The Design Space of Kirchhoff Rods
Cheat Sheet

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Some of the symbols refer to quantities that vary across the length of a beam with arc-length parameter $s \in (0, \ell)$. In the paper, we will often omit the parameter s for brevity, whenever we make an argument that is true at every parameter location. Sometimes, we will also write, e.g., $I \in S^2_{++}$ instead of $I : (0, \ell) \to S^2_{++}$, when it is clear from context that a choice $I(s) \in S^2_{++}$ is made for every $s \in (0, \ell)$.

Sym.	Type	Description
$(\cdot)'$	$(\cdot)' : C^d \to C^{d-1}$	First derivative with respect to arc-length parameter s
$[\cdot]_\times$	$[\cdot]_\times : \mathbb{R}^3 \to \mathbb{R}^{3\times 3}$	Transforms a vector $v \in \mathbb{R}^3$ into its "cross product matrix", the skew-symmetric matrix $[v]_\times$ such that $[v]_\times x = v \times x$ for all $x \in \mathbb{R}^3$
a	$a \in \mathbb{R}_{>0}$	Radius of an ellipse, associated with the first semi-axis $(\cos\varphi, \sin\varphi)^t$
b	$b \in \mathbb{R}_{>0}$	Radius of an ellipse, associated with the second semi-axis $(-\sin\varphi, \cos\varphi)^t$
β	$\beta : (0, \ell) \to \mathbb{R}$	Rotation of the normal plane, relating two frames F and F_β adapted to the same curve γ via $F_{\beta,n} = F_n Q_\beta$, with $Q_\beta = \begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix}$
$(c, \bar{c})$	$c, \bar{c} \in \mathbb{R}^3$	Homogeneous coordinates of the linear complex C
C	$C \subset \Lambda_{\mathrm{kl}}$	Set of all lines in $\mathbb{R}^3$ whose Plücker coordinates $(l, \bar{l})$ satisfy $\langle l, \bar{c}\rangle + \langle \bar{l}, c\rangle = 0$
$\mathcal{D}$	$\mathcal{D}(s) \subset \mathbb{R}^2$	Cross section of the Kirchhoff rod at a particular $s \in (0, \ell)$; often assumed to be elliptical
E	$E \in \mathbb{R}_{>0}$	Young's modulus of the base material
e_i	$e_i \in \mathbb{R}^3$	Standard basis vectors $e_1 = (1, 0, 0)^t$, $e_2 = (0, 1, 0)^t$, and $e_3 = (0, 0, 1)^t$
F	$F : (0, \ell) \to SO(3)$	Moving frame adapted to γ; encodes the twist of the Kirchhoff rod deformation; the columns of F are given by $F(s) = (n_1(s), n_2(s), \gamma'(s))$
F_n	$F_n : (0, \ell) \to \mathbb{R}^{3\times 2}$	The matrix of material normals of F, so $F_n = FS = (n_1, n_2)$
f_i	$f_i \in \mathbb{R}^3, i = 1, \ldots, n$	Concentrated point load f_i is applied to the centerline of a rod at $\gamma(s_i)$
γ	$\gamma : (0, \ell) \to \mathbb{R}^3$	Arc-length parametrized curve that gives the centerline of a deformed Kirchhoff rod; assumed at least twice continuously differentiable

