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ABSTRACT

We consider the eigenvalues of a large dimensional real or complex Ginibre matrix in the region of the complex plane where their real parts reach their maximum value. This maximum follows the Gumbel distribution and that these extreme eigenvalues form a Poisson point process as the dimension asymptotically tends to infinity. In the complex case, these facts have already been established by Bender [Probab. Theory Relat. Fields **147**, 241 (2010)] and in the real case by Akemann and Phillips [J. Stat. Phys. **155**, 421 (2014)] even for the more general elliptic ensemble with a sophisticated saddle point analysis. The purpose of this article is to give a very short direct proof in the Ginibre case with an effective error term. Moreover, our estimates on the correlation kernel in this regime serve as a key input for accurately locating $\max \Re \mathrm{Spec}(X)$ for any large matrix X with i.i.d. entries in the companion paper [G. Cipolloni *et al.*, arXiv:2206.04448 (2022)].

I. INTRODUCTION

The Ginibre matrix ensemble[1] is the simplest and most commonly used prototype of non-Hermitian random matrices. It consists of $n \times n$ matrices X with independent, identically distributed (i.i.d.) Gaussian entries x_{ij}. We use the normalization $\mathbf{E}x_{ij} = 0$, $\mathbf{E}|x_{ij}|^2 = \frac{1}{n}$, i.e., $\sqrt{n}x_{ij}$ is a standard real or complex normal random variable. Correspondingly, we talk about real or complex Ginibre matrices. The empirical density of eigenvalues converges to the uniform distribution on the unit disk in the complex plane, known as *Girko's circular law* and proven in increasing generality even without the Gaussian assumption in Refs. 2–4, while the spectral radius converges to 1 (see Refs. 5–8) with an explicit speed of convergence.[9] For the Gaussian case, the eigenvalues form a determinantal (or Pfaffian) point process with an explicit correlation kernel $K_n(z, w)$ [see (7) and (50) later]. This kernel was computed by Ginibre[1] in the complex case and later by Borodin and Sinclair[10,11] for the more complicated real case based on earlier works on Pfaffian formulas[12,13] (some special cases have been solved earlier in Refs. 14–18, and see also Ref. 19 for a comprehensive summary of all known related kernels). While the eigenvalue distribution is rotationally symmetric in the complex case, the main complication in the real case stems from the fact that the real axis plays a special role; in fact, there are many real eigenvalues.[15]

The explicit formula for the eigenvalue correlation function allows one, in principle, to compute the distribution of any interesting statistics of the eigenvalues. In reality, these calculations may require very precise asymptotic analysis of certain special functions where the complex and real cases may differ substantially. For example, the distribution of $\rho(X) := \max|\mathrm{Spec}(X)|$, the spectral radius of X (i.e., the largest eigenvalue in modulus), can still be easily identified in the complex case by using Kostlan's observation[20] on the moduli of the complex Ginibre eigenvalues. The precise result, stated in this form in Ref. 21, asserts that

$$\rho(X) \overset{\mathrm{d}}{=} 1 + \sqrt{\frac{\alpha_n}{4n}} + \frac{1}{\sqrt{4n\alpha_n}} G_n, \qquad \alpha_n := \log n - 2\log\log n - \log(2\pi), \tag{1}$$

where G_n converges in distribution to a standard *Gumbel random variable*, i.e.,

$$\lim_{n\to\infty} \mathbf{P}(G_n \le t) = \exp(-e^{-t})$$

for any fixed $t \in \mathbf{R}$. On the other hand, lacking radial symmetry, which is the key element of Kostlan's observation, the analogous result for the real Ginibre ensemble required a much more sophisticated analysis by Rider and Sinclair.[22] They showed that (1) also holds for the real case with the same scaling factor α_n, but G_n converges to a slightly rescaled Gumbel law with distribution function $\exp(-\frac{1}{2}e^{-t})$. The additional factor 1/2 stems from the fact that the spectrum of a real Ginibre matrix is symmetric with respect to the real axis.

In the current paper, we investigate a related quantity, the maximum real part of the spectrum of X, where radial symmetry does not help even in the complex case. It turns out that a similar asymptotic holds but with a new scaling factor,

$$\max \Re\mathrm{Spec}(X) \overset{\mathrm{d}}{=} 1 + \sqrt{\frac{\gamma}{4n}} + \frac{1}{\sqrt{4n\gamma}} G_n, \qquad \gamma = \gamma_n := \frac{\log n - 5\log\log n - \log(2\pi^4)}{2}, \tag{2}$$

with G_n still converging to a Gumbel variable. More precisely, we have the following.

Theorem 1 (Gumbel distribution). *Let $\sigma_1, \ldots, \sigma_n$ denote the eigenvalues of a real ($\beta = 1$) or complex ($\beta = 2$) $n \times n$ Ginibre matrix. Then, for any fixed*[23] *$t \in \mathbf{R}$, it holds that*

$$\mathbf{P}\bigg(\max_i \Re\sigma_i < 1 + \sqrt{\frac{\gamma}{4n}} + \frac{t}{\sqrt{4\gamma n}}\bigg) = \exp\bigg(-\frac{\beta}{2}\exp(-t)\bigg) + \mathcal{O}\bigg(\frac{(\log\log n)^2}{\log n}\bigg) \tag{3}$$

as $n \to \infty$.

In the complex case, (3) as a limit statement was proven by Bender[24] and in the real case by Akemann and Phillips[25] even for the more involved elliptic Ginibre ensemble where the kernel K_n is expressed by a contour integral (later, it was extended to the chiral two-matrix model with complex entries in Ref. 26). Here, we give a short alternative proof that also provides an effective estimate on the speed of convergence. In Theorem 1, we only considered the eigenvalue with the largest real part for simplicity; however, a similar result holds for the largest eigenvalue in any chosen direction. More precisely, in the complex case, the distribution of $\max_i \Re(e^{\mathrm{i}\theta}\sigma_i)$ is independent of $\theta \in \mathbf{R}$ by rotational symmetry. For real Ginibre matrices and for any fixed $\theta \neq 0$ independent of n, $\max_i \Re(e^{\mathrm{i}\theta}\sigma_i)$ still satisfies (3) but with $\beta = 2$. Our proof can easily be extended to cover this more general case using that the local eigenvalue correlation functions for real and complex Ginibre matrices practically coincide away from the real axis.

