Eliminating Higher-Multiplicity Intersections, II. The Deleted Product Criterion in the r-Metastable Range[*]

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Abstract

Motivated by Tverberg-type problems in topological combinatorics and by classical results about embeddings (maps without double points), we study the question whether a finite simplicial complex K can be mapped into $\mathbb{R}^d$ without higher-multiplicity intersections. We focus on conditions for the existence of *almost r-embeddings*, i.e., maps $f\colon K \to \mathbb{R}^d$ such that $f(\sigma_1) \cap \cdots \cap f(\sigma_r) = \emptyset$ whenever $\sigma_1, \ldots, \sigma_r$ are pairwise disjoint simplices of K.

Generalizing the classical Haefliger-Weber embeddability criterion, we show that a well-known necessary *deleted product condition* for the existence of almost r-embeddings is sufficient in a suitable *r-metastable range* of dimensions: If $rd \geq (r+1)\dim K + 3$, then there exists an almost r-embedding $K \to \mathbb{R}^d$ if and only if there exists an equivariant map $(K)^r_\Delta \to_{\mathfrak{S}_r} S^{d(r-1)-1}$, where $(K)^r_\Delta$ is the deleted r-fold product of K, the target $S^{d(r-1)-1}$ is the sphere of dimension $d(r-1) - 1$, and $\mathfrak{S}_r$ is the symmetric group. This significantly extends one of the main results of our previous paper (which treated the special case where $d = rk$ and $\dim K = (r-1)k$ for some $k \geq 3$), and settles an open question raised there.

1   Introduction

Let K be a finite simplicial complex, and let $f\colon K \to \mathbb{R}^d$ be a continuous map.[1] Given an integer $r \geq 2$, we say that $y \in \mathbb{R}^d$ is an r-**fold point** or r-**intersection point** of f if it has r pairwise distinct preimages, i.e., if there exist $y_1, \ldots, y_r \in K$ such that $f(y_1) = \ldots = f(y_r) = y$ and $y_i \neq y_j$ for $1 \leq i < j \leq r$. We will pay particular attention to r-fold points that are **global**[2] in the sense that their preimages lie in r *pairwise disjoint simplices* of K, i.e., $y \in f(\sigma_1) \cap \ldots \cap f(\sigma_r)$, where $\sigma_i \cap \sigma_j = \emptyset$ for $1 \leq i < j \leq r$.

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[1] For simplicity, throughout most of the paper we use the same notation for a simplicial complex K and its *underlying topological space*, relying on context to distinguish between the two when necessary.

[2] In our previous paper [17], we used the terminology "r-*Tverberg point*" instead of "*global r-fold point*."

We say that a map $f\colon K \to \mathbb{R}^d$ is an **r-embedding** if it has no r-fold points, and we say that f is an **almost r-embedding** if it has no *global* r-fold points.[3]

The most fundamental case $r = 2$ is that of **embeddings** (=2-embeddings), i.e., injective continuous maps $f\colon K \to \mathbb{R}^d$. Finding conditions for a simplicial complex K to be embeddable into $\mathbb{R}^d$ — a higher-dimensional generalization of graph planarity — is a classical problem in topology (see [27, 33] for surveys) and has recently also become the subject of systematic study from a viewpoint of algorithms and computational complexity (see [23, 22, 9]).

Here, we are interested in necessary and sufficient conditions for the existence of almost r-embeddings. One motivation are *Tverberg-type problems* in topological combinatorics (see the corresponding subsection below). Another motivation is that, in the classical case $r = 2$, embeddability is often proved in two steps: in the first step, the existence of an **almost embedding** (=almost 2-embedding) is established; in the second step this almost embedding is transformed into a honest embedding, by removing *local* self-intersections. Similarly, we expect the existence of an almost r-embedding to be not only an obvious necessary condition but a useful stepping stone towards the existence of r-embeddings and, in a further step, towards the existence of embeddings in certain ranges of dimensions.

The Deleted Product Criterion for Almost r-Embeddings

There is a natural *necessary condition* for the existence of almost r-embeddings. Given a simplicial complex K and $r \geq 2$, the (combinatorial) **deleted r-fold product**[4] of K is defined as

$$(K)^r_\Delta := \{(x_1, \ldots, x_r) \in \sigma_1 \times \cdots \times \sigma_r \mid \sigma_i \text{ a simplex of } K, \sigma_i \cap \sigma_j = \emptyset \text{ for } 1 \leq i < j \leq r\}.$$

The deleted product is a regular polytopal cell complex (a subcomplex of the cartesian product), whose cells are products of r-tuples of pairwise disjoint simplices of K.

