EXTERIOR ALGEBRA AND COMBINATORICS

by

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1 Introduction

The extension of extremal combinatorics to the setting of exterior algebra is a work in progress that gained attention recently. The incorporation of exterior algebra into combinatorics has yielded substantial advancements in various areas, including hypergraph theory, convexity, and extremal set theory [1, 6, 13, 14]. In 1977, Lovász [1] employed exterior algebra technique to prove an extension of Bollobás' two families theorem [2]. The original two families theorem states the following:

Theorem 1.1 (Two Families Theorem [2]). *Let $A_1, \dots, A_m$ be k-element sets and $B_1, \dots, B_m$ be ℓ-element sets such that*

1. $A_i \cap B_i = \emptyset$ *for* $i = 1, \dots, m$;
2. $A_i \cap B_j \neq \emptyset$ *for* $i \neq j$.

Then $m \leq \binom{k+\ell}{k}$. Furthermore, if $m = \binom{k+\ell}{k}$ then there is some set S of cardinality $k+\ell$ such that the A_i are all subsets of S of size k, and $B_i = S \setminus A_i$ for each i.

Lovász proved the following extension of Theorem 1.1 using exterior algebra method which allows the relaxation of condition (2): instead of requiring A_i and B_j to intersect for all pairs with $i \neq j$, it only requires that the intersection is non-trivial when $i < j$.

Theorem 1.2 (Skew Two Families [1]). *Let $A_1, \dots, A_m$ be k-element sets and $B_1, \dots, B_m$ be ℓ-element sets such that*

1. $A_i \cap B_i = \emptyset$ *for* $i = 1, \dots, m$;
2. $A_i \cap B_j \neq \emptyset$ *for* $1 \leq i < j \leq m$.

Then $m \leq \binom{k+\ell}{k}$.

We will present Lovász's proof of Theorem 1.2 in Section 3.

An exemplary application of exterior algebra in tackling combinatorial problems can be observed in the study of intersecting convex sets. Kalai [14] made notable contributions by employing the exterior algebra method to investigate the properties of intersecting convex sets. Building upon this approach, a recent paper by Bulavka, Goodarzi, and Tancer [6] further utilized the exterior algebra framework introduced by Kalai in their work on finding optimal bounds for the standard fractional Helly theorem. However, most of these results use exterior algebra as a tool to study combinatorics rather than using combinatorics as a tool to study the geometric structure of the subspaces of exterior algebra. Our approach focuses on exploring the connection between exterior algebra and combinatorics by utilizing combinatorial results to investigate the geometric properties inherent in exterior algebra.

In recent results, Scott–Willmer [4] and Woodroofe [5] explored the application of combinatorics in exterior algebra. Through this connection, they proved an analog for the upper bound in Erdös–Ko–Rado theorem. Let us first formulate the original Erdös–Ko–Rado theorem.

Theorem 1.3 (Erdös–Ko–Rado [3]). *Suppose $k \leq n/2$. If $\mathcal{A}$ is an intersecting family of k-element subsets of $\{1, \dots, n\}$ i.e. any two subsets in $\mathcal{A}$ shares at least one common element, then $|\mathcal{A}| \leq \binom{n-1}{k-1}$. If $k < n/2$, then the equality holds only if all sets in $\mathcal{A}$ share a common element.*

It was later shown by Hilton and Milner [9] that the upper bound decreases significantly when the intersecting family is not maximal.

Theorem 1.4 (Hilton–Milner [9]). *Suppose that $k < n/2$. If $\mathcal{A}$ is an intersecting family of k-element subsets of $\{1, \dots, n\}$ such that there is no common element for all sets of $\mathcal{A}$, then $|\mathcal{A}| \leq \binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1$.*

Building upon these theorems, recent papers by Scott–Willmer [4] and Woodroofe [5] showed the following result.

Theorem 1.5 (Exterior Algebra E.K.R. [4, 5]). *Suppose that $k \leq n/2$. If W is a subspace of $\bigwedge^k \mathbb{R}^n$ such that $v \wedge w = 0$ for all $v, w \in W$, then $\dim W \leq \binom{n-1}{k-1}$.*

We will define the operator $\wedge$ and the space of k-forms $\bigwedge^k \mathbb{R}^n$ in Section 2 and later we will explain why Theorem 1.5 is an extension of Erdös–Ko–Rado theorem. However, unlike the original Erdös–Ko–Rado theorem which has the characterization of the equality case, the equality case in Theorem 1.5 is an open problem. A desired result for this problem is the following conjecture.

Conjecture 1.6. *Suppose $k < n/2$ and let W be a subspace of $\bigwedge^k \mathbb{R}^n$ such that for all $v, w \in W$ $v \wedge w = 0$. Then $\dim W = \binom{n-1}{k-1}$ if and only if there is an $a \in \mathbb{R}^n$ such that for all $w \in W$, $a \wedge w = 0$.*

In this thesis, we aim to study the combinatorial structure of exterior algebra by introducing a dictionary that translates the notions from the set systems into the framework of exterior algebra. We prove the conjecture regarding subspaces of two-forms and further extend the result to Hilton–Milner theorem in the exterior algebra setting. Specifically, we prove the following theorem.

Theorem 1.7 (EKR and Hilton–Milner for Two Forms [10]). *Suppose $n \geq 5$, and W is a subspace of $\bigwedge^2 \mathbb{R}^n$ such that for all $v, w \in W$ $v \wedge w = 0$. Then*

(i) $\dim W \leq n - 1$, *and the equality holds if and only if* $\exists a \in \mathbb{R}^n$ *s.t.* $a \wedge w = 0$ $\forall w \in W$.

(ii) If there is no $a \in \mathbb{R}^n$ such that $a \wedge w = 0$ for all $w \in W$, then $\dim W \leq 3$ *and the equality holds if and only if $W = span\{x_1 \wedge x_2, x_2 \wedge x_3, x_3 \wedge x_1\}$ for some linearly independent $x_1, x_2, x_3 \in \mathbb{R}^n$.*

This thesis is organized as follows: In Section 2, we introduce notations and preliminaries on exterior algebra and provide some results on the geometric structure of exterior algebra. In Section 3, we build the connection between combinatorics and exterior algebra. Section 4 is dedicated to Erdös–Ko–Rado theorem and its exterior algebra analog. We present a proof of the original Erdös–Ko–Rado theorem and give a potential proof idea for the equality case of the Exterior Algebra Erdös–Ko–Rado theorem, which closely follows the methodology used in the proof of the original Erdös–Ko–Rado theorem. In Section 5, we present our main result, which is the proof of Theorem 1.7. Finally in Section 6, we discuss exterior algebra analogs of other well-known combinatorial problems such as deBruijn–Erdös theorem and Frankl's conjecture.

2 Notations and preliminaries

We denote the set $\{1, 2, \ldots, n\}$ by $[n]$ and the set of all k-sets of $I \subseteq [n]$ by $\binom{I}{k}$. The linear hull of a set S of vectors is denoted by $span\{S\}$.

We will use $\bigwedge^k \mathbb{R}^n$ to denote the space of k-forms over $\mathbb{R}^n$. For a sequence of k vectors $\{v_1, ..., v_k\} \in \mathbb{R}^n$, a k-form $v_1 \wedge ... \wedge v_k$ is defined by its evaluation on vectors $\{w_1, ..., w_k\} \in \mathbb{R}^n$ given by

$$v_1 \wedge ... \wedge v_k(w_1, \ldots, w_k) = \det(v_i(w_j)), \quad \text{where } i, j \in [k].$$

Here we only provide concise definitions, intending to give a brief overview of exterior algebra. We refer the reader to Greub's book [17] for a proper introduction to multilinear algebra.