As a motivation, we remark that $\max \Re\mathrm{Spec}(X)$ is the basic quantity determining the exponential growth rate of the long time asymptotics of the solution of the linear system of differential equations,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{u}(t) = X\mathbf{u}(t).$$

Starting from the pioneering work of May[27] (see also the more recent review in Ref. 28), this equation is frequently used in phenomenological models to describe the evolution of many interacting agents with random couplings both in theoretical neuroscience[29,30] and in mathematical ecology.[31,32] Moreover, $\max \Re\mathrm{Spec}(X)$ is also important in counting the number of stable equilibria in a system of randomly coupled non-linear differential equations in Ref. 33.

The appearance of the universal Gumbel distribution in (1) and (2) is typical for extreme value statistics of independent random variables as one of the three main cases described in the Fisher–Tippet–Gnedenko theorem. While nearby Ginibre eigenvalues inside the unit disk are strongly correlated, the extreme eigenvalues are essentially independent, which heuristically explains the Gumbel law. The key point is that the correlation length of the eigenvalues is of order $n^{-1/2}$ as the scaling of the Ginibre kernel $K_n(z, w)$ indicates, but in the extreme regime, the few eigenvalues that may contribute to $\rho(X)$ or $\max \Re\mathrm{Spec}(X)$ are much farther away from each other than $n^{-1/2}$. In fact, the scaling factor $\gamma = \gamma_n$ is chosen in such a way that there are typically finitely many (independent of n) eigenvalues in an elongated box of size $(4\gamma n)^{-1/2} \times \mathrm{i}(\gamma n)^{-1/4}$ around $1 + \sqrt{\gamma/4n}$ (see Fig. 1). The height of this box, which is essentially the square root of its width, is determined by the curvature of the boundary of the circular law: above or below this box, there are no eigenvalues since their modulus would be too large. Given this heuristic picture, the typical distance between the eigenvalues in the relevant box is of order $n^{-1/4}$ modulo logarithmic factors, so they are well beyond the correlation scale and, hence, independent. As a second result, we also establish this independence rigorously; in fact, we show that within this box, the eigenvalues form a Poisson point process in the $n \to \infty$ limit. Again, as a pure limit statement, this result has already been proven in Ref. 24 for the complex Ginibre ensemble and in Ref. 25 for the real case; our contribution is to give an alternative direct proof with an effective error bound.

Theorem 2 (Poisson point process). *Let $\sigma_1, \ldots, \sigma_n$ denote the eigenvalues of a real or complex $n \times n$ Ginibre matrix. Fix any $t \in \mathbf{R}$ and any function $f : \mathbf{C} \to [0, \infty)$ supported on $[t, \infty) \times \mathrm{i}\mathbf{R}$, which, additionally, is assumed to be symmetric $f(z) = f(\bar{z})$ in the real case.*[34] *Then, we*

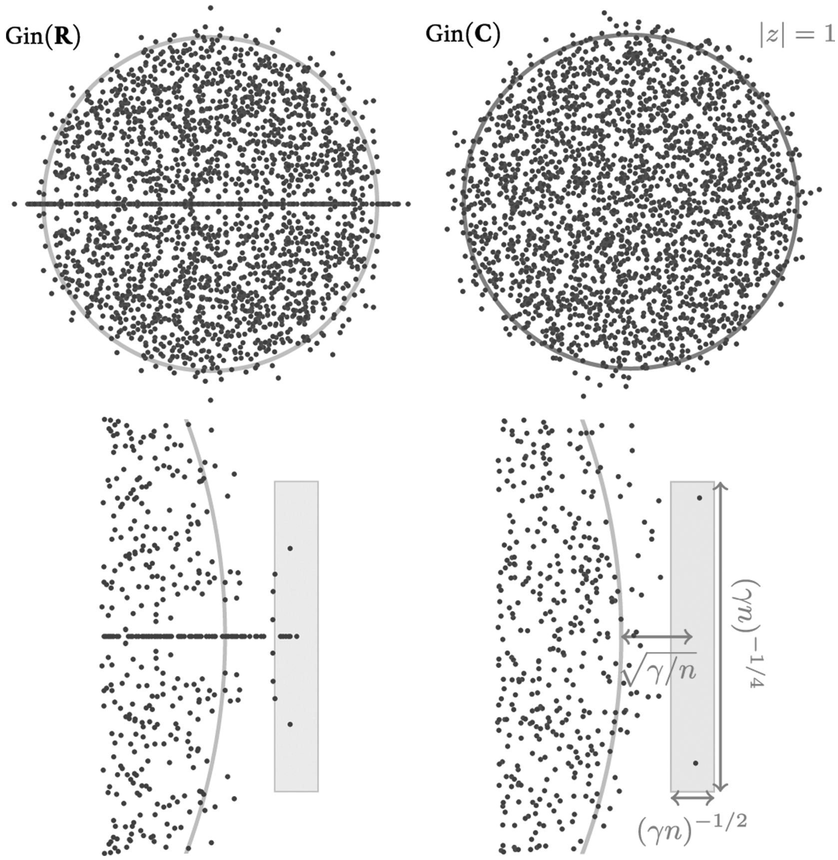

FIG. 1. The eigenvalues of real and complex Ginibre matrices. The eigenvalues for the top panels have been computed for 50 independent Ginibre matrices of size 50 × 50, while for the bottom panel, 100 independent matrices of size 100 × 100 have been sampled. Note that the eigenvalues of the real Ginibre matrix are symmetric with respect to the real axis and that some (in fact, $\sim\sqrt{n}$) eigenvalues are on the axis itself. Furthermore, the top left panel misleadingly hints that the rightmost eigenvalue is real. This is a finite n effect (see Ref. 22 for a detailed discussion of this so-called "Saturn effect"); we actually prove [see (60) to (62)] that in the large n limit, the largest real eigenvalue is much smaller than the real part of the rightmost complex eigenvalue.

have

$$\mathbf{E}e^{-\sum_{i=1}^n f(x_i+\mathrm{i}y_i)} = \exp\left(-\int_{\mathbf{F}}\left(1-e^{-f(x+\mathrm{i}y)}\right)\frac{e^{-x-y^2}}{\sqrt{\pi}}\,\mathrm{d}y\mathrm{d}x\right) + \mathcal{O}\left(\frac{(\log\log n)^2}{\log n}\right), \tag{4}$$

where we introduced the eigenvalue rescaling

$$\sigma_i = 1+\sqrt{\frac{\gamma}{4n}}+\frac{x_i}{\sqrt{4\gamma n}}+\frac{\mathrm{i}y_i}{(\gamma n)^{1/4}}, \tag{5}$$

and we set $\mathbf{F} = \mathbf{H} := \{z\in\mathbf{C}|\Im z\geq 0\}$ *in the real case and* $\mathbf{F}=\mathbf{C}$ *in the complex case.*

Both our main results follow from precise asymptotics of the rescaled Ginibre kernel $K_n(z,w)$ in the relevant box combined with the idea of the regularized Fredholm determinant also used in Ref. 22. The compact form of K_n in the Ginibre case makes the calculations considerably shorter than the saddle point analysis for its contour integral representation used for the elliptic ensemble in Refs. 24 and 25. In particular, we obtain an effective bound on the speed of convergence unlike in Refs. 24 and 25 that rely on dominated convergence. As a by-product, we also obtain the concentration result with an effective error term for the linear statistics (in particular, the number) of eigenvalues on a slightly larger box. This result is crucially used in our companion paper[35] in which we accurately identify the size of $\max\Re\mathrm{Spec}(X)$ for matrices with general i.i.d. entries, going well beyond the explicitly solvable models.

