



Ergodic decompositions of Dirichlet forms under order isomorphisms

LORENZO DELLO SCHIAVO AND MELCHIOR WIRTH

Abstract. We study ergodic decompositions of Dirichlet spaces under intertwining via unitary order isomorphisms. We show that the ergodic decomposition of a quasi-regular Dirichlet space is unique up to a unique isomorphism of the indexing space. Furthermore, every unitary order isomorphism intertwining two quasi-regular Dirichlet spaces is decomposable over their ergodic decompositions up to conjugation via an isomorphism of the corresponding indexing spaces.

1. Introduction

How much geometric information can be recovered from associated analytic or probabilistic objects is the underlying issue for a range of questions most prominently embodied by M. Kac's *Can One Hear the Shape of a Drum?* [9]. Here we take up this issue and investigate how much of the geometric structure of a space is encoded by an associated Markov semigroup or, equivalently, by a Markov process. To be more precise, let us briefly introduce the setting. For $i = 1, 2$, let $(X^i, \mathcal{X}^i, \mathfrak{T}^i, \mu^i)$ be a locally compact Polish σ -finite Radon-measure space and $(T_t^i)_{t \geq 0}$ be a sub-Markovian semigroup on $L^2(\mu^i)$. They are *intertwined* by $U: L^2(\mu^1) \rightarrow L^2(\mu^2)$ if

$$U \circ T_t^1 = T_t^2 \circ U, \quad t \geq 0.$$

In [13] Lenz, Schmidt, and the second named author investigated the properties of intertwined sub-Markovian semigroups in the case when the intertwining operator U is additionally an *order isomorphism*, that is, an invertible linear operator satisfying $Uf \geq 0$ if and only if $f \geq 0$. Suppose that $(T_t^i)_{t \geq 0}$ is the semigroup of a quasi-regular symmetric Dirichlet form $(E^i, \mathcal{D}(E^i))$ on $L^2(\mu^i)$, and thus it is associated with a

Mathematics Subject Classification: 31C25, 37A30, 47D07, 60J35

Keywords: Dirichlet forms, Direct integral, Ergodic decomposition, Intertwining, Order isomorphism.

Research supported by the Austrian Science Fund (FWF) grant F65 at the Institute of Science and Technology Austria and by the European Research Council (ERC) (Grant agreement No. 716117 awarded to Prof. Dr. Jan Maas). L.D.S. gratefully acknowledges funding of his current position by the Austrian Science Fund (FWF) through the ESPRIT Programme (Grant No. 208). M.W. gratefully acknowledges funding of his current position by the Austrian Science Fund (FWF) through the ESPRIT Programme (Grant No. 156).

symmetric Markov process \mathbf{M}^i with state space X^i . It is then the main result of [13] that \mathbf{M}^i (quasi-)determines the topology \mathfrak{T}^i , in the following sense.

Theorem 1.1. *If the irreducible semigroups $(T_t^i)_{t \geq 0}$ are intertwined by an order isomorphism, then the corresponding Dirichlet spaces $(X^i, \mathfrak{T}^i, \mu^i, E^i)$ are quasi-homeomorphic.*

Here, by a *quasi-homeomorphism*¹ $j: (X^1, \mathfrak{T}^1, \mu^1, E^1) \rightarrow (X^2, \mathfrak{T}^2, \mu^2, E^2)$ we mean a map such that for every $\varepsilon > 0$ there exist closed sets $F^i \subset X^i$ with $\text{Cap}(X^i \setminus F^i) < \varepsilon$ and j restricts to a homeomorphism $j|_{F^1}: F^1 \rightarrow F^2$ (cf. [7, Sect. A.4, p. 429]).

Importantly, this completely characterizes how a Markov process \mathbf{M} identifies the topological properties of its state space (X, \mathfrak{T}, μ) , and constitutes a sharp affirmative result in the spirit of M. Kac’s isospectrality question. For example,—as previously established by Arendt, Biegert, and ter Elst in [1]—a Brownian motion on a complete Riemannian manifold (M, g) characterizes both the differential and the Riemannian structure of (M, g) .