▶ **Lemma 1** (Necessity of the Deleted Product Criterion). *Let K be a finite simplicial complex, and let $d \geq 1$ and $r \geq 2$ be integers. If there exists an almost r-embedding $f : K \to \mathbb{R}^d$ then there exists an equivariant map*[5]

$$\widetilde{f}\colon (K)^r_\Delta \to_{\mathfrak{S}_r} S^{d(r-1)-1},$$

where $S^{d(r-1)-1} = \left\{(y_1, \ldots, y_r) \in (\mathbb{R}^d)^r \mid \sum_{i=1}^r y_i = 0, \sum_{i=1}^r \|y_i\|_2^2 = 1\right\}$, and the symmetric group $\mathfrak{S}_r$ acts on both spaces by permuting components.

Proof. Given $f\colon K \to \mathbb{R}^d$, define $f^r\colon (K)^r_\Delta \to (\mathbb{R}^d)^r$ by $f^r(x_1, \ldots, x_r) := (f(x_1), \ldots f(x_r))$. Then f is an almost r-embedding iff its image avoids the *thin diagonal* $\delta_r(\mathbb{R}^d) := \{(y, \ldots, y) \mid y \in \mathbb{R}^d\} \subset (\mathbb{R}^d)^r$. Moreover, $S^{d(r-1)-1}$ is the unit sphere in the orthogonal complement $\delta_r(\mathbb{R}^d)^\perp \cong \mathbb{R}^{d(r-1)}$, and there is a straightforward homotopy equivalence $\rho\colon (\mathbb{R}^d)^r \setminus \delta_r(\mathbb{R}^d) \simeq S^{d(r-1)-1}$. Both f^r and ρ are equivariant hence so is their composition

$$\widetilde{f} := \rho \circ f^r\colon (K)^r_\Delta \to_{\mathfrak{S}_r} S^{d(r-1)-1}. \qquad\qquad \blacktriangleleft$$

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[3] We emphasize that the definitions of global r-fold points and of almost r-embeddings depend on the actual simplicial complex K (the specific triangulation), not just the underlying topological space.

[4] For more background on deleted products and the broader *configuration space/test map* framework, see, e.g., [21] or [39, 40].

[5] Here and in what follows, if X and Y are spaces on which a finite group G acts (all group actions will be from the right) then we will use the notation $F\colon X \to_G Y$ for maps that are **equivariant**, i.e., that satisfy $F(x \cdot g) = F(x) \cdot g$ for all $x \in X$ and $g \in G$.

Our main result is that the converse of Lemma 1 holds in a wide range of dimensions.

▶ **Theorem 2** (Sufficiency of the Deleted Product Criterion in the r-Metastable Range)**.** *Let* $m, d \geq 1$ *and* $r \geq 2$ *be integers satisfying*

$$rd \geq (r+1)m + 3. \tag{1}$$

Suppose that K is a finite m-dimensional simplicial complex and that there exists an equivariant map $F : (K)_\Delta^r \to_{\mathfrak{S}_r} S^{d(r-1)-1}$. Then there exists an almost r-embedding $f : K \to \mathbb{R}^d$.

▶ Remark.
a. When studying almost r-embeddings, it suffices to consider maps $f : K \to \mathbb{R}^d$ that are *piecewise-linear*[6] *(PL)* and *in general position.*[7]
b. Theorem 2 is trivial for *codimension* $d - m \leq 2$. Indeed, if r, d, m satisfy (1) and, additionally, $d - m \leq 2$ then a straightforward calculation shows that $(r-1)d > rm$, so that a map $K \to \mathbb{R}^d$ in general position has no r-fold points.
c. The special case $r = 2$ of Theorem 2 corresponds to the classical *Haefliger–Weber Theorem* [16, 37], which guarantees that for $2d \geq 3m + 3$ the existence of an equivariant map $(K)_\Delta^2 \to_{\mathfrak{S}_2} S^{d-1}$ guarantees the existence of an almost embedding $f : K \to \mathbb{R}^d$. An almost embedding can be then be turned into an embedding by a delicate construction of Skopenkov [32] or Weber [37]. The condition $2d \geq 3m + 3$ is often referred to as the **metastable range**; correspondingly, we call Condition (1) the **r-metastable range**.
d. Theorem 2 significantly extends one of the main results of our previous paper [18, Thm. 7] and [17, Thm. 3], which treated the special case $(r-1)d = rm$, $d - m \geq 3$. That special case corresponds to the situation when the set Σ^r of global r-fold points is 0-dimensional, i.e., consists of a finite number of points. In the present paper, we deal with the case where Σ^r is of higher dimension.
e. The r-metastable range is very close to the condition $rd > (r+1)m$ that guarantees that a map $f : K \to \mathbb{R}^d$ in general position does not have any $(r+1)$-fold points.

Background and Motivation: Topological Tverberg-Type Problems

Tverberg's classical theorem [35] in convex geometry can be rephrased as follows: if $N = (d+1)(r-1)$ then any *affine* map from the N-dimensional simplex σ^N to $\mathbb{R}^d$ has a global r-fold point, i.e., there does not exist an *affine* almost r-embedding of σ^N in $\mathbb{R}^d$.