Following directly from the definition of a k-form, we have the following properties:

(i) $v_1 \wedge ... \wedge v_k = 0$ whenever the vectors $v_1, ..., v_k \in \mathbb{R}^n$ are linearly dependent .

(ii) If σ is a permutation of the integers $\{1, \ldots, k\}$,

$$v_{\sigma(1)} \wedge ... \wedge v_{\sigma(k)} = sgn(\sigma) v_1 \wedge ... \wedge v_k$$

where $sgn(\sigma)$ is the sign of the permutation σ.

The space of k-forms $\bigwedge^k \mathbb{R}^n$ is a vector space spanned by k-forms. Let $W \subset \mathbb{R}^n$ be a subspace. Then it follows that $\bigwedge^k(W)$ is a subspace of $\bigwedge^k \mathbb{R}^n$. We define $W^\perp$ as the subspace of $\mathbb{R}^n$ consisting of all vectors v that are orthogonal to W, i.e., $\langle v, w\rangle = 0$ for all $w \in W$. Extending this notion to k-forms, we have $\bigwedge^k(W^\perp) \subseteq \bigwedge^k \mathbb{R}^n$.

A k-form $v \in \bigwedge^k \mathbb{R}^n$ is called *decomposable* if it can be written in the form $v_1 \wedge ... \wedge v_k$ for some $v_1, ..., v_k \in \mathbb{R}^n$.

Example. For a given orthogonal basis $f_1, ..., f_n$ of $\mathbb{R}^n$, a 2-form $f_1 \wedge f_2$ is decomposable whereas $f_1 \wedge f_2 + f_3 \wedge f_4$ is not. To see that, assume $f = f_1 \wedge f_2 + f_3 \wedge f_4$ is decomposable. Then f can be written in the form $v \wedge w$ for some $v, w \in \mathbb{R}^n$. Then $f \wedge f = v \wedge w \wedge v \wedge w$ should be zero. However, we have

$$f \wedge f = (f_1 \wedge f_2 + f_3 \wedge f_4) \wedge (f_1 \wedge f_2 + f_3 \wedge f_4) = 2(f_1 \wedge f_2 \wedge f_3 \wedge f_4) \neq 0.$$

Remark. Note that $f' = f \wedge f_5 = f_1 \wedge f_2 \wedge f_5 + f_3 \wedge f_4 \wedge f_5$ is still non-decomposable, however $f' \wedge f' = 0$.

To gain intuition regarding the geometric meaning of k-forms, we provide the following lemma.

Lemma 2.1 (Geometric Interpretation of decomposable forms)**.** *Let $v_1, \ldots, v_k, w_1, \ldots, w_k \in \mathbb{R}^n$. If $span\{v_1, \ldots, v_k\} = span\{w_1, \ldots, w_k\}$, then*

$$v_1 \wedge \cdots \wedge v_k = \lambda\, w_1 \wedge \cdots \wedge w_k$$

for some $\lambda \in \mathbb{R}$.

Proof. We can express each v_i as a linear combination of w_j's. Expanding each component linearly, we find that $v_1 \wedge \cdots \wedge v_k$ is a linear combination of the forms $w_{i_1} \wedge \cdots \wedge w_{i_k}$ where $i_1, \ldots, i_k \in [k]$. The ones whose indices are not all different vanish by the property of k-forms. The rest $k!$ many forms are $\pm\, w_1 \wedge \cdots \wedge w_k$, the sign depending on the parity of the permutation $(i_1, \ldots, i_k)$. ☐

By Lemma 2.1, we observe that decomposable k-forms in $\bigwedge^k \mathbb{R}^n$ correspond to k-dimensional linear subspaces of $\mathbb{R}^n$. This implies that a decomposable form $v_1 \wedge \cdots \wedge v_k$ is the volume form on $span\{v_1, \ldots, v_k\}$ up to a multiplicative factor. In other words, it captures the k-dimensional volume of the subspace spanned by the vectors $v_1, \ldots, v_k$.

For a fixed basis $\{v_1, \ldots, v_n\} \in \mathbb{R}^n$ and a set $I = \{i_1, ..., i_k\} \in \binom{[n]}{k}$, we will use v_I to denote the k-form $v_{i_1} \wedge ... \wedge v_{i_k}$ for the sake of convenience. Fix the standard basis $\{e_1, \ldots, e_n\}$. Then the set $\{e_I \mid I \in \binom{[n]}{k}\}$ is a standard basis of $\bigwedge^k \mathbb{R}^n$. If $\{f_1, ..., f_n\}$ is another basis of $\mathbb{R}^n$, then it is easy to see that $\{f_I \mid I \in \binom{[n]}{k}\}$ is also a basis of $\bigwedge^k \mathbb{R}^n$. It follows that dim $\bigwedge^k \mathbb{R}^n = \binom{n}{k}$.

In order to further explore the properties and applications of k-forms, we will introduce additional fundamental notions of multilinear algebra.

2.1 Inner Product

We can define the inner product of two k-forms $v_1 \wedge \cdots \wedge v_k$ and $w_1 \wedge \cdots \wedge w_k$ by

$$\langle v_1 \wedge \cdots \wedge v_k, w_1 \wedge \cdots \wedge w_k \rangle = det(\langle v_i \, , w_j \rangle),$$

where $\langle v_i \, , w_j \rangle$ is the inner product of vectors v_i and w_j in $\mathbb{R}^n$. We can also extend the definition of the inner product to the entire space of k-forms by linearity.

The inner product on the space of k-forms is a fundamental tool in the study of exterior algebra, and it plays a crucial role in defining other important notions such as the interior product and the Hodge star operator.

2.2 Interior Product

Definition 2.2. *The interior product with a form a is defined to be the transpose of the wedge product with a, that is*

$$\langle v \llcorner a, w \rangle = \langle v, w \wedge a \rangle$$

for any $a \in \bigwedge^l \mathbb{R}^n$, $v \in \bigwedge^k \mathbb{R}^n$ and $w \in \bigwedge^{k-l} \mathbb{R}^n$.

Corollary 2.3. *For $v = v_1 \wedge .... \wedge v_k$ and $a = a_1 \wedge ... \wedge a_l$*

$$v \llcorner a = \sum_{\sigma} sgn(\sigma) a_1(v_{\sigma(1)}) \cdot \cdots \cdot a_l(v_{\sigma(l)}) \cdot v_{\sigma(l+1)} \wedge ... \wedge v_{\sigma(k)}.$$

where the sum over all permutations that preserve the order of $\{l+1, ..., k\}$.