We close this Introduction with a remark about eigenvectors. For many Hermitian random matrices or operators originating from disordered quantum systems, the general prediction is that Poisson eigenvalue statistics entails localized eigenvectors (while strongly correlated eigenvalue statistics, e.g., Wigner–Dyson, imply delocalized eigenvectors). This is not the case here: all eigenvectors and even those corresponding to extreme eigenvalues in the Poisson regime are fully delocalized (Ref. 9, Corollary 2.4).

II. COMPLEX GINIBRE

We recall a few basic facts about the correlation functions. The joint probability density of the eigenvalues of a complex Ginibre matrix is given by[21]

$$\rho_n(\boldsymbol{z}) = \rho_n(z_1,\ldots,z_n) := \frac{n^n}{\pi^n 1!\cdots n!}\exp\Big(-n\sum_i |z_i|^2\Big)\prod_{i<j}\big(n|z_i - z_j|^2\big). \tag{6}$$

The product can be written as a product of Vandermonde determinants, and we obtain

$$\begin{aligned}\prod_{i<j}\big(n|z_i - z_j|^2\big) &= \det\begin{pmatrix} 1 & \sqrt{n}z_1 & \cdots & (\sqrt{n}z_1)^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \sqrt{n}z_n & \cdots & (\sqrt{n}z_n)^{n-1}\end{pmatrix}\begin{pmatrix} 1 & \cdots & 1 \\ \sqrt{n}\overline{z_1} & \cdots & \sqrt{n}\overline{z_n} \\ \vdots & \ddots & \vdots \\ (\sqrt{n}\overline{z_1})^{n-1} & \cdots & (\sqrt{n}\overline{z_n})^{n-1}\end{pmatrix} \\ &= 1!\cdots(n-1)!\det\big(K_n(z_i,z_j)\big)_{i,j=1}^n, \quad K_n(z,w) := \sum_{l=0}^{n-1}\frac{(nz\overline{w})^l}{l!}\end{aligned} \tag{7}$$

so that we conclude

$$\rho_n(\boldsymbol{z}) = \frac{n^n}{\pi^n n!} e^{-n|\boldsymbol{z}|^2}\det\big(K_n(z_i,z_j)\big)_{i,j=1}^n, \tag{8}$$

i.e., the eigenvalues form a *determinantal process*. Note that K_n is the kernel of a positive operator of rank n; in particular, its off-diagonal terms are estimated by the diagonal ones via the Cauchy–Schwarz inequality,

$$|K_n(z,w)|^2 \le K_n(z,z)K_n(w,w), \tag{9}$$

which also follows directly from the formula for $K_n(z,w)$. In order to integrate out variables, we rely on the following well-known identities:

$$\frac{n}{\pi}\int_{\mathbf{C}} e^{-n|z|^2} K_n(z,z)\,\mathrm{d}^2 z = n, \tag{10}$$

and for any fixed $w_1, w_2 \in \mathbf{C}$,

$$\frac{n}{\pi}\int_{\mathbf{C}} e^{-n|z|^2} K_n(w_1,z)K_n(z,w_2)\,\mathrm{d}^2 z = K_n(w_1,w_2). \tag{11}$$

We recall that both claims follow directly from the identity

$$\frac{n}{\pi}\int_{\mathbf{C}} e^{-n|z|^2}(\sqrt{n}z)^a(\sqrt{n}\overline{z})^b\,\mathrm{d}^2 z = \delta_{ab}a! \tag{12}$$

for any $a, b \in \mathbf{N}$ and the definition of K_n. As a consequence of these identities, an arbitrary number of variables can be integrated out and we obtain the following standard formula for the correlation functions.

Lemma 3 (*k-point correlation function*). *For*

$$\rho_n^k(z_1,\ldots,z_k) := \int_{\mathbf{C}^{n-k}} \rho_n(\boldsymbol{z})\,\mathrm{d}^2 z_{k+1}\cdots\mathrm{d}^2 z_n, \tag{13}$$

it holds that

$$\rho_n^k(\boldsymbol{z}) = \frac{n^k(n-k)!}{\pi^n n!} e^{-n|\boldsymbol{z}|^2}\det\big(K_n(z_i,z_j)\big)_{i,j=1}^k. \tag{14}$$

Consider a function $g : \mathbf{C} \to [0,1]$ and evaluate

$$\begin{aligned}
\mathbf{E}\prod_{i=1}^{n}(1-g(\sigma_i)) &= \int_{\mathbf{C}^n}\rho_n(\boldsymbol{z})\prod_{i=1}^{n}(1-g(z_i))\mathrm{d}^2\boldsymbol{z} \\
&= \sum_{k=0}^{n}(-1)^k\binom{n}{k}\int_{\mathbf{C}^k}\rho_n^k(\boldsymbol{z})\prod_{i=1}^{k}g(z_i)\mathrm{d}^2\boldsymbol{z} \\
&= \sum_{k=0}^{n}\frac{(-1)^k}{k!}\frac{n^k}{\pi^k}\int_{\mathbf{C}^n}e^{-n|z|^2}\det\left(K_n(z_i,z_j)\right)_{i,\;\;j=1}^{n}\prod_{i=1}^{k}g(z_i)\mathrm{d}^2\boldsymbol{z} \\
&= \sum_{k=0}^{n}\frac{(-1)^k}{k!}\int_{\mathbf{C}^k}\det\left(\sqrt{g(z_i)}\widetilde{K}_n(z_i,z_j)\sqrt{g(z_j)}\right)_{i,\;\;j=1}^{k}\mathrm{d}^2\boldsymbol{z} \\
&= \det(1-\sqrt{g}\widetilde{K}_n\sqrt{g}),
\end{aligned} \tag{15}$$

which we recognize as the Fredholm determinant of $1-\sqrt{g}\widetilde{K}_n\sqrt{g}$ (see Definition 4, and recall that $\widetilde{K}_n$ has rank n), where

$$\widetilde{K}_n(z,w) := \frac{n}{\pi}e^{-n(|z|^2+|w|^2)/2}K_n(z,w) = \frac{n}{\pi}e^{-n(|z|^2+|w|^2-2z\overline{w})/2}\frac{\Gamma(n,nz\overline{w})}{\Gamma(n)}. \tag{16}$$

Here, $\Gamma(\cdot,\cdot)$ denotes the incomplete Gamma function defined as

$$\Gamma(s,z) := \int_z^\infty t^{s-1}e^{-t}\mathrm{d}t, \tag{17}$$

where $s \in \mathbf{N}$ and the integration contour goes from $z \in \mathbf{C}$ to real infinity.