Bajmoczy and Bárány [2] and Tverberg [15, Problem 84] raised the question whether the conclusion holds true, more generally, for arbitrary continuous maps:

▶ **Conjecture 3** (Topological Tverberg Conjecture)**.** *Let $r \geq 2$, $d \geq 1$, and $N = (d+1)(r-1)$. Then there is no almost r-embedding $\sigma^N \to \mathbb{R}^d$.*

This was proved by Bajmoczy and Bárány [2] for $r = 2$, by Bárány, Shlosman, and Szűcs [4] for all primes r, and by Özaydin [24] for prime powers r, but the case of arbitrary r remained open and was considered a central unsolved problem of topological combinatorics.

There are numerous close relatives and other variants of (topological) Tverberg-type problems and results. These can be seen as *generalized nonembeddability results* or *problems* and typically state that a particular complex K (or family of complexes) does not admit an almost r-embedding into $\mathbb{R}^d$. Well-known examples are the *Colored Tverberg Problem*

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[6] Recall that f is PL if there is some subdivision K' of K such that $f|_\sigma$ is affine for each simplex σ of K'.
[7] Every continuous map $g : K \to \mathbb{R}^d$ can be approximated arbitrarily closely by PL maps in general position, and if g is an almost r-embedding, then the same holds for any map sufficiently close to g.

[5, 3, 41, 40, 6] and generalized *Van Kampen–Flores-type results* [29, 36]. Theorem 2 provides a general necessary and sufficient condition for topological Tverberg-type results in the r-metastable range.

The topological Tverberg conjecture and the subsequent developments played an important role in the introduction and use of methods from *equivariant topology* in discrete and computational geometry. The prime and prime power cases of Conjecture 3 were proved via Lemma 1, $(\sigma^N)^r_\Delta \to_{\mathfrak{S}_r} S^{d(r-1)-1}$. However, this fails in the remaining cases: Özaydin [24, Thm. 4.2] showed that if r is *not* a prime power then there exists an equivariant map $F\colon (\Delta^N)^r_\Delta \to_{\mathfrak{S}_r} S^{d(r-1)-1}$.

In the extended abstract of our previous paper [17], we proposed a new approach to the conjecture, based on the idea of combining Özaydin's result with the sufficiency of the deleted product product ([17, Thm 3]) to construct counterexamples, i.e., almost r-embeddings $\sigma^N \to \mathbb{R}^d$, whenever r is not a prime power. At the time we suggested this in [17], there remained what seemed a very serious obstacle to completing this approach: Our theory required the assumption of *codimension $d - \dim K \geq 3$*, which is not satisfied for $K = \sigma^N$.

In a recent breakthrough, Frick [12] was the first to find a way to overcome this "codimension 3 barrier" and to construct counterexamples to the topological Tverberg conjecture for all parameters (d, r) with $d \geq 3r + 1$ and r not a prime power, by a clever reduction (using a combinatorial trick discovered independently in [14, p. 445-446] and [7, Thm. 6.3]) to a suitable lower-dimensional skeleton for which the codimension condition is satisfied and the required almost r-embedding exists by Özaydin's result and ours.

A different solution to the codimension 3 obstacle (based on the notion of *prismatic maps*) is given in the full version of our paper [18], leading to counterexamples for $d \geq 3r$. In joint work with Avvakumov and Skopenkov [1], we recently improved this further and obtained counterexamples for $d \geq 2r$, using an extension (for $r \geq 3$) of [18, Thm. 7] to *codimension 2*.

In conclusion, methods from equivariant topology and the general framework of *configuration spaces* and *test maps* [39, 40] have been very successfully used in discrete and computational geometry. In particular, *equivariant obstruction theory* and, more generally *equivariant homotopy theory*, provide powerful tools for deciding whether suitable test maps exist. However in cases where the existence of a test map does not settle the problem (as with the topological Tverberg conjecture), further geometric ideas are needed. The general philosophy and underlying idea here and in the two companion papers [18, 1] is to complement equivariant methods by methods from *geometric topology*, in particular *piecewise-linear topology*, and we hope that these will find further applications.

▶ Remark (Further Questions and Future Research).

a. *Beyond the r-Metastable Range.* Is condition (1) in Theorem 2 necessary? In the case $r = 2$, it is known that for $d \geq 3$, the Haefliger–Weber Theorem fails outside the metastable range: for every pair (m, d) with $2d < 3m + 3$ and $d \geq 3$, there are examples [20, 31, 11, 30, 13] of m-dimensional complexes K such that $(K)^2_\Delta \to_{\mathfrak{S}_2} S^{d-1}$ but K does not embed into $\mathbb{R}^d$. Moreover, in the case $r = 2$, $m = 2$ and $d = 4$, the examples do not even admit an almost embedding into $\mathbb{R}^4$, see [1].