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I	$I : (0, \ell) \to S^2_{++}$	Area moment of inertia tensor of the cross section of the Kirchhoff rod, at a particular $s \in (0, \ell)$, given by $I(s) = \int_{\mathcal{D}(s)} \begin{pmatrix} y^2 & -xy \\ -xy & x^2 \end{pmatrix} \mathrm{d}(x, y)$
J	$J : (0, \ell) \to \mathbb{R}_{>0}$	Torsional rigidity of the cross section of the Kirchhoff rod, at a particular $s \in (0, \ell)$; computed as $J(s) = 4 \int_{D(s)} \|\nabla \chi\|^2$, where χ is the solution to $\Delta \chi = -1$ in $\mathcal{D}(s)$, and $\chi = 0$ on $\partial \mathcal{D}(s)$
K	$K : (0, \ell) \to \mathbb{R}^{3\times3}$	Stiffness matrix of the Kirchhoff rod, at a particular $s \in (0, \ell)$; the upper-left two-by-two block is given by EI, and the lower-right entry by μJ; we often use K and the pair (I, J) interchangeably, because E and μ are assumed fixed
k	$k : (0, \ell) \to \mathbb{R}^3$	Curvature vector of the framed curve (γ, F), with components $k = (\kappa_1, \kappa_2, \tau)$; related to F and ω via $\omega = Fk$ and $[k]_\times = F^t F'$
k_n	$k_n : (0, \ell) \to \mathbb{R}^2$	Vector of material curvatures of F, so $k_n = S^t k = (\kappa_1, \kappa_2)^t$
$\mathcal{K}$	$\mathcal{K} \subset S^2_{++} \times \mathbb{R}$	Set of admissible stiffnesses (I, J) that satisfy $0 < J \leq 4\psi(I)$; by abuse of notation, we write $K \in \mathcal{K}$ and $(I, J) \in \mathcal{K}$ interchangeably
$\mathcal{K}^*$	$\mathcal{K}^* \subset \mathcal{K}$	Set of stiffnesses induced by elliptical cross sections, i.e., $J = 4\psi(I)$
κ_i	$\kappa_i : (0, \ell) \to \mathbb{R}$	Material curvatures κ_1 and κ_2 of F; measure bending of the Kirchhoff rod around the material normals n_1 and n_2, respectively
κ	$\kappa : (0, \ell) \to \mathbb{R}_{\geq 0}$	Total (Frenet) curvature of γ, given by $\kappa = \|\gamma''\|$; for any frame F adapted to γ, it holds that $\kappa = \sqrt{\kappa_1^2 + \kappa_2^2}$
ℓ	$\ell \in \mathbb{R}_{>0}$	Length of the Kirchhoff rod
λ	$\lambda(l, \bar{l}) \subset \mathbb{R}^3$	Map from the Plücker coordinates $l, \bar{l} \in \mathbb{R}^3$ with $\langle l, \bar{l} \rangle = 0$ to the line in $\mathbb{R}^3$ incident to the point $\frac{l \times \bar{l}}{\langle l, l \rangle}$ and with direction l
Λ_{kl}	$\Lambda_{\mathrm{kl}} \subset \mathbb{P}^5$	Klein quadric, the set of all points with homogeneous coordinates $(l, \bar{l}) \in \mathbb{R}^6$ satisfying $\langle l, \bar{l} \rangle = 0$; we interpret these points as Plücker coordinates of a line in $\mathbb{R}^3$ with direction v and incident to a point x, such that $(l, \bar{l}) = (v, x \times v)$
M	$M : (0, \ell) \to \mathbb{R}^3$	Accumulated moment on a deformed rod, given by $M = \int_0^s \gamma \times q$
μ	$\mu \in \mathbb{R}_{>0}$	Shear modulus of the base material
n_i	$n_i : (0, \ell) \to \mathbb{R}^3$	Material normals of the moving frame F, so $n_i = Fe_i$ for $i = 1, 2$
ν	$\nu \in (-1, 1/2)$	Poisson's ratio of the base material
ω	$\omega : (0, \ell) \to \mathbb{R}^3$	Darboux vector of the moving frame F; related to F and k via $\omega = Fk$ and $F' = [\omega]_\times F$
p	$p : (0, \ell) \to \mathbb{R}^3$	Line load applied to the centerline of a rod, where $p(s)$ gives the load density at $\gamma(s)$
φ	$\varphi \in \mathbb{R}$	Orientation of ellipse with respect to reference frame; first and second semi-axes are given by $(\cos \varphi, \sin \varphi)^t$ and $(-\sin \varphi, \cos \varphi)^t$ respectively
ψ	$\psi : S^2_{++} \to \mathbb{R}_{>0}$	The determinant-over-trace function $\psi(X) := \frac{\det X}{\operatorname{tr} X}$
Q	$Q : (0, \ell) \to \mathbb{R}^3$	Accumulated load on a rod, given by $Q(s) = \int_0^s q$
q	$q \in \mathscr{D}'((0, \ell); \mathbb{R}^3)$	Load distribution applied to the centerline of a rod, consisting of a line load p and point loads f_i
S	$S \in \mathbb{R}^{3\times2}$	Selection matrix $S = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$ that extracts the first two columns of a three-column matrix by multiplication from the right, i.e., $(x_1, x_2, x_3)S = (x_1, x_2)$

S^2_{++}	$S^2_{++} \subset \mathbb{R}^{2\times 2}$	Set of all symmetric positive-definite 2-by-2 matrices
$SO(3)$	$SO(3) \subset \mathbb{R}^{3\times 3}$	Set of all rotations of $\mathbb{R}^3$ about the origin
s	$s \in (0, \ell)$	Arc-length parameter of γ
s_i	$s_i \in (0, \ell), i = 1, \ldots, n$	Concentrated point load f_i is applied to the centerline of a rod at $\gamma(s_i)$
τ	$\tau : (0, \ell) \to \mathbb{R}$	Twist of the moving frame F; measures rotation per arc-length unit around γ'