 0$. By continuity (Lemma 5.4.1) there is an $\epsilon > 0$ such that $\inf \sigma(H_T^1) < 0$ for all $T < T_c^0(\lambda) + \epsilon$. Therefore, $T_c^1(\lambda) > T_c^0(\lambda)$.

To prove that $\inf \sigma(\tilde{H}^1_{T_c^0(\lambda)}) < 0$ for small enough λ, we pick a suitable family of trial states $\psi_\epsilon(r, z_1)$. Let λ be such that $H^0_{T_c^0(\lambda)}$ has a unique (and hence radial) ground state Φ_λ. According to Remark 5.3.4, this is the case for $0 < \lambda \leq \lambda_0$. We choose the trial states

$$\psi_\epsilon(r, z_1) = \Phi_\lambda(r) e^{-\epsilon|z_1|} \mp \Phi_\lambda(z_1, \tilde{r}) e^{-\epsilon|r_1|}, \tag{5.59}$$

with the $-$ sign for Dirichlet and $+$ for Neumann boundary conditions. Since $\Phi_\lambda(r) = \Phi_\lambda(-r) = \Phi_\lambda(-r_1, \tilde{r})$, these trial states satisfy the symmetry constraints and lie in the form domain of $\tilde{H}^1_T$. The norm of ψ_ϵ diverges as $\epsilon \to 0$.

Remark 5.4.2. The trial state is the (anti-)symmetrization of $\Phi_\lambda(r)e^{-\epsilon|z_1|}$, i.e. the projection of $\Phi_\lambda(r)e^{-\epsilon|z_1|}$ onto the domain of $\tilde{H}^1_T$. The intuition behind our choice is that, as we will see in Section 5.6, at weak coupling the Birman-Schwinger operator corresponding to $H^{\Omega_1}_T$ approximately looks like A^0_T (defined in (5.22)) on a restricted domain. This is why we want our trial state to look like the ground state Φ_λ of H^0_T projected onto the domain of $\tilde{H}^1_T$.

We shall prove that $\lim_{\epsilon\to 0}\langle \psi_\epsilon, \tilde{H}^1_{T_c^0(\lambda)} \psi_\epsilon\rangle$ is negative for weak enough coupling. This is the content of the next two Lemmas, which are proved in Sections 5.4.2 and 5.4.3, respectively.

Lemma 5.4.3. *Let $d \in \{1, 2, 3\}$, $\mu > 0$ and assume 5.1.1. Let λ be such that $H^0_{T_c^0(\lambda)}$ has a unique ground state Φ_λ. Then,*

$$\lim_{\epsilon\to 0}\langle \psi_\epsilon, \tilde{H}^1_{T_c^0(\lambda)} \psi_\epsilon\rangle = -2\lambda\Bigg(\int_{\mathbb{R}^{d+1}} V(r)\left[-|\Phi_\lambda(r) \mp \Phi_\lambda(z_1, \tilde{r})|^2 \chi_{|z_1|<|r_1|} + |\Phi_\lambda(z_1, \tilde{r})|^2\right] \mathrm{d}r\mathrm{d}z_1 \\ \mp 2\pi \int_{\mathbb{R}^{d-1}} \overline{\widehat{\Phi}_\lambda(0, \tilde{p})}\widehat{V\Phi}_\lambda(0, \tilde{p})\mathrm{d}\tilde{p}\Bigg), \tag{5.60}$$

where the upper signs correspond to Dirichlet and the lower signs to Neumann boundary conditions. For $d = 1$, the last term in (5.60) is to be understood as $\mp 2\pi \overline{\widehat{\Phi}_\lambda(0)}\widehat{V\Phi}_\lambda(0)$.

For small λ we shall prove that the expression in the round bracket in (5.60) is positive.

Lemma 5.4.4. *Let $d \in \{1, 2, 3\}$, $\mu > 0$ and let V satisfy 5.1.1. Let λ_0 be as in Remark 5.3.4. Assume Dirichlet or Neumann boundary conditions. For $d = 3$ assume that $\int_{\mathbb{R}^3} V(r)\widetilde{m}_3^{D/N}(r)\mathrm{d}r > 0$, where $\widetilde{m}_3^{D/N}$ was defined in (5.6). Then there is a $\lambda_0 \geq \lambda_1 > 0$ such that for $\lambda \leq \lambda_1$ the right hand side in (5.60) is negative.*

Therefore, for small enough ϵ, $\langle \psi_\epsilon, \tilde{H}^1_{T_c^0(\lambda)} \psi_\epsilon\rangle < 0$ proving that $\inf \sigma(\tilde{H}^1_{T_c^0(\lambda)}) < 0$. This concludes the proof of Theorem 5.1.3.

Remark 5.4.5. The additional condition $\int_{\mathbb{R}^3} V(r)\widetilde{m}_3^{D/N}(r)\mathrm{d}r > 0$ for $d = 3$ is exactly the limit of the terms in the round brackets in (5.60) for $\lambda \to 0$. Taking the limit amounts to replacing Φ_λ by j_3 (cf. Lemma 5.3.6).

5.4.1 Proof of Lemma 5.4.1

Proof of Lemma 5.4.1. Let $0 < T_0 < T_1 < \infty$. We claim that there exists a constant C_{T_0,T_1} such that $|K_T(p, q) - K_{T'}(p, q)| \leq C_{T_0,T_1}|T - T'|(1 + p^2 + q^2)$ for all $T_0 \leq T, T' \leq T_1$. To see this, compute

$$\frac{\partial}{\partial T}K_T(p,q) = \frac{K_T(p,q)}{2T^2}\frac{\operatorname{sech}\left(\frac{p^2-\mu}{2T}\right)^2(p^2-\mu)+\operatorname{sech}\left(\frac{q^2-\mu}{2T}\right)^2(q^2-\mu)}{\tanh\left(\frac{p^2-\mu}{2T}\right)+\tanh\left(\frac{q^2-\mu}{2T}\right)}. \tag{5.61}$$

K_T can be estimated using Lemma 5.2.1 and the remaining term is bounded.

The kinetic part K_T^0 of H_T^0 acts as multiplication by $K_T(p,0)$ in momentum space. For $T_0 < T, T' < T_1$ and ψ in the Sobolev space $H^1(\mathbb{R}^d)$, therefore

$$\langle \psi, (K_T^0 - K_{T'}^0)\psi\rangle \leq C_{T_0,T_1}|T-T'|\|\psi\|_{H^1(\mathbb{R}^d)}. \tag{5.62}$$

Similarly, for $T_0 < T, T' < T_1$ and $\psi \in H^1(\mathbb{R}^{2d})$,

$$\langle \psi, (K_T^{\mathbb{R}^d} - K_{T'}^{\mathbb{R}^d})\psi\rangle \leq C_{T_0,T_1}|T-T'|\|\psi\|_{H^1(\mathbb{R}^{2d})}. \tag{5.63}$$

Set $D_0 = H^1(\mathbb{R}^d)$ and $D_1 := \{\psi \in H^1(\mathbb{R}^{2d}) | \psi(x,y) = \psi(y,x) = \mp\psi((-x_1,\tilde{x}),y)\}$, where $-/+$ corresponds to Dirichlet/Neumann boundary conditions, respectively. Let $j \in \{0,1\}$ and $\epsilon > 0$. There is a family $\{\psi_T\}$ of functions in D_j such that $\|\psi_T\|_2 = 1$ and $\langle \psi_T, H_T^j \psi_T\rangle \leq \inf\sigma(H_T^j) + \epsilon$.

We first argue that there is a constant $C > 0$ such that for all $T \in [T_0, T_1] : \|\psi_T\|_{H^1} < C$. Recall that $2T$ lies in the essential spectrum of H_T^0. Together with Lemma 5.2.3, $\langle \psi_T, H_T^j\psi_T\rangle \leq 2T_1 + \epsilon$. Furthermore, by Lemma 5.2.1, the kinetic part of H_T^j is bounded below by some constant $C_1(T_0)(1-\Delta)$, where Δ denotes the Laplacian in all variables. Since the interaction is infinitesimally form bounded with respect to the Laplacian, there is a finite constant $C_2(T_0)$, such that for all $\psi \in D_j$ with $\|\psi\|_2 = 1$, $\langle \psi, H_T^j\psi\rangle \geq \frac{C_1(T_0)}{2}\langle \psi, (1-\Delta)\psi\rangle - C_2(T_0) = \frac{C(T_0)}{2}\|\psi\|_{H^1} - C_2(T_0)$. In particular, $\|\psi_T\|_{H^1} \leq \frac{2}{C_1(T_0)}(2T_1 + \epsilon + C_2(T_0)) =: C$.

Let $T, T' \in [T_0, T_1]$. Then

$$\begin{aligned}\inf\sigma(H_T^j) + \epsilon \geq \langle \psi_T, H_T^j\psi_T\rangle &= \langle \psi_T, H_{T'}^j\psi_T\rangle + \langle \psi_T, K_T - K_{T'}\psi_T\rangle \\ &\geq \inf\sigma(H_{T'}^j) - |T-T'|C_{T_0,T_1}C.\end{aligned} \tag{5.64}$$

Swapping the roles of T, T', we obtain

$$\inf\sigma(H_T^j) - \epsilon - |T-T'|C_{T_0,T_1}C \leq \inf\sigma(H_{T'}^j) \leq \inf\sigma(H_T^j) + \epsilon + |T-T'|C_{T_0,T_1}C \tag{5.65}$$

and thus

$$\inf\sigma(H_T^j) - \epsilon \leq \lim_{T'\to T}\inf\sigma(H_{T'}^j) \leq \inf\sigma(H_T^j) + \epsilon. \tag{5.66}$$

Since ϵ was arbitrary, equality follows. Hence $\inf\sigma(H_T^j)$ is continuous in T for $T > 0$. □

5.4.2 Proof of Lemma 5.4.3

The following technical lemma will be helpful for $d = 3$.

Lemma 5.4.6. *Let $V, W \in L^1 \cap L^{3/2}(\mathbb{R}^3)$, let W be radial and let $\psi \in L^2(\mathbb{R}^3)$. Then*

$$\begin{aligned}\int_{\mathbb{R}^5} |\widehat{V^{1/2}\psi}(p)|\frac{1}{1+p^2}\widehat{W}(0,\tilde{p}-\tilde{q})\frac{1}{1+p_1^2+\tilde{q}^2}|\widehat{V^{1/2}\psi}(p_1,\tilde{q})|\mathrm{d}p\mathrm{d}\tilde{q} \\ \leq C\|\widehat{W}(0,\cdot)\|_{L^3(\mathbb{R}^2)}\|V\|_{3/2}\|\psi\|_2^2 < \infty\end{aligned} \tag{5.67}$$

for some constant C independent of V, W and ψ.

Proof of Lemma 5.4.6. By Lemma 5.3.74, $\widehat{W}(0,\cdot) \in L^3(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$. By Young's inequality, the integral is bounded by

$$C\|\widehat{W}(0,\cdot)\|_{L^3(\mathbb{R}^2)} \int_{\mathbb{R}} \left| \int_{\mathbb{R}^2} \left| \frac{1}{1+p^2} \widehat{V^{1/2}\psi}(p) \right|^{6/5} \mathrm{d}\tilde{p} \right|^{5/3} \mathrm{d}p_1 \tag{5.68}$$

By Lemma 5.3.73, $\|\widehat{V^{1/2}\psi}\|_6 \leq C\|V\|_{3/2}^{1/2}\|\psi\|_2$. Applying Hölder's inequality in the $\tilde{p}$ variables, we obtain the bound

$$C\|\widehat{W}(0,\cdot)\|_{L^3(\mathbb{R}^2)} \int_{\mathbb{R}} \left| \int_{\mathbb{R}^2} \frac{1}{(1+p^2)^{3/2}} \mathrm{d}\tilde{p} \right|^{4/3} \left| \int_{\mathbb{R}^2} |\widehat{V^{1/2}\psi}(p)|^6 \mathrm{d}\tilde{p} \right|^{1/3} \mathrm{d}p_1 \tag{5.69}$$

Applying Hölder's inequality in p_1, we further obtain

$$C\|\widehat{W}(0,\cdot)\|_{L^3(\mathbb{R}^2)} \left(\int_{\mathbb{R}} \left| \int_{\mathbb{R}^2} \frac{1}{(1+p^2)^{3/2}} \mathrm{d}\tilde{p} \right|^2 \mathrm{d}p_1 \right)^{2/3} \|\widehat{V^{1/2}\psi}\|_6^2 \tag{5.70}$$

The remaining integral is finite. □

Proof of Lemma 5.4.3. Plugging in the trial state and regrouping terms we obtain

$$\begin{aligned}
\langle \psi_\epsilon, \tilde{H}^1_{T_c^0(\lambda)} \psi_\epsilon \rangle = 2 \int_{\mathbb{R}^{2d+2}} \Bigg[& \overline{\Phi_\lambda(r)} e^{-\epsilon|z_1|} (K_T(r,z_1;r',z_1') - \lambda V(r)\delta(r-r')\delta(z_1-z_1'))\Phi_\lambda(r')e^{-\epsilon|z_1'|} \\
\mp \overline{\Phi_\lambda(r)} e^{-\epsilon|z_1|} & (K_T(r,z_1;r',z_1') - \lambda V(r)\delta(r-r')\delta(z_1-z_1'))e^{-\epsilon|r_1'|}\Phi_\lambda(z_1',\tilde{r}') \Bigg] \mathrm{d}r\mathrm{d}z_1\mathrm{d}r'\mathrm{d}z_1' \\
+ 2\int_{\mathbb{R}^{d+1}} \Bigg[& \lambda V(r)\chi_{|z_1|<|r_1|}|\Phi_\lambda(r)|^2 e^{-2\epsilon|z_1|} \mp \overline{\Phi_\lambda(r)} e^{-\epsilon|z_1|}\lambda V(r)\chi_{|z_1|<|r_1|}e^{-\epsilon|r_1|}\Phi_\lambda(z_1,\tilde{r}) \\
& + \lambda V(z_1,\tilde{r})\chi_{|r_1|<|z_1|}|\Phi_\lambda(r)|^2 e^{-2\epsilon|z_1|} \mp \overline{\Phi_\lambda(r)} e^{-\epsilon|z_1|}\lambda V(z_1,\tilde{r})\chi_{|z_1|>|r_1|}e^{-\epsilon|r_1|}\Phi_\lambda(z_1,\tilde{r}) \\
& - \lambda V(z_1,\tilde{r})|\Phi_\lambda(r)|^2 e^{-2\epsilon|z_1|} \pm \overline{\Phi_\lambda(r)} e^{-\epsilon|z_1|}\lambda V(z_1,\tilde{r})e^{-\epsilon|r_1|}\Phi_\lambda(z_1,\tilde{r}) \Bigg] \mathrm{d}r\mathrm{d}z_1
\end{aligned} \tag{5.71}$$

We will prove that the first integral vanishes due to the eigenvalue equation $H^0_{T_c^0(\lambda)}\Phi_\lambda = 0$ as $\epsilon \to 0$. For the second integral in (5.71), we will show that it is bounded as $\epsilon \to 0$ and argue that it is possible to interchange limit and integration. The limit of the second integral is exactly the right hand side of (5.60).

The first two terms in the integrand of the second integral in (5.71) can be bounded by $\lambda\|\Phi_\lambda\|_\infty^2 |V(r)|\chi_{|z_1|<|r_1|}$. This is an L^1 function, since $|\cdot|V \in L^1$ and $\|\Phi_\lambda\|_\infty < \infty$ by Lemma 5.3.5. The same argument applies to the next two terms as well.

For the fifth term in the second integral, we can interchange limit and integration by dominated convergence if $\int_{\mathbb{R}^{d+1}} |V(r)||\Phi_\lambda(z_1,\tilde{r})|^2 \mathrm{d}r\mathrm{d}z_1 < \infty$. Observe that

$$\int_{\mathbb{R}^{d+1}} |V(r)||\Phi_\lambda(z_1,\tilde{r})|^2 \mathrm{d}r\mathrm{d}z_1 = (2\pi)^{1-d/2} \int_{\mathbb{R}^{2d-1}} \overline{\widehat{\Phi}_\lambda(p)} \widehat{|V|}(0,\tilde{p}-\tilde{q})\widehat{\Phi}_\lambda(p_1,\tilde{q})\mathrm{d}p\mathrm{d}\tilde{q} \tag{5.72}$$

According to Lemma 5.3.5(1) the latter is bounded by

$$C\int_{\mathbb{R}^{2d-1}} |\widehat{V^{1/2}\Psi_{T_c^0(\lambda)}}(p)| \frac{1}{1+p^2} \widehat{|V|}(0,\tilde{p}-\tilde{q}) \frac{1}{1+p_1^2+\tilde{q}^2} |\widehat{V^{1/2}\Psi_{T_c^0(\lambda)}}(p_1,\tilde{q})|\mathrm{d}p\mathrm{d}\tilde{q} \tag{5.73}$$

For $d = 1, 2$ we bound this by

$$C\|V\|_1^2\|\Psi\|_2^2 \int_{\mathbb{R}^{2d-1}} \frac{1}{(1+p_1^2+\tilde{p}^2)(1+p_1^2+\tilde{q}^2)} \mathrm{d}p\mathrm{d}\tilde{q}, \tag{5.74}$$

which is finite. For $d = 3$, (5.74) is finite by Lemma 5.4.6 since $W = |V|$ is radial and in $L^1 \cap L^{3/2}$. Hence, limit and integration can be interchanged for the fifth term in the second integral in (5.71).

For the last term in (5.71) we have

$$\begin{aligned}\int_{\mathbb{R}^{d+1}} &\overline{\Phi_\lambda(r)} e^{-\epsilon|z_1|} V(z_1,\tilde{r}) e^{-\epsilon|r_1|} \Phi_\lambda(z_1,\tilde{r}) \mathrm{d}z_1 \mathrm{d}r \\ &= \frac{2}{\pi} \int_{\mathbb{R}^{d+1}} \overline{\widehat{\Phi}_\lambda(p)} \frac{\epsilon}{\epsilon^2+q_1^2} \frac{\epsilon}{\epsilon^2+p_1^2} \widehat{V\Phi_\lambda}(q_1,\tilde{p}) \mathrm{d}p\mathrm{d}q_1 \\ &= \frac{2}{\pi} \int_{\mathbb{R}^{d+1}} \overline{\widehat{\Phi}_\lambda(\epsilon p_1,\tilde{p})} \frac{1}{1+q_1^2} \frac{1}{1+p_1^2} \widehat{V\Phi_\lambda}(\epsilon q_1,\tilde{p}) \mathrm{d}p\mathrm{d}q_1. \end{aligned} \tag{5.75}$$

According to Lemma 5.3.5(1) and Lemma 5.3.73, the integrand is bounded by $\frac{C(\lambda)}{1+\tilde{p}^2} \frac{\|V\|_1\|\Psi\|_2^2}{(1+q_1^2)(1+p_1^2)}$. For $d = 1, 2$ this is integrable, so by dominated convergence and since $\int_{\mathbb{R}} \frac{1}{1+x^2} \mathrm{d}x = \pi$, this term converges to the last term in (5.60). For $d = 3$, the following result will be useful.

Lemma 5.4.7. *Let $\lambda, T, \mu > 0$ and $d = 3$ and let V satisfy 5.1.1. The functions*

$$f(p_1,q_1) = \int_{\mathbb{R}^2} \overline{\widehat{\Phi}_\lambda(p)} \widehat{V\Phi_\lambda}(q_1,\tilde{p}) \mathrm{d}\tilde{p} \tag{5.76}$$

and

$$g(p_1,q_1) = \int_{\mathbb{R}^2} B_T^{-1}((p_1,\tilde{p}),(q_1,\tilde{0})) \overline{\widehat{\Phi}_\lambda(p_1,\tilde{p})} \widehat{\Phi}_\lambda(q_1,\tilde{p}) \mathrm{d}\tilde{p} \tag{5.77}$$

are bounded and continuous.

Its proof can be found after the end of the current proof.

We write the term in (5.75) as

$$\frac{2}{\pi} \int_{\mathbb{R}^2} \frac{f(\epsilon p_1, \epsilon q_1)}{(1+q_1^2)(1+p_1^2)} \mathrm{d}p_1 \mathrm{d}q_1. \tag{5.78}$$

By Lemma 5.4.7 we can exchange limit and integration by dominated convergence and (5.78) converges to the last term in (5.60).

For the second summand in the first integral in (5.71) we also want to argue using dominated convergence. The interaction term agrees with (5.75). The kinetic term can be written as

$$\begin{aligned}\frac{4}{\pi} \int_{\mathbb{R}^{d+1}} &\frac{1}{(1+q_1^2)(1+p_1^2)} B_T^{-1}((\epsilon p_1,\tilde{p}),(\epsilon q_1,\tilde{0})) \overline{\widehat{\Phi}_\lambda(\epsilon p_1,\tilde{p})} \widehat{\Phi}_\lambda(\epsilon q_1,\tilde{p}) \mathrm{d}p\mathrm{d}q_1 \\ &= \frac{4}{\pi} \int_{\mathbb{R}^2} \frac{1}{(1+q_1^2)(1+p_1^2)} g(\epsilon p_1, \epsilon q_1) \mathrm{d}p_1 \mathrm{d}q_1 \end{aligned} \tag{5.79}$$

For $d = 3$, we can apply dominated convergence according to Lemma 5.4.7. For $d = 1, 2$ note that by Lemma 5.3.5 and Lemma 5.2.1,

$$\begin{aligned}B_T^{-1}(p,(q_1,\tilde{0}))|\widehat{\Phi}_\lambda(p)||\widehat{\Phi}_\lambda(q_1,\tilde{p})| &\leq C_{T,\mu,\lambda} \frac{1+p^2+q_1^2}{(1+p^2)(1+q_1^2+\tilde{p}^2)} \|V\|_1\|\Psi\|_2^2 \\ &\leq 2C_{T,\mu,\lambda} \frac{\|V\|_1\|\Psi\|_2^2}{1+\tilde{p}^2} \end{aligned} \tag{5.80}$$

Therefore, the integrand is bounded by $\frac{C\|V\|_1\|\Psi\|_2^2}{(1+q_1^2)(1+p_1^2)(1+\tilde{p}^2)}$. For $d=1,2$ this is integrable and we can apply dominated convergence. We conclude that the limit of the second summand in the first integral in (5.71) as $\epsilon \to 0$ equals

$$4\pi \int_{\mathbb{R}^{d-1}} \left(\frac{|\widehat{\Phi}_\lambda(0,\tilde{p})|^2}{B_T((0,\tilde{p}),0)} - \lambda \overline{\widehat{\Phi}_\lambda(0,\tilde{p})} \widehat{V\Phi_\lambda}(0,\tilde{p}) \right) \mathrm{d}\tilde{p} = 0 \tag{5.81}$$

where we used that $\int_{\mathbb{R}} \frac{1}{1+x^2} \mathrm{d}x = \pi$ and (5.43).

To see that the first summand in the first integral in (5.71) vanishes as $\epsilon \to 0$, we use (5.43) to obtain

$$\frac{2}{\epsilon}\lambda \int_{\mathbb{R}^d} V(r)|\Phi_\lambda(r)|^2 \mathrm{d}r = \frac{2}{\epsilon}\int_{\mathbb{R}^d} B_T^{-1}(p,0)|\widehat{\Phi}_\lambda(p)|^2 \mathrm{d}p = \frac{4}{\pi}\int_{\mathbb{R}^{d+1}} \frac{\epsilon^2}{(\epsilon^2+q_1^2)^2} B_T^{-1}(p,0)|\widehat{\Phi}_\lambda(p)|^2 \mathrm{d}p\mathrm{d}q_1. \tag{5.82}$$

Hence, we need to prove that

$$\lim_{\epsilon\to 0}\int_{\mathbb{R}^{d+1}} \frac{\epsilon^2}{(\epsilon^2+q_1^2)^2}(B_T^{-1}(p,(q_1,\tilde{0})) - B_T^{-1}(p,0))|\widehat{\Phi}_\lambda(p)|^2 \mathrm{d}p\mathrm{d}q_1 = 0 \tag{5.83}$$

We split the integration into two regions with $|q_1| > C_1$ and $|q_1| < C_1$, respectively. By Lemma 5.2.1, we have $B_T^{-1}(p,q) \leq C_2(1+p^2+q^2)$. Together with $\Phi_\lambda \in H^1(\mathbb{R}^d)$ therefore

$$\begin{aligned}\int_{\mathbb{R}^{d+1},|q_1|>C_1} &\frac{\epsilon^2}{(\epsilon^2+q_1^2)^2}|B_T^{-1}(p,(q_1,\tilde{0})) - B_T^{-1}(p,0)||\widehat{\Phi}_\lambda(p)|^2 \mathrm{d}p\mathrm{d}q_1 \\ &\leq 2C_2 \int_{\mathbb{R}^2,|q_1|>C_1} \frac{\epsilon^2(1+p^2+q_1^2)|\widehat{\Phi}_\lambda(p)|^2}{q_1^4}\mathrm{d}p\mathrm{d}q_1 < C_3\epsilon^2\|\Phi_\lambda\|_{H^1}^2, \end{aligned} \tag{5.84}$$

which vanishes in the limit $\epsilon \to 0$. For the case $|q_1| < C$, the following Lemma is useful. Its proof can be found at the end of this Section.

Lemma 5.4.8. *Let $T,\mu > 0$, $d \in \{1,2,3\}$. The function*

$$k(p,q) := \frac{1}{|q|}(B_T(p,q) - B_T(p,0)) \tag{5.85}$$

is continuous at $q=0$ and satisfies $k(p,0)=0$ for all $p \in \mathbb{R}^d$. Furthermore, there is a constant C depending only on T,μ,d such that $|k(p,q)| < \frac{C}{1+p^2}$ for all $p,q \in \mathbb{R}^d$.

Since $B_T^{-1}(p,q) - B_T^{-1}(p,0) = -\frac{|q|k(p,q)}{B_T(p,q)B_T(p,0)}$, we have

$$\begin{aligned}\int_{\mathbb{R}^{d+1},|q_1|<C_1} &\frac{\epsilon^2}{(\epsilon^2+q_1^2)^2}(B_T^{-1}(p,(q_1,\tilde{0})) - B_T^{-1}(p,0))|\widehat{\Phi}_\lambda(p)|^2 \mathrm{d}p\mathrm{d}q_1 \\ &= -\int_{\mathbb{R}^{d+1}} \frac{|q_1|\chi_{|q_1|<C_1/\epsilon}}{(1+q_1^2)^2} \frac{k(p,(\epsilon q_1,\tilde{0}))}{B_T(p,(\epsilon q_1,\tilde{0}))B_T(p,0)}|\widehat{\Phi}_\lambda(p)|^2 \mathrm{d}p\mathrm{d}q_1 \end{aligned} \tag{5.86}$$

By Lemma 5.2.1 and Lemma 5.4.8, we can bound the absolute value of the integrand by

$$C\frac{|q_1|\chi_{|q_1|<C_1/\epsilon}}{(1+q_1^2)^2}(1+p^2+\epsilon^2q_1^2)|\widehat{\Phi}_\lambda(p)|^2 \leq C\frac{|q_1|}{(1+q_1^2)^2}(1+p^2+C_1^2)|\widehat{\Phi}_\lambda(p)|^2 \tag{5.87}$$

The latter is integrable since $\Phi_\lambda \in H^1(\mathbb{R}^d)$. Thus, by dominated convergence and since $k(p,0)=0$, the integral vanishes in the limit $\epsilon \to 0$. □

Proof of Lemma 5.4.7. For convenience, we introduce the notation $D_f(p,q_1) = \lambda B_T(p,0)$ and

$$D_g(p,q_1) = \lambda^2 B_T(p,0) B_T(p,(q_1,\tilde{0}))^{-1} B_T((q_1,\tilde{p}),0). \tag{5.88}$$

For $h \in \{f,g\}$, $D_f(p,q_1), D_g(p,q_1) \leq \frac{C}{1+\tilde{p}^2}$ by Lemma 5.2.1 and (5.11). Furthermore,

$$h(p_1,q_1) = \int_{\mathbb{R}^{d-1}} \overline{\widehat{V\Phi_\lambda}(p_1,\tilde{p})} D_h(p,q_1) \widehat{V\Phi_\lambda}(q_1,\tilde{p}) \mathrm{d}\tilde{p} \tag{5.89}$$

using (5.43).

Lemma 5.4.9. *For $h \in \{f,g\}$,*

$$\sup_{p_1,q_1,w_1\in\mathbb{R}} \| D_h((p_1,\cdot),q_1)\widehat{V\Phi_\lambda}(w_1,\cdot)\|_{L^1(\mathbb{R}^2)} \leq \sup_{w_1\in\mathbb{R}} \left\| \frac{C}{1+|\cdot|^2} \widehat{V\Phi_\lambda}(w_1,\cdot)\right\|_{L^1(\mathbb{R}^2)} < \infty. \tag{5.90}$$

Proof. Using Hölder's inequality,

$$\begin{aligned}
\| D_h((p_1,\cdot),q_1)\widehat{V\Phi_\lambda}(w_1,\cdot)\|_{L^1(\mathbb{R}^2)} &\leq \left\| \frac{C}{1+|\cdot|^2} \widehat{V\Phi_\lambda}(w_1,\cdot)\right\|_{L^1(\mathbb{R}^2)} \\
&\leq \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^2}\int_{\mathbb{R}^3} \frac{C}{1+\tilde{p}^2} |\widehat{V}((w_1,\tilde{p})-k)||\widehat{\Phi_\lambda}(k)| \mathrm{d}k \mathrm{d}\tilde{p} \\
\leq C \left\| \frac{1}{1+|\cdot|^2}\right\|_{L^r(\mathbb{R}^2)} \int_{\mathbb{R}} \left(\int_{\mathbb{R}^2} |\widehat{V}(w_1-k_1,\tilde{p})|^s \mathrm{d}\tilde{p}\right)^{1/s} &\left(\int_{\mathbb{R}^2} |\widehat{\Phi_\lambda}(k)|\mathrm{d}\tilde{k}\right) \mathrm{d}k_1 \\
&\leq C \left\| \frac{1}{1+|\cdot|^2}\right\|_{L^r(\mathbb{R}^2)} \sup_{k_1} \|\widehat{V}(k_1,\cdot)\|_s \|\widehat{\Phi}_\lambda\|_1,
\end{aligned} \tag{5.91}$$

where $1 = 1/r + 1/s$. For this to be finite we need $r > 1$, i.e. $s < \infty$. By Lemma 5.3.74, $\sup_{q_1} \|\widehat{V}(q_1,\cdot)\|_3 < \infty$. Furthermore $\|\widehat{\Phi}_\lambda\|_1$ is bounded by Lemma 5.3.5. □

The functions f and g are bounded, as can be seen using that $\|\widehat{V\Phi_\lambda}\|_\infty \leq C\|V\|_1^{1/2} \|\Psi_{T_c^0(\lambda)}\|_2$ by Lemma 5.3.73 and $\|\Psi_{T_c^0(\lambda)}\|_2 = 1$, hence we get that for $h \in \{f,g\}$

$$|h(p_1,q_1)| \leq C\|V\|_1^{1/2} \sup_{p_1,q_1} \|D_h((p_1,\cdot),q_1)\widehat{V\Phi_\lambda}(q_1,\cdot)\|_{L^1(\mathbb{R}^2)}, \tag{5.92}$$

which is finite by Lemma 5.4.9. To see continuity, we write for $h \in \{f,g\}$

$$\begin{aligned}
&|h(p_1+\epsilon_1,q_1+\epsilon_2) - h(p_1,q_1)| \leq \\
&\Bigg[\left| \int_{\mathbb{R}^2} \overline{(\widehat{V\Phi_\lambda}(p_1+\epsilon_1,\tilde{p}) - \widehat{V\phi_{T,\lambda}(p)})} D_h((p_1+\epsilon_1,\tilde{p}),q_1+\epsilon_2)\widehat{V\Phi_\lambda}(q_1+\epsilon_2,\tilde{p})\mathrm{d}\tilde{p}\right| \\
&\quad + \left| \int_{\mathbb{R}^2} \overline{\widehat{V\Phi_\lambda}(p)} D_h((p_1+\epsilon_1,\tilde{p}),q_1+\epsilon_2)(\widehat{V\Phi_\lambda}(q_1+\epsilon_2,\tilde{p}) - \widehat{V\Phi_\lambda}(q_1,\tilde{p}))\mathrm{d}\tilde{p}\mathrm{d}k\right| \\
&\qquad + \left| \int_{\mathbb{R}^2} \overline{\widehat{V\Phi_\lambda}(p)} (D_h((p_1+\epsilon_1,\tilde{p}),q_1+\epsilon_2) - D_h(p,q_1))\widehat{V\Phi_\lambda}(q_1,\tilde{p})\mathrm{d}\tilde{p}\right|\Bigg]
\end{aligned} \tag{5.93}$$

Observe that

$$|\widehat{V\Phi_\lambda}(p_1+\epsilon_1,\tilde{p}) - \widehat{V\Phi_\lambda}(p)| \leq \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} |e^{i\epsilon_1 r_1} - 1||V(r)||\Phi_\lambda(r)|\mathrm{d}r \leq \frac{\epsilon_1 \|\Phi_\lambda\|_\infty \||\cdot|V\|_1}{(2\pi)^{d/2}} \tag{5.94}$$

With Lemma 5.4.9 and Lemma 5.3.5, we bound the first two terms in (5.93) by $C\epsilon_1$ and $C\epsilon_2$, respectively. Hence they vanish as $\epsilon_1, \epsilon_2 \to 0$. The absolute value of the integrand in the last term in (5.93) is bounded by $\|\widehat{V\Phi_\lambda}\|_\infty \frac{2C}{1+\tilde{p}^2}\widehat{V\Phi_\lambda}(q_1, \tilde{p})$. By Lemma 5.4.9, this is an L^1 function. Hence, when taking the limit $\epsilon_1, \epsilon_2 \to 0$, we are allowed to pull the limit into the integral by dominated convergence, showing that also the last term vanishes. Therefore, the functions f and g are continuous. □

Proof of Lemma 5.4.8. This Lemma is a generalization of [34, Lemma 3.2] and its proof follows the same ideas. For $|q| > 1$, Lemma 5.2.1 implies the bound $|k(p,q)| < \frac{C}{1+p^2}$. For $|q| < 1$, we use the partial fraction expansion (see [34, (2.2)])

$$k(p,q) = 2T \sum_{n\in\mathbb{Z}} \frac{|q|(2\mu - q^2 - 2p^2 + 4(p\cdot \frac{q}{|q|})^2) - 4iw_n p \cdot \frac{q}{|q|}}{\left((p+q)^2 - \mu - iw_n\right)\left((p-q)^2 - \mu + iw_n\right)(p^2 - \mu - iw_n)(p^2 - \mu + iw_n)} \tag{5.95}$$

where $w_n = (2n+1)\pi T$. Continuity of k follows e.g. using the Weierstrass M-test. Noting that $w_n = -w_{-n-1}$, it is easy to see that $k(p,0) = 0$.

With the estimates

$$\begin{aligned}\sup_{(p,q)\in\mathbb{R}^{2d}, |q|<1} &\left| \frac{|q|(2\mu - q^2 - 2p^2 + 4(p\cdot\frac{q}{|q|})^2)}{\left((p+q)^2 - \mu - iw_n\right)\left((p-q)^2 - \mu + iw_n\right)} \right| \\ &\leq \sup_{(p,q)\in\mathbb{R}^{2d}, |q|<1} \frac{|q|(2\mu + q^2 + 6p^2)}{\sqrt{\left[(p+q)^2 - \mu\right]^2 + w_0^2}\sqrt{\left[(p-q)^2 - \mu\right]^2 + w_0^2}} =: c_1 < \infty \end{aligned} \tag{5.96}$$

and

$$\begin{aligned}\sup_{(p,q)\in\mathbb{R}^{2d}, |q|<1} &\left| \frac{4iw_n p}{\left((p+q)^2 - \mu - iw_n\right)\left((p-q)^2 - \mu + iw_n\right)} \right| \\ &\leq \sup_{(p,q)\in\mathbb{R}^{2d}, |q|<1} \frac{4|p|}{\sqrt{\left[(p+q)^2 - \mu\right]^2 + w_0^2}} =: c_2 < \infty \end{aligned} \tag{5.97}$$

one obtains

$$|k(p,q)| \leq 2T(c_1 + c_2) \sum_{n\in\mathbb{Z}} \frac{1}{(p^2 - \mu)^2 + w_n^2} \tag{5.98}$$

Using that the summands are decreasing in n, we can estimate the sum by an integral

$$\begin{aligned}|k(p,q)| &\leq 4T(c_1 + c_2)\left[\frac{1}{(p^2-\mu)^2 + w_0^2} + \int_{1/2}^{\infty} \frac{1}{(p^2-\mu)^2 + 4\pi^2T^2x^2} \mathrm{d}x\right] \\ &= 4T(c_1 + c_2)\left[\frac{1}{(p^2-\mu)^2 + w_0^2} + \frac{\arctan\left(\frac{|p^2-\mu|}{\pi T}\right)}{2\pi T|p^2-\mu|}\right] < C\frac{1}{1+p^2} \end{aligned} \tag{5.99}$$

for some constant C independent of p and q. □

5.4.3 Proof of Lemma 5.4.4

Proof of Lemma 5.4.4. Recall that $\Psi_{T_c^0(\lambda)} = V^{1/2}\Phi_\lambda$ with normalization $\|\Psi_{T_c^0(\lambda)}\|_2^2 = \|\Psi\|_2^2 = \int_{\mathbb{R}^d} V(r) j_d(r)^2 \mathrm{d}r$, where j_d was defined in (5.5). Recall from (5.60) that

$$
\begin{aligned}
&-\frac{1}{2\lambda}\lim_{\epsilon\to 0}\langle\psi_\epsilon, H^1_{T_c^0(\lambda),\lambda}\psi_\epsilon\rangle = \int_{\mathbb{R}^{d+1}} V(r)|\Phi_\lambda(z_1,\tilde{r})|^2 \mathrm{d}r\mathrm{d}z_1 \\
&-\int_{\mathbb{R}^{d+1}} V(r)|\Phi_\lambda(z_1,\tilde{r}) \mp \Phi_\lambda(r)|^2 \chi_{|z_1|<|r_1|}\mathrm{d}r\mathrm{d}z_1 \mp 2\pi \int_{\mathbb{R}^{d-1}} \overline{\widehat{\Phi}_\lambda(0,\tilde{p})}\widehat{V\Phi}_\lambda(0,\tilde{p})\mathrm{d}\tilde{p} \quad (5.100)
\end{aligned}
$$

The claim follows, if we prove that the right hand side is positive in the limit $\lambda \to 0$. For $d \in \{1,2\}$ we prove that the terms on the second line are bounded and the first term diverges as $\lambda \to 0$. For $d = 3$ the first term is bounded too, so we need to compute the limit of all terms. The idea is that in the limit, one would like to replace Φ_λ by j_3 using Lemmas 5.3.3 and 5.3.6. We consider each of the three summands in (5.100) separately.

Second term: The second term is bounded by $4\||\cdot|V\|_1\|\Phi_\lambda\|_\infty^2$, which is bounded for small λ by Lemma 5.3.5. For $d = 3$ we want to compute the limit. By Lemma 5.3.6 the integrand is bounded by $8|V(r)|\|j_3\|_\infty^2\chi_{|z_1|<|r_1|}$ for λ small enough, which is integrable. By dominated convergence, the term thus converges to

$$
-\int_{\mathbb{R}^4} V(r)|j_3(z_1,\tilde{r}) \mp j_3(r)|^2\chi_{|z_1|<|r_1|}\mathrm{d}r\mathrm{d}z_1. \quad (5.101)
$$

Third term: Using (5.43) the third term in (5.100) equals

$$
\mp 2\pi\lambda \int_{\mathbb{R}^{d-1}} |\widehat{V^{1/2}\Psi_{T_c^0(\lambda)}}(0,\tilde{p})|^2 B_{T_c^0(\lambda)}((0,\tilde{p}),0)\mathrm{d}\tilde{p} \quad (5.102)
$$

For $d = 1$, this is bounded by $2\pi\lambda B_{T_c^0(\lambda)}(0,0)\|\widehat{V^{1/2}\Psi_{T_c^0(\lambda)}}\|_\infty^2$. By Lemma 5.3.73 and since $\sup_T B_T(0,0) = \frac{1}{\mu}$, this is $O(\lambda)$ as $\lambda \to 0$. For $d = 2$ we use (5.11) to bound (5.102) by

$$
2\pi\lambda \int_{|\tilde{p}|^2<2\mu} B_{T_c^0(\lambda)}((0,\tilde{p}),0)\mathrm{d}\tilde{p}\|\widehat{V^{1/2}\Psi_{T_c^0(\lambda)}}\|_\infty^2 + C\lambda \int_{|\tilde{p}|^2>2\mu} \frac{1}{1+\tilde{p}^2}\mathrm{d}\tilde{p}\|\widehat{V^{1/2}\Psi_{T_c^0(\lambda)}}\|_\infty^2, \quad (5.103)
$$

where C is independent of λ. By Lemma 5.3.73 $\|\widehat{V^{1/2}\Psi_{T_c^0(\lambda)}}\|_\infty$ is bounded as $\lambda \to 0$. The second term in (5.103) thus vanishes as $\lambda \to 0$. For the first term, recall from (5.23) that $\int_{|\tilde{p}|^2<2\mu} B_{T_c^0,\mu}((0,\tilde{p}),0)\mathrm{d}\tilde{p} = 2\pi m_\mu^{d=2}(T_c^0(\lambda))$. By Lemma 5.3.2 the first term is bounded for small λ. For $d = 3$, we rewrite (5.102) as

$$
\begin{aligned}
&\mp 2\pi\lambda \int_{\tilde{p}^2>2\mu} |\widehat{V^{1/2}\Psi_{T_c^0(\lambda)}}(0,\tilde{p})|^2 B_{T_c^0}((0,\tilde{p}),0)\mathrm{d}\tilde{p} \\
&\mp \lambda \int_{\tilde{p}^2<2\mu}\int_{\mathbb{R}^6} \overline{V^{1/2}\Psi_{T_c^0(\lambda)}(x)} \frac{e^{i\tilde{p}\cdot(\tilde{x}-\tilde{y})} - e^{i\sqrt{\mu}\frac{\tilde{p}}{|\tilde{p}|}\cdot(\tilde{x}-\tilde{y})}}{(2\pi)^2} B_{T_c^0}((0,\tilde{p}),0)V^{1/2}\Psi_{T_c^0(\lambda)}(y)\mathrm{d}x\mathrm{d}y\mathrm{d}\tilde{p} \\
&\mp \lambda \int_{\tilde{p}^2<2\mu}\int_{\mathbb{R}^6} \overline{\left(V^{1/2}\Psi_{T_c^0(\lambda)}(x) - Vj_3(x)\right)} \frac{e^{i\sqrt{\mu}\frac{\tilde{p}}{|\tilde{p}|}\cdot(\tilde{x}-\tilde{y})}}{(2\pi)^2} B_{T_c^0}((0,\tilde{p}),0)V^{1/2}\Psi_{T_c^0(\lambda)}(y)\mathrm{d}x\mathrm{d}y\mathrm{d}\tilde{p} \\
&\mp \lambda \int_{\tilde{p}^2<2\mu}\int_{\mathbb{R}^6} V(x)j_3(x) \frac{e^{i\sqrt{\mu}\frac{\tilde{p}}{|\tilde{p}|}\cdot(\tilde{x}-\tilde{y})}}{(2\pi)^2} B_{T_c^0}((0,\tilde{p}),0)\left(V^{1/2}\Psi_{T_c^0(\lambda)}(y) - Vj_3(y)\right)\mathrm{d}x\mathrm{d}y\mathrm{d}\tilde{p} \\
&\mp \lambda \int_{\tilde{p}^2<2\mu}\int_{\mathbb{R}^6} V(x)j_3(x) \frac{e^{i\sqrt{\mu}\frac{\tilde{p}}{|\tilde{p}|}\cdot(\tilde{x}-\tilde{y})}}{(2\pi)^2} B_{T_c^0}((0,\tilde{p}),0)V(y)j_3(y)\mathrm{d}x\mathrm{d}y\mathrm{d}\tilde{p}. \quad (5.104)
\end{aligned}
$$

We prove that the first four integrals vanish as $\lambda \to 0$ and compute the limit of the expression in the last line.

Using (5.11), Lemma 5.3.73 and $\Psi_{T_c^0(\lambda)} = V^{1/2}\Phi_\lambda$ the first term in (5.104) is bounded by

$$C\lambda \|V\|_1^{1/2} \|\Psi_{T_c^0(\lambda)}\|_2 \left\| \frac{1}{1+|\cdot|^2} \widehat{V\Phi}_\lambda(0,\cdot) \right\|_{L^1(\mathbb{R}^2)} \tag{5.105}$$

where C is independent of λ. By (5.91),

$$\left\| \frac{1}{1+|\cdot|^2} \widehat{V\Phi}_\lambda(0,\cdot) \right\|_{L^1(\mathbb{R}^2)} \leq \left\| \frac{1}{1+|\cdot|^2} \right\|_{L^{3/2}(\mathbb{R}^2)} \sup_{k_1} \|\widehat{V}(k_1,\cdot)\|_3 \|\widehat{\Phi}_\lambda\|_1 \tag{5.106}$$

By Lemma 5.3.74, $\sup_{k_1} \|\widehat{V}(k_1,\cdot)\|_3 < \infty$. Furthermore $\|\widehat{\Phi}_\lambda\|_1$ is bounded uniformly in λ by Lemma 5.3.5. In total, the first term in (5.104) is $O(\lambda)$ as $\lambda \to 0$.

For the second line of (5.104) we use that

$$\sup_{\lambda>0} \sup_{\tilde{x},\tilde{y}\in\mathbb{R}^2} \left| \int_{\mathbb{R}^2, \tilde{p}^2<2\mu} \frac{e^{i\tilde{p}\cdot(\tilde{x}-\tilde{y})} - e^{i\sqrt{\mu}\frac{\tilde{p}}{|\tilde{p}|}\cdot(\tilde{x}-\tilde{y})}}{(2\pi)^3} B_{T_c^0(\lambda)}((0,\tilde{p}),0) \mathrm{d}\tilde{p} \right| < \infty, \tag{5.107}$$

as was shown in the proof of [40, Lemma 3.4]. Applying the Schwarz inequality, the second line is bounded by $C\lambda \|V\|_1 \|\Psi_{T_c^0(\lambda)}\|_2^2$ for some constant C and vanishes for $\lambda \to 0$.

We bound the third line of (5.104) by

$$\begin{aligned} &\frac{\lambda}{(2\pi)^2} \int_{\mathbb{R}^2, \tilde{p}^2<2\mu} \int_{\mathbb{R}^6} |\overline{(V^{1/2}\Psi_\lambda(x) - Vj_3(x))}| B_{T_c^0(\lambda)}((0,\tilde{p}),0) |V^{1/2}\Psi_{T_c^0(\lambda)}(y)| \mathrm{d}x\mathrm{d}y\mathrm{d}\tilde{p} \\ &\qquad \leq \lambda \frac{|\mathbb{S}^1|}{(2\pi)^2} m_\mu^{d=2}(T_c^0(\lambda)) \|V\|_1 \|\Psi_{T_c^0(\lambda)}\|_2 \|\Psi_{T_c^0(\lambda)} - \Psi\|_2, \end{aligned} \tag{5.108}$$

where in the second step we carried out the $\tilde{p}$ integration and used the Schwarz inequality in x and y. By Lemma 5.3.2, $\lambda m_\mu^{d=2}(T_c^0(\lambda))$ is bounded and by Lemma 5.3.3, $\|\Psi_{T_c^0(\lambda)} - \Psi\|_2$ decays like $\lambda^{1/2}$. Hence, this vanishes for $\lambda \to 0$. Similarly, the fourth integral in (5.104) is bounded by

$$\lambda \frac{|\mathbb{S}^1|}{(2\pi)^2} m_\mu^{d=2}(T_c^0(\lambda)) \|V\|_1 \|V^{1/2} j_3\|_2 \|\Psi_{T_c^0(\lambda)} - \Psi\|_2, \tag{5.109}$$

which vanishes for $\lambda \to 0$.

For the last line of (5.104) we first carry out the integration over x, y and the radial part of $\tilde{p}$, and then use that $\widehat{Vj_3}$ is a radial function. This way we obtain

$$\mp\lambda m_\mu^{d=2}(T_c^0(\lambda)) 2\pi \int_{\mathbb{S}^1} |\widehat{Vj_3}(0,\sqrt{\mu}w)|^2 \mathrm{d}w = \mp\lambda m_\mu^{d=2}(T_c^0(\lambda)) \pi \int_{\mathbb{S}^2} |\widehat{Vj_3}(\sqrt{\mu}w)|^2 \mathrm{d}w \tag{5.110}$$

The latter integral equals $\langle |V|^{1/2} j_3, O_\mu V^{1/2} j_3 \rangle = e_\mu \int_{\mathbb{R}^3} V(x) j_3(x)^2 \mathrm{d}x$. By Lemma 5.3.2,

$$\lim_{\lambda\to 0} \lambda m_\mu^{d=2}(T_c^0(\lambda)) e_\mu = \lim_{\lambda\to 0} \lambda \ln(\mu/T_c^0(\lambda)) e_\mu = \frac{1}{\mu^{1/2}}. \tag{5.111}$$

Therefore, the limit of the last line of (5.104) for $\lambda \to 0$ equals

$$\mp \frac{\pi}{\mu^{1/2}} \int_{\mathbb{R}^3} V(x) j_3(x)^2 \mathrm{d}x. \tag{5.112}$$

First term: It remains to consider the first term in (5.100). If $V \geq 0$, one could argue directly using the convergence of Φ_λ in Lemma 5.3.6 for $d = 3$. However, the analogue of Lemma 5.3.6 does not hold for $d = 1$. Instead, the strategy is to use the L^2-convergence of the ground state in the Birman-Schwinger picture, Lemma 5.3.3. This approach also allows us to treat V that take negative values.

Switching to momentum space and using the eigenvalue equation (5.43), we rewrite the first term in (5.100) as

$$(2\pi)^{1-\frac{d}{2}} \int_{\mathbb{R}^{2d-1}} \overline{\widehat{\Phi}_\lambda(p)} \widehat{V}(0, \tilde{p} - \tilde{q}) \widehat{\Phi}_\lambda(p_1, \tilde{q}) \mathrm{d}p \mathrm{d}\tilde{q} = (2\pi)^{1-\frac{d}{2}} \lambda^2 \langle \Psi_{T_c^0(\lambda)}, D_{T_c^0(\lambda)} \Psi_{T_c^0(\lambda)} \rangle, \quad (5.113)$$

where D_T is the operator given by

$$\langle \psi, D_T \psi \rangle = \int_{\mathbb{R}^{2d-1}} \overline{\widehat{|V|^{1/2}\psi}(p)} B_T(p,0) \widehat{V}(0, \tilde{p} - \tilde{q}) B_T((p_1, \tilde{q}), 0) \widehat{|V|^{1/2}\psi}(p_1, \tilde{q}) \mathrm{d}p \mathrm{d}\tilde{q} \quad (5.114)$$

for $\psi \in L^2(\mathbb{R}^d)$. We decompose (5.113) as

$$\begin{aligned}(2\pi)^{1-\frac{d}{2}} \lambda^2 \langle \Psi_{T_c^0(\lambda)}, D_{T_c^0(\lambda)} \Psi_{T_c^0(\lambda)} \rangle = (2\pi)^{1-\frac{d}{2}} \lambda^2 \Big(&\langle \Psi_{T_c^0(\lambda)} - \Psi, D_{T_c^0(\lambda)} \Psi_{T_c^0(\lambda)} \rangle \\ &+ \langle \Psi, D_{T_c^0(\lambda)} (\Psi_{T_c^0(\lambda)} - \Psi) \rangle + \langle \Psi, D_{T_c^0(\lambda)} \Psi \rangle \Big). \quad (5.115)\end{aligned}$$

Recall that by Lemma 5.3.3, $\|\Psi_{T_c^0} - \Psi\|_2 = O(\lambda^{1/2})$. The strategy is to prove that $\|D_T\|$ and $\langle \Psi, D_T \Psi \rangle$ are of the same order for $T \to 0$. Then, the positive term $\langle \Psi, D_{T_c^0(\lambda)} \Psi \rangle$ will be the leading order term in (5.115) as $\lambda \to 0$. The asymptotic behavior of $\|D_T\|$ and $\langle \Psi, D_T \Psi \rangle$ is the content of the following two Lemmas. These asymptotics strongly depend on the dimension and this is where the different treatment of $d = 3$ versus $d \in \{1, 2\}$ in Theorem 5.1.3 originates.

It will be convenient to introduce the operator $D_T^<$ as

$$\langle \psi, D_T^< \psi \rangle = \int_{|p|^2 < 2\mu, |(p_1, \tilde{q})|^2 < 2\mu, p_1^2 < \mu} \overline{\widehat{|V|^{1/2}\psi}(p)} B_T(p,0) \widehat{V}(0, \tilde{p} - \tilde{q}) B_T((p_1, \tilde{q}), 0) \widehat{|V|^{1/2}\psi}(p_1, \tilde{q}) \mathrm{d}p \mathrm{d}\tilde{q} \quad (5.116)$$

for $\psi \in L^2(\mathbb{R}^d)$. Furthermore, for $d = 2$ we define for $0 < \delta < \mu$ the operator D_T^δ as

$$\langle \psi, D_T^\delta \psi \rangle = \int_{\mu - \delta < p_1^2 < \mu, p_2^2 < 2\delta, q_2^2 < 2\delta} \overline{\widehat{|V|^{1/2}\psi}(p)} B_T(p,0) \widehat{V}(0, p_2 - q_2) B_T((p_1, \tilde{q}), 0) \widehat{|V|^{1/2}\psi}(p_1, q_2) \mathrm{d}p \mathrm{d}q_2 \quad (5.117)$$

for $\psi \in L^2(\mathbb{R}^2)$.

Lemma 5.4.10. *Let $\mu > \delta > 0$ and let V satisfy 5.1.1. There are constants $C, T_0 > 0$ such that for all $0 < T < T_0$ for $d = 1$ $\|D_T\| \leq C/T$, for $d = 2$ $\|D_T\| \leq C(\ln \mu/T)^3$ and $\|D_T - D_T^\delta\| \leq C(\ln \mu/T)^2$, and for $d = 3$ $\|D_T\| \leq C(\ln \mu/T)^2$ and $\|D_T - D_T^<\| \leq C \ln \mu/T$.*

Lemma 5.4.11. *Let $\mu > 0$ and let V satisfy 5.1.1. Recall that $\Psi = V^{1/2} j_d$. There are constants $C, T_0 > 0$ such that for all $0 < T < T_0$, $\langle \Psi, D_T \Psi \rangle \geq C/T$ for $d = 1$ and $\geq C(\ln \mu/T)^3$ for $d = 2$. For $d = 3$, $\lim_{\lambda \to 0} (2\pi)^{-1/2} \lambda^2 \langle \Psi, D_{T_c^0(\lambda)} \Psi \rangle = \int_{\mathbb{R}^4} V(r) j_3(z_1, \tilde{r}; \mu)^2 \mathrm{d}r \mathrm{d}z_1$.*

For $\lambda \to 0$, by Lemma 5.3.2, $\ln(\mu/T_c^0(\lambda))$ is of order $1/\lambda$, hence the last term in (5.115) diverges for $d = 1, 2$. For $d = 3$ we get the desired constant by Lemma 5.4.11. □

Proof of Lemma 5.4.10. Assume that $T/\mu < 1/2$. We treat the different dimensions d separately.