Definition 4 (Fredholm determinant). Let $(\Omega,\ \mu)$ denote a measure space, and let $K(z,w)$ be a kernel on Ω. Then, the Fredholm determinant of $1-K$ is defined as

$$\det(1-K) := \sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\int_{\Omega^k}\det\left(K(z_i,z_j)\right)_{i,\;\;j=1}^{k}\mathrm{d}\mu(z_1)\cdots\mathrm{d}\mu(z_k). \tag{18}$$

A. Scaling limit for max $\Re\sigma_i$

We now consider the scaling limit for the part of the complex plane in which the eigenvalue with the largest real part is located; cf. Fig. 1. We will show that the eigenvalue with the largest real part lives on a scale $(4\gamma n)^{-1/2}\times \mathrm{i}(\gamma n)^{-1/4}$ around $1+\sqrt{\gamma/4n}$.

The fact that outside the unit circle the kernel $\widetilde{K}_n$ has a small Hilbert–Schmidt norm prompts the introduction of the regularized determinant [Ref. 36, IV.(7.8)],

$$\det_2(1-K) := \det\big((1-K)e^K\big), \tag{19}$$

which for finite-rank K allows us to write $\det(1-K) = \det_2(1-K)\exp(-\mathrm{Tr}K)$. From Ref. 36 [IV.(7.11)], we, thus, conclude

$$|\det(1-K) - \exp(-\operatorname{Tr}K)| \le \|K\|_2 e^{(\|K\|_2+1)^2/2-\operatorname{Tr}K}, \tag{20}$$

where

$$\operatorname{Tr}K = \int_\Omega K(x,x)\mathrm{d}\mu(x), \quad \|K\|_2^2 = \int_{\Omega^2}|K(x,y)|^2\mathrm{d}\mu(x)\mathrm{d}\mu(y). \tag{21}$$

The regularized determinant as a technical tool was used in Ref. 22 in a very similar context for the spectral radius of real Ginibre matrices.

Proposition 5. Let $|t| \le \sqrt{\log n}/10$, and define the set

$$A = A(t) := \left\{z\in\mathbf{C}\,\middle|\,\Re z \ge 1+\sqrt{\frac{\gamma}{4n}}+\frac{t}{\sqrt{4\gamma n}}\right\}. \tag{22}$$

Then, for $g : \mathbf{C} \to [0,1]$ *supported on* $\operatorname{supp} g \subset A(t)$ *and for n large enough so that* $\gamma > 0$*, it holds that*

$$\begin{aligned} \operatorname{Tr} \sqrt{g}\widetilde{K}_n\sqrt{g} &= \int_t^\infty \int_{\mathbf{R}} g(z) \frac{e^{-x-y^2}}{\sqrt{\pi}} \mathrm{d}y\mathrm{d}x + \mathcal{O}\bigg(e^{-t}\frac{(\log\log n)^2 + |t|^2}{\log n}\bigg), \\ z &= 1 + \sqrt{\frac{\gamma}{4n}} + \frac{x}{\sqrt{4\gamma n}} + \frac{\mathrm{i}y}{(\gamma n)^{1/4}}, \end{aligned} \tag{23}$$

and

$$\|\sqrt{g}\widetilde{K}_n\sqrt{g}\|_2 \lesssim e^{-\sqrt{\log n}/32}. \tag{24}$$

The unspecified constants in $\lesssim$ *and* $\mathcal{O}(\cdot)$ *are uniform in n and in* $|t| \le \sqrt{\log n}/10$.

In particular, (15) and (20) combined with Proposition 5 for any fixed t gives

$$\begin{aligned} \mathbf{P}\bigg(\max_i \Re\sigma_i < 1 + \sqrt{\frac{\gamma}{4n}} + \frac{t}{\sqrt{4\gamma n}}\bigg) &= \mathbf{P}\big(\sigma_1,\dots,\sigma_n \in A(t)^c\big) \\ &= \det(1 - \chi_{A(t)}\widetilde{K}_n\chi_{A(t)}) \xrightarrow{n\to\infty} e^{-e^{-t}}, \end{aligned} \tag{25}$$

with χ_A denoting the characteristic function of the set A, completing the Proof of Theorem 1 in the complex case. Moreover, for any function $f : \mathbf{C} \to [0,\infty)$ supported in $A(t)$, we also have that

$$\begin{aligned} \mathbf{E}\exp\bigg(-\sum_{i=1}^n f(\sigma_i)\bigg) &= \det\Big(1 - \sqrt{1-e^{-f}}\widetilde{K}_n\sqrt{1-e^{-f}}\Big) \\ &\xrightarrow{n\to\infty} \exp\bigg(-\int_t^\infty \int_{\mathbf{R}} (1 - e^{-f(z)}) \frac{e^{-x-y^2}}{\sqrt{\pi}} \mathrm{d}y\mathrm{d}x\bigg) \end{aligned} \tag{26}$$

with z as in (23), proving the complex case of Theorem 2 after the change of variables. The error terms in (3) and (4) can easily be obtained from (23) and (24).

Hence, the remaining task is to prove Proposition 5, which will be an easy consequence of Lemma 6.