The arguments in [13] rely however on the additional technical assumption that the semigroups $(T_t^i)_{t \geq 0}$ (or the Markov processes \mathbf{M}^i) be *irreducible*. In terms of the semigroups, this is to say that

$$T_t \mathbf{1}_A = \mathbf{1}_A T_t \text{ for some } t > 0 \iff \text{either } A \text{ or } A^c \text{ is } \mu\text{-negligible.}$$

Our main contribution in this work is that to extend the results in [13] by removing the irreducibility assumption, thus proving Theorem 1.1 in its full generality. To this end, we employ the machinery of *ergodic decompositions* of Dirichlet spaces developed by the first named author in [3]. An ergodic decomposition of a Dirichlet space $(E^i, \mathscr{D}(E^i))$ is a collection of irreducible Dirichlet spaces $(E_\zeta, \mathscr{D}(E_\zeta))$ indexed by a measure space (Z, \mathcal{Z}, ν) decomposing $(E, \mathscr{D}(E))$ as the direct integral

$$E = \int_Z^\oplus E_\zeta \, d\nu(\zeta).$$

Extending the uniqueness results in [3], we show that ergodic decompositions are unique up to unique automorphisms of their indexing spaces Z (Theorem 4.4). Profiting this fact, we prove as our main result (Theorem 4.5) that two Dirichlet spaces $(E^i, \mathscr{D}(E^i))$ are intertwined by order isomorphisms U if and only if their ergodic decompositions $\zeta \mapsto (E_\zeta^i, \mathscr{D}(E_\zeta^i))$ are intertwined by an order isomorphism U_ζ , in which case U is decomposable over the ergodic decompositions and represented as a direct integral of the operators U_ζ .

¹In the rest of the paper, by a *quasi-homeomorphism* we shall mean a map satisfying the more restrictive conditions in [2, Dfn. 3.1].

2. Direct integrals

In this section, we review some of the basic theory of direct integrals of Hilbert spaces and quadratic forms for later use. More detailed accounts of the material discussed here can be found in [3, 5]. We refer to [6] for all measure-theoretical statements.

2.1. Measure spaces

To formulate the ergodic decomposition of Dirichlet forms in full generality, some rather technical aspects of measure theory are required. For the reader's convenience, let us recall the relevant definitions, starting with some classes of measurable spaces that we will use.

Definition 2.1. (Measurable spaces) A measurable space (X, \mathcal{X}) is

- *separable* if \mathcal{X} contains all singletons in X ;
- *countably separated* if there exists a countable family of sets in \mathcal{X} separating points in X ;
- *countably generated* if there exists a countable family of sets in \mathcal{X} generating \mathcal{X} as a σ -algebra;
- a *standard Borel space* if there exists a Polish topology \mathfrak{T} on X so that \mathcal{X} coincides with the Borel σ -algebra induced by \mathfrak{T} .

A σ -finite measure space (X, \mathcal{X}, μ) is *standard* if there exists a μ -conegligible set $X_0 \in \mathcal{X}$ such that X_0 is a standard Borel space when regarded as a measurable subspace of (X, \mathcal{X}) . We denote by $(X, \mathcal{X}^\mu, \hat{\mu})$ the (Carathéodory) completion of (X, \mathcal{X}, μ) . A Hausdorff topological measure space $(X, \mathfrak{T}, \mathcal{X}, \mu)$ is a *Radon-measure space* if \mathcal{X}^μ coincides with the μ -completion of the Borel σ -algebra and $\hat{\mu}$ is a Radon measure, i.e., it is locally finite and inner regular with respect to the compact sets.

A $[-\infty, \infty]$ -valued function is called μ -measurable if it is measurable w.r.t. \mathcal{X}^μ . For measures μ_1, μ_2 we write $\mu_1 \sim \mu_2$ to indicate that μ_1 and μ_2 are equivalent, i.e., mutually absolutely continuous.