Dimension one: Note that

$$|\langle \psi, D_T\psi\rangle| = |\widehat{V}(0)| \int_{\mathbb{R}} B_T(p,0)^2 |\widehat{|V|^{1/2}\psi}(p)|^2 \mathrm{d}p \leq \|V\|_1^2 \int_{\mathbb{R}} B_T(p,0)^2 \mathrm{d}p \|\psi\|_2^2, \quad (5.118)$$

where we used Lemma 5.3.7. Recall from (5.11) that $B_T(p,0) \leq \min\left\{\frac{1}{|p^2-\mu|}, \frac{1}{2T}\right\}$. We estimate the integral

$$\int_{\mathbb{R}} B_T(p,0)^2 \mathrm{d}p \leq \int_{\sqrt{\mu}-\frac{T}{\sqrt{\mu}}<|p|<\sqrt{\mu}+\frac{T}{\sqrt{\mu}}} \frac{1}{4T^2}\mathrm{d}p + \int_{\mathbb{R}} \frac{\chi_{|p|<\sqrt{\mu}-\frac{T}{\sqrt{\mu}}} + \chi_{\sqrt{\mu}+\frac{T}{\sqrt{\mu}}<p<2\sqrt{\mu}}}{\mu(|p|-\sqrt{\mu})^2}\mathrm{d}p + \int_{p>2\sqrt{\mu}} \frac{1}{(p^2-\mu)^2}\mathrm{d}p \quad (5.119)$$

The first term equals $(\sqrt{\mu}T)^{-1}$. The last term is a finite constant independent of T. In the second term we substitute $||p|-\sqrt{\mu}|$ by x and get the bound

$$2\int_{\frac{T}{\sqrt{\mu}}}^{\sqrt{\mu}} \frac{1}{\mu x^2}\mathrm{d}x = \frac{2}{\sqrt{\mu}}(1/T - 1/\mu) \quad (5.120)$$

Dimension two: Using the Schwarz inequality we have

$$\langle \psi, D_T\psi\rangle \leq C\|V\|_1^2 \int_{\mathbb{R}^3} B_{T,\mu}(p,0) B_{T,\mu}((p_1,\tilde{q}),0)\mathrm{d}p\mathrm{d}\tilde{q} \|\psi\|_2^2 \quad (5.121)$$

The integral can be rewritten as

$$\int_{\mathbb{R}} \left(\int_{\mathbb{R}} B_{T,\mu-p_1^2}(p_2,0)\mathrm{d}p_2\right)^2 \mathrm{d}p_1, \quad (5.122)$$

where $B_{T,\mu}$ here is understood as the function on $\mathbb{R}\times\mathbb{R}$ instead of $\mathbb{R}^2\times\mathbb{R}^2$. Similarly,

$$|\langle \psi, (D_T - D_T^\delta)\psi\rangle| \leq C\|V\|_1^2 \int_{\mathbb{R}^3} (1-\chi_{\mu-\delta<p_1^2<\mu}\chi_{p_2^2<2\delta}\chi_{p_2'^2<2\delta}) B_{T,\mu}(p,0) B_{T,\mu}((p_1,\tilde{q}),0)\mathrm{d}p\mathrm{d}\tilde{q} \|\psi\|_2^2 \quad (5.123)$$

We prove that (5.122) and (5.123) are of order $O(\ln(\mu/T)^3)$ and $O(\ln(\mu/T)^2)$ for $T \to 0$, respectively. To bound the integrals we consider three regimes, $p_1^2 < \mu - T$, $\mu - T < p_1^2 < \mu + T$, and $\mu + T < p_1^2$. Corresponding to these regimes, we need to understand $\int_{\mathbb{R}} B_{T,\mu}(p,0)\mathrm{d}p$ for $T/\mu < 1$, $-1 < \mu/T < 1$, and $\mu/T < -1$.

In the first regime, there is a constant C_1, such that for all $T/\mu < 1$

$$\left|\sqrt{\mu}\int_{\mathbb{R}} B_{T,\mu}(p,0)\chi_{p^2<2\mu}\mathrm{d}p - 2\ln\frac{\mu}{T}\right| + \left|\sqrt{\mu}\int_{\mathbb{R}} B_{T,\mu}(p,0)\chi_{p^2>2\mu}\mathrm{d}p\right| \leq C_1 \quad (5.124)$$

This follows from rescaling $\sqrt{\mu}\int_{\mathbb{R}} B_{T,\mu}(p,0)\mathrm{d}p = \int_{\mathbb{R}} B_{T/\mu,1}(p,0)\mathrm{d}p$ and applying [34, Lemma 3.5]. For the second regime, we rewrite

$$\int_{\mathbb{R}} B_{T,\mu}(p,0)\mathrm{d}p = \frac{1}{\sqrt{T}}\int_{\mathbb{R}} \frac{\tanh((p^2-\mu/T)/2)}{p^2-\mu/T}\mathrm{d}p \quad (5.125)$$

Since $\tanh(x)/x \le \min\{1, 1/|x|\}$ the latter integral is uniformly bounded for $|\mu/T| < 1$,

$$\int_{\mathbb{R}} B_{T,\mu}(p,0)\mathrm{d}p \le \frac{C_2}{\sqrt{T}}. \tag{5.126}$$

For the third regime, it follows from (5.125) that

$$\int_{\mathbb{R}} B_{T,\mu}(p,0)\mathrm{d}p \le \frac{1}{\sqrt{T}}\int_{\mathbb{R}} \frac{1}{p^2-\mu/T}\mathrm{d}p = \frac{1}{\sqrt{-\mu}}\int_{\mathbb{R}} \frac{1}{p^2+1}\mathrm{d}p =: \frac{C_3}{\sqrt{-\mu}}. \tag{5.127}$$

Combining the bounds in the three regimes, we bound (5.122) from above by

$$\int_{|p_1|<\sqrt{\mu-T}} \frac{\left(2\ln\left(\frac{\mu-p_1^2}{T}\right)+C_1\right)^2}{\mu-p_1^2}\mathrm{d}p_1 + \int_{\sqrt{\mu-T}<|p_1|<\sqrt{\mu+T}} \frac{C_2^2}{T}\mathrm{d}p_1 + \int_{\sqrt{\mu+T}<|p_1|} \frac{C_3^2}{p_1^2-\mu}\mathrm{d}p_1 \tag{5.128}$$

The first integral is bounded above by

$$\left(2\ln\left(\frac{\mu}{T}\right)+C_1\right)^2 \int_{|p_1|<\sqrt{\mu-T}} \frac{1}{\mu-p_1^2}\mathrm{d}p_1. \tag{5.129}$$

Since

$$\int_{|p_1|<\sqrt{\mu-T}} \frac{1}{\mu-p_1^2}\mathrm{d}p_1 = \frac{1}{\sqrt{\mu}}\ln\left(\frac{2\mu-T+\sqrt{\mu(\mu-T)}}{T}\right) = O(\ln(\mu/T)), \tag{5.130}$$

the first integral in (5.128) is of order $O(\ln(\mu/T)^3)$. In the second integral, the size of the integration domain is $2T/\sqrt{\mu}+O(T^2)$, so the integral is bounded as $T\to 0$. The third integral equals

$$\frac{C_3^2}{\sqrt{\mu}}\ln\left(\frac{2\mu+T+\sqrt{\mu(\mu+T)}}{T}\right) = O(\ln\mu/T). \tag{5.131}$$

In total (5.122) is of order $O(\ln(\mu/T)^3)$.

For the integral in (5.123) we obtain the upper bound similar to (5.128). The main difference is that in the regime $\sqrt{\mu-\delta} < |p_1| < \sqrt{\mu-T}$, at least one of the variables p_2, p_2' is constrained to absolute values larger than $\sqrt{2\delta} \ge \sqrt{2(\mu-p_1^2)}$, and thus for the integration over this variable there will be no $\ln\left(\frac{\mu-p_1^2}{T}\right)$ contribution from (5.124). The upper bound for (5.123) is

$$\int_{|p_1|<\sqrt{\mu-\delta}} \frac{\left(2\ln\left(\frac{\mu-p_1^2}{T}\right)+C_1\right)^2}{\mu-p_1^2}\mathrm{d}p_1 + \int_{\sqrt{\mu-\delta}<|p_1|<\sqrt{\mu-T}} \frac{2\left(2\ln\left(\frac{\mu-p_1^2}{T}\right)+C_1\right)C_1}{\mu-p_1^2}\mathrm{d}p_1$$
$$+ \int_{\sqrt{\mu-T}<|p_1|<\sqrt{\mu+T}} \frac{C_2^2}{T}\mathrm{d}p_1 + \int_{\sqrt{\mu+T}<|p_1|} \frac{C_3^2}{p_1^2-\mu}\mathrm{d}p_1 \tag{5.132}$$

We have already seen above that the last two integrals are of order $O(1)$ and $O(\ln\mu/T)$, respectively. The first integral in (5.132) is bounded above by $\left(2\ln\left(\frac{\mu}{T}\right)+C_1\right)^2 \int_{|p_1|<\sqrt{\mu-\delta}} \frac{1}{\mu-p_1^2}\mathrm{d}p_1 = O(\ln(\mu/T)^2)$. Similarly, the second integral in (5.132) is of order $O(\ln(\mu/T)^2)$ by (5.130).

Dimension three: For $d=3$, we first prove that $\|D_T^{<}\| = O(\ln(\mu/T)^2)$. We bound (5.116) using the Schwarz inequality

$$\langle\psi, D_T^{<}\psi\rangle \le \|V\|_1^2\|\psi\|_2^2 \int_{\mathbb{R}^5} \chi_{|p|^2<2\mu,|(p_1,\tilde{q})|<2\mu,p_1^2<\mu} B_{T,\mu}(p,0)B_{T,\mu}((p_1,\tilde{q}),0)\mathrm{d}p\mathrm{d}\tilde{q}. \tag{5.133}$$

The integral can be rewritten as

$$4\pi^2 \int_0^{\sqrt{\mu}} \left(\int_0^{\sqrt{2\mu - p_1^2}} B_{T,\mu-p_1^2}(t,0) t \mathrm{d}t \right)^2 \mathrm{d}p_1 \tag{5.134}$$

Substituting $s = (t^2 + p_1^2 - \mu)/T$ gives

$$\pi^2 \int_0^{\sqrt{\mu}} \left(\int_{-(\mu-p_1^2)/T}^{\mu/T} \frac{\tanh(s)}{s} \mathrm{d}s \right)^2 \mathrm{d}p_1 \leq \sqrt{\mu}\pi^2 \left(\int_{-\mu/T}^{\mu/T} \frac{\tanh(s)}{s} \mathrm{d}s \right)^2 \tag{5.135}$$

Since $\tanh(x)/x \leq \min\{1, 1/|x|\}$, this is bounded by

$$\sqrt{\mu} 4\pi^2 \left(1 + \ln(\mu/T)\right)^2. \tag{5.136}$$

To bound $\|D_T - D_T^<\|$, we distinguish the cases were p^2 and $(p_1, \tilde{q})^2$ are larger or smaller than 2μ. Using (5.11) we bound

$$\begin{aligned}|\langle \psi, (D_T - D_T^<)\psi \rangle| \leq \|V\|_1^2 \|\psi\|_2^2 \int_{\mathbb{R}^5} \chi_{|p|^2<2\mu, |(p_1,\tilde{q})|<2\mu, p_1^2>\mu} B_{T,\mu}(p,0) B_{T,\mu}((p_1,\tilde{q}),0) \mathrm{d}p \mathrm{d}\tilde{q} \\ + 2\|V\|_1 \|\psi\|_2^2 \int_{\mathbb{R}^5} \frac{C}{\tilde{p}^2+1} |\widehat{V}(0, \tilde{p}-\tilde{q})| B_{T,\mu}((p_1,\tilde{q}),0) \chi_{|(p_1,\tilde{q})|^2<2\mu} \mathrm{d}p \mathrm{d}\tilde{q} \\ + \int_{\mathbb{R}^5} \overline{\widehat{|V|^{1/2}\psi}(p)} \frac{C}{p^2+1} |\widehat{V}(0,\tilde{p}-\tilde{q})| \frac{C}{p_1^2+\tilde{q}^2+1} \widehat{|V|^{1/2}\psi}(p_1,\tilde{q}) \mathrm{d}p \mathrm{d}\tilde{q},\end{aligned} \tag{5.137}$$

where C is a constant independent of T. For the first term, proceeding similarly to (5.134)–(5.136), the integral equals

$$\pi^2 \int_{\sqrt{\mu}}^{\sqrt{2\mu}} \left(\int_{(p_1^2-\mu)/T}^{\mu/T} \frac{\tanh(s)}{s} \mathrm{d}s \right)^2 \mathrm{d}p_1 \leq \pi^2 \int_{\sqrt{\mu}}^{\sqrt{2\mu}} \ln\left(\frac{\mu}{p_1^2-\mu}\right)^2 \mathrm{d}p_1 < \infty \tag{5.138}$$

For the second term in (5.137) we apply Young's inequality to bound the integral by

$$C \left\| \frac{1}{1+|\cdot|^2} \right\|_{L^{3/2}(\mathbb{R}^2)} \|\widehat{V}(0,\cdot)\|_{L^3(\mathbb{R}^2)} |\mathbb{S}^2| m_\mu(T) \tag{5.139}$$

which is $O(\ln \mu/T)$. The third term in (5.137) is bounded by $C\|\psi\|_2^2$ by Lemma 5.4.6. □

Proof of Lemma 5.4.11. By assumption, $0 < e_\mu = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{S}^{d-1}} \widehat{V}(p - \sqrt{\mu}\omega) \mathrm{d}\Omega(\omega) = \widehat{Vj_d}(|p| = \sqrt{\mu})$. By continuity of $\widehat{Vj_d}(p)$ in p, there is an $\epsilon > 0$ such that $\widehat{Vj_d}(p) > \frac{1}{2}\widehat{Vj_d}(|p| = \sqrt{\mu}) > 0$ for all $\sqrt{\mu} - \epsilon < |p| < \sqrt{\mu} + \epsilon$. In the following we treat the different dimensions separately.

Dimension one: Suppose $T < \epsilon$. Since $\widehat{V}(0) > 0$,

$$\langle V^{1/2} j_1, D_T V^{1/2} j_1 \rangle = \widehat{V}(0) \int_{\mathbb{R}} B_T(p,0)^2 |\widehat{Vj_1}(p)|^2 \mathrm{d}p \geq \frac{1}{4} \widehat{V}(0) |\widehat{Vj_1}(\sqrt{\mu})|^2 \int_{\sqrt{\mu}+T}^{\sqrt{\mu}+\epsilon} B_T(p,0)^2 \mathrm{d}p \tag{5.140}$$

For $p \in [\sqrt{\mu} + T, \sqrt{\mu} + \epsilon]$, $B_T(p,0) \geq \frac{\tanh(\sqrt{\mu})}{p^2-\mu} \geq \frac{\tanh(\sqrt{\mu})}{(2\sqrt{\mu}+\epsilon)(p-\sqrt{\mu})}$. Since $\int_{\sqrt{\mu}+T}^{\sqrt{\mu}+\epsilon} \frac{1}{(p-\sqrt{\mu})^2} \mathrm{d}p = 1/T - 1/\epsilon$, we obtain the lower bound

$$\langle V^{1/2} j_1, D_T V^{1/2} j_1 \rangle \geq \frac{1}{4} \widehat{V}(0) |\widehat{Vj_1}(\sqrt{\mu})|^2 \frac{\tanh(\sqrt{\mu})^2}{(2\sqrt{\mu}+\epsilon)^2} \left(\frac{1}{T} - \frac{1}{\epsilon} \right) \tag{5.141}$$

and the claim follows.

Dimension two: Since $\widehat{V}(0) > 0$, by continuity also $\widehat{V}(p) > 0$ for small $|p|$. Therefore, there are constants $0 < \delta < \mu$ and $C > 0$ such that for all $\sqrt{\mu - \delta} < p_1 \leq \sqrt{\mu}$ and $|p_2|, |q_2| < (2\delta)^{1/2}$

$$\overline{\widehat{Vj_2}(p_1, p_2)}\widehat{V}(0, p_2 - q_2)\widehat{Vj_2}(p_1, q_2) > C. \tag{5.142}$$

By Lemma 5.4.10, we have $\langle V^{1/2}j_2, D_T V^{1/2}j_2\rangle = \langle V^{1/2}j_2, D_T^\delta V^{1/2}j_2\rangle + O((\ln \mu/T)^2)$. It hence suffices to show that $\langle V^{1/2}j_2, D_T^\delta V^{1/2}j_2\rangle$ grows like $(\ln \mu/T)^3$. Let $A := \{(p_1, p_2, q_2) \in \mathbb{R}^3 | \sqrt{\mu - \delta} < p_1 < \sqrt{\mu}, 0 < p_2, q_2 < \delta^{1/2}, p_1^2 + p_2^2 > \mu + T, p_1^2 + q_2^2 > \mu + T\}$. This is a subset of the support in D_T^δ. Using that all terms in the integrand of $\langle V^{1/2}j_2, D_T^\delta V^{1/2}j_2\rangle$ are positive, we estimate

$$\langle V^{1/2}j_2, D_T^\delta V^{1/2}j_2\rangle \geq C\int_A B_T(p, 0)B_T((p_1, q_2), 0)\mathrm{d}p\mathrm{d}q_2. \tag{5.143}$$

For $(p_1, p_2, q_2) \in A$ we have $p_1^2 + p_2^2 - \mu > T$ and thus

$$B_T(p, 0) \geq \frac{\tanh\left(\frac{1}{2}\right)}{p_1^2 + p_2^2 - \mu} \tag{5.144}$$

For $p_1^2 > \mu + T - \delta$

$$\int_{\sqrt{\mu+T-p_1^2}}^{\delta^{1/2}} \frac{1}{p_1^2 + p_2^2 - \mu}\mathrm{d}p_2 = \frac{1}{\sqrt{\mu - p_1^2}}\left[\operatorname{artanh}\left(\sqrt{1 - \frac{T}{\mu + T - p_1^2}}\right) - \operatorname{artanh}\left(\sqrt{\frac{\mu - p_1^2}{\delta}}\right)\right]. \tag{5.145}$$

Hence, the integral in (5.143) is bounded below by

$$\tanh\left(\frac{1}{2}\right)^2 \int_{\sqrt{\mu+T-\delta}}^{\sqrt{\mu}} \frac{1}{\mu - p_1^2}\left[\operatorname{artanh}\left(\sqrt{1 - \frac{T}{\mu + T - p_1^2}}\right) - \operatorname{artanh}\left(\sqrt{\frac{\mu - p_1^2}{\delta}}\right)\right]^2 \mathrm{d}p_1 \tag{5.146}$$

Assume that $T < \delta/2$. For a lower bound, we further restrict the p_1-integration to the interval $\left(\sqrt{\mu - \delta/2}, \sqrt{\mu - \mu^{1/2}T^{1/2}}\right)$. For these values of p_1, we have

$$\operatorname{artanh}\left(\sqrt{\frac{\mu - p_1^2}{\delta}}\right) \leq \operatorname{artanh}\left(\frac{1}{\sqrt{2}}\right) \leq \operatorname{artanh}\left(\sqrt{1 - \frac{T^{1/2}}{\mu^{1/2}}}\right) \leq \operatorname{artanh}\left(\sqrt{1 - \frac{T}{\mu + T - p_1^2}}\right). \tag{5.147}$$

Furthermore,

$$\int_{\sqrt{\mu-\delta/2}}^{\sqrt{\mu-\mu^{1/2}T^{1/2}}} \frac{1}{\mu - p_1^2}\mathrm{d}p_1 = \frac{1}{\sqrt{\mu}}\operatorname{artanh}\left(1 - \frac{(\sqrt{\mu}/a + 1)(1 - b/\sqrt{\mu})}{\sqrt{\mu}/a - b/\sqrt{\mu}}\right), \tag{5.148}$$

where $a = \sqrt{\mu - \delta/2}$ and $b = \sqrt{\mu - \mu^{1/2}T^{1/2}} \leq \sqrt{\mu}$. This is bounded below by

$$\frac{1}{\sqrt{\mu}}\operatorname{artanh}\left(1 - \frac{(\sqrt{\mu}/a + 1)(1 - b/\sqrt{\mu})}{\sqrt{\mu}/a - 1}\right). \tag{5.149}$$

In total, (5.146) is bounded from below by

$$\frac{1}{\sqrt{\mu}}\tanh\left(\frac{1}{2}\right)^2\left(\operatorname{artanh}\left(\sqrt{1-\frac{T^{1/2}}{\mu^{1/2}}}\right)-\operatorname{artanh}\left(\frac{1}{\sqrt{2}}\right)\right)^2\times$$
$$\operatorname{artanh}\left(1-\frac{(\sqrt{\mu}/a+1)(1-\sqrt{1-(T/\mu)^{1/2}})}{\sqrt{\mu}/a-1}\right) \quad (5.150)$$

With $\operatorname{artanh}(1-x)=\frac{1}{2}\ln 2/x+o(1)$ as $x\to 0$, we obtain that for $T\to 0$

$$\operatorname{artanh}\left(\sqrt{1-\frac{T^{1/2}}{\mu^{1/2}}}\right)=\frac{1}{4}\ln\left(16\frac{\mu}{T}\right)+o(1) \quad (5.151)$$

and

$$\operatorname{artanh}\left(1-\frac{(\sqrt{\mu}/a+1)(1-\sqrt{1-(T/\mu)^{1/2}})}{\sqrt{\mu}/a-1}\right)=\frac{1}{4}\ln\left(16\left(\frac{\sqrt{\mu}/a-1}{\sqrt{\mu}/a+1}\right)^2\frac{\mu}{T}\right)+o(1) \quad (5.152)$$

In particular, we obtain

$$\langle V^{1/2}j_2, D_T V^{1/2}j_2\rangle \geq \frac{C}{\sqrt{\mu}}\ln\left(\frac{\mu}{T}\right)^3+O\left(\ln\left(\frac{\mu}{T}\right)^2\right) \quad (5.153)$$

for some $C>0$ which implies the claim.

Dimension three: Using Lemma 5.4.10 and that $\ln \mu/T_c^0(\lambda)\sim 1/\lambda$ by Lemma 5.3.2,

$$\lim_{\lambda\to 0}\lambda^2\langle V^{1/2}j_3, D_{T_c^0(\lambda)}V^{1/2}j_3\rangle=\lim_{\lambda\to 0}\lambda^2\langle V^{1/2}j_3, D^<_{T_c^0(\lambda)}V^{1/2}j_3\rangle. \quad (5.154)$$

By integrating out the angular variables $\int_{\mathbb{R}^3}V(r)j_3(r;\mu)\frac{e^{i\sqrt{\mu}r\cdot p/|p|}}{(2\pi)^{3/2}}\mathrm{d}r=\frac{1}{|\mathbb{S}^2|}\int_{\mathbb{R}^3}V(r)j_3(r;\mu)^2=e_\mu$. Therefore, we can write

$$\langle V^{1/2}j_3, D^<_{T_c^0(\lambda)}V^{1/2}j_3\rangle=\frac{1}{(2\pi)^3}\int_{\mathbb{R}^{11};\tilde p^2,\tilde q^2<2\mu-p_1^2,p_1^2<\mu}\Big(Vj_3(r;\mu)(e^{ir\cdot p}-e^{i\sqrt{\mu}r\cdot p/|p|})\times$$
$$B_{T_c^0(\lambda)}(p,0)\widehat{V}(0,\tilde p-\tilde q)B_{T_c^0(\lambda)}((p_1,\tilde q),0)e^{-ip\cdot r'}Vj_3(r';\mu)$$
$$+Vj_3(r;\mu)e^{i\sqrt{\mu}r\cdot p/|p|}B_{T_c^0(\lambda)}(p,0)\widehat{V}(0,\tilde p-\tilde q)B_{T_c^0(\lambda)}((p_1,\tilde q),0)(e^{-ip\cdot r'}-e^{-i\sqrt{\mu}r'\cdot p/|p|})Vj_3(r';\mu)\Big)\mathrm{d}p\mathrm{d}\tilde q\mathrm{d}r\mathrm{d}r'$$
$$+e_\mu^2\int_{\mathbb{R}^8;\tilde p^2,\tilde q^2<2\mu-p_1^2,p_1^2<\mu}B_{T_c^0(\lambda)}(p,0)\frac{e^{i(\tilde p-\tilde q)\tilde r}}{(2\pi)^{3/2}}V(r)B_{T_c^0(\lambda)}((p_1,\tilde q),0)\mathrm{d}p\mathrm{d}\tilde q\mathrm{d}r \quad (5.155)$$

By [32, Proof of Lemma 3.1]

$$\left|\int_{\mathbb{S}^2}e^{i|r|w\cdot p}-e^{i\sqrt{\mu}|r|w\cdot p/|p|}\mathrm{d}w\right|\leq C\frac{|p|-\sqrt{\mu}}{|p|+\sqrt{\mu}}. \quad (5.156)$$

Furthermore, note that $B_T(p,0)\frac{|p|-\sqrt{\mu}}{|p|+\sqrt{\mu}}\leq\frac{1}{\mu}$. Hence, the first integral in (5.155) is bounded by

$$\frac{C}{\mu}\|Vj_3\|_1^2\|\widehat{V}\|_\infty\int_{p_1^2+\tilde q^2<2\mu,\tilde p^2<2\mu}B_{T_c^0(\lambda)}((p_1,\tilde q),0)\mathrm{d}p_1\mathrm{d}\tilde p\mathrm{d}\tilde q\leq C\|Vj_3\|_1^2\|\widehat{V}\|_\infty m_\mu(T_c^0(\lambda)), \quad (5.157)$$

which is of order $1/\lambda$ by Lemma 5.3.2.

Changing to angular coordinates for the $\tilde{p}$ and $\tilde{q}$ integration, the integral on the last line of (5.155) can be rewritten as

$$2\int_{\mathbb{R}^3} \mathrm{d}r \int_0^{\sqrt{\mu}} \mathrm{d}p_1 \int_0^{\sqrt{2\mu-p_1^2}} \mathrm{d}t \int_0^{\sqrt{2\mu-p_1^2}} \mathrm{d}s \int_{\mathbb{S}^1} \mathrm{d}w \int_{\mathbb{S}^1} \mathrm{d}w' B_{T_c^0(\lambda)}(\sqrt{p_1^2+t^2},0)t \frac{e^{i(tw-sw')\cdot\tilde{r}}}{(2\pi)^{3/2}} \times$$
$$V(r) B_{T_c^0(\lambda)}(\sqrt{p_1^2+s^2},0)s$$
$$= 2\int_{\mathbb{R}^3} \mathrm{d}r \int_0^{\sqrt{\mu}} \mathrm{d}p_1 \int_{p_1}^{\sqrt{2\mu}} \mathrm{d}x \int_{p_1}^{\sqrt{2\mu}} \mathrm{d}y \int_{\mathbb{S}^1} \mathrm{d}w \int_{\mathbb{S}^1} \mathrm{d}w' B_{T_c^0(\lambda)}(x,0)x \frac{e^{i(\sqrt{x^2-p_1^2}w-\sqrt{y^2-p_1^2}w')\cdot\tilde{r}}}{(2\pi)^{3/2}} \times$$
$$V(r) B_{T_c^0(\lambda)}(y,0)y \quad (5.158)$$

where we substituted $x=\sqrt{p_1^2+t^2}, y=\sqrt{p_1^2+s^2}$. Next, we want to replace the x^2 and y^2 in the exponent by μ. We rewrite (5.158) as

$$2\int B_{T_c^0(\lambda)}(x,0)x \frac{\left(e^{i\sqrt{x^2-p_1^2}w\cdot\tilde{r}} - e^{i\sqrt{\mu-p_1^2}w\cdot\tilde{r}}\right)}{(2\pi)^{3/2}} V(r) e^{-i\sqrt{y^2-p_1^2}w'\cdot\tilde{r}} B_{T_c^0(\lambda)}(y,0)y \mathrm{d}p_1 \mathrm{d}r \mathrm{d}x \mathrm{d}y \mathrm{d}w \mathrm{d}w'$$
$$+2\int B_{T_c^0(\lambda)}(x,0)x e^{i\sqrt{\mu-p_1^2}w\cdot\tilde{r}} V(r) \frac{\left(e^{-i\sqrt{y^2-p_1^2}w'\cdot\tilde{r}} - e^{i\sqrt{\mu-p_1^2}w'\cdot\tilde{r}}\right)}{(2\pi)^{3/2}} B_{T_c^0(\lambda)}(y,0)y \mathrm{d}p_1 \mathrm{d}r \mathrm{d}x \mathrm{d}y \mathrm{d}w \mathrm{d}w'$$
$$+2\int B_{T_c^0(\lambda)}(x,0)x \frac{e^{i\sqrt{\mu-p_1^2}(w-w')\cdot\tilde{r}}}{(2\pi)^{3/2}} V(r) B_{T_c^0(\lambda)}(y,0)y \mathrm{d}p_1 \mathrm{d}r \mathrm{d}x \mathrm{d}y \mathrm{d}w \mathrm{d}w' \quad (5.159)$$

By [40, Proof of Lemma 3.4]

$$\left|\int_{\mathbb{S}^1} \frac{e^{i\sqrt{x-p_1^2}w\cdot\tilde{r}} - e^{i\sqrt{\mu-p_1^2}w\cdot\tilde{r}}}{(2\pi)^2} \mathrm{d}w\right| \leq C\left|\sqrt{x^2-p_1^2} - \sqrt{\mu-p_1^2}\right|^{1/3} \left|(x^2-p_1^2)^{-1/6} + (\mu-p_1^2)^{-1/6}\right| \quad (5.160)$$

We bound this further by $C\,|x^2-\mu|^{1/3}\left((x^2-p_1^2)^{-1/3}+(\mu-p_1^2)^{-1/3}\right)$. Using that $B_{T_c^0(\lambda)}(x,0) \leq 1/|x^2-\mu|$ by (5.11) and recalling the definition of m_μ in (5.23) we bound the first two lines in (5.159) by

$$C\|V\|_1 m_\mu^{d=2}(T_c^0(\lambda)) \int_0^{\sqrt{\mu}} \mathrm{d}p_1 \int_{p_1}^{\sqrt{2\mu}} \mathrm{d}x \frac{1}{|x-\sqrt{\mu}|^{2/3}(x+\sqrt{\mu})^{2/3}} \left(\frac{1}{(x^2-p_1^2)^{1/3}} + \frac{1}{(\mu-p_1^2)^{1/3}}\right) \quad (5.161)$$

The integral is bounded by

$$\sqrt{\mu}\int_0^{\sqrt{2}} \mathrm{d}x \int_0^x \mathrm{d}p_1 \frac{1}{|x-1|^{2/3}} \left(\frac{1}{x^{1/3}(x-p_1)^{1/3}} + \frac{1}{(1-p_1)^{1/3}}\right) < \infty \quad (5.162)$$

Hence, the first two lines in (5.159) are of order $O(1/\lambda)$ by Lemma 5.3.2. For the third line we carry out the r-integration and obtain

$$2\int_0^{\sqrt{\mu}} \left(\int_{p_1}^{\sqrt{2\mu}} B_{T_c^0(\lambda)}(x,0)x\mathrm{d}x\right)^2 \left(\int_{\mathbb{S}^1}\int_{\mathbb{S}^1} \widehat{V}\left(0, \sqrt{\mu-p_1^2}(w-w')\right) \mathrm{d}w\mathrm{d}w'\right) \mathrm{d}p_1. \quad (5.163)$$

Note that $\int_{p_1}^{\sqrt{2\mu}} B_{T_c^0(\lambda)}(x,0)x\mathrm{d}x = m_\mu^{d=2}(T_c^0(\lambda)) - \int_0^{p_1} B_{T_c^0(\lambda)}(x,0)x\mathrm{d}x$ and

$$\int_0^{p_1} B_{T_c^0(\lambda)}(x,0)x\mathrm{d}x = \frac{1}{2}\int_{(\mu-p_1^2)/T_c^0(\lambda)}^{\mu/T_c^0(\lambda)} \frac{\tanh s}{s}\mathrm{d}s \leq \frac{1}{2}\ln\frac{\mu}{\mu-p_1^2} \quad (5.164)$$

where we substituted $s = (\mu - x^2)/T_c^0(\lambda)$. In particular,

$$\Bigg| 2\int_0^{\sqrt{\mu}} \left[\left(\int_{p_1}^{\sqrt{2\mu}} B_{T_c^0(\lambda)}(x,0) x \mathrm{d}x \right)^2 - m_\mu^{d=2}(T_c^0(\lambda))^2 \right] \times \left(\int_{\mathbb{S}^1}\int_{\mathbb{S}^1} \widehat{V}\left(0, \sqrt{\mu - p_1^2}(w-w')\right) \mathrm{d}w \mathrm{d}w' \right) \mathrm{d}p_1 \Bigg|$$
$$\leq 2|\mathbb{S}^1|^2 \|\widehat{V}\|_\infty \int_0^{\sqrt{\mu}} \left(\frac{1}{4}\left(\ln \frac{\mu}{\mu - p_1^2} \right)^2 + \ln \frac{\mu}{\mu - p_1^2} m_\mu^{d=2}(T_c^0(\lambda)) \right) \mathrm{d}p_1 \leq C(1 + m_\mu^{d=2}(T_c^0(\lambda))) \tag{5.165}$$

which is of order $O(1/\lambda)$ by Lemma 5.3.2. In total, we thus obtain

$$\lim_{\lambda\to 0} \lambda^2 \langle V^{1/2} j_3, D_{T_c^0(\lambda)} V^{1/2} j_3 \rangle = \lim_{\lambda \to 0} 2\lambda^2 m_\mu^{d=2}(T_c^0)^2 \sqrt{\mu} e_\mu^2 \int_0^1 \left(\int_{\mathbb{S}^1}\int_{\mathbb{S}^1} \widehat{V}\left(0, \sqrt{\mu}\sqrt{1-p_1^2}(w-w')\right) \mathrm{d}w\mathrm{d}w' \right) \mathrm{d}p_1 \tag{5.166}$$

By writing out the definition of j_3 and then switching to spherical coordinates and carrying out the r integration, we have

$$\int_{\mathbb{R}^4} V(r) j_3(z_1, \tilde{r}; \mu)^2 \mathrm{d}r\mathrm{d}z_1 = \int_{\mathbb{S}^2} \mathrm{d}u \int_{\mathbb{S}^2} \mathrm{d}v \int_{\mathbb{R}^7} \mathrm{d}p\mathrm{d}r\mathrm{d}z_1 \frac{e^{ip\cdot r}\widehat{V}(p)}{(2\pi)^{3/2}} \frac{e^{i\sqrt{\mu}(z_1,\tilde{r})\cdot(u-v)}}{(2\pi)^3} = \frac{1}{(2\pi)^{3/2}} \times$$
$$\int_{\mathbb{R}} \left(\int_0^\pi \sin\theta \mathrm{d}\theta \int_0^\pi \sin\theta' \mathrm{d}\theta' \int_{\mathbb{S}^1} \mathrm{d}w \int_{\mathbb{S}^1} \mathrm{d}w' \widehat{V}(0, \sqrt{\mu}(\sin\theta w - \sin\theta' w') e^{i\sqrt{\mu} z_1 (\cos\theta - \cos\theta')} \right) \mathrm{d}z_1$$
$$= \frac{1}{\sqrt{\mu}(2\pi)^{1/2}} \int_{-1}^1 \mathrm{d}t \int_{-1}^1 \mathrm{d}s \int_{\mathbb{S}^1} \mathrm{d}w \int_{\mathbb{S}^1} \mathrm{d}w' \widehat{V}(0, \sqrt{\mu}(\sqrt{1-t^2}w - \sqrt{1-s^2}w')\delta(s-t), \tag{5.167}$$

where in the last step we substituted $t = \cos\theta, s = \cos\theta'$ and carried out the z_1 integration. Furthermore, according to Lemma 5.3.2, $\lim_{\lambda\to 0} \lambda m_\mu^{d=2}(T_c^0) e_\mu = \frac{1}{\sqrt{\mu}}$. This gives the desired

$$\lim_{\lambda \to 0} \lambda^2 \langle V^{1/2} j_3, D_{T_c^0(\lambda)} V^{1/2} j_3 \rangle = (2\pi)^{1/2} \int_{\mathbb{R}^4} V(r) j_3(z_1, \tilde{r}; \mu)^2 \mathrm{d}r\mathrm{d}z_1 \tag{5.168}$$

□

5.5 Boundary Superconductivity in 3d

In this section we shall prove Theorem 5.1.4, which provides sufficient conditions for (5.7) to hold. Due to rotation invariance, we consider the spherical average of $\tilde{m}_3^{D/N}$ (defined in (5.6)). With

$$m_3^{D/N}(|r|;\mu) := \frac{1}{4\pi} \int_{\mathbb{S}^2} \tilde{m}_3^{D/N}(|r|\omega;\mu)\mathrm{d}\omega \tag{5.169}$$

we have $\int_{\mathbb{R}^3} V(r)\tilde{m}_3^{D/N}(r;\mu)\mathrm{d}r = \int_{\mathbb{R}^3} V(r) m_3^{D/N}(|r|;\mu)\mathrm{d}r$. Furthermore, we have the scaling property

$$m_3^{D/N}(|r|;\mu) = \frac{1}{\sqrt{\mu}} m_3^{D/N}(\sqrt{\mu}|r|;1). \tag{5.170}$$

We shall derive the following, more explicit, expression for $m_3^{D/N}$ in Section 5.5.1.

Lemma 5.5.1. *For $x \geq 0$ we can write $m_3^D(x;1) = \sum_{j=1}^4 t_j(x)$ and $m_3^N(x;1) = \sum_{j=1}^2 t_j(x) - \sum_{j=3}^4 t_j(x)$, where*

$$\begin{aligned} t_1(x) &= \frac{4}{\pi x}\int_1^\infty \frac{\sin^2(xk)}{k}\operatorname{arcoth}(k)\mathrm{d}k \\ t_2(x) &= -\frac{2}{\pi}\frac{\sin^2(x)}{x} \\ t_3(x) &= -2\frac{\sin^2(x)}{x^2} \\ t_4(x) &= \frac{4\sin x}{\pi x^2}\left(\sin x \operatorname{Si} 2x - \cos x \operatorname{Cin} 2x\right) \\ &= \frac{\sin x}{2\pi^3 x}\int_{\mathbb{S}^2}\int_{\mathbb{S}^2} \frac{\sin(x\omega_1|\omega_1'|)e^{-ix\tilde{\omega}\cdot\tilde{\omega}'}}{\omega_1}\mathrm{d}\omega\mathrm{d}\omega' \end{aligned}$$

where $\operatorname{Cin}(x) = \int_0^x \frac{1-\cos t}{t}\mathrm{d}t$ and $\operatorname{Si}(x) = \int_0^x \frac{\sin t}{t}\mathrm{d}t$.

To determine for which interactions $\int_{\mathbb{R}^3} V(r) m_3^{D/N}(|r|;\mu)\mathrm{d}r > 0$ holds, we need to understand $m_3^{D/N}(|r|;\mu)$. In Figures 5.1 and 5.2 we plot m_3^D and m_3^N for $\mu = 1$, respectively. The

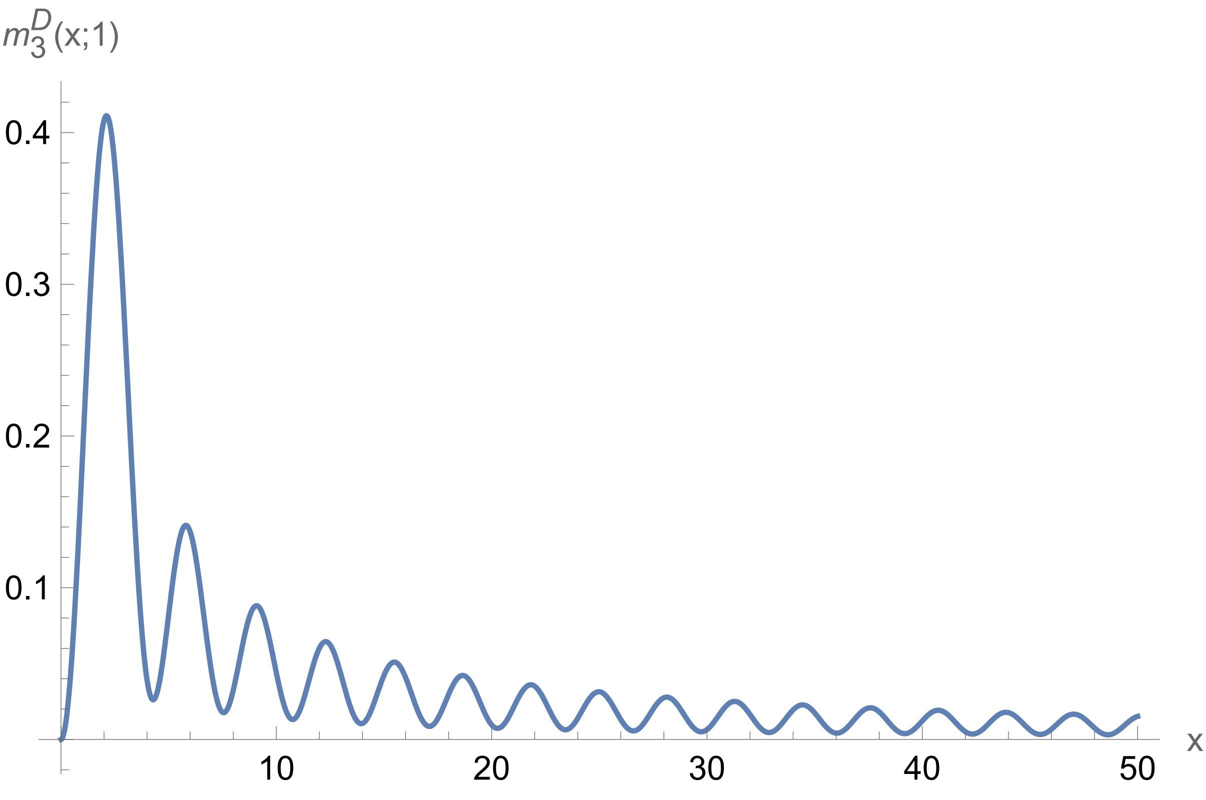

Figure 5.1: Plot of m_3^D for $\mu = 1$, created using [53].

function m_3^D seems to be nonnegative. If one could prove that $m_3^D \geq 0$, then Theorem 5.1.3 would apply to all $V \geq 0$ satisfying 5.1.1. Unfortunately, this is beyond our reach. On the other hand, the function m_3^N changes sign, but is positive in a neighborhood of zero.

Remark 5.5.2. To create the plots, it is computationally more efficient to use the first expression for t_4, whereas for the following analytic computations the second expression is more convenient.

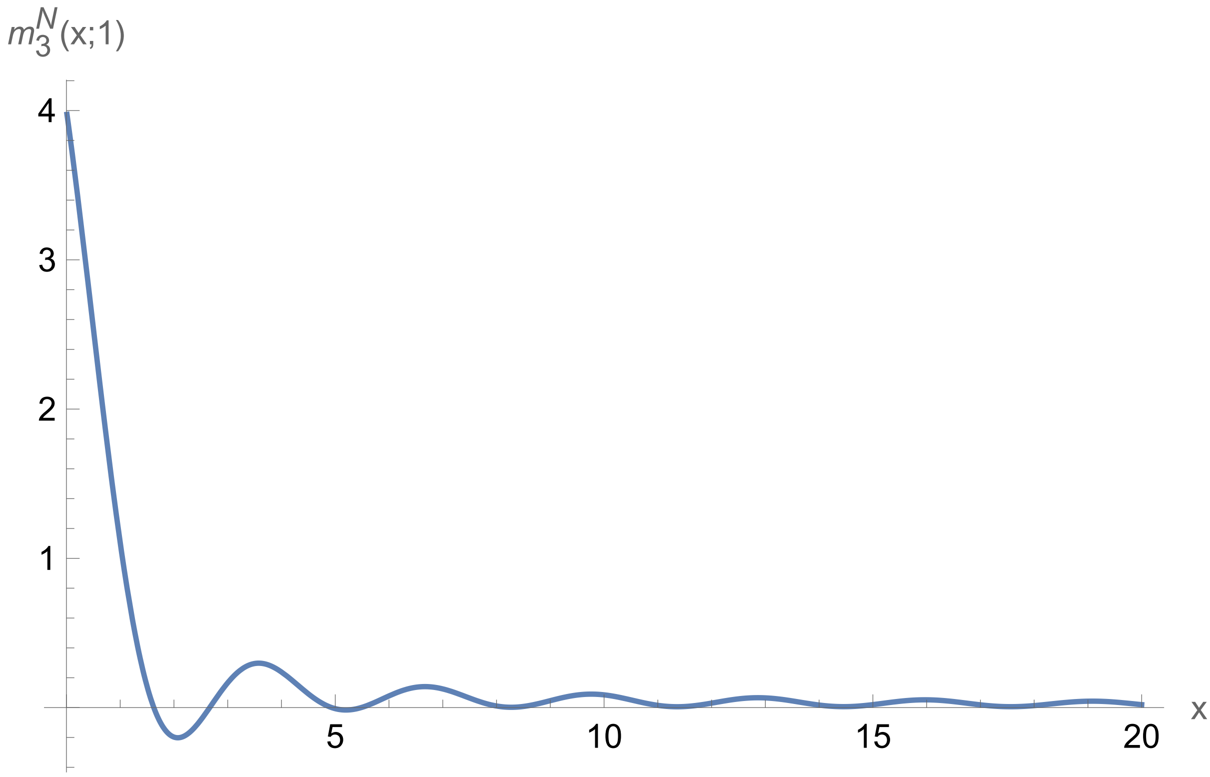

Figure 5.2: Plot of m_3^N for $\mu = 1$, created using [53].

Intuitively, if we let $\mu \to 0$, due to the scaling (5.170) the sign of $\int_{\mathbb{R}^3} V(r) m_3^{D/N}(|r|;\mu)\mathrm{d}r$ is determined by the values of $m_3^{D/N}(|r|;1)$ for r in the vicinity of zero. To obtain Theorem 5.1.4, we prove that both functions $m_3^{D/N}(|r|;1)$ are non-negative in a neighborhood of zero.

The following is proved in Section 5.5.2.

Lemma 5.5.3. *The functions t_j for $j = 1,2,3,4$ are bounded and twice continuously differentiable. The values of the functions and their derivatives at zero are listed in Table 5.1.*

f	t_1	t_2	t_3	t_4	$m_3^D(\cdot;1)$	$m_3^N(\cdot;1)$
$f(0)$	2	0	-2	0	0	4
$f'(0)$	$-2/\pi$	$-2/\pi$	0	$4/\pi$	0	
$f''(0)$	$-8/9$	0	$4/3$	0	$4/9$	

Table 5.1: Values of the functions t_j and $m_3^{D/N}$ and their derivatives at zero. The missing entries are not needed.

Proof of Theorem 5.1.4. We start with the case of Neumann boundary condition. By (5.170), it suffices to prove that $\lim_{\mu\to 0} \int_{\mathbb{R}^3} V(r) m_3^N(\sqrt{\mu}|r|;1)\mathrm{d}r > 0$. With $V \in L^1$ and Lemma 5.5.3 it follows by dominated convergence that $\lim_{\mu\to 0} \int_{\mathbb{R}^3} V(r) m_3^N(\sqrt{\mu}|r|;1)\mathrm{d}r = m_3^N(0;1)\int_{\mathbb{R}^3} V(r)\mathrm{d}r = 4\int_{\mathbb{R}^3} V(r)\mathrm{d}r$. Since $\widehat{V}(0) > 0$ by assumption, this is positive.

For Dirichlet boundary conditions, according to Lemma 5.5.3, $m_3^D(0;1)$ and its first derivative vanish. Thus, we consider $I(\sqrt{\mu}) := \frac{1}{\mu}\int_{\mathbb{R}^3} m_3^D(\sqrt{\mu}|r|;1)V(r)\mathrm{d}r$. Since $m_3^D(\cdot;1)$ is bounded, I is continuous away from 0. It suffices to prove that $\lim_{\mu\to 0} I(\sqrt{\mu}) > 0$. According to

Lemma 5.5.3 and Taylor's theorem, we have $m_3^D(x;1) = \frac{1}{2}(m_3^D)''(0;1)x^2 + R(x)$, where R is continuous with $\lim_{x\to 0} \frac{|R(x)|}{x^2} = 0$. Let $\epsilon > 0$ and $c := \sup_{0\leq x<\epsilon} \frac{|R(x)|}{x^2} < \infty$. One can bound

$$\left|\frac{1}{\mu} m_3^D(\sqrt{\mu}|r|;1)V(r)\right| \leq \chi_{\sqrt{\mu}|r|<\epsilon}\left(\frac{1}{2}(m_3^D)''(0;1)+c\right)|r^2V(r)| + \chi_{\sqrt{\mu}|r|>\epsilon}\frac{\|m_3^D\|_\infty}{\epsilon^2}|r^2V(r)|$$
$$\leq \left(\frac{1}{2}(m_3^D)''(0;1)+c+\frac{\|m_3^D\|_\infty}{\epsilon^2}\right)|r^2V(r)|, \quad (5.171)$$

which is integrable by the assumptions on V. By dominated convergence

$$\lim_{\mu\to 0} I(\sqrt{\mu}) = \int_{\mathbb{R}^3} \lim_{\mu\to 0}\frac{m_3^D(\sqrt{\mu}|r|;1)}{\mu|r|^2}V(r)|r|^2\mathrm{d}r = \frac{1}{2}\int_{\mathbb{R}^3}(m_3^D)''(0;1)V(r)|r|^2\mathrm{d}r = \frac{2}{9}\int_{\mathbb{R}^3}V(r)|r|^2\mathrm{d}r, \quad (5.172)$$

which is positive by assumption. □

5.5.1 Proof of Lemma 5.5.1

Proof of Lemma 5.5.1. With

$$\tilde{t}_1(r) = \int_{\mathbb{R}} j_3(z_1,r_2,r_3;1)^2\chi_{|z_1|>|r_1|}\mathrm{d}z_1$$
$$\tilde{t}_2(r) = -j_3(r;1)^2\int_{\mathbb{R}}\chi_{|z_1|<|r_1|}\mathrm{d}z_1$$
$$\tilde{t}_3(r) = \mp\pi j_3(r;1)^2$$
$$\tilde{t}_4(r) = \pm 2 j_3(r;1)\int_{\mathbb{R}} j_3(z_1,r_2,r_3;1)\chi_{|z_1|<|r_1|}\mathrm{d}z_1$$

one can write $\widetilde{m}_3^D(r;1) = \sum_{j=1}^4 \tilde{t}_j(r)$ and $\widetilde{m}_3^N(r;1) = \sum_{j=1}^2 \tilde{t}_j(r) - \sum_{j=3}^4 \tilde{t}_j(r)$. Let $t_j(|r|) = \frac{1}{4\pi}\int_{\mathbb{S}^2}\tilde{t}_j^{D/N}(|r|\omega;\mu)\mathrm{d}\omega$. The following explicit computations show that the t_j agree with the claimed expressions.

Recall that $j_3(r;1) = \sqrt{\frac{2}{\pi}}\frac{\sin|r|}{|r|}$. For t_1 we write out the integral in spherical coordinates and substitute $z_1 = xy$ and $s = \cos\theta$

$$t_1(x) = \frac{1}{\pi}\frac{2\pi}{4\pi}\int_0^\pi\int_{\mathbb{R}}\frac{\sin^2\sqrt{z_1^2+(x\sin\theta)^2}}{z_1^2+(x\sin\theta)^2}\chi_{|z_1|>x|\cos\theta|}\sin\theta\mathrm{d}z_1\mathrm{d}\theta$$
$$= \frac{1}{\pi x}\int_{-1}^1\int_{\mathbb{R}}\frac{\sin^2 x\sqrt{y^2+1-s^2}}{y^2+1-s^2}\chi_{|y|>|s|}\mathrm{d}y\mathrm{d}s \quad (5.173)$$

Next, we use the reflection symmetry of the integrand in s and y, substitute y by $k = \sqrt{y^2+1-s^2}$ and then carry out the s integration to obtain

$$t_1(x) = \frac{4}{\pi x}\int_0^1\int_1^\infty\frac{\sin^2 xk}{k\sqrt{k^2+s^2-1}}\mathrm{d}k\mathrm{d}s = \frac{4}{\pi x}\int_1^\infty\frac{\sin^2 xk}{k}\operatorname{arcoth}(k)\mathrm{d}k. \quad (5.174)$$

For t_2, we have

$$t_2(x) = -\frac{2}{\pi}\frac{\sin^2 x}{x^2}\frac{1}{4\pi}\int_{\mathbb{S}^2}2x|\omega_1|\mathrm{d}\omega = -\frac{2}{\pi}\frac{\sin^2 x}{x}. \quad (5.175)$$

Since $\tilde{t}_3$ is radial, we have $t_3 = \tilde{t}_3$. For t_4 we want to derive two expressions. For the first, we perform the same substitutions as for t_1

$$t_4(x) = \frac{4}{\pi}\frac{\sin x}{x}\frac{2\pi}{4\pi}\int_0^\pi \int_{\mathbb{R}} \frac{\sin\sqrt{z_1^2+(x\sin\theta)^2}}{\sqrt{z_1^2+(x\sin\theta)^2}}\chi_{|z_1|<|x|\cos\theta|}\sin\theta \mathrm{d}z_1\mathrm{d}\theta$$
$$= \frac{2}{\pi}\frac{\sin x}{x}\int_{-1}^1\int_{\mathbb{R}}\frac{\sin x\sqrt{y^2+1-s^2}}{\sqrt{y^2+1-s^2}}\chi_{|y|<|s|}\mathrm{d}y\mathrm{d}s = \frac{8}{\pi}\frac{\sin x}{x}\int_0^1\int_0^1\frac{\sin xk}{\sqrt{k^2+s^2-1}}\chi_{k^2+s^2>1}\mathrm{d}k\mathrm{d}s$$
$$= \frac{8}{\pi}\frac{\sin x}{x}\int_0^1 \sin xk \operatorname{artanh} k\mathrm{d}k = \pm\frac{4\sin x}{\pi x^2}\left(\sin x \operatorname{Si} 2x - \cos x \operatorname{Cin} 2x\right) \quad (5.176)$$

To obtain the second expression for t_4, note that $\int_{\mathbb{R}} e^{-i\omega_1 z_1}\chi_{|z_1|<|r_1|}\mathrm{d}z_1 = \frac{2\sin\omega_1|r_1|}{\omega_1}$. Therefore,

$$t_4(x) = 2\sqrt{\frac{2}{\pi}}\frac{\sin x}{x}\frac{1}{4\pi}\int_{\mathbb{S}^2}\int_{\mathbb{R}}\int_{\mathbb{S}^2}\frac{e^{-i\omega\cdot(z_1,x\tilde{\omega}')}}{(2\pi)^{3/2}}\chi_{|z_1|<x|\omega_1'|}\mathrm{d}\omega\mathrm{d}z_1\mathrm{d}\omega'$$
$$= \frac{1}{2\pi^3}\frac{\sin x}{x}\int_{\mathbb{S}^2}\int_{\mathbb{S}^2}\frac{\sin x\omega_1|\omega_1'|}{\omega_1}e^{-ix\tilde{\omega}\cdot\tilde{\omega}'}\mathrm{d}\omega\mathrm{d}\omega' \quad (5.177)$$

□

5.5.2 Proof of Lemma 5.5.3

Proof of Lemma 5.5.3. Since $\sin(x)/x$ is a bounded and smooth function, also t_2 and t_3 are bounded and smooth. Elementary computations give the entries in Table 5.1.

For t_4 use the second expression in Lemma 5.5.1. Since the integrand is bounded and smooth and the domain of integration is compact, the integral is bounded and we can exchange integration and taking limits and derivatives. In particular, t_4 is bounded and smooth and it is then an elementary computation to verify the entries in Table 5.1. For instance,

$$t_4'(0) = \frac{1}{2\pi^3}\int_{\mathbb{S}^2}\int_{\mathbb{S}^2}|\omega_1'|\mathrm{d}\omega\mathrm{d}\omega' = \frac{4}{\pi}. \quad (5.178)$$

To study t_1 we define auxiliary functions $f(x) = \frac{4}{\pi x}\operatorname{artanh}(x)$ and $g(x) = \frac{\sin(x)^2}{x^2}$. Note that $f(x)$ diverges logarithmically for $x \to 1$ and is continuous otherwise with $f(0) = \frac{4}{\pi}$. Furthermore, $f(x)$ is increasing on $[0,1)$ and for every $0<\epsilon<1$, $\sup_{0\le x<\epsilon}\frac{f'(x)}{x} = \frac{f'(\epsilon)}{\epsilon} < \infty$ since all coefficients in the Taylor series of $\operatorname{artanh}(x)$ are positive.

We can write

$$t_1(x) = \int_1^\infty xg(xk)f(1/k)\mathrm{d}k = \int_1^c xg(xk)f(1/k)\mathrm{d}k + \int_{cx}^\infty g(k)f(x/k)\mathrm{d}k \quad (5.179)$$

for any constant $c>1$. The first integrand is bounded by $Cx\operatorname{arcoth}(k)$, the second one by $C\frac{1}{k^2}$ (since f is bounded on the integration domain). By dominated convergence we obtain that t_1 is continuous and $t_1(0) = \frac{4}{\pi}\int_0^\infty g(k)\mathrm{d}k = 2$.