Lemma 6. Rescale the kernel variables as

$$z = 1 + \sqrt{\frac{\gamma}{4n}} + \frac{x_1}{\sqrt{4\gamma n}} + \frac{\mathrm{i}y_1}{(\gamma n)^{1/4}}, \quad w = 1 + \sqrt{\frac{\gamma}{4n}} + \frac{x_2}{\sqrt{4\gamma n}} + \frac{\mathrm{i}y_2}{(\gamma n)^{1/4}}, \tag{27}$$

with $\boldsymbol{x} := (x_1, x_2)$, $\boldsymbol{y} := (y_1, y_2)$ *being real vectors. In the regime* $|\boldsymbol{x}| + |\boldsymbol{y}|^2 \le \sqrt{\log n}/2$ *and for* $|y_1 - y_2| < n^{1/10}n^{-1/4}$*, we have the asymptotics*

$$\frac{|\widetilde{K}_n(z,w)|^2}{4(\gamma n)^{3/2}} = \frac{\gamma e^{-x_1-x_2-y_1^2-y_2^2}}{\pi(\gamma + \sqrt{n/\gamma}(y_1-y_2)^2)}\bigg(1 + \mathcal{O}\bigg(\frac{\log\log n + |\boldsymbol{x}|^2 + |\boldsymbol{y}|^4}{\log n}\bigg)\bigg). \tag{28}$$

On the other hand, for $|\boldsymbol{x}| + |\boldsymbol{y}|^2 \le \sqrt{\log n}/2$ *and* $|y_1 - y_2| \ge Cn^{-1/4}$ *for some* $C \ge 1$*, we have the estimate*

$$\frac{|\widetilde{K}_n(z,w)|^2}{(\gamma n)^{3/2}} \lesssim \frac{\gamma e^{-x_1-y_1^2-x_2-y_2^2}}{\gamma + \sqrt{n/\gamma}(y_1-y_2)^2}\bigg(1 + \mathcal{O}\bigg(\frac{\sqrt{\gamma}}{C^2} + \frac{|\boldsymbol{x}|^2 + |\boldsymbol{y}|^4}{\log n}\bigg)\bigg). \tag{29}$$

Finally, for $x_1 + y_1^2 \ge 0$, $x_2 + y_2^2 \ge 0$*, we have the uniform bound*

$$\frac{|\widetilde{K}_n(z,w)|^2}{(\gamma n)^{3/2}} \lesssim |z|^2|w|^2 e^{-(x_1+y_1^2)/3} e^{-(x_2+y_2^2)/3}. \tag{30}$$

Proof of Proposition 5. Set $t_0 := 4(\log n + |t|)$, and estimate the trace in (23) as follows:

$$\begin{aligned}
\operatorname{Tr}\sqrt{g}\widetilde{K}_n\sqrt{g} &= \int_{A(t)} g(z)\widetilde{K}_n(z,z)\mathrm{d}^2 z\\
&= \left(\int_t^{t_0}\int_{y^2<2t_0} + \int_t^{t_0}\int_{y^2\ge 2t_0} + \int_{t_0}^{\infty}\int_{\mathbf{R}}\right) g(z)\frac{\widetilde{K}_n(z,z)}{2(\gamma n)^{3/4}}\mathrm{d}y\,\mathrm{d}x\\
&= \int_t^{t_0}\int_{y^2<t_0} g(z)\frac{e^{-x-y^2}}{\sqrt{\pi}}\mathrm{d}y\,\mathrm{d}x\left(1+\mathcal{O}\left(\frac{(\log\log n)^2+|t|^2}{\log n}\right)\right) + \mathcal{O}\left(e^{-|t_0|/4}\right)\\
&= \int_t^{\infty}\int_{\mathbf{R}} g(z)\frac{e^{-x-y^2}}{\sqrt{\pi}}\mathrm{d}y\,\mathrm{d}x + \mathcal{O}\left(e^{-t}\frac{(\log\log n)^2+|t|^2}{\log n}\right),
\end{aligned} \tag{31}$$

where we used (28) for the first integral and (30) for the remaining two integrals.

For the bound on (23), we estimate

$$\operatorname{Tr}\left(\sqrt{g}\widetilde{K}_n\sqrt{g}\right)^2 \le \iint_{A(t)} |\widetilde{K}_n(z,w)|^2 \mathrm{d}^2 z\,\mathrm{d}^2 w, \tag{32}$$

and after a change of variables from (z,w) to $(\boldsymbol{x},\boldsymbol{y})$ using (27), we split the integral into two parts. First, estimate the part where $|\boldsymbol{x}|+|\boldsymbol{y}|^2 > \sqrt{\log n}/2$ and obtain

$$\begin{aligned}
&\iint_t^{\infty}\iint_{\mathbf{R}} \frac{|\widetilde{K}_n(z,w)|^2}{4(\gamma n)^{3/2}}\mathbf{1}\left(|\boldsymbol{x}|+|\boldsymbol{y}|^2 > \frac{\sqrt{\log n}}{2}\right)\mathrm{d}\boldsymbol{y}\,\mathrm{d}\boldsymbol{x}\\
&\le \int_t^{\infty}\int_{\mathbf{R}}\iint_{\mathbf{R}} \frac{|\widetilde{K}_n(z,w)|^2}{4(\gamma n)^{3/2}}\mathbf{1}\left(|x_1|+y_1^2 > \frac{\sqrt{\log n}}{4}\right)\mathrm{d}\boldsymbol{y}\,\mathrm{d}x_2\,\mathrm{d}x_1\\
&= \int_t^{\infty}\int_{\mathbf{R}} \frac{\widetilde{K}_n(z,z)}{2(\gamma n)^{3/4}}\mathbf{1}\left(|x_1|+y_1^2 > \frac{\sqrt{\log n}}{4}\right)\mathrm{d}y_1\,\mathrm{d}x_1\\
&\lesssim \int_t^{\infty}\int_{\mathbf{R}} e^{-(x+y^2)/4}\mathbf{1}\left(|x|+y^2 > \frac{\sqrt{\log n}}{4}\right)\mathrm{d}y\,\mathrm{d}x \lesssim e^{-\sqrt{\log n}/16}
\end{aligned} \tag{33}$$

due to (11) in the second step and (30) in the last step. In the remaining integral, we use (28) whenever $|y_1-y_2| \le n^{-1/6}$ and (29) otherwise to find

$$\begin{aligned}
&\iint_t^{\infty}\iint_{\mathbf{R}} \frac{|\widetilde{K}_n(z,w)|^2}{(\gamma n)^{3/2}}\mathbf{1}\left(|\boldsymbol{x}|+|\boldsymbol{y}|^2 \le \frac{\sqrt{\log n}}{2}\right)\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y}\\
&\lesssim \iint_t^{\infty}\iint_{\mathbf{R}} e^{-x_1-x_2-y_1^2-y_2^2}\left(\mathbf{1}(|y_1-y_2|\le n^{-1/6}) + \frac{\mathbf{1}(|y_1-y_2|>n^{-1/6})}{\gamma^{-3/2}n^{1/6}}\right)\mathrm{d}\boldsymbol{y}\mathrm{d}\boldsymbol{x}\\
&\lesssim e^{-2t}n^{-1/6}\gamma^{3/2},
\end{aligned} \tag{34}$$

concluding the proof. □

Proof of Lemma 6. For (30), by the Cauchy–Schwarz inequality, it is sufficient to prove that

$$\frac{\widetilde{K}_n(z,z)}{(\gamma n)^{3/4}} \lesssim |z|^2 e^{-(x+y^2)/3}. \tag{35}$$