Lemma 2.4. *Let $v \in \bigwedge^k \mathbb{R}^n$, $v' \in \bigwedge^l \mathbb{R}^n$ for some $k, l \in [n]$ and let $a, b \in \mathbb{R}^n$ be unit vectors. Then the following properties hold.*

i. For any orthonormal basis $e_1, ..., e_n$ *of* $\mathbb{R}^n$,

$$e_I \llcorner e_J = \begin{cases} \pm e_{I \setminus J} & \text{if } J \subset I \\ 0 & \text{if } J \not\subset I. \end{cases}$$

ii. $(v \wedge v') \llcorner a = (v \llcorner a) \wedge v' + (-1)^k v \wedge (v' \llcorner a)$

iii. $v \llcorner (a \wedge b) = (v \llcorner b) \llcorner a.$

iv. $((v \llcorner a) \wedge a) \llcorner a = (-1)^{k+1} v \llcorner a.$

Proof.

i. For any subset L, $\langle e_I \llcorner e_J, e_L \rangle = \langle e_I, e_L \wedge e_J \rangle$. If $J \subset I$, then $\langle e_I, e_L \wedge e_J \rangle$ is non-zero if and only if $L \cup J = I$, and hence $L = I \setminus J$. Thus, we have $e_I \llcorner e_J = \pm e_{I \setminus J}$. If $J \not\subset I$, then $\langle e_I, e_L \wedge e_J \rangle = 0$ for any L, meaning that $e_I \llcorner e_J = 0$.

ii. It suffices to prove this for decomposable forms since the operator $\llcorner$ is linear on $\bigwedge^k \mathbb{R}^n$. Let $v = v_1 \wedge \cdots \wedge v_k$ and $v' = v_{k+1} \wedge \cdots \wedge v_{k+l}$. By Corollary 2.3,

$$(v \wedge v') \llcorner a = \sum_{\sigma \in \mathcal{S}_n} a(v_{\sigma(1)}) \cdot v_{\sigma(2)} \wedge \cdots \wedge v_{\sigma(k+l)}.$$

Observe that we can split the permutations into two as $S_1 = \{\sigma \in \mathcal{S}_n \mid \sigma(1) \in \{1, \ldots k\}\}$ and $S_2 = \{\sigma \in \mathcal{S}_n \mid \sigma(1) \in \{k+1, \ldots, k+1\}\}$ where $\mathcal{S}_n$ denotes the set of all permutations of $\{1, \ldots, n\}$. Then we have,

$$\begin{aligned} (v \wedge v') \llcorner a &= \sum_{\sigma \in S_1} a(v_{\sigma(1)}) \cdot v_{\sigma(2)} \wedge \cdots \wedge v_{\sigma(k+l)} + \sum_{\sigma \in S_2} a(v_{\sigma(1)}) \cdot v_{\sigma(2)} \wedge \cdots \wedge v_{\sigma(k+l)} \\ &= \sum_{\sigma \in S_1} a(v_{\sigma(1)}) \cdot v_{\sigma(2)} \wedge \cdots \wedge v_{\sigma(k)} \wedge v' + \sum_{\sigma \in S_2} a(v_{\sigma(1)}) \cdot v_{\sigma(2)} \wedge \cdots \wedge v_{\sigma(l)} \wedge v \\ &= (v \llcorner a) \wedge v' + (-1)^k v \wedge (v' \llcorner a). \end{aligned}$$

iii. For any w, $\langle v \llcorner (a \wedge b) \, , w \rangle = \langle v \, , w \wedge (a \wedge b) \rangle = \langle v \llcorner b \, , w \wedge a \rangle = \langle (v \llcorner b) \llcorner a \, , w \rangle$.

iv. For any w such that $w \wedge a \neq 0$, we have:

$$\langle ((v \llcorner a) \wedge a) \llcorner a \, , w \rangle = \langle (v \llcorner a) \wedge a \, , w \wedge a \rangle = (-1)^{k+1} \langle v \llcorner a \, , w \rangle.$$

$\square$

Remark. Note that in general, $(v \wedge a) \llcorner a \neq (v \llcorner a) \wedge a$. This can be easily seen by the following counter-example,

$$((e_1 \wedge e_2) \wedge e_3) \llcorner e_3 = e_1 \wedge e_2 \neq 0 = ((e_1 \wedge e_2) \llcorner e_3) \wedge e_3.$$

Claim 2.5. *Let $v \in \bigwedge^k \mathbb{R}^n$ be a k-form and $a \in \mathbb{R}^n$ be a unit vector, then $v \llcorner a \in \bigwedge^{k-1}(a^\perp) \subseteq \bigwedge^{k-1} \mathbb{R}^n$.*

Proof. It suffices to prove the claim for decomposable forms since the operator $\llcorner$ is linear on $\bigwedge^k \mathbb{R}^n$. Let $v = v_1 \wedge ... \wedge v_k$. If $v_i \in a^\perp$ for all i, then $a(v_i) = 0$ for all i and so $v \llcorner a \in \bigwedge^{k-1}(a^\perp)$ by the definition of interior product. If not, since $\{v_1, ..., v_k\}$ is a linearly independent set in V, we can assume without loss of generality that $v_1 = a + \lambda_1 b$ where $b \in a^\perp$ and $v_i = \lambda_i b$ for $i \neq 1$ where λ_i, $1 \leq i \leq k$ are some real constants. Then,

$$\begin{aligned} v \llcorner a &= \sum_\sigma sgn(\sigma) a(v_{\sigma(1)}) \cdot v_{\sigma(2)} \wedge ... \wedge v_{\sigma(k)} \\ &= \pm v_2 \wedge ... \wedge v_k \in \bigwedge^{k-1}(a^\perp). \end{aligned}$$

□

Definition 2.6. *Let a be a unit vector in $\mathbb{R}^n$ and k be a non-negative integer. Define the operator $A^{\uparrow a} : \bigwedge^k \mathbb{R}^n \to \bigwedge^k \mathbb{R}^n$ by:*

$$A^{\uparrow a}(v) = (v \llcorner a) \wedge a;$$

and $A^{\downarrow a} : \bigwedge^k \mathbb{R}^n \to \bigwedge^k \mathbb{R}^n$ by:

$$A^{\downarrow a}(v) = (v \wedge a) \llcorner a.$$

Now, we will show that $(-1)^k A^{\downarrow a}(v)$ is the orthogonal projection of v onto the subspace $\bigwedge^k(a^\perp)$, which consists of k-forms generated by all vectors orthogonal to a.

Lemma 2.7 (Geometric meaning of $A^{\downarrow a}$)**.** *For a unit vector $a \in \mathbb{R}^n$, $(-1)^k A^{\downarrow a}$ is the orthogonal projection onto $\bigwedge^k(a^\perp)$.*

Proof.

$$\begin{aligned} (-1)^k A^{\downarrow a}((-1)^k A^{\downarrow a}(v)) &= (((v \wedge a) \llcorner a) \wedge a) \llcorner a \\ &= (-1)^k (v \wedge a) \llcorner a \\ &= (-1)^k A^{\downarrow a}(v) \end{aligned}$$

This proves that $(-1)^k A^{\downarrow a}$ is a projection. Notice that it is also self-adjoint,

$$\begin{aligned}\langle(-1)^k A^{\downarrow a}(v),\, w\rangle &= \langle(-1)^k (v \wedge a) \llcorner a,\, w\rangle \\ &= (-1)^k \langle v \wedge a,\, w \wedge a\rangle \\ &= \langle v,\, (-1)^k (w \wedge a) \llcorner a\rangle \\ &= \langle v,\, (-1)^k A^{\downarrow a}(w)\rangle.\end{aligned}$$

Hence, $(-1)^k A^{\downarrow a}$ is an orthogonal projection of v onto $\bigwedge^k(a^\perp)$. □

Lemma 2.8. $(\bigwedge^k(a^\perp))^\perp = \{a \wedge v \in \bigwedge^k \mathbb{R}^n \mid v \in \bigwedge^{k-1}(a^\perp)\}$.