For $x>0$ we compute the derivative

$$t_1'(x) = \int_1^c (g(xk)+xkg'(xk))f(1/k)\mathrm{d}k - cg(cx)f(1/c) + \int_{cx}^\infty g(k)f'(x/k)\frac{1}{k}\mathrm{d}k$$
$$= \int_1^c (g(xk)+xkg'(xk))f(1/k)\mathrm{d}k - cg(cx)f(1/c) + \int_c^\infty g(kx)f'(1/k)\frac{1}{k}\mathrm{d}k, \quad (5.180)$$

where we could apply the Leibnitz integral rule since $f'(1/k)$ decays like $1/k$ for $k \to \infty$. By dominated convergence, t_1' is continuous for $x > 0$. By continuity of t_1 and the mean value theorem, $t_1'(0) = \lim_{x\to 0} \frac{t_1(x)-t_1(0)}{x} = \lim_{x\to 0}\lim_{y\to 0} \frac{t_1(x)-t_1(y)}{x-y} = \lim_{x\to 0} t_1'(x)$. We evaluate

$$\begin{aligned} t_1'(0) = \int_1^c f(1/k)\mathrm{d}k - cf(1/c) + \int_c^\infty f'(1/k)\frac{1}{k}\mathrm{d}k \\ = \int_1^c (f(1/k) - f(1/c))\,\mathrm{d}k - f(1/c) + \int_c^\infty f'(1/k)\frac{1}{k}\mathrm{d}k \end{aligned} \tag{5.181}$$

This is a number independent of c. To compute the number, we let $c \to \infty$, and by monotone convergence

$$t_1'(0) = \int_1^\infty (f(1/k) - f(0))\,\mathrm{d}k - f(0) = \frac{2}{\pi} - \frac{4}{\pi} = -\frac{2}{\pi}. \tag{5.182}$$

Note that $g'(k) = 2(\cos k - \frac{\sin k}{k})\frac{\sin k}{k^2}$ has a zero of order one at $k = 0$. Therefore, $|g'(kx)f'(1/k)| < \frac{C}{x^2k^3}$ and for $x > 0$ the second derivative is

$$\begin{aligned} t_1''(x) = \int_1^c (2xg'(xk) + xk^2g''(xk))f(1/k)\mathrm{d}k - c^2g'(cx)f(1/c) + \int_c^\infty g'(kx)f'(1/k)\mathrm{d}k \\ = \int_1^c (2xg'(xk) + xk^2g''(xk))f(1/k)\mathrm{d}k - c^2g'(cx)f(1/c) + \int_{cx}^\infty \frac{g'(y)}{y}\frac{f'(x/y)}{x/y}\mathrm{d}y \end{aligned} \tag{5.183}$$

We can bound $\frac{g'(y)}{y} \le \frac{C}{1+y^3}$ and $\sup_y |\frac{f'(x/y)}{x/y}\chi_{y>cx}| = cf'(1/c) < \infty$. By dominated convergence, the function above is continuous (also at zero). We have

$$t_1''(0) = \int_0^\infty \frac{g'(y)}{y}\mathrm{d}y \lim_{x\to 0}\frac{f'(x)}{x} \tag{5.184}$$

Since $\int_0^\infty \frac{g'(y)}{y}\mathrm{d}y = -\frac{\pi}{3}$ and $\lim_{x\to 0}\frac{f'(x)}{x} = \frac{8}{3\pi}$ we obtain

$$t_1''(0) = -\frac{8}{9}. \tag{5.185}$$

□

5.6 Relative Temperature Shift

In this section we shall prove Theorem 5.1.7, which states that the relative temperature shift vanishes in the weak coupling limit. We proceed similarly to the δ-interaction case in one dimension analyzed in [34]. For this, we switch to the Birman-Schwinger formulation. Recall the Birman-Schwinger operator A_T^0 corresponding to H_T^0 from (5.22). Let $\tilde{\Omega}_1 = \{(r,z) \in \mathbb{R}^{2d} | |r_1| < z_1\}$. Define the operator A_T^1 on $\psi \in L^2_{\mathrm{s}}(\tilde{\Omega}_1) = \{\psi \in L^2(\tilde{\Omega}_1) | \psi(r,z) = \psi(-r,z)\}$ via

$$\begin{aligned} \langle\psi, A_T^1\psi\rangle = \int_{\mathbb{R}^{4d+2(d-1)}} \mathrm{d}r\mathrm{d}r'\mathrm{d}p\mathrm{d}q\mathrm{d}\tilde{z}\mathrm{d}\tilde{z}' \int_{|r_1|<z_1} \mathrm{d}z_1 \int_{|r_1'|<z_1'} \mathrm{d}z_1' \frac{1}{(2\pi)^{2d}} \overline{\psi(r,z)} V(r)^{1/2} e^{i(p\cdot z + q\cdot r)} \times \\ B_T(p,q) \left(e^{-i(p_1z_1'+q_1r_1')} + e^{i(p_1z_1'+q_1r_1')} \mp e^{-i(q_1z_1'+p_1r_1')} \mp e^{i(q_1z_1'+p_1r_1')} \right) e^{-i(\tilde{p}\cdot\tilde{z}'+\tilde{q}\cdot\tilde{r}')} |V(r')|^{1/2}\psi(r',z'), \end{aligned} \tag{5.186}$$

where the upper signs correspond to Dirichlet and the lower signs to Neumann boundary conditions, respectively. It follows from a computation analogous to [34, Lemma 2.4] that the operator A_T^1 is the Birman-Schwinger operator corresponding to $H_T^{\Omega_1}$ in relative and center of mass variables. The Birman-Schwinger principle implies that $\operatorname{sgn}\inf\sigma(H_T^{\Omega_1}) = \operatorname{sgn}(1/\lambda - \sup\sigma(A_T^1))$, where we use the convention that $\operatorname{sgn} 0 = 0$.

One can reformulate the claim of Theorem 5.1.7 in terms of the Birman-Schwinger operators. For $j = 0, 1$ let $a_T^j = \sup\sigma(A_T^j)$. Then

$$\lim_{\lambda\to 0}\frac{T_c^1(\lambda) - T_c^0(\lambda)}{T_c^0(\lambda)} = 0 \Leftrightarrow \lim_{T\to 0}\left(a_T^0 - a_T^1\right) = 0. \tag{5.187}$$

This is a straightforward generalization of [34, Lemma 4.1] and we refer to [34, Lemma 4.1] for its proof.

Proof of Theorem 5.1.7. First we will argue that $a_T^0 \leq a_T^1$ for all $T > 0$. If $\inf\sigma(K_T^0 - \lambda V) < 2T$, then $\inf\sigma(K_T^0 - \lambda' V) < \inf\sigma(K_T^0 - \lambda V)$ for all $\lambda' > \lambda$. Furthermore, $\inf\sigma(K_T^0 - (a_T^0)^{-1}V) = 0 = \inf\sigma(K_T^{\Omega_1} - (a_T^1)^{-1}V) \leq \inf\sigma(K_T^{\Omega_0} - (a_T^1)^{-1}V)$, where we used Lemma 5.2.3 in the last step. In particular, $a_T^0 \leq a_T^1$.

It remains to show that $\lim_{T\to 0}(a_T^0 - a_T^1) \geq 0$. Let $\iota : L^2(\tilde{\Omega}_1) \to L^2(\mathbb{R}^{2d})$ be the isometry

$$\iota\psi(r_1, \tilde{r}, z_1, \tilde{z}) = \frac{1}{\sqrt{2}}(\psi(r_1, \tilde{r}, z_1, \tilde{z})\chi_{\tilde{\Omega}_1}(r, z) + \psi(-r_1, \tilde{r}, -z_1, \tilde{z})\chi_{\tilde{\Omega}_1}(-r_1, \tilde{r}, -z_1, \tilde{z})). \tag{5.188}$$

Let F_2 denote the Fourier transform in the second variable $F_2\psi(r, q) = \frac{1}{(2\pi)^{d/2}}\int_{\mathbb{R}^d} e^{-iq\cdot z}\psi(r, z)\mathrm{d}z$ and F_1 the Fourier transform in the first variable $F_1\psi(p, q) = \frac{1}{(2\pi)^{d/2}}\int_{\mathbb{R}^d} e^{-ip\cdot r}\psi(r, q)\mathrm{d}r$. Recall that by assumption $V \geq 0$ and for functions $\psi \in L^2(\mathbb{R}^d \times \mathbb{R}^d)$ we have $V^{1/2}\psi(r, q) = V^{1/2}(r)\psi(r, q)$. We define self-adjoint operators $\tilde{E}_T$ and G_T on $L^2(\mathbb{R}^{2d})$ through

$$\langle\psi, \tilde{E}_T\psi\rangle = a_T^0\|\psi\|_2^2 - \int_{\mathbb{R}^{2d}} B_T(p, q)|F_1V^{1/2}\psi(p, q)|^2\mathrm{d}p\mathrm{d}q \tag{5.189}$$

and

$$\langle\psi, G_T\psi\rangle = \int_{\mathbb{R}^{2d}} \overline{F_1V^{1/2}\psi((q_1, \tilde{p}), (p_1, \tilde{q}))}B_T(p, q)F_1V^{1/2}\psi(p, q)\mathrm{d}p\mathrm{d}q. \tag{5.190}$$

With this notation, we have $a_T^0\mathbb{I} - A_T^1 = \iota^\dagger F_2^\dagger(\tilde{E}_T \pm G_T)F_2\iota$, where $\mathbb{I}$ denotes the identity operator on $L^2_{\mathrm{s}}(\tilde{\Omega}_1)$. In particular,

$$a_T^0 - a_T^1 = \inf_{\psi\in L^2_{\mathrm{s}}(\tilde{\Omega}_1), \|\psi\|_2=1}\langle F_2\iota\psi, (\tilde{E}_T \pm G_T)F_2\iota\psi\rangle \geq \inf_{\psi\in L^2_{\mathrm{s}}(\mathbb{R}^{2d}), \|\psi\|_2=1}\langle\psi, (\tilde{E}_T \pm G_T)\psi\rangle, \tag{5.191}$$

where we used that $\|F_2\iota\psi\|_2 = \|\psi\|_2$. Define the function

$$E_T(q) = a_T^0 - \|V^{1/2}B_T(\cdot, q)V^{1/2}\|. \tag{5.192}$$

We claim that $\|V^{1/2}B_T(\cdot, q)V^{1/2}\| \leq a_T^0$. The Birman Schwinger operator $\tilde{A}_T$ corresponding to $H_T^{\mathbb{R}^d}$ satisfies $\sup\sigma(\tilde{A}_T) = \sup_q\|V^{1/2}B_T(\cdot, q)V^{1/2}\|$. Pick $\lambda = \sup\sigma(\tilde{A}_T)^{-1}$. According to the Birman Schwinger principle and Lemma 5.2.4, $0 = \inf\sigma(H_T^{\Omega_0}) = \inf\sigma(H_T^0)$. Using the Birman Schwinger principle for H_T^0, we obtain $a_T^0 = \sup\sigma(\tilde{A}_T) \geq \|V^{1/2}B_T(\cdot, q)V^{1/2}\|$. Hence, $E_T(q) \geq 0$. Let E_T act on $L^2(\mathbb{R}^{2d})$ as $E_T\psi(r, q) = E_T(q)\psi(r, q)$. Then

$$a_T^0 - a_T^1 \geq \inf_{\psi\in L^2_s(\mathbb{R}^{2d}), \|\psi\|_2=1}\langle\psi, (E_T \pm G_T)\psi\rangle. \tag{5.193}$$

It thus suffices to prove that $\lim_{T\to 0}\inf\sigma(E_T \pm G_T) \geq 0$. With the following three Lemmas, which are proved in the next sections, the claim follows completely analogously to the proof of [34, Theorem 1.2 (ii)]. For completeness, we provide a sketch of the argument in [34, Theorem 1.2 (ii)] after the statement of the Lemmas.

Lemma 5.6.1. *Let $\mu > 0$, $d \in \{1,2,3\}$ and let $V \geq 0$ satisfy 5.1.1. Then* $\sup_{T>0}\|G_T\| < \infty$.

Lemma 5.6.2. *Let $\mu > 0$, $d \in \{1,2,3\}$ and let $V \geq 0$ satisfy 5.1.1. Let $\mathbb{I}_{\leq\epsilon}$ act on $L^2(\mathbb{R}^{2d})$ as $\mathbb{I}_{\leq\epsilon}\psi(r,p) = \psi(r,p)\chi_{|p|\leq\epsilon}$. Then $\lim_{\epsilon\to 0}\sup_{T>0}\|\mathbb{I}_{\leq\epsilon}G_T\mathbb{I}_{\leq\epsilon}\| = 0$.*

Lemma 5.6.3. *Let $\mu > 0$, $d \in \{1,2,3\}$ and let $V \geq 0$ satisfy 5.1.1. Let $0 < \epsilon < \sqrt{\mu}$. There are constants $c_1, c_2, T_0 > 0$ such that for $0 < T < T_0$ and $|q| > \epsilon$ we have $E_T(q) > c_1|\ln(c_2/T)|$.*

Since $E_T(q) \geq 0$, we can write

$$E_T \pm G_T + \delta = \sqrt{E_T + \delta}\left(\mathbb{I} \pm \frac{1}{\sqrt{E_T+\delta}}G_T\frac{1}{\sqrt{E_T+\delta}}\right)\sqrt{E_T+\delta} \tag{5.194}$$

for any $\delta > 0$. It suffices to prove that for all $\delta > 0$

$$\lim_{T\to 0}\left\|\frac{1}{\sqrt{E_T+\delta}}G_T\frac{1}{\sqrt{E_T+\delta}}\right\| = 0\,. \tag{5.195}$$

To prove (5.195), with the notation introduced in Lemma 5.6.2 we have for all $0 < \epsilon < \sqrt{\mu}$

$$\begin{aligned}\left\|\frac{1}{\sqrt{E_T+\delta}}G_T\frac{1}{\sqrt{E_T+\delta}}\right\| &\leq \left\|\mathbb{I}_{\leq\epsilon}\frac{1}{\sqrt{E_T+\delta}}G_T\frac{1}{\sqrt{E_T+\delta}}\mathbb{I}_{\leq\epsilon}\right\| \\ &\quad + \left\|\mathbb{I}_{\leq\epsilon}\frac{1}{\sqrt{E_T+\delta}}G_T\frac{1}{\sqrt{E_T+\delta}}\mathbb{I}_{>\epsilon}\right\| + \left\|\mathbb{I}_{>\epsilon}\frac{1}{\sqrt{E_T+\delta}}G_T\frac{1}{\sqrt{E_T+\delta}}\right\|\,. \end{aligned} \tag{5.196}$$

With $E_T \geq 0$ and Lemma 5.6.3 we obtain

$$\lim_{T\to 0}\left\|\frac{1}{\sqrt{E_T+\delta}}G_T\frac{1}{\sqrt{E_T+\delta}}\right\| \leq \sup_{T>0}\frac{1}{\delta}\|\mathbb{I}_{\leq\epsilon}G_T\mathbb{I}_{\leq\epsilon}\| + \lim_{T\to 0}\frac{2}{(\delta c_1|\ln(c_2/T)|)^{1/2}}\|G_T\|. \tag{5.197}$$

The second term vanishes by Lemma 5.6.1 and the first term can be made arbitrarily small by Lemma 5.6.2. Hence, (5.195) follows. □

Remark 5.6.4. The variational argument above relies on A_T^1 being self-adjoint. This is why we assume $V \geq 0$ in Theorem 5.1.7.

5.6.1 Proof of Lemma 5.6.1

Proof of Lemma 5.6.1. We have $\|G_T\| \leq \|G_T^<\| + \|G_T^>\|$, where for $d \in \{2,3\}$

$$\langle\psi, G_T^<\psi\rangle = \int_{\mathbb{R}^{2d}}\overline{F_1V^{1/2}\psi((q_1,\tilde{p}),(p_1,\tilde{q}))}B_T(p,q)\chi_{|\tilde{p}|<2\sqrt{\mu}}F_1V^{1/2}\psi(p,q)\mathrm{d}p\mathrm{d}q, \tag{5.198}$$

and for $G_T^>$ change $\chi_{|\tilde{p}|<2\sqrt{\mu}}$ to $\chi_{|\tilde{p}|>2\sqrt{\mu}}$. For $d = 1$ set $G_T^< = G_T$ and $G_T^> = 0$. We will prove that $G_T^<$ and $G_T^>$ are bounded uniformly in T.

To bound $G_T^{>}$ in $d=2,3$ we use the Schwarz inequality in p_1, q_1 to obtain

$$\|G_T^{>}\| \leq \sup_{\psi\in L^2(\mathbb{R}^{2d}),\|\psi\|=1} \int_{\mathbb{R}^{2d}} B_T\left(p,q\right)\chi_{|\tilde{p}|>2\sqrt{\mu}}|F_1 V^{1/2}\psi(p,q)|^2 \mathrm{d}q\mathrm{d}p \tag{5.199}$$

The right hand side defines a multiplication operator in q. By (5.11) there is a constant $C>0$ independent of T such that $\|G_T^{>}\|\leq C\|M\|$, where $M := V^{1/2}\frac{1}{1-\Delta}V^{1/2}$ on $L^2(\mathbb{R}^d)$. It follows from the Hardy-Littlewood-Sobolev and the Hölder inequalities that M is a bounded operator [32, 40, 50].

To bound $G_T^{<}$ note that for fixed q, $\|F_1 V^{1/2}\psi(\cdot,q)\|_\infty \leq C\|V\|_1^{1/2}\|\psi(\cdot,q)\|_2$ by Lemma 5.3.73. Therefore, we estimate

$$\|G_T^{<}\| \leq C^2\|V\|_1 \sup_{\psi\in L^2(\mathbb{R}^{2d}),\|\psi\|=1} \int_{\mathbb{R}^{2d}} \|\psi(\cdot,(p_1,\tilde{q}))\|_2 B_T(p,q)\chi_{\tilde{p}^2<2\mu}\|\psi(\cdot,q)\|_2 \mathrm{d}p\mathrm{d}q \tag{5.200}$$

Since the right hand side defines a multiplication operator in $\tilde{q}$, we obtain

$$\|G_T^{<}\| \leq C^2\|V\|_1 \sup_{\tilde{q}\in\mathbb{R}^{d-1}} \sup_{\psi\in L^2(\mathbb{R}),\|\psi\|=1} \int_{\mathbb{R}^{d+1}} \overline{\psi(p_1)} B_T(p,q)\chi_{\tilde{p}^2<2\mu}\psi(q_1)\mathrm{d}p\mathrm{d}q_1, \tag{5.201}$$

where for $d=1$ the supremum over $\tilde{q}$ is absent. For $d=1$, the operator with integral kernel $B_T(p,q)$ is bounded uniformly in T according to [34, Lemma 4.2], and thus the claim follows. For $d\in\{2,3\}$ we need to prove that the operators with integral kernel $\int_{\mathbb{R}^{d-1}} B_T(p,q)\chi_{|\tilde{p}|<2\sqrt{\mu}}\mathrm{d}\tilde{p}$ are bounded uniformly in $\tilde{q}$ and T. We apply the bound [34, Lemma 4.6.]

$$B_T(p,q) \leq \frac{2}{|(p+q)^2-\mu|+|(p-q)^2-\mu|} \tag{5.202}$$

Then, we scale out μ and estimate the expression by pulling the supremum over ψ into the $\tilde{p}$-integral

$$\begin{aligned}
&\sup_{\tilde{q}\in\mathbb{R}^{d-1}} \sup_{\psi\in L^2(\mathbb{R}),\|\psi\|=1} \int_{\mathbb{R}^{d+1}} \frac{2\chi_{|\tilde{p}|<2\sqrt{\mu}}\overline{\psi(p_1)}\psi(q_1)}{|(p+q)^2-\mu|+|(p-q)^2-\mu|}\,\mathrm{d}p\mathrm{d}q_1 \\
&\quad = \mu^{d/2-1} \sup_{\tilde{q}\in\mathbb{R}^{d-1}} \sup_{\psi\in L^2(\mathbb{R}),\|\psi\|=1} \int_{\mathbb{R}^{d+1}} \frac{2\chi_{|\tilde{p}|<2}\overline{\psi(p_1)}\psi(q_1)}{|(p+q)^2-1|+|(p-q)^2-1|}\,\mathrm{d}p\mathrm{d}q_1 \\
&\leq \mu^{d/2-1} \sup_{\tilde{q}\in\mathbb{R}^{d-1}} \int_{\mathbb{R}^{d-1}} \chi_{|\tilde{p}|<2} \left[\sup_{\psi\in L^2(\mathbb{R}),\|\psi\|=1} \int_{\mathbb{R}^2} \frac{2\overline{\psi(p_1)}\psi(q_1)}{|(p+q)^2-1|+|(p-q)^2-1|}\,\mathrm{d}p_1\mathrm{d}q_1\right]\mathrm{d}\tilde{p}
\end{aligned} \tag{5.203}$$

Let $\mu_1 = 1-(\tilde{p}+\tilde{q})^2$ and $\mu_2 = 1-(\tilde{p}-\tilde{q})^2$. For fixed μ_1,μ_2 we need to bound the operator with integral kernel

$$D_{\mu_1,\mu_2}(p_1,q_1) = \frac{2}{|(p_1+q_1)^2-\mu_1|+|(p_1-q_1)^2-\mu_2|}. \tag{5.204}$$

Lemma 5.6.5. *Let $\mu_1,\mu_2\leq 1$ with $\min\{\mu_1,\mu_2\}\neq 0$. The operator D_{μ_1,μ_2} on $L^2(\mathbb{R})$ with integral kernel given by* (5.204) *satisfies*

$$\|D_{\mu_1,\mu_2}\| \leq C(1+d(\mu_1,\mu_2)|\min\{\mu_1,\mu_2\}|^{-1/2}) \tag{5.205}$$

for some finite C independent of μ_1,μ_2, where

$$d(\mu_1,\mu_2) = \begin{cases} 1+\ln\left(1+\frac{\max\{\mu_1,\mu_2\}}{|\min\{\mu_1,\mu_2\}|}\right) & \text{if} \quad \min\{\mu_1,\mu_2\}<0\leq\max\{\mu_1,\mu_2\}, \\ 1 & \text{otherwise.}\end{cases} \tag{5.206}$$

This is a generalization of [34, Lemma 4.2]. The proof of Lemma 5.6.5 is based on the Schur test and can be found in Section 5.7.1. Since $\max\{\mu_1, \mu_2\} \leq 1$, it follows from Lemma 5.6.5 that for any $\alpha > 1/2$ one has $\|D_{\mu_1,\mu_2}\| \leq C\,(1 + |\min\{\mu_1, \mu_2\}|^{-\alpha})$ for a constant C independent of μ_1, μ_2. The following Lemma concludes the proof of $\sup_{T>0}\|G_T^{<}\| < \infty$.

Lemma 5.6.6. *Let* $d \in \{2, 3\}$ *and* $0 \leq \alpha < 1$. *Let* $\mu_1 = 1 - (\tilde{p} + \tilde{q})^2$ *and* $\mu_2 = 1 - (\tilde{p} - \tilde{q})^2$. *Then*

$$\sup_{\tilde{q} \in \mathbb{R}^{d-1}} \int_{\mathbb{R}^{d-1}} \frac{\chi_{|\tilde{p}|<2}}{|\min\{\mu_1, \mu_2\}|^{\alpha}} \mathrm{d}\tilde{p} < \infty. \tag{5.207}$$

Lemma 5.6.6 follows from elementary computations which are carried out in Section 5.7.2. □

5.6.2 Proof of Lemma 5.6.2

Proof of Lemma 5.6.2. With the notation introduced in the proof of Lemma 5.6.1 we have $\|\mathbb{I}_{\leq\epsilon} G_T \mathbb{I}_{\leq\epsilon}\| \leq \|\mathbb{I}_{\leq\epsilon} G_T^{<} \mathbb{I}_{\leq\epsilon}\| + \|\mathbb{I}_{\leq\epsilon} G_T^{>} \mathbb{I}_{\leq\epsilon}\|$.

For $d = 2, 3$ we have analogously to (5.199)

$$\|\mathbb{I}_{\leq\epsilon} G_T^{>} \mathbb{I}_{\leq\epsilon}\| \leq \sup_{\psi \in L^2(\mathbb{R}^{2d}), \|\psi\|=1} \int_{\mathbb{R}^{2d}} \chi_{|q|<\epsilon} \chi_{|(p_1,\tilde{q})|<\epsilon} B_T\,(p, q)\, \chi_{|\tilde{p}|>2\sqrt{\mu}} |F_1 V^{1/2} \psi(p, q)|^2 \mathrm{d}q \mathrm{d}p. \tag{5.208}$$

Let $1 < t < \infty$ such that $V \in L^t(\mathbb{R}^d)$. According to Lemma 5.3.72, for fixed q we have

$$\|F_1 V^{1/2} \psi(\cdot, q)\|_{L^s(\mathbb{R}^d)} \leq C \|V\|_t^{1/2} \|\psi(\cdot, q)\|_{L^2(\mathbb{R}^d)}, \tag{5.209}$$

where $2 \leq s = 2t/(t-1) < \infty$. By (5.11) and Hölder's inequality in p, there is a constant C independent of T such that

$$\begin{aligned} \|\mathbb{I}_{\leq\epsilon} G_T^{>} \mathbb{I}_{\leq\epsilon}\| &\leq C \sup_{\psi \in L^2(\mathbb{R}^{2d}), \|\psi\|=1} \int_{\mathbb{R}^{2d}} \frac{\chi_{|p_1|<\epsilon}}{1 + \tilde{p}^2} |F_1 V^{1/2} \psi(p, q)|^2 \mathrm{d}p \mathrm{d}q \\ &\leq C \|V\|_t \left(\int_{\mathbb{R}^d} \frac{\chi_{|p_1|<\epsilon}}{(1 + \tilde{p}^2)^t} \mathrm{d}p \right)^{1/t}. \end{aligned} \tag{5.210}$$

In particular, the remaining integral is of order $O(\epsilon^{1/t})$ and vanishes as $\epsilon \to 0$.

To estimate $\|\mathbb{I}_{\leq\epsilon} G_T^{<} \mathbb{I}_{\leq\epsilon}\|$ we proceed as in the derivation of the bound on $\|G_T^{<}\|$ from (5.200) until the first line of (5.203) and obtain

$$\|\mathbb{I}_{\leq\epsilon} G_T^{<} \mathbb{I}_{\leq\epsilon}\| \leq C \|V\|_1 \sup_{|\tilde{q}|<\epsilon} \sup_{\psi \in L^2(\mathbb{R}), \|\psi\|=1} \int_{\mathbb{R}^{d+1}} \frac{2\chi_{|p_1|,|q_1|<\epsilon} \chi_{|\tilde{p}|<2\sqrt{\mu}} \overline{\psi(p_1)} \psi(q_1)}{|(p+q)^2 - \mu| + |(p-q)^2 - \mu|} \mathrm{d}p \mathrm{d}q_1 \tag{5.211}$$

Hence, we need that the norm of the operator on $L^2(\mathbb{R})$ with integral kernel

$$\int_{\mathbb{R}^{d-1}} \frac{2\chi_{|p_1|,|q_1|<\epsilon} \chi_{|\tilde{p}|<2\sqrt{\mu}}}{|(p+q)^2 - \mu| + |(p-q)^2 - \mu|} \mathrm{d}\tilde{p} \tag{5.212}$$

vanishes uniformly in $\tilde{q}$ as $\epsilon \to 0$. In $d = 1$, the Hilbert-Schmidt norm clearly vanishes as $\epsilon \to 0$. Similarly for $d = 2, 3$ the following Lemma implies that the Hilbert-Schmidt norm vanishes uniformly in $\tilde{q}$ as $\epsilon \to 0$.

Lemma 5.6.7. *Let $d \in \{2,3\}$. Then*

$$\lim_{\epsilon\to 0}\sup_{|\tilde{q}|<\epsilon}\int_{\mathbb{R}^2}\chi_{|p_1|,|q_1|<\epsilon}\left[\int_{\mathbb{R}^{d-1}}\frac{2\chi_{\tilde{p}^2<2}}{|(p+q)^2-1|+|(p-q)^2-1|}\mathrm{d}\tilde{p}\right]^2\mathrm{d}p_1\mathrm{d}q_1 = 0 \qquad (5.213)$$

The proof can be found in Section 5.7.3. We give the proof for $d=2$ only; the one for $d=3$ works analogously and is left to the reader. □

5.6.3 Proof of Lemma 5.6.3

Proof of Lemma 5.6.3. Since a_T^0 diverges like $e_\mu \mu^{d/2-1}\ln(\mu/T)$ as $T\to 0$, the claim follows if we prove that $\sup_{T>0}\sup_{|q|>\epsilon}\|V^{1/2}B_T(\cdot,q)V^{1/2}\| < \infty$. For $d=1$ we have

$$\begin{aligned}&\|V^{1/2}B_T(\cdot,q)V^{1/2}\|^2 \le \|V^{1/2}B_T(\cdot,q)V^{1/2}\|_{\mathrm{HS}}^2\\ &= \int_{\mathbb{R}^2}V(r)V(r')\left(\int_{\mathbb{R}}B_T(p,q)\frac{e^{ip(r-r')}}{2\pi}\mathrm{d}p\right)^2\mathrm{d}r\mathrm{d}r' \le \frac{1}{(2\pi)^2}\|V\|_1^2\left(\int_{\mathbb{R}}B_T(p,q)\mathrm{d}p\right)^2 \end{aligned}\qquad (5.214)$$

It was shown in the proof of [34, Lemma 4.4] that $\sup_{T>0,|q|>\epsilon}\int_{\mathbb{R}}B_T(p,q)\mathrm{d}p<\infty$.

For $d\in\{2,3\}$, the claim follows from the following Lemma which is proved below.

Lemma 5.6.8. *Let $d\in\{2,3\}$ and $\mu>0$. Let V satisfy Assumption 5.1.1 and $V\ge 0$. Recall that $O_\mu = V^{1/2}\mathcal{F}^\dagger\mathcal{F}V^{1/2}$ (defined above (5.23)). Let $f(x)=\chi_{(0,1/2)}(x)\ln(1/x)$. There is a constant $C(d,\mu,V)$ such that for all $T>0$, $q\in\mathbb{R}^d$, and $\psi\in L^2(\mathbb{R}^d)$ with $\|\psi\|_2=1$*

$$\langle\psi, V^{1/2}B_T(\cdot,q)V^{1/2}\psi\rangle \le \mu^{d/2-1}\langle\psi,O_\mu\psi\rangle f(\max\{T/\mu,|q|/\sqrt{\mu}\}) + C(d,\mu,V). \qquad (5.215)$$

This concludes the proof. □

Proof of Lemma 5.6.8. Note that if we set $q=0$, and optimize over ψ, the left hand side would have the asymptotics $a_{T,\mu}^0 \sim e_\mu\mu^{d/2-1}\ln(1/T)$ as $T\to 0$. Intuitively, keeping q away from 0 on a scale larger than T will slow down the divergence. In the case $q=0$, divergence comes from the singularity on the set $|p|=\sqrt{\mu}$. For $|q|>0$, there will be two relevant sets, $(p+q)^2=\mu$ and $(p-q)^2=\mu$. These sets are circles or spheres in 2d and 3d, respectively. The function B_T is very small on the region which lies inside exactly one of the disks or balls (see the shaded area in Figure 5.3). The part lying inside or outside both disks (the white area in Figure 5.3) will be relevant for the asymptotics. Define the family of operators $Q_T(q): L^1(\mathbb{R}^d)\to L^\infty(\mathbb{R}^d)$ for $q\in\mathbb{R}^d$ through

$$\langle\psi,Q_T(q)\psi\rangle = \chi_{\max\left\{\frac{T}{\mu},\frac{|q|}{\sqrt{\mu}}\right\}<\frac{1}{2}}\int_{\mathbb{R}^d}\left|\widehat{\psi}\left(\sqrt{\mu}p/|p|\right)\right|^2 B_T(p,q)\chi_{((p+q)^2-\mu)((p-q)^2-\mu)>0}\chi_{p^2<3\mu}\mathrm{d}p. \qquad (5.216)$$

We claim that Q_T captures the divergence of B_T.

Lemma 5.6.9. *Let $d\in\{2,3\}$ and $\mu>0$. Let V satisfy Assumption 5.1.1. Then*

$$\sup_{T>0}\sup_{q\in\mathbb{R}^d}\|V^{1/2}B_T(\cdot,q)|V|^{1/2} - V^{1/2}Q_T(q)|V|^{1/2}\| < \infty. \qquad (5.217)$$

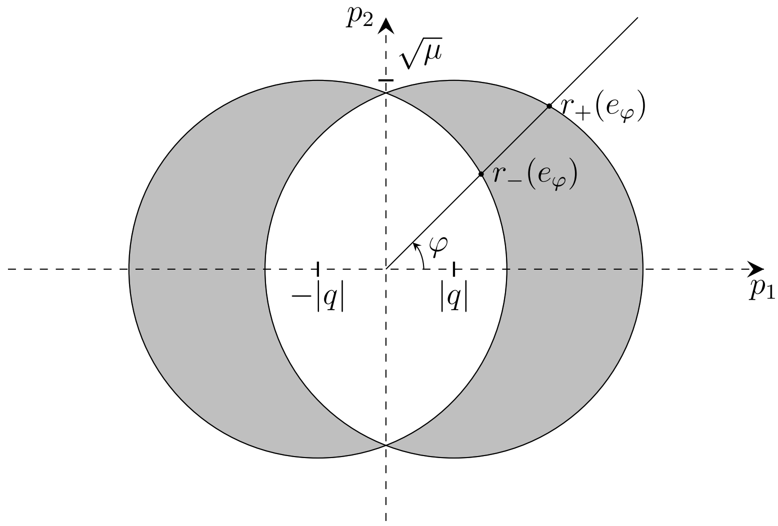

Figure 5.3: Two circles of radius $\sqrt{\mu}$, centered at $(-|q|,0)$ and $(|q|,0)$. In $d=2$ the function $B_T(p,(|q|,0))$ diverges on the two circles as $T\to 0$ and approaches zero in the shaded area. Given an angle φ, the numbers $r_\pm(e_\varphi)$ are the distances between zero and the intersections of the circles with the ray tilted by an angle φ with respect to the p_1-axis.

The proof of Lemma 5.6.9 can be found in Section 5.7.4. It now suffices to prove that there is a constant C such that for all $T>0$ and $q\in\mathbb{R}^d$

$$\langle\psi, Q_T(q)\psi\rangle \leq \langle\psi,\mathcal{F}^\dagger\mathcal{F}\psi\rangle f(\max\{T/\mu,|q|/\sqrt{\mu}\}) + C\|\psi\|_1^2. \tag{5.218}$$

Then for all $\psi\in L^2(\mathbb{R}^d)$ with $\|\psi\|_2=1$

$$\langle\psi, V^{1/2}Q_T(q)V^{1/2}\psi\rangle \leq \langle\psi,O_\mu\psi\rangle f(\max\{T/\mu,|q|/\sqrt{\mu}\}) + C\|V\|_1 \tag{5.219}$$

and the claim follows with Lemma 5.6.9.

We are left with proving (5.218). By the definition of Q_T, it suffices to restrict to $|q|<\sqrt{\mu}/2, T<\mu/2$. Let R be the rotation in $\mathbb{R}^d$ around the origin such that $q=R(|q|,\tilde{0})$. For $d=2$ the condition $((p+(|q|,0))^2-\mu)((p-(|q|,0))^2-\mu)>0$ holds exactly in the white region sketched in Figure 5.3. The inner white region is characterized by $(|p_1|+|q|)^2+\tilde{p}^2<\mu$, and the outer region by $(|p_1|-|q|)^2+\tilde{p}^2>\mu$. Thus,

$$\langle\psi, Q_T(q)\psi\rangle = \int_{\mathbb{R}^d}\left|\widehat{\psi}\left(\sqrt{\mu}Rp/|p|\right)\right|^2\left[\chi_{(|p_1|+|q|)^2+\tilde{p}^2<\mu}+\chi_{(|p_1|-|q|)^2+\tilde{p}^2>\mu}\right]B_T(p,(|q|,\tilde{0}))\chi_{p^2<3\mu}\mathrm{d}p, \tag{5.220}$$

where we substituted p by Rp.

Let us use the notation $r_\pm(e)=\pm|e_1||q|+\sqrt{\mu-e_2^2|q|^2}$ and $e_\varphi=(\cos\varphi,\sin\varphi)$, where the choice of $r_\pm$ is motivated in Figure 5.3. For $d=2$ rewriting the integral (5.220) in angular coordinates gives

$$\int_0^{2\pi}\left|\widehat{\psi}\left(\sqrt{\mu}Re_\varphi|\right)\right|^2\left[\int_0^{r_-(e_\varphi)}B_T(re_\varphi,(|q|,0))r\mathrm{d}r+\int_{r_+(e_\varphi)}^{\sqrt{3\mu}}B_{T,\mu}(re_\varphi,(|q|,0))r\mathrm{d}r\right]\mathrm{d}\varphi. \tag{5.221}$$

For $d=3$ with the notation $e_{\varphi,\theta}=(\cos\varphi,\sin\varphi\cos\theta,\sin\varphi\sin\theta)$ and using that $B_T(re_{\varphi,\theta},(|q|,0,0))=B_T(re_\varphi,(|q|,0))$, (5.220) equals

$$\int_0^{\pi}\left(\int_0^{2\pi}\left|\widehat{\psi}\left(\sqrt{\mu}re_{\varphi,\theta}|\right)\right|^2\mathrm{d}\theta\right)\left[\int_0^{r_-(e_\varphi)}B_T(re_\varphi,(|q|,0))r^2\mathrm{d}r+\int_{r_+(e_\varphi)}^{\sqrt{3\mu}}B_T(re_\varphi,(|q|,0))r^2\mathrm{d}r\right]\sin\varphi\mathrm{d}\varphi. \tag{5.222}$$

We distinguish two cases depending on whether r is within distance $T/\sqrt{\mu}$ to $r_\pm$ or not. Note that $r_-(e) \geq -|q| + \sqrt{\mu} \geq \frac{\sqrt{\mu}}{2} \geq \frac{T}{\sqrt{\mu}}$ and $r_+(e) + \frac{T}{\sqrt{\mu}} \leq |q| + \sqrt{\mu} + T \leq 2\sqrt{\mu}$. If r is close to $r_\pm$ we use that $B_T(p,q) \leq 1/2T$. Otherwise we use (5.202). The expressions in the square brackets in (5.221) and (5.222) are thus bounded by

$$\int_0^{r_-(e_\varphi)-\frac{T}{\sqrt{\mu}}} \frac{r^{d-1}}{\mu - r^2 - q^2}\mathrm{d}r + \int_{r_-(e_\varphi)-\frac{T}{\sqrt{\mu}}}^{r_-(e_\varphi)} \frac{r^{d-1}}{2T}\mathrm{d}r + \int_{r_+(e_\varphi)}^{r_+(e_\varphi)+\frac{T}{\sqrt{\mu}}} \frac{r^{d-1}}{2T}\mathrm{d}r + \int_{r_+(e_\varphi)+\frac{T}{\sqrt{\mu}}}^{\sqrt{3\mu}} \frac{r^{d-1}}{r^2+q^2-\mu}\mathrm{d}r \tag{5.223}$$

The second and third term are clearly bounded for $T < \mu/2$. Since $\|\widehat{\psi}\|_\infty \leq (2\pi)^{-d/2}\|\psi\|_1$, they contribute $C\|\psi\|_1$ to the upper bound on $\langle \psi, Q_T(q)\psi\rangle$.

To bound the contributions of the first and the last term in (5.223) we treat $d=2$ and $d=3$ separately.

Case $d=2$: The sum of the two integrals equals

$$\ln\left(\sqrt{\frac{(\mu-q^2)(2\mu+q^2)}{(\mu - q^2 - (r_-(e_\varphi) - \frac{T}{\sqrt{\mu}})^2)((r_+(e_\varphi)+\frac{T}{\sqrt{\mu}})^2 + q^2 - \mu)}}\right) \tag{5.224}$$

To bound this expression, we first make a few observations. Note that

$$\begin{aligned}\mu - q^2 - \left(r_-(e_\varphi) - \frac{T}{\sqrt{\mu}}\right)^2 &= 2|e_1||q|(\sqrt{\mu - e_2^2|q|^2} - |e_1||q|) + \frac{T}{\sqrt{\mu}}\left(2r_-(e_\varphi) - \frac{T}{\sqrt{\mu}}\right)\\ &\geq (\sqrt{3}-1)\sqrt{\mu}|e_1||q| + \frac{T}{2},\end{aligned} \tag{5.225}$$

where we used that $r_-(e_\varphi) \geq \sqrt{\mu} - |q|$ and $|q|, T/\sqrt{\mu} \leq \sqrt{\mu}/2$. Similarly,

$$\begin{aligned}\left(r_+(e_\varphi) + \frac{T}{\sqrt{\mu}}\right)^2 + q^2 - \mu &= 2|e_1||q|(\sqrt{\mu - e_2^2|q|^2} + |e_1||q|) + \frac{T}{\sqrt{\mu}}\left(2r_+(e_\varphi) + \frac{T}{\sqrt{\mu}}\right)\\ &\geq \sqrt{3}\sqrt{\mu}|e_1||q| + \sqrt{3}T\end{aligned} \tag{5.226}$$

Furthermore, note that $2\mu + q^2 \leq \frac{5\mu}{4}$. The expression under the square root in (5.224) is therefore bounded above by

$$\frac{5\mu^2}{4((\sqrt{3}-1)\sqrt{\mu}|e_1||q| + \frac{T}{2})(\sqrt{3}\sqrt{\mu}|e_1||q| + \sqrt{3}T)} \tag{5.227}$$

We now bound this from above in two ways. First we drop the T terms in the denominator, and second we drop the other terms in the denominator, which gives $\frac{5\mu}{4\sqrt{3}(\sqrt{3}-1)|e_1|^2|q|^2}$ and $\frac{5\mu^2}{2\sqrt{3}T^2}$, respectively. Thus, (5.224) is bounded above by $f(\max\{T/\mu, |q|/\sqrt{\mu}\}) + \ln(1/|e_1|) + C$. The contribution to the upper bound on $\langle \psi, Q_T(q)\psi\rangle$ is

$$\int_0^{2\pi} \left|\widehat{\psi}\left(\sqrt{\mu}e_\varphi|\right)\right|^2 f(\max\{T/\mu, |q|/\sqrt{\mu}\})\mathrm{d}\varphi + (2\pi)^{-2}\|\psi\|_1^2 \int_0^{2\pi} \left(\ln\left(1/|\cos\varphi|\right) + C\right)\mathrm{d}\varphi, \tag{5.228}$$

where for the second term we used that $|\widehat{\psi}\left(\sqrt{\mu}e_\varphi|\right)|^2 \leq (2\pi)^{-2}\|\psi\|_1^2$. Note that the first summand equals $\langle \psi, \mathcal{F}^\dagger\mathcal{F}\psi\rangle f(\max\{T/\mu, |q|/\sqrt{\mu}\})$ and that the integral in the second summand is finite. In total, we have obtained (5.218) for $d=2$.

Case $d = 3$**:** Note that $\frac{\mathrm{d}}{\mathrm{d}r}(-r + a\,\mathrm{artanh}(r/a)) = r^2/(a^2 - r^2)$ and $\frac{\mathrm{d}}{\mathrm{d}r}(r - a\,\mathrm{arcoth}(r/a)) = r^2/(r^2 - a^2)$. The sum of the first and the last integral in (5.223) hence equals

$$\sqrt{3\mu} - r_+(e_\varphi) - r_-(e_\varphi) - \frac{\sqrt{\mu - q^2}}{2}\ln\left(\frac{(\sqrt{\mu - q^2} + \sqrt{3\mu})}{(\sqrt{3\mu} - \sqrt{\mu - q^2})}\right) + \frac{\sqrt{\mu - q^2}}{2}\ln\left(\frac{(\sqrt{\mu - q^2} + r_-(e_\varphi) - \frac{T}{\sqrt{\mu}})}{(\sqrt{\mu - q^2} - r_-(e_\varphi) + \frac{T}{\sqrt{\mu}})}\frac{(\sqrt{\mu - q^2} + r_+(e_\varphi) + \frac{T}{\sqrt{\mu}})}{(r_+(e_\varphi) + \frac{T}{\sqrt{\mu}} - \sqrt{\mu - q^2})}\right) \tag{5.229}$$

The terms in the first line are bounded. The argument of the logarithm in the second line equals

$$\frac{(\sqrt{\mu - q^2} + r_-(e_\varphi) - \frac{T}{\sqrt{\mu}})^2}{(\mu - q^2 - (r_-(e_\varphi) - \frac{T}{\sqrt{\mu}})^2)}\frac{(\sqrt{\mu - q^2} + r_+(e_\varphi) + \frac{T}{\sqrt{\mu}})^2}{((r_+(e_\varphi) + \frac{T}{\sqrt{\mu}})^2 - \mu + q^2)} \leq \frac{C\mu^2}{((\sqrt{3} - 1)\sqrt{\mu}|e_1||q| + \frac{T}{2})(\sqrt{3}\sqrt{\mu}|e_1||q| + \sqrt{3}T))} \tag{5.230}$$

where we used (5.225) and (5.226). Analogously to the case $d = 2$ the contribution to the upper bound on $\langle\psi, Q_T(q)\psi\rangle$ is

$$\int_0^\pi \left(\int_0^{2\pi} \left|\widehat{\psi}\,(\sqrt{\mu}e_{\varphi,\theta}|)\right|^2 \mathrm{d}\theta\right) f(\max\{T/\mu, |q|/\sqrt{\mu}\}) \sin\varphi \mathrm{d}\varphi + (2\pi)^{-2}\|\psi\|_1^2 \int_0^\pi (\ln(1/|\cos\varphi|) + C) \sin\varphi \mathrm{d}\varphi. \tag{5.231}$$

and (5.218) follows. □

5.7 Proofs of Auxiliary Lemmas

5.7.1 Proof of Lemma 5.6.5

Proof of Lemma 5.6.5. If we write D_{μ_1,μ_2} as a sum $D_{\mu_1,\mu_2} = \sum_{j=1}^n D^j_{\mu_1,\mu_2}$ a.e. for some integral kernels $D^j_{\mu_1,\mu_2}$, then $\|D_{\mu_1,\mu_2}\| \leq \sum_{j=1}^n \|D^j_{\mu_1,\mu_2}\|$. We will choose the $D^j_{\mu_1,\mu_2}$ as localized versions of D_{μ_1,μ_2} in different regions (by multiplying D_{μ_1,μ_2} by characteristic functions).

Let $D^1_{\mu_1,\mu_2} = D_{\mu_1,\mu_2}\chi_{\max\{|p_1|,|q_1|\}>2}$ and $D^2_{\mu_1,\mu_2} = D_{\mu_1,\mu_2}\chi_{\max\{|p_1|,|q_1|\}<2}$. We first prove that the Hilbert-Schmidt norm of $\|D^1_{\mu_1,\mu_2}\|$ is bounded uniformly in μ_1, μ_2. Note that if $\max\{|p_1|, |q_1|\} > 2$, we have $\max\{(p_1 \pm q_1)^2\} = (|p_1| + |q_1|)^2 > 4$ and $\mu_1, \mu_2 \leq 1$. Hence,

$$D^1_{\mu_1,\mu_2}(p_1, q_1) \leq \frac{2\chi_{\max\{|p_1|,|q_1|\}>2}}{(|p_1| + |q_1|)^2 - 1} \leq \frac{2\chi_{\max\{|p_1|,|q_1|\}>2}}{p_1^2 + q_1^2 - 1}. \tag{5.232}$$

For the Hilbert-Schmidt norm we obtain

$$\|D^1_{\mu_1,\mu_2}\|^2_{\mathrm{HS}} \leq 4\int_{\mathbb{R}^2} \frac{\chi_{\max\{|p_1|,|q_1|\}>2}}{(p_1^2 + q_1^2 - 1)^2}\mathrm{d}p_1\mathrm{d}q_1 \leq 8\pi \int_2^\infty \frac{r}{(r^2 - 1)^2}\mathrm{d}r = \frac{4\pi}{3}, \tag{5.233}$$

and therefore $\|D^1_{\mu_1,\mu_2}\|$ is indeed bounded uniformly in μ_1, μ_2.

For $D^2_{\mu_1,\mu_2}$ we first observe that $\|D^2_{\mu_2,\mu_1}\| = \|D^2_{\mu_1,\mu_2}\|$ since $D^2_{\mu_1,\mu_2}(p_1, q_1) = D^2_{\mu_2,\mu_1}(p_1, -q_1)$. Hence, without loss of generality we may assume $\mu_1 \leq \mu_2$ from now on. To bound the norm of $D^2_{\mu_1,\mu_2}$ we distinguish the cases $\mu_1 < 0$ and $\mu_1 > 0$ and continue localizing.

Case $\mu_1 < 0$**:** We localize in the regions $|p_1 - q_1|^2 < \mu_2$ and $|p_1 - q_1|^2 > \mu_2$, where the first one only occurs if $\mu_2 > 0$. Let $D^3_{\mu_1,\mu_2} = D^2_{\mu_1,\mu_2}\chi_{|p_1-q_1|^2<\mu_2}$ and $D^4_{\mu_1,\mu_2} = D^2_{\mu_1,\mu_2}\chi_{|p_1-q_1|^2>\mu_2}$.

For $D^3_{\mu_1,\mu_2}$ we do a Schur test with test function $h(p_1) = |p_1|^{1/2}$. Using the symmetry of the integrand under $(p_1, q_1) \to -(p_1, q_1)$, we have

$$\begin{aligned}\|D^3_{\mu_1,\mu_2}\| &\le \sup_{-2<p_1<2} |p_1|^{1/2} \int_{-2}^{2} \frac{1}{2} \frac{\chi_{|p_1-q_1|^2<\mu_2}}{p_1 q_1 + (\mu_2 - \mu_1)/4} \frac{1}{|q_1|^{1/2}} \mathrm{d}q_1 \\ &= \chi_{0<\mu_2} \sup_{0\le p_1<2} |p_1|^{1/2} \int_{p_1-\sqrt{\mu_2}}^{p_1+\sqrt{\mu_2}} \frac{1}{2} \frac{1}{p_1 q_1 + (\mu_2 - \mu_1)/4} \frac{1}{|q_1|^{1/2}} \mathrm{d}q_1. \end{aligned} \tag{5.234}$$

For $\mu_2 > 0$, carrying out the integration we obtain

$$\begin{aligned}\|D^3_{\mu_1,\mu_2}\| \le \sup_{0\le p_1<2} \frac{2}{\sqrt{\mu_2 - \mu_1}} \Bigg[&\arctan\left(\sqrt{\frac{4p_1(p_1 + \sqrt{\mu_2})}{\mu_2 - \mu_1}}\right) \\ -\chi_{p_1>\sqrt{\mu_2}} \arctan\left(\sqrt{\frac{4p_1(p_1 - \sqrt{\mu_2})}{\mu_2 - \mu_1}}\right) &+ \chi_{p_1<\sqrt{\mu_2}} \operatorname{artanh}\left(\sqrt{\frac{4p_1(\sqrt{\mu_2} - p_1)}{\mu_2 - \mu_1}}\right)\Bigg] \\ &\le \frac{2}{\sqrt{\mu_2 - \mu_1}} \left[\frac{\pi}{2} + \operatorname{artanh}\left(\sqrt{\frac{\mu_2}{\mu_2 - \mu_1}}\right)\right], \end{aligned} \tag{5.235}$$

where we used the monotonicity of artanh. Note that for $x \ge 0$,

$$\operatorname{artanh}\left(\sqrt{\frac{1}{1+x}}\right) = \ln\left(\sqrt{\frac{1}{x} + 1} + \sqrt{\frac{1}{x}}\right) \le \ln\left(2\sqrt{\frac{1}{x} + 1}\right) = \ln(2) + \frac{1}{2}\ln\left(1 + \frac{1}{x}\right). \tag{5.236}$$

In total, we obtain

$$\|D^3_{\mu_1,\mu_2}\| \le \frac{C}{\sqrt{-\mu_1}} \left(1 + \ln\left(1 + \frac{\mu_2}{-\mu_1}\right)\right) \tag{5.237}$$

for some constant C.

We bound the Hilbert-Schmidt norm of $D^4_{\mu_1,\mu_2}$ as

$$\|D^4_{\mu_1,\mu_2}\|_{\mathrm{HS}} = \left(\int_{(-2,2)^2} \frac{\chi_{|p_1-q_1|^2>\mu_2}}{(p_1^2 + q_1^2 - \frac{\mu_1+\mu_2}{2})^2} \mathrm{d}p_1 \mathrm{d}q_1\right)^{1/2} \tag{5.238}$$

For $\mu_2 < 0$, we clearly have $\|D^4_{\mu_1,\mu_2}\|_{\mathrm{HS}} \le \|D^4_{\mu_1,0}\|_{\mathrm{HS}}$. For $\mu_2 \ge 0$ observe that the constraint $|p_1 - q_1|^2 > \mu_2$ implies $p_1^2 + q_1^2 > \frac{\mu_2}{2}$. Hence,

$$\|D^4_{\mu_1,\mu_2}\|_{\mathrm{HS}} \le \left(2\pi \int_{\sqrt{\frac{\mu_2}{2}}}^{\infty} \frac{r}{(r^2 - \frac{\mu_1+\mu_2}{2})^2} \mathrm{d}r\right)^{1/2} = \left(\frac{2\pi}{-\mu_1}\right)^{1/2}. \tag{5.239}$$

Case $\mu_1 > 0$**:** We are left with estimating $D^2_{\mu_1,\mu_2}$ in the case that $\mu_1 > 0$. First we sketch the location of the singularities of $D^2_{\mu_1,\mu_2}(p_1, q_1)$. On each of the diagonal lines in Figure 5.4, one of the two terms $|(p_1 + q_1)^2 - \mu_1|, |(p_1 - q_1)^2 - \mu_2|$ in the denominator of $D^2_{\mu_1,\mu_2}(p_1, q_1)$ vanishes. The function $D^2_{\mu_1,\mu_2}(p_1, q_1)$ thus has four singularities located at the crossings of the diagonal lines Figure 5.4. The coordinates of the singularities are $(p_1, q_1) \in \{(s_1, -s_2), (s_2, -s_1), (-s_1, s_2), (-s_2, s_1)\}$, where $s_1 = \frac{\sqrt{\mu_1}+\sqrt{\mu_2}}{2}$, $s_2 = \frac{\sqrt{\mu_2}-\sqrt{\mu_1}}{2}$. Note that $s_1^2 + s_2^2 = \frac{\mu_1+\mu_2}{2}$ and $s_1 s_2 = \frac{\mu_2-\mu_1}{4}$.

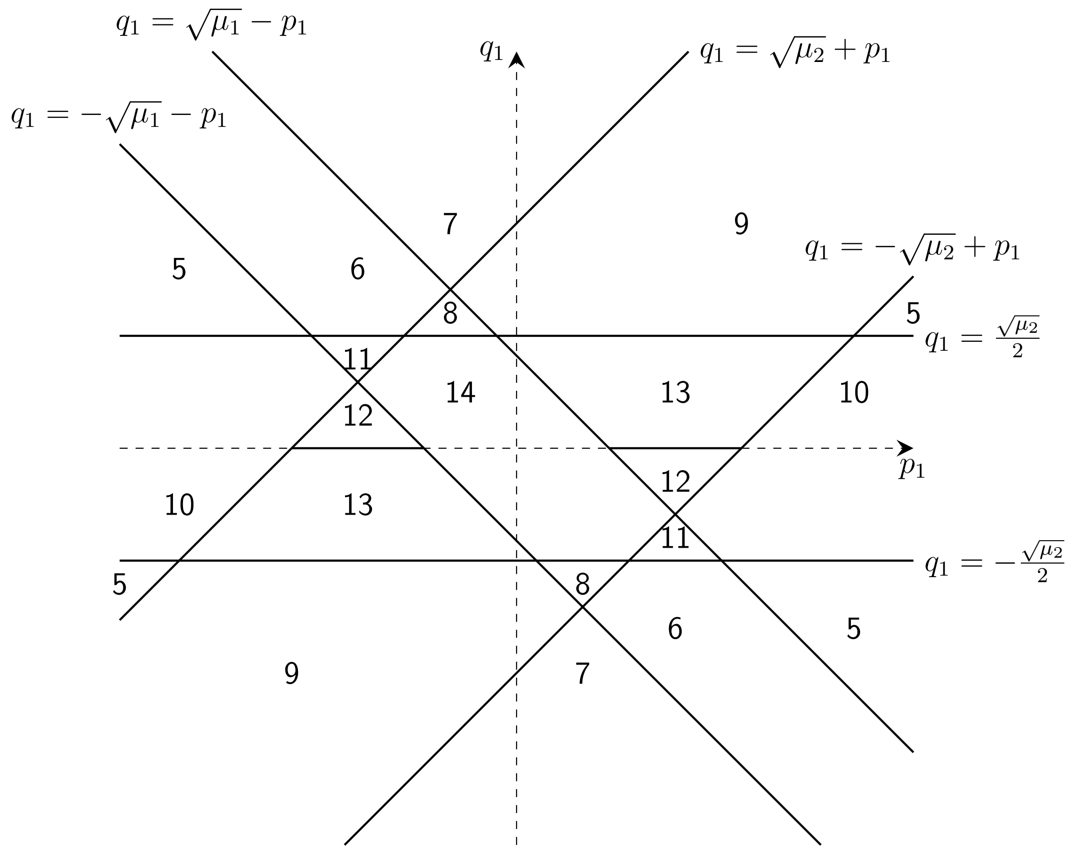

Figure 5.4: In the proof of Lemma 5.6.5, in the case $0 < \mu_1 \leq \mu_2$ we split the domain of p_1, q_1 into ten different regions. The solid lines indicate the boundaries between these regions.