For the proof of (35), we recall the asymptotics[22] (Lemma 3.2) of the incomplete Γ function,

$$\frac{\Gamma(n,nt)}{\Gamma(n)} = \frac{t\mu(t)\operatorname{erfc}(\sqrt{n}\mu(t))}{\sqrt{2}(t-1)}\left(1+\mathcal{O}\left(n^{-1/2}\right)\right), \quad \mu(t) := \sqrt{t-\log(t)-1}, \tag{36}$$

which holds uniformly in $t > 1$, and note that

$$|z|^2 = 1 + \frac{\sqrt{\gamma}+(x+y^2)/\sqrt{\gamma}}{\sqrt{n}} + \frac{(\gamma+x)^2}{4\gamma n} \ge 1 + \frac{\sqrt{\gamma}+(x+y^2)/\sqrt{\gamma}}{\sqrt{n}}. \tag{37}$$

Then, for (35), we use $\operatorname{erfc}(x) \lesssim e^{-x^2}/x$ to estimate

$$\frac{1}{(\gamma n)^{3/4}} \widetilde{K}_n(z,z) \lesssim \frac{n^{1/4}}{\gamma^{5/4}} |z|^2 e^{-n\mu(|z|^2)^2} \leq \frac{n^{1/4}}{\gamma^{5/4}} |z|^2 e^{-\gamma(1-\sqrt{\gamma/n})/2} e^{-(x+y^2)/3} \tag{38}$$

using the elementary bound $t - \log t - 1 \geq \delta(1-\delta)(t-1)/2$ for $t \geq 1+\delta$ and $\delta \in [0,1)$, implying

$$\mu(|z|^2)^2 = |z|^2 - 2\log|z| - 1 \geq \frac{\gamma + x + y^2}{2n}\left(1 - \sqrt{\frac{\gamma}{n}}\right) \geq \frac{\gamma}{2n}\left(1 - \sqrt{\frac{\gamma}{n}}\right) + \frac{x+y^2}{3n} \tag{39}$$

due to $\gamma/n \ll 1$ in the last step. Now, (35) follows from

$$e^{-\gamma/2} = \exp\left(-\frac{1}{4}\log\frac{n}{2\pi^4(\log n)^5}\right) = \frac{2^{1/4}\pi(\log n)^{5/4}}{n^{1/4}} = \frac{2^{3/2}\pi\gamma^{5/4}}{n^{1/4}}\left(1 + \mathcal{O}\left(\frac{\log\log n}{\log n}\right)\right). \tag{40}$$

For (29), we first note

$$z\overline{w} = 1 + \frac{\sqrt{\gamma} + \left(\frac{x_1+x_2}{2} + y_1 y_2\right)/\sqrt{\gamma}}{\sqrt{n}} + \mathrm{i}\frac{y_1 - y_2}{(\gamma n)^{1/4}} + \mathrm{i}\frac{y_1(\gamma + x_2/\gamma) - y_2(\gamma + x_1/\gamma)}{(\gamma n)^{3/4}}, \tag{41}$$

and hence, $|1 - z\overline{w}| \gtrsim (|y_1 - y_2|(n/\gamma)^{1/4} + \sqrt{\gamma})/\sqrt{n}$. Now, we use the asymptotics[22] (Lemma 3.4)

$$\frac{\Gamma(n, nz\overline{w})}{\Gamma(n)} = e^{-nz\overline{w}} \frac{e^n (z\overline{w})^n}{\sqrt{2\pi n}(1 - z\overline{w})}\left(1 + \mathcal{O}\left(\frac{1}{n|1 - z\overline{w}|^2}\right)\right) \tag{42}$$

to estimate

$$\frac{|\widetilde{K}_n(z,w)|}{(n\gamma)^{3/4}} \lesssim \frac{n^{1/4} e^{n(1-|z|^2/2-|w|^2/2+\log|z\overline{w}|)}}{\gamma^{3/4}(|y_1 - y_2|(n/\gamma)^{1/4} + \sqrt{\gamma})}\left(1 + \mathcal{O}\left(\frac{\sqrt{\gamma}}{C^2}\right)\right). \tag{43}$$

In the exponent, we use

$$\begin{aligned}1 - |z|^2/2 - |w|^2/2 + \log|z| + \log|w| &= -\frac{(|z|^2-1)^2}{4} - \frac{(|w|^2-1)^2}{4} + \mathcal{O}\left((\gamma/n)^{3/2}\right) \\ &= -\frac{\gamma + x_1 + y_2^2 + x_2 + y_2^2}{2n} + \mathcal{O}\left(\frac{1 + |\boldsymbol{x}|^2 + |\boldsymbol{y}|^4}{n\gamma}\right)\end{aligned} \tag{44}$$

to conclude

$$\frac{|\widetilde{K}_n(z,w)|}{(n\gamma)^{3/4}} \lesssim \frac{\sqrt{\gamma} e^{-(x_1+y_1^2)/2} e^{-(x_2+y_2^2)/2}}{|y_1 - y_2|(n/\gamma)^{1/4} + \sqrt{\gamma}}\left(1 + \mathcal{O}\left(\frac{\sqrt{\gamma}}{C^2} + \frac{\log\log n + x^2 + y^4}{\log n}\right)\right). \tag{45}$$

It remains to consider (28) where we use Ref. 22, Lemma 3.3 in the form

$$\frac{\Gamma(n, nz\overline{w})}{\Gamma(n)} = \frac{z\overline{w}\mu(z\overline{w})\operatorname{erfc}(\sqrt{n}\mu(z\overline{w}))}{\sqrt{2}(z\overline{w} - 1)}\left(1 + \mathcal{O}\left(\frac{1}{n|1 - z\overline{w}|}\right)\right), \quad \mu(z) := \sqrt{z - \log(z) - 1}. \tag{46}$$