Proof. Consider the definition of the orthogonal complement:

$$(\bigwedge^k(a^\perp))^\perp = \{v \in \bigwedge^k \mathbb{R}^n \mid \langle v, w\rangle = 0 \;\; \forall w \in \bigwedge^k(a^\perp)\}.$$

For any $a \wedge v \in \bigwedge^k \mathbb{R}^n$ and $w \in \bigwedge^k(a^\perp)$, it is clear that $\langle a \wedge v, w\rangle = 0$ for all $w \in \bigwedge^k(a^\perp)$. Note that the dimension of $\bigwedge^k(a^\perp)$ is $\binom{n-1}{k}$, while the dimension of $\{a \wedge v \in \bigwedge^k \mathbb{R}^n \mid v \in \bigwedge^{k-1}(a^\perp)\}$ is $\binom{n-1}{k-1}$. These dimensions sum up to $\binom{n}{k}$, which is the dimension of the entire space $\bigwedge^k \mathbb{R}^n$. Therefore, we conclude that $\{a \wedge v \in \bigwedge^k \mathbb{R}^n \mid v \in \bigwedge^{k-1}(a^\perp)\}$ is the orthogonal complement of $\bigwedge^k(a^\perp)$.

□

It follows that $(-1)^{k+1} A^{\uparrow a}(v)$ is the orthogonal projection of v onto the subspace $\{a \wedge w \in \bigwedge^k \mathbb{R}^n \mid w \in \bigwedge^{k-1}(a^\perp)\} \subseteq \bigwedge^k \mathbb{R}^n$. Then, we can show that any k-form can be written as the sum of its projections $A^{\uparrow a}(v)$ and $A^{\downarrow a}(v)$.

Corollary 2.9. *For any unit vector $a \in \mathbb{R}^n$ and any k-form $v \in \bigwedge^k \mathbb{R}^n$, we have the following partitioning:*

$$v = (-1)^{k+1}(A^{\uparrow a}(v) - A^{\downarrow a}(v)).$$

Proof. We will prove it for decomposable forms as before since it can be generalized for non-decomposable forms by the linearity of $\llcorner$. Then,

$$\begin{aligned}(v \llcorner a) \wedge a - (v \wedge a) \llcorner a &= (\sum_{\sigma} sgn(\sigma) a(v_{\sigma(1)}) \cdot v_{\sigma(2)} \wedge ... \wedge v_{\sigma(k)}) \wedge a \\ &\quad - (\sum_{\sigma'} sgn(\sigma') a(v_{\sigma'(1)}) \cdot v_{\sigma'(2)} \wedge ... \wedge v_{\sigma'(k)} \wedge a + (-1)^k a(a) \cdot v_1 \wedge ... \wedge v_k) \\ &= 0 - (-1)^k a(a) \cdot v_1 \wedge ... \wedge v_k \\ &= (-1)^{k+1} v.\end{aligned}$$

$\square$

2.3 Hodge Star Operator

The Hodge star operator is a linear operator defined on the space endowed with Euclidean structure. Recall from Lemma 2.1 that decomposable k-forms represent the k-dimensional subspaces of $\mathbb{R}^n$. The Hodge star operator maps k-dimensional volume forms (k-forms) to $(n-k)$-dimensional volume forms in the complementary space. To define the Hodge star operator, we start by defining a mapping ϕ_w for each $w \in \bigwedge^{n-k} \mathbb{R}^n$, which takes a k-form and maps it to a scalar. Using this mapping, we construct a linear map ϕ that maps $(n-k)$-forms to the dual space of k-forms. We then show that this map is an isomorphism, which allows us to define the Hodge star operator as an isometry map from k-forms to $(n-k)$-forms.

For each $w \in \bigwedge^{n-k} \mathbb{R}^n$, define the mapping $\phi_w : \bigwedge^k \mathbb{R}^n \to \mathbb{R}$ such that for all $v \in \bigwedge^k \mathbb{R}^n$,

$$v \wedge w = \phi_w(v)\, e_{[n]}$$

In other words, $\phi_w(v)$ is the coefficient of $e_{[n]}$ in the expansion of $v \wedge w$. Note that the map ϕ_w is well-defined because $v \wedge w$ is a scalar multiple of the basis element $e_{[n]}$, which implies that the coefficients of the wedge product of v and w with respect to $e_{[n]}$ are uniquely determined. The linearity of ϕ_w follows from the multilinearity of the wedge product. Then, the map

$$\phi : \bigwedge^{n-k} \mathbb{R}^n \to \left(\bigwedge^{k} \mathbb{R}^n\right)^* \\ w \mapsto \phi_w$$

is also linear. The map ϕ allows us to relate the space of k-forms to the dual space of $\bigwedge^k \mathbb{R}^n$, which will be useful in defining the Hodge star operator. Now we prove that ϕ is an isomorphism.

Claim 2.10. *ϕ is injective.*

Proof. Suppose $\phi_w = 0$. Then for all $v \in \bigwedge^k \mathbb{R}^n$, $\phi_w(v) = 0$. By the definition of ϕ_w, it implies that for all $v \in \bigwedge^k \mathbb{R}^n$, $v \wedge w = 0$. But this is true if and only if $w = 0$ because for any $w = \sum_{I \subset \binom{[n]}{k}} c_I\, e_{[n]\setminus I}$ where $c_I \in \mathbb{R}$, we can take $v = \sum_{I \subset \binom{[n]}{k}} c_I\, e_I$. Then $v \wedge w = \sum_I (c_I)^2$ which is zero if and only if $c_I = 0$ for all I i.e. $w = 0$. $\square$

Since $\dim \bigwedge^{n-k} \mathbb{R}^n = \dim \bigwedge^k \mathbb{R}^n = \dim(\bigwedge^k \mathbb{R}^n)^*$, we obtain that ϕ is an isomorphism.

We can define a linear map from k-forms to $(n-k)$-forms as follows:

Definition 2.11. *The **Hodge star operator** $* : \bigwedge^k \mathbb{R}^n \to \bigwedge^{n-k} \mathbb{R}^n$ is the unique linear operator such that*

$$v \wedge (*w) = \langle v, w \rangle \, e_{[n]}$$

for all $v, w \in \bigwedge^k \mathbb{R}^n$.

In particular, $*w$ is the unique element in $\bigwedge^{n-k} \mathbb{R}^n$ such that

$$\phi_{*w} = \langle \cdot \, , w \rangle.$$

Then for any $v, w \in \bigwedge^k \mathbb{R}^n$,

$$v \wedge *w = \phi_{*w}(v) \, e_{[n]} = \langle v \, , w \rangle \, e_{[n]}.$$

Lemma 2.12. *Let I be a k-element subset of $[n]$. Then, $*e_I = \pm e_{[n]\setminus I}$, where the sign depends on the order of indices in I.*

Proof. By the definition above,

$$e_I \wedge (*e_I) = \langle e_I \, , e_I \rangle e_{[n]}.$$

Since $\langle e_I \, , e_I \rangle = 1$, we have $e_I \wedge (*e_I) = e_{[n]}$. Let $*e_I = e_J$ for some $J \in \binom{[n]}{n-k}$. Then $e_I \wedge (*e_I) = e_I \wedge e_J \neq 0$ if and only if $J = [n] \setminus I$ which implies $*e_I = \pm \, e_{[n]\setminus I}$. □

Lemma 2.13. *The Hodge star operator satisfies the following relation:*

$$* \circ * : \bigwedge^k \mathbb{R}^n \to \bigwedge^k \mathbb{R}^n = (-1)^{k(n-k)} \mathbb{1}.$$

Proof. For $e_I \in \bigwedge^k \mathbb{R}^n$ and $e_J = e_{[n]\setminus I} \in \bigwedge^{n-k} \mathbb{R}^n$ let $\varepsilon_{I,J} \in \{\pm 1\}$ be a sign function such that $e_I \wedge e_J = \varepsilon_{I,J} \, e_1 \wedge \ldots \wedge e_n$. Notice that $\varepsilon_{J,I} \, \varepsilon_{I,J} = (-1)^{k(n-k)}$. Then by Lemma 2.12, $*e_J = \varepsilon_{I,J} e_{[n]\setminus I}$ which implies

$$* * e_J = \varepsilon_{I,J} * e_{[n]\setminus I} = \varepsilon_{J,I} \, \varepsilon_{I,J} e_J = (-1)^{k(n-k)} e_J.$$

□

Lemma 2.14. *Let $w \in \bigwedge^k \mathbb{R}^n$. Then the following identities hold for any form v and vector $a \in \mathbb{R}^n$.*

i. $*w = e_{[n]} \llcorner w$.

ii. $v \wedge *(w \wedge a) = (v \llcorner a) \wedge *w$.

iii. $v \wedge *(w \llcorner a) = v \wedge a \wedge *w$.