To bound $\|D^2_{\mu_1,\mu_2}\|$, the idea is to perform a Schur test with test function $h(p_1) = \min\{||p_1| - s_1|^{1/2}, ||p_1| - s_2|^{1/2}\}$. Since the behavior of $D^2_{\mu_1,\mu_2}(p_1, q_1)$ strongly depends on whether $|p_1 + q_1| \gtrless \sqrt{\mu_1}, |p_1 - q_1| \gtrless \sqrt{\mu_2}$ and which singularity of $D^2_{\mu_1,\mu_2}$ is close to p_1, q_1, we distinguish the ten different regions sketched in Figure 5.4. For $5 \leq j \leq 14$, we define the operator $D^j_{\mu_1,\mu_2}$ to be localized in region j, $D^j_{\mu_1,\mu_2} = D^2_{\mu_1,\mu_2}\chi_j$. According to the Schur test,

$$\|D^j_{\mu_1,\mu_2}\| \leq \sup_{|p_1|<2} h(p_1)^{-1} \int_{-2}^{2} D^j_{\mu_1,\mu_2}(p_1, q_1) h(q_1) \mathrm{d}q_1. \tag{5.240}$$

The bounds on $\|D^j_{\mu_1,\mu_2}\|$ we obtain from the Schur test are listed in Table 5.2. In the following we prove all the bounds.

Operator	Upper bound	Proof
D^5	$\frac{16}{\mu_1^{1/2}}$	(5.241)-(5.243)
D^6	$\frac{6}{\mu_1^{1/2}}$	(5.244)-(5.249)
D^7	$\frac{(6+2\sqrt{2})^{1/2}}{\mu_1^{1/2}}$	(5.250)-(5.252)
D^8	$\frac{2^{1/2}4}{\mu_1^{1/2}}$	(5.253) -(5.257)
D^9	$\frac{8}{\mu_1^{1/2}}$	(5.258)-(5.260)
D^{10}	$\frac{4}{\mu_1^{1/2}}$	(5.261)-(5.264)
D^{11}	$\frac{4}{\mu_1^{1/2}}$	(5.265)- (5.268)
D^{12}	$\frac{2}{\mu_1^{1/2}}$	(5.269)- (5.271)
D^{13}	$\frac{4(\operatorname{artanh}(1/\sqrt{2})+\pi)}{\mu_1^{1/2}}$	(5.272)- (5.280)
D^{14}	$\frac{4(\sqrt{3}+1)}{\mu_1^{1/2}}$	(5.281)-(5.285)

Table 5.2: Overview of the estimates used in the proof of Lemma 5.6.5.

Region 5: By symmetry of the integrand under $(p_1, q_1) \to -(p_1, q_1)$ we have

$$
\begin{aligned}
\|D^5_{\mu_1,\mu_2}\| &\leq \sup_{-2<p_1<2} h(p_1) \int_{-2}^{2} \frac{\chi_5}{p_1^2+q_1^2-s_1^2-s_2^2} \frac{1}{h(q_1)} \mathrm{d}q_1 \\
&= \sup_{\sqrt{\mu_1}+\frac{\sqrt{\mu_2}}{2}<p_1<2} |p_1-s_1|^{1/2} \Bigg[\int_{\sqrt{\mu_1}-p_1}^{-\frac{\sqrt{\mu_2}}{2}} \frac{1}{p_1^2+q_1^2-s_1^2-s_2^2} \frac{1}{|q_1+s_1|^{1/2}} \mathrm{d}q_1 \\
&\quad + \int_{\sqrt{\mu_2}/2}^{p_1-\sqrt{\mu_2}} \frac{1}{p_1^2+q_1^2-s_1^2-s_2^2} \frac{1}{|q_1-s_1|^{1/2}} \mathrm{d}q_1 \Bigg] \\
&\leq 2 \sup_{\sqrt{\mu_1}+\frac{\sqrt{\mu_2}}{2}<p_1<2} |p_1-s_1|^{1/2} \int_{\sqrt{\mu_2}/2}^{p_1-\sqrt{\mu_1}} \frac{1}{p_1^2+q_1^2-s_1^2-s_2^2} \frac{1}{|q_1-s_1|^{1/2}} \mathrm{d}q_1 \\
&\leq 2 \sup_{\sqrt{\mu_1}+\frac{\sqrt{\mu_2}}{2}<p_1<2} \frac{|p_1-s_1|^{1/2}}{p_1^2+\frac{\mu_2}{4}-s_1^2-s_2^2} \int_{\sqrt{\mu_2}/2}^{p_1-\sqrt{\mu_1}} \frac{1}{|q_1-s_1|^{1/2}} \mathrm{d}q_1
\end{aligned} \tag{5.241}
$$

Note that $p_1^2+\frac{\mu_2}{4}-s_1^2-s_2^2 = p_1^2-\frac{\mu_1}{2}-\frac{\mu_2}{4} \geq \frac{\sqrt{\mu_2}}{2}(p_1-\sqrt{\frac{\mu_1}{2}+\frac{\mu_2}{4}})$. Carrying out the integration, (5.241) is bounded above by

$$
\frac{8}{\sqrt{\mu_2}} \sup_{\sqrt{\mu_1}+\frac{\sqrt{\mu_2}}{2}<p_1<2} \frac{|p_1-s_1|^{1/2}}{p_1-\sqrt{\frac{\mu_1}{2}+\frac{\mu_2}{4}}} \left(\left(\frac{\sqrt{\mu_1}}{2}\right)^{1/2} + \chi_{p_1>s_1+\sqrt{\mu_1}} |p_1-s_1-\sqrt{\mu_1}|^{1/2} \right) \tag{5.242}
$$

Note that $s_1 > \sqrt{\frac{\mu_1}{2}+\frac{\mu_2}{4}}$. Using that for $x \geq a \geq b$, $(x-a)/(x-b) \leq 1$ we bound (5.242) above by

$$
\frac{8}{\sqrt{\mu_2}} \left(\frac{\left(\frac{\sqrt{\mu_1}}{2}\right)^{1/2}}{|\sqrt{\mu_1}+\frac{\sqrt{\mu_2}}{2}-\sqrt{\frac{\mu_1}{2}+\frac{\mu_2}{4}}|^{1/2}} + 1 \right) \leq \frac{8}{\sqrt{\mu_2}} \left(\frac{\sqrt{\mu_1}+\frac{\sqrt{\mu_2}}{2}+\sqrt{\frac{\mu_1}{2}+\frac{\mu_2}{4}}}{\sqrt{\mu_1}+2\sqrt{\mu_2}} \right)^{1/2} + \frac{8}{\sqrt{\mu_1}} \leq \frac{16}{\sqrt{\mu_1}}. \tag{5.243}
$$

Region 6: By symmetry under $(p_1, q_1) \to -(p_1, q_1)$, we obtain

$$\begin{aligned} \|D^6_{\mu_1,\mu_2}\| &\leq \sup_{-2<p_1<2} h(p_1) \int_{-2}^{2} \frac{1}{2} \frac{\chi_6}{-p_1 q_1 - s_1 s_2} \frac{1}{h(q_1)} \mathrm{d}q_1 \\ &\leq \sup_{-2<p_1<-s_2} h(p_1) \int_{\max\{-\sqrt{\mu_1}-p_1, \frac{\sqrt{\mu_2}}{2}, \sqrt{\mu_2}+p_1\}}^{\min\{-p_1+\sqrt{\mu_1}, 2\}} \frac{1}{-p_1 q_1 - s_1 s_2} \frac{1}{|q_1 - s_1|^{1/2}} \mathrm{d}q_1 \end{aligned} \tag{5.244}$$

We split the integral into the sum of the integral over $q_1 > s_1$ and $q_1 < s_1$. For $p_1 < -s_2$ and $q_1 > s_1$ we have $-p_1 q_1 - s_1 s_2 > -(p_1 + s_2) s_1$. Hence,

$$\begin{aligned} &\sup_{-2<p_1<-s_2} h(p_1) \int_{s_1}^{\min\{-p_1+\sqrt{\mu_1}, 2\}} \frac{1}{-p_1 q_1 - s_1 s_2} \frac{1}{|q_1 - s_1|^{1/2}} \mathrm{d}q_1 \\ &\qquad \leq \sup_{-2<p_1<-s_2} \frac{1}{|p_1 + s_2|^{1/2} s_1} \int_{s_1}^{-p_1+\sqrt{\mu_1}} \frac{1}{|q_1 - s_1|^{1/2}} \mathrm{d}q_1 = \frac{2}{s_1} \leq \frac{2}{\sqrt{\mu_1}} \end{aligned} \tag{5.245}$$

The case $q_1 < s_1$ only occurs for $p_1 > -s_1 - \sqrt{\mu_1}$. For $-\frac{\sqrt{\mu_2}}{2} < p_1 < -s_2$ and $\sqrt{\mu_2} + p_1 < q_1 < s_1$ note that $-p_1 q_1 - s_1 s_2 \geq -p_1(\sqrt{\mu_2} + p_1) - s_1 s_2 = |p_1 + s_2|(p_1 + s_1) \geq |p_1 + s_2| \frac{\sqrt{\mu_1}}{2}$. Hence,

$$\begin{aligned} &\sup_{-\frac{\sqrt{\mu_2}}{2}<p_1<-s_2} h(p_1) \int_{\sqrt{\mu_2}+p_1}^{s_1} \frac{1}{-p_1 q_1 - s_1 s_2} \frac{1}{|q_1 - s_1|^{1/2}} \mathrm{d}q_1 \\ &\qquad \leq \sup_{-\frac{\sqrt{\mu_2}}{2}<p_1<-s_2} \frac{2}{\sqrt{\mu_1} |p_1 + s_2|^{1/2}} \int_{\sqrt{\mu_2}+p_1}^{s_1} \frac{1}{|q_1 - s_1|^{1/2}} \mathrm{d}q_1 = \frac{4}{\sqrt{\mu_1}} \end{aligned} \tag{5.246}$$

For $-s_1 - \frac{\sqrt{\mu_1}}{2} < p_1 < -\frac{\sqrt{\mu_2}}{2}$ and $\frac{\sqrt{\mu_2}}{2} < q_1 < s_1$, we have $-p_1 q_1 - s_1 s_2 \geq \frac{\mu_2}{4} - s_1 s_2 = \frac{\mu_1}{4}$. Therefore,

$$\begin{aligned} &\sup_{-s_1-\frac{\sqrt{\mu_1}}{2}<p_1<-\frac{\sqrt{\mu_2}}{2}} h(p_1) \int_{\frac{\sqrt{\mu_2}}{2}}^{s_1} \frac{1}{-p_1 q_1 - s_1 s_2} \frac{1}{|q_1 - s_1|^{1/2}} \mathrm{d}q_1 \\ &\leq \sup_{-s_1-\frac{\sqrt{\mu_1}}{2}<p_1<-\frac{\sqrt{\mu_2}}{2}} \frac{4|p_1 + s_1|^{1/2}}{\mu_1} \int_{\frac{\sqrt{\mu_2}}{2}}^{s_1} \frac{1}{|q_1 - s_1|^{1/2}} \mathrm{d}q_1 \leq \frac{8\left(\frac{\sqrt{\mu_1}}{2}\right)^{1/2}}{\mu_1} \left(\frac{\sqrt{\mu_1}}{2}\right)^{1/2} = \frac{4}{\mu_1^{1/2}} \end{aligned} \tag{5.247}$$

For $-s_1 - \sqrt{\mu_1} < p_1 < -s_1 - \frac{\sqrt{\mu_1}}{2}$ and $-p_1 - \sqrt{\mu_1} < q_1 < s_1$, we have $-p_1 q_1 - s_1 s_2 \geq p_1(p_1 + \sqrt{\mu_1}) - s_1 s_2 = -(p_1 + s_1)(s_2 - p_1)$. Hence,

$$\begin{aligned} &\sup_{-s_1-\sqrt{\mu_1}<p_1<-s_1-\frac{\sqrt{\mu_1}}{2}} h(p_1) \int_{-p_1-\sqrt{\mu_1}}^{s_1} \frac{1}{-p_1 q_1 - s_1 s_2} \frac{1}{|q_1 - s_1|^{1/2}} \mathrm{d}q_1 \\ &\qquad \leq \sup_{-s_1-\sqrt{\mu_1}<p_1<-s_1-\frac{\sqrt{\mu_1}}{2}} \frac{2|p_1 + \sqrt{\mu_1} + s_1|^{1/2}}{|p_1 + s_1|^{1/2}(s_2 - p_1)} = \frac{2}{s_2 + s_1 + \frac{\sqrt{\mu_1}}{2}} \leq \frac{4}{\sqrt{\mu_1}} \end{aligned} \tag{5.248}$$

In total, summing the contributions from $q_1 > s_1$ and $q_1 < s_1$ gives

$$\|D^6_{\mu_1,\mu_2}\| \leq \frac{6}{\sqrt{\mu_1}} \tag{5.249}$$

Region 7: By symmetry of the two components of region 7 we have

$$\begin{aligned}\|D^7_{\mu_1,\mu_2}\| &\leq \sup_{-2<p_1<2} h(p_1)\int_{-2}^2 \frac{\chi_7}{p_1^2+q_1^2-s_1^2-s_2^2}\frac{1}{h(q_1)}\mathrm{d}q_1\\ &\leq 2\sup_{-2<p_1<2}||p_1|-s_2|^{1/2}\int_{\max\{\sqrt{\mu_1}-p_1,\sqrt{\mu_2}+p_1\}}^2 \frac{1}{p_1^2+q_1^2-s_1^2-s_2^2}\frac{1}{|q_1-s_1|^{1/2}}\mathrm{d}q_1\end{aligned} \tag{5.250}$$

For $|p_1|>s_2$, $q_1>s_1$ we observe $p_1^2+q_1^2-s_1^2-s_2^2 \geq (q_1+s_1)(q_1-s_1)\geq 2s_1(q_1-s_1)$. Therefore,

$$\begin{aligned}&\sup_{s_2<|p_1|<2}||p_1|-s_2|^{1/2}\int_{\max\{\sqrt{\mu_1}-p_1,\sqrt{\mu_2}+p_1\}}^2 \frac{1}{p_1^2+q_1^2-s_1^2-s_2^2}\frac{1}{|q_1-s_1|^{1/2}}\mathrm{d}q_1\\ &\quad\leq \sup_{s_2<|p_1|<2}\frac{||p_1|-s_2|^{1/2}}{2s_1}\int_{\max\{\sqrt{\mu_1}-p_1,\sqrt{\mu_2}+p_1\}}^\infty \frac{1}{(q_1-s_1)^{3/2}}\mathrm{d}q_1\\ &\quad= \sup_{s_2<|p_1|<2}\frac{||p_1|-s_2|^{1/2}}{s_1(\max\{\sqrt{\mu_1}-p_1,\sqrt{\mu_2}+p_1\}-s_1)^{1/2}} = \frac{1}{s_1}\leq\frac{1}{\sqrt{\mu_1}}.\end{aligned} \tag{5.251}$$

For $|p_1|<s_2$, $q_1>s_1$ we have $(p_1^2+q_1^2-s_1^2-s_2^2)(q_1-s_1)^{1/2}\geq (q_1+\sqrt{s_1^2+s_2^2-p_1^2})(q_1-\sqrt{s_1^2+s_2^2-p_1^2})^{3/2}\geq 2s_1(q_1-\sqrt{s_1^2+s_2^2-p_1^2})^{3/2}$. Hence,

$$\begin{aligned}&\sup_{|p_1|<s_2}||p_1|-s_2|^{1/2}\int_{\max\{\sqrt{\mu_1}-p_1,\sqrt{\mu_2}+p_1\}}^2 \frac{1}{p_1^2+q_1^2-s_1^2-s_2^2}\frac{1}{|q_1-s_1|^{1/2}}\mathrm{d}q_1\\ &\quad\leq \sup_{|p_1|<s_2}\frac{|p_1+s_2|^{1/2}}{2s_1}\int_{\sqrt{\mu_2}+p_1}^\infty \frac{1}{(q_1-\sqrt{s_1^2+s_2^2-p_1^2})^{3/2}}\mathrm{d}q_1\\ &\quad= \sup_{|p_1|<s_2}\frac{|p_1+s_2|^{1/2}}{s_1}\frac{1}{(\sqrt{\mu_2}+p_1-\sqrt{s_1^2+s_2^2-p_1^2})^{1/2}}\\ &\quad= \sup_{|p_1|<s_2}\frac{1}{s_1}\frac{|p_1+s_2|^{1/2}(\sqrt{\mu_2}+p_1+\sqrt{s_1^2+s_2^2-p_1^2})^{1/2}}{(p_1+s_1)^{1/2}(p_1+s_2)^{1/2}}\\ &\quad= \sup_{|p_1|<s_2}\frac{1}{s_1}\frac{(\sqrt{\mu_2}+p_1+\sqrt{s_1^2+s_2^2-p_1^2})^{1/2}}{(p_1+s_1)^{1/2}}\leq\frac{(\frac{3}{2}+\sqrt{2})^{1/2}s_1^{1/2}}{s_1\mu_1^{1/4}}\leq\frac{(\frac{3}{2}+\sqrt{2})^{1/2}}{\mu_1^{1/2}}\end{aligned} \tag{5.252}$$

In total, we obtain $\|D^7\|\leq\frac{(6+2\sqrt{2})^{1/2}}{\sqrt{\mu_1}}$.

Region 8: Taking the supremum separately over the two symmetric components of region 8, we have

$$\begin{aligned}\|D^8_{\mu_1,\mu_2}\| &\leq \sup_{-2<p_1<2} h(p_1)\int_{-2}^2 \frac{\chi_8}{s_1^2+s_2^2-p_1^2-q_1^2}\frac{1}{h(q_1)}\mathrm{d}q_1\\ &\leq 2\sup_{-\frac{\sqrt{\mu_2}}{2}<p_1<\sqrt{\mu_1}-\frac{\sqrt{\mu_2}}{2}} h(p_1)\int_{\sqrt{\mu_2}/2}^{\min\{\sqrt{\mu_2}+p_1,\sqrt{\mu_1}-p_1\}} \frac{1}{s_1^2+s_2^2-p_1^2-q_1^2}\frac{1}{|s_1-q_1|^{1/2}}\mathrm{d}q_1\\ &\leq 2\sup_{-\frac{\sqrt{\mu_2}}{2}<p_1<\sqrt{\mu_1}-\frac{\sqrt{\mu_2}}{2}} \frac{h(p_1)}{\sqrt{\mu_2}}\int_{\sqrt{\mu_2}/2}^{\min\{\sqrt{\mu_2}+p_1,\sqrt{\mu_1}-p_1\}} \frac{1}{\sqrt{s_1^2+s_2^2-p_1^2}-q_1}\frac{1}{|s_1-q_1|^{1/2}}\mathrm{d}q_1,\end{aligned} \tag{5.253}$$

since $\sqrt{s_1^2+s_2^2-p_1^2}+q_1 > \sqrt{\frac{\mu_1}{2}+\frac{\mu_2}{2}-\frac{\mu_2}{4}}+\frac{\sqrt{\mu_2}}{2} \geq \sqrt{\mu_2}$. For $|p_1| > s_2$ we have $s_1 > \sqrt{s_1^2+s_2^2-p_1^2}$, whereas for $|p_1| < s_2$, $s_1 < \sqrt{s_1^2+s_2^2-p_1^2}$. For $p_1 < -s_2$ we obtain

$$\begin{aligned}
&\sup_{-\frac{\sqrt{\mu_2}}{2}<p_1<-s_2} \frac{2h(p_1)}{\sqrt{\mu_2}} \int_{\sqrt{\mu_2}/2}^{\min\{\sqrt{\mu_2}+p_1,\sqrt{\mu_1}-p_1\}} \frac{1}{\sqrt{s_1^2+s_2^2-p_1^2}-q_1} \frac{1}{|s_1-q_1|^{1/2}} \mathrm{d}q_1 \\
&\leq \sup_{-\frac{\sqrt{\mu_2}}{2}<p_1<-s_2} \frac{2|p_1+s_2|^{1/2}}{\sqrt{\mu_2}} \int_{-\infty}^{\sqrt{\mu_2}+p_1} \frac{1}{(\sqrt{s_1^2+s_2^2-p_1^2}-q_1)^{3/2}} \mathrm{d}q_1 \\
&= \sup_{-\frac{\sqrt{\mu_2}}{2}<p_1<-s_2} \frac{4|p_1+s_2|^{1/2}}{\sqrt{\mu_2}(\sqrt{s_1^2+s_2^2-p_1^2}-\sqrt{\mu_2}-p_1)^{1/2}} \\
&\leq \sup_{-\frac{\sqrt{\mu_2}}{2}<p_1<-s_2} \frac{4(\sqrt{s_1^2+s_2^2-p_1^2}+\sqrt{\mu_2}+p_1)^{1/2}}{2^{1/2}\sqrt{\mu_2}(p_1+s_1)^{1/2}} \leq \frac{2^{1/2}4 s_1^{1/2}}{\sqrt{\mu_2}\mu_1^{1/4}} \leq \frac{2^{1/2}4}{\mu_1^{1/2}}
\end{aligned} \tag{5.254}$$

Similarly, for $p_1 > s_2$ (which only occurs if $2\sqrt{\mu_2} < 3\sqrt{\mu_1}$),

$$\begin{aligned}
&\sup_{s_2<p_1<\sqrt{\mu_1}-\frac{\sqrt{\mu_2}}{2}} \frac{2h(p_1)}{\sqrt{\mu_2}} \int_{\sqrt{\mu_2}/2}^{\min\{\sqrt{\mu_2}+p_1,\sqrt{\mu_1}-p_1\}} \frac{1}{\sqrt{s_1^2+s_2^2-p_1^2}-q_1} \frac{1}{|s_1-q_1|^{1/2}} \mathrm{d}q_1 \\
&\leq \sup_{s_2<p_1<\frac{\sqrt{\mu_2}}{2}} \frac{2|p_1-s_2|^{1/2}}{\sqrt{\mu_2}} \int_{-\infty}^{\sqrt{\mu_2}-p_1} \frac{1}{(\sqrt{s_1^2+s_2^2-p_1^2}-q_1)^{3/2}} \mathrm{d}q_1 \leq \frac{2^{1/2}4}{\mu_1^{1/2}},
\end{aligned} \tag{5.255}$$

by (5.254). For $|p_1| < s_2$,

$$\begin{aligned}
&\sup_{-s_2<p_1<s_2} \frac{2h(p_1)}{\sqrt{\mu_2}} \int_{\sqrt{\mu_2}/2}^{\sqrt{\mu_1}-p_1} \frac{1}{\sqrt{s_1^2+s_2^2-p_1^2}-q_1} \frac{1}{|s_1-q_1|^{1/2}} \mathrm{d}q_1 \\
&\leq \sup_{-s_2<p_1<s_2} \frac{2||p_1|-s_2|^{1/2}}{\sqrt{\mu_2}} \int_{-\infty}^{\sqrt{\mu_1}-p_1} \frac{1}{|s_1-q_1|^{3/2}} \mathrm{d}q_1 \\
&= \sup_{-s_2<p_1<s_2} \frac{4||p_1|-s_2|^{1/2}}{\sqrt{\mu_2}|s_2+p_1|^{1/2}} = \frac{4}{\sqrt{\mu_2}}
\end{aligned} \tag{5.256}$$

In total, we have

$$\|D^8_{\mu_1,\mu_2}\| \leq \frac{2^{1/2}4}{\mu_1^{1/2}}. \tag{5.257}$$

Region 9: By taking the supremum separately over the two components of region 9 and using the symmetry in $(p_1, q_1) \to -(p_1, q_1)$, we obtain

$$\begin{aligned}
\|D^9_{\mu_1,\mu_2}\| &\leq \sup_{-2<p_1<2} h(p_1) \int_{-2}^{2} \frac{1}{2} \frac{\chi_9}{p_1 q_1 + s_1 s_2} \frac{1}{h(q_1)} \mathrm{d}q_1 \\
&\leq \sup_{-s_2<p_1<2} h(p_1) \int_{\max\{\sqrt{\mu_1}-p_1,\sqrt{\mu_2}/2,p_1-\sqrt{\mu_2}\}}^{\min\{p_1+\sqrt{\mu_2},2\}} \frac{1}{p_1 q_1 + s_1 s_2} \frac{1}{|q_1-s_1|^{1/2}} \mathrm{d}q_1
\end{aligned} \tag{5.258}$$

For $p_1 > -s_2$ and $\max\{\sqrt{\mu_1} - p_1, \frac{\sqrt{\mu_2}}{2}\} < q_1 < \sqrt{\mu_2} + p_1$ note that

$$p_1 q_1 + s_1 s_2 \geq \begin{cases} p_1(\sqrt{\mu_2} + p_1) + s_1 s_2 = (p_1 + s_2)(p_1 + s_1) & \text{if } p_1 \leq 0 \\ p_1(\sqrt{\mu_1} - p_1) + s_1 s_2 = (p_1 + s_2)(s_1 - p_1) & \text{if } \sqrt{\mu_1} - \frac{\sqrt{\mu_2}}{2} \geq p_1 \geq 0 \\ p_1 \frac{\sqrt{\mu_2}}{2} + s_1 s_2 & \text{if } p_1 \geq \max\{\sqrt{\mu_1} - \frac{\sqrt{\mu_2}}{2}, 0\} \end{cases} \geq \frac{\sqrt{\mu_1}}{2}(p_1 + s_2) \quad (5.259)$$

Hence,

$$\|D^9_{\mu_1,\mu_2}\| \leq \sup_{-s_2 < p_1 < 2} \frac{2}{\sqrt{\mu_1}(p_1 + s_2)^{1/2}} \int_{\sqrt{\mu_1} - p_1}^{p_1 + \sqrt{\mu_2}} \frac{1}{|q_1 - s_1|^{1/2}} \mathrm{d}q_1 = \frac{8}{\sqrt{\mu_1}} \quad (5.260)$$

Region 10: By symmetry in p_1, we have

$$\begin{aligned} \|D^{10}_{\mu_1,\mu_2}\| &\leq \sup_{-2<p_1<2} h(p_1) \int_{-2}^{2} \frac{\chi_{10}}{p_1^2 + q_1^2 - s_1^2 - s_2^2} \frac{1}{h(q_1)} \mathrm{d}q_1 \\ &= \sup_{s_1 < p_1 < 2} |p_1 - s_1|^{1/2} \int_{\max\{\sqrt{\mu_1} - p_1, -\frac{\sqrt{\mu_2}}{2}\}}^{\min\{p_1 - \sqrt{\mu_2}, \frac{\sqrt{\mu_2}}{2}\}} \frac{1}{p_1^2 + q_1^2 - s_1^2 - s_2^2} \frac{1}{||q_1| - s_2|^{1/2}} \mathrm{d}q_1 \quad (5.261) \end{aligned}$$

If we mirror the part of region 10 with $p_1 > 0, q_1 < 0$ along $q_1 = 0$, its image contains the part of region 10 with $p_1 > 0, q_1 > 0$. Since the integrand is symmetric in q_1, we can thus bound

$$\|D^{10}_{\mu_1,\mu_2}\| \leq \sup_{s_1 < p_1 < 2} 2|p_1 - s_1|^{1/2} \int_{\max\{\sqrt{\mu_2} - p_1, 0\}}^{\min\{p_1 - \sqrt{\mu_1}, \frac{\sqrt{\mu_2}}{2}\}} \frac{1}{p_1^2 + q_1^2 - s_1^2 - s_2^2} \frac{1}{|q_1 - s_2|^{1/2}} \mathrm{d}q_1 \quad (5.262)$$

Note that for $q_1 \geq \sqrt{\mu_2} - p_1$, $p_1 > s_1$ we have

$$\begin{aligned} p_1^2 + q_1^2 - s_1^2 - s_2^2 &= (p_1 - s_1)^2 + (q_1 - s_2)^2 + 2s_1(p_1 - s_1) + 2s_2(q_1 - s_2) \\ &\geq 2s_1(p_1 - s_1) + 2s_2(s_1 - p_1) = 2\sqrt{\mu_1}(p_1 - s_1). \quad (5.263) \end{aligned}$$

Therefore,

$$\|D^{10}_{\mu_1,\mu_2}\| \leq \sup_{s_1 < p_1 < 2} \frac{1}{\sqrt{\mu_1}|p_1 - s_1|^{1/2}} \int_{\sqrt{\mu_2} - p_1}^{p_1 - \sqrt{\mu_1}} \frac{1}{|q_1 - s_2|^{1/2}} \mathrm{d}q_1 = \frac{4}{\sqrt{\mu_1}}. \quad (5.264)$$

Region 11: By symmetry in p_1, we obtain

$$\begin{aligned} \|D^{11}_{\mu_1,\mu_2}\| &\leq \sup_{-2<p_1<2} h(p_1) \int_{-2}^{2} \frac{1}{2} \frac{\chi_{11}}{-p_1 q_1 - s_1 s_2} \frac{1}{h(q_1)} \mathrm{d}q_1 \\ &= \sup_{-\mu_1 - \frac{\sqrt{\mu_2}}{2} < p_1 < -\frac{\sqrt{\mu_2}}{2}} \frac{1}{2}|p_1 + s_1|^{1/2} \int_{\max\{-\sqrt{\mu_1} - p_1, \sqrt{\mu_2} + p_1\}}^{\frac{\sqrt{\mu_2}}{2}} \frac{1}{-p_1 q_1 - s_1 s_2} \frac{1}{|q_1 - s_2|^{1/2}} \mathrm{d}q_1 \end{aligned} \quad (5.265)$$

For $p_1 < -s_1$ we have $-p_1 q_1 - s_1 s_2 > s_1(q_1 - s_2)$. Hence,

$$\begin{aligned} &\sup_{-\mu_1 - \frac{\sqrt{\mu_2}}{2} < p_1 < -s_1} \frac{1}{2}|p_1 + s_1|^{1/2} \int_{-\sqrt{\mu_1} - p_1}^{\frac{\sqrt{\mu_2}}{2}} \frac{1}{-p_1 q_1 - s_1 s_2} \frac{1}{|q_1 - s_2|^{1/2}} \mathrm{d}q_1 \\ &\leq \sup_{-\mu_1 - \frac{\sqrt{\mu_2}}{2} < p_1 < -s_1} \frac{|p_1 + s_1|^{1/2}}{2s_1} \int_{-\sqrt{\mu_1} - p_1}^{\infty} \frac{1}{|q_1 - s_2|^{3/2}} \mathrm{d}q_1 = \frac{1}{s_1} \leq \frac{1}{\sqrt{\mu_1}} \quad (5.266) \end{aligned}$$

For $p_1 > -s_1$, we carry out the integration

$$\begin{aligned}&\sup_{-s_1<p_1<-\frac{\sqrt{\mu_2}}{2}} \frac{1}{2}|p_1+s_1|^{1/2}\int_{\sqrt{\mu_2}+p_1}^{\frac{\sqrt{\mu_2}}{2}} \frac{1}{-p_1q_1-s_1s_2}\frac{1}{|q_1-s_2|^{1/2}}\mathrm{d}q_1\\ &\quad\leq \sup_{-s_1<p_1<-\frac{\sqrt{\mu_2}}{2}} \frac{1}{|p_1|^{1/2}s_2^{1/2}}\operatorname{artanh}\left(\frac{s_2^{1/2}}{|p_1|^{1/2}}\right) = \frac{2^{1/2}}{\mu_2^{1/4}s_2^{1/2}}\operatorname{artanh}\left(\frac{2^{1/2}s_2^{1/2}}{\mu_2^{1/4}}\right) \end{aligned} \tag{5.267}$$

With $\operatorname{artanh}(x) \leq \frac{x}{1-x}$, we obtain

$$\frac{2^{1/2}}{\mu_2^{1/4}s_2^{1/2}}\operatorname{artanh}\left(\frac{2^{1/2}s_2^{1/2}}{\mu_2^{1/4}}\right) \leq \frac{2^{1/2}}{\mu_2^{1/4}s_2^{1/2}}\frac{s_2^{1/2}}{\frac{\mu_2^{1/4}}{2^{1/2}}-s_2^{1/2}} = \frac{2^{1/2}}{\mu_2^{1/4}}\frac{\frac{\mu_2^{1/4}}{2^{1/2}}+s_2^{1/2}}{\frac{\mu_1^{1/2}}{2}} \leq \frac{4}{\mu_1^{1/2}} \tag{5.268}$$

Therefore, $\|D^{11}_{\mu_1,\mu_2}\| \leq \frac{4}{\mu_1^{1/2}}$.

Region 12: By symmetry in p_1, we obtain

$$\begin{aligned}\|D^{12}_{\mu_1,\mu_2}\| &\leq \sup_{-2<p_1<2} h(p_1)\int_{-2}^{2}\frac{1}{2}\frac{\chi_{12}}{p_1q_1+s_1s_2}\frac{1}{h(q_1)}\mathrm{d}q_1\\ &= \sup_{-\sqrt{\mu_2}<p_1<-\sqrt{\mu_1}} \frac{1}{2}h(p_1)\int_0^{\min\{p_1+\sqrt{\mu_2},-\sqrt{\mu_1}-p_1\}} \frac{1}{p_1q_1+s_1s_2}\frac{1}{|s_2-q_1|^{1/2}}\mathrm{d}q_1 \end{aligned} \tag{5.269}$$

For $p_1 \geq -s_1$ note that $p_1q_1+s_1s_2 \geq s_1(s_2-q_1) \geq \frac{\sqrt{\mu_1}}{2}(s_2-q_2)$. For $p_1 \leq -s_1$ and $q_1 < p_1+\sqrt{\mu_2}$ observe that

$$\begin{aligned}&p_1q_1+s_1s_2 = (-p_1-s_1)(s_2-q_1)+s_1(s_2-q_1)+s_2(p_1+s_1)\\ &\geq \frac{\sqrt{\mu_1}}{2}(s_2-q_1)+\frac{\sqrt{\mu_2}}{2}(s_2-q_1)+s_2(q_1-\sqrt{\mu_2}+s_1) = \frac{\sqrt{\mu_1}}{2}(s_2-q_1)+\frac{\sqrt{\mu_2}}{2}(s_2-q_1)-s_2(s_2-q_1)\\ &\geq \frac{\sqrt{\mu_1}}{2}(s_2-q_1) \end{aligned} \tag{5.270}$$

Therefore,

$$\|D^{12}_{\mu_1,\mu_2}\| \leq \sup_{-\sqrt{\mu_2}<p_1<-\sqrt{\mu_1}} \frac{|p_1+s_1|^{1/2}}{\sqrt{\mu_1}}\int_{-\infty}^{\min\{p_1+\sqrt{\mu_2},-\sqrt{\mu_1}-p_1\}} \frac{1}{|s_2-q_1|^{3/2}}\mathrm{d}q_1 = \frac{2}{\sqrt{\mu_1}} \tag{5.271}$$

Region 13: By symmetry under $(p_1,q_1) \to -(p_1,q_1)$, we obtain

$$\begin{aligned}\|D^{13}_{\mu_1,\mu_2}\| &\leq \sup_{-2<p_1<2} h(p_1)\int_{-2}^{2}\frac{1}{2}\frac{\chi_{13}}{p_1q_1+s_1s_2}\frac{1}{h(q_1)}\mathrm{d}q_1\\ &= \sup_{-\frac{\sqrt{\mu_2}}{2}+\sqrt{\mu_1}<p_1<\frac{3\sqrt{\mu_2}}{2}} h(p_1)\int_{\max\{\sqrt{\mu_1}-p_1,0,-\sqrt{\mu_2}+p_1\}}^{\frac{\sqrt{\mu_2}}{2}} \frac{1}{p_1q_1+s_1s_2}\frac{1}{|s_2-q_1|^{1/2}}\mathrm{d}q_1 \end{aligned} \tag{5.272}$$

For $p_1 > \sqrt{\mu_1}, q_1 > 0$, we have $p_1 q_1 + s_1 s_2 \geq \sqrt{\mu_1}(q_1 + s_2)$. Therefore,

$$\sup_{\sqrt{\mu_1}<p_1<\sqrt{\mu_2}+s_2} h(p_1) \int_{\max\{\sqrt{\mu_1}-p_1,0,-\sqrt{\mu_2}+p_1\}}^{\frac{\sqrt{\mu_2}}{2}} \frac{1}{p_1 q_1 + s_1 s_2} \frac{1}{|s_2 - q_1|^{1/2}} \mathrm{d}q_1$$
$$\leq \sup_{\sqrt{\mu_1}<p_1<\sqrt{\mu_2}+s_2} \frac{|p_1 - s_1|^{1/2}}{\sqrt{\mu_1}} \int_0^\infty \frac{1}{q_1 + s_2} \frac{1}{|s_2 - q_1|^{1/2}} \mathrm{d}q_1$$
$$= \frac{2^{1/2}}{\sqrt{\mu_1}} \int_0^\infty \frac{1}{q_1+1} \frac{1}{|1-q_1|^{1/2}} \mathrm{d}q_1 \leq \frac{2^{1/2}}{\sqrt{\mu_1}} \left[\int_0^2 \frac{1}{|1-q_1|^{1/2}} \mathrm{d}q_1 + \int_2^\infty \frac{1}{|q_1 - 1|^{3/2}} \mathrm{d}q_1 \right] = \frac{2^{1/2} 6}{\sqrt{\mu_1}} \tag{5.273}$$

and

$$\sup_{\sqrt{\mu_2}+s_2<p_1<\frac{3\sqrt{\mu_2}}{2}} h(p_1) \int_{\max\{\sqrt{\mu_1}-p_1,0,-\sqrt{\mu_2}+p_1\}}^{\frac{\sqrt{\mu_2}}{2}} \frac{1}{p_1 q_1 + s_1 s_2} \frac{1}{|s_2 - q_1|^{1/2}} \mathrm{d}q_1$$
$$\leq \sup_{\sqrt{\mu_2}+s_2<p_1<\frac{3\sqrt{\mu_2}}{2}} \frac{|p_1 - s_1|^{1/2}}{\sqrt{\mu_1}} \int_{-\sqrt{\mu_2}+p_1}^\infty \frac{1}{q_1 + s_2} \frac{1}{|q_1 - s_2|^{1/2}} \mathrm{d}q_1$$
$$= \sup_{2s_2<x<\mu_2-\frac{\sqrt{\mu_1}}{2}} \frac{|x|^{1/2}}{\sqrt{\mu_1}} \int_x^\infty \frac{1}{y} \frac{1}{|y-2s_2|^{1/2}} \mathrm{d}y = \sup_{2s_2<x<\mu_2-\frac{\sqrt{\mu_1}}{2}} \frac{1}{\sqrt{\mu_1}} \int_1^\infty \frac{1}{y} \frac{1}{|y - \frac{2s_2}{x}|^{1/2}} \mathrm{d}y$$
$$= \frac{1}{\sqrt{\mu_1}} \int_1^\infty \frac{1}{y} \frac{1}{|y-1|^{1/2}} \mathrm{d}y \leq \frac{1}{\sqrt{\mu_1}} \left[\int_1^2 \frac{1}{|y-1|^{1/2}} \mathrm{d}y + \int_2^\infty \frac{1}{|y-1|^{3/2}} \mathrm{d}y \right] = \frac{4}{\sqrt{\mu_1}}, \tag{5.274}$$

where we substituted $x = p_1 - s_1$ and $y = q_1 + s_2$. Next, we consider the case $p_1 < \frac{\sqrt{\mu_1}}{2}$. For $\frac{\sqrt{\mu_2}}{2} \geq q_1 \geq \sqrt{\mu_1} - p_1$ and $-s_2 < p_1 < \frac{\sqrt{\mu_1}}{2}$ we have

$$p_1 q_1 + s_1 s_2 \geq \begin{cases} \frac{\sqrt{\mu_1}}{2}(p_1 + s_2) & \text{if } p_1 > 0 \\ (s_1 - q_1)(p_1 + s_2) - p_1(s_1 - q_1) + q_1(p_1 + s_2) & \text{if } p_1 < 0 \end{cases}$$
$$\geq \begin{cases} \frac{\sqrt{\mu_1}}{2}(p_1 + s_2) & \text{if } p_1 > 0 \\ (s_1 - q_1)(p_1 + s_2) & \text{if } p_1 < 0 \end{cases} \geq \frac{\sqrt{\mu_1}}{2}(p_1 + s_2) \tag{5.275}$$

Therefore,

$$\sup_{-\frac{\sqrt{\mu_2}}{2}+\sqrt{\mu_1}<p_1<\frac{\sqrt{\mu_1}}{2}} h(p_1) \int_{\max\{\sqrt{\mu_1}-p_1,0,-\sqrt{\mu_2}+p_1\}}^{\frac{\sqrt{\mu_2}}{2}} \frac{1}{p_1 q_1 + s_1 s_2} \frac{1}{|s_2 - q_1|^{1/2}} \mathrm{d}q_1$$
$$\leq \sup_{-\frac{\sqrt{\mu_2}}{2}+\sqrt{\mu_1}<p_1<\frac{\sqrt{\mu_1}}{2}} \frac{2h(p_1)}{\sqrt{\mu_1}(p_1 + s_2)} \int_{\sqrt{\mu_1}-p_1}^{\frac{\sqrt{\mu_2}}{2}} \frac{1}{|s_2 - q_1|^{1/2}} \mathrm{d}q_1$$
$$\leq \sup_{-\frac{\sqrt{\mu_2}}{2}+\sqrt{\mu_1}<p_1<\frac{\sqrt{\mu_1}}{2}} \frac{4}{\sqrt{\mu_1}(p_1 + s_2)^{1/2}} \begin{cases} \left(\frac{\sqrt{\mu_1}}{2}\right)^{1/2} & \text{if } \sqrt{\mu_1} - p_1 > s_2 \\ \left(\frac{\sqrt{\mu_1}}{2}\right)^{1/2} + (s_2 - \sqrt{\mu_1} + p_1)^{1/2} & \text{if } \sqrt{\mu_1} - p_1 < s_2 \end{cases} \tag{5.276}$$

Note that $\sup_{-\frac{\sqrt{\mu_2}}{2}+\sqrt{\mu_1}<p_1<\frac{\sqrt{\mu_1}}{2}} (p_1 + s_2)^{-1/2} \left(\frac{\sqrt{\mu_1}}{2}\right)^{1/2} = 1$ and that for $p_1 > \sqrt{\mu_1} - s_2$ we have $\left|\frac{s_2 - \sqrt{\mu_1} + p_1}{p_1 + s_2}\right| \leq 1$. One can hence bound (5.276) above by $\frac{8}{\sqrt{\mu_1}}$.

For $q_1 \geq 0$ and $p_1 > \frac{\sqrt{\mu_1}}{2}$ we have $p_1 q_1 + s_1 s_2 \geq \frac{\sqrt{\mu_1}}{2}(q_1 + s_2)$. Therefore,

$$\begin{aligned}
&\sup_{\frac{\sqrt{\mu_1}}{2}<p_1<\sqrt{\mu_1}} h(p_1) \int_{\max\{\sqrt{\mu_1}-p_1,0,-\sqrt{\mu_2}+p_1\}}^{\frac{\sqrt{\mu_2}}{2}} \frac{1}{p_1 q_1 + s_1 s_2} \frac{1}{|s_2 - q_1|^{1/2}} \mathrm{d}q_1 \\
&\quad\leq \sup_{\frac{\sqrt{\mu_1}}{2}<p_1<\sqrt{\mu_1}} \frac{2h(p_1)}{\sqrt{\mu_1}} \int_{\sqrt{\mu_1}-p_1}^{\infty} \frac{1}{(q_1+s_2)|s_2-q_1|^{1/2}} \mathrm{d}q_1 \\
&\quad= \sup_{\frac{\sqrt{\mu_1}}{2}<p_1<\sqrt{\mu_1}} \frac{4h(p_1)}{\sqrt{\mu_1}\sqrt{2s_2}} \begin{cases} \operatorname{artanh}\left(\sqrt{\frac{s_2-\sqrt{\mu_1}+p_1}{2s_2}}\right) + \pi & \text{if } s_2 > \sqrt{\mu_1} - p_1 \\ \arctan\left(\sqrt{\frac{2s_2}{\sqrt{\mu_1}-p_1-s_2}}\right) & \text{if } s_2 < \sqrt{\mu_1} - p_1 \end{cases}
\end{aligned} \tag{5.277}$$

We estimate the two cases separately:

$$\begin{aligned}
&\sup_{\sqrt{\mu_1}-s_2<p_1<\sqrt{\mu_1}} \frac{4h(p_1)}{\sqrt{\mu_1}\sqrt{2s_2}} \left[\operatorname{artanh}\left(\sqrt{\frac{s_2-\sqrt{\mu_1}+p_1}{2s_2}}\right) + \pi\right] \\
&\quad\leq \frac{4|s_1 - \sqrt{\mu_1} + s_2|^{1/2}}{\sqrt{\mu_1}\sqrt{2s_2}} \left[\operatorname{artanh}\left(\frac{1}{\sqrt{2}}\right) + \pi\right] = 4\frac{\operatorname{artanh}\left(\frac{1}{\sqrt{2}}\right) + \pi}{\sqrt{\mu_1}}
\end{aligned} \tag{5.278}$$

and

$$\begin{aligned}
&\sup_{\frac{\sqrt{\mu_1}}{2}<p_1<\sqrt{\mu_1}-s_2} \frac{4h(p_1)}{\sqrt{\mu_1}\sqrt{2s_2}} \arctan\left(\sqrt{\frac{2s_2}{\sqrt{\mu_1}-p_1-s_2}}\right) \\
&\quad\leq \frac{4}{\sqrt{\mu_1}} \sup_{\frac{\sqrt{\mu_1}}{2}<p_1<\sqrt{\mu_1}-s_2} \Bigg[\frac{|s_1-p_1|^{1/2} - |\sqrt{\mu_1}-p_1-s_2|^{1/2}}{\sqrt{2s_2}}\frac{\pi}{2} \\
&\qquad + \frac{|\sqrt{\mu_1}-p_1-s_2|^{1/2}}{\sqrt{2s_2}} \arctan\left(\sqrt{\frac{2s_2}{\sqrt{\mu_1}-p_1-s_2}}\right)\Bigg] \\
&\quad\leq \frac{4}{\sqrt{\mu_1}} \sup_{\frac{\sqrt{\mu_1}}{2}<p_1<\sqrt{\mu_1}-s_2} \frac{\sqrt{2s_2}}{|s_1-p_1|^{1/2} + |\sqrt{\mu_1}-p_1-s_2|^{1/2}}\frac{\pi}{2} + 1 \leq \frac{4(\frac{\pi}{2}+1)}{\sqrt{\mu_1}}
\end{aligned} \tag{5.279}$$

In total, we obtain

$$\begin{aligned}
\|D^{13}_{\mu_1,\mu_2}\| &\leq \max\left\{\frac{2^{1/2}6}{\mu_1^{1/2}}, \frac{4}{\mu_1^{1/2}}, \frac{8}{\mu_1^{1/2}}, \frac{4(\operatorname{artanh}(1/\sqrt{2})+\pi)}{\mu_1^{1/2}}, \frac{4(\pi/2+1)}{\mu_1^{1/2}}\right\} \\
&= \frac{4(\operatorname{artanh}(1/\sqrt{2})+\pi)}{\mu_1^{1/2}}
\end{aligned} \tag{5.280}$$

Region 14: By symmetry in p_1, we have

$$\begin{aligned}
\|D^{14}_{\mu_1,\mu_2}\| &\leq \sup_{-2<p_1<2} h(p_1) \int_{-2}^{2} \frac{\chi_{14}}{s_1^2+s_2^2-p_1^2-q_1^2} \frac{1}{h(q_1)} \mathrm{d}q_1 \\
&= \sup_{0<p_1<s_1} h(p_1) \int_{\max\{-\sqrt{\mu_1}-p_1,-\sqrt{\mu_2}/2,-\sqrt{\mu_2}+p_1\}}^{\min\{\sqrt{\mu_1}-p_1,\frac{\sqrt{\mu_2}}{2}\}} \frac{1}{s_1^2+s_2^2-p_1^2-q_1^2} \frac{1}{||q_1|-s_2|^{1/2}} \mathrm{d}q_1 \\
&\leq \sup_{0<p_1<s_1} 2h(p_1) \int_{\max\{0,p_1-\sqrt{\mu_1}\}}^{\min\{\sqrt{\mu_1}+p_1,\frac{\sqrt{\mu_2}}{2},\sqrt{\mu_2}-p_1\}} \frac{1}{s_1^2+s_2^2-p_1^2-q_1^2} \frac{1}{||q_1|-s_2|^{1/2}} \mathrm{d}q_1,
\end{aligned} \tag{5.281}$$

where in the last inequality we increased the domain to be symmetric in q_1 and used the symmetry of the integrand.

For $p_1 \leq s_2$ and $\sqrt{\mu_1}+p_1 > q_1$ we have $s_1^2+s_2^2-p_1^2-q_1^2 \geq s_1^2+s_2^2-p_1^2-(\sqrt{\mu_1}+p_1)^2 = 2(s_2-p_1)(p_1+s_1)$. Hence,

$$\begin{aligned}
&\sup_{0<p_1<\frac{\sqrt{\mu_2}}{2}-\sqrt{\mu_1}} 2h(p_1)\int_0^{\sqrt{\mu_1}+p_1} \frac{1}{s_1^2+s_2^2-p_1^2-q_1^2}\frac{1}{||q_1|-s_2|^{1/2}}\mathrm{d}q_1 \\
&\qquad \leq \sup_{0<p_1<\frac{\sqrt{\mu_2}}{2}-\sqrt{\mu_1}} \frac{1}{(s_2-p_1)^{1/2}(p_1+s_1)}\int_0^{\sqrt{\mu_1}+p_1} \frac{1}{||q_1|-s_2|^{1/2}}\mathrm{d}q_1 \\
&= \sup_{0<p_1<\frac{\sqrt{\mu_2}}{2}-\sqrt{\mu_1}} \frac{2(s_2^{1/2}+(p_1+\sqrt{\mu_1}-s_2)^{1/2})}{(s_2-p_1)^{1/2}(p_1+s_1)} \leq \frac{2(s_2^{1/2}+\left(\frac{\sqrt{\mu_1}}{2}\right)^{1/2})}{\left(\frac{\sqrt{\mu_1}}{2}\right)^{1/2}s_1} \leq \frac{4\left(\frac{\sqrt{\mu_2}}{2}\right)^{1/2}}{\left(\frac{\sqrt{\mu_1}}{2}\right)^{1/2}\frac{\sqrt{\mu_2}}{2}} \leq \frac{8}{\sqrt{\mu_1}}
\end{aligned} \tag{5.282}$$

Similarly, for $p_1 \geq s_2$ and $\sqrt{\mu_2}-p_1 > q_1$ we have $s_1^2+s_2^2-p_1^2-q_1^2 \geq s_1^2+s_2^2-p_1^2-(\sqrt{\mu_2}-p_1)^2 = 2(s_1-p_1)(p_1-s_2)$. Therefore,

$$\begin{aligned}
&\sup_{\frac{\sqrt{\mu_2}}{2}<p_1<s_1} 2h(p_1)\int_{p_1-\sqrt{\mu_1}}^{\sqrt{\mu_2}-p_1} \frac{1}{s_1^2+s_2^2-p_1^2-q_1^2}\frac{1}{|q_1-s_2|^{1/2}}\mathrm{d}q_1 \\
&\leq \sup_{\frac{\sqrt{\mu_2}}{2}<p_1<s_1} \frac{1}{(s_1-p_1)^{1/2}(p_1-s_2)}\int_{p_1-\sqrt{\mu_1}}^{\sqrt{\mu_2}-p_1} \frac{1}{|q_1-s_2|^{1/2}}\mathrm{d}q_1 = \sup_{\frac{\sqrt{\mu_2}}{2}<p_1<s_1} \frac{4}{p_1-s_2} = \frac{8}{\sqrt{\mu_1}}.
\end{aligned} \tag{5.283}$$

For $\frac{\sqrt{\mu_2}}{2}-\sqrt{\mu_1} \leq p_1 \leq \frac{\sqrt{\mu_2}}{2}$ and $q_1 < \frac{\sqrt{\mu_2}}{2}$, we have $s_1^2+s_2^2-p_1^2-q_1^2 \geq \frac{\mu_1}{2}$. Thus,

$$\begin{aligned}
&\sup_{\frac{\sqrt{\mu_2}}{2}-\sqrt{\mu_1}<p_1<\frac{\sqrt{\mu_2}}{2}} 2h(p_1)\int_{\max\{0,p_1-\sqrt{\mu_1}\}}^{\frac{\sqrt{\mu_2}}{2}} \frac{1}{s_1^2+s_2^2-p_1^2-q_1^2}\frac{1}{|q_1-s_2|^{1/2}}\mathrm{d}q_1 \\
&\leq \frac{4}{\mu_1}\left(\frac{\sqrt{\mu_1}}{2}\right)^{1/2}\int_{\frac{\sqrt{\mu_2}}{2}-2\sqrt{\mu_1}}^{\frac{\sqrt{\mu_2}}{2}} \frac{1}{|q_1-s_2|^{1/2}}\mathrm{d}q_1 = \frac{8\left(\left(\frac{3\sqrt{\mu_1}}{2}\right)^{1/2}+\left(\frac{\sqrt{\mu_1}}{2}\right)^{1/2}\right)}{2^{1/2}\mu_1^{3/4}} \leq \frac{4}{\mu_1^{1/2}}\left(\sqrt{3}+1\right)
\end{aligned} \tag{5.284}$$

In total, we have

$$\|D^{14}_{\mu_1,\mu_2}\| \leq \frac{4}{\mu_1^{1/2}}\left(\sqrt{3}+1\right) \tag{5.285}$$

□

5.7.2 Proof of Lemma 5.6.6

Proof of Lemma 5.6.6. The integral in (5.207) is invariant under rotations of $\tilde{q}$. Therefore, it suffices to take the supremum over $\tilde{q} = q_2 \geq 0$ for $d = 2$ and $\tilde{q} = (q_2, 0)$ with $q_2 \geq 0$ for $d = 3$. Furthermore, it suffices to restrict to $p_2 \geq 0$ since the integrand is invariant under $\tilde{p} \to -\tilde{p}$. Note that under these conditions $\mu_1 \leq \mu_2$. We split the domain of integration in (5.207) into two regions according to $\mu_1 = \min\{\mu_1, \mu_2\} \lesseqgtr 0$.

Dimension three: We first consider the case $\mu_1 < 0$, i.e. $|p_2 + q_2|^2 > 1 - p_3^2$. In this case,

$$\sup_{\tilde{q}=(q_2,0),q_2\geq 0} \int_{\mathbb{R}^2} \chi_{|\tilde{p}|<2}\chi_{p_2\geq 0} \frac{\chi_{\min\{\mu_1,\mu_2\}<0}}{(-\min\{\mu_1,\mu_2\})^\alpha} \mathrm{d}\tilde{p}$$
$$= \sup_{q_2\geq 0} \Bigg[\int_{-1}^{1} \mathrm{d}p_3 \int_{\max\{\sqrt{1-p_3^2}-q_2,0\}}^{\sqrt{4-p_3^2}} \frac{1}{((p_2+q_2)^2+p_3^2-1)^\alpha} \mathrm{d}p_2$$
$$+ \int_{1<|p_3|<2} \mathrm{d}p_3 \int_0^{\sqrt{4-p_3^2}} \frac{1}{((p_2+q_2)^2+p_3^2-1)^\alpha} \mathrm{d}p_2 \Bigg] \quad (5.286)$$

Let q_2 and $|p_3| \leq 1$ be fixed. By substituting $x = p_2 + q_2 - \sqrt{1-p_3^2}$ if $q_2 \leq \sqrt{1-p_3^2}$ one obtains

$$\int_{\max\{\sqrt{1-p_3^2}-q_2,0\}}^{2} \frac{1}{((p_2+q_2)^2+p_3^2-1)^\alpha} \mathrm{d}p_2$$
$$\leq \int_0^2 \frac{\chi_{q_2\leq\sqrt{1-p_3^2}}}{(x+\sqrt{1-p_3^2})^2+p_3^2-1)^\alpha} \mathrm{d}x + \int_0^2 \frac{\chi_{q_2>\sqrt{1-p_3^2}}}{((p_2+\sqrt{1-p_3^2})^2+p_3^2-1)^\alpha} \mathrm{d}p_2$$
$$\leq \int_0^2 \frac{1}{(2p_2\sqrt{1-p_3^2})^\alpha} \mathrm{d}p_2 \leq \frac{C}{(1-p_3^2)^{\alpha/2}} \quad (5.287)$$

for some finite constant C. Since $\int_{-1}^1 (1-p_3^2)^{-\alpha/2} \mathrm{d}p_3 < \infty$, the first term in (5.286) is bounded. The second term is bounded by

$$\int_{1<|p_3|<2} \mathrm{d}p_3 \int_0^2 \frac{1}{(p_3^2-1)^\alpha} \mathrm{d}p_2 < \infty. \quad (5.288)$$

For the case $\mu_1 > 0$ we have $|p_2 + q_2|^2 < 1 - p_3^2$. Hence,

$$\sup_{\tilde{q}=(q_2,0),q_2\geq 0} \int_{\mathbb{R}^2} \chi_{|\tilde{p}|<2}\chi_{p_2\geq 0} \frac{\chi_{0<\min\{\mu_1,\mu_2\}}}{\min\{\mu_1,\mu_2\}^\alpha} \mathrm{d}\tilde{p}$$
$$= \sup_{q_2\geq 0} \int_{-1}^1 \mathrm{d}p_3 \chi_{q_2\leq\sqrt{1-p_3^2}} \int_0^{\sqrt{1-p_3^2}-q_2} \frac{1}{(1-(p_2+q_2)^2-p_3^2)^\alpha} \mathrm{d}p_2 \quad (5.289)$$

For fixed $|p_3| < 1$ and $q_2 \leq \sqrt{1-p_3^2}$ substituting $x = \sqrt{1-p_3^2} - q_2 - p_2$ gives

$$\int_0^{\sqrt{1-p_3^2}-q_2} \frac{1}{(1-(p_2+q_2)^2-p_3^2)^\alpha} \mathrm{d}p_2 = \int_0^{\sqrt{1-p_3^2}-q_2} \frac{1}{(1-(\sqrt{1-p_3^2}-x)^2-p_3^2)^\alpha} \mathrm{d}x$$
$$= \int_0^{\sqrt{1-p_3^2}-q_2} \frac{1}{x^\alpha(2\sqrt{1-p_3^2}-x)^\alpha} \mathrm{d}x. \quad (5.290)$$

Thus the expression in (5.289) is bounded by

$$\sup_{q_2\geq 0} \int_{-1}^1 \mathrm{d}p_3 \chi_{q_2\leq\sqrt{1-p_3^2}} \int_0^{\sqrt{1-p_3^2}-q_2} \frac{1}{x^\alpha(\sqrt{1-p_3^2}+q_2)^\alpha} \mathrm{d}x$$
$$\leq \int_{-1}^1 \mathrm{d}p_3 \int_0^1 \frac{1}{x^\alpha(\sqrt{1-p_3^2})^\alpha} \mathrm{d}x < \infty \quad (5.291)$$

Dimension two: For the case $\mu_1 < 0$ we have $|p_2 + q_2| > 1$. Hence,

$$\sup_{q_2 \geq 0} \int_0^2 \frac{\chi_{\min\{\mu_1,\mu_2\}<0}}{(-\min\{\mu_1,\mu_2\})^\alpha} \mathrm{d}p_2 = \sup_{q_2 \geq 0} \int_{\max\{1-q_2,0\}}^2 \frac{1}{((p_2+q_2)^2-1)^\alpha} \mathrm{d}p_2. \tag{5.292}$$

This is finite according to (5.287).