Let us now recall the notion of a perfect measure space together with some of its properties. A measure space (X, \mathcal{X}, μ) is *perfect* (e.g., [6, 342K]) if whenever $f: X \rightarrow \mathbb{R}$ is measurable and $A \in \mathcal{X}$ with $\mu A > 0$, then there exists a compact set $K \subset f(A)$ with $f_\# \mu(K) > 0$. Here, and everywhere in the following, $f_\# \mu$ is the push-forward measure of μ via f .

The following properties of perfect measure spaces are easily verified (cf. [6, 342X]). For a proof of (v), see [6, 416W(a)].

Lemma 2.2. *Let (X, \mathcal{X}, μ) be any measure space. Then,*

- (i) (X, \mathcal{X}, μ) is perfect if and only if so is its completion;
- (ii) if (X, \mathcal{X}, μ) is perfect and \mathcal{X}_0 is a σ -subalgebra of \mathcal{X} , then the restricted measure space $(X, \mathcal{X}_0, \mu|_{\mathcal{X}_0})$ is perfect;
- (iii) if (X, \mathcal{X}, μ) is perfect and λ has a density w.r.t. μ , then $(X, \mathcal{X}, \lambda)$ is perfect;

- (iv) if (Z, \mathcal{Z}) is a measurable space, $f: X \rightarrow Z$ is \mathcal{X}/\mathcal{Z} -measurable and (X, \mathcal{X}, μ) is perfect, then $(Z, \mathcal{Z}, f_{\#}\mu)$ is perfect;
- (v) every Radon-measure space is perfect.

Definition 2.3. Let (X, \mathcal{X}, μ) be a σ -finite measure space, and $\mathcal{X}^* \subset \mathcal{X}$ be a countably generated σ -subalgebra. We say that:

- \mathcal{X} is μ -essentially countably generated by \mathcal{X}^* if for each $A \in \mathcal{X}$ there is $A^* \in \mathcal{X}^*$ with $\mu(A \Delta A^*) = 0$;
- \mathcal{X} is μ -essentially countably generated if it is so by some \mathcal{X}^* as above.

Definition 2.4. (Almost isomorphisms) For $i \in \{1, 2\}$ let (X^i, \mathcal{X}^i) be a measurable space, and \mathcal{N}^i be a σ -ideal of $\mathcal{X}^i \cup \mathcal{N}^i := \{A \cup N : A \in \mathcal{X}^i, N \in \mathcal{N}^i\}$. We say that the triples $(X^i, \mathcal{X}^i, \mathcal{N}^i)$ are *almost isomorphic* if there exist sets $N^i \in \mathcal{N}^i$ so that the spaces $X^i \setminus N^i$ are strictly isomorphic when endowed with the restriction to $X^i \setminus N^i$ of the σ -algebra $\mathcal{X}^i \cup \mathcal{N}^i$ and of the σ -ideal \mathcal{N}^i .

For $i \in \{1, 2\}$ let $(X^i, \mathcal{X}^i, \mu^i)$ be measure spaces. We say that they are *almost isomorphic* if there exist μ^i -negligible sets $N^i \subset X^i$ so that the spaces $X^i \setminus N^i$ are strictly isomorphic when endowed with the restriction to $X^i \setminus N^i$ of the completed σ -algebra $(\mathcal{X}^i)^{\mu^i}$, of the completed measure $\hat{\mu}^i$, and of the σ -ideal of μ^i -negligible sets \mathcal{N}^{μ^i} .

2.2. Quadratic forms

Every Hilbert space is assumed to be *separable* and a real vector space.

Let $(H, \|\cdot\|)$ be a Hilbert space. For a bounded operator $B: H \rightarrow H$ we write $\|B\|_{\text{op}, H}$ for the operator norm of B ; the subscript H is omitted whenever apparent from context. A *quadratic form* Q on H is a map $Q: H \rightarrow [0, \infty]$ such that

$$Q(u + v) + Q(u - v) = 2Q(u) + 2Q(v), \quad u, v \in H.$$

We always assume that the domain $\mathcal{D}(Q) = \{u \in H : Q(u) < \infty\}$ is dense in H and write $(Q, \mathcal{D}(Q))$ for Q when we want to make the domain explicit.