On the other hand, as remarked above, in [1] the following extension of [18, Thm. 7] is proved: if $r \geq 3$ $d = 2r$, and $m = 2(r - 1)$, then a finite m-dimensional complex K admits an almost r-embedding if and only if there exists an equivariant map $(K)^r_\Delta \to_{\mathfrak{S}_r} S^{d(r-1)-1}$. It would be interesting to know whether there is analogous extension (for $r \geq 3$) of Theorem 2 that is nontrivial in codimension $d - m = 2$.

b. *The Planar Case and Hanani–Tutte.* In the classical setting ($r = 2$) of embeddings, the case $d = 2$, $m = 1$ of *graph planarity* is somewhat exceptional: the parameters lie outside

the (2-fold) metastable range, but the existence of an equivariant map $F \colon (K)^2_\Delta \to_{\mathfrak{S}_2} S^1$ is sufficient for a graph K to be planar, by the *Hanani–Tutte Theorem*[8] [10, 34]. The classical proofs of that theorem rely on *Kuratowski's Theorem*, but recently [25, 26], more direct proofs have been found that do not use forbidden minors. It would be interesting to know whether there is an analogue of the Hanani–Tutte theorem for almost r-embeddings of 2-dimensional complexes in $\mathbb{R}^2$, as an approach to constructing counterexamples to the topological Tverberg conjecture in dimension $d = 2$. We plan to investigate this in a future paper.

Structure of the Paper

The remainder of the paper is devoted to the proof of Theorem 2. By Lemma 1, we only need to show that the existence of an equivariant map $(K)^r_\Delta \to_{\mathfrak{S}_r} S^{d(r-1)-1}$ implies the existence of an almost r-embedding $K \to \mathbb{R}^d$. Moreover, by Remarks 1 (b) and (d), we may assume, in addition to the parameters being in the r-fold metastable range, that the *codimension* $d - m$ of the image of K in $\mathbb{R}^d$ is at least 3, and that the intersection multiplicity r is also at least 3. Thus, we will work under the following hypothesis:

$$rd \geq (r+1)m + 3, \quad d - m \geq 3, \quad \text{and} \quad r \geq 3. \tag{2}$$

The proof of Theorem 2 is based on two main lemmas: Lemma 5 (*Reduction Lemma*) reduces the situation to a single r-tuple of pairwise disjoint simplices of K, and Lemma 7 (*generalized Weber–Whitney Trick*) solves that reduced situation. In Section 2, we give the precise (and somewhat technical) statements of these lemmas, along with some background, and prove the Reduction Lemma 5. In Section 3, we show how to prove Theorem 2 using these lemmas.

Due to the page limit, the proof of Lemma 7 is omitted from this *extended abstract*; we refer to the full version of this paper [19] for the details.

2   The Two Main Lemmas

In this section, we formulate the two main lemmas on which the proof of Theorem 2 rests.

We work in the *piecewise-linear* (*PL*) category (standard references are [38, 28]). All manifolds (possibly with boundary) are PL-manifolds (can be triangulated as locally finite simplicial complexes such that the link of every nonempty face is either a PL-sphere or a PL-ball), and all maps between **polyhedra** (geometric realizations of simplicial complexes) are PL-maps (*i.e.*, simplicial on sufficiently fine subdivisions).[9] In particular, all balls are PL-ball and all spheres are PL-spheres (PL-homeomorphic to a simplex and the boundary of a simplex, respectively).

A submanifold P of a manifold Q is **properly embedded** if $\partial P = P \cap \partial Q$. The **singular set** of a PL-map f defined on a polyhedron K is the closure in K of the set of points at which f is not injective.

One basic fact that we will use for the proofs of both Lemmas 5 and 7 is the following version of *engulfing* [38, Ch. VII]:

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[8] The existence of an equivariant map implies, via standard equivariant obstruction theory, that there exists a map from the graph K into $\mathbb{R}^2$ such that the images of any two disjoint (*independent*) edges intersect an even number of times, which is the hypothesis of the Hanani–Tutte Theorem.

[9] The PL assumption is no loss of generality: if K is a finite simplicial complex and $f \colon K \to \mathbb{R}^d$ is an almost r-embedding then f can be slightly perturbed to a PL map with the same property.
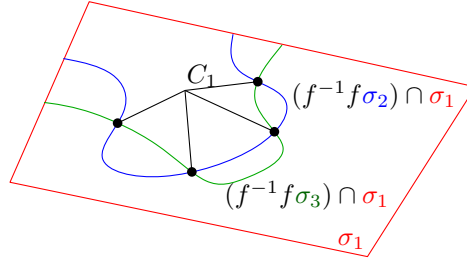
■ **Figure 1** For $r = 3$, the construction of C_1 inside of σ_1. The collapsible polyhedron C_1 is a "cone" over the triple intersection set S_1 (which consists of four isolated points in the picture).