For the case $\mu_1 > 0$,

$$\sup_{q_2 \geq 0} \int_0^2 \frac{\chi_{0<\min\{\mu_1,\mu_2\}}}{\min\{\mu_1,\mu_2\}^\alpha} \mathrm{d}p_2 = \sup_{0 \leq q_2 \leq 1} \int_0^{1-q_2} \frac{1}{(1-(p_2+q_2)^2)^\alpha} \mathrm{d}p_2 = \int_0^1 \frac{1}{x^\alpha (2-x)^\alpha} \mathrm{d}x < \infty, \tag{5.293}$$

where we used (5.290) in the second equality. □

5.7.3 Proof of Lemma 5.6.7

Proof of Lemma 5.6.7. The proof follows from elementary computations. We carry out the case $d = 2$ and leave the case $d = 3$, where one additional integration over q_3 needs to be performed, to the reader.

By symmetry, we may restrict to $p_1, q_1, p_2 \geq 0$. Furthermore, we will partition the remaining domain of p_2, q_2 into nine subdomains. Let χ_j be the characteristic function of domain j. Since $(a+b)^2 \leq 2(a^2+b^2)$, there is a constant C such that the expression in (5.213) is bounded above by $C \sum_{j=1}^9 \lim_{\epsilon \to 0} I_j$, where

$$I_j = \sup_{0 \leq p_2 < \epsilon} \int_{\mathbb{R}^2} \chi_{0<p_1,q_1<\epsilon} \left[\int_{-\sqrt{2}}^{\sqrt{2}} \frac{2\chi_j}{|(p+q)^2-1| + |(p-q)^2-1|} \mathrm{d}q_2 \right]^2 \mathrm{d}p_1 \mathrm{d}q_1. \tag{5.294}$$

Hence, we can consider the domains case by case and prove that $\lim_{\epsilon \to 0} I_j = 0$ for each of them.

We use the notation $\mu_1 = 1 - (p_1+q_1)^2$ and $\mu_2 = 1 - (p_1 - q_1)^2$. (Note that this differs from the notation in Lemma 5.6.5). Since $p_1, q_1 \geq 0$ we have $\mu_2 \geq \mu_1$. We assume that $\epsilon < 1/4$, and thus for $p_1, q_1 < \epsilon$ we have $\mu_1, \mu_2 > 1 - 4\epsilon^2 > 3/4$.

For fixed $0 < p_1, q_1 < \epsilon$, we choose the subdomains for p_2, q_2 as sketched in Figure 5.5. The subdomains are chosen according to the signs of $(p_2+q_2)^2 - \mu_1$ and $(p_2-q_2)^2 - \mu_2$, and to distinguish which of $-\sqrt{\mu_1} - p_2, -\sqrt{\mu_2} + p_2$ is larger.

We start with domains 1 to 4, where $(p+q)^2 - 1 = (p_2+q_2)^2 - \mu_1 > 0$ and $(p-q)^2 - 1 = (p_2-q_2)^2 - \mu_2 > 0$. Note that in domain 4, $p_2 \geq \frac{\sqrt{\mu_1}+\sqrt{\mu_2}}{2} \geq \sqrt{1-4\epsilon^2}$, which is larger than ϵ. Hence $\chi_4 = 0$ for $p_2 < \sqrt{1-4\epsilon^2}$, giving $I_4 = 0$. For domains 2 and 3, we have

$$I_2 = \sup_{0 \leq p_2 < \epsilon} \int_{\mathbb{R}^2} \chi_{0<p_1,q_1<\epsilon} \chi_{2p_2 < \sqrt{\mu_2} - \sqrt{\mu_1}} \left[\int_{a_2}^2 \frac{1}{p_1^2+q_1^2+p_2^2+q_2^2-1} \mathrm{d}q_2 \right]^2 \mathrm{d}p_1 \mathrm{d}q_1, \tag{5.295}$$

$$I_3 = \sup_{0 \leq p_2 < \epsilon} \int_{\mathbb{R}^2} \chi_{0<p_1,q_1<\epsilon} \chi_{2p_2 > \sqrt{\mu_2} - \sqrt{\mu_1}} \left[\int_{a_3}^2 \frac{1}{p_1^2+q_1^2+p_2^2+q_2^2-1} \mathrm{d}q_2 \right]^2 \mathrm{d}p_1 \mathrm{d}q_1, \tag{5.296}$$

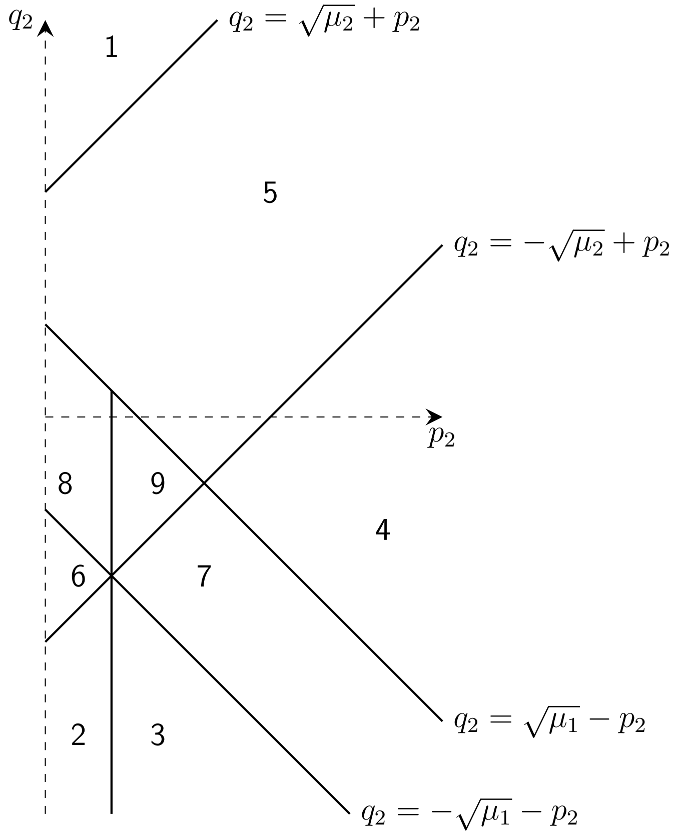

Figure 5.5: Domains occurring in the proof of Lemma 5.6.7.

where $a_2 = \sqrt{\mu_2} - p_2$ and $a_3 = \sqrt{\mu_1} + p_2$. Since $0 \le p_1, q_1, p_2 < \epsilon$ we have $1 - p_1^2 - q_1^2 - p_2^2 > 3/4$ and thus

$$\int_{a_j}^{2} \frac{1}{p_1^2 + q_1^2 + p_2^2 + q_2^2 - 1} = \frac{\operatorname{artanh} \frac{\sqrt{1-p_1^2-q_1^2-p_2^2}}{a_j} - \operatorname{artanh} \frac{\sqrt{1-p_1^2-q_1^2-p_2^2}}{2}}{\sqrt{1-p_1^2-q_1^2-p_2^2}}$$
$$\le C \operatorname{artanh} \frac{\sqrt{1-p_1^2-q_1^2-p_2^2}}{a_j}. \quad (5.297)$$

Since $\operatorname{artanh}(x/y) = \ln((y+x)^2/(y^2-x^2))/2$ and $\sqrt{1-p_1^2-q_1^2-p_2^2} + a_j \le 3$ we get

$$I_2 \le \frac{C}{2} \sup_{0 \le p_2 < \epsilon} \int_{\mathbb{R}^2} \chi_{0<p_1,q_1<\epsilon} \chi_{2p_2<\sqrt{\mu_2}-\sqrt{\mu_1}} \left[\ln \frac{9}{2(p_1 q_1 - p_2(\sqrt{\mu_2} - p_2))} \right]^2 \mathrm{d}p_1 \mathrm{d}q_1 \quad (5.298)$$

and

$$I_3 \le \frac{C}{2} \sup_{0 \le p_2 < \epsilon} \int_{\mathbb{R}^2} \chi_{0<p_1,q_1<\epsilon} \chi_{2p_2>\sqrt{\mu_2}-\sqrt{\mu_1}} \left[\ln \frac{9}{2(p_1 q_1 + p_2(\sqrt{\mu_1} + p_2))} \right]^2 \mathrm{d}p_1 \mathrm{d}q_1. \quad (5.299)$$

For domain 2, we substitute $z = p_1 + q_1$ and $r = p_1 - q_1$ and obtain the bound

$$I_2 \le \frac{C}{4} \sup_{0 \le p_2 < \epsilon} \int_{\mathbb{R}^2} \chi_{|r|<z<2\epsilon} \chi_{2p_2<\sqrt{\mu_2}-\sqrt{\mu_1}} \left[\ln \frac{18}{z^2 - r^2 - 4p_2(\sqrt{1-r^2} - p_2)} \right]^2 \mathrm{d}r \mathrm{d}z \quad (5.300)$$

The condition $2p_2 < \sqrt{\mu_2} - \sqrt{\mu_1}$ implies that $x := z^2 - r^2 - 4p_2(\sqrt{1-r^2} - p_2) \geq 0$. Substituting z by x gives

$$I_2 \leq \frac{C}{4} \sup_{0\leq p_2<\epsilon} \int_{-2\epsilon}^{2\epsilon} \mathrm{d}r \int_0^{\epsilon^2} \mathrm{d}x \left[\ln \frac{18}{x}\right]^2 \frac{1}{2\sqrt{x + r^2 + 4p_2(\sqrt{1-r^2} - p_2)}}$$
$$\leq \frac{C}{2}\epsilon \int_0^{\epsilon^2} \left[\ln \frac{18}{x}\right]^2 \frac{1}{\sqrt{x}} \mathrm{d}x \quad (5.301)$$

This vanishes as $\epsilon \to 0$. For domain 3 we bound (5.299) by

$$I_3 \leq C \int_{\mathbb{R}^2} \chi_{0<p_1,q_1<\epsilon} \left[\ln \frac{9}{2p_1q_1}\right]^2 \mathrm{d}p_1 \mathrm{d}q_1, \quad (5.302)$$

which vanishes in the limit $\epsilon \to 0$. For domain 1 note that since $\sqrt{\mu_2} + p_2 \geq a_2, a_3$, we have $I_1 \leq I_2 + I_3$.

Now consider domain 5, where $(p+q)^2 - 1 = (p_2+q_2)^2 - \mu_1 > 0$ and $(p-q)^2 - 1 = (p_2 - q_2)^2 - \mu_2 < 0$. We have

$$I_5 = \sup_{0\leq p_2<\epsilon} \int_{\mathbb{R}^2} \chi_{0<p_1,q_1<\epsilon} \left[\int_{\sqrt{\mu_1}-p_2}^{\sqrt{\mu_2}+p_2} \frac{1}{2(p_1q_1 + p_2q_2)} \mathrm{d}q_2\right]^2 \mathrm{d}p_1 \mathrm{d}q_1. \quad (5.303)$$

Integration over q_2 gives

$$\int_{\sqrt{\mu_1}-p_2}^{\sqrt{\mu_2}+p_2} \frac{1}{2(p_1q_1 + p_2q_2)} \mathrm{d}q_2 = \frac{1}{2p_2} \ln\left(1 + \frac{(\sqrt{\mu_2} - \sqrt{\mu_1})p_2 + 2p_2^2}{p_1q_1 + (\sqrt{\mu_1} - p_2)p_2}\right). \quad (5.304)$$

Note that $\sqrt{\mu_2} - \sqrt{\mu_1} = 4p_1q_1/(\sqrt{\mu_2} + \sqrt{\mu_1}) \leq 2p_1q_1/\sqrt{1-4\epsilon^2}$ and $\sqrt{\mu_1} - p_2 \geq \sqrt{1-4\epsilon^2} - \epsilon$. We can therefore bound the previous expression from above by

$$\frac{1}{2p_2} \ln\left(1 + \frac{2p_2}{\sqrt{1-4\epsilon^2}} + \frac{2p_2}{\sqrt{1-4\epsilon^2} - \epsilon}\right) \leq \frac{1}{\sqrt{1-4\epsilon^2}} + \frac{1}{\sqrt{1-4\epsilon^2} - \epsilon} < C, \quad (5.305)$$

where we used that $\ln(1+x)/x \leq 1$ for $x \geq 0$. Therefore $I_5 \leq C^2\epsilon^2$ vanishes as $\epsilon \to 0$.

For region 6 we have

$$I_6 = \sup_{0\leq p_2<\epsilon} \int_{\mathbb{R}^2} \chi_{0<p_1,q_1<\epsilon} \chi_{2p_2 \leq \sqrt{\mu_2} - \sqrt{\mu_1}} \left[\int_{-\sqrt{\mu_2}+p_2}^{-\sqrt{\mu_1}-p_2} \frac{1}{2(p_1q_1 + p_2q_2)} \mathrm{d}q_2\right]^2 \mathrm{d}p_1 \mathrm{d}q_1. \quad (5.306)$$

Integration over q_2 gives

$$\int_{-\sqrt{\mu_2}+p_2}^{-\sqrt{\mu_1}-p_2} \frac{1}{2(p_1q_1 + p_2q_2)} \mathrm{d}q_2 = \frac{1}{2p_2} \ln\left(1 + p_2 \frac{(\sqrt{\mu_2} - \sqrt{\mu_1} - 2p_2)}{p_1q_1 - (\sqrt{\mu_2} - p_2)p_2}\right). \quad (5.307)$$

One can compute that

$$\frac{\partial}{\partial p_2} \frac{\sqrt{\mu_2} - \sqrt{\mu_1} - 2p_2}{p_1q_1 - (\sqrt{\mu_2} - p_2)p_2} = \frac{8}{(\sqrt{\mu_2} + \sqrt{\mu_1} - 2p_2)^2} > 0. \quad (5.308)$$

Thus, for $\chi_{2p_2 \leq \sqrt{\mu_2} - \sqrt{\mu_1}}$ we have

$$\frac{\sqrt{\mu_2} - \sqrt{\mu_1} - 2p_2}{p_1q_1 - (\sqrt{\mu_2} - p_2)p_2} \leq \lim_{p_2 \to (\sqrt{\mu_2} - \sqrt{\mu_1})/2} \frac{\sqrt{\mu_2} - \sqrt{\mu_1} - 2p_2}{p_1q_1 - (\sqrt{\mu_2} - p_2)p_2} = \frac{2}{\sqrt{\mu_1}}. \quad (5.309)$$

The expression in (5.307) is thus bounded above by

$$\frac{1}{2p_2}\ln\left(1+p_2\frac{2}{\sqrt{\mu_1}}\right)\leq\frac{1}{\sqrt{\mu_1}}\leq\frac{1}{\sqrt{1-4\epsilon^2}}, \tag{5.310}$$

which is bounded as $\epsilon\to 0$. In total, we have $I_6\leq C\epsilon^2$, which vanishes in the limit $\epsilon\to 0$.

For region 7,

$$I_7=\sup_{0\leq p_2<\epsilon}\int_{\mathbb{R}^2}\chi_{0<p_1,q_1<\epsilon}\chi_{2p_2\geq\sqrt{\mu_2}-\sqrt{\mu_1}}\left[\int_{-\sqrt{\mu_1}-p_2}^{-\sqrt{\mu_2}+p_2}\frac{1}{-2(p_1q_1+p_2q_2)}\mathrm{d}q_2\right]^2\mathrm{d}p_1\mathrm{d}q_1. \tag{5.311}$$

Integration over q_2 gives

$$\int_{-\sqrt{\mu_1}-p_2}^{-\sqrt{\mu_2}+p_2}\frac{1}{-2(p_1q_1+p_2q_2)}\mathrm{d}q_2=\frac{1}{2p_2}\ln\left(1+p_2\frac{(\sqrt{\mu_2}-\sqrt{\mu_1}-2p_2)}{p_1q_1-(\sqrt{\mu_2}-p_2)p_2}\right). \tag{5.312}$$

According to (5.308), for $(\sqrt{\mu_2}-\sqrt{\mu_1})/2\leq p_2<\epsilon$ this is bounded by

$$\frac{1}{2p_2}\ln\left(1+p_2\frac{(\sqrt{\mu_2}-\sqrt{\mu_1}-2\epsilon)}{p_1q_1-(\sqrt{\mu_2}-\epsilon)\epsilon}\right)\leq\frac{1}{2}\frac{2\epsilon-(\sqrt{\mu_2}-\sqrt{\mu_1})}{(\sqrt{\mu_2}-\epsilon)\epsilon-p_1q_1}. \tag{5.313}$$

For $p_1,q_1<\epsilon$ this can be further estimated by

$$\frac{1}{2}\frac{2\epsilon}{(\sqrt{\mu_2}-\epsilon)\epsilon-\epsilon^2}\leq\frac{1}{\sqrt{1-4\epsilon^2}-2\epsilon}, \tag{5.314}$$

which is bounded for $\epsilon\to 0$. Hence, $I_7\leq C\epsilon^2$ vanishes for $\epsilon\to 0$.

For domains 8 and 9, we have

$$I_8=\sup_{0\leq p_2<\epsilon}\int_{\mathbb{R}^2}\chi_{0<p_1,q_1<\epsilon}\chi_{2p_2<\sqrt{\mu_2}-\sqrt{\mu_1}}\left[\int_{-\sqrt{\mu_1}-p_2}^{\sqrt{\mu_1}-p_2}\frac{1}{1-p_1^2-q_1^2-p_2^2-q_2^2}\mathrm{d}q_2\right]^2\mathrm{d}p_1\mathrm{d}q_1, \tag{5.315}$$

$$I_9=\sup_{0\leq p_2<\epsilon}\int_{\mathbb{R}^2}\chi_{0<p_1,q_1<\epsilon}\chi_{2p_2>\sqrt{\mu_2}-\sqrt{\mu_1}}\left[\int_{-\sqrt{\mu_2}+p_2}^{\sqrt{\mu_1}-p_2}\frac{1}{1-p_1^2-q_1^2-p_2^2-q_2^2}\mathrm{d}q_2\right]^2\mathrm{d}p_1\mathrm{d}q_1. \tag{5.316}$$

We bound

$$\begin{aligned}\int_{-\sqrt{\mu_1}-p_2}^{\sqrt{\mu_1}-p_2}\frac{1}{1-p_1^2-q_1^2-p_2^2-q_2^2}\mathrm{d}q_2&\leq 2\int_0^{\sqrt{\mu_1}+p_2}\frac{1}{1-p_1^2-q_1^2-p_2^2-q_2^2}\mathrm{d}q_2\\&=\frac{1}{\sqrt{1-p_1^2-q_1^2-p_2^2}}\ln\left(\frac{\sqrt{1-p_1^2-q_1^2-p_2^2}+\sqrt{\mu_1}+p_2}{\sqrt{1-p_1^2-q_1^2-p_2^2}-\sqrt{\mu_1}-p_2}\right)\\&=\frac{1}{\sqrt{1-p_1^2-q_1^2-p_2^2}}\ln\left(\frac{(\sqrt{1-p_1^2-q_1^2-p_2^2}+\sqrt{\mu_1}+p_2)^2}{2(p_1q_1-p_2(\sqrt{\mu_1}+p_2))}\right)\\&\leq\frac{1}{\sqrt{1-3\epsilon^2}}\ln\left(\frac{9}{2(p_1q_1-p_2(\sqrt{\mu_1}+p_2))}\right)\end{aligned} \tag{5.317}$$

Substituting $z = p_1 + q_1$ and $r = p_1 - q_1$ we obtain

$$\begin{aligned} I_8 &\leq C \sup_{0 \leq p_2 < \epsilon} \int_{\mathbb{R}^2} \chi_{0<p_1,q_1<\epsilon} \chi_{2p_2 < \sqrt{\mu_2} - \sqrt{\mu_1}} \ln\left(\frac{9}{2(p_1 q_1 - p_2(\sqrt{\mu_1} + p_2))} \right)^2 \mathrm{d}p_1 \mathrm{d}q_1 \\ &\leq \frac{C}{2} \sup_{0 \leq p_2 < \epsilon} \int_0^{2\epsilon} \mathrm{d}z \int_{-\epsilon}^{\epsilon} \mathrm{d}r \chi_{|r|<z} \chi_{2p_2 < \sqrt{1-r^2} - \sqrt{1-z^2}} \ln\left(\frac{18}{z^2 - r^2 - 4p_2(\sqrt{1-z^2} + p_2)} \right)^2. \end{aligned} \tag{5.318}$$

Substituting r by $x = z^2 - r^2 - 4p_2(\sqrt{1-z^2} + p_2)$ and using Hölder's inequality we obtain

$$\begin{aligned} I_8 \leq \frac{C}{4} \sup_{0 \leq p_2 < \epsilon} \int_{4p_2 - 4p_2^2}^{2\epsilon} \mathrm{d}z \int_0^{z^2 - 4p_2(p_2 + \sqrt{1-z^2})} \ln\left(\frac{18}{x} \right)^2 \frac{1}{\sqrt{z^2 - 4p_2(\sqrt{1-z^2} + p_2) - x}} \mathrm{d}x \\ \leq \frac{C}{4} \sup_{0 \leq p_2 < \epsilon} \int_{4p_2 - 4p_2^2}^{2\epsilon} \mathrm{d}z \Bigg[\left(\int_0^{z^2 - 4p_2(p_2 + \sqrt{1-z^2})} \ln\left(\frac{18}{x} \right)^6 \mathrm{d}x \right)^{1/3} \times \\ \left(\int_0^{z^2 - 4p_2(p_2 + \sqrt{1-z^2})} \frac{1}{(z^2 - 4p_2(\sqrt{1-z^2} + p_2) - x)^{3/4}} \mathrm{d}x \right)^{2/3} \Bigg]. \end{aligned} \tag{5.319}$$

In the last line we substitute $y = z^2 - 4p_2(\sqrt{1-z^2} + p_2) - x$, and then we use $z^2 - 4p_2(\sqrt{1-z^2} + p_2) - x \leq 4\epsilon^2$ to arrive at the bound

$$\begin{aligned} I_8 &\leq \frac{C}{4} \sup_{0 \leq p_2 < \epsilon} \int_{4p_2 - 4p_2^2}^{2\epsilon} \mathrm{d}z \left[\left(\int_0^{4\epsilon^2} \ln\left(\frac{18}{x} \right)^6 \mathrm{d}x \right)^{1/3} \left(\int_0^{4\epsilon^2} \frac{1}{y^{3/4}} \mathrm{d}y \right)^{2/3} \right] \\ &\leq \frac{C}{2} \epsilon \left(\int_0^{4\epsilon^2} \ln\left(\frac{18}{x} \right)^6 \mathrm{d}x \right)^{1/3} \left(\int_0^{4\epsilon^2} \frac{1}{y^{3/4}} \mathrm{d}y \right)^{2/3}, \end{aligned} \tag{5.320}$$

which vanishes as $\epsilon \to 0$. For I_9 we bound (analogously to (5.317))

$$\begin{aligned} \int_{-\sqrt{\mu_2}+p_2}^{\sqrt{\mu_1}-p_2} \frac{1}{1 - p_1^2 - q_1^2 - p_2^2 - q_2^2} \mathrm{d}q_2 &\leq 2 \int_0^{\sqrt{\mu_2}-p_2} \frac{1}{1 - p_1^2 - q_1^2 - p_2^2 - q_2^2} \mathrm{d}q_2 \\ &= \frac{1}{\sqrt{1 - p_1^2 - q_1^2 - p_2^2}} \ln\left(\frac{(\sqrt{1 - p_1^2 - q_1^2 - p_2^2} + \sqrt{\mu_2} - p_2)^2}{2(p_2(\sqrt{\mu_2} - p_2) - p_1 q_1)} \right) \\ &\leq \frac{1}{\sqrt{1 - 3\epsilon^2}} \ln\left(\frac{4}{2(p_2(\sqrt{\mu_2} - p_2) - p_1 q_1)} \right) \end{aligned} \tag{5.321}$$

Substituting $z = p_1 + q_1$ and $r = p_1 - q_1$ we obtain

$$\begin{aligned} I_9 &\leq C \sup_{0 \leq p_2 < \epsilon} \int_{\mathbb{R}^2} \chi_{0<p_1,q_1<\epsilon} \chi_{2p_2 > \sqrt{\mu_2} - \sqrt{\mu_1}} \ln\left(\frac{4}{2(p_2(\sqrt{\mu_2} - p_2) - p_1 q_1)} \right)^2 \mathrm{d}p_1 \mathrm{d}q_1 \\ &\leq \frac{C}{2} \sup_{0 \leq p_2 < \epsilon} \int_{-\epsilon}^{\epsilon} \mathrm{d}r \int_0^{2\epsilon} \mathrm{d}z \chi_{|r|<z} \chi_{2p_2 > \sqrt{1-r^2} - \sqrt{1-z^2}} \ln\left(\frac{8}{4p_2(\sqrt{1-r^2} - p_2) - z^2 + r^2} \right)^2. \end{aligned} \tag{5.322}$$

Substituting z by $x = 4p_2(\sqrt{1-r^2} - p_2) - z^2 + r^2$ and using Hölder's inequality we obtain

$$I_9 \leq \frac{C}{4} \sup_{0\leq p_2<\epsilon} \int_{-\epsilon}^{\epsilon} \mathrm{d}r \int_0^{4p_2(\sqrt{1-r^2}-p_2)+r^2} \ln\left(\frac{8}{x}\right)^2 \frac{1}{\sqrt{4p_2(\sqrt{1-r^2}-p_2)+r^2-x}} \mathrm{d}x$$

$$\leq \frac{C}{4} \sup_{0\leq p_2<\epsilon} \int_{-\epsilon}^{\epsilon} \mathrm{d}r \left(\int_0^{4p_2(\sqrt{1-r^2}-p_2)+r^2} \ln\left(\frac{8}{x}\right)^6 \mathrm{d}x\right)^{1/3} \times$$

$$\left(\int_0^{4p_2(\sqrt{1-r^2}-p_2)+r^2} \frac{1}{(4p_2(\sqrt{1-r^2}-p_2)+r^2-x)^{3/4}} \mathrm{d}x\right)^{2/3}$$

$$\leq \frac{C}{2}\epsilon \left(\int_0^{4\epsilon+\epsilon^2} \ln\left(\frac{8}{x}\right)^6 \mathrm{d}x\right)^{1/3} \left(\int_0^{4\epsilon+\epsilon^2} \frac{1}{y^{3/4}} \mathrm{d}y\right)^{2/3}, \quad (5.323)$$

which vanishes for $\epsilon \to 0$. □

5.7.4 Proof of Lemma 5.6.9

Proof of Lemma 5.6.9. To prove Lemma 5.6.9 we show that the following expressions are finite.

1. $\sup_{T>\mu/2} \sup_{q\in\mathbb{R}^d} \|V^{1/2} B_T(\cdot, q)|V|^{1/2}\|$
2. $\sup_T \sup_{q\in\mathbb{R}^d} \|V^{1/2} B_T(\cdot, q)\chi_{|\cdot|^2>3\mu}|V|^{1/2}\|$
3. $\sup_T \sup_{q\in\mathbb{R}^d} \|V^{1/2} B_T(\cdot, q)\chi_{((\cdot+q)^2-\mu)((\cdot-q)^2-\mu)<0}|V|^{1/2}\|$
4. $\sup_T \sup_{|q|>\frac{\sqrt{\mu}}{2}} \|V^{1/2} B_T(\cdot, q)\chi_{p^2<3\mu}\chi_{((\cdot+q)^2-\mu)((\cdot-q)^2-\mu)>0}|V|^{1/2}\|$
5. $\sup_T \sup_{|q|<\frac{\sqrt{\mu}}{2}} \left\|V^{1/2}\left[B_T(\cdot, q)\chi_{|\cdot|^2<3\mu}\chi_{((\cdot+q)^2-\mu)((\cdot-q)^2-\mu)>0} - Q_T(q)\right]|V|^{1/2}\right\|$

In combination, they prove (5.217).

For part 1, note that $\|V^{1/2} B_T(\cdot, q)|V|^{1/2}\| = \||V|^{1/2} B_T(\cdot, q)|V|^{1/2}\|$ and by Lemma 5.2.4 this is maximal for $q = 0$, i.e.

$$\sup_{T>\mu/2} \sup_{q\in\mathbb{R}^d} \|V^{1/2} B_T(\cdot, q)|V|^{1/2}\| = \sup_{T>\mu/2} \||V|^{1/2} B_T(\cdot, 0)|V|^{1/2}\|. \quad (5.324)$$

By Lemma 5.24, there is a constant C depending only on μ and V such that

$$\sup_{T>\mu/2} \||V|^{1/2} B_T(\cdot, 0)|V|^{1/2}\| \leq \sup_{T>\mu/2} e_\mu(|V|) m_\mu(T) + C < \infty, \quad (5.325)$$

where $e_\mu(|V|) = \sup \sigma(|V|^{1/2}\mathcal{F}^\dagger\mathcal{F}|V|^{1/2})$.

Part 2 follows using (5.11) and that $\||V|^{1/2}\frac{1}{1-\Delta}|V|^{1/2}\|$ is bounded [32, 40, 50].

For part 3, it suffices to prove that

$$Y = \sup_T \sup_{q\in\mathbb{R}^d} \int_{\mathbb{R}^d} B_T(p, q)\chi_{((p+q)^2-\mu)((p-q)^2-\mu)<0} \mathrm{d}p < \infty \quad (5.326)$$

since (3) is bounded by $\|V\|_1 Y$. The integrand is invariant under rotation of $(p, q) \to (Rp, Rq)$ around the origin. Hence, the integral only depends on the absolute value of q and we may take the supremum over q of the form $q = (|q|, 0)$ only. For $p, (q_1, 0)$ satisfying $((p + (q_1, 0))^2 - \mu)((p - (q_1, 0))^2 - \mu) < 0$, we can estimate by [34, Lemma 4.7]

$$B_T(p, (q_1, 0)) \leq \frac{2}{T} \exp\left(-\frac{1}{T} \min\{(|p_1| + |q_1|)^2 + \tilde{p}^2 - \mu, \mu - (|p_1| - |q_1|)^2 - \tilde{p}^2\}\right) \tag{5.327}$$

Note that $(|p_1| + |q_1|)^2 + \tilde{p}^2 - \mu < \mu - (|p_1| - |q_1|)^2 - \tilde{p}^2 \leftrightarrow p^2 + q_1^2 < \mu$. We can therefore further estimate

$$\begin{aligned} &B_T(p, (q_1, 0))\chi_{(|p_1|+|q_1|)^2+\tilde{p}^2>\mu>(|p_1|-|q_1|)^2+\tilde{p}^2} \\ &\quad \leq \frac{2}{T} \exp\left(-\frac{1}{T}((|p_1| + |q_1|)^2 + \tilde{p}^2 - \mu)\right) \chi_{(|p_1|+|q_1|)^2+\tilde{p}^2>\mu}\chi_{p^2+q_1^2<\mu} \\ &\qquad + \frac{2}{T} \exp\left(-\frac{1}{T}(\mu - (|p_1| - |q_1|)^2 - \tilde{p}^2)\right) \chi_{\mu>(|p_1|-|q_1|)^2+\tilde{p}^2} \end{aligned} \tag{5.328}$$

We now integrate the bound over p and use the symmetry in p_1 to restrict to $p_1 > 0$, replace $|p_1|$ by p_1 and then extend the domain to $p_1 \in \mathbb{R}$. We obtain

$$\begin{aligned} Y \leq \sup_T \sup_{q_1 \in \mathbb{R}} \frac{4}{T} \Bigg[&\int_{\mathbb{R}^d} \exp\left(-\frac{1}{T}((p_1 + |q_1|)^2 + \tilde{p}^2 - \mu)\right) \chi_{(p_1+|q_1|)^2+\tilde{p}^2>\mu}\chi_{p^2+q_1^2<\mu} \mathrm{d}p \\ &+ \int_{\mathbb{R}^d} \exp\left(-\frac{1}{T}(\mu - (p_1 - |q_1|)^2 - \tilde{p}^2)\right) \chi_{\mu>(p_1-|q_1|)^2+\tilde{p}^2} \mathrm{d}p \Bigg]. \end{aligned} \tag{5.329}$$

Now we substitute $p_1 \pm |q_1|$ by p_1 and obtain

$$\begin{aligned} Y &\leq \sup_T \sup_{|q_1|<\sqrt{\mu}} \frac{4}{T} \int_{\mathbb{R}^d} \exp\left(-\frac{1}{T}(p_1^2 + \tilde{p}^2 - \mu)\right) \chi_{p_1^2+\tilde{p}^2>\mu}\chi_{(p_1-|q_1|)^2+\tilde{p}^2+q_1^2<\mu} \mathrm{d}p \\ &\quad + \sup_T \frac{4}{T} \int_{\mathbb{R}^d} \exp\left(-\frac{1}{T}(\mu - p_1^2 - \tilde{p}^2)\right) \chi_{\mu>p_1^2+\tilde{p}^2} \mathrm{d}p \\ &\leq \sup_T \frac{4|\mathbb{S}^{d-1}|(2\sqrt{\mu})^{d-1} e^{\mu/T}}{T} \int_{\sqrt{\mu}}^{\infty} e^{-r^2/T} \mathrm{d}r + \sup_T \frac{4|\mathbb{S}^{d-1}|\sqrt{\mu}^{d-1} e^{-\mu/T}}{T} \int_0^{\sqrt{\mu}} e^{r^2/T} \mathrm{d}r, \end{aligned} \tag{5.330}$$

where we used that $(p_1 - |q_1|)^2 + \tilde{p}^2 + q_1^2 < \mu \Rightarrow p^2 < 2\mu$. Note that

$$\frac{\sqrt{\mu} e^{\mu/T}}{T} \int_{\sqrt{\mu}}^{\infty} e^{-r^2/T} \mathrm{d}r = \frac{\pi^{1/2}}{2} \sqrt{\frac{\mu}{T}} e^{\mu/T} \mathrm{erfc}\left(\sqrt{\frac{\mu}{T}}\right) \tag{5.331}$$

and

$$\frac{\sqrt{\mu} e^{-\mu/T}}{T} \int_0^{\sqrt{\mu}} e^{r^2/T} \mathrm{d}r = \frac{\pi^{1/2}}{2} \sqrt{\frac{\mu}{T}} e^{-\mu/T} \mathrm{erfi}\left(\sqrt{\frac{\mu}{T}}\right) \tag{5.332}$$

As in the proof of [34, Lemma 4.4], we conclude that $Y < \infty$ since the functions $x e^{x^2} \mathrm{erfc}(x)$ and $x e^{-x^2} \mathrm{erfi}(x)$ are bounded for $x \geq 0$.

For part 4, it again suffices to prove that

$$X = \sup_T \sup_{|q|>\frac{\sqrt{\mu}}{2}} \int_{\mathbb{R}^d} B_T(p, q) \chi_{p^2<3\mu} \chi_{((p+q)^2-\mu)((p-q)^2-\mu)>0} \mathrm{d}p < \infty, \tag{5.333}$$

since (4) is bounded by $\|V\|_1 X$. Again we can restrict to q of the form $q = (|q|, 0)$. The idea is to split the integrand in X into four terms localized in different regions. The integrand is supported on the intersection and the complement of the two disks/balls with radius $\sqrt{\mu}$ centered at $(\pm q_1, 0)$. (For $d = 2$ this is the white region in Figure 5.3).

- The first term covers the domain with $|\tilde{p}| > \sqrt{\mu}$ outside the disks/balls:
  $X_1 = \sup_T \sup_{|q_1|>\frac{\sqrt{\mu}}{2}} \int_{\mathbb{R}^d} B_T(p,(q_1,0))\chi_{p^2<3\mu}\chi_{\tilde{p}^2>\mu}\mathrm{d}p$
- The second term covers the remaining domain with $|p_1| > |q_1|$ outside of the two disks/balls:
  $X_2 = \sup_T \sup_{|q_1|>\frac{\sqrt{\mu}}{2}} \int_{\tilde{p}^2<\mu} \mathrm{d}\tilde{p} \int_{|p_1|>\sqrt{\mu-\tilde{p}^2}+|q_1|} \mathrm{d}p_1 B_T(p,(q_1,0))\chi_{p^2<3\mu}$
- The third term covers the remaining domain with $|p_1| < |q_1|$ outside of the two disks/balls:
  $X_3 = \sup_T \sup_{|q_1|>\frac{\sqrt{\mu}}{2}} \int_{\mu-q_1^2<\tilde{p}^2<\mu} \mathrm{d}\tilde{p} \int_{|p_1|<-\sqrt{\mu-\tilde{p}^2}+|q_1|} \mathrm{d}p_1 B_T(p,(q_1,0))\chi_{p^2<3\mu}$
- The fourth term covers the domain in the intersection of the two disks/balls:
  $X_4 = \sup_T \sup_{|q_1|>\frac{\sqrt{\mu}}{2}} \int_{\tilde{p}^2<\mu-q_1^2} \mathrm{d}\tilde{p} \int_{|p_1|<\sqrt{\mu-\tilde{p}^2}-|q_1|} \mathrm{d}p_1 B_T(p,(q_1,0))\chi_{p^2<3\mu}$

We prove that each X_j is finite. It then follows that $X \leq X_1 + X_2 + X_3 + X_4$ is finite. We use the bounds

$$B_T(p,(q_1,0)) \leq \begin{cases} \frac{1}{p^2+q_1^2-\mu} & \text{if } (|p_1|-|q_1|)^2+\tilde{p}^2 > \mu, \\ \frac{1}{\mu-p^2-q_1^2} & \text{if } (|p_1|+|q_1|)^2+\tilde{p}^2 < \mu, \end{cases} \tag{5.334}$$

which follow from (5.202). The first line applies to X_1, X_2, X_3, the second line to X_4. For X_1, we have $p^2+q_1^2-\mu > q_1^2 > \mu/4$ and thus $X_1 < \infty$. Similarly, for X_2, we have $p^2+q_1^2-\mu = (\sqrt{q_1^2+p_1^2}+\sqrt{\mu-\tilde{p}^2})(\sqrt{q_1^2+p_1^2}-\sqrt{\mu-\tilde{p}^2}) \geq |q_1|(|p_1|-\sqrt{\mu-\tilde{p}^2}) \geq q_1^2 \geq \mu/4$ and thus $X_2 < \infty$. For X_3, we have $p^2+q_1^2-\mu \geq |q_1|(|q_1|-\sqrt{\mu-\tilde{p}^2}) \geq \frac{\sqrt{\mu}}{2}(|q_1|-\sqrt{\mu-\tilde{p}^2})$. Hence, $X_3 \leq \sup_{|q_1|>\frac{\sqrt{\mu}}{2}} \frac{4}{\sqrt{\mu}} \int_{\mu-q_1^2<\tilde{p}^2<\mu} \mathrm{d}\tilde{p} < \infty$. For X_4 we have $\mu-p^2-q_1^2 \geq \mu-(\sqrt{\mu-\tilde{p}^2}-|q_1|)^2-\tilde{p}^2-q_1^2 = 2|q_1|\sqrt{\mu-\tilde{p}^2} \geq \sqrt{\mu}\sqrt{\mu-\tilde{p}^2}$. Thus,

$$X_4 \leq \sup_{|q_1|>\frac{\sqrt{\mu}}{2}} \frac{2}{\sqrt{\mu}} \int_{\tilde{p}^2<\mu-q_1^2} \frac{\sqrt{\mu-\tilde{p}^2}-|q_1|}{\sqrt{\mu-\tilde{p}^2}} \mathrm{d}\tilde{p} < \infty. \tag{5.335}$$

To prove that (5) is finite, let $S_{T,d}(q) : L^1(\mathbb{R}^d) \to L^\infty(\mathbb{R}^d)$ be the operator with integral kernel

$$S_{T,d}(q)(x,y) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \left[e^{i(x-y)\cdot p} - e^{i\sqrt{\mu}(x-y)\cdot p/|p|} \right] B_T(p,q)\chi_{((p+q)^2-\mu)((p-q)^2-\mu)>0}\chi_{p^2<3\mu}\mathrm{d}p \tag{5.336}$$

Then (5) equals $\sup_T \sup_{|q|<\frac{\sqrt{\mu}}{2}} \left\| V^{1/2} S_{T,d}(q) |V|^{1/2} \right\|$. With (5.11) and $|e^{ix}-e^{iy}| \leq \min\{|x-y|,2\}$ we obtain

$$\begin{aligned} |S_{T,d}(q)(x,y)| &\leq \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \frac{\min\left\{|(|p|-\sqrt{\mu})(x-y)\cdot p/|p||, 2\right\}}{|p^2+q^2-\mu|} \chi_{((p+q)^2-\mu)((p-q)^2-\mu)>0}\chi_{p^2<3\mu}\mathrm{d}p \\ &\leq \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \frac{\min\left\{||p|-\sqrt{\mu}||x-y|, 2\right\}}{|p^2+q^2-\mu|} \chi_{((p+q)^2-\mu)((p-q)^2-\mu)>0}\chi_{p^2<3\mu}\mathrm{d}p \end{aligned} \tag{5.337}$$

Again, the integral only depends on $|q|$, so we may restrict to $q = (|q|, 0)$. We now switch to angular coordinates. Recall the notation $r_\pm$ and e_φ introduced before (5.221) and that

$(|r\cos\varphi| \mp |q_1|)^2 + r^2\sin\varphi^2 \gtrless \mu \leftrightarrow r \gtrless r_\pm(e_\varphi)$. For $d=2$ we have

$$|S_{T,2}((q_1,0))(x,y)| \leq \frac{1}{(2\pi)^2}\int_0^{2\pi}\left[\int_{r_+(e_\varphi)}^{\sqrt{3\mu}} \frac{\min\left\{|r-\sqrt{\mu}||x-y|,2\right\}}{r^2+q_1^2-\mu} r\mathrm{d}r + \int_0^{r_-(e_\varphi)} \frac{\min\left\{(\sqrt{\mu}-r)|x-y|,2\right\}}{\mu-r^2-q_1^2} r\mathrm{d}r\right]\mathrm{d}\varphi =: g(x,y,q_1) \quad (5.338)$$

and for $d=3$

$$|S_{T,3}((q_1,0))(x,y)| \leq \frac{1}{(2\pi)^2}\int_0^{\pi}\left[\int_{r_+(e_\theta)}^{\sqrt{3\mu}} \frac{\min\left\{|r-\sqrt{\mu}||x-y|,2\right\}}{r^2+q_1^2-\mu} \sin\theta r^2\mathrm{d}r + \int_0^{r_-(e_\theta)} \frac{\min\left\{(\sqrt{\mu}-r)|x-y|,2\right\}}{\mu-r^2-q_1^2} \sin\theta r^2\mathrm{d}r\right]\mathrm{d}\theta \leq \frac{\sqrt{3\mu}}{2} g(x,y,q_1). \quad (5.339)$$

We bound g by

$$|g(x,y,q_1)| \leq \frac{1}{(2\pi)^2}\int_0^{2\pi}\left[\int_{r_+(e_\varphi)}^{\sqrt{3\mu}} \frac{\min\left\{(r-r_+(e_\varphi))|x-y|,2\right\}+\min\left\{|\sqrt{\mu}-r_+(e_\varphi)||x-y|,2\right\}}{r^2+q_1^2-\mu} r\mathrm{d}r + \int_0^{r_-(e_\varphi)} \frac{\min\left\{(r_-(e_\varphi)-r)|x-y|,2\right\}+\min\left\{(\sqrt{\mu}-r_-(e_\varphi))|x-y|,2\right\}}{\mu-r^2-q_1^2} r\mathrm{d}r\right]\mathrm{d}\varphi \quad (5.340)$$

Note that $r_+(e_\varphi)$ attains the minimal value $\sqrt{\mu-q_1^2}$ at $|\varphi| = \frac{\pi}{2}$ and the maximal value $\sqrt{\mu}+|q_1|$ at $|\varphi|=0$. Similarly, $r_-(e_\varphi)$ attain the maximal value $\sqrt{\mu-q_1^2}$ at $|\varphi| = \frac{\pi}{2}$ and the minimal value $\sqrt{\mu}-|q_1|$ at $|\varphi|=0$. For the first summand in both integrals we take the supremum over the angular variable. For the second summand in both integrals, we carry out the integration over r and use that $|\sqrt{\mu}-r_-(e_\varphi)|, |\sqrt{\mu}-r_+(e_\varphi)| \leq |q_1|$. We obtain the bound

$$|g(x,y,q_1)| \leq \frac{1}{2\pi}\int_0^{\sqrt{3\mu}} \frac{\min\left\{|r-\sqrt{\mu-q_1^2}||x-y|,2\right\} r}{|r^2+q_1^2-\mu|}\mathrm{d}r + \frac{\min\left\{|q_1||x-y|,2\right\}}{2(2\pi)^2}\int_0^{2\pi}\left[\ln\left(\frac{2\mu+q_1^2}{r_+(e_\varphi)^2+q_1^2-\mu}\right)+\ln\left(\frac{\mu-q_1^2}{\mu-q_1^2-r_-(e_\varphi)^2}\right)\right]\mathrm{d}\varphi \quad (5.341)$$

Recall that we are only interested in $|q_1| < \sqrt{\mu}/2$. For the first term, we use that $r \leq \sqrt{3\mu}$ and $|r^2+q_1^2-\mu| = |r-\sqrt{\mu-q_1^2}||r+\sqrt{\mu-q_1^2}| \geq |r-\sqrt{\mu-q_1^2}|\sqrt{\mu-q_1^2}$. This gives the following bound, where we first carry out the r-integration and then use that $\sqrt{\mu-q_1^2} \geq \sqrt{3\mu}/2$:

$$\begin{aligned}
&\frac{\sqrt{3\mu}}{\pi\sqrt{\mu-q_1^2}}\int_0^{\sqrt{3\mu}} \min\left\{\frac{|x-y|}{2}, \frac{1}{|r-\sqrt{\mu-q_1^2}|}\right\}\mathrm{d}r \\
&\leq \frac{\sqrt{3\mu}}{\pi\sqrt{\mu-q_1^2}}\left[\ln\left(\max\left\{1,\frac{\sqrt{\mu-q_1^2}|x-y|}{2}\right\}\right)+2+\ln\left(\max\left\{1,\frac{(\sqrt{3\mu}-\sqrt{\mu-q_1^2})|x-y|}{2}\right\}\right)\right] \\
&\leq C\left[1+\ln\left(1+\frac{\sqrt{3\mu}|x-y|}{2}\right)\right] \quad (5.342)
\end{aligned}$$

For the second term, we use that

$$\frac{2\mu + q_1^2}{r_+(e_\varphi)^2 + q_1^2 - \mu} \frac{\mu - q_1^2}{\mu - q_1^2 - r_-(e_\varphi)^2} = \frac{2\mu + q_1^2}{4|e_{\varphi,1}|^2|q_1|^2} \tag{5.343}$$

and $|q_1| < \sqrt{\mu}/2$ as well as $|e_{\varphi,1}| = |\cos\varphi| \geq \frac{1}{2}\min\{|\frac{\pi}{2} - \varphi|, |\frac{3\pi}{2} - \varphi|\}$ to arrive at the bound

$$\begin{aligned}\frac{\min\{|q_1||x-y|, 2\}}{(2\pi)^2} \int_0^{2\pi} \ln\left(\frac{\sqrt{3\mu}}{2|e_{\varphi,1}q_1|}\right) \mathrm{d}\varphi &\leq \frac{4\min\{|q_1||x-y|, 2\}}{(2\pi)^2} \int_0^{\pi/2} \ln\left(\frac{\sqrt{3\mu}}{|\varphi q_1|}\right) \mathrm{d}\varphi \\ &= \frac{\min\{|q_1||x-y|, 2\}}{2\pi} \left(1 + \ln\left(\frac{2\sqrt{3\mu}}{\pi|q_1|}\right)\right) \\ &= \frac{\min\{|q_1||x-y|, 2\}}{2\pi} \left(1 + \ln\left(\sqrt{3\mu}|x-y|\right) + \ln\left(\frac{2\pi}{|x-y||q_1|}\right)\right),\end{aligned} \tag{5.344}$$

where we used $\int \ln(1/x)\mathrm{d}x = x + x\ln(1/x)$. Since $x\ln(1/x) \leq C$, this is bounded above by

$$\frac{1}{\pi}\left(1 + \max\left\{\ln\left(\sqrt{3\mu}|x-y|\right), 0\right\}\right) + C. \tag{5.345}$$

In total, we obtain the bound

$$\sup_{|q_1| < \frac{\sqrt{\mu}}{2}} |g(x, y, q_1)| \leq C\left[1 + \ln\left(1 + \sqrt{\mu}|x-y|\right)\right]. \tag{5.346}$$

Let $M : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ be the operator with integral kernel $M(x,y) = |V|^{1/2}(x)(1 + \ln\left(1 + \sqrt{\mu}|x-y|\right))|V|^{1/2}(y)$. We have

$$\sup_T \sup_{|q| < \frac{\sqrt{\mu}}{2}} \left\| V^{1/2} S_{T,d}(q) |V|^{1/2} \right\| \leq C(\mu, d)\|M\| \tag{5.347}$$

for some constant $C(\mu, d) < \infty$. The operator M is Hilbert-Schmidt since the function $x \mapsto (1 + \ln(1+|x|)^2)|V(x)|$ is in $L^1(\mathbb{R}^d)$. □

CHAPTER 6

Enhanced BCS Superconductivity at a Corner

Abstract We consider the critical temperature for superconductivity, defined via the linear BCS equation. We prove that at weak coupling the critical temperature for a sample confined to a quadrant in two dimensions is strictly larger than the one for a half-space, which in turn is strictly larger than the one for $\mathbb{R}^2$. Furthermore, we prove that the relative difference of the critical temperatures vanishes in the weak coupling limit.

6.1 Introduction

Recent work [6, 7, 62, 63, 64, 68] predicts the occurrence of boundary superconductivity in the BCS model. Close to edges superconductivity sets in at higher temperatures than in the bulk, and at corners the critical temperature appears to be even higher than at edges. Our goal is to provide a mathematically rigorous justification of these results. It was proved in [34, 60] that the system on half-spaces in dimensions $d \in \{1, 2, 3\}$ can have higher critical temperatures than on $\mathbb{R}^d$. Here, we extend this result for $d = 2$ and show that a quadrant has a higher critical temperature than a half-space, at least at weak coupling.

We consider the full plane, and the half- and quarter-spaces $\Omega_k = (0, \infty)^k \times \mathbb{R}^{2-k}$ for $k \in \{0, 1, 2\}$. We define the critical temperature as in [34, 60] using the operator

$$H_T^\Omega = \frac{-\Delta_x - \Delta_y - 2\mu}{\tanh\left(\frac{-\Delta_x - \mu}{2T}\right) + \tanh\left(\frac{-\Delta_y - \mu}{2T}\right)} - \lambda V(x - y) \tag{6.1}$$

acting in $L^2_{\text{sym}}(\Omega \times \Omega) = \{\psi \in L^2(\Omega \times \Omega) | \psi(x, y) = \psi(y, x) \text{ for all } x, y \in \Omega\}$, where $-\Delta$ denotes the Dirichlet or Neumann Laplacian and the subscript indicates on which variable it acts, T is the temperature, μ is the chemical potential, V is the interaction, and λ is the coupling constant. The first term is defined through functional calculus. For $V \in L^t(\mathbb{R}^2)$ with $t > 1$, the $H_T^{\Omega_k}$ are self-adjoint operators defined via the KLMN theorem [60, Remark 2.2].

The critical temperatures are defined as

$$T_c^k(\lambda) := \inf\{T \in (0, \infty) | \inf \sigma(H_T^{\Omega_k}) \geq 0\}. \tag{6.2}$$

The operator $H_T^{\Omega_k}$ is the Hessian of the BCS functional at the normal state [21]. In particular, the system is superconducting for $T < T_c^k(\lambda)$, when the normal state is not a minimizer of the BCS functional. A priori, superconductivity may also occur at temperatures $T > T_c^k(\lambda)$, when the normal state is a local minimum of the BCS functional, but not a global one. For translation invariant systems, in particular for $\Omega_0 = \mathbb{R}^2$, this is not the case and the system is in the normal state if $T > T_c^0(\lambda)$ [32, 33], hence T_c^0 separates the normal and the superconducting phase. However, it remains an open question whether the same is true for T_c^1 and T_c^2.

We prove that for small enough λ, the critical temperatures defined through the linear criterion (6.2) satisfy $T_c^2(\lambda) > T_c^1(\lambda)$. Together with the result from [60], we get the strictly decreasing sequence $T_c^2(\lambda) > T_c^1(\lambda) > T_c^0(\lambda)$ of critical temperatures at weak coupling.

Similarly to [60, Lemma 2.3], where it was shown that $T_c^1(\lambda) \geq T_c^0(\lambda)$ for all λ, the following Lemma is relatively easy to prove.

Lemma 6.1.1. *Let* $\lambda, T > 0$ *and* $V \in L^t(\mathbb{R}^2)$ *for some* $t > 1$. *Then* $\inf \sigma(H_T^{\Omega_2}) \leq \inf \sigma(H_T^{\Omega_1})$.

Its proof can be found in Section 6.2. In particular it follows that for all $\lambda > 0$, we have $T_c^2(\lambda) \geq T_c^1(\lambda)$. The difficulty lies in proving a strict inequality, which the rest of the paper will be devoted to. In order to prove $T_c^2(\lambda) > T_c^1(\lambda)$, we shall give a precise analysis of the asymptotic behavior of $H^{\Omega_1}_{T_c^1(\lambda)}$ as $\lambda \to 0$.

For $\mu > 0$ let $\mathcal{F} : L^1(\mathbb{R}^2) \to L^2(\mathbb{S}^1)$ act as the restriction of the Fourier transform to a sphere of radius $\sqrt{\mu}$, i.e., $\mathcal{F}\psi(\omega) = \widehat{\psi}(\sqrt{\mu}\omega)$ and for $V \geq 0$ define $O_\mu = V^{1/2}\mathcal{F}^\dagger \mathcal{F} V^{1/2}$ on $L^2_{\mathsf{s}}(\mathbb{R}^2) = \{\psi \in L^2(\mathbb{R}^2) | \psi(r) = \psi(-r)\}$. The operator O_μ is compact. For the desired asymptotic behavior of $H^{\Omega_1}_{T_c^1(\lambda)}$ we need that O_μ has a non-degenerate eigenvalue $e_\mu = \sup \sigma(O_\mu) > 0$ at the top of its spectrum [32, 40].

We require the following assumptions for our main result.

Assumption 6.1.2. Let $\mu > 0$. Assume that

1. $V \in L^1(\mathbb{R}^2) \cap L^t(\mathbb{R}^2)$ for some $t > 1$,
2. V is radial, $V \not\equiv 0$,
3. $|\cdot| V \in L^1(\mathbb{R}^2)$,
4. $V \geq 0$,
5. $e_\mu = \sup \sigma(O_\mu)$ is a non-degenerate eigenvalue.