For every quadratic form Q there exists a unique symmetric bilinear form whose diagonal coincides with Q . This bilinear form will also be denoted by Q , and we will use the term quadratic form for both objects interchangeably.

Additionally, for every $\alpha > 0$ we set

$$Q_\alpha(u) := Q(u) + \alpha \|u\|^2, \quad u \in H.$$

For $\alpha > 0$, we let $\mathcal{D}(Q)_\alpha$ be the completion of $\mathcal{D}(Q)$, endowed with the Hilbert norm $Q_\alpha^{1/2}$.

To every closed quadratic form $(Q, \mathcal{D}(Q))$ one can associate a unique non-negative self-adjoint operator $-L$ such that $\mathcal{D}(\sqrt{-L}) = \mathcal{D}(Q)$ and $Q(u, v) = \langle -Lu \mid v \rangle$ for all $u, v \in \mathcal{D}(L)$. We denote the associated strongly continuous contraction semigroup by $T_t := e^{tL}$, $t > 0$.

2.3. Direct integrals of Hilbert spaces

We recall the main definitions concerning direct integrals of separable Hilbert spaces, referring to [5, Sect. II.1, II.2] for a systematic treatment.

Definition 2.5. (Measurable fields, [5, Sect. II.1.3, Dfn. 1, p. 164]) Let (Z, \mathcal{Z}, ν) be a σ -finite measure space, $(H_\zeta)_{\zeta \in Z}$ be a family of separable Hilbert spaces, and F be the linear space $F := \prod_{\zeta \in Z} H_\zeta$. We say that $\zeta \mapsto H_\zeta$ is a ν -measurable field of Hilbert spaces (with underlying space S) if there exists a linear subspace S of F with

- (a) for every $u \in S$, the function $\zeta \mapsto \|u_\zeta\|_\zeta$ is ν -measurable;
- (b) if $v \in F$ is such that $\zeta \mapsto \langle u_\zeta | v_\zeta \rangle_\zeta$ is ν -measurable for every $u \in S$, then $v \in S$;
- (c) there exists a sequence $(u_n)_n \subset S$ such that $(u_{n,\zeta})_n$ is a total sequence² in H_ζ for every $\zeta \in Z$.

Any such S is called a space of ν -measurable vector fields. Any sequence in S possessing property (c) is called a fundamental sequence.

Proposition 2.6. ([5, Sect. II.1.4, Prop. 4, p. 167]) Let \mathcal{S} be a family of functions satisfying Definition 2.5 (a) and (c). Then, there exists exactly one space of ν -measurable vector fields S so that $\mathcal{S} \subset S$.

Definition 2.7. (Direct integrals, [5, Sect. II.1.5, Prop. 5, p. 169]) A ν -measurable vector field u is called (ν -)square-integrable if

$$\|u\| := \left(\int_Z \|u_\zeta\|_\zeta^2 \, d\nu(\zeta) \right)^{1/2} < \infty. \tag{2.1}$$

Two square-integrable vector fields u, v are called (ν -)equivalent if $\|u - v\| = 0$. The space H of equivalence classes of square-integrable vector fields, endowed with the non-reabeled quotient norm $\|\cdot\|$, is a Hilbert space [5, Sect. II.1.5, Prop. 5(i), p. 169], called the direct integral of $\zeta \mapsto H_\zeta$ (with underlying space S) and denoted by

$$H = \int_Z^{S^\oplus} H_\zeta \, d\nu(\zeta). \tag{2.2}$$

The superscript ‘ S ’ is omitted whenever S is clear from context.

In the following, it will occasionally be necessary to distinguish an element u of H from one of its representatives modulo ν -equivalence, say \hat{u} in S . In this case, we shall write $u = [\hat{u}]_H$.