Proof.

i. For any $v \in \bigwedge^{n-k} \mathbb{R}^n$, we have

$$\langle v\,, e_{[n]} \llcorner w \rangle = \langle v \wedge w\,, e_{[n]} \rangle = \langle\, \langle v\,, *w \rangle\, e_{[n]}\,, e_{[n]} \rangle = \langle v\,, *w \rangle \langle e_{[n]}\,, e_{[n]} \rangle = \langle v\,, *w \rangle$$

implying $*w = e_{[n]} \llcorner w$.

ii. $v \wedge *(w \wedge a) = \langle v\,, w \wedge a \rangle\, e_{[n]} = \langle v \llcorner a\,, w \rangle\, e_{[n]} = (v \llcorner a) \wedge *w$.

iii. $v \wedge *(w \llcorner a) = \langle v\,, w \llcorner a \rangle\, e_{[n]} = \langle v \wedge a\,, w \rangle\, e_{[n]} = v \wedge a \wedge *w$.

□

3 Combinatorics and Exterior Algebra

In this section, we aim to establish a connection between combinatorics and exterior algebra. We will translate the combinatorial objects and operations into the geometric notions in exterior algebra, which we have discussed in the previous section.

3.1 Sets to Subspaces

Fix a basis $\{v_1, \ldots, v_n\} \in \mathbb{R}^n$. With every k-set $I \subseteq [n]$, we associate the wedge product

$$v_I = \bigwedge_{i \in I} v_i \in \bigwedge^k \mathbb{R}^n. \tag{1}$$

In other words, we have an embedding from set systems into the space of forms. This relation allows us to build the correspondence between combinatorial structures and geometrical structures in exterior algebra. To establish the link between wedge products and intersection conditions, we introduce the following lemma:

Lemma 3.1. *Let $A, B \subseteq [n]$. Then,*

$$v_A \wedge v_B \begin{cases} \neq 0 & A \cap B = \emptyset \\ = 0 & A \cap B \neq \emptyset \end{cases}$$

Proof. In the first case, we have the wedge product of linearly independent vectors. In the second, the vectors corresponding to the elements in the intersection $A \cap B$ repeat, hence the terms are linearly dependent. Consequently, the wedge product is zero. □

Drawing upon the connection between intersecting set systems and exterior algebra, Lovász proved the skew version of Bollobás' two families theorem (Theorem 1.2), using exterior algebra methods. This method justifies our choice of embedding in (1). Lovász's proof of Theorem 1.2 is as follows.

Proof. (Lovász, 1977) Let X be a finite set containing all the sets A_i and B_j. Associate each $i \in X$ with vectors $v_i \in \mathbb{R}^{k+l}$ such that the vectors are in general position. By the conditions of the theorem and Lemma 3.1, we have

$$v_{A_i} \wedge v_{B_j} \begin{cases} \neq 0 & i = j \\ = 0 & i < j. \end{cases}$$

Now we show that $v_{A_1}, \ldots, v_{A_m}$ are linearly independent. Let $\sum_{i=1}^{m} \lambda_i v_{A_i} = 0$ and assume that for some j, $\lambda_i = 0$ for all $i > j$. Consider the exterior product $0 \wedge v_{B_j}$ and rewrite it as

$$0 \wedge v_{B_j} = (\sum_{i=1}^{m} \lambda_i v_{A_i}) \wedge v_{B_j} = \sum_{i=1}^{m} \lambda_i (v_{A_i} \wedge v_{B_j}) = \lambda_j v_{A_j} \wedge v_{B_j}.$$

Since $v_{A_j} \wedge v_{B_j} \neq 0$, $\lambda_j = 0$. This implies that $v_{A_1}, \ldots, v_{A_m}$ are linearly independent. Consequently, $m \leq \dim \bigwedge^k \mathbb{R}^{k+l} = \binom{k+l}{k}$. □

It is worth noting that this proof admits a generalization to subspaces of a linear subspace.

Theorem 3.2 (Subspace Two Families [1])**.** *Let $V_1, \ldots, V_m$ and $W_1 \ldots, W_m$ be two families of linear subspaces such that*

1. $\dim V_i \leq k$*, and* $\dim W_i \leq \ell$ *for* $i = 1, \ldots, m$*;*

2. $V_i \cap W_i = \emptyset$ *for* $i = 1, \dots, m$*;*

3. $V_i \cap W_j \neq \emptyset$ *for* $1 \leq i < j \leq m$*.*

Then $m \leq \binom{k+\ell}{k}$*.*

3.2 Intersection and Annihilation

Using the idea in Lemma 3.1, we can identify a property of subspaces in exterior algebra that mirrors the concept of intersection in set systems. In exterior algebra, two k-forms $v, w \in \bigwedge^k \mathbb{R}^n$ are *annihilating* if their wedge product vanishes, i.e., $v \wedge w = 0$. A subspace $W \subseteq \bigwedge^k \mathbb{R}^n$ is *self-annihilating* if every pair of k-forms in W is annihilating. This notion of self-annihilating subspaces was introduced in previous works such as [4, 5] as an exterior analog of intersection in set systems.

Recall that Theorem 1.5 states that for $k \leq n/2$, if W is a subspace of $\bigwedge^k \mathbb{R}^n$ such that $v \wedge w = 0$ for all $v, w \in W$, then the dimension of W is bounded by $\binom{n-1}{k-1}$. Using the embedding in (1), we can associate each subset $A \in \mathcal{A}$ of an intersecting family of k-element subsets of $[n]$ with the k-form $v_A \in \bigwedge^k \mathbb{R}^n$. Then, we observe that the subspace spanned by $\{v_A \mid A \in \mathcal{A}\}$ is a self-annihilating subspace of $\bigwedge^k \mathbb{R}^n$. Therefore, Theorem 1.5 implies the inequality in the original Erdös–Ko–Rado theorem, that is $|\mathcal{A}| \leq \binom{n-1}{k-1}$.

This analogy becomes more tricky when considering non-decomposable forms, as there is no clear correspondence between non-decomposable k-forms and k-sets of $[n]$. One issue is that the property $v \wedge v = 0$ does not hold for some non-decomposable forms, as exemplified by $e_1 \wedge e_2 + e_3 \wedge e_4$. Intuitively one would expect a form to "intersect" with itself analogically but in this case, self-annihilation may not always have a clear geometric interpretation.

Therefore, a more nuanced approach is needed when studying this relationship. To address this issue, we introduce another annihilation property, called *strong annihilation.*

Definition 3.3. *Two* k*-forms* $v, w \in \bigwedge^k V$ *are strongly annihilating if there exists a vector* $a \in \mathbb{R}^n$ *such that* $v \wedge a = w \wedge a = 0$*. A subspace* $W \subseteq \bigwedge^k V$ *is strongly self-annihilating if every pair of* k*-forms in* W *is strongly annihilating.*

It is worth noting that strong annihilation implies annihilation, but not the other way around. However, the self-annihilating subspaces spanned by decomposable forms with respect to a certain basis, like $\{v_A \mid A \in \mathcal{A}\}$, are also strongly annihilating as each pair has to have a common factor. Therefore, strong annihilation is compatible with the embedding defined in (1).