Remark 6.1.3. Similarly to the three dimensional case discussed in [32, Section III.B.1], because of rotation invariance the eigenfunctions of O_μ are given, in radial coordinates $r \equiv (|r|, \varphi)$, by $V^{1/2}(r) e^{im\varphi} J_m(\sqrt{\mu}|r|)$, where J_m denote the Bessel functions and $m \in 2\mathbb{Z}$ since the functions must be even in r. The corresponding eigenvalues are

$$e_\mu^{(m)} = \frac{1}{2\pi} \int_{\mathbb{R}^2} V(r) |J_m(\sqrt{\mu}|r|)|^2 \mathrm{d}r \tag{6.3}$$

and in particular $e_\mu^{(m)} = e_\mu^{(-m)}$. Assumption 5 therefore means that $e_\mu = e_\mu^{(0)}$ and that all other eigenvalues $e_\mu^{(m)}$ are strictly smaller. Hence, the eigenstate corresponding to e_μ has zero angular momentum. Analogously to the three dimensional case, a sufficient condition for Assumption 5 to hold is that $\widehat{V} \geq 0$.

Our first main result is:

Theorem 6.1.4. *Let $\mu > 0$ and let V satisfy Assumption 6.1.2. Assume the same boundary conditions, either Dirichlet or Neumann, on Ω_1 and Ω_2. Then there is a $\lambda_1 > 0$, such that for all $0 < \lambda < \lambda_1$, $T_c^2(\lambda) > T_c^1(\lambda)$.*

The second main result is that the relative difference in critical temperatures vanishes in the weak coupling limit.

Theorem 6.1.5. *Let $\mu > 0$ and let V satisfy Assumption 6.1.2. Assume either Dirichlet or Neumann boundary conditions on Ω_2. Then*

$$\lim_{\lambda\to 0} \frac{T_c^2(\lambda) - T_c^0(\lambda)}{T_c^0(\lambda)} = 0. \tag{6.4}$$

Since $T_c^2(\lambda) \geq T_c^1(\lambda) \geq T_c^0(\lambda)$, this implies $\lim_{\lambda\to 0} \frac{T_c^2(\lambda)-T_c^1(\lambda)}{T_c^1(\lambda)} = 0$ and $\lim_{\lambda\to 0} \frac{T_c^1(\lambda)-T_c^0(\lambda)}{T_c^0(\lambda)} = 0$. The latter was already shown in [60] and we closely follow [60] to prove Theorem 6.1.5.

The paper is structured as follows. In Section 6.1.1 we explain the proof strategy for Theorem 6.1.4. Section 6.2 contains the proofs of some basic properties of H_T^Ω. Section 6.3 discusses the regularity and asymptotic behavior of the ground state of $H_T^{\Omega_1}$. In Section 6.4 we prove Lemma 6.1.8, the first key step in the proof of Theorem 6.1.4. The second key step, Lemma 6.1.9 is proved in Section 6.5. In Section 6.6 we prove Theorem 6.1.5. Section 6.7 contains the proofs of auxiliary Lemmas.

6.1.1 Proof strategy for Theorem 6.1.4

The proof of Theorem 6.1.4 is based on the variational principle. The idea is to construct a trial state for $H^{\Omega_2}_{T_c^1(\lambda)}$ involving the ground state of $H^{\Omega_1}_{T_c^1(\lambda)}$. However, the latter operator is translation invariant in the second component of the center of mass variable and therefore has purely essential spectrum. To work with an operator that has eigenvalues, we fix the momentum in the translation invariant direction, and choose it in order to minimize the energy.

Let $U : L^2(\mathbb{R}^2 \times \mathbb{R}^2) \to L^2(\mathbb{R}^2 \times \mathbb{R}^2)$ be the unitary operator switching to relative and center of mass coordinates $r = x - y$ and $z = x + y$, i.e. $U\psi(r, z) = \frac{1}{2}\psi((r + z)/2, (z - r)/2)$. We shall apply U to functions defined on a subset of $\Omega \subset \mathbb{R}^2 \times \mathbb{R}^2$, by identifying $L^2(\Omega)$ with the set of functions in $L^2(\mathbb{R}^2 \times \mathbb{R}^2)$ supported in Ω. The operator $UH_T^{\Omega_1}U^\dagger$, which is $H_T^{\Omega_1}$ transformed to relative and center of mass coordinates, acts on functions on $\tilde{\Omega}_1 \times \mathbb{R}$, where $\tilde{\Omega}_1 = \{(r, z_1) \in \mathbb{R}^3 | |r_1| < z_1\}$, and is translation invariant in z_2. For every $q_2 \in \mathbb{R}$ let $H_T^1(q_2)$ be the operator obtained from $UH_T^{\Omega_1}U^\dagger$ by restricting to momentum q_2 in the z_2 direction. The operator $H_T^1(q_2)$ acts in $L^2_{\mathrm{s}}(\tilde{\Omega}_1) = \{\psi \in L^2(\tilde{\Omega}_1) | \psi(r, z_1) = \psi(-r, z_1)\}$ and we have $\inf\sigma(H^{\Omega_1}_{T_c^1(\lambda)}) = \inf_{q_2\in\mathbb{R}} \inf\sigma(H^1_{T_c^1(\lambda)}(q_2))$. We want to choose q_2 to be optimal. That this can be done is a consequence of the following Lemma, whose proof will be given in Section 6.2.2.

Lemma 6.1.6. *Let $T, \lambda, \mu > 0$ and $V \in L^t(\mathbb{R}^2)$ for some $t > 1$. The function $q_2 \mapsto \inf\sigma(H_T^1(q_2))$ is continuous, even and diverges to $+\infty$ as $|q_2| \to \infty$.*

Therefore, the infimum is attained and we can define $\eta(\lambda)$ to be the minimal number in $[0, \infty)$ such that $\inf\sigma(H^1_{T_c^1(\lambda)}(\eta(\lambda))) = \inf\sigma(H^{\Omega_1}_{T_c^1(\lambda)})$.

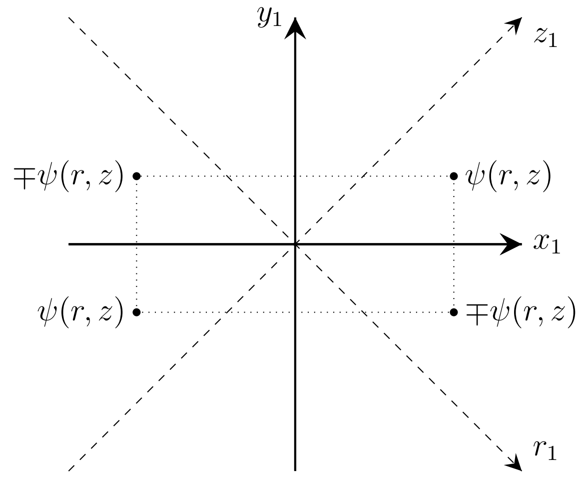

Figure 6.1: Sketch of the (anti)symmetric extension of a function ψ defined on the upper right quadrant in the (r_1, z_1)-coordinates. The extension is defined by mirroring along the x_1 and y_1-axes and multiplying by -1 for Dirichlet boundary conditions.

Next, we shall argue that $H^1_{T_c^1(\lambda)}(\eta(\lambda))$ indeed has a ground state, at least for small enough coupling, which allows us to construct the desired trial state. Let $\lambda_0 > 0$ be such that $T_c^1(\lambda) > T_c^0(\lambda)$ for $\lambda \leq \lambda_0$. Such a λ_0 exists by [60, Theorem 1.3].

Lemma 6.1.7. *Let $\mu > 0$, let V satisfy Assumption 6.1.2 and let $0 < \lambda \leq \lambda_0$. Then $H^1_{T_c^1(\lambda)}(\eta(\lambda))$ has an eigenvalue at the bottom of its spectrum.*

The proof of Lemma 6.1.7 can be found in Section 6.2.3. For $\lambda \leq \lambda_0$ let $\tilde{\Phi}_\lambda$ be the ground state of $H^1_{T_c^1(\lambda)}(\eta(\lambda))$. In the case $\eta(\lambda) = 0$, the operator $H^1_{T_c^1(\lambda)}(\eta(\lambda))$ commutes with reflections $r_2 \to -r_2$ and we may assume that $\tilde{\Phi}_\lambda$ is even or odd under this reflection. We extend the function $\tilde{\Phi}_\lambda$ (anti)symmetrically from $\tilde{\Omega}_1$ to $\mathbb{R}^3$, such that the extended function Φ_λ satisfies $\Phi_\lambda((-r_1, r_2), -z_1) = \Phi_\lambda(r, z_1)$ and $\mp\Phi_\lambda((z_1, r_2), r_1) = \Phi_\lambda(r, z_1)$, where $-/+$ corresponds to Dirichlet/Neumann boundary conditions (see Figure 6.1 for an illustration). The function Φ_λ is the key ingredient for our trial state. Let $\chi_{\tilde{\Omega}_1}$ denote multiplication by the characteristic function of $\tilde{\Omega}_1$; then $\tilde{\Phi}_\lambda = \chi_{\tilde{\Omega}_1}\Phi_\lambda$. We choose the normalization such that $\|V^{1/2}\chi_{\tilde{\Omega}_1}\Phi_\lambda\|_2 = 1$, where $V^{1/2}\psi(r,z) = V^{1/2}(r)\psi(r,z)$. (Since $V \in L^t(\mathbb{R}^2)$ for some $t > 1$ and $\Phi_\lambda \in H^1(\mathbb{R}^3)$, it follows by the Hölder and Sobolev inequalities that $V^{1/2}\Phi_\lambda$ is an L^2 function [50].)

Our choice of trial state is

$$\begin{aligned}\psi_\lambda^\epsilon(r_1, r_2, z_1, z_2) = {} & (\Phi_\lambda(r_1, r_2, z_1)e^{i\eta(\lambda)z_2} + \Phi_\lambda(r_1, -r_2, z_1)e^{-i\eta(\lambda)z_2})e^{-\epsilon|z_2|} \\ & \mp (\Phi_\lambda(r_1, z_2, z_1)e^{i\eta(\lambda)r_2} + \Phi_\lambda(r_1, -z_2, z_1)e^{-i\eta(\lambda)r_2})e^{-\epsilon|r_2|} \end{aligned} \tag{6.5}$$

for some $\epsilon > 0$. Here and throughout the paper we use the convention that upper signs correspond to Dirichlet and lower signs to Neumann boundary conditions, unless stated otherwise. The function (6.5) is the natural generalization of the trial state for a half-space used in [60]. Note that ψ_λ^ϵ is the (anti)symmetrization of $\Phi_\lambda(r, z_1)e^{i\eta(\lambda)z_2 - \epsilon|z_2|}$ and satisfies the boundary conditions. The trial state vanishes if $\eta = 0$ and Φ_λ is odd under $r_2 \to -r_2$; our proof will implicitly show that at weak coupling Φ_λ must be even if $\eta = 0$. We shall prove the following two Lemmas in Sections 6.4 and 6.5, respectively.

Lemma 6.1.8. *Let $\mu > 0$, let V satisfy Assumption 6.1.2 and let $0 < \lambda \leq \lambda_0$. Then*

$$\lim_{\epsilon\to 0}\langle \psi_\lambda^\epsilon, UH^{\Omega_2}_{T_c^1(\lambda)}U^\dagger \psi_\lambda^\epsilon\rangle = \lambda(L_1+L_2) \tag{6.6}$$

with

$$\begin{aligned} L_1 = \int_{\tilde\Omega_1\times\mathbb{R}} \chi_{|z_2|<|r_2|}V(r)\Bigg(& |\Phi_\lambda(r_1,r_2,z_1)|^2 + |\Phi_\lambda(r_1,z_2,z_1)|^2 \\ & + \overline{\Phi_\lambda(r_1,r_2,z_1)}\Phi_\lambda(r_1,-r_2,z_1)e^{-2i\eta(\lambda)z_2} + \overline{\Phi_\lambda(r_1,z_2,z_1)}\Phi_\lambda(r_1,-z_2,z_1)e^{-2i\eta(\lambda)r_2} \\ & \mp \overline{\Phi_\lambda(r_1,r_2,z_1)}\Phi_\lambda(r_1,z_2,z_1)e^{i\eta(\lambda)(r_2-z_2)} \mp \overline{\Phi_\lambda(r_1,z_2,z_1)}\Phi_\lambda(r_1,r_2,z_1)e^{-i\eta(\lambda)(r_2-z_2)} \\ \mp\overline{\Phi_\lambda(r_1,r_2,z_1)}\Phi_\lambda(r_1,-z_2,z_1)e^{-i\eta(\lambda)(r_2+z_2)} & \mp\overline{\Phi_\lambda(r_1,z_2,z_1)}\Phi_\lambda(r_1,-r_2,z_1)e^{i\eta(\lambda)(-r_2+z_2)}\Bigg)\mathrm{d}r\mathrm{d}z \end{aligned} \tag{6.7}$$

and

$$\begin{aligned} L_2 = -\int_{\tilde\Omega_1\times\mathbb{R}} V(r)\left(|\Phi_\lambda(r_1,z_2,z_1)|^2 + \overline{\Phi_\lambda(r_1,z_2,z_1)}\Phi_\lambda(r_1,-z_2,z_1)e^{-2i\eta(\lambda)r_2}\right)\mathrm{d}r\mathrm{d}z \\ \mp 2\pi\int_{\mathbb{R}^2}\left(\overline{\widehat{\Phi_\lambda}(p_1,\eta(\lambda),q_1)}\widehat{V\chi_{\tilde\Omega_1}\Phi_\lambda}(p_1,\eta(\lambda),q_1) + \overline{\widehat{\Phi_\lambda}(p_1,-\eta(\lambda),q_1)}\widehat{V\chi_{\tilde\Omega_1}\Phi_\lambda}(p_1,-\eta(\lambda),q_1)\right)\mathrm{d}p_1\mathrm{d}q_1, \end{aligned} \tag{6.8}$$

where $\widehat{\psi}(p,q_1) = \int_{\mathbb{R}^3}\frac{e^{-ip\cdot r - iq_1z_1}}{(2\pi)^{3/2}}\psi(r,z_1)\mathrm{d}r\mathrm{d}z_1$ denotes the Fourier transform and $\chi_{\tilde\Omega_1}$ denotes multiplication by the characteristic function of $\tilde\Omega_1$.

Lemma 6.1.9. *Let $\mu > 0$ and let V satisfy Assumption 6.1.2. As $\lambda \to 0$ we have $L_1 = O(1)$ and $L_2 \leq -\frac{C}{\lambda}$ for some constant $C > 0$.*

In particular, there is a $\lambda_1 > 0$ such that for all $0 < \lambda \leq \lambda_1$, $\lim_{\epsilon\to0}\langle\psi_\lambda^\epsilon, UH^{\Omega_2}_{T_c^1(\lambda)}U^\dagger\psi_\lambda^\epsilon\rangle < 0$ and hence also $\inf\sigma(H^{\Omega_2}_{T_c^1(\lambda)}) < 0$. The final ingredient is the continuity of $\inf\sigma(H^{\Omega_2}_T)$ in T, which can be proved analogously to [60, Lemma 4.1]. For $\lambda \leq \lambda_1$ we have for $T < T_c^1(\lambda)$ by Lemma 6.1.1 and the definition of T_c^1 that $\inf\sigma(H^{\Omega_2}_T) \leq \inf\sigma(H^{\Omega_1}_T) < 0$. We saw that $\inf\sigma(H^{\Omega_2}_{T_c^1(\lambda)}) < 0$ and thus by continuity there is an $\epsilon > 0$ such that for all $T \in (0, T_c^1(\lambda)+\epsilon]$ we have $\inf\sigma(H^{\Omega_2}_T) < 0$. In particular, $T_c^2(\lambda) > T_c^1(\lambda)$. This concludes the proof of Theorem 6.1.4.

Remark 6.1.10. Compared to the proof of $T_c^1(\lambda) > T_c^0(\lambda)$ in [60] there are two main differences and additional difficulties here. The first difference is that Φ_λ here depends on r and z_1, and not just r. In particular, we need to understand the dependence and regularity of Φ_λ in z_1. The second difference is that for the full space minimizer it was possible to prove that the optimal momentum in the translation invariant center of mass direction is zero, whereas here we have to work with the momentum $\eta(\lambda)$, which potentially is non-zero, and we need knowledge about its asymptotics for $\lambda \to 0$. As a consequence, we may have that $\Phi_\lambda(r_1,r_2,z_1)e^{i\eta(\lambda)z_2} \neq \Phi_\lambda(r_1,-r_2,z_1)e^{-i\eta(\lambda)z_2}$, which is why the expressions in Lemma 6.1.8 are twice as long as in the analogous ones in [60, Lemma 4.3].

Remark 6.1.11. The Assumptions 6.1.2 are almost identical to the assumptions for proving $T_c^1(\lambda) > T_c^0(\lambda)$ in dimension two in [60]. Here we additionally assume $V \geq 0$ because to compute the asymptotics of Φ_λ we apply [60, Theorem 1.7] and several Lemmas used in the proof thereof, which require $V \geq 0$. However, we do not expect this assumption to be necessary.

Remark 6.1.12. We expect that our method of proof can also be applied in the three-dimensional case. For a quarter space in $d = 3$, we conjecture that similarly to the case of a half-space [60], the three-dimensional analogues of L_1 and L_2 in Lemma 6.1.8 are of equal order and converge to some finite numbers as $\lambda \to 0$. The limits of L_1 and L_2 then need to be computed to determine whether $\lim_{\lambda\to 0}(L_1 + L_2) < 0$. In [60], the ground state on the full space could effectively be replaced by $\Phi_0 = (\int_{\mathbb{R}^3} V(r) j_3(r)^2 \mathrm{d}r)^{-1} j_3$, with $j_3(r) = (2\pi)^{-3/2} \int_{\mathbb{S}^2} e^{i\sqrt{\mu}w\cdot r}\mathrm{d}\omega$, in the limit $\lambda \to 0$. Motivated by the asymptotics of the half-space minimizer Φ_λ in two dimensions proved in Lemma 6.3.2, we expect that as $\lambda \to 0$, $\eta(\lambda) \to 0$ and the function Φ_λ behaves like Φ_0 in the r-variable, and concentrates at zero momentum in the z_1 direction. A combination of the methods used in [60] and the methods developed in this paper should then allow to compute the limit, and the expected result is

$$\lim_{\lambda\to 0} L_1 = 2\int_{\mathbb{R}^4} \chi_{|z_2|<|r_2|} V(r) |\Phi_0(r) \mp \Phi_0(r_1, z_2, r_3)|^2 \mathrm{d}r\mathrm{d}z_2 \tag{6.9}$$

and

$$\lim_{\lambda\to 0} L_2 = -2\int_{\mathbb{R}^4} V(r)|\Phi_0(r_1, z_2, r_3)|^2 \mathrm{d}r\mathrm{d}z_2 \mp \frac{2\pi}{\mu^{1/2}} \int_{\mathbb{R}^3} V(r)|\Phi_0(r)|^2 \mathrm{d}r. \tag{6.10}$$

We therefore expect $T_c^2(\lambda) > T_c^1(\lambda)$ at weak enough coupling if V satisfies $\lim_{\lambda\to 0}(L_1 + L_2) < 0$, which due to radiality of V and Φ_0 is the same condition as for $T_c^1(\lambda) > T_c^0(\lambda)$ in [60, Theorem 1.3]. In [60, Theorem 1.4 and Remark 1.5] this condition on V is further analyzed.

6.2 Basic properties of $H_T^{\Omega_1}$ and $H_T^{\Omega_2}$

In this section we shall introduce some notation that will be useful later on, and prove Lemmas 6.1.1, 6.1.6 and 6.1.7. The following functions will be important:

$$K_T(p, q) = \frac{p^2 + q^2 - 2\mu}{\tanh\left(\frac{p^2-\mu}{2T}\right) + \tanh\left(\frac{q^2-\mu}{2T}\right)}, \quad \text{and} \quad B_T(p, q) = \frac{1}{K_T(p + q, p - q)}. \tag{6.11}$$

We may write $B_{T,\mu}$ when the dependence on μ matters. The function K_T satisfies the following bounds [34, Lemma 2.1].

Lemma 6.2.1. *For every $T > 0$ there are constants $C_1(T, \mu), C_2(T, \mu) > 0$ such that $C_1(1 + p^2 + q^2) \leq K_T(p, q) \leq C_2(1 + p^2 + q^2)$.*

We will frequently use the following estimates for B_T [60, Eq. (2.3)]:

$$B_T(p, q) \leq \frac{1}{\max\{|p^2 + q^2 - \mu|, 2T\}} \quad \text{and} \quad B_T(p, q)\chi_{p^2+q^2>2\mu>0} \leq \frac{C(\mu)}{1 + p^2 + q^2}, \tag{6.12}$$

where $C(\mu)$ depends only on μ.

We use the notation $H_0^1(\Omega)$ for the Sobolev space of functions vanishing at the boundary of Ω. In the case of Dirichlet boundary conditions, the form domain corresponding to $H_T^{\Omega_k}$ is

$D_k^D := \{\psi \in H_0^1(\Omega_k \times \Omega_k) | \psi(x,y) = \psi(y,x)\}$. For Neumann boundary conditions, one needs to replace the Sobolev space H_0^1 by H^1 to obtain D_k^N. Let K_T^Ω be the kinetic term in H_T^Ω. The corresponding quadratic form acts as

$$\langle \psi, K_T^\Omega \psi \rangle = \int_{\mathbb{R}^4} K_T(p,q) \left| \int_{\Omega^2} T_\Omega(x,p) T_\Omega(y,q) \psi(x,y) \mathrm{d}x \mathrm{d}y \right|^2 \mathrm{d}p \mathrm{d}q, \tag{6.13}$$

with

$$T_{\Omega_1}(x,p) = \frac{(e^{-ip_1x_1} \mp e^{ip_1x_1})e^{-ip_2x_2}}{2^{1/2}2\pi}, \quad \text{and} \quad T_{\Omega_2}(x,p) = \frac{(e^{-ip_1x_1} \mp e^{ip_1x_1})(e^{-ip_2x_2} \mp e^{ip_2x_2})}{4\pi}. \tag{6.14}$$

As already mentioned in the Introduction, we shall use the convention that upper signs correspond to Dirichlet and lower signs to Neumann boundary conditions, unless stated otherwise. For the switch to relative and center of mass coordinates, it is convenient to define

$$t(p,q_1,r,z_1) = \frac{1}{2}\left(e^{-i(p_1r_1+q_1z_1)} + e^{i(p_1r_1+q_1z_1)} \mp e^{-i(p_1z_1+q_1r_1)} \mp e^{i(p_1z_1+q_1r_1)}\right) e^{-ip_2r_2}. \tag{6.15}$$

Note that with $r = x-y$, $z = x+y$, $p' = (p-q)/2$ and $q' = (p+q)/2$ we have

$$T_{\Omega_1}(x,p)T_{\Omega_1}(y,p) = \frac{1}{(2\pi)^2} t(p',q_1',r,z_1)e^{-iq_2'z_2} \tag{6.16}$$

and therefore

$$\langle \psi, UK_T^{\Omega_1}U^\dagger \psi \rangle = \int_{\mathbb{R}^4} B_T(p',q')^{-1} \left| \int_{\tilde{\Omega}_1 \times \mathbb{R}} \frac{1}{(2\pi)^2} t(p',q_1',r,z_1)e^{-iq_2'z_2}\psi(r,z)\mathrm{d}r\mathrm{d}z \right|^2 \mathrm{d}p'\mathrm{d}q'. \tag{6.17}$$

The operators $H_T^1(q_2)$ defined by restricting $UH_T^{\Omega_1}U^\dagger$ to momentum q_2 in z_2-direction can thus be expressed as

$$\langle \psi, H_T^1(q_2)\psi \rangle = \langle \psi, K_T^1(q_2)\psi \rangle - \lambda \int_{\tilde{\Omega}_1} V(r)|\psi(r,z_1)|^2 \mathrm{d}r\mathrm{d}z_1 \tag{6.18}$$

where the kinetic term $K_T^1(q_2)$ on $L^2_{\mathrm{s}}(\tilde{\Omega}_1)$ is given by

$$\langle \psi, K_T^1(q_2)\psi \rangle = \int_{\mathbb{R}^3} B_T(p,(q_1,q_2))^{-1} \left| \int_{\tilde{\Omega}_1} \frac{1}{(2\pi)^{3/2}} t(p,q_1,r,z_1)\psi(r,z_1)\mathrm{d}r\mathrm{d}z_1 \right|^2 \mathrm{d}p\mathrm{d}q_1. \tag{6.19}$$

It is convenient to introduce the Birman-Schwinger operators A_T^0 and A_T^1 corresponding to $H_T^{\Omega_0}$ and $H_T^{\Omega_1}$, respectively. Let A_T^0 be the operator with domain $L^2(\mathbb{R}^2 \times \mathbb{R}^2)$ restricted to functions satisfying $\psi(r,z) = \psi(-r,z)$ and given by

$$\langle \psi, A_T^0 \psi \rangle = \int_{\mathbb{R}^4} B_T(p,q)|\widehat{V^{1/2}\psi}(p,q)|^2 \mathrm{d}p\mathrm{d}q. \tag{6.20}$$

Define the operator A_T^1 on $\psi \in L^2_{\mathrm{s}}(\tilde{\Omega}_1 \times \mathbb{R}) = \{\psi \in L^2(\tilde{\Omega}_1 \times \mathbb{R}) | \psi(r,z) = \psi(-r,z)\}$ via

$$\langle \psi, A_T^1 \psi \rangle = \int_{\mathbb{R}^4} B_T(p,q) \left| \int_{\tilde{\Omega}_1 \times \mathbb{R}} \frac{1}{(2\pi)^2} t(p,q_1,r,z_1)e^{-iq_2z_2}V^{1/2}(r)\psi(r,z)\mathrm{d}r\mathrm{d}z \right|^2 \mathrm{d}p\mathrm{d}q. \tag{6.21}$$

For $j \in \{0,1\}$, the operator A_T^j is the Birman-Schwinger operator corresponding to $H_T^{\Omega_j}$ in relative and center of mass variables [60, Section 6]. The Birman-Schwinger principle implies that $\operatorname{sgn}\inf\sigma(H_T^{\Omega_j}) = \operatorname{sgn}(1/\lambda - \sup\sigma(A_T^j))$, where we use the convention that $\operatorname{sgn} 0 = 0$.

Due to translation invariance in z_2, for fixed momentum q_2 in this direction, we obtain the operators $A_T^1(q_2)$ on $\psi \in L^2_{\mathrm{s}}(\tilde{\Omega}_1)$ given by

$$\langle \psi, A_T^1(q_2)\psi\rangle = \int_{\mathbb{R}^3} B_T(p,(q_1,q_2)) \left| \int_{\tilde{\Omega}_1} \frac{1}{(2\pi)^{3/2}} t(p,q_1,r,z_1) V^{1/2}(r)\psi(r,z_1) \mathrm{d}r\mathrm{d}z_1 \right|^2 \mathrm{d}p\mathrm{d}q_1. \tag{6.22}$$

The operator $A_T^1(q_2)$ is the Birman-Schwinger version of $H_T^1(q_2)$. In particular, $H^1_{T_c^1(\lambda)}(\eta(\lambda))$ has the eigenvalue zero at the bottom of its spectrum if and only if $1/\lambda$ is the largest eigenvalue of $A^1_{T_c^1(\lambda)}(\eta(\lambda))$.

Let $\iota : L^2(\tilde{\Omega}_1) \to L^2(\mathbb{R}^3)$ be the isometry

$$\iota\psi(r_1,r_2,z_1) = \frac{1}{\sqrt{2}}(\psi(r_1,r_2,z_1)\chi_{\tilde{\Omega}_1}(r,z_1) + \psi(-r_1,r_2,-z_1)\chi_{\tilde{\Omega}_1}(-r_1,r_2,-z_1)). \tag{6.23}$$

Using the definition of t in (6.15) and evenness of V in r_2 one can rewrite (6.22) as

$$\langle \psi, A_T^1(q_2)\psi\rangle = \int_{\mathbb{R}^3} B_T(p,q) \left| \frac{1}{\sqrt{2}} (\widehat{V^{1/2}\iota\psi}(p,q_1) \mp \widehat{V^{1/2}\iota\psi}((q_1,p_2),p_1)) \right|^2 \mathrm{d}p\mathrm{d}q_1 \tag{6.24}$$

Let F_2 denote the Fourier transform in the second variable $F_2\psi(r,q_1) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-iq_1z_1}\psi(r,z_1)\mathrm{d}z_1$ and F_1 the Fourier transform in the first variable $F_1\psi(p,q) = \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{-ip\cdot r}\psi(r,q)\mathrm{d}r$. Define the operators $G_T(q_2)$ on $L^2(\mathbb{R}^3)$ through

$$\langle \psi, G_T(q_2)\psi\rangle = \int_{\mathbb{R}^3} \overline{F_1 V^{1/2}\psi((q_1,p_2),p_1)} B_T(p,q) F_1 V^{1/2}\psi(p,q_1)\mathrm{d}p\mathrm{d}q_1. \tag{6.25}$$

Let $A_T^0(q_2)$ acting on $L^2_{\mathsf{s}}(\mathbb{R}^2\times\mathbb{R})$ be given by $\langle\psi, A_T^0(q_2)\psi\rangle = \int_{\mathbb{R}^3} B_T(p,q)|\widehat{V^{1/2}\psi}(p,q_1)|^2\mathrm{d}p\mathrm{d}q_1$. It follows from (6.24) and $B_T(p,q) = B_T((q_1,p_2),(p_1,q_2))$ that

$$A_T^1(q_2) = \iota^\dagger(A_T^0(q_2) \mp F_2^\dagger G_T(q_2)F_2)\iota. \tag{6.26}$$

6.2.1 Proof of Lemma 6.1.1

Proof of Lemma 6.1.1. We proceed analogously to the proof of [60, Lemma 2.3]. Let S_l be the shift by l in the second component, i.e. $S_l\psi(x,y) = \psi((x_1,x_2-l),(y_1,y_2-l))$. Let ψ be a function in $D_1^{D/N}$ with bounded support, for the case of Dirichlet/Neumann boundary conditions, respectively. For l big enough, $S_l\psi$ is supported on $\Omega_2\times\Omega_2$ and satisfies the boundary conditions. The goal is to prove that $\lim_{l\to\infty}\langle S_l\psi, H_T^{\Omega_2}S_l\psi\rangle = \langle\psi, H_T^{\Omega_1}\psi\rangle$. Then, since functions with bounded support are dense in $D_1^{D/N}$ (with respect to the Sobolev norm), the claim follows.

Note that $\langle S_l\psi, VS_l\psi\rangle = \langle\psi, V\psi\rangle$. Let $\tilde{\psi}$ be the (anti-)symmetric continuation of ψ from $\Omega_1\times\Omega_1$ to $\mathbb{R}^2\times\mathbb{R}^2$ as in Figure 6.1, giving $\tilde{\psi}\in H^1(\mathbb{R}^4)$. Furthermore, using symmetry of K_T in p_2 and q_2 one obtains

$$\langle S_l\psi, K_T^{\Omega_2}S_l\psi\rangle = \frac{1}{4}\int_{\mathbb{R}^4} \overline{\hat{\tilde{\psi}}(p,q)} K_T(p,q) \Big[\hat{\tilde{\psi}}(p,q) \mp \hat{\tilde{\psi}}((p_1,-p_2),q)e^{i2lp_2} \mp \hat{\tilde{\psi}}(p,(q_1,-q_2))e^{i2lq_2} + \hat{\tilde{\psi}}((p_1,-p_2),(q_1,-q_2))e^{i2l(p_2+q_2)}\Big]\mathrm{d}p\mathrm{d}q \tag{6.27}$$

for l big enough such that $S_l\psi$ is supported on $\Omega_2 \times \Omega_2$. The first term is exactly $\langle \psi, K_T^{\Omega_1}\psi\rangle$. Note that by the Schwarz inequality and Lemma 6.2.1, the function

$$(p,q) \mapsto \overline{\widehat{\tilde{\psi}}(p,q)} K_T(p,q) \widehat{\tilde{\psi}}((p_1,-p_2),q) \tag{6.28}$$

is in $L^1(\mathbb{R}^{2d})$ since $\tilde{\psi} \in H^1(\mathbb{R}^4)$. By the Riemann-Lebesgue Lemma, the second term in (6.27) vanishes for $l \to \infty$. By the same argument, also the remaining terms vanish in the limit. □

6.2.2 Proof of Lemma 6.1.6

Proof of Lemma 6.1.6. For continuity, it suffices to prove that for all $T > 0$ and $\mu, Q_0, Q_1 \in \mathbb{R}$ there is a constant $C(T,\mu,Q_0,Q_1)$ such that for all $Q_0 < q_2, q_2' < Q_1$ we have $|B_T(p,q)^{-1} - B_T(p,(q_1,q_2'))^{-1}| \leq C(T,\mu,Q_0,Q_1)|q_2 - q_2'|(1+p^2+q_1^2)$. The claim then follows analogously to the proof of [60, Lemma 4.1].

We write $B_T(p,q)^{-1} - B_T(p,(q_1,q_2'))^{-1} = (q_2' - q_2) f(p,q,q_2' - q_2) B_T^{-1}(p,(q_1,q_2')) B_T^{-1}(p,q)$, where f is defined as in the following Lemma.

Lemma 6.2.2. *Let $T,\mu,Q_1 > 0$ and define the function $f : \mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ through*

$$f(p,q,x) = \frac{1}{x}\left(B_T(p,(q_1,q_2+x)) - B_T(p,q)\right) \tag{6.29}$$

for $x \neq 0$ and $f(p,q,0) = \partial_{q_2} B_T(p,q)$. Then f is continuous and for $|q_2| < Q_1$ there is a constant C depending only on T,μ and Q_1 such that

$$|f(p,q,x)| \leq \frac{C}{1+p_1^2+p_2^2+q_1^2}. \tag{6.30}$$

The proof is provided in Section 6.7.1. Together with Lemma 6.2.1 the desired bound on $|B_T(p,q)^{-1} - B_T(p,(q_1,q_2'))^{-1}|$ follows.

The function $q_2 \to \inf\sigma(H_T^1(q_2))$ is even since by (6.18), (6.19) and radiality of V we have $\langle \psi, H_T^1(-q_2)\psi\rangle = \langle \tilde{\psi}, H_T^1(q_2)\tilde{\psi}\rangle$, where $\tilde{\psi}(r,z_1) = \psi((r_1,-r_2),z_1)$. The divergence of $\inf\sigma(H_T^1(q_2))$ as $|q_2| \to \infty$ follows from (6.19) and (6.12). □

6.2.3 Proof of Lemma 6.1.7

Proof of Lemma 6.1.7. According to (6.26), the half-space Birman-Schwinger operator $A_T^1(q_2)$ for $q_2 \in \mathbb{R}$ can be decomposed into a term involving $A_T^0(q_2)$ and a perturbation involving $G_T(q_2)$. The operator $A_T^0(q_2)$ has purely essential spectrum and $a_T^0 := \sup\sigma(A_T^0) = \sup\sigma(A_T^0(0))$ [60, Lemma 2.4]. Below we shall prove that $G_T(q_2)$ is compact. The part of the spectrum of A_T^1 that lies above a_T^0 hence consists of eigenvalues.

Since $\sup\sigma(A_T^0)$ is strictly decreasing in T and $T_c^1(\lambda) > T_c^0(\lambda)$, $\sup\sigma(A^1_{T_c^1(\lambda)}(\eta(\lambda))) = \lambda^{-1} > a^0_{T_c^1(\lambda)}$. Hence λ^{-1} is an eigenvalue of $A^1_{T_c^1(\lambda)}(\eta(\lambda))$ and by the Birman-Schwinger principle $H^1_{T_c^1(\lambda)}(\eta(\lambda))$ has an eigenvalue at the bottom of the spectrum.

To prove compactness of $G_T(q_2)$ defined in (6.25), we prove that its Hilbert-Schmidt norm is finite. Writing out the Hilbert-Schmidt norm in terms of the integral kernel of $G_T(q_2)$ and carrying out the integrations over relative and center of mass coordinates, one obtains

$$\|G_T(q_2)\|^2_{\mathrm{HS}} = \int_{\mathbb{R}^4} |\widehat{V}(0,p_2-p_2')|^2 B_T(p,q) B_T((p_1,p_2'),q) \mathrm{d}p_1 \mathrm{d}q_1 \mathrm{d}p_2 \mathrm{d}p_2'. \tag{6.31}$$

By (6.12) and Young's inequality, this is bounded above by

$$C(T,\mu)\left(\int_{\mathbb{R}}|\widehat{V}(0,|p_2|)|^{2r}\mathrm{d}p_2\right)^{1/r}\int_{\mathbb{R}}\left(\int_{\mathbb{R}}\left(\frac{1}{1+p_1^2+q_1^2+p_2^2}\right)^s\mathrm{d}p_2\right)^{2/s}\mathrm{d}p_1\mathrm{d}q_1 \tag{6.32}$$

where $2=1/r+2/s$. By assumption $V\in L^1\cap L^t$ for some $t>1$. Note that $\widehat{V}$ is continuous by Riemann-Lebesgue and $\widehat{V}\in L^{t'}\cap L^\infty$ for some $t'<\infty$ by the Hausdorff-Young inequality. In particular, due to the radiality of V, we can bound $\left(\int_{\mathbb{R}}|\widehat{V}(0,|p_2|)|^{2r}\right)^{1/r}\leq\|V\|_\infty^2+\frac{1}{2\pi}\|\widehat{V}\|_{2r}^2$, which is finite for the choice $r=t'/2$. With this choice, we have $s>1$. Note that $\left(\int_{\mathbb{R}}\left(\frac{1}{1+p_1^2+q_1^2+p_2^2}\right)^s\mathrm{d}p_2\right)^{2/s}=\frac{C}{(1+p_1^2+q_1^2)^{2-1/s}}$ for some constant C. Hence the integral over p_1,q_1 in (6.32) is finite for $s>1$. □

6.3 Regularity and asymptotic behavior of the half-space ground state

In this section we prove some regularity and convergence results for Φ_λ (defined in Section 6.1.1), which we shall use later to prove Lemmas 6.1.8 and 6.1.9. The asymptotics of $T_c^0(\lambda)$ and $T_c^1(\lambda)$ for $\lambda\to 0$ are known:

Remark 6.3.1. It follows from [40, Theorem 2.5] that $|\lambda^{-1}-e_\mu\ln\frac{\mu}{T_c^0(\lambda)}|=O(1)$ for $\lambda\to 0$. Furthermore, [60, Theorem 1.7] implies that $\ln\frac{\mu}{T_c^0(\lambda)}-\ln\frac{\mu}{T_c^1(\lambda)}=o(1)$ for $\lambda\to 0$. Therefore, $|\lambda^{-1}-e_\mu\ln\frac{\mu}{T_c^1(\lambda)}|=O(1)$ as well. In particular, both $T_c^0(\lambda)$ and $T_c^1(\lambda)\to 0$ as $\lambda\to 0$ exponentially fast.

Let $\Psi_\lambda(r,z_1):=\frac{1}{\sqrt{2}}V^{1/2}(r)\Phi_\lambda(r,z_1)\chi_{|r_1|<|z_1|}$ as function on $\mathbb{R}^3$. Note that $\|\Psi_\lambda\|_2=1$ due to the symmetry under $(r_1,z_1)\to-(r_1,z_1)$ and the normalization $\|V^{1/2}\chi_{\tilde{\Omega}_1}\Phi_\lambda\|_2=1$. The first convergence result describes the asymptotic behavior of $\eta(\lambda)$ and Ψ_λ as $\lambda\to 0$. According to the Birman-Schwinger principle, $\chi_{\tilde{\Omega}_1}\Psi_\lambda$ is an eigenvector of $A_{T_c^1(\lambda)}(\eta(\lambda))$ corresponding to the largest eigenvalue.

Let

$$j_2(r):=\frac{1}{2\pi}\int_{\mathbb{S}^1}e^{i\omega\cdot r\sqrt{\mu}}\mathrm{d}\omega. \tag{6.33}$$

Due to assumptions 6.1.22 and 5, the eigenvector corresponding to the largest eigenvalue e_μ of O_μ has angular momentum zero and is given by [60]

$$\psi^0(r)=\frac{V^{1/2}(r)j_2(r)}{\left(\int_{\mathbb{R}^2}V(r')j_2(r')^2\mathrm{d}r'\right)^{1/2}}. \tag{6.34}$$

Let $\mathbb{P}:L^2(\mathbb{R}^3)\to L^2(\mathbb{R}^3)$ denote the projection onto ψ^0 in the r-variable, i.e. $\mathbb{P}\psi(r,q_1)=\psi^0(r)\int_{\mathbb{R}^2}\overline{\psi^0(r')}\psi(r',q_1)\mathrm{d}r'$. For $0\leq\beta<1$ let $\mathbb{Q}_\beta$ denote the projection onto small momenta in q_1, i.e. $\mathbb{Q}_\beta\psi(r,q_1)=\psi(r,q_1)\chi_{\frac{|q_1|}{\sqrt{\mu}}<\left(\frac{T_c^1(\lambda)}{\mu}\right)^\beta}$. Let $\mathbb{P}^\perp=\mathbb{I}-\mathbb{P}$ and $\mathbb{Q}_\beta^\perp=1-\mathbb{Q}_\beta$.

Our first convergence result is that for $\lambda\to 0$ the minimizer of $H^{\Omega_1}_{T_c^1(\lambda)}$ concentrates at momentum zero in the center of mass variable. More precisely, $\eta(\lambda)\to 0$ and Ψ_λ concentrates at momentum zero in z_1 direction and approaches ψ^0 in the r-variables. This is made precise in the following Lemma, whose proof can be found in Section 6.3.1.

Lemma 6.3.2. *Let $\mu > 0$, V satisfy Assumption 6.1.2 and let $0 \leq \beta < 1$. For $\lambda \to 0$ we have*

1. $\eta(\lambda) = O(T_c^1(\lambda))$
2. $\|\mathbb{P}^\perp F_2 \Psi_\lambda\|_2^2 = O(\lambda)$
3. $\|\mathbb{Q}_\beta^\perp F_2 \Psi_\lambda\|_2^2 = O(\lambda)$

For the following regularity and convergence results we need to introduce some more notation. For a function f depending on two variables we define the mixed Lebesgue norm $\|f\|_{L_i^p L_j^q}$ for $\{i,j\} = \{1,2\}$, as first taking the L^q-norm in the j-th variable and then taking the L^p-norm in the i-th variable. The following estimate is analogous to [60, Lemma 3.7] and follows from the Cauchy-Schwarz inequality.

Lemma 6.3.3. *Let $V \in L^1(\mathbb{R}^2)$ and $\psi \in L^2(\mathbb{R}^2 \times \mathbb{R})$. Then*

$$\begin{aligned}\|\widehat{V^{1/2}\psi}\|_{L_1^\infty L_2^2} &\leq \sup_p \left(\int_{\mathbb{R}} |\widehat{V^{1/2}\psi}(p,q_1)|^2 \mathrm{d}q_1 \right)^{1/2} \\ &\leq \|\widehat{V^{1/2}\psi}\|_{L_2^2 L_1^\infty} = \left(\int_{\mathbb{R}} \sup_p |\widehat{V^{1/2}\psi}(p,q_1)|^2 \mathrm{d}q_1 \right)^{1/2} \leq \frac{\|V\|_1^{1/2}}{2\pi} \|\psi\|_2. \quad (6.35)\end{aligned}$$

To simplify notation, we shall sometimes write T_c^1, η instead of $T_c^1(\lambda), \eta(\lambda)$. The eigenvalue equation $\chi_{\tilde{\Omega}_1}\Phi_\lambda = \lambda(K^1_{T_c^1(\lambda)}(\eta(\lambda))^{-1} V \chi_{\tilde{\Omega}_1}\Phi_\lambda$ combined with

$$\frac{1}{(2\pi)^{3/2}} \int_{\tilde{\Omega}_1} t(p,q_1,r,z_1)\Phi_\lambda(r,z_1)\mathrm{d}r\mathrm{d}z_1 = \frac{1}{2}\widehat{\Phi_\lambda}(p,q_1) \quad (6.36)$$

gives

$$\widehat{\Phi_\lambda}(p,q_1) = \frac{2\lambda}{(2\pi)^{3/2}} \int_{\tilde{\Omega}_1} B_{T_c^1(\lambda)}(p,(q_1,\eta(\lambda)))t(p,q_1,r',z_1')V(r')\Phi_\lambda(r',z_1')\mathrm{d}r'\mathrm{d}z_1' \quad (6.37)$$

for $(p,q_1) \in \mathbb{R}^3$. We use (6.37) together with (6.15) and the definition of Ψ_λ to split Φ_λ into the sum $\Phi_\lambda^d \mp \Phi_\lambda^{ex}$, where

$$\Phi_\lambda^d(r,z_1) = \sqrt{2}\lambda \int_{\mathbb{R}^3} \frac{e^{i(p\cdot r + q_1 z_1)}}{(2\pi)^{3/2}} B_{T_c^1}(p,(q_1,\eta))\widehat{V^{1/2}\Psi}_\lambda(p,q_1)\mathrm{d}p\mathrm{d}q_1 \quad (6.38)$$

and

$$\Phi_\lambda^{ex}(r,z_1) = \sqrt{2}\lambda \int_{\mathbb{R}^3} \frac{e^{i(p\cdot r + q_1 z_1)}}{(2\pi)^{3/2}} B_{T_c^1}(p,(q_1,\eta))\widehat{V^{1/2}\Psi}_\lambda((q_1,p_2),p_1)\mathrm{d}p\mathrm{d}q_1. \quad (6.39)$$

For $j \in \{d, ex\}$ we further split $\Phi_\lambda^j = \Phi_\lambda^{j,<} + \Phi_\lambda^{j,>}$, where $\Phi^{j,\#}$ for $\# \in \{<,>\}$ has the characteristic function $\chi_{p^2+q_1^2 \# 2\mu}$ in the integrand. Furthermore, let $\Phi^\# = \Phi^{d,\#} \mp \Phi^{ex,\#}$.

The following three Lemmas contain regularity properties for Φ_λ, which are later used for dominated convergence arguments in the proof of Lemma 6.1.8. Furthermore, they also contain information about the weak coupling behavior of the different $\Phi_\lambda^{j,\#}$, which is important for the proof of Lemma 6.1.9. The first Lemma is useful to prove that L_1 is of order $O(1)$.

Lemma 6.3.4. *Let $\mu > 0$, let V satisfy Assumption 6.1.2 and let $0 < \lambda \leq \lambda_0$. Then $\|\Phi_\lambda\|_{L_1^\infty L_2^2} < \infty$. Furthermore, $\|\Phi_\lambda^d\|_{L_1^\infty L_2^2} = O(1)$ and $\|\Phi_\lambda^{ex,>}\|_{L_1^\infty L_2^2} = O(\lambda)$ as $\lambda \to 0$.*

To understand the asymptotics of L_2 the following result comes in handy.

Lemma 6.3.5. *Let $\mu > 0$, let V satisfy Assumption 6.1.2 and let $0 < \lambda \leq \lambda_0$. The function $(r, z) \mapsto V^{1/2}(r)|\Phi_\lambda(r_1, z_2, z_1)|$ is in $L^2(\mathbb{R}^4)$. Furthermore, as $\lambda \to 0$, the $L^2(\mathbb{R}^4)$-norms of the functions $V^{1/2}(r)|\Phi_\lambda^>(r_1, z_2, z_1)|$, $V^{1/2}(r)|\Phi_\lambda^{d,<}(r_1, z_2, z_1)|$ and $V^{1/2}(r)|\Phi_\lambda^{ex,<}(r_1, z_2, z_1)|$ are of order $O(\lambda)$, $O(\lambda^{-1/2})$, and $O(\lambda^{1/2})$, respectively.*

This suggests that the only possible origin for divergence in L_2 lies in contributions from $V^{1/2}(r)|\Phi_\lambda^{d,<}(r_1, z_2, z_1)|$. In the proof of Lemma 6.1.9 we shall show that the L^2 norm of this term indeed grows as $\lambda^{-1/2}$, resulting in the $1/\lambda$ divergence of L_2. Furthermore, we need the following for the proof of Lemma 6.1.8.

Lemma 6.3.6. *Let $\mu > 0$, let V satisfy Assumption 6.1.2 and let $0 < \lambda \leq \lambda_0$. Define the functions g_0, g_+ and g_- on $\mathbb{R}^2$ as*

$$g_0(p_2, q_2) := \int_{\mathbb{R}^2} \overline{\widehat{\Phi}_\lambda(p, q_1)} \widehat{V\chi_{\tilde{\Omega}_1}\Phi_\lambda}(p_1, q_2, q_1) \mathrm{d}p_1 \mathrm{d}q_1 \tag{6.40}$$

and

$$g_\pm(p_2, q_2) := \int_{\mathbb{R}^2} \overline{\widehat{\Phi}_\lambda(p, q_1)} \Big[B_{T_c^1}^{-1}(p, q) - B_{T_c^1}^{-1}(p, (q_1, \eta)) \Big] \widehat{\Phi}_\lambda((p_1, \pm q_2), q_1) \mathrm{d}p_1 \mathrm{d}q_1. \tag{6.41}$$

The functions g_0 and $g_\pm$ are continuous and bounded and $g_\pm(p_2, \eta) = 0$ for all $p_2 \in \mathbb{R}$.

The proofs of these three Lemmas are given in Sections 6.3.2 – 6.3.4.

6.3.1 Proof of Lemma 6.3.2

Proof of Lemma 6.3.2. Recall the operators A_T^0, and A_T^1 from Section 6.2 and let $a_T^j = \sup \sigma(A_T^j)$. It follows from Lemma 6.1.1 and the Birman-Schwinger principle that $a_T^0 \leq a_T^1$ (for details see the proof of [60, Theorem 1.7]). According to [40, Lemma 3.4] for $T \to 0$ the asymptotic behavior of a_T^0 is given by $a_T^0 = e_\mu \ln(\mu/T) + O(1)$. Recall the decomposition of $A_T^1(q_2)$ in (6.26). The operator norm of $G_T(q_2)$ is bounded uniformly in T and q_2 according to [60, Lemma 6.1]. Recall that $\sqrt{2}\chi_{\tilde{\Omega}_1}\Psi_\lambda$ is a normalized eigenvector of $A^1_{T_c^1(\lambda)}(\eta(\lambda))$ and note that $\iota\sqrt{2}\chi_{\tilde{\Omega}_1}\Psi_\lambda = \Psi_\lambda$, where ι was defined in (6.23). With Remark 6.3.1, we have for $\lambda \to 0$

$$e_\mu \ln \mu/T_c^1(\lambda) + O(1) = a^0_{T_c^1(\lambda)} \leq a^1_{T_c^1(\lambda)} = \langle \Psi_\lambda, A^0_{T_c^1(\lambda)}(\eta(\lambda))\Psi_\lambda \rangle + O(1) \tag{6.42}$$

For $q \in \mathbb{R}^2$ let $B_T(\cdot, q)$ denote the operator on $L^2(\mathbb{R}^2)$ which acts as multiplication by $B_T(p, q)$ (defined in (6.11)) in momentum space. Note that

$$\langle \Psi_\lambda, A^0_{T_c^1(\lambda)}(\eta(\lambda))\Psi_\lambda \rangle = \int_{\mathbb{R}} \langle F_2\Psi_\lambda(\cdot, q_1), V^{1/2} B_{T_c^1(\lambda)}(\cdot, (q_1, \eta(\lambda))) V^{1/2} F_2 \Psi_\lambda(\cdot, q_1)\rangle \mathrm{d}q_1 \tag{6.43}$$

According to [60, Lemma 6.8], there is a constant $C(\mu, V)$, such that for all $q \in \mathbb{R}^2$ and $\psi \in L^2_{\mathrm{s}}(\mathbb{R}^2)$ with $\|\psi\|_2 = 1$

$$\langle \psi, V^{1/2} B_T(\cdot, q) V^{1/2} \psi \rangle \leq \langle \psi, O_\mu \psi \rangle \ln\left(\min\left\{ \frac{\sqrt{\mu}}{|q|}, \frac{\mu}{T} \right\} \right) \chi_{2 < \min\{\mu/T, \sqrt{\mu}/|q|\}} + C(\mu, V). \tag{6.44}$$

In combination, we have for $\lambda \to 0$

$$e_\mu \ln \mu/T_c^1(\lambda) \leq \int_{|q_1|<\sqrt{\mu}/2} \langle F_2\Psi_\lambda(\cdot,q_1), O_\mu F_2\Psi_\lambda(\cdot,q_1)\rangle \ln\left(\min\left\{\frac{\sqrt{\mu}}{\sqrt{q_1^2+\eta(\lambda)^2}}, \frac{\mu}{T_c^1(\lambda)}\right\}\right) \mathrm{d}q_1 + O(1) \tag{6.45}$$

We will use this to prove the three parts of the claim.

Part 1: By definition of e_μ, we can bound $\langle F_2\Psi_\lambda(\cdot,q_1), O_\mu F_2\Psi_\lambda(\cdot,q_1)\rangle \leq e_\mu \|F_2\Psi_\lambda(\cdot,q_1)\|_2^2$. Moreover, clearly $\ln\left(\min\left\{\frac{\sqrt{\mu}}{\sqrt{q_1^2+\eta(\lambda)^2}}, \frac{\mu}{T_c^1(\lambda)}\right\}\right) \leq \ln(\sqrt{\mu}/\eta(\lambda))$. By (6.45) and since $\|F_2\Psi_\lambda\|_2 = 1$, there is a constant c such that $e_\mu \ln(\mu/T_c^1(\lambda)) \leq e_\mu \ln(\sqrt{\mu}/\eta(\lambda)) + c$ for small λ. In particular, $|\eta(T)| \leq \frac{\exp(c/e_\mu)}{\sqrt{\mu}} T_c^1(\lambda)$, i.e. $\eta(\lambda) = O(T_c^1(\lambda))$.

Part 2: Denote the ratio of the second highest and the highest eigenvalue of O_μ by α, where $\alpha < 1$ by Assumption 6.1.25. Then

$$\begin{aligned}\int_{\mathbb{R}} \langle F_2\Psi_\lambda(\cdot,q_1), O_\mu F_2\Psi_\lambda(\cdot,q_1)\rangle \mathrm{d}q_1 &\leq e_\mu \left(\|\mathbb{P}F_2\Psi_\lambda\|^2 + \alpha\|\mathbb{P}^\perp F_2\Psi_\lambda\|^2\right) \\ &= e_\mu \left(\|F_2\Psi_\lambda\|^2 - (1-\alpha)\|\mathbb{P}^\perp F_2\Psi_\lambda\|^2\right)\end{aligned} \tag{6.46}$$

Therefore, by (6.45)

$$\ln \mu/T_c^1(\lambda) \leq \left(1-(1-\alpha)\|\mathbb{P}^\perp F_2\Psi_\lambda\|^2\right) \ln \mu/T_c^1(\lambda) + O(1) \tag{6.47}$$

for $\lambda \to 0$. This means that $\|\mathbb{P}^\perp F_2\Psi_\lambda\|^2 = O(1/\ln \mu/T_c^1(\lambda))$. According to Remark 6.3.1, $\lim_{\lambda\to 0} \lambda \ln \mu/T_c^1(\lambda) = e_\mu^{-1}$ and thus $\|\mathbb{P}^\perp F_2\Psi_\lambda\|^2 = O(\lambda)$.

Part 3: Let $\epsilon(\lambda) = \|\mathbb{Q}_\beta^\perp F_2\Psi_\lambda\|^2 = \int_{\mathbb{R}^3} |F_2\Psi_\lambda(r,q_1)|^2 \chi_{|q_1|>\sqrt{\mu}\left(\frac{T_c^1(\lambda)}{\mu}\right)^\beta} \mathrm{d}r\mathrm{d}q_1$. By (6.45), we have for small λ

$$e_\mu \ln \mu/T_c^1(\lambda) \leq (1-\epsilon(\lambda)) e_\mu \ln \mu/T_c^1(\lambda) + \epsilon(\lambda) e_\mu \ln \frac{\mu^\beta}{T_c^1(\lambda)^\beta} + C \tag{6.48}$$

for some constant C. Hence

$$\epsilon(\lambda) \leq \frac{C}{(1-\beta)e_\mu \ln \mu/T_c^1(\lambda)} = O(\lambda) \tag{6.49}$$

where we used Remark 6.3.1 in the last step. □

6.3.2 Proof of Lemma 6.3.4

Proof of Lemma 6.3.4. If we show $\|\Phi_\lambda^d\|_{L_1^\infty(\mathbb{R}^2)L_2^2(\mathbb{R})} < \infty$ and $\|\Phi_\lambda^{ex}\|_{L_1^\infty(\mathbb{R}^2)L_2^2(\mathbb{R})} < \infty$, the Schwarz inequality implies $\|\Phi_\lambda\|_{L_1^\infty(\mathbb{R}^2)L_2^2(\mathbb{R})} < \infty$.