Lemma 2.8. ([3, Lem. 2.6]) If (Z, \mathcal{Z}) is σ -finite countably generated, then H in (2.2) is separable.

We now turn to measurable fields of bounded operators.

²A sequence in a Banach space B is called total if the strong closure of its linear span coincides with B .

Definition 2.9. (Measurable fields of bounded operators, decomposable operators) Let H be as in (2.2). A field of bounded operators $\zeta \mapsto B_\zeta \in \mathcal{B}(H_\zeta)$ is called ν -measurable (with underlying space S) if $\zeta \mapsto B_\zeta u_\zeta \in H_\zeta$ is a ν -measurable vector field for every ν -measurable vector field u . Set $\|B_\zeta\|_{\text{op},\zeta} := \|B_\zeta\|_{\text{op},H_\zeta}$. A ν -measurable vector field of bounded operators is called ν -essentially bounded if ν -esssup $_{\zeta \in Z} \|B_\zeta\|_{\text{op},\zeta} < \infty$. A bounded operator $B \in \mathcal{B}(H)$ is called decomposable if Bu is represented by a ν -essentially bounded ν -measurable field of bounded operators $\zeta \mapsto B_\zeta$, in which case we write

$$B = \int_Z^\oplus B_\zeta \, d\nu(\zeta).$$

2.4. Direct integrals of quadratic forms

We briefly recall all the relevant notions concerning direct integrals of quadratic forms according to [3].

Definition 2.10. (Direct integral of quadratic forms) Let (Z, \mathcal{Z}, ν) be a σ -finite countably generated measure space. For $\zeta \in Z$ let (Q_ζ, D_ζ) be a closable (densely defined) quadratic form on a Hilbert space H_ζ . We say that $\zeta \mapsto (Q_\zeta, D_\zeta)$ is a ν -measurable field of quadratic forms on Z if

- (a) $\zeta \mapsto H_\zeta$ is a ν -measurable field of Hilbert spaces on Z with underlying space S_H ;
- (b) $\zeta \mapsto \mathcal{D}(Q_\zeta)_1$ is a ν -measurable field of Hilbert spaces on Z with underlying space S_Q ;
- (c) S_Q is a linear subspace of S_H under the canonical identification of $\mathcal{D}(Q_\zeta)$ with a subspace of H_ζ .

We further denote by

$$Q = \int_Z^{S_Q \oplus} Q_\zeta \, d\nu(\zeta)$$

the direct integral of $\zeta \mapsto (Q_\zeta, \mathcal{D}(Q_\zeta))$, i.e., the quadratic form defined on H as in (2.2) given by

$$\begin{aligned} \mathcal{D}(Q) &:= \left\{ [\hat{u}]_H : \hat{u} \in S_Q, \int_Z Q_{\zeta,1}(\hat{u}_\zeta) \, d\nu(\zeta) < \infty \right\}, \\ Q(u, v) &:= \int_Z Q_\zeta(u_\zeta, v_\zeta) \, d\nu(\zeta), \quad u, v \in \mathcal{D}(Q). \end{aligned} \tag{2.3}$$

Remark 2.11. (Separability) It is implicit in our definition of ν -measurable field of Hilbert spaces that H_ζ is separable for every $\zeta \in Z$. As a consequence, $\mathcal{D}(Q_\zeta)_1$ is $(Q_\zeta)_1^{1/2}$ -separable.

Proposition 2.12. ([3, Prop. 2.13]) Let $(Q, \mathcal{D}(Q))$ be a direct integral of quadratic forms. Then,

- (i) $(Q, \mathcal{D}(Q))$ is a densely defined closed quadratic form on H ;

- (ii) $\zeta \mapsto T_{\zeta,t}$ is ν -measurable fields of bounded operators for every $t > 0$;
- (iii) the semigroup associated with Q is given by

$$T_t := \int_Z^{S_H \oplus} T_{\zeta,t} \, d\nu(\zeta), \quad t > 0. \tag{2.4}$$

Under the assumptions of Proposition 2.12, assertion (i) of the same Proposition implies that the space of ν -measurable vector fields S_H is uniquely determined by S_Q as a consequence of Proposition 2.6. Thus, everywhere in the following when referring to a direct integral of quadratic forms we shall—with abuse of notation—write S in place of both S_H and S_Q .