▶ **Theorem 4** (Engulfing, [38, Ch. VII, Thm. 20]). *Let M be an m-dimensional k-connected manifold with $k \leq m - 3$. Let X a compact x-dimensional subpolyhedron in the interior of M. If $x \leq k$, then there exists a collapsible subpolyhedron C in the interior of M with $X \subseteq C$ and $\dim(C) \leq x + 1$.*

The collapsible polyhedron C can be thought of as an analogue of a "cone" over X.

▶ **Lemma 5** (Reduction Lemma). *Let m, d, r be three positive integers satisfying* (2). *Suppose $f \colon K \to \mathbb{R}^d$ is a map in general position, and $\sigma_1, \ldots, \sigma_r$ be pairwise disjoint simplices of K of dimension $s_1, \ldots, s_r \leq m$ such that $f^{-1}(f(\sigma_1) \cap \cdots \cap f(\sigma_r)) \cap \sigma_i$ is contained in the interior of σ_i. Then there exists a ball B^d in $\mathbb{R}^d$ such that*
1. *B^d intersects each $f(\sigma_i)$ in a ball that is properly embedded in B^d, and that ball avoids the image of the singular set of $f|_{\sigma_i}$, as well as $f(\partial \sigma_i)$;*
2. *B^d contains $f(\sigma_1) \cap \cdots \cap f(\sigma_r)$ in its interior; and*
3. *B^d does not intersect any other parts of the image $f(K)$.*

Proof. Let us consider $S_i := f^{-1}(f(\sigma_1) \cap \cdots \cap f(\sigma_r)) \cap \sigma_i$. By general position [28, Thm 5.4] this is a polyhedron of dimension at most $s_1 + \cdots + s_r - (r-1)d \leq rm - (r-1)d$. By Theorem 4, we find $C_i \subseteq \sigma_i$ collapsible, containing S_i, and of dimension at most $rm - (r-1)d + 1$. Figure 1 illustrates the case $r = 3$.

The dimension of the singular set of $f|_{\sigma_i}$ is at most $2s_i - d$. Hence, C_i is disjoint from it since $(rm - (r-1)d + 1) + (2s_i - d) - s_i \leq (r+1)m - rd + 1$, which is negative in the metastable range. Thus, f is injective in a neighbourhood of C_i.

Again by Theorem 4, we find in $\mathbb{R}^d$ a collapsible polyhedron $C_{\mathbb{R}^d}$ of dimension at most $rm - (r-1)d + 2$ and containing $f(C_1) \cup \cdots \cup f(C_r)$. Figure 2 illustrates the construction for $r = 3$. By general position we have the following properties:
1. $C_{\mathbb{R}^d}$ intersects $f(\sigma_i)$ exactly in $f(C_i)$. Indeed, in the metastable range, $rm - (r-1)d + 2 + s_i - d \leq (r+1)m - rd + 2 < 0$.
2. $C_{\mathbb{R}^d}$ does not intersect any other part of $f(K)$ (by a similar computation).

We take a small **regular neighbourhood** [28, Ch. 3] B of $C_{\mathbb{R}^d}$, which still avoids the singular set of each $f|_{\sigma_i}$ as well as other parts of $f(K)$. This regular neighbourhood is a ball, since $C_{\mathbb{R}^d}$ is collapsible. The intersection $B \cap f(\sigma_i)$ is a regular neighbourhood of $f(C_i)$ which is also a collapsible space, hence $B \cap f(\sigma_i)$ is a ball (properly contained in B).   ◀

An **ambient isotopy** H of a PL-manifold X is a collection of homeomorphisms $H_t \colon X \to X$ for $t \in [0,1]$, which vary continuously with t, and with $H_0 = \text{id}$. We say that an ambient isotopy H **throws** a subspace $Y \subseteq X$ onto Z if $H_1(Y) = Z$, see [38, Ch. V].

We say that an ambient isotopy H of X is **proper** if $H_t|_{\partial X} = \text{id}_{\partial X}$ for all t.
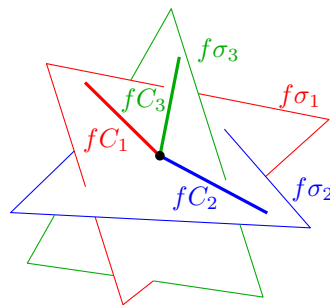
Figure 2 For $r = 3$, the polyhedron $C_{\mathbb{R}^d}$ is a "cone" over $fC_1 \cup fC_2 \cup fC_3$.