We shall first prove that $\|\Phi_\lambda^d\|_{L_1^\infty L_2^2}$ is finite and of order $O(1)$ for $\lambda \to 0$. Using (6.38) we have

$$\begin{aligned}&\|\Phi_\lambda^d(r,\cdot)\|_2^2 \\ &= 2\lambda^2 \int_{\mathbb{R}^5} \overline{\widehat{V^{1/2}\Psi}_\lambda(p',q_1)} B_{T_c^1}(p',(q_1,\eta)) \frac{e^{i(p-p')\cdot r}}{(2\pi)^2} B_{T_c^1}(p,(q_1,\eta)) \widehat{V^{1/2}\Psi}_\lambda(p,q_1) \mathrm{d}p\mathrm{d}p'\mathrm{d}q_1 \\ &\leq 2\lambda^2 \sup_{q_1\in\mathbb{R}} \sup_{\psi\in L^2(\mathbb{R}^2),\|\psi\|_2=1} \int_{\mathbb{R}^4} \overline{\widehat{V^{1/2}\psi}(p')} B_{T_c^1}(p',(q_1,\eta)) \frac{e^{i(p-p')\cdot r}}{(2\pi)^2} B_{T_c^1}(p,(q_1,\eta)) \widehat{V^{1/2}\psi}(p) \mathrm{d}p\mathrm{d}p'\end{aligned} \tag{6.50}$$

The latter integral is the quadratic form corresponding to the projection onto the function $\phi_{q_1}(r') = \frac{1}{2\pi} F_1 B_{T_c^1}(r-r',(q_1,\eta))V^{1/2}(r')$. Hence, taking the supremum over ψ, (6.50) equals

$$2\lambda^2 \sup_{q_1\in\mathbb{R}} \|\phi_{q_1}\|_2^2 = 2\lambda^2 \sup_{q_1\in\mathbb{R}} \int_{\mathbb{R}^4} \frac{e^{i(p-p')\cdot r}}{(2\pi)^3} B_{T_c^1}(p,(q_1,\eta))\widehat{V}(p-p')B_{T_c^1}(p',(q_1,\eta))\mathrm{d}p\mathrm{d}p'. \quad (6.51)$$

We split the integration into $p^2 > 2\mu, p^2 < 2\mu$ and $p'^2 > 2\mu, p'^2 < 2\mu$. Using (6.12) leads to the bound

$$\begin{aligned}\|\Phi_\lambda^d(r,\cdot)\|_2^2 \leq \frac{2\lambda^2}{(2\pi)^3}\Bigg[&\|\widehat{V}\|_\infty \sup_{q_1}\left(\int_{\mathbb{R}^2} B_{T_c^1}(p,(q_1,\eta))\chi_{p^2<2\mu}\mathrm{d}p\right)^2 \\ &+ 2\sup_{q_1}\int_{\mathbb{R}^4} B_{T_c^1}(p,(q_1,\eta))\chi_{p^2<2\mu}|\widehat{V}(p-p')|\frac{C}{1+p'^2}\mathrm{d}p\mathrm{d}p' \\ &+ \int_{\mathbb{R}^4}\frac{C}{1+p^2}|\widehat{V}(p-p')|\frac{C}{1+p'^2}\mathrm{d}p\mathrm{d}p'\Bigg] \quad (6.52)\end{aligned}$$

for a constant C independent of λ. We start by considering the first term in the square bracket. Note that $\|\widehat{V}\|_\infty < \frac{\|V\|_1}{2\pi} < \infty$. For fixed $T > 0$, the function $B_T(p,q)$ is bounded, hence the term is finite for fixed λ. For $T \to 0$ we have $\sup_{q\in\mathbb{R}^2}\int_{\mathbb{R}^2} B_T(p,q)\chi_{p^2<2\mu}\mathrm{d}p = O(\ln\mu/T)$. To see this, we first apply the inequality [34, (6.1)]

$$B_T(p,q) \leq \frac{1}{2}(B_T(p+q,0) + B_T(p-q,0)). \quad (6.53)$$

This gives the upper bound $\sup_{q\in\mathbb{R}^2}\int_{\mathbb{R}^2} B_T(p,0)\chi_{(p-q)^2<2\mu}\mathrm{d}p$. The vector q shifts the disk-shaped domain of integration, but does not change its size. In particular, the contribution with $p^2 < 2\mu$ is bounded above by $\int_{\mathbb{R}^2} B_T(p,0)\chi_{p^2<2\mu}\mathrm{d}p = O(\ln\mu/T)$ [40, Proposition 3.1] while the contribution with $p^2 > 2\mu$ is uniformly bounded in T since by (6.12) the integrand is uniformly bounded. Since for $\lambda \to 0$ we have $\ln\mu/T_c^1(\lambda) = O(1/\lambda)$ by Remark 6.3.1, the first term in the square bracket in (6.52) is of order $1/\lambda^2$ as $\lambda \to 0$. For the second term in the square bracket we use Hölder's inequality in p'. Since $V \in L^t$ for some $t > 0$, by the Hausdorff-Young inequality we have $\widehat{V} \in L^{t'}$ where $1 = 1/t' + 1/t$. Hence, the second term is bounded by

$$2\sup_{q_1}\int_{\mathbb{R}^4} B_{T_c^1}(p,(q_1,\eta))\chi_{p^2<2\mu}\mathrm{d}p\|\widehat{V}\|_{t'}\left\|\frac{C}{1+|\cdot|^2}\right\|_{L^t(\mathbb{R}^2)}, \quad (6.54)$$

which is finite for fixed λ and of order $O(1/\lambda)$ for $\lambda \to 0$. Using Young's inequality, one sees that the third term in the square bracket is bounded. Taking into account the factor λ^2 in front of the square bracket, we conclude that $\|\Phi_\lambda^d(r,\cdot)\|_2^2 = O(1)$ uniformly in r.

We shall now show that for fixed λ, $\|\Phi_\lambda^{ex}\|_{L_1^\infty L_2^2} < \infty$ and $\|\Phi_\lambda^{ex,>}\|_{L_1^\infty L_2^2} = O(\lambda)$ as $\lambda \to 0$. We have

$$\begin{aligned}\|\Phi_\lambda^{ex}(r,\cdot)\|_2^2 = 2\lambda^2\int_{\mathbb{R}^{2d+1}} &\overline{\widehat{V^{1/2}\Psi}_\lambda((q_1,p_2'),p_1')} B_{T_c^1}(p',(q_1,\eta))\frac{e^{i(p-p')\cdot r}}{(2\pi)^d}B_{T_c^1}(p,(q_1,\eta)) \\ &\times \widehat{V^{1/2}\Psi}_\lambda((q_1,p_2),p_1)\mathrm{d}p\mathrm{d}p'\mathrm{d}q_1 \quad (6.55)\end{aligned}$$

Similarly, we get an expression for $\|\Phi_\lambda^{ex,>}(r,\cdot)\|_2^2$ if we multiply the above integrand by the characteristic functions $\chi_{p^2+q_1^2>2\mu}\chi_{p'^2+q_1^2>2\mu}$. Using (6.12), we bound $\|\Phi_\lambda^{ex}\|_{L_1^\infty L_2^2}^2$ and

$\|\Phi_\lambda^{ex,>}\|^2_{L_1^\infty L_2^2}$ above by

$$C\lambda^2 \int_{\mathbb{R}^{2d+1}} |\overline{\widehat{V^{1/2}\Psi}_\lambda((q_1,p_2'),p_1')}| \frac{1}{1+p'^2+q_1^2}\frac{1}{1+p^2+q_1^2} |\widehat{V^{1/2}\Psi}_\lambda((q_1,p_2),p_1)| \mathrm{d}p\mathrm{d}p'\mathrm{d}q_1 \tag{6.56}$$

where the constant C depends on μ and λ for the bound on $\|\Phi_\lambda^{ex}\|^2_{L_1^\infty L_2^2}$, but is independent of λ for the bound on $\|\Phi_\lambda^{ex,>}\|^2_{L_1^\infty L_2^2}$. Using the Schwarz inequality in p_1 and p_1' and then Lemma 6.3.3 we get the upper bound

$$C\lambda^2 \|\widehat{V^{1/2}\Psi}_\lambda\|^2_{L_1^\infty L_2^2} \int_{\mathbb{R}^{2d+1}} \left(\int_{\mathbb{R}} \frac{1}{(1+p'^2+q_1^2)^2}\mathrm{d}p_1'\right)^{1/2} \left(\int_{\mathbb{R}} \frac{1}{(1+p^2+q_1^2)^2}\mathrm{d}p_1\right)^{1/2} \mathrm{d}p_2\mathrm{d}p_2'\mathrm{d}q_1 \\ \leq \tilde{C}\lambda^2\|V\|_1\|\Psi_\lambda\|_2^2 \tag{6.57}$$

Therefore, $\|\Phi_\lambda^{ex}\|_{L_1^\infty L_2^2}$ is finite and $\|\Phi_\lambda^{ex,>}\|_{L_1^\infty L_2^2} = O(\lambda)$. □

6.3.3 Proof of Lemma 6.3.5

Proof of Lemma 6.3.5. By the Schwarz inequality, it suffices to prove that for $j \in \{d, ex\}$ and $\# \in \{<,>\}$ the integrals $\int_{\mathbb{R}^4} V(r)|\Phi_\lambda^{j,\#}(r_1,z_2,z_1)|^2\mathrm{d}r\mathrm{d}z$ are finite for all $\lambda_0 \geq \lambda > 0$ and that as $\lambda \to 0$ we have $\int_{\mathbb{R}^4} V(r)|\Phi_\lambda^{j,>}(r_1,z_2,z_1)|^2\mathrm{d}r\mathrm{d}z = O(\lambda^2)$ for $j \in \{d, ex\}$, $\int_{\mathbb{R}^4} V(r)|\Phi_\lambda^{d,<}(r_1,z_2,z_1)|^2\mathrm{d}r\mathrm{d}z = O(\lambda^{-1})$ and $\int_{\mathbb{R}^4} V(r)|\Phi_\lambda^{ex,<}(r_1,z_2,z_1)|^2\mathrm{d}r\mathrm{d}z = O(\lambda)$.

Using the definitions (see (6.38) and (6.39)) one can rewrite for $\# \in \{<,>\}$

$$\int_{\mathbb{R}^4} V(r)|\Phi_\lambda^{d,\#}(r_1,z_2,z_1)|^2\mathrm{d}r\mathrm{d}z = 2\lambda^2\int_{\mathbb{R}^4} \widehat{V}(p_1-p_1',0)B_{T_c^1}((p_1',p_2),(q_1,\eta))\overline{\widehat{V^{1/2}\Psi}_\lambda(p_1',p_2,q_1)} \\ \times B_{T_c^1}(p,(q_1,\eta))\widehat{V^{1/2}\Psi}_\lambda(p,q_1)\chi_{p^2+q_1^2\#2\mu}\chi_{p_1'^2+p_2^2+q_1^2\#2\mu}\mathrm{d}p_1\mathrm{d}p_1'\mathrm{d}p_2\mathrm{d}q_1 \tag{6.58}$$

and

$$\int_{\mathbb{R}^4} V(r)|\Phi_\lambda^{ex,\#}(r_1,z_2,z_1)|^2\mathrm{d}r\mathrm{d}z = 2\lambda^2\int_{\mathbb{R}^4} \widehat{V}(p_1-p_1',0)B_{T_c^1}((p_1',p_2),(q_1,\eta))\overline{\widehat{V^{1/2}\Psi}_\lambda(q_1,p_2,p_1')} \\ \times B_{T_c^1}(p,(q_1,\eta))\widehat{V^{1/2}\Psi}_\lambda(q_1,p_2,p_1)\chi_{p^2+q_1^2\#2\mu}\chi_{p_1'^2+p_2^2+q_1^2\#2\mu}\mathrm{d}p_1\mathrm{d}p_1'\mathrm{d}p_2\mathrm{d}q_1. \tag{6.59}$$

For $\Phi_\lambda^{d,>}$, with the aid of (6.12) and Lemma 6.3.3 the expression is bounded by

$$C\lambda^2\|V\|_1\int_{\mathbb{R}^4} \frac{1}{1+p_1'^2+p_2^2}\frac{1}{1+p_1^2+p_2^2}\|\widehat{V^{1/2}\Psi}_\lambda(\cdot,q_1)\|_\infty^2\mathrm{d}q_1\mathrm{d}p_1'\mathrm{d}p_1\mathrm{d}p_2 \\ \leq \tilde{C}\lambda^2\|V\|_1^2\|\Psi_\lambda\|_2^2 < \infty \tag{6.60}$$

where the constants $C, \tilde{C}$ depend only on μ. For $\Phi_\lambda^{ex,>}$ we use (6.12) and the Schwarz inequality in p_1 and p_1' to bound (6.59) by

$$C\lambda^2\|V\|_1\int_{\mathbb{R}^2}\left\|\frac{1}{1+|\cdot|^2+p_2^2+q_1^2}\right\|^2_{L^2(\mathbb{R})}\mathrm{d}p_2\mathrm{d}q_1\|\widehat{V^{1/2}\Psi}_\lambda\|^2_{L_p^\infty L_q^2} \leq \tilde{C}\lambda^2\|V\|_1^2\|\Psi_\lambda\|_2^2 \tag{6.61}$$

where we used Lemma 6.3.3 in the second step. Again, the constants $C, \tilde{C}$ depend only on μ.

For $\Phi_\lambda^{d,<}$ we bound (6.58) above by

$$\frac{\|V\|_1}{\pi}\lambda^2\int_{\mathbb{R}^4} B_{T_c^1}(p,(q_1,\eta))B_{T_c^1}((p_1',p_2),(q_1,\eta))\|\widehat{V^{1/2}\Psi}_\lambda(\cdot,q_1)\|_\infty^2\chi_{p^2+q_1^2<2\mu}\chi_{p_1'^2+p_2^2+q_1^2<2\mu}\mathrm{d}p\mathrm{d}p_1'\mathrm{d}q_1$$
$$\leq \frac{\|V\|_1^2}{4\pi^3}\lambda^2\sup_{q_1\in\mathbb{R}}\int_{\mathbb{R}^3} B_{T_c^1}(p,(q_1,\eta))B_{T_c^1}((p_1',p_2),(q_1,\eta))\chi_{p^2+q_1^2<2\mu}\chi_{p_1'^2+p_2^2+q_1^2<2\mu}\mathrm{d}p\mathrm{d}p_1' \quad (6.62)$$

where we used Lemma 6.3.3 and $\|\Psi_\lambda\|_2=1$ in the second step. For fixed λ this is finite because $B_{T_c^1}$ is a bounded function. For $\lambda\to 0$ the following Lemma together with Remark 6.3.1 imply that this is of order $O(\lambda^{-1})$.

Lemma 6.3.7. *Let $\mu, C>0$. For $T\to 0$ we have*

$$\sup_{q,q'\in\mathbb{R}^2}\int_{\mathbb{R}^3} B_T(p,q)B_T((p_1',p_2),q')\mathrm{d}p_1\mathrm{d}p_1'\mathrm{d}p_2 = O(\ln\mu/T)^3. \quad (6.63)$$

Furthermore, for every $0<\delta_1<\mu$ there is a $\delta_2>0$ such that for $T\to 0$

$$\sup_{|q|,|q'|<\delta_2}\int_{\mathbb{R}^3}(1-\chi_{\mu-\delta_1<p_2^2<\mu+\delta_1}\chi_{p_1^2<4\delta_1}\chi_{p_1'^2<4\delta_1})B_T(p,q)B_T((p_1',p_2),q')\mathrm{d}p_1\mathrm{d}p_1'\mathrm{d}p_2$$
$$= O(\ln\mu/T)^{5/2}. \quad (6.64)$$

The second part of this Lemma will be used in the proof of Lemma 6.1.9 to compute the asymptotics of L_2. The proof of Lemma 6.3.7 can be found in Section 6.7.2.

For $\Phi_\lambda^{ex,<}$ we bound (6.59) above using Lemma 6.3.3 and $\|\Psi_\lambda\|_2=1$, which gives

$$\frac{\lambda^2}{2\pi^2}\|V\|_1^2\|B_{T_c^1}^{ex,2}(\eta)\| \quad (6.65)$$

where $B_T^{ex,2}(\xi)$ is the operator acting on $L^2(-\sqrt{2\mu},\sqrt{2\mu})$ with integral kernel

$$B_T^{ex,2}(\xi)(p_1',p_1)=\int_{\mathbb{R}^2} B_T((p_1',p_2),(q_1,\xi))B_T(p,(q_1,\xi))\chi_{q_1^2+p_2^2<2\mu}\mathrm{d}q_1\mathrm{d}p_2. \quad (6.66)$$

The superscript 2 indicates that there are two factors of B_T, as opposed to B_T^{ex} which is defined later in (6.113). The following Lemma together with Remark 6.3.1 and Lemma 6.3.21 implies that (6.65) is bounded for fixed λ and of order $O(\lambda)$ for $\lambda\to 0$.

Lemma 6.3.8. *Let $c,\mu>0$. Then $\sup_{|\xi|<cT}\|B_T^{ex,2}(\xi)\|$ is finite for all $T>0$ and of order $O(\ln\mu/T)$ as $T\to 0$.*

The proof of Lemma 6.3.8 is given in Section 6.7.3. □

6.3.4 Proof of Lemma 6.3.6

Proof of Lemma 6.3.6. For functions ψ on $\mathbb{R}^3$ let $S\psi(p_1,p_2,q_1)=\psi(p,q_1)+\psi(-p_1,p_2,-q_1)\mp\psi(q_1,p_2,p_1)\mp\psi(-q_1,p_2,-p_1)$. For $p,q\in\mathbb{R}^2$ let

$$L^0(p,q):=\lambda B_{T_c^1}(p,(q_1,\eta)), \quad (6.67)$$
$$L^\pm(p,q):=\lambda^2 B_{T_c^1}(p,(q_1,\eta))\Big[B_{T_c^1}^{-1}(p,q)-B_{T_c^1}^{-1}(p,(q_1,\eta))\Big]B_{T_c^1}((p_1,\pm q_2),(q_1,\eta) \quad (6.68)$$

Using (6.37) we have

$$g_0(p_2,q_2) = \int_{\mathbb{R}^2} \overline{S\widehat{V\chi_{\tilde{\Omega}_1^c}\Phi}_\lambda(p,q_1)} L^0(p,q) \widehat{V\chi_{\tilde{\Omega}_1^c}\Phi}_\lambda(p_1,q_2,q_1)\mathrm{d}p_1\mathrm{d}q_1 \tag{6.69}$$

and

$$g_\pm(p_2,q_2) = \int_{\mathbb{R}^2} \overline{S\widehat{V\chi_{\tilde{\Omega}_1^c}\Phi}_\lambda(p,q_1)} L^\pm(p,q) S\widehat{V\chi_{\tilde{\Omega}_1^c}\Phi}_\lambda(p_1,\pm q_2,q_1)\mathrm{d}p_1\mathrm{d}q_1. \tag{6.70}$$

Note that $g_\pm(p_2,\eta)=0$ since $L^\pm(p,(q_1,\eta))=0$. For measurable functions ψ_1,ψ_2 on $\mathbb{R}^3$ and $p_2,q_2\in\mathbb{R}$ we obtain using the Schwarz inequality in q_1

$$\begin{aligned}\int_{\mathbb{R}^2} |\psi_1(p_1,p_2,q_1)| \frac{1}{1+p_1^2} |\psi_2(p_1,q_2,q_1)|\mathrm{d}p_1\mathrm{d}q_1 \\ \le \int_{\mathbb{R}} \frac{1}{1+p_1^2}\mathrm{d}p_1 \sup_{p\in\mathbb{R}^2}\|\psi_1(p,\cdot)\|_{L^2(\mathbb{R})} \sup_{p\in\mathbb{R}^2}\|\psi_2(p,\cdot)\|_{L^2(\mathbb{R})}\end{aligned} \tag{6.71}$$

and using the Schwarz inequality in q_1,p_1

$$\begin{aligned}\int_{\mathbb{R}^2} |\psi_1(p_1,p_2,q_1)| \frac{1}{1+p_1^2+q_1^2} |\psi_2(q_1,q_2,p_1)|\mathrm{d}p_1\mathrm{d}q_1 \\ \le \int_{\mathbb{R}} \frac{1}{1+p_1^2}\mathrm{d}p_1 \sup_{p\in\mathbb{R}^2}\|\psi_1(p,\cdot)\|_{L^2(\mathbb{R})} \sup_{p\in\mathbb{R}^2}\|\psi_2(p,\cdot)\|_{L^2(\mathbb{R})}.\end{aligned} \tag{6.72}$$

By (6.12) there is a constant C independent of p,q (but dependent on λ) such that $L^0(p,q)\le \frac{C}{1+p_1^2+q_1^2}$. Similarly, by (6.12) and Lemma 6.2.1 there is a constant C independent of p,q but dependent on λ such that

$$L^\pm(p,q) \le \frac{C(1+p^2+q^2)}{(1+p^2+q_1^2)(1+p_1^2+q^2)} \le \frac{2C}{1+p_1^2+q_1^2} \tag{6.73}$$

It follows from (6.71) and (6.72) that there is a constant C such that for all measurable functions ψ_1,ψ_2 on $\mathbb{R}^3$ and $p_2,p_2',q_2,q_2'\in\mathbb{R}$

$$\left|\int_{\mathbb{R}^2} \overline{S\psi_1(p,q_1)} L^0(p_1,p_2',q_1,q_2')\psi_2(p_1,q_2,q_1)\mathrm{d}p_1\mathrm{d}q_1\right| \le C\sup_{p\in\mathbb{R}^2}\|\psi_1(p,\cdot)\|_{L^2(\mathbb{R})}\sup_{p\in\mathbb{R}^2}\|\psi_2(p,\cdot)\|_{L^2(\mathbb{R})}, \tag{6.74}$$

and similarly

$$\left|\int_{\mathbb{R}^2} \overline{S\psi_1(p,q_1)} L^\pm(p_1,p_2',q_1,q_2')S\psi_2(p_1,\pm q_2,q_1)\mathrm{d}p_1\mathrm{d}q_1\right| \le C\sup_{p\in\mathbb{R}^2}\|\psi_1(p,\cdot)\|_{L^2(\mathbb{R})}\sup_{p\in\mathbb{R}^2}\|\psi_2(p,\cdot)\|_{L^2(\mathbb{R})}. \tag{6.75}$$

In particular it follows from (6.69) and (6.70) with Lemma 6.3.3 and the normalization $\|V^{1/2}\chi_{\tilde{\Omega}_1}\Phi_\lambda\|_2=1$ that g_0 and $g_\pm$ are bounded.

To prove continuity, first note that

$$\widehat{V\chi_{\tilde{\Omega}_1^c}\Phi}_\lambda(p_1,p_2+\epsilon,q_1) - \widehat{V\chi_{\tilde{\Omega}_1^c}\Phi}_\lambda(p,q_1) = \widehat{W_\epsilon\chi_{\tilde{\Omega}_1^c}\Phi}_\lambda(p,q_1) \tag{6.76}$$

where $W_\epsilon(r) = V(r)(e^{-i\epsilon r_2} - 1)$. We only spell out the proof for $g_\pm$, the argument for g_0 is analogous. For all $p_2, q_2 \in \mathbb{R}$ we have

$$\begin{aligned} &g_\pm(p_2+\epsilon, q_2+\epsilon') - g_\pm(p_2,q_2) \\ &\quad = \int_{\mathbb{R}^2} \overline{\widehat{SV\chi_{\tilde\Omega_1}\Phi}_\lambda(p_1,p_2+\epsilon,q_1)} L^\pm(p_1,p_2+\epsilon,q_1,q_2+\epsilon') \widehat{SW_\epsilon\chi_{\tilde\Omega_1}\Phi}_\lambda(p_1,\pm q_2,q_1)\mathrm{d}p_1\mathrm{d}q_1 \\ &\qquad + \int_{\mathbb{R}^2} \overline{\widehat{SW_\epsilon\chi_{\tilde\Omega_1}\Phi}_\lambda(p,q_1)} L^\pm(p_1,p_2+\epsilon,q_1,q_2+\epsilon') \widehat{SV\chi_{\tilde\Omega_1}\Phi}_\lambda(p_1,\pm q_2,q_1)\mathrm{d}p_1\mathrm{d}q_1 \\ &+ \int_{\mathbb{R}^2} \overline{\widehat{SV\chi_{\tilde\Omega_1}\Phi}_\lambda(p,q_1)}(L^\pm(p_1,p_2+\epsilon,q_1,q_2+\epsilon') - L^\pm(p,q))\widehat{SV\chi_{\tilde\Omega_1}\Phi}_\lambda(p_1,\pm q_2,q_1)\mathrm{d}p_1\mathrm{d}q_1 \end{aligned} \tag{6.77}$$

Using (6.73) it follows by dominated convergence that the last line vanishes as $\epsilon, \epsilon' \to 0$. Furthermore, note that by Lemma 6.3.3

$$\|\widehat{W_\epsilon\chi_{\tilde\Omega_1}\Phi}_\lambda\|_{L^\infty_p L^2_{q_1}} \leq \frac{\|W_\epsilon\|_1^{1/2}}{2\pi}\|W_\epsilon^{1/2}\chi_{\tilde\Omega_1}\Phi_\lambda\|_2 \leq \frac{\|W_\epsilon\|_1}{2\pi}\|\Phi_\lambda\|_{L^\infty_r L^2_{z_1}} \tag{6.78}$$

where $\|\Phi_\lambda\|_{L^\infty_r L^2_{z_1}} < \infty$ by Lemma 6.3.4. Since $\|W_\epsilon\|_1 \leq |\epsilon| \| | \cdot |V\|_1$ it follows from (6.75) that the first two lines in (6.77) vanish as $\epsilon, \epsilon' \to 0$. In particular, $g_\pm$ are continuous. □

6.4 Proof of Lemma 6.1.8

This section contains the proof of Lemma 6.1.8. Recall the definition of t from (6.15) and let $\tilde t(p_1,q_1,r_1,z_1) = t((p_1,0),q_1,(r_1,0),z_1)$. Let $\tilde\Omega_2 = \{(r,z) \in \mathbb{R}^2 \times \mathbb{R}^2 | |r_1| < z_1, |r_2| < z_2\}$. Analogously to (6.17) we have

$$\begin{aligned} \langle \psi_\lambda^\epsilon, UH_T^{\Omega_2}U^\dagger \psi_\lambda^\epsilon\rangle = \int_{\mathbb{R}^4} B_T(p,q)^{-1}\left|\int_{\tilde\Omega_2} \frac{1}{(2\pi)^2}\tilde t(p_1,q_1,r_1,z_1)\tilde t(p_2,q_2,r_2,z_2)\psi_\lambda^\epsilon(r,z)\mathrm{d}r\mathrm{d}z\right|^2 \mathrm{d}p\mathrm{d}q \\ - \lambda\int_{\tilde\Omega_2} V(r)|\psi_\lambda^\epsilon(r,z)|^2\mathrm{d}r\mathrm{d}z. \end{aligned} \tag{6.79}$$

Since the function ψ_λ^ϵ defined in (6.5) is symmetric under $(r_2,z_2) \to -(r_2,z_2)$ and (anti)symmetric under $(r_2,z_2) \to (z_2,r_2)$, we have

$$\int_{|r_2|<z_2} \tilde t(p_2,q_2,r_2,z_2)\psi_\lambda^\epsilon(r,z)\mathrm{d}r_2\mathrm{d}z_2 = \frac{1}{2}\int_{\mathbb{R}^2} e^{-ip_2r_2 - iq_2z_2}\psi_\lambda^\epsilon(r,z)\mathrm{d}r_2\mathrm{d}z_2 \tag{6.80}$$

and

$$\int_{|r_2|<z_2} V(r)|\psi_\lambda^\epsilon(r,z)|^2\mathrm{d}r_2\mathrm{d}z_2 = \frac{1}{4}\int_{\mathbb{R}^2}(V(r)\chi_{|r_2|<|z_2|} + V(r_1,z_2)\chi_{|z_2|<|r_2|})|\psi_\lambda^\epsilon(r,z)|^2\mathrm{d}r_2\mathrm{d}z_2. \tag{6.81}$$

Together with (6.17) we obtain $\langle \psi_\lambda^\epsilon, UH_{T_c^1(\lambda)}^{\Omega_2}U^\dagger\psi_\lambda^\epsilon\rangle = \frac{1}{4}\langle\psi_\lambda^\epsilon, H_{T_c^1(\lambda)}^2\psi_\lambda^\epsilon\rangle$, where the operator H_T^2 is given by

$$H_T^2 = UK_T^{\Omega_1}U^\dagger - \lambda V(r)\chi_{|r_2|<|z_2|} - \lambda V(r_1,z_2)\chi_{|z_2|<|r_2|} \tag{6.82}$$

acting on $L^2(\tilde\Omega_1 \times \mathbb{R})$ functions symmetric in r and antisymmetric/symmetric under swapping $r_2 \leftrightarrow z_2$ for Dirichlet/Neumann boundary conditions, respectively. Let us define $K_T^2 := UK_T^{\Omega_1}U^\dagger$.

The trial state ψ_λ^ϵ has four summands, which we number from one to four in the order they appear in (6.5) and refer to as $|j\rangle$ for $j \in \{1,2,3,4\}$. By symmetry under $(z_2, r_2) \to -(z_2, r_2)$ and $(r_2, z_2) \to (z_2, r_2)$ we have

$$\langle \psi_\lambda^\epsilon, H_{T_c^1}^2 \psi_\lambda^\epsilon \rangle = 4 \sum_{j=1}^{4} \langle 1, H_{T_c^1}^2 j \rangle \tag{6.83}$$

For each $j \in \{1,2,3,4\}$ we write

$$\langle 1, H_{T_c^1}^2 j\rangle = \langle 1, (K_{T_c^1}^2 - \lambda V(r)) j\rangle + \langle 1, (\lambda V(r)\chi_{|z_2|<|r_2|} + \lambda V(r_1, z_2)\chi_{|r_2|<|z_2|}) j\rangle - \langle 1, \lambda V(r_1, z_2)\, j\rangle \tag{6.84}$$

We shall prove that

$$\lim_{\epsilon\to 0} \sum_{j=1}^{4} \langle 1, (K_{T_c^1}^2 - \lambda V(r)) j\rangle = 0, \tag{6.85}$$

$$L_1 = \lim_{\epsilon\to 0} \sum_{j=1}^{4} \langle 1, (V(r)\chi_{|z_2|<|r_2|} + V(r_1, z_2)\chi_{|r_2|<|z_2|}) j\rangle, \tag{6.86}$$

and

$$L_2 = -\lim_{\epsilon\to 0} \sum_{j=1}^{4} \langle 1, V(r_1, z_2)\, j\rangle. \tag{6.87}$$

In particular, it follows that $\lim_{\epsilon\to 0}\langle \psi_\lambda^\epsilon, U H_{T_c^1}^{\Omega_2} U^\dagger \psi_\lambda^\epsilon\rangle = \lambda(L_1 + L_2)$.

6.4.1 Proof of (6.85):

We argue that all summands vanish as $\epsilon \to 0$.

j=1: We first show that

$$\langle 1, (K_{T_c^1}^2 - \lambda V(r)) 1\rangle = \frac{1}{2\pi} \int_{\mathbb{R}^4} \left[B_{T_c^1}^{-1}(p, (q_1, q_2+\eta)) - B_{T_c^1}^{-1}(p, (q_1, \eta)) \right] \frac{\epsilon^2}{(\epsilon^2 + q_2^2)^2} |\widehat{\Phi_\lambda}(p, q_1)|^2 \mathrm{d}p\mathrm{d}q \tag{6.88}$$

Using eigenvalue equation $K_{T_c^1}^1(\eta)\chi_{\tilde{\Omega}_1}\Phi_\lambda = \lambda V \chi_{\tilde{\Omega}_1}\Phi_\lambda$ together with the expressions (6.17) and (6.19) for $K_T^{\Omega_1}$ and $K_T^1(q_2)$, respectively, we observe that

$$\begin{aligned} &\langle 1, (K_{T_c^1}^2 - \lambda V(r)) 1\rangle \\ &= \frac{1}{(2\pi)^4} \int_{(\tilde{\Omega}_1\times\mathbb{R})^2\times\mathbb{R}^3} \overline{\Phi_\lambda}(r, z_1) \overline{t(p, q_1, r, z_1)} \Bigg[\int_{\mathbb{R}} B_{T_c^1}^{-1}(p, q) e^{i(\eta - q_2)(z_2' - z_2) - \epsilon(|z_2| + |z_2'|)} \mathrm{d}q_2 \\ &\quad - B_{T_c^1}^{-1}(p, (q_1, \eta)) e^{-2\epsilon|z_2|} 2\pi\delta(z_2 - z_2') \Bigg] t(p, q_1, r', z_1') \Phi_\lambda(r', z_1') \mathrm{d}r\mathrm{d}z\mathrm{d}r'\mathrm{d}z'\mathrm{d}p\mathrm{d}q_1 \end{aligned} \tag{6.89}$$

We shall carry out the r, r', z, z' integrations. With $\int_{\mathbb{R}} e^{i(\eta-q_2)z_2 - \epsilon|z_2|}\mathrm{d}z_2 = \frac{2\epsilon}{\epsilon^2 + (\eta - q_2)^2}$, $2\pi\int_{\mathbb{R}} e^{-2\epsilon|z_2|} = 2\pi\epsilon^{-1} = \int_{\mathbb{R}} \frac{4\epsilon^2}{(\epsilon^2+(\eta-q_2)^2)^2}\mathrm{d}q_2$ and (6.36) we obtain

$$\langle 1, (K_{T_c^1}^2 - \lambda V(r)) 1\rangle = \frac{1}{2\pi} \int_{\mathbb{R}^4} \left[B_{T_c^1}^{-1}(p, q) - B_{T_c^1}^{-1}(p, (q_1, \eta)) \right] \frac{\epsilon^2}{(\epsilon^2 + (\eta - q_2)^2)^2} |\widehat{\Phi_\lambda}(p, q_1)|^2 \mathrm{d}p\mathrm{d}q \tag{6.90}$$

and substituting $q_2 \to q_2 + \eta$ we arrive at (6.88).

For $|q_2| > 1$, we bound the integrand in (6.88) by $\frac{C\epsilon^2(1+p^2+q_1^2)}{q_2^2}|\widehat{\Phi_\lambda}(p,q_1)|^2$ using Lemma 6.2.1. Since $\Phi_\lambda \in H^1(\mathbb{R}^3)$, the integral vanishes as $\epsilon \to 0$. For $|q_2| < 1$ substitute $q_2 \to \epsilon q_2$ and use that $q_2^{-1}(B_{T_c^1}^{-1}(p,(q_1,q_2+\eta)) - B_{T_c^1}^{-1}(p,(q_1,\eta))) = -f(p,(q_1,\eta),q_2)B_{T_c^1}^{-1}(p,(q_1,q_2+\eta)B_{T_c^1}^{-1}(p,(q_1,\eta))$ where f is defined as in Lemma 6.2.2. The integral then equals

$$-\frac{1}{2\pi}\int_{\mathbb{R}^4} \chi_{|q_2|<\epsilon^{-1}} f(p,(q_1,\eta),\epsilon q_2)B_{T_c^1}^{-1}(p,(q_1,\epsilon q_2+\eta)B_{T_c^1}^{-1}(p,(q_1,\eta))\frac{q_2}{(1+q_2^2)^2}|\widehat{\Phi_\lambda}(p,q_1)|^2\mathrm{d}p\mathrm{d}q. \tag{6.91}$$

By Lemma 6.2.2 and Lemma 6.2.1 the integrand is bounded above by the integrable function

$$C(1+p^2+q_1^2)\frac{|q_2|}{(1+q_2^2)^2}|\widehat{\Phi_\lambda}(p,q_1)|^2. \tag{6.92}$$

Thus by dominated convergence, continuity of f and B_T and since $\int_{\mathbb{R}} \frac{q_2}{(1+q_2^2)^2}\mathrm{d}q_2 = 0$ we have $\lim_{\epsilon\to 0}\langle 1, K_{T_c^1}^2 - \lambda V(r)1\rangle = 0$.

j=2: We distinguish the cases $\eta(\lambda) = 0$ and $\eta(\lambda) \neq 0$. If $\eta(\lambda) = 0$, $\Phi_\lambda(r,z_1)$ is either even or odd in r_2. The term for $j = 2$ hence agrees with the term for $j = 1$ or its negative and hence vanishes in the limit. For $\eta(\lambda) \neq 0$, the intuition is that integration over z_2, z_2' approximately gives a product of delta functions $\delta(q_2 - \eta)\delta(q_2 + \eta) = 0$. Using (6.36) and $t(p,q_1,(r_1,-r_2),z_1) = t((p_1,-p_2),q_1,r,z_1)$ we have

$$\begin{aligned}&\langle 1, (K_{T_c^1}^2 - \lambda V(r))2\rangle \\ &= \frac{1}{8\pi}\int_{\mathbb{R}^6} \overline{\widehat{\Phi}_\lambda(p,q_1)}B_{T_c^1}^{-1}(p,q)e^{-i(\eta-q_2)z_2 - i(\eta+q_2)z_2' - \epsilon(|z_2|+|z_2'|)}\widehat{\Phi}_\lambda((p_1,-p_2),q_1)\mathrm{d}z_2\mathrm{d}z_2'\mathrm{d}p\mathrm{d}q \\ &\qquad - \int_{\tilde{\Omega}_1\times\mathbb{R}} \overline{\Phi_\lambda(r,z_1)}\lambda V(r)\Phi_\lambda(r_1,-r_2,z_1)e^{-2i\eta z_2 - 2\epsilon|z_2|}\mathrm{d}r\mathrm{d}z\end{aligned} \tag{6.93}$$

Carrying out the z_2 and z_2' integrations gives

$$\begin{aligned}&\langle 1, (K_{T_c^1}^2 - \lambda V(r))2\rangle \\ &= \frac{1}{2\pi}\int_{\mathbb{R}^4} \overline{\widehat{\Phi}_\lambda(p,q_1)}B_{T_c^1}^{-1}(p,q)\frac{\epsilon^2}{(\epsilon^2+(\eta-q_2)^2)(\epsilon^2+(\eta+q_2)^2)}\widehat{\Phi}_\lambda((p_1,-p_2),q_1)\mathrm{d}p\mathrm{d}q \\ &\qquad - \int_{\tilde{\Omega}_1} \overline{\Phi_\lambda(r,z_1)}\lambda V(r)\Phi_\lambda(r_1,-r_2,z_1)\frac{\epsilon}{\epsilon^2+\eta^2}\mathrm{d}r\mathrm{d}z_1\end{aligned} \tag{6.94}$$

Using the Schwarz inequality in the r_2 variable, we bound the absolute value of the second term by $\frac{\epsilon\lambda}{\eta^2}\int_{\tilde{\Omega}_1} V(r)|\Phi_\lambda(r,z_1)|^2\mathrm{d}r\mathrm{d}z_1 \leq \frac{\epsilon\lambda}{\eta^2}\|V\|_1\|\Phi_\lambda\|_{L_1^\infty L_2^2}^2$. According to Lemma 6.3.4, $\|\Phi_\lambda\|_{L_1^\infty L_2^2} < \infty$ and hence the term vanishes for $\epsilon \to 0$. To bound the absolute value of the first term in (6.94), we first use Lemma 6.2.1 and the Schwarz inequality in the p_2 variable, and then use symmetry to restrict to $q_2 > 0$ and distinguish the cases $|q_2 - \eta| \lessgtr \epsilon$:

$$\begin{aligned}&C\int_{\mathbb{R}^4} \frac{\epsilon^2(1+p^2+q^2)}{(\epsilon^2+(\eta-q_2)^2)(\epsilon^2+(\eta+q_2)^2)}|\widehat{\Phi}_\lambda(p,q_1)|^2\mathrm{d}p\mathrm{d}q \\ &\leq 2C\int_{\mathbb{R}^3}\left(\int_0^\infty \left[\frac{\chi_{|q_2-\eta|<\epsilon}(1+p^2+q^2)}{(\eta-q_2)^2+(\eta+q_2)^2} + \frac{\chi_{|q_2-\eta|>\epsilon}\epsilon^2(1+p^2+q^2)}{(\eta-q_2)^2(\eta+q_2)^2}\right]\mathrm{d}q_2\right)|\widehat{\Phi}_\lambda(p,q_1)|^2\mathrm{d}p\mathrm{d}q_1.\end{aligned} \tag{6.95}$$

There is a constant $C(\eta)$ such that the first term in the square brackets is bounded above by $C(\eta)\chi_{|q_2-\eta|<\epsilon}(1+p^2+q_1^2)$, and the second term is bounded by $C(\eta)\frac{\chi_{|q_2-\eta|>\epsilon}\epsilon^2(1+p^2+q_1^2)}{(\eta-q_2)^2}$. This gives the upper bound

$$\tilde{C}\left(\int_0^\infty \left[\chi_{|q_2-\eta|<\epsilon} + \frac{\chi_{|q_2-\eta|>\epsilon}\epsilon^2}{(\eta-q_2)^2}\right]\mathrm{d}q_2\right)\|\Phi_\lambda\|^2_{H^1(\mathbb{R}^3)} \tag{6.96}$$

The remaining integral is of order $O(\epsilon)$ as $\epsilon\to 0$, and thus the term vanishes in the limit $\epsilon\to 0$.

j=3,4: Using the eigenvalue equation $K^1_{T_c^1(\lambda)}(\eta)\chi_{\tilde{\Omega}_1}\Phi_\lambda = \lambda V\chi_{\tilde{\Omega}_1}\Phi_\lambda$ and (6.36) we have

$$\begin{aligned}&|\langle 1,(K^2_{T_c^1}-\lambda V(r))j\rangle|\\ &\quad=\left|\frac{1}{8\pi}\int_{\mathbb{R}^6}\overline{\widehat{\Phi}_\lambda(p,q_1)}\left(B^{-1}_{T_c^1}(p,q)-B^{-1}_{T_c^1}(p,(q_1,\eta))\right)e^{-i(\eta-q_2)z_2-i(\mp\eta+p_2)r_2'-\epsilon(|z_2|+|r_2'|)}\right.\\ &\qquad\qquad\left.\times\widehat{\Phi}_\lambda((p_1,\pm q_2),q_1)\mathrm{d}z_2\mathrm{d}r_2'\mathrm{d}p\mathrm{d}q\right|\end{aligned} \tag{6.97}$$

where the upper signs correspond to $j=3$ and the lower ones to $j=4$, respectively. Carrying out the integration over r_2' and z_2 and substituting $q_2\to\epsilon q_2+\eta, p_2\to\epsilon p_2\pm\eta$ we obtain

$$\begin{aligned}&|\langle 1,(K^2_{T_c^1}-\lambda V(r))j\rangle|\\ &\quad=\left|\frac{1}{2\pi}\int_{\mathbb{R}^4}\overline{\widehat{\Phi}_\lambda((p_1,\epsilon p_2\pm\eta),q_1)}\frac{1}{1+p_2^2}\frac{1}{1+q_2^2}\Big[B^{-1}_{T_c^1}((p_1,\epsilon p_2\pm\eta),(q_1,\epsilon q_2+\eta))\right.\\ &\qquad\left.-B^{-1}_{T_c^1}((p_1,\epsilon p_2\pm\eta),(q_1,\eta))\Big]\widehat{\Phi}_\lambda((p_1,\pm(\epsilon q_2+\eta)),q_1)\mathrm{d}p\mathrm{d}q\right|\end{aligned} \tag{6.98}$$

With the definition of $g_\pm$ as in Lemma 6.3.6, the latter equals

$$\left|\frac{1}{2\pi}\int_{\mathbb{R}^2}\frac{g_\pm(\epsilon p_2\pm\eta,\epsilon q_2+\eta)}{(1+p_2^2)(1+q_2^2)}\mathrm{d}p_2\mathrm{d}q_2\right| \tag{6.99}$$

With Lemma 6.3.6 it follows by dominated convergence that $\lim_{\epsilon\to 0}\langle 1,(K^2_{T_c^1}-\lambda V(r))j\rangle=0$.

6.4.2 Proof of (6.86):

We have

$$\begin{aligned}&\sum_{j=1}^4\langle 1,(V(r)\chi_{|z_2|<|r_2|}+V(r_1,z_2)\chi_{|r_2|<|z_2|})j\rangle=\int_{\tilde{\Omega}_1\times\mathbb{R}}(V(r)\chi_{|z_2|<|r_2|}+V(r_1,z_2)\chi_{|r_2|<|z_2|})\overline{\Phi_\lambda(r,z_1)}\\ &\times\Big(\Phi_\lambda(r,z_1)e^{-2\epsilon|z_2|}+\Phi_\lambda(r_1,-r_2,z_1)e^{-2\epsilon|z_2|-2i\eta z_2}\mp\Phi_\lambda(r_1,z_2,z_1)e^{-\epsilon(|r_2|+|z_2|)-i\eta(z_2-r_2)}\\ &\qquad\mp\Phi_\lambda(r_1,-z_2,z_1)e^{-\epsilon(|r_2|+|z_2|)-i\eta(z_2+r_2)}\Big)\mathrm{d}r\mathrm{d}z\end{aligned} \tag{6.100}$$

The claim follows from dominated convergence provided that

$$\begin{aligned}&\int_{\mathbb{R}^4}(V(r)\chi_{|z_2|<|r_2|}+V(r_1,z_2)\chi_{|r_2|<|z_2|})|\Phi_\lambda(r,z_1)|\Big(|\Phi_\lambda(r,z_1)|+|\Phi_\lambda(r_1,-r_2,z_1)|\\ &\qquad+|\Phi_\lambda(r_1,z_2,z_1)|+|\Phi_\lambda(r_1,-z_2,z_1)|\Big)\mathrm{d}r\mathrm{d}z\end{aligned} \tag{6.101}$$

is finite. Using the Schwarz inequality in z_1 and carrying out the integration over z_2, this is bounded above by

$$4\int_{\mathbb{R}^3}(V(r)\chi_{|z_2|<|r_2|}+V(r_1,z_2)\chi_{|r_2|<|z_2|})\|\Phi_\lambda\|_{L_1^\infty L_2^2}\mathrm{d}r\mathrm{d}z_2 \leq 16\int_{\mathbb{R}^2}V(r)|r_2|\mathrm{d}r\|\Phi_\lambda\|_{L_1^\infty L_2^2} \tag{6.102}$$

This is finite by Lemma 6.3.4 and since $|\cdot|V\in L^1$ by assumption.

6.4.3 Proof of (6.87):

j=1,2: We have

$$\langle 1, V(r_1,z_2)1\rangle = \int_{\tilde{\Omega}_1\times\mathbb{R}} V(r_1,z_2)|\Phi_\lambda(r,z_1)|^2 e^{-2\epsilon|z_2|}\mathrm{d}r\mathrm{d}z \tag{6.103}$$

and

$$\langle 1, V(r_1,z_2)2\rangle = \int_{\tilde{\Omega}_1\times\mathbb{R}} V(r_1,z_2)\overline{\Phi_\lambda(r,z_1)}\Phi_\lambda(r_1,-r_2,z_1)e^{-2\epsilon|z_2|-2i\eta z_2}\mathrm{d}r\mathrm{d}z \tag{6.104}$$

In both cases we can apply dominated convergence by Lemma 6.3.5 (and the Schwarz inequality in the second case) and obtain the first two terms in L_2.

j=3,4: We start with the case of Neumann boundary conditions. Rewriting the expression in momentum space we have

$$\begin{aligned}\langle 1, V(r_1,z_2)j\rangle &= \int_{\mathbb{R}^4} V(r_1,z_2)\chi_{\tilde{\Omega}_1}\overline{\Phi_\lambda(r,z_1)}\Phi_\lambda(r_1,\pm z_2,z_1)e^{-\epsilon|z_2|-i\eta z_2}e^{-\epsilon|r_2|\pm i\eta r_2}\mathrm{d}r\mathrm{d}z\\ &= \frac{2}{\pi}\int_{\mathbb{R}^4}\overline{\widehat{\Phi_\lambda}(p,q_1)}\widehat{V\chi_{\tilde{\Omega}_1}\Phi_\lambda}(p_1,p_2',q_1)\frac{\epsilon^2}{(\epsilon^2+(p_2\mp\eta)^2)(\epsilon^2+(p_2'\mp\eta)^2)}\mathrm{d}p_1\mathrm{d}p_2\mathrm{d}p_2'\mathrm{d}q_1\\ &= \frac{2}{\pi}\int_{\mathbb{R}^2} g_0(\epsilon p_2\pm\eta,\epsilon p_2'\pm\eta)\frac{1}{(1+p_2^2)(1+p_2'^2)}\mathrm{d}p_2\mathrm{d}p_2'\end{aligned} \tag{6.105}$$

where the upper/lower signs correspond to $j=3$ and $j=4$, respectively, and g_0 is defined as in Lemma 6.3.6. It follows from Lemma 6.3.6, dominated convergence and $\int_{\mathbb{R}}\frac{1}{1+x^2}\mathrm{d}x=\pi$ that

$$\lim_{\epsilon\to 0}\langle 1, V(r_1,z_2)j\rangle = 2\pi g_0(\pm\eta,\pm\eta) \tag{6.106}$$

For Dirichlet boundary conditions this comes with a minus sign.

6.5 Weak coupling asymptotics

In this section we shall prove Lemma 6.1.9. We prove the desired asymptotic bounds for L_1 and L_2 in Sections 6.5.1 and 6.5.2, respectively.

6.5.1 Asymptotics of L_1

The goal is to show that L_1 defined in (6.7) is of order $O(1)$ as $\lambda\to 0$. By the Schwarz inequality, it suffices to prove that $\int_{\tilde{\Omega}_1\times\mathbb{R}}\chi_{|z_2|<|r_2|}V(r)(|\Phi_\lambda(r_1,r_2,z_1)|^2+|\Phi_\lambda(r_1,z_2,z_1)|^2)\mathrm{d}r\mathrm{d}z = O(1)$. Furthermore, since $\Phi_\lambda=\Phi_\lambda^d\mp\Phi_\lambda^{ex,<}\mp\Phi_\lambda^{ex,>}$ (see (6.38) and (6.39) for the definitions), again by the Schwarz inequality it suffices to prove

$$\int_{\tilde{\Omega}_1\times\mathbb{R}}\chi_{|z_2|<|r_2|}V(r)|\Phi_\lambda^j(r_1,r_2,z_1)|^2\mathrm{d}r\mathrm{d}z = O(1) \tag{6.107}$$

and

$$\int_{\tilde{\Omega}_1\times\mathbb{R}} \chi_{|z_2|<|r_2|} V(r) |\Phi^j_\lambda(r_1, z_2, z_1)|^2 \mathrm{d}r\mathrm{d}z = O(1) \tag{6.108}$$

for $j \in \{d, (ex, <), (ex, >)\}$.

Case $j \in \{d, (ex, >)\}$**:** According to Lemma 6.3.4, $\sup_{r\in\mathbb{R}^2} \int_{\mathbb{R}} |\Phi^j_\lambda(r, z_1)|^2 \mathrm{d}z_1 = O(1)$. Both (6.107) and (6.108) follow since $|\cdot|V \in L^1$.

Case $j = (ex, <)$**:** Let $W_1(r) := 2|r_2|V(r)$ and $W_2(r) := \int_{\mathbb{R}} V(r_1, z_2)\chi_{|r_2|<|z_2|}\mathrm{d}z_2$. We have $W_1, W_2 \in L^1(\mathbb{R}^2)$. Note that

$$\int_{\tilde{\Omega}_1\times\mathbb{R}} \chi_{|z_2|<|r_2|} V(r) |\Phi^{ex,<}_\lambda(r_1, r_2, z_1)|^2 \mathrm{d}r\mathrm{d}z = \int_{\tilde{\Omega}_1} W_1(r) |\Phi^{ex,<}_\lambda(r_1, r_2, z_1)|^2 \mathrm{d}r\mathrm{d}z_1 \tag{6.109}$$

and

$$\int_{\tilde{\Omega}_1\times\mathbb{R}} \chi_{|z_2|<|r_2|} V(r) |\Phi^{ex,<}_\lambda(r_1, z_2, z_1)|^2 \mathrm{d}r\mathrm{d}z = \int_{\tilde{\Omega}_1} W_2(r) |\Phi^{ex,<}_\lambda(r_1, r_2, z_1)|^2 \mathrm{d}r\mathrm{d}z_1, \tag{6.110}$$

where we renamed $z_2 \leftrightarrow r_2$. For any L^1-function $W \geq 0$ we have

$$\begin{aligned}\left(\int_{\tilde{\Omega}_1} W(r) |\Phi^{ex,<}_\lambda(r_1, r_2, z_1)|^2 \mathrm{d}r\mathrm{d}z_1\right)^{1/2} &= \|W^{1/2}\Phi^{ex,<}_\lambda\|_2 = \sup_{\psi\in L^2(\tilde{\Omega}_1), \|\psi\|_2=1} |\langle \psi, W^{1/2}\Phi^{ex,<}_\lambda\rangle| \\ &\leq \sqrt{2}\lambda \sup_{\psi_1,\psi_2\in L^2(\mathbb{R}^3), \|\psi_1\|=\|\psi_2\|=1} \int_{\mathbb{R}^3} \Big|\widehat{W^{1/2}\psi_1}(p, q_1) B_{T^1_c}(p, (q_1, \eta))\chi_{p_2^2<2\mu} \\ &\qquad \times \widehat{V^{1/2}\psi_2}((q_1, p_2), p_1)\Big| \mathrm{d}p\mathrm{d}q_1 \end{aligned}\tag{6.111}$$

where we used (6.39) and the normalization $\|\Psi_\lambda\| = 1$ in the last step. We bound $|\widehat{W^{1/2}\psi_1}(p, q_1)| \leq \|W\|_1^{1/2} \|F_2\psi_1(\cdot, q_1)\|_2$, and similarly for $|\widehat{V^{1/2}\psi_2}(p, q_1)|$. Thus (6.111) is bounded above by

$$\sqrt{2}\lambda \|W\|_1^{1/2} \|V\|_1^{1/2} \|B^{ex}_T(\eta)\| \tag{6.112}$$

where $B^{ex}_T(q_2)$ is the operator on $L^2(\mathbb{R})$ with integral kernel

$$B^{ex}_T(q_2)(p_1, q_1) = \int_{\mathbb{R}} B_T(p, q)\chi_{p_2^2<2\mu}\mathrm{d}p_2. \tag{6.113}$$

It was shown in [60, Proof of Lemma 6.1] (see Eq. (6.16) and rest of argument), that

$$\sup_T \sup_{q_2} \|B^{ex}_T(q_2)\| < \infty. \tag{6.114}$$

In particular, we conclude that $\int_{\tilde{\Omega}_1} W_k(r) |\Phi^{ex,<}_\lambda(r_1, r_2, z_1)|^2 \mathrm{d}r\mathrm{d}z_1 = O(\lambda^2)$ for $k \in \{1, 2\}$.

6.5.2 Asymptotics of L_2

The goal is to prove that L_2 defined in (6.8) diverges like $-\lambda^{-1}$ to negative infinity as $\lambda \to 0$. We shall prove that the second line in (6.8) is of order $O(1)$ as $\lambda \to 0$. For the first line in (6.8) we shall prove that it is bounded above by $-c\lambda^{-1}$ for some $c > 0$ as $\lambda \to 0$.

Second line of (6.8)**:** Let $\xi \in \{\eta, -\eta\}$. Combining (6.40), (6.69) and the definitions of L^0 and S at the beginning of Section 6.3.4 we have

$$\begin{aligned}\Big|\int_{\mathbb{R}^2} \overline{\widehat{\Phi}_\lambda(p_1, \xi, q_1)} \widehat{V\chi_{\tilde{\Omega}_1}\Phi}_\lambda(p_1, \xi, q_1)\mathrm{d}p_1\mathrm{d}q_1\Big| &\leq \lambda \int_{\mathbb{R}^2} (|\widehat{V\chi_{\tilde{\Omega}_1}\Phi}_\lambda(p_1, \xi, q_1)| + |\widehat{V\chi_{\tilde{\Omega}_1}\Phi}_\lambda(-p_1, \xi, -q_1)| \\ + |\widehat{V\chi_{\tilde{\Omega}_1}\Phi}_\lambda(q_1, \xi, p_1)|) + |\widehat{V\chi_{\tilde{\Omega}_1}\Phi}_\lambda(-q_1, \xi, -p_1)|) &B_{T^1_c}((p_1, \xi), (q_1, \eta)) |\widehat{V\chi_{\tilde{\Omega}_1}\Phi}_\lambda(p_1, \xi, q_1)| \mathrm{d}p_1\mathrm{d}q_1 \end{aligned}\tag{6.115}$$

Using the Schwarz inequality and $|\widehat{V\chi_{\tilde{\Omega}_1^c}\Phi}_\lambda(p_1,\xi,q_1)| \leq \|\widehat{V\chi_{\tilde{\Omega}_1^c}\Phi}_\lambda(\cdot,q_1)\|_\infty$ this is bounded above by

$$4\lambda\int_{\mathbb{R}^2} B_{T_c^1}((p_1,\xi),(q_1,\eta))\|\widehat{V\chi_{\tilde{\Omega}_1^c}\Phi}_\lambda(\cdot,q_1)\|_\infty^2 \mathrm{d}p_1\mathrm{d}q_1$$
$$\leq 4\lambda \sup_{q_1\in\mathbb{R}}\int_{\mathbb{R}} B_{T_c^1}((p_1,\xi),(q_1,\eta))\mathrm{d}p_1 \|\widehat{V\chi_{\tilde{\Omega}_1^c}\Phi}_\lambda\|^2_{L_2^2(\mathbb{R})L_1^\infty(\mathbb{R}^2)}, \quad (6.116)$$

where in the second step we used that $\int_{\mathbb{R}} B_{T_c^1}((p_1,\xi),(q_1,\eta))\mathrm{d}p_1$ acts as multiplication operator on $\|\widehat{V\chi_{\tilde{\Omega}_1^c}\Phi}_\lambda(\cdot,q_1)\|_\infty$. By Lemma 6.3.3 and since $\|V^{1/2}\chi_{\tilde{\Omega}_1}\Phi_\lambda\|_2 = 1$ we have $\|\widehat{V\chi_{\tilde{\Omega}_1^c}\Phi}_\lambda\|^2_{L_2^2(\mathbb{R})L_1^\infty(\mathbb{R}^2)} \leq \|V\|_1$. The following Lemma together with Remark 6.3.1 and Lemma 6.3.21 implies that (6.116) is of order $O(1)$.