3.3 Subtraction and Interior Product

The interior product is a fundamental operation in exterior algebra that allows us to extract a lower-dimensional form from a higher-dimensional one. Interestingly, when we restrict our attention to decomposable k-forms, the interior product has a close correspondence with set subtraction which was first discovered by Kalai in [14].

In the context of the interior product, the projection operation acts as an analogy for set subtraction. Recall from the properties of the interior product in Lemma 2.4 that for an orthonormal basis $\{e_1, ..., e_n\}$ of V, we have $e_I \llcorner e_i = e_{I\setminus\{i\}}$ where $i \in I \subseteq [n]$. Thus, the interior product of e_I with e_i corresponds to removing the i-th element from the set I if $i \in I$.

Geometrically, the interior product with a vector a represents the contraction of a k-form v with a along the direction of v. In the case of a decomposable form, this corresponds to removing the component in the direction of a. The result is a $(k-1)$-form that lives in the subspace spanned by the remaining basis vectors. This makes intuitive sense as the subspace spanned by these remaining basis vectors corresponds to the hyperplane orthogonal to a as we have seen in Claim 2.5.

Building on this intuition we observe that the images of the forms under the operators $A^{\downarrow a}$ and $A^{\uparrow a}$ correspond to the partitioning of a family of subsets. Take a vector $e_i \in \mathbb{R}^n$ from the standard basis $\{e_1, \ldots, e_n\}$ and consider the subspace $V = \{e_A \mid A \in \mathcal{A}\} \subseteq \bigwedge^k \mathbb{R}^n$, where $\mathcal{A}$ is a family of sets. When we apply the operator $A^{\downarrow e_i}$ to V, it gives us the subspace spanned by $\{e_A \mid i \notin A\}$. On the other hand, the operator $A^{\uparrow e_i}$ applied to V yields the subspace spanned by $\{e_A \mid i \in A\}$. Therefore, these operators correspond to partitioning the family $\mathcal{A}$ into two parts: $\{A \in \mathcal{A} \mid i \notin A\}$ and $\{A \in \mathcal{A} \mid i \in A\}$.

It's worth noting that these correspondences are not so trivial for non-decomposable forms again. Nevertheless, the interior product has a rich geometric interpretation even for non-decomposable forms, allowing us to extract lower-dimensional parts of these forms that capture important geometric information. Therefore we believe that it can be still useful to extend the natural correspondence to non-decomposable forms.

Following this relation, we can also define the "shadow" of a subspace. We first introduce the definition of a shadow for a subset in a set system.

Definition 3.4. *For a family of sets $\mathcal{A}$, its s-shadow $\partial_s\mathcal{A}$ denotes the family of s-subsets of its members $\partial_s\mathcal{A} := \{S \mid |S| = s, \exists A \in \mathcal{A}, S \subseteq A\}$.*

In exterior algebra, we can define the shadow of a subspace. More precisely, given a subspace of k-forms W, the s-shadow of W where $s < k$, denoted as $\partial_s W$ is the subspace of s-forms that comprises all possible contractions of elements from W. More formally, we can express $\partial_s W$ as follows:

$$\partial_s W := span\{w \llcorner e_I \mid I \in \binom{[n]}{k-s},\ w \in W\}.$$

It easily follows that this definition is again consistent with our embedding (1).

3.4 Complement and Hodge Star Operator

We observe that the Hodge star operator is intimately related to the set complement operation since the Lemma 2.12 shows that for any k-element subset $I \subseteq [n]$, the Hodge star of the k-form e_I is given by $*e_I = \pm\, e_{[n]\setminus I}$.

Intuitively, the Hodge star operator maps a k-form to its complementary $(n-k)$-form, much like set complement maps a subset of $[n]$ to its complementary subset. Therefore we believe that extending this relation to all k-forms can be still meaningful.

4 Erdös–Ko–Rado Theorem in Exterior Algebra

In this section, we will present the proof of the original Erdös–Ko–Rado theorem and state our conjecture on the exterior algebra extension of Erdös–Ko–Rado theorem. Using the correspondences we built in the previous section, we will give a potential proof idea that follows the same methods as the proof of the original Erdös–Ko–Rado theorem. We will begin by providing proof of the original EKR (Theorem 1.3), as presented in [16]. The proof utilizes Katona's shadow theorem for intersecting families.

Theorem 4.1 (Katona, [15])**.** *Let $\mathcal{A}$ be a family of m-sets such that $|A \cap A'| \geq l > 0$ for all $A, A' \in \mathcal{A}$. Then $|\mathcal{A}| \leq |\partial_{m-l}\mathcal{A}|$. Equality holds if and only if $m = l$ or $\mathcal{A} = \emptyset$ or $\mathcal{A} \equiv \binom{[2m-l]}{m}$.*

We will skip the proof of Theorem 4.1 and proceed to prove Theorem 1.3.

Proof of Theorem 1.3. [16] Define the partitions of $\mathcal{A}$ such that $\mathcal{A}_0 := \{A \in \mathcal{A} \mid 1 \notin A\}$ and $\mathcal{A}_1 := \{A \in \mathcal{A} \mid 1 \in A\}$.

Now let $\mathcal{B}_1 := \{A \setminus \{1\} \mid A \in \mathcal{A}_1\}$ and $\mathcal{B}_0 := \{[2,n] \setminus A \mid A \in \mathcal{A}_0\}$. Let $B \in \mathcal{B}_1 \cap \partial_{k-1}\mathcal{B}_0$. Then $B = A_1 \setminus \{1\} = [2,n] \setminus A_0$ for some $A_0 \in \mathcal{A}_0$ and $A_1 \in \mathcal{A}_1$. This implies $A_0 \cap A_1 = 0$ but this contradicts with the intersection property of $\mathcal{A}$. Hence we obtain

$$\mathcal{B}_1 \cap \partial_{k-1}\mathcal{B}_0 = \emptyset$$

$\mathcal{B}_0$ and $\partial_{k-1}\mathcal{B}_1$ are $(k-1)$-uniform, non-intersecting subfamilies of $[2, n]$ so we have

$$|\mathcal{B}_1| + |\partial_{k-1}\mathcal{B}_0| \le \binom{n-1}{k-1}$$

For any B, $B' \in \mathcal{B}_0$,

$$|B \cap B'| = |([2, n] \setminus A_0) \cap ([2, n] \setminus A'_0)| = (n-1) - 2k + |A_0 \cap A'_0| \ge n - 2k.$$

Then taking $m = n - k - 1$ and $l = n - 2k \ge 0$ in Theorem 4.1, we get

$$|\mathcal{B}_0| \le |\partial_{k-1}\mathcal{B}_0|.$$

To conclude,

$$|\mathcal{A}| = |\mathcal{A}_0| + |\mathcal{A}_1| = |\mathcal{B}_0| + |\mathcal{B}_1| \le |\partial_{k-1}\mathcal{B}_0| + |\mathcal{B}_1| \le \binom{n-1}{k-1}.$$

Equality implies $|\mathcal{B}_0| = |\partial_{k-1}\mathcal{B}_0|$. By Theorem 4.1, for $n > 2k$ this is true if and only if either $\mathcal{B}_0 = \emptyset$ and so each set in $\mathcal{A}$ contains 1, or $\mathcal{B}_0 \equiv \binom{[2,n-1]}{n-1-k}$ and so each set in $\mathcal{A}$ contains n. □

Theorem 1.5, which is the extension of Erdös–Ko–Rado theorem, was proven by Scott and Willmer in [4] and also independently by Woodroofe in [5]. While both papers provide different approaches to the proof, they both fail on explaining the behavior in the equality case. To approach this problem, we propose to change the self-annihilation property in the conjecture to the strong annihilation we defined in the previous chapter.