We use the Taylor expansion $\mu(1+z) = z/\sqrt{2} + \mathcal{O}(|z|^2)$ (for small enough $|z|$) and the asymptotics [Ref. 37, Eq. (7.12.1)] of the error function $\operatorname{erfc}(z) = e^{-z^2}/(\sqrt{\pi}z)(1 + \mathcal{O}(|z|^{-2}))$ for $|\arg z| < 3\pi/4$ to obtain

$$\frac{\Gamma(n, nz\overline{w})}{\Gamma(n)} = \frac{e^{-n(z\overline{w}-1)^2/2}}{\sqrt{2\pi}\sqrt{n}(z\overline{w} - 1)}\left(1 + \mathcal{O}\left(|z\overline{w} - 1| + n|z\overline{w} - 1|^3 + \frac{1}{n|z\overline{w} - 1|^2}\right)\right), \tag{47}$$

and thereby,

$$\begin{aligned}\frac{|\widetilde{K}_n(z,w)|^2}{4(n\gamma)^{3/2}} &= \frac{n^{1/2}}{\gamma^{3/2}(2\pi)^3} \frac{e^{-\gamma - x_1 - x_2 - y_1^2 - y_2^2}}{\gamma + \sqrt{n/\gamma}(y_1 - y_2)^2}\left(1 + \mathcal{O}\left(\frac{1 + |\boldsymbol{x}|^2 + |\boldsymbol{y}|^4}{\gamma}\right)\right) \\ &= \frac{\gamma e^{-x_1 - x_2 - y_1^2 - y_2^2}}{\pi(\gamma + \sqrt{n/\gamma}(y_1 - y_2)^2)}\left(1 + \mathcal{O}\left(\frac{\log\log n + |\boldsymbol{x}|^2 + |\boldsymbol{y}|^4}{\log n}\right)\right).\end{aligned} \tag{48}$$

Here, we also used the upper bound on $|y_1 - y_2| \leq n^{1/10} n^{-1/4}$ in order to estimate $\sqrt{\gamma/n} \lesssim |1 - z\overline{w}| \lesssim n^{-1/2}(\sqrt{\gamma} + n^{1/10}/\gamma^{1/4})$. □

III. REAL GINIBRE

We now consider the real case. The analog of (15) for test functions $g : \mathbf{C} \to [0,1]$ invariant under complex conjugation, $g(\overline{z}) = g(z)$, and vanishing on the real line, $g(x) = 0$, $x \in \mathbf{R}$, is given by[22]

$$\mathbf{E}\prod_{i=1}^{n}(1 - g(\sigma_i)) = \left[\det(1 - \sqrt{g}K_n^{\mathbf{C},\mathbf{C}}\sqrt{g})\right]^{1/2}, \tag{49}$$

where

$$K_n^{\mathbf{C},\mathbf{C}}(z,w) := \begin{pmatrix} S_n(z,w) & -\mathrm{i}S_n(z,\overline{w}) \\ -\mathrm{i}S_n(\overline{z},w) & S_n(w,z) \end{pmatrix} \tag{50}$$

with

$$\begin{aligned} S_n(z,w) &:= \frac{\mathrm{i}ne^{-n(z-\overline{w})^2/2}}{\sqrt{2\pi}}\sqrt{n}(\overline{w}-z)\sqrt{\mathrm{erfc}(\sqrt{2n}|\Im z|)\mathrm{erfc}(\sqrt{2n}|\Im w|)}e^{-nz\overline{w}}K_n(z,w) \\ &= \Phi_n(z,w)\widetilde{K}_n(z,w), \\ \Phi_n(z,w) &:= e^{n(|z|^2+|w|^2-2z\overline{w})/2}\frac{\mathrm{i}\sqrt{\pi}e^{-n(z-\overline{w})^2/2}}{\sqrt{2}}\sqrt{n}(\overline{w}-z)\sqrt{\mathrm{erfc}(\sqrt{2n}|\Im z|)\mathrm{erfc}(\sqrt{2n}|\Im w|)}. \end{aligned} \tag{51}$$

The analog to Proposition 5 is the following result.

Proposition 7. Let $|t| \le \sqrt{\log n}/10$, let $A(t)$ be as in (22), and recall $\gamma = \gamma_n$ from (2). Consider any function $g : \mathbf{C} \to [0,1]$ supported on $\mathrm{supp}\, g \subset A(t)$ *that is symmetric in the sense $g(z) = g(\overline{z})$, and let n be large enough such that $\gamma > 0$. Then, we have*

$$\begin{aligned} \mathrm{Tr}\sqrt{g}K_n^{\mathbf{C},\mathbf{C}}\sqrt{g} &= 2\int_t^\infty \int_0^\infty g(z)\frac{e^{-x-y^2}}{\sqrt{\pi}}\mathrm{d}y\mathrm{d}x + \mathcal{O}\left(e^{-t}\frac{(\log\log n)^2 + |t|^2}{\log n}\right), \\ z &= 1 + \sqrt{\frac{\gamma}{4n}} + \frac{x}{\sqrt{4\gamma n}} + \frac{\mathrm{i}y}{(\gamma n)^{1/4}}, \end{aligned} \tag{52}$$

and

$$\|\sqrt{g}K_n^{\mathbf{C},\mathbf{C}}\sqrt{g}\|_2 \lesssim e^{-\sqrt{\log n}/32}. \tag{53}$$

The unspecified constants in $\lesssim$ and $\mathcal{O}(\cdot)$ are uniform in n and in $|t| \le \sqrt{\log n}/10$.

Proof. We estimate

$$\Phi_n(z,z) = \sqrt{\pi}e^{2n(\Im z)^2}\sqrt{2n}\Im z\,\mathrm{erfc}(\sqrt{2n}|\Im z|) = 1 + \left(\min\left\{1, \frac{1}{n(\Im z)^2}\right\}\right), \tag{54}$$

where we used the asymptotic $\mathrm{erfc}(x) = e^{-x^2}/(\sqrt{\pi}x)(1 + \mathcal{O}(x^{-2}))$ and the bound $\mathrm{erfc}(x) \le e^{-x^2}/(\sqrt{\pi}x)$. Thus, the tracial computation essentially reduces to the complex case (31) and we obtain

$$\begin{aligned} \mathrm{Tr}\sqrt{g}K_n^{\mathbf{C},\mathbf{C}}(z,z)\sqrt{g} &= 2\int_{A(t)_+} g(z)S_n(z,z)\mathrm{d}^2z \\ &= 2\int_{A(t)_+} g(z)\widetilde{K}_n(z,z)\mathbf{1}(\Im z > n^{-5/12})\mathrm{d}^2z\left(1 + \mathcal{O}\left(n^{-5/12}\right)\right) \\ &\quad + \mathcal{O}\left(\int_{A(t)_+}\widetilde{K}_n(z,z)\mathbf{1}(\Im z \le n^{-5/12})\mathrm{d}^2z\right) \\ &= 2\int_t^\infty\int_0^\infty \frac{e^{-x-y^2}}{\sqrt{\pi}}g(z)\mathrm{d}y\mathrm{d}x + \mathcal{O}\left(e^{-t}\frac{(\log\log n)^2 + |t|^2}{\log n}\right), \end{aligned} \tag{55}$$

where $A(t)_+ := A(t) \cap \mathbf{H}$. We parameterized z with x, y as in (23), and we used (28) and (30).