Lemma 6.5.1. *Let $\xi(T),\xi'(T)$ be functions of T with $\lim_{T\to 0}\xi(T) = \lim_{T\to 0}\xi'(T) = 0$. Then as $T\to 0$,*

$$\sup_{q_1}\int_{\mathbb{R}} B_T((p_1,\xi(T)),(q_1,\xi'(T)))\mathrm{d}p_1 = O(\ln\mu/T). \quad (6.117)$$

The proof can be found in Section 6.7.4.

First line of (6.8): Recall from Section 6.3 that $\Phi_\lambda = \Phi_\lambda^{>} + \Phi_\lambda^{d,<} \mp \Phi_\lambda^{ex,<}$. By Lemma 6.3.5 the L^2-norms of $V^{1/2}(r)\Phi_\lambda^{>}(r_1,z_2,z_1)$, $V^{1/2}(r)\Phi_\lambda^{d,<}(r_1,z_2,z_1)$, and $V^{1/2}(r)\Phi_\lambda^{ex,<}(r_1,z_2,z_1)$ are of order $O(\lambda)$, $O(\lambda^{-1/2})$, and $O(\lambda^{1/2})$, respectively. It follows with the Schwarz inequality that the first line of L_2 in (6.8) equals

$$-\int_{\tilde{\Omega}_1\times\mathbb{R}} V(r)\left(|\Phi_\lambda^{d,<}(r_1,z_2,z_1)|^2 + \overline{\Phi_\lambda^{d,<}(r_1,z_2,z_1)}\Phi_\lambda^{d,<}(r_1,-z_2,z_1)e^{-2i\eta(\lambda)r_2}\right)\mathrm{d}r\mathrm{d}z + O(1) \quad (6.118)$$

Note that $\Phi_\lambda^{d,<}(r_1,z_2,z_1) = \Phi_\lambda^{d,<}(-r_1,z_2,-z_1)$. We rewrite the expression in (6.118) as

$$-\frac{1}{2}\int_{\mathbb{R}^4} V(r)\overline{\Phi_\lambda^{d,<}(r_1,z_2,z_1)}\left(\Phi_\lambda^{d,<}(r_1,z_2,z_1) + \Phi_\lambda^{d,<}(r_1,-z_2,z_1)e^{-2i\eta(\lambda)r_2}\right)\chi_{|r_1|<|z_1|}\mathrm{d}r\mathrm{d}z$$
$$= -\frac{1}{2}\int_{\mathbb{R}^4} V(r)\overline{\Phi_\lambda^{d,<}(r_1,z_2,z_1)}\left(\Phi_\lambda^{d,<}(r_1,z_2,z_1) + \Phi_\lambda^{d,<}(r_1,-z_2,z_1)e^{-2i\eta(\lambda)r_2}\right)\mathrm{d}r\mathrm{d}z$$
$$+\frac{1}{2}\int_{\mathbb{R}^4} V(r)\overline{\Phi_\lambda^{d,<}(r_1,z_2,z_1)}\left(\Phi_\lambda^{d,<}(r_1,z_2,z_1) + \Phi_\lambda^{d,<}(r_1,-z_2,z_1)e^{-2i\eta(\lambda)r_2}\right)\chi_{|z_1|<|r_1|}\mathrm{d}r\mathrm{d}z \quad (6.119)$$

We first consider the last line in (6.119) with the restriction to $|z_1|<|r_1|$. We prove that this term is of order $O(1)$ as $\lambda\to 0$. Second, we will prove that the expression on the second line in (6.119) is bounded above by $-c\lambda^{-1}$ for some constant $c>0$ as $\lambda\to 0$.

Asymptotics of third line in (6.119): Define $W\in L^1(\mathbb{R}^3)$ by $W(r,z_1) := V(r)\chi_{|z_1|<|r_1|}$. By the Schwarz inequality it suffices to prove that $\int_{\mathbb{R}^4} W(r,z_1)|\Phi_\lambda^{d,<}(r_1,z_2,z_1)|^2\mathrm{d}r\mathrm{d}z = O(1)$ for $\lambda\to 0$. Using the definition of $\Phi_\lambda^{d,<}$ we have

$$\int_{\mathbb{R}^4} W(r,z_1)|\Phi_\lambda^{d,<}(r_1,z_2,z_1)|^2\mathrm{d}r\mathrm{d}z = \frac{2\lambda^2}{(2\pi)^{1/2}}\int_{\mathbb{R}^5}\widehat{W}((p_1-p_1',0),q_1-q_1')B_{T_c^1}(p,(q_1,\eta))$$
$$\times\overline{\widehat{V^{1/2}\Psi}_\lambda(p,q_1)}B_{T_c^1}((p_1',p_2),(q_1',\eta))\widehat{V^{1/2}\Psi}_\lambda(p_1',p_2,q_1')\chi_{p^2+q_1^2<2\mu}\chi_{p_1'^2+p_2^2+q_1'^2<2\mu}\mathrm{d}p\mathrm{d}p_1'\mathrm{d}q_1\mathrm{d}q_1' \quad (6.120)$$

Using $|\widehat{W}(p,q_1)| \leq \frac{\|W\|_1}{(2\pi)^{3/2}}$ and $\|\widehat{V^{1/2}\Psi}_\lambda(\cdot,q_1)\|_\infty \leq \|V\|_1^{1/2}\|F_2\Psi_\lambda(\cdot,q_1)\|_2$ we bound this from above by

$$\begin{aligned}
&\frac{\lambda^2}{2\pi^2}\|W\|_1\|V\|_1\int_{\mathbb{R}^5} B_{T_c^1}(p,(q_1,\eta))B_{T_c^1}((p_1',p_2),(q_1',\eta))\chi_{p^2+q_1^2<2\mu}\chi_{p_1'^2+p_2^2+q_1'^2<2\mu}\\
&\qquad\qquad \times \|F_2\Psi_\lambda(\cdot,q_1)\|_2\|F_2\Psi_\lambda(\cdot,q_1')\|_2 \mathrm{d}p\mathrm{d}p_1'\mathrm{d}q_1\mathrm{d}q_1'\\
&\leq \frac{\lambda^2}{2\pi^2}\|W\|_1\|V\|_1\left[\sup_{q_1,q_1'\in\mathbb{R}}\int_{\mathbb{R}^3} B_{T_c^1}(p,(q_1,\eta))B_{T_c^1}((p_1',p_2),(q_1',\eta))\chi_{p_1'^2+q_1'^2+p_2^2<2\mu}\chi_{p^2+q_1^2<2\mu}\mathrm{d}p\mathrm{d}p_1'\right]\\
&\qquad\qquad\qquad\qquad \times\left(\int_{\mathbb{R}}\|F_2\Psi_\lambda(\cdot,q_1)\|_2\chi_{q_1^2<2\mu}\mathrm{d}q_1\right)^2 \quad (6.121)
\end{aligned}$$

By Lemma 6.3.7 and Remark 6.3.1, the term in the square bracket in (6.121) is of order $O(\lambda^{-3})$. Splitting the domain of integration into $|q_1|/\sqrt{\mu} \gtrless (T_c^1/\mu)^\beta$ for some $0<\beta<1$ and using the Schwarz inequality we observe that

$$\int_{\mathbb{R}}\|F_2\Psi_\lambda(\cdot,q_1)\|_2\chi_{q_1^2<2\mu}\mathrm{d}q_1 \leq (2\sqrt{\mu}(T_c^1/\mu)^\beta)^{1/2}\|\Psi_\lambda\|_2 + (2\sqrt{2\mu})^{1/2}\|F_2\Psi_\lambda\chi_{|q_1|/\sqrt{\mu}>(T_c^1/\mu)^\beta}\|_2 \quad (6.122)$$

By Lemma 6.3.23, $\|F_2\Psi_\lambda\chi_{|q_1|/\sqrt{\mu}>(T_c^1/\mu)^\beta}\|_2 = O(\lambda^{1/2})$ and by Remark 6.3.1 we have $(T_c^1/\mu)^{\beta/2} \leq O((\ln\mu/T_c^1)^{-1}) = O(\lambda)$. Thus, $\left(\int_{\mathbb{R}}\|F_2\Psi_\lambda(\cdot,q_1)\|_2\chi_{q_1^2<2\mu}\mathrm{d}q_1\right)^2 = O(\lambda)$ and (6.121) is of order $O(1)$.

Asymptotics of second line in (6.119)**:** Analogously to (6.58) we have

$$\begin{aligned}
&\int_{\mathbb{R}^4} V(r)\overline{\Phi_\lambda^{d,<}(r_1,z_2,z_1)}\Phi_\lambda^{d,<}(r_1,-z_2,z_1)e^{-2i\eta(\lambda)r_2}\mathrm{d}r\mathrm{d}z = 2\lambda^2\int_{\mathbb{R}^4}\widehat{V}(p_1+p_1',2\eta)B_{T_c^1}(p,(q_1,\eta))\\
&\times\overline{\widehat{V^{1/2}\Psi}_\lambda(p,q_1)}B_{T_c^1}((p_1',p_2),(q_1,\eta))\widehat{V^{1/2}\Psi}_\lambda(p_1',p_2,q_1)\chi_{p^2+q_1^2<2\mu}\chi_{p_1'^2+p_2^2+q_1^2<2\mu}\mathrm{d}p\mathrm{d}p_1'\mathrm{d}q_1
\end{aligned} \quad (6.123)$$

We can thus write

$$\frac{1}{2}\int_{\mathbb{R}^4} V(r)\overline{\Phi_\lambda^{<,d}(r_1,z_2,z_1)}\Big(\Phi_\lambda^{<,d}(r_1,z_2,z_1)+\Phi_\lambda^{<,d}(r_1,-z_2,z_1)e^{-2i\eta(\lambda)r_2}\Big)\mathrm{d}r\mathrm{d}z = \langle F_2\Psi_\lambda, M_\lambda F_2\Psi_\lambda\rangle, \quad (6.124)$$

where M_λ is the operator acting on $L^2(\mathbb{R}^3)$ given by

$$\begin{aligned}
\langle\psi, M_\lambda\psi\rangle = \lambda^2\int_{\mathbb{R}^4}&(\widehat{V}(p_1-p_1',0)+\widehat{V}(p_1+p_1',2\eta))B_{T_c^1}(p,(q_1,\eta))\overline{F_1V^{1/2}\psi(p,q_1)}\chi_{p^2+q_1^2<2\mu}\\
&\times B_{T_c^1}((p_1',p_2),(q_1,\eta))\chi_{p_1'^2+p_2^2+q_1^2<2\mu}F_1V^{1/2}\psi(p_1',p_2,q_1)\mathrm{d}p\mathrm{d}p_1'\mathrm{d}q_1 \quad (6.125)
\end{aligned}$$

By the same argument as in the proof of $\int_{\mathbb{R}^4} V(r)|\Phi_\lambda^{d,<}(r_1,z_2,z_1)|^2\mathrm{d}r\mathrm{d}z = O(\lambda^{-1})$ in Lemma 6.3.5 (see (6.62)) we have $\|M_\lambda\| = O(\lambda^{-1})$. Recall the projections $\mathbb{P}$ and $\mathbb{Q}_\beta$ from Section 6.3. Let $\mathbb{T}$ be the projection $\mathbb{T}=\mathbb{P}\mathbb{Q}_\beta$ for some $0<\beta<1$ and $\mathbb{T}^\perp = 1-\mathbb{T}$. We have

$$\langle F_2\Psi_\lambda, M_\lambda F_2\Psi_\lambda\rangle = \langle\mathbb{T}F_2\Psi_\lambda, M_\lambda\mathbb{T}F_2\Psi_\lambda\rangle + \langle\mathbb{T}F_2\Psi_\lambda, M_\lambda\mathbb{T}^\perp F_2\Psi_\lambda\rangle + \langle\mathbb{T}^\perp F_2\Psi_\lambda, M_\lambda F_2\Psi_\lambda\rangle \quad (6.126)$$

Since $\mathbb{P}$ and $\mathbb{Q}_\beta$ commute, we have $\|\mathbb{T}^\perp F_2\Psi_\lambda\| = \|\mathbb{Q}_\beta^\perp F_2\Psi_\lambda + \mathbb{Q}_\beta\mathbb{P}^\perp F_2\Psi_\lambda\| = O(\lambda^{1/2})$ according to Lemma 6.3.22 and 3. In particular, the last two terms in (6.126) are of order

$O(\lambda^{-1/2})$. The remaining term in (6.126) is bounded below by

$$\begin{aligned}&\langle \mathbb{T}F_2\Psi_\lambda, M_\lambda \mathbb{T}F_2\Psi_\lambda\rangle\\ &\geq \inf_{|q_1|/\sqrt{\mu}<(T_c^1/\mu)^\beta} \lambda^2 \int_{\mathbb{R}^3} (\widehat{V}(p_1-p_1',0)+\widehat{V}(p_1+p_1',2\eta))B_{T_c^1}((p_1,p_2),(q_1,\eta))\widehat{Vj_2}(p)\chi_{p^2+q_1^2<2\mu}\\ &\qquad \times B_{T_c^1}((p_1',p_2),(q_1,\eta))\chi_{p_1'^2+p_2^2+q_1^2<2\mu}\widehat{Vj_2}(p_1',p_2)\mathrm{d}p\mathrm{d}p_1' \|\mathbb{T}F_2\Psi_\lambda\|_2^2\|V^{1/2}j_2\|_2^{-2} \end{aligned} \tag{6.127}$$

The remainder of the proof follows the same ideas as the proof of [60, Lemma 4.11]. Since $V\geq 0$ we have $\widehat{V}(0)>0$. Furthermore, the eigenvalue equation $e_\mu V^{1/2}j_2 = O_\mu V^{1/2}j_2 = \widehat{Vj_2}(|p|=\sqrt{\mu})V^{1/2}j_2$ implies that $\widehat{Vj_2}(|p|=\sqrt{\mu})=e_\mu>0$. By continuity of $\widehat{V}$ and $\widehat{Vj_2}$ and Lemma 6.3.21, there exist $\tilde{\lambda}>0$, $0<\delta<\mu$ and $c_1>0$ such that for all $\sqrt{\mu-\delta}<p_2<\sqrt{\mu+\delta}, p_1^2<4\delta, p_1'^2<4\delta$ and $\lambda<\tilde{\lambda}$ we have

$$(\widehat{V}(p_1-p_1',0)+\widehat{V}(p_1+p_1',2\eta))\widehat{Vj_2}(p)\widehat{Vj_2}(p_1',p_2)\chi_{p^2+q_1^2<2\mu}\chi_{p_1'^2+p_2^2+q_1^2<2\mu}\|V^{1/2}j_2\|_2^{-2}>c_1. \tag{6.128}$$

Using the second part of Lemma 6.3.7, Lemma 6.3.21 and the boundedness of $\widehat{V},\widehat{Vj_2}$, it follows that up to an error of order $O(\lambda^2(\ln\mu/T_c^1)^{5/2})=O(\lambda^{-1/2})$ we may restrict the domain of integration in (6.127) to $\sqrt{\mu-\delta}<p_2<\sqrt{\mu+\delta}, p_1^2<4\delta, p_1'^2<4\delta$. Since $\|\mathbb{T}F_2\Psi_\lambda\|_2^2 = 1-O(\lambda)\geq \frac{1}{2}$ for small λ, we obtain

$$\begin{aligned}\langle \mathbb{T}F_2\Psi_\lambda, M_\lambda \mathbb{T}F_2\Psi_\lambda\rangle \geq \frac{c_1}{2}\inf_{|q_1|/\sqrt{\mu}<(T_c^1/\mu)^\beta}\lambda^2\int_{\mathbb{R}^3} B_{T_c^1}((p_1,p_2),(q_1,\eta))\\ \times B_{T_c^1}((p_1',p_2),(q_1,\eta))\chi_{\mu-\delta<p_2^2<\mu+\delta}\chi_{p_1^2<4\delta}\chi_{p_1'^2<4\delta}\mathrm{d}p\mathrm{d}p_1' + O(\lambda^{-1/2})\end{aligned} \tag{6.129}$$

Using Lemma 6.3.7 once more, we may leave away the characteristic functions at the expense of an error of order $O(\lambda^{-1/2})$. Since $\eta(\lambda)=O(T_c^1(\lambda))$, there is a $c_2>0$ such that $\eta^2+(\sqrt{\mu}(T_c^1/\mu)^\beta)^2\leq c_2^2\mu(T_c^1/\mu)^{2\beta}$ for $T_c^1<\mu$. The following Lemma, whose proof is given in Section 6.7.5, thus concludes the proof of Lemma 6.1.9.

Lemma 6.5.2. *Let $\mu, c_2>0$, $0<\beta<1$ and $\epsilon := c_2\sqrt{\mu}(T/\mu)^\beta$ for $T>0$. Then there are constants $T_0, C>0$ such that*

$$\inf_{|q|<\epsilon}\int_{\mathbb{R}}\left(\int_{\mathbb{R}} B_T(p,q)\mathrm{d}p_1\right)^2\mathrm{d}p_2 \geq C(\ln\mu/T)^3 \tag{6.130}$$

for all $0<T<T_0$.

6.6 Proof of Theorem 6.1.5

This Section is dedicated to the proof of Theorem 6.1.5, which states that the relative difference of T_c^2 and T_c^0 vanishes in the weak coupling limit. It has been shown in [60, Theorem 1.7] that the relative difference of T_c^1 and T_c^0 vanishes in the weak coupling limit and we follow the same proof strategy here. We first switch to the Birman-Schwinger picture. Recall the Birman-Schwinger operator A_T^0 corresponding to $H_T^{\Omega_0}$ defined in (6.20). Furthermore, recall the notation $\tilde{t}$, $\tilde{\Omega}_2$ and the representation of $UH_T^{\Omega_2}U^\dagger$ in (6.79) from Section 6.4. The corresponding Birman-Schwinger operator $A_T^2: L^2(\tilde{\Omega}_2)\to L^2(\tilde{\Omega}_2)$ is given by

$$\langle\psi, A_T^2\psi\rangle = \int_{\mathbb{R}^4} B_T(p,q)\left|\int_{\tilde{\Omega}_2}\frac{1}{(2\pi)^2}\tilde{t}(p_1,q_1,r_1,z_1)\tilde{t}(p_2,q_2,r_2,z_2)V^{1/2}(r)\psi(r,z)\mathrm{d}r\mathrm{d}z\right|^2\mathrm{d}p\mathrm{d}q \tag{6.131}$$

and it follows from the Birman-Schwinger principle that $\operatorname{sgn}\inf\sigma(H_T^{\Omega_2}) = \operatorname{sgn}(1/\lambda - \sup\sigma(A_T^2))$. Let $a_T^j = \sup\sigma(A_T^j)$. It is a straightforward generalization of [34, Lemma 4.1] that the claim (6.4) is equivalent to

$$\lim_{T\to 0}(a_T^0 - a_T^2) = 0 \tag{6.132}$$

and we refer to [34] for the proof.

To verify (6.132), the first step is to argue that $a_T^2 \geq a_T^0$ for all $T > 0$. Lemma 6.1.1 together with [60, Lemma 2.3] imply that $\inf\sigma(H_T^{\Omega_2}) \leq \inf\sigma(H_T^{\Omega_0})$ for all $\lambda, T > 0$. Using the Birman-Schwinger principle, it follows that $a_T^2 \geq a_T^0$ for all $T > 0$. For details we refer to the proof of [60, Theorem 1.7].

It remains to show that $\lim_{T\to 0}(a_T^0 - a_T^2) \geq 0$. We decompose A_T^2 in the same spirit as we decomposed $A_T^1(q_2)$ in (6.26). For A_T^1, the decomposition consisted of the "unperturbed" term A_T^0 and the "perturbation term" G_T, where the first components of the momentum variables were swapped. For A_T^2 we additionally get the terms arising from swapping the variables in the second component, which leads to four terms in total. Let $\tilde{\iota} : L^2(\tilde{\Omega}_2) \to L^2(\mathbb{R}^4)$ be the isometry

$$\begin{aligned}\tilde{\iota}\psi(r,z) = \frac{1}{2}\Big(&\psi(r,z)\chi_{\tilde{\Omega}_2}(r,z) + \psi(-r_1,r_2,-z_1,z_2)\chi_{\tilde{\Omega}_2}(-r_1,r_2,-z_1,z_2)\\ &+ \psi(r_1,-r_2,z_1,-z_2)\chi_{\tilde{\Omega}_2}(r_1,-r_2,z_1,-z_2) + \psi(-r,-z)\chi_{\tilde{\Omega}_2}(-r,-z)\Big).\end{aligned} \tag{6.133}$$

Using the definition of $\tilde{t}$ and evenness of V in r_1 and r_2 we rewrite (6.131) as

$$\begin{aligned}\langle\psi, A_T^2\psi\rangle = \int_{\mathbb{R}^4} B_T(p,q)\Big|\frac{1}{2}(&\widehat{V^{1/2}\tilde{\iota}\psi}(p,q) \mp \widehat{V^{1/2}\tilde{\iota}\psi}((q_1,p_2),(p_1,q_2))\\ &\mp \widehat{V^{1/2}\tilde{\iota}\psi}((p_1,q_2),(q_1,p_2)) + \widehat{V^{1/2}\tilde{\iota}\psi}(q,p))\Big|^2 \mathrm{d}p\mathrm{d}q\end{aligned} \tag{6.134}$$

Define the self-adjoint operators G_T^1, G_T^2, and N_T on $L^2(\mathbb{R}^4)$ through

$$\langle\psi, G_T^1\psi\rangle = \int_{\mathbb{R}^4} \overline{F_1V^{1/2}\psi((q_1,p_2),(p_1,q_2))}B_T(p,q)F_1V^{1/2}\psi(p,q)\mathrm{d}p\mathrm{d}q, \tag{6.135}$$

$$\langle\psi, G_T^2\psi\rangle = \int_{\mathbb{R}^4} \overline{F_1V^{1/2}\psi((p_1,q_2),(q_1,p_2))}B_T(p,q)F_1V^{1/2}\psi(p,q)\mathrm{d}p\mathrm{d}q, \text{ and} \tag{6.136}$$

$$\langle\psi, N_T\psi\rangle = \int_{\mathbb{R}^4} \overline{F_1V^{1/2}\psi(q,p)}B_T(p,q)F_1V^{1/2}\psi(p,q)\mathrm{d}p\mathrm{d}q. \tag{6.137}$$

We slightly abuse notation and write F_2 for the Fourier transform in the second variable also when the second variable has two components, i.e. $F_2\psi(r,q) = \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{-iq\cdot z}\psi(r,z)\mathrm{d}z$. It follows from (6.134) and $B_T(p,q) = B_T((q_1,p_2),(p_1,q_2)) = B_T(q,p)$ that

$$A_T^2 = \tilde{\iota}^\dagger(A_T^0 - F_2^\dagger R_T F_2)\tilde{\iota}, \tag{6.138}$$

where $R_T = \pm G_T^1 \pm G_T^2 - N_T$. Let $B_T(\cdot,q) : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ denote multiplication by $B_T(p,q)$ in momentum space and define the function $E_T(q)$ on $\mathbb{R}^2$ through

$$E_T(q) := a_T^0 - \|V^{1/2}B_T(\cdot,q)V^{1/2}\|. \tag{6.139}$$

Note that $a_T^0 = \sup_{q\in\mathbb{R}^2}\|V^{1/2}B_T(\cdot,q)V^{1/2}\|$ and therefore $E_T(q) \geq 0$. For $\psi \in L^2(\mathbb{R}^4)$ let $E_T\psi(r,q) = E_T(q)\psi(r,q)$. We get the operator inequality $a_T^0\mathbb{I} - A_T^0 \geq F_2^\dagger E_T F_2$, where

$\mathbb{I}$ denotes the identity operator on $L^2(\mathbb{R}^4)$. Using (6.138), the above inequality and that $\|F_2\tilde{\iota}\psi\|_2 = \|\psi\|_2$ we obtain

$$a_T^0 - a_T^2 \geq \inf_{\psi \in L_s^2(\tilde{\Omega}_2), \|\psi\|_2 = 1} \langle F_2\tilde{\iota}\psi, (E_T + R_T)F_2\tilde{\iota}\psi\rangle \geq \inf_{\psi \in L^2(\mathbb{R}^4), \|\psi\|_2 = 1} \langle \psi, (E_T + R_T)\psi\rangle. \tag{6.140}$$

Therefore, it suffices to show that $\lim_{T\to 0} \inf \sigma(E_T + R_T) \geq 0$. The proof relies on the following three Lemmas.

Lemma 6.6.1. *Let $\mu > 0$ and let V satisfy Assumption 6.1.2. Then $\sup_{T>0}\|R_T\| < \infty$.*

Lemma 6.6.2. *Let $\mu > 0$ and let V satisfy Assumption 6.1.2. Let $\mathbb{I}_{\leq\epsilon}$ act on $L^2(\mathbb{R}^4)$ as $\mathbb{I}_{\leq\epsilon}\psi(r,q) = \psi(r,q)\chi_{|q|\leq\epsilon}$. Then $\lim_{\epsilon\to 0}\sup_{T>0}\|\mathbb{I}_{\leq\epsilon}R_T\mathbb{I}_{\leq\epsilon}\| = 0$.*

Lemma 6.6.3. *Let $\mu > 0$ and let V satisfy Assumption 6.1.2. Let $0 < \epsilon < \sqrt{\mu}$. There are constants $c_1, c_2, T_0 > 0$ such that for $0 < T < T_0$ and $|q| > \epsilon$ we have $E_T(q) > c_1|\ln(c_2/T)|$.*

The first two Lemmas are extensions of [60, Lemma 6.1 and Lemma 6.2] and proved in Sections 6.7.6 and 6.7.7, respectively. The third Lemma is contained in [60, Lemma 6.3].

With these Lemmas, the claim follows completely analogously to the proof of [34, Theorem 1.2 (ii)] and we provide a sketch for completeness. Using that $E_T(q) \geq 0$, we write

$$E_T + R_T + \delta = \sqrt{E_T + \delta}\left(\mathbb{I} + \frac{1}{\sqrt{E_T + \delta}}R_T\frac{1}{\sqrt{E_T + \delta}}\right)\sqrt{E_T + \delta} \tag{6.141}$$

for any $\delta > 0$. It suffices to prove that for all $\delta > 0$ the norm of the second term in the bracket vanishes in the limit $T \to 0$. With the notation from Lemma 6.6.2 we estimate for all $0 < \epsilon < \sqrt{\mu}$

$$\begin{aligned}\left\|\frac{1}{\sqrt{E_T+\delta}}R_T\frac{1}{\sqrt{E_T+\delta}}\right\| &\leq \left\|\mathbb{I}_{\leq\epsilon}\frac{1}{\sqrt{E_T+\delta}}R_T\frac{1}{\sqrt{E_T+\delta}}\mathbb{I}_{\leq\epsilon}\right\| \\ &\quad + \left\|\mathbb{I}_{\leq\epsilon}\frac{1}{\sqrt{E_T+\delta}}R_T\frac{1}{\sqrt{E_T+\delta}}\mathbb{I}_{>\epsilon}\right\| + \left\|\mathbb{I}_{>\epsilon}\frac{1}{\sqrt{E_T+\delta}}R_T\frac{1}{\sqrt{E_T+\delta}}\right\|.\end{aligned} \tag{6.142}$$

Lemma 6.6.3 and $E_T \geq 0$ imply

$$\lim_{T\to 0}\left\|\frac{1}{\sqrt{E_T+\delta}}R_T\frac{1}{\sqrt{E_T+\delta}}\right\| \leq \sup_{T>0}\frac{1}{\delta}\|\mathbb{I}_{\leq\epsilon}R_T\mathbb{I}_{\leq\epsilon}\| + \lim_{T\to 0}\frac{2}{(\delta c_1|\ln(c_2/T)|)^{1/2}}\|R_T\|. \tag{6.143}$$

The first term can be made arbitrarily small by Lemma 6.6.2 and the second term vanishes by Lemma 6.6.1. Hence, Theorem 6.1.5 follows.

6.7 Proofs of Auxiliary Lemmas

6.7.1 Proof of Lemma 6.2.2

Proof of Lemma 6.2.2. Using the Mittag-Leffler series (as in [34, (2.1)]) one can write

$$\begin{aligned}f(p,q,x) = 2T\sum_{n\in\mathbb{Z}}\Xi_n^{-1}\Big[&(2q_2 + x)(2\mu - 2q^2 - 2p^2 - x^2 + 2(p_2 - q_2)x) \\ &+ 2p_2(4p\cdot q - 2iw_n + 2(p_2 - q_2)x - x^2)\Big]\end{aligned} \tag{6.144}$$

where

$$\Xi_n = \left((p+q+(0,x))^2 - \mu - iw_n\right)\left((p-q-(0,x))^2 - \mu + iw_n\right) \times \left((p+q)^2 - \mu - iw_n\right)\left((p-q)^2 - \mu + iw_n\right) \quad (6.145)$$

and $w_n = (2n+1)\pi T$. Continuity of f follows from dominated convergence. For $x > \sqrt{\mu}/4$ the bound on f follows from (6.12). Let $Q_2 = Q_1 + \sqrt{\mu}/4$. For $x < \sqrt{\mu}/4$ we have

$$|f(p,q,x)| \leq \sup_{|q_2| \leq Q_2} |\frac{\partial}{\partial q_2} B_T(p,q)| = \sup_{|q_2|\leq Q_2} |f(p,q,0)|. \quad (6.146)$$

To bound $|f(p,q,0)|$, first note that for $x=0$ with the notation $y=(p+q)^2-\mu$, $z=(p-q)^2-\mu$ and $v = \max\{(|p_1|+|q_1|)^2 + (|p_2|-|q_2|)^2 - \mu, 0\}$,

$$|\Xi_n| = \left(y^2 + w_n^2\right)\left(z^2 + w_n^2\right) \geq \left(v^2 + w_n^2\right)\left(\max\{(|p_2|-|q_2|)^2 - \mu, 0\})^2 + w_n^2\right). \quad (6.147)$$

Furthermore,

$$\sup_{(p,q)\in\mathbb{R}^4, |q_2|\leq Q_2} \left|\frac{4iw_n p_2}{\max\{(|p_2|-|q_2|)^2-\mu,0\})^2 + w_n^2}\right| \leq \sup_{(p,q)\in\mathbb{R}^4,|q_2|<Q_2} \frac{4|p_2|}{\sqrt{\max\{(|p_2|-|q_2|)^2-\mu,0\})^2+w_0^2}} =: c_1 < \infty \quad (6.148)$$

There is a constant $c_2 > \mu$ depending only on μ and Q_2 such that $|p_2|^2 \leq 4(\min\{y,z\}+c_2)$ for $|q_2| \leq Q_2$ and all $p_1, q_1 \in \mathbb{R}$. One obtains that for $|q_2| \leq Q_2$

$$|f(p,q,0)| \leq 2T \sum_{n\in\mathbb{Z}} \frac{2Q_2|y+z| + 4\sqrt{\min\{y,z\}+c_2}|y-z|}{(y^2+w_n^2)(z^2+w_n^2)} + 2T\sum_{n\in\mathbb{Z}} \frac{c_1}{v^2+w_n^2} \quad (6.149)$$

Since the summands are decreasing in n, we can estimate the sums by integrals. The second term is bounded by

$$4Tc_1\left[\frac{1}{v^2+w_0^2} + \int_{1/2}^{\infty} \frac{1}{v^2+4\pi^2T^2x^2}\mathrm{d}x\right] = 4Tc_1\left[\frac{1}{v^2+w_0^2} + \frac{\arctan\left(\frac{v}{\pi T}\right)}{2\pi T v}\right] < \frac{C}{1+p_1^2+q_1^2+p_2^2} \quad (6.150)$$

for some constant C independent of p and q_1, since $\sup_{(p,q)\in\mathbb{R}^4,|q_2|\leq Q_2} \frac{1+p_1^2+q_1^2+p_2^2}{1+v} < \infty$. The first term in (6.149) is bounded by

$$16T(Q_2 + 2\sqrt{\min\{|y|,|z|\}+c_2})\max\{|y|,|z|\}\left[\frac{1}{(y^2+w_0^2)(z^2+w_0^2)} + \int_{1/2}^{\infty}\frac{1}{(y^2+4\pi^2T^2x^2)(z^2+4\pi^2T^2x^2)}\mathrm{d}x\right] \quad (6.151)$$

Note that $y+z+2\mu+1 = 1+2p^2+2q^2$. The claim thus follows if we prove that for $c_3 > 0$

$$\sup_{y>z>0}(1+y+z)(1+\sqrt{z+1})y\left[\frac{1}{(y^2+1)(z^2+1)} + \int_{c_3}^{\infty}\frac{1}{(y^2+x^2)(z^2+x^2)}\mathrm{d}x\right] < \infty \quad (6.152)$$

The supremum over the first summand is obviously finite. The supremum over the second summand is bounded by

$$\sup_{y>z>0} \frac{(1+2y)y}{y^2+c_3^2} \frac{1+\sqrt{z+1}}{(z^2+c_3^2)^{1/4}} \int_{c_3}^{\infty} \frac{1}{x^{3/2}} \mathrm{d}x < \infty. \tag{6.153}$$

□

6.7.2 Proof of Lemma 6.3.7

Proof of Lemma 6.3.7. Using (6.53) and substituting $p_1 \pm q_1 \to p_1, p_1' \pm q_1' \to p_1'$, we have

$$\int_{\mathbb{R}^3} B_T(p,q) B_T((p_1',p_2),q') \mathrm{d}p_1 \mathrm{d}p_1' \mathrm{d}p_2 \leq \frac{1}{4} \int_{\mathbb{R}^3} (B_T((p_1,p_2+q_2),0) + B_T((p_1,p_2-q_2),0)) \times (B_T((p_1',p_2+q_2'),0) + B_T((p_1',p_2-q_2'),0)) \mathrm{d}p_1 \mathrm{d}p_1' \mathrm{d}p_2 \tag{6.154}$$

One can bound this from above by

$$\begin{aligned} &\sup_{q_2,q_2' \in \mathbb{R}} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} B_T((p_1,p_2+q_2),0) \mathrm{d}p_1 \right) \left(\int_{\mathbb{R}} B_T((p_1',p_2+q_2'),0) \mathrm{d}p_1' \right) \mathrm{d}p_2 \\ &\leq \sup_{q_2 \in \mathbb{R}} \int_{\mathbb{R}^3} B_T((p_1,p_2+q_2),0) B_T((p_1',p_2+q_2),0) \mathrm{d}p_1 \mathrm{d}p_1' \mathrm{d}p_2 \\ &= \int_{\mathbb{R}^3} B_T((p_1,p_2),0) B_T((p_1',p_2),0) \mathrm{d}p_1 \mathrm{d}p_1' \mathrm{d}p_2 \end{aligned} \tag{6.155}$$

where in the second step we used the Schwarz inequality in p_2. The latter expression is of order $O(\ln(\mu/T)^3)$ for $T \to 0$, as was shown in the proof of [60, Lemma 4.10].

To prove the second statement, we shall use that for fixed $0 < \delta < \mu$

$$\int_{\mathbb{R}^3} (1 - \chi_{\mu-\delta<p_2^2<\mu} \chi_{p_1^2<2\delta} \chi_{p_1'^2<2\delta}) B_T(p,0) B_T((p_1',p_2),0) \mathrm{d}p_1 \mathrm{d}p_1' \mathrm{d}p_2 = O((\ln \mu/T)^2) \tag{6.156}$$

for $T \to 0$ as was shown in the proof of [60, Lemma 4.10]. We choose δ_2 and δ small enough, such that for all $q^2 < \delta_2$, if $p_1^2 > 4\delta_1$ we have $(p_1+q_1)^2 > 2\delta$ and if $p_2^2 < \mu - \delta_1$ or $p_2^2 > \mu + \delta_1$ we have $(p_2+q_2)^2 < \mu - \delta$ or $(p_2+q_2)^2 > \mu$, respectively. Using (6.53) as above, we have

$$\begin{aligned} &\sup_{q^2,q'^2<\delta_2} \int_{\mathbb{R}^3} (1 - \chi_{\mu-\delta_1<p_2^2<\mu+\delta_1} \chi_{p_1^2<4\delta_1} \chi_{p_1'^2<4\delta_1}) B_T(p,q) B_T((p_1',p_2),q') \mathrm{d}p_1 \mathrm{d}p_1' \mathrm{d}p_2 \\ &\leq \sup_{q^2,q'^2<\delta_2} \int_{\mathbb{R}^3} (1 - \chi_{\mu-\delta_1<p_2^2<\mu+\delta_1} \chi_{p_1^2<4\delta_1} \chi_{p_1'^2<4\delta_1}) B_T(p+q,0) B_T((p_1',p_2)+q',0) \mathrm{d}p_1 \mathrm{d}p_1' \mathrm{d}p_2 \end{aligned} \tag{6.157}$$

Note that $1 - \chi_{\mu-\delta_1<p_2^2<\mu+\delta_1} \chi_{p_1^2<4\delta_1} \chi_{p_1'^2<4\delta_1} \leq \chi_{\mu-\delta_1>p_2^2} + \chi_{\mu+\delta_1<p_2^2} + \chi_{p_1^2>4\delta_1} + \chi_{p_1'^2>4\delta_1}$. Using the Schwarz inequality in p_2 we bound (6.157) above by

$$\begin{aligned} &\sup_{q^2<\delta_2} \int_{\mathbb{R}^3} (\chi_{\mu-\delta_1>p_2^2} + \chi_{\mu+\delta_1<p_2^2}) B_T((p_1+q_1,p_2+q_2),0) B_T((p_1'+q_1,p_2+q_2),0) \mathrm{d}p_1 \mathrm{d}p_1' \mathrm{d}p_2 \\ &+2 \sup_{q^2,q'^2<\delta_2} \left(\int_{\mathbb{R}^3} B_T((p_1+q_1,p_2+q_2),0) B_T((p_1'+q_1,p_2+q_2),0) \chi_{p_1^2>4\delta_1} \chi_{p_1'^2>4\delta_1} \mathrm{d}p_1 \mathrm{d}p_1' \mathrm{d}p_2 \right)^{1/2} \\ &\times \left(\int_{\mathbb{R}^3} B_T((p_1,p_2+q_2),0) B_T((p_1',p_2+q_2),0) \mathrm{d}p_1 \mathrm{d}p_1' \mathrm{d}p_2 \right)^{1/2} \end{aligned} \tag{6.158}$$

Substituting $p_j + q_j \to p_j$ and by choice of δ_2 and δ, this is bounded above by

$$\int_{\mathbb{R}^3} (\chi_{\mu-\delta>p_2^2} + \chi_{\mu<p_2^2}) B_T(p,0) B_T((p_1',p_2),0) \mathrm{d}p_1 \mathrm{d}p_1' \mathrm{d}p_2 \\ + 2\Big(\int_{\mathbb{R}^3} B_T(p,0) B_T((p_1',p_2),0) \chi_{p_1^2>2\delta} \chi_{p_1'^2>2\delta} \mathrm{d}p_1 \mathrm{d}p_1' \mathrm{d}p_2 \Big)^{1/2} \\ \times \Big(\int_{\mathbb{R}^3} B_T(p,0) B_T((p_1',p_2),0) \mathrm{d}p_1 \mathrm{d}p_1' \mathrm{d}p_2 \Big)^{1/2} \quad (6.159)$$

By (6.156) and the first part of this Lemma, this is of order $O((\ln \mu/T)^2) + O((\ln \mu/T)(\ln \mu/T)^{3/2}) = O((\ln \mu/T)^{5/2})$. □

6.7.3 Proof of Lemma 6.3.8

Proof of Lemma 6.3.8. For $p_2, q_2 \in \mathbb{R}$ let $B_T((\cdot,p_2),(\cdot,q_2))$ denote the self-adjoint operator on $L^2((-\sqrt{2\mu},\sqrt{2\mu}))$ acting as $\langle \psi, B_T((\cdot,p_2),(\cdot,q_2))\psi \rangle = \int_{-\sqrt{2\mu}}^{\sqrt{2\mu}} \int_{-\sqrt{2\mu}}^{\sqrt{2\mu}} \overline{\psi(p_1)} B_T(p,q) \psi(q_1) \mathrm{d}p_1 \mathrm{d}q_1$. Enlarging the domain of integration for (q_1,p_2) from a disk to square we have

$$\|B_T^{ex,2}(\xi)\| \leq \sup_{\|\psi\|_2=1} \int_{(-\sqrt{2\mu},\sqrt{2\mu})^4} \overline{\psi(p_1')} B_T((p_1',p_2),(q_1,\xi)) B_T(p,(q_1,\xi)) \psi(p_1) \mathrm{d}p_1 \mathrm{d}p_1' \mathrm{d}q_1 \mathrm{d}p_2 \\ = \sup_{\|\psi\|_2=1} \int_{-\sqrt{2\mu}}^{\sqrt{2\mu}} \langle \psi, B_T((\cdot,p_2),(\cdot,\xi))^2 \psi \rangle \mathrm{d}p_2. \quad (6.160)$$

By the triangle inequality,

$$\|B_T^{ex,2}(\xi)\| \leq \int_{-\sqrt{2\mu}}^{\sqrt{2\mu}} \|B_T((\cdot,\xi),(\cdot,p_2))\|^2 \mathrm{d}p_2. \quad (6.161)$$

For fixed p_2, q_2 we derive two bounds on $\|B_T((\cdot,p_2),(\cdot,q_2))\|^2$. For the first bound we estimate the Hilbert-Schmidt norm using (6.12):

$$\|B_T((\cdot,p_2),(\cdot,q_2))\|^2 \leq \|B_{T,\mu}((\cdot,p_2),(\cdot,q_2))\|_{\mathrm{HS}}^2 \\ \leq \int_{-\sqrt{2\mu}}^{\sqrt{2\mu}} \int_{-\sqrt{2\mu}}^{\sqrt{2\mu}} \frac{1}{\max\{|p_1^2+q_1^2+p_2^2+q_2^2-\mu|^2, T^2\}} \mathrm{d}p_1 \mathrm{d}q_1 \\ \leq 2\pi \int_0^{2\sqrt{\mu}} \frac{r}{\max\{|r^2+p_2^2+q_2^2-\mu|^2, T^2\}} \mathrm{d}r \leq \pi \int_{\mathbb{R}} \frac{1}{\max\{x^2,T^2\}} \mathrm{d}x = \frac{4\pi}{T} \quad (6.162)$$

where we first switched to angular coordinates and then substituted $x = r^2 + p_2^2 + q_2^2 - \mu$.

For the second bound the idea is to apply [60, Lemma 6.5]. For $\mu_1, \mu_2 \in \mathbb{R}$ let D_{μ_1,μ_2} be the operator on $L^2(\mathbb{R})$ with integral kernel

$$D_{\mu_1,\mu_2}(p_1,q_1) = \frac{2}{|(p_1+q_1)^2-\mu_1| + |(p_1-q_1)^2-\mu_2|}. \quad (6.163)$$

It was shown in [34, Lemma 4.6] that

$$B_T(p,q) \leq \frac{2}{|(p+q)^2-\mu| + |(p-q)^2-\mu|}. \quad (6.164)$$

In particular, we have $\|B_T((\cdot,p_2),(\cdot,q_2))\| \leq \|D_{\mu-(p_2+q_2)^2,\mu-(p_2-q_2)^2}\|$ and

$$\|B_T^{ex,2}(\xi)\| \leq \int_{-\sqrt{2\mu}}^{\sqrt{2\mu}} \min\Big\{\frac{4\pi}{T}, \|D_{\mu-(\xi+q_2)^2,\mu-(\xi-q_2)^2}\|^2\Big\} \mathrm{d}q_2 \quad (6.165)$$

According to [60, Lemma 6.5], for $\mu_1,\mu_2 \leq \mu$ there is a constant $C>0$ such that

$$\|D_{\mu_1,\mu_2}\| \leq C + \frac{C\mu^{1/2}}{|\min\{\mu_1,\mu_2\}|^{1/2}}\left[1+\chi_{\min\{\mu_1,\mu_2\}<0<\max\{\mu_1,\mu_2\}} \ln\left(1+\frac{\max\{\mu_1,\mu_2\}}{|\min\{\mu_1,\mu_2\}|}\right)\right] \tag{6.166}$$

The condition $\mu-(|q_2|+|\xi|)^2<0<\mu-(|q_2|-|\xi|)^2$ can only be satisfied for $\sqrt{\mu}-|\xi| \leq |q_2| \leq \sqrt{\mu}+|\xi|$. We get the bound

$$\begin{aligned}\sup_{|\xi|<cT} \|B_T^{ex,2}(\xi)\| \leq C\Bigg(&\int_{||q_2|-\sqrt{\mu}|<2cT} \frac{1}{T}\mathrm{d}q_2 \\ &+ \sup_{|\xi|<cT}\int_{-\sqrt{2\mu}}^{\sqrt{2\mu}} \chi_{||q_2|-\sqrt{\mu}|>2cT}\left[1+\frac{1}{|\mu-(|q_2|+|\xi|)^2|^{1/2}}\right]^2 \mathrm{d}q_2\Bigg) \leq \tilde{C}(1+\ln\mu/T)\end{aligned} \tag{6.167}$$

□

6.7.4 Proof of Lemma 6.5.1

Proof of Lemma 6.5.1. Applying (6.53), we have

$$\begin{aligned}&\sup_{q_1}\int_{\mathbb{R}} B_{T,\mu}((p_1,\xi(T)),(q_1,\xi'(T)))\mathrm{d}p_1 \\ &\quad\leq \frac{1}{2}\left[\int_{\mathbb{R}} B_{T,\mu}((p_1,\xi(T)+\xi'(T)),0)\mathrm{d}p_1 + \int_{\mathbb{R}} B_{T,\mu}((p_1,\xi(T)-\xi'(T)),0)\mathrm{d}p_1\right]\end{aligned} \tag{6.168}$$

The first integral equals

$$\int_{\mathbb{R}} B_{T,\mu-(\xi(T)+\xi'(T))^2}(p_1,0)\mathrm{d}p_1, \tag{6.169}$$

where here $B_{T,\mu}$ is understood as the function defined through (6.11) on $\mathbb{R}\times\mathbb{R}$ instead of $\mathbb{R}^2\times\mathbb{R}^2$. For the second integral replace $\xi'(T)$ by $-\xi'(T)$. The claim follows from the asymptotics

$$\int_{\mathbb{R}} B_{T,\mu}(p_1,0)\mathrm{d}p_1 = \frac{2}{\sqrt{\mu}}(\ln(\mu/T)+O(1)) \tag{6.170}$$

for $T/\mu\to 0$, see e.g. [34, Lemma 3.5]. □

6.7.5 Proof of Lemma 6.5.2

Proof of Lemma 6.5.2. Let $\gamma=\mu(T/\mu)^{\beta/2}$. By invariance of $B_T(p,q)$ under $(p_j,q_j)\to -(p_j,q_j)$ for $j\in\{1,2\}$, we may assume without loss of generality that $q\in[0,\infty)^2$. For a lower bound, we restrict the integration to $p_1,p_2>0$, $p_2^2<\mu-\epsilon^2-\gamma$ and $p_1^2>(\sqrt{\mu}+\epsilon)^2+T-p_2^2$. For $p,q\in[0,\infty)^2$ with $|q|<\epsilon$ and $p^2>(\sqrt{\mu}+\epsilon)^2+T$, we have $(p-q)^2-\mu\geq ||p|-|q||^2-\mu\geq 0$ and $(p+q)^2-\mu\geq p^2+q^2-\mu\geq T$. Therefore, in this regime

$$B_T(p,q)\geq \frac{1}{2}\frac{\tanh(1/2)}{p^2+q^2-\mu}. \tag{6.171}$$

This is minimal if $|q|=\epsilon$. Since $\int_a^\infty \frac{1}{p_1^2-b^2}\mathrm{d}p_1 = \frac{\operatorname{artanh}(b/a)}{b} = \frac{1}{b}\operatorname{artanh}\left(\sqrt{1-(a^2-b^2)/a^2}\right)$ for $a>b>0$, the left hand side of (6.130) is bounded below by

$$\frac{\tanh(1/2)^2}{4}\int_{\sqrt{\mu-\delta}}^{\sqrt{\mu-\epsilon^2-\gamma}} \frac{\operatorname{artanh}\left(\sqrt{1-\frac{2\sqrt{\mu}\epsilon+2\epsilon^2+T}{(\sqrt{\mu}+\epsilon)^2+T-p_2^2}}\right)^2}{\mu-\epsilon^2-p_2^2}\mathrm{d}p_2 \tag{6.172}$$

By monotonicity of artanh, the artanh term in the integrand is minimal for $p_2 = \sqrt{\mu - \epsilon^2 - \gamma}$. Since $\int_{\sqrt{\mu-\delta}}^{\sqrt{\mu-\epsilon^2-\gamma}} \frac{1}{\mu-\epsilon^2-p_2^2} \mathrm{d}p_2 = \frac{1}{\mu-\epsilon^2}(\mathrm{artanh}(\sqrt{1-(\epsilon^2+\gamma)/\mu}) - \mathrm{artanh}(\sqrt{1-\delta/\mu}))$, the left hand side of (6.130) is bounded below by

$$\frac{\tanh(1/2)^2}{4(\mu-\epsilon^2)} \mathrm{artanh}\left(\sqrt{1-\frac{2\sqrt{\mu}\epsilon+2\epsilon^2+T}{2\sqrt{\mu}\epsilon+2\epsilon^2+T+\gamma}}\right)^2 \left[\mathrm{artanh}\left(\sqrt{1-\frac{\epsilon^2+\gamma}{\mu}}\right) - \mathrm{artanh}\left(\sqrt{1-\frac{\delta}{\mu}}\right)\right] \tag{6.173}$$

With $\mathrm{artanh}(\sqrt{1-x}) = \frac{1}{2}\ln(4/x) + o(1)$ as $x \to 0$, we have for $T \to 0$

$$\mathrm{artanh}\left(\sqrt{1-\frac{2\sqrt{\mu}\epsilon+2\epsilon^2+T}{2\sqrt{\mu}\epsilon+2\epsilon^2+T+\gamma}}\right) = \frac{\beta}{4}\ln(\mu/T) + O(1) \tag{6.174}$$

and

$$\mathrm{artanh}\left(\sqrt{1-\frac{\epsilon^2+\gamma}{\mu}}\right) = \frac{\beta}{4}\ln(\mu/T) + O(1). \tag{6.175}$$

Hence, the left hand side of (6.130) is bounded below by $\frac{\tanh(1/2)^2}{4^3}\frac{\beta^3}{\mu}(\ln\mu/T)^3 + O(\ln\mu/T)^2$, and the claim follows. □

6.7.6 Proof of Lemma 6.6.1

Proof of Lemma 6.6.1. According to [60, Lemma 6.1], $\sup_T \|G_T^j\| < \infty$ for $j \in \{1,2\}$ and it suffices to prove $\sup_T \|N_T\| < \infty$. We have $\|N_T\| \leq \|N_T^<\| + \|N_T^>\|$, where

$$\langle\psi, N_T^<\psi\rangle = \int_{\mathbb{R}^4} \overline{F_1 V^{1/2}\psi(q,p)} B_T(p,q)\chi_{p^2,q^2<2\mu} F_1 V^{1/2}\psi(p,q)\mathrm{d}p\mathrm{d}q \tag{6.176}$$

and for $N_T^>$ replace the characteristic function by $1 - \chi_{p^2,q^2<2\mu}$.

To bound $\|N_T^>\|$, we first use the Schwarz inequality to obtain

$$\|N_T^>\| \leq \sup_{\psi\in L^2(\mathbb{R}^4),\|\psi\|_2=1} \int_{\mathbb{R}^4} B_T(p,q)(1-\chi_{p^2,q^2<2\mu})|F_1 V^{1/2}\psi(p,q)|^2\mathrm{d}p\mathrm{d}q \tag{6.177}$$

By (6.12), there is a constant $C > 0$ independent of T such that $\|N_T^>\| \leq C\|M\|$, where $M := V^{1/2}\frac{1}{1-\Delta}V^{1/2}$ on $L^2(\mathbb{R}^2)$. The Young and Hölder inequalities imply that M is a bounded operator [50].

To bound $\|N_T^<\|$, we use that $\|F_1 V^{1/2}\psi(\cdot,q)\|_\infty \leq \|V\|_1^{1/2}\|\psi(\cdot,q)\|_2$ by the Schwarz inequality and (6.164) to obtain

$$\langle\psi, N_T^<\psi\rangle \leq 2\|V\|_1 \int_{\mathbb{R}^4} \frac{\|\psi(\cdot,q)\|_2\|\psi(\cdot,p)\|_2}{|(p+q)^2-\mu| + |(p-q)^2-\mu|}\chi_{p^2,q^2<2\mu}\mathrm{d}p\mathrm{d}q \tag{6.178}$$

Recalling the definition of the operator D_{μ_1,μ_2} from (6.163), this is further bounded by

$$2\|V\|_1 \int_{\mathbb{R}^2} \|\psi(\cdot,(\cdot,q_2))\|_2 \|D_{\mu-(p_2+q_2)^2,\mu-(p_2-q_2)^2}\| \|\psi(\cdot,(\cdot,p_2))\|_2 \chi_{p_2^2,q_2^2<2\mu}\mathrm{d}p\mathrm{d}q \tag{6.179}$$

It follows from (6.166) that for any $\alpha > 0$ there is a constant C_α independent of p_2, q_2 such that $\|D_{\mu-(p_2+q_2)^2,\mu-(p_2-q_2)^2}\| \leq C_\alpha(1+|\mu-(|p_2|+|q_2|)^2|^{-1/2-\alpha})$. Let $\tilde{D}_\alpha$ denote the operator on $L^2((-\sqrt{2\mu},\sqrt{2\mu}))$ with integral kernel $\tilde{D}_\alpha(q_2,p_2) = (1+|\mu-(|p_2|+|q_2|)^2|^{-1/2-\alpha})$. Then we have $\|N_T^{<}\| \leq 2C_\alpha \|V\|_1 \|\tilde{D}_\alpha\|$ and it remains to prove that $\|\tilde{D}_\alpha\| < \infty$ for a suitable choice of α. Applying the Schur test with constant test function gives

$$\|\tilde{D}_\alpha\| \leq \sup_{|q_2|<\sqrt{2\mu}} \int_{-\sqrt{2\mu}}^{\sqrt{2\mu}} (1 + |\mu - (|p_2| + |q_2|)^2|^{-1/2-\alpha}) \mathrm{d}p_2, \tag{6.180}$$

which is finite for $\alpha < 1/2$. □

6.7.7 Proof of Lemma 6.6.2

Proof of Lemma 6.6.2. It was shown in [60, Lemma 6.2] that $\lim_{\epsilon\to 0}\sup_{T>0}\|\mathbb{I}_{\leq\epsilon} G_T^j \mathbb{I}_{\leq\epsilon}\| = 0$ for $j \in \{1,2\}$ and it remains to prove $\lim_{\epsilon\to 0}\sup_{T>0}\|\mathbb{I}_{\leq\epsilon} N_T \mathbb{I}_{\leq\epsilon}\| = 0$. We use the Schwarz inequality twice to bound

$$\begin{aligned}\|\mathbb{I}_{\leq\epsilon} N_T \mathbb{I}_{\leq\epsilon}\| &\leq \|V\|_1 \sup_{\psi\in L^2(\mathbb{R}^4), \|\psi\|_2=1} \int_{\mathbb{R}^4} \|\psi(\cdot,p)\|_2 B_T(p,q) \chi_{|p|,|q|\leq\epsilon} \|\psi(\cdot,q)\|_2 \mathrm{d}p\mathrm{d}q \\ &\leq \|V\|_1 \sup_{\psi\in L^2(\mathbb{R}^4), \|\psi\|_2=1} \int_{\mathbb{R}^4} B_T(p,q) \chi_{|p|,|q|\leq\epsilon} \|\psi(\cdot,q)\|_2^2 \mathrm{d}p\mathrm{d}q \leq \|V\|_1 \sup_{|q|\leq\epsilon} \int_{|p|\leq\epsilon} B_T(p,q)\mathrm{d}p.\end{aligned} \tag{6.181}$$

Applying (6.12), for $\epsilon < \sqrt{\mu/2}$ one can bound the right hand side uniformly in T by

$$\|V\|_1 \sup_{|q|\leq\epsilon} \int_{|p|\leq\epsilon} \frac{1}{\mu - p^2 - q^2} \mathrm{d}p, \tag{6.182}$$

which vanishes as $\epsilon \to 0$. The claim follows. □

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