▶ **Definition 6.** Let m, d, r be three positive integers satisfying (2). Let $\sigma_1, \ldots, \sigma_r$ be balls of dimensions $s_1, \ldots, s_r \leq m$. We define

$$s := s_1 + \ldots + s_r.$$

Let f be a continuous map, mapping the disjoint union of the σ_i to a d-dimensional ball B^d, i.e.,

$$f : \sigma_1 \sqcup \cdots \sqcup \sigma_r \to B^d.$$

We define the **test map** $\widetilde{f}$ associated to f

$$\widetilde{f} : \sigma_1 \times \cdots \times \sigma_r \to B^d \times \cdots \times B^d, \quad \text{by} \quad (x_1, ..., x_r) \mapsto (fx_1, ..., fx_r).$$

If, for each $i = 1, ..., r$,

$$f\sigma_1 \cap \cdots \cap f\partial\sigma_i \cap \cdots \cap f\sigma_r = \emptyset,$$

then $\widetilde{f}\partial(\sigma_1 \times \cdots \times \sigma_r) \subset B^d \times \cdots \times B^d$, avoids the **thin diagonal** $\delta_r(B^d) = \{(x, \ldots, x) \mid x \in B^d\}$ of B^d. Thus,

$$\partial(\sigma_1 \times \cdots \times \sigma_r) \to (B^d \times \cdots \times B^d) \setminus \delta_r(B^d). \tag{3}$$

Observe that $\partial(\sigma_1 \times \cdots \times \sigma_r) \cong S^{s-1}$, where $s := \sum_i s_i$, and $(B^d \times \cdots \times B^d) \setminus \delta_r(B^d)$ is homotopy equivalent to $S^{d(r-1)-1}$. Therefore, the map (3) defines an element

$$\alpha(f) \in \pi_{s-1}(S^{d(r-1)-1}),$$

which we call **intersection class of f**.

▶ **Lemma 7** (Generalized Weber-Whitney Trick)**.** *Let m, d, r be three positive integers satisfying (2). Let $\sigma_1, \ldots, \sigma_r$ be balls of dimensions $s_1, \ldots, s_r \leq m$ properly contained in a d-dimensional ball B and with $\sigma_1 \cap \cdots \cap \sigma_r$ in the* interior *of B.*

1. *Let us denote by α the intersection class of the map $\sigma_1 \sqcup \cdots \sqcup \sigma_r \to B^d$.*
    If $\alpha = 0$, then there exists $(r-1)$ proper ambient isotopies of B that we can apply to $\sigma_1, \ldots, \sigma_{r-1}$, respectively, to remove the r-intersection set; i.e., there exist $(r-1)$ proper isotopies $H_t^1, \ldots, H_t^{r-1}$ of B throwing σ_i onto $\sigma_i' := H_1^i \sigma_i$ and such that

$$\sigma_1' \cap \cdots \cap \sigma_{r-1}' \cap \sigma_r = \emptyset.$$

2. *Let us assume that $\sigma_1 \cap \cdots \cap \sigma_r = \emptyset$ and $\sigma_2 \cap \cdots \cap \sigma_r \neq \emptyset$, and let $z \in \pi_s(S^{d(r-1)-1})$. There exists J_t a proper ambient isotopy of B such that*

- $J_1\sigma_1 \cap \sigma_2 \cap \cdots \cap \sigma_{r-1} \cap \sigma_r = \emptyset,$
- *The intersection class of f is z, where*

$$f : (\sigma_1 \times I) \sqcup \sigma_2 \sqcup \cdots \sqcup \sigma_r \to B^d$$

  is defined as the inclusion on σ_i for $i \geq 2$, and for $(x,t) \in \sigma_1 \times I$, $f(x,t) = J_t(x)$.

▶ Remark.
- For $r = 2$, Lemma 7 already appears in Section 4 of Weber's thesis [37].
- Roughly speaking, Part 1 of Lemma 7 means that if the intersection class vanishes, then one can resolve the r-intersection set.
  Part 2 means that each element of $\pi_s(S^{d(r-1)-1})$ can be obtained by moving from a fixed solution to a new solution.

3 Proof of Theorem 2

Here, we show how to use Lemmas 5 and 7 to prove the main theorem. The inductive argument used in the proof mirrors that of Section 5 in Weber's thesis [37], where Theorem 2 is proven for $r = 2$.

Proof of Theorem 2. We are given $F : (K)^r_\Delta \to_{\mathfrak{S}_r} S^{d(r-1)-1}$, and we want to construct $f : K \to \mathbb{R}^d$ without global r-fold point.

We start with a map $f : K \to \mathbb{R}^d$ in general position. Inductively, we will redefine f on the skeleta of K as to get the desired property. There are two levels in the induction. To describe these, let us fix a total ordering of the simplices of K that extends the partial ordering by dimension, *i.e.*,

$$K = \{\tau_1, \ldots, \tau_N\}, \qquad \dim \tau_i \leq \dim \tau_{i+1} \text{ for } 1 \leq i \leq N-1.$$

First, we give a very informal plan of the "double induction" that we are going to use in the proof: we go over the list of simplices $\tau_1, ..., \tau_N$, and for each simplex τ_i we consider all the global r-fold points of τ_i with all the simplices *before* τ_i in the list. More precisely, we consider the list l_i of all r-tuples of pairwise disjoint simplices containing τ_i and simplices *before* τ_i in the list $\tau_1, ..., \tau_N$. For each r-tuple in l_i, we need to eliminate its global r-fold points.