Conjecture 4.2. *Suppose $k \le n/2$ and let W a strongly annihilating subspace of $\bigwedge^k \mathbb{R}^n$. Then $\dim W = \binom{n-1}{k-1}$ if and only if there is an $a \in \mathbb{R}^n$ such that for all $w \in W$, $a \wedge w = 0$.*

4.1 A Reduction for the Exterior EKR Conjecture

Based on the dictionary we have constructed, we propose an approach that follows the same methods as Frankl and Füredi's proof of the original Erdös-Ko-Rado (EKR) theorem.

Let $V \subset \bigwedge^k \mathbb{R}^n$ be a strongly intersecting subspace i.e. for any $u, v \in V$, there exists a vector $a \in \mathbb{R}^n$ such that $u \wedge a = v \wedge a = 0$. Let $a \in \mathbb{R}^n$ be a fixed unit vector. We define the following partitions:

1. $V_0 := \mathrm{span}\{(v \wedge a) \llcorner a \mid v \in V\} \subseteq \bigwedge^k (a^\perp)$:

   The subspace V_0 captures the part of the elements of V that do not have a as a linear factor, i.e., V_0 is the orthogonal projection of $v \in V$ onto $\bigwedge^k (a^\perp)$. Hence, it is the image of V under the map $A^{\downarrow a}$. Notice that this corresponds to the set $\mathcal{A}_0$ in the proof of the original EKR.

2. $V_1 := \mathrm{span}\{(v \llcorner a) \wedge a \mid v \in V\}$:

   V_1 is the image of V under the map $A^{\uparrow a}$, and it corresponds to the set $\mathcal{A}_1$ in the original proof.

3. $W_0 := *V_0 \subseteq \bigwedge^{n-k-1} (a^\perp)$:

   W_0 is obtained by applying the Hodge star operator to V_0 with respect to the space $\bigwedge^k (a^\perp)$. Therefore, it consists of $(n-k-1)$-forms. It corresponds to the "complement" of V_0 in the subspace $\bigwedge^k (a^\perp)$, and hence it corresponds to the set $\mathcal{B}_0$ in the original proof.

4. $W_1 := \mathrm{span}\{v \llcorner a \mid v \in V\} \subseteq \bigwedge^{k-1} (a^\perp)$:

   W_1 captures the parts of the elements in V that have a as a factor without including a itself. Naturally, it corresponds to the set $\mathcal{B}_1$.

5. $\partial_{k-1} W_0 := \mathrm{span}\{(*v_0) \llcorner e_I \mid I \in \binom{[n]}{n-2k},\ v_0 \in V_0\}$:

   $\partial_{k-1} W_0$ is the space spanned by all $(k-1)$-forms obtained by taking the interior product with all $(n-2k)$-forms. The resulting space corresponds to the set $\partial_{k-1} \mathcal{B}_0$ in the original proof.

Notice that we have $V = V_0 \oplus V_1$ by Lemma 2.9. We would like to show that there is a unit vector $a \in \mathbb{R}^n$ such that $V = V_1$, which would imply the Conjecture 4.2. We begin with the following lemma that can be easily proven by the initial ideal method of Scott and Wilmer in [4].

Lemma 4.3. $\dim W_0 \leq \dim (\partial_{k-1} W_0)$ *and equality holds if and only if* $W_0 = \{0\}$ *or* $W_0 = \bigwedge^{n-k-1}(a^{\perp})$.

We propose the following conjecture that is a reduction of the Conjecture 4.2.

Conjecture 4.4. $W_1 \bigcap \partial_{k-1} W_0 = \emptyset$ *for some unit vector* $a \in \mathbb{R}^n$.

Lemma 4.5. *Conjecture 4.4 implies Conjecture 4.2.*

Proof. Define the partitions V_0, V_1, W_0, W_1 as above. Since $\dim W_1, \dim \partial_{k-1} W_0 \leq \binom{n-1}{k-1}$ and Conjecture 4.4 implies that $\dim (\partial_{k-1} W_0) = \dim W_0$, we have

$$\dim W_1 + \dim \partial_{k-1} W_0 \leq \binom{n-1}{k-1}.$$

Then,

$$\dim V = \dim V_0 + \dim V_1 = \dim W_0 + \dim W_1 \leq \dim (\partial_{k-1} W_0) + \dim W_1 \leq \binom{n-1}{k-1}.$$

Equality holds if and only if $W_0 = \{0\}$ or $W_0 = \bigwedge^{n-k-1}(a^{\perp})$ which implies that $v \wedge a = 0$ for all $v \in V$. $\square$

5 Main Result

In this section, we present our main result on Exterior EKR and Hilton–Milner for two-forms. Specifically, we provide the proof of Theorem 1.7. We show that for the case of two-forms, both the characterization of the extremal case in the Erdös–Ko–Rado theorem and the extension of the Hilton–Milner theorem follow.

The proof of the Theorem 1.7 uses a result from symplectic geometry that gives a precise way to express two forms.

Theorem 5.1 ([11])**.** *Any two-form can be written in the standard form* $f_1 \wedge f_2 + \cdots + f_{2k+1} \wedge f_{2k+2}$ *in some orthogonal basis* $\{f_1, \ldots, f_n\}$.

We are now able to present the proof of our main result.

Proof of Theorem 1.7. By Lemma 5.1, a two-form w can be written in the form $w = \sum_{1 \leq i \leq m} f_i \wedge f_{i+1}$ where i is odd and $m \leq n$, in some orthogonal basis $\{f_1, \ldots, f_n\}$. If w satisfies $w \wedge w = 0$, then we have

$$w \wedge w = 2 \sum_{i \neq j} f_i \wedge f_{i+1} \wedge f_j \wedge f_{j+1}.$$

Since $f_i, f_{i+1}, f_j, f_{j+1}$ are basis elements, the wedge product $f_i \wedge f_{i+1} \wedge f_j \wedge f_{j+1} \neq 0$. Moreover, since $\{f_1, \dots, f_n\}$ is an orthogonal basis, the set of forms $\{f_i \wedge f_{i+1} \wedge f_j \wedge f_{j+1}, |\ \ i \neq j\}$ are linearly independent. Therefore, $w \wedge w = 0$ if and only if w is decomposable, that is, $w = v_1 \wedge v_2$ for some $v_1, v_2 \in \mathbb{R}^n$. Consequently, all elements of W are decomposable. That is, they correspond to two-dimensional subspaces of $\mathbb{R}^n$ or, equivalently, to lines in $\mathbb{R}P^{n-1}$. Then, the proof follows from the classical result on the structure of sets of lines in projective space:

Folklore lemma. Let L be a set of lines in $\mathbb{R}P^{n-1}$ such that any two of them intersect. Then either all lines pass through one point, or all lines lie in a two-dimensional subspace.

Proof of Folklore Lemma. Let ℓ_1 and ℓ_2 be two arbitrary lines in L and L_2 be the two-dimensional subspace containing both ℓ_1 and ℓ_2. Any line that doesn't intersect ℓ_1 and ℓ_2 at their intersection point must have at least two points on L_2. This implies that the line lies entirely within L_2. □

By Folklore lemma, there are two cases:

1. All the lines pass through one point. Then there are at most $n - 1$ linearly independent of them, which easily yields the first part of the theorem, (i).
2. All the lines belong to some two-dimensional subspace. Then there are at most 3 of them that can be linearly independent, and we have (ii).