For the Hilbert–Schmidt norm, we estimate, analogously to (33),

$$\begin{aligned}\|\sqrt{g}K_n^{\mathbf{C},\mathbf{C}}\sqrt{g}\|_2 &= \iint g(z)g(w)\operatorname{Tr}K_n^{\mathbf{C},\mathbf{C}}(z,w)K_n^{\mathbf{C},\mathbf{C}}(w,z)\mathrm{d}^2z\mathrm{d}^2w \\ &\le \iint_{\Re\ge t}\operatorname{Tr}K_n^{\mathbf{C},\mathbf{C}}(z,w)K_n^{\mathbf{C},\mathbf{C}}(w,z)\mathbf{1}\left(|\boldsymbol{x}|+|\boldsymbol{y}|^2\le\frac{\sqrt{\log n}}{2}\right)\mathrm{d}^2z\mathrm{d}^2w \\ &\quad+\int\operatorname{Tr}K_n^{\mathbf{C},\mathbf{C}}(z,z)\mathbf{1}\left(|x|+y^2>\frac{\sqrt{\log n}}{4}\right)\mathrm{d}^2z \\ &\lesssim \iint_{\Re\ge t}|S_n(z,w)|^2\mathbf{1}\left(|\boldsymbol{x}|+|\boldsymbol{y}|^2\le\frac{\sqrt{\log n}}{2}\right)\mathrm{d}^2z\mathrm{d}^2w+e^{-\sqrt{\log n}/16},\end{aligned} \tag{56}$$

where we used that the integrals of $|S_n(z,w)|^2$ and $|S_n(z,\overline{w})|^2$ are equal by symmetry of the integration region and $\Re\ge t$ indicates the integration region $\{\Re z\ge t\}\cap\{\Re w\ge t\}$. Now, we use (29) together with the elementary bound

$$|\Phi_n(z,w)|^2\lesssim\frac{n|z-\overline{w}|^2}{(1\vee\sqrt{n}\Im z)(1\vee\sqrt{n}\Im w)}\lesssim\frac{(x_1-x_2)^2/\sqrt{\gamma}+\sqrt{n}(y_1+y_2)^2}{(\gamma^{1/4}\vee n^{1/4}y_1)(\gamma^{1/4}\vee n^{1/4}y_2)} \tag{57}$$

to estimate

$$\frac{|S_n(z,w)|^2}{(\gamma n)^{3/2}}\lesssim\frac{\gamma e^{-x_1-x_2-y_1^2-y_2^2}}{\gamma+\sqrt{n/\gamma}(y_1-y_2)^2}\frac{(x_1-x_2)^2/\sqrt{\gamma}+\sqrt{n}(y_1+y_2)^2}{(\gamma^{1/4}\vee n^{1/4}y_1)(\gamma^{1/4}\vee n^{1/4}y_2)} \tag{58}$$

and conclude, similarly to (34), that

$$\iint_{\Re\ge t}|S_n(z,w)|^2\boldsymbol{I}\left(|\boldsymbol{x}|+|\boldsymbol{y}|^2\le\frac{\sqrt{\log n}}{2}\right)\mathrm{d}^2z\mathrm{d}^2w\lesssim e^{-2t}n^{-1/6}\gamma^{3/2}. \tag{59}$$

□

As a consequence of (20) and (49) and Proposition 7, we obtain that for any fixed t, it holds that

$$\begin{aligned}\mathbf{P}\left(\max_{i:\sigma_i\notin\mathbf{R}}\Re\sigma_i<1+\sqrt{\frac{\gamma}{4n}}+\frac{t}{\sqrt{4\gamma n}}\right)&=\mathbf{P}(\sigma_1,\dots,\sigma_n\in\mathbf{R}\cup[\mathbf{C}\backslash(A(t)\cup\mathbf{R})]) \\ &=\left[\det(1-\chi_{A(t)}K_n^{\mathbf{C},\mathbf{C}}\chi_{A(t)})\right]^{1/2}\xrightarrow{n\to\infty}e^{-e^{-t}/2},\end{aligned} \tag{60}$$

with χ_A denoting the characteristic function of the set A, using that $\int_0^\infty e^{-y^2}\mathrm{d}y=\sqrt{\pi}/2$. Moreover, for any symmetric function $f:\mathbf{C}\to[0,\infty)$ supported in $A(t)$, we also have that

$$\begin{aligned}\mathbf{E}\exp\left(-\sum_{i:\sigma_i\notin\mathbf{R}}f(\sigma_i)\right)&=\det\left(1-\sqrt{1-e^{-f}}K_n^{\mathbf{C},\mathbf{C}}\sqrt{1-e^{-f}}\right)^{1/2} \\ &\xrightarrow{n\to\infty}\exp\left(-\int_t^\infty\int_0^\infty(1-e^{-f(z)})\frac{e^{-x-y^2}}{\sqrt{\pi}}\mathrm{d}y\mathrm{d}x\right).\end{aligned} \tag{61}$$

In order to complete the Proof of Theorems 1 and 2, it remains to estimate the real eigenvalues. However, the real eigenvalues affect neither of these results since the largest real eigenvalue lives on a smaller scale, $1+\mathcal{O}(1/\sqrt{n})$, than the largest real part of complex eigenvalues, $1+\mathcal{O}\left(\sqrt{\log n/n}\right)$. Indeed, the main result of Ref. 38 is that for large t,

$$\lim_{n\to\infty}\mathbf{P}\left(\max_{i:\sigma_i\in\mathbf{R}}\sigma_i\le 1+\frac{t}{\sqrt{n}}\right)=1-\frac{1}{4}\operatorname{erfc}(t)+\mathcal{O}\left(e^{-2t^2}\right). \tag{62}$$

Together with (60) and (61), this also concludes the Proof of Theorems 1 and 2 in the real case.

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AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

Author Contributions

Giorgio Cipolloni: Writing – original draft (equal). **László Erdős**: Writing – original draft (equal). **Dominik Schröder**: Writing – original draft (equal). **Yuanyuan Xu**: Writing – original draft (equal).

DATA AVAILABILITY

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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