Therefore, once τ_i is fixed, we have a *new list l_i*. We are going to *order l_i* (by a notion of dimension), and then inductively scan over it and remove the global r-fold points for each r-tuple in l_i.

Let us describe now the first level of the inductive argument. We have to prove the following: Suppose we are given a map $f : K \to \mathbb{R}^d$ in general position with the following two properties:

1. Restricted to the subcomplex $L = \{\tau_1, \ldots, \tau_{N-1}\}$ the map $f|_L$ does not have any r-fold points between disjoint r-tuples of simplices;
2. $\widetilde{f}$ restricted to $(L)^r_\Delta$ is $\mathfrak{S}_r$-equivariantly homotopic to F, where $\widetilde{f}$ is the map defined in Lemma 1.

Then we can redefine f as to have these two properties on the whole of K. This is the first level of induction.

For the second level of the induction, let us define the **dimension** of a finite set of simplices as the sum of their individual dimensions. For the the purposes of this proof, we use the terminology k-**collection** for a set of cardinality k. Consider those $(r-1)$-collections

t of simplices of L that, together with τ_N, form an r-collection of pairwise disjoint simplices. We fix a total ordering of these $(r-1)$-collections that extends the partial ordering given by dimension, *i.e.*, we list them as

$$t_1, \ldots, t_M,$$

with $\dim t_i \leq \dim t_{i+1}$ for $1 \leq i < M$. (Thus, each t_i is an $(r-1)$-collection of simplices of L, and t_i joined with τ_N is a r-collection of pairwise disjoint simplices.) Once again, inductively, it suffices to prove the following: Assuming that f has the two properties

1. For each $(r-1)$-collection t_i in the list $t_1, \ldots, t_{M-1}$, the map f does not have any r-fold points with preimages in the r-collection formed by adjoining τ_N to t_i.
2. the map $\widetilde{f}$ is $\mathfrak{S}_r$-equivariantly homotopic to F on the complex

$$(L)_\Delta^r \cup \bigcup_{i \leq M-1} [t_i \cup \{\tau_N\}] \subseteq (K)_\Delta^r,$$

where the operator $[-]$ converts an unordered r-collection of pairwise disjoint simplices of K into the set of its corresponding cells[10] in $(K)_\Delta^r$.

Then we can modify f as to have these two properties on the list $t_1, \ldots, t_M$.

In order to do so, let us consider the r-collection $t_M \cup \{\tau_N\}$. We rename its elements as

$$t_M \cup \{\tau_N\} = \{\sigma_1, \ldots, \sigma_r\}, \quad (\text{with } \tau_N = \sigma_r).$$

By the induction hypothesis (namely the order on the τ_i and the t_i), for each $i = 1, \ldots, r$, $f^{-1}(f\sigma_1 \cap \cdots \cap f\sigma_r) \cap \sigma_i$ is contained in the *interior* of σ_i (since the induction has already "worked" on the simplices in $\partial \sigma_i$). Furthermore, the map $\widetilde{f} : \partial(\sigma_1 \times \cdots \times \sigma_r) \to S^{d(r-1)-1}$ is homotopic to F, this also follows from the ordering on the τ_i and the t_i (the homotopy is already defined on all the cells of $\partial(\sigma_1 \times \cdots \times \sigma_r)$).

We are in position to apply Lemma 5: we find a ball B^d in $\mathbb{R}^d$ with the three properties listed in the Lemma. Let us call σ_i' the sub-ball in σ_i properly embedded into B^d, *i.e.*, $\sigma_i' \overset{f}{\hookrightarrow} B^d$, and $f\partial\sigma_i' = \partial B^d \cap f\sigma_i'$.

By the Combinatorial Annulus Theorem [8, 3.10], there exists an isotopy of σ_i in itself that progressively retracts σ_i to σ_i'. I.e., there exists $G_t^i : \sigma_i \to \sigma_i$ with G_0^i being the identity and G_1^i being a homeomorphism between σ_i and σ_i'. We define a homotopy by

$$
\begin{array}{cccc}
G : & \partial(I \times \sigma_1 \times \cdots \times \sigma_r) & \xrightarrow{fG^1 \times \cdots \times fG^r} & \mathbb{R}^d \times \cdots \times \mathbb{R}^d \setminus \delta_r \mathbb{R}^d \\
& (t, x_1, \ldots, x_r) & \longmapsto & (fG_t^1 x_1, \ldots fG_t^r x_r).
\end{array}
\tag{4}
$$