□

Note that in the case of $k > 2$, such a relation does not exist because most self-annihilating subspaces of k-forms for $k > 3$ are spanned by non-decomposable forms.

6 Other extremal problems

In this section, we propose conjectures analog to known extremal combinatorics problems.

6.1 deBruijn–Erdös Theorem

Theorem 6.1. *If $\mathcal{A}$ is an intersecting family of subsets of $[n]$ such that for each $A, B \in \mathcal{A}, |A \cap B| = 1$, then $|\mathcal{A}| \leq n$.*

Proof of Theorem 6.1. Let $|\mathcal{A}| = m$. For the m sets in the family $A_1, ..., A_m$ let $(a_{ij})_{m\times n}$ be a matrix such that $a_{ij} = 1$ if $i \in A_j$ and $a_{ij} = 0$ otherwise for all $1 \leq i \leq n$ and $1 \leq j \leq m$. Note that if there exists a set A_j such that $|A_j| = 1$, it implies that all sets have one common element. Then maximum possible family would look like $\{\{1\}, \{1, 2\}, \{1, 3\}, ..., \{1, n\}\}$ which has cardinality exactly n.

Claim 6.2. *The columns of the matrix $(a_{ij})_{m\times n}$ are linearly independent.*

Proof of Claim 6.2. Let $c_1, ..., c_m$ be the columns of $(a_{ij})_{m\times n}$. We can assume that each set has a cardinality of at least 2. Let $s = \sum_i \alpha_i c_i$. Then,

$$\langle s, s\rangle = \sum_j \sum_i \alpha_i \alpha_j \langle c_i, c_j\rangle.$$

Note that $\langle c_i, c_j\rangle = 1$ for $i \neq j$ and $\langle c_i, c_i\rangle = |A_i|$. Therefore,

$$\langle s, s\rangle = (\sum_{i=1}^{m} \alpha_i)^2 + \sum_{i=1}^{m} \alpha_i^2(|A_i| - 1)$$

This equation is equal to zero if and only if $\alpha_i = 0$ for all $1 \leq i \leq m$ which implies that $c_1, ..., c_m$ are linearly independent. This proves the claim. □

As a corollary of the Claim 6.2, we get $m \leq n$ and this proves Theorem 6.1. □

In the exterior algebra setting, an analogy of Theorem 6.1 can be stated as follows.

Conjecture 6.3. *Let V be a strongly annihilating graded vector subspace of $\bigwedge \mathbb{R}^n$ such that for each pair $v, w \in V$, there exists a unit vector $a \in \mathbb{R}^n$ such that $(v \llcorner a) \wedge (w \llcorner a) \neq 0$. Then $\dim V \leq n$.*

6.2 Cross-union and Frankl's Conjecture

Definition 6.4 (Cross Union). *A collection of families $\mathcal{F}_0, ..., \mathcal{F}_s$ of subsets in $\binom{[n]}{k}$ are called $(s+1)$-cross union if there is no choice of $F_0, ..., F_s$ where $F_i \in \mathcal{F}_i$ for all $0 \leq i \leq s$ such that $F_0 \cup ... \cup F_s = [n]$.*

Frankl proposed the following conjecture in [12].

Conjecture 6.5 (Frankl, [12]). *Let $n = sk + \ell$ with $1 \leq \ell \leq k$. Suppose that $\mathcal{F}_0, ..., \mathcal{F}_s \subset \binom{[n]}{k}$ are non-empty and $(s+1)$-cross-union. Then there exists $s_0 = s_0(\ell)$ such that the following holds for all $s \geq s_0$:*

$$\frac{|\mathcal{F}_0| + |\mathcal{F}_1| + ... + |\mathcal{F}_s|}{s+1} \leq \binom{n-1}{k-1}.$$

We can define *cross-union* for the subspaces of $\bigwedge^k \mathbb{R}^n$ similarly.

Definition 6.6 (Exterior Cross Union). *The subspaces $V_0, ..., V_s$ of $\bigwedge^k \mathbb{R}^n$ are called cross-union if there is no choice of $v_0 \in V_0, ..., v_s \in V_s$ such that for some basis $\{f_1, ..., f_n\}$ of $\mathbb{R}^n$, for all f_i there exists a v_j such that $v_j \wedge f_j = 0$.*

A natural analogy of Frankl's conjecture on exterior algebra can be stated as follows.

Conjecture 6.7. *Let $n = sk + \ell$ with $1 \leq \ell \leq k$. Supposed that $V_0, ..., V_s$ are cross-union proper subspaces of $\bigwedge^k \mathbb{R}^n$. Then there exists $s_0 = s_0(\ell)$ such that the following holds for all $s \geq s_0$:*

$$\frac{dim V_0 + dim V_1 + ... + dim V_s}{s+1} \leq \binom{n-1}{k-1}.$$

References

[1] Lászlo Lovász, *Flats in matroids and geometric graphs*, Combinatorial Surveys (Proc. Sixth British Combinatorial Conf., Royal Holloway Coll., Egham, 1977), Academic Press, London (1977), 45–86.

[2] Béla Bollobás, *On Generalized Graphs*, Acta. Math. Acad. Sci. Hungar. **16** (1965), 447–452.

[3] Paul Erdös, Chao Ko and Richard Rado, *Intersection theorems for systems of finite sets*, Quart. J. Math. Oxford Ser. (2) **12** (1961), 313–320.

[4] Alex Scott and Elizabeth Wilmer, *Combinatorics in the exterior algebra and the Bollobás two families theorem*, Journal of the London Mathematical Society, **104(4)** (2021), 1812–1839.

[5] Russ Woodroofe, *An algebraic groups perspective on Erdös–Ko–Rado*, arXiv: 2007.03707.

[6] Denys Bulavka, Afshin Goodarzi and Martin Tancer, *Optimal Bounds for the Colorful Fractional Helly Theorem*, 37th International Symposium on Computational Geometry (2021), 19:1–19:14.

[7] Ibrahim Dibag, *Factorization in exterior algebras*, J. Algebra **30** (1974), 259-262.

[8] Peter Frankl and Richard M. Wilson, *The Erdös–Ko–Rado theorem for vector spaces*, J. Comb. Theory, Ser. A, **43** (1986), 228–236.

[9] A. J. W. Hilton and E. C. Milner, *Some intersection theorems for systems of finite sets,* Quart. J. Math. Oxford Ser. (2) **18** (1967), 369–384.

[10] Grigory Ivanov, Seyda Köse, *Erdös—Ko—Rado and Hilton—Milner theorems for two-forms,* Discrete Mathematics, **346(6)** (2023).

[11] Ana Cannas Da Silva, *Lectures on symplectic geometry*, Springer, 2008.

[12] Peter Frankl, *On the arithmetic mean of the size of cross-union families* Acta Math. Hungar, **164(1)** (2021), 312–325.

[13] Gil Kalai, *Algebraic Shifting*, Computational commutative algebra and combinatorics (Osaka, 1999), Adv. Stud. Pure Math., vol. 33, Math. Soc. Japan, Tokyo, 2002, 121–163.

[14] Gil Kalai, *Intersection patterns of convex sets*, Israel J. Math., **48** (1984), 161–174.

[15] G. Katona, *Intersection theorems for systems of finite sets*, Acta Math. Acad. Sci. Hungar. **15** (1964), 329–337.

[16] Peter Frankl, Zoltán Füredi, *A new short proof of the EKR theorem*, J. Comb. Theory, Ser. A, **119** (2012), 1388-1390.

[17] Werner Greub, *Multilinear Algebra*, Springer, 1978.