By the induction hypothesis,

$$\partial(\sigma_1 \times \cdots \times \sigma_r) \xrightarrow{f \times \cdots \times f} \mathbb{R}^d \times \cdots \times \mathbb{R}^d \setminus \delta_r \mathbb{R}^d \tag{5}$$

is homotopic to F, and F is defined over $\sigma_1 \times \cdots \times \sigma_r$. Therefore, the homotopy class of

$$\partial(\sigma_1' \times \cdots \times \sigma_r') \xrightarrow{f \times \cdots \times f} B^d \times \cdots \times B^d \setminus \delta_r B^d$$

is trivial. Hence, we can use the first part of the Lemma 7 to find $(r-1)$ proper ambient isotopies of B, say $H_t^1, \ldots, H_t^{r-1}$, such that $H_1^1(f\sigma_1') \cap \cdots \cap H_1^{r-1}(f\sigma_{r-1}') \cap f\sigma_r' = \emptyset$. This removes the r-fold points.

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[10] E.g., $[\{\alpha, \beta, \gamma\}] = \{\alpha \times \beta \times \gamma, \alpha \times \gamma \times \beta, \beta \times \alpha \times \gamma, \beta \times \gamma \times \alpha, \gamma \times \alpha \times \beta, \gamma \times \beta \times \alpha\}$.

To finish the induction, we also need to extend the equivariant homotopy between $\widetilde{f}$ and F on the cell $\sigma_1 \times \cdots \times \sigma_r$, as the homotopy is already defined on $\partial(\sigma_1 \times \cdots \times \sigma_r)$. This is when the second part of Lemma 7 becomes useful.

We define a map on $\partial(I \times \sigma_1 \times \cdots \times \sigma_r) \to \mathbb{R}^d \times \cdots \times \mathbb{R}^d \setminus \delta_r \mathbb{R}^d$ in the following way:

1. on $\{0\} \times \sigma_1 \times \cdots \times \sigma_r$, we use F,
2. on $[0, \frac{1}{3}] \times \partial(\sigma_1 \times \cdots \times \sigma_r)$, we use the homotopy from F to (5),
3. on $[\frac{1}{3}, \frac{2}{3}] \times \partial(\sigma_1 \times \cdots \times \sigma_r)$, we use G,
4. on $[\frac{2}{3}, 1] \times \partial(\sigma_1 \times \cdots \times \sigma_r)$, we use $(H_t^1 \times \cdots \times H_t^{r-1} \times \mathrm{id}) \circ (fG_1^1 \times \cdots \times fG_1^r)$,
5. $\{1\} \times \sigma_1 \times \cdots \times \sigma_r$, we use $(H_1^1 \times \cdots \times H_1^{r-1} \times \mathrm{id}) \circ (fG_1^1 \times \cdots \times fG_1^r)$.

This defines a class $\theta \in \pi_{\sum \dim \sigma_i}(S^{d(r-1)-1})$. To conclude, we need to have $\theta = 0$ (this is the condition to be able to extend to homotopy between $\widetilde{f}$ and F).

By the second part of Lemma 7, we can[11] perform a "second move" on σ_1 with an ambient isotopy J_t of B such that

$$\partial(I \times \sigma_1 \times \cdots \times \sigma_r) \xrightarrow{(J_t \times \mathrm{id} \times \cdots \times \mathrm{id}) \circ (H_1^1 \times \cdots \times H_1^{r-1} \times \mathrm{id}) \circ (fG_1^1 \times \cdots \times fG_1^r)} \mathbb{R}^d \times \cdots \times \mathbb{R}^d \setminus \delta_r \mathbb{R}^d$$

represents exactly $-\theta$. Therefore, by using this last move, we can assume that $\theta = 0$, *i.e.*, we can extend the equivariant homotopy between $\widetilde{f}$ and F, as needed for the induction. ◀

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[11] We can always obtain the assumption $\sigma_2 \cap \cdots \cap \sigma_r \neq \emptyset$ by modifying the map f as follows [18, "Finger moves" in the proof of Lemma 43]: we pick $r - 1$ spheres $S^{s_2}, ..., S^{s_r}$ in the interior of B^d of dimension $s_2, ..., s_r$ in general position and such that $S^{s_2} \cap \cdots \cap S^{s_r}$ is a sphere S. Then, for $i = 2, ..., r$, we pipe σ_i' to S^{s_i}. The resulting map has the desired property.

This "piping" change can be absorbed by a slight modification (and renumbering) of the H_t^i. The support of these modifications is a collection of regular neighborhoods of 1-polyhedra (= paths used for piping).

Also, note that the cases when, by general position, $\dim S < 0$ corresponds the trivial cases $\theta = 0$. Indeed, $\dim S < 0$ corresponds to $(d - s_2) + \cdots + (d - s_r) > d$, i.e., $(r - 1)d + s_1 - d > \sum s_i$, and since $s_1 - d \leq -3$, we have $(r - 1)d - 1 > \sum s_i$, and so $\pi_{\sum \dim \sigma_i}(S^{d(r-1)-1}) = 0$.

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