2.5. Direct-integral representation of L^2 -spaces

In order to introduce direct-integral representations of Dirichlet forms, we need to construct direct integrals of concrete Hilbert spaces in such a way to additionally preserve the Riesz structure of Lebesgue spaces implicitly used to phrase the sub-Markovianity property (2.11). To this end, we shall need the concept of a disintegration of measures.

Disintegrations. Let (X, \mathcal{X}, μ) and (Z, \mathcal{Z}, ν) be (nontrivial) measure spaces. A map $s : (X, \mathcal{X}) \rightarrow (Z, \mathcal{Z})$ is *inverse-measure-preserving* if $s_{\#}\mu := \mu \circ s^{-1} = \nu$.

Definition 2.13. (Disintegrations [6, 452E]) A *pseudo-disintegration* of μ over ν is any family of non-zero measures $(\mu_\zeta)_{\zeta \in Z}$ on (X, \mathcal{X}) so that $\zeta \mapsto \mu_\zeta A$ is ν -measurable and

$$\mu A = \int_Z \mu_\zeta A \, d\nu(\zeta), \quad A \in \mathcal{X}.$$

A pseudo-disintegration is

- *separated* if there exists a family of pairwise disjoint sets $\{A_\zeta\}_{\zeta \in Z} \subset \mathcal{X}^\mu$ so that A_ζ is μ_ζ -conegligible for ν -a.e. $\zeta \in Z$, henceforth called a *separating family* for $(\mu_\zeta)_{\zeta \in Z}$;
- *s-separated* if it is separated and there exists a \mathcal{X}/\mathcal{Z} -measurable map $s : X \rightarrow Z$ so that $\{s^{-1}(\zeta)\}_{\zeta \in Z}$ is a separating family;
- *consistent with s* if

$$\mu(A \cap s^{-1}(B)) = \int_B \mu_\zeta A \, d\nu(\zeta), \quad A \in \mathcal{X}, B \in \mathcal{Z} \tag{2.5}$$

- *strongly consistent with s* if it is consistent with s and $s^{-1}(\zeta)$ is μ_ζ -conegligible for ν -a.e. $\zeta \in Z$.

A *disintegration* of μ over ν is a pseudo-disintegration additionally so that μ_ζ is a subprobability measure for every $\zeta \in Z$. A disintegration is

- *ν -essentially unique* if the measures μ_ζ are uniquely determined for ν -a.e. $\zeta \in Z$.

If $(\mu_\zeta)_{\zeta \in Z}$ is a pseudo-disintegration of μ over ν , then

$$\int_X g \, d\mu = \int_Z \int_X g(x) \, d\mu_\zeta(x) \, d\nu(\zeta) \tag{2.6}$$

whenever the left-hand side makes sense, [6, 452F]. We note that a disintegration $(\mu_\zeta)_{\zeta \in Z}$ of μ over ν strongly consistent with a map s is automatically separated, with separating family $\{s^{-1}(\zeta)\}_{\zeta \in Z}$.

Direct integrals and disintegrations. Let (X, \mathcal{X}, μ) be σ -finite standard, (Z, \mathcal{Z}, ν) be σ -finite countably generated, and $(\mu_\zeta)_{\zeta \in Z}$ be a pseudo-disintegration of μ over ν . Denote by

- $\mathcal{L}^0(\mu)$ the space of μ -measurable real-valued functions (*not*: μ -classes) on X ;
- $\mathcal{L}^\infty(\mu)$ the space of uniformly bounded (*not*: μ -essentially uniformly bounded) functions in $\mathcal{L}^0(\mu)$;
- $\mathcal{L}^p(\mu)$ the space of p -integrable functions in $\mathcal{L}^0(\mu)$.

For a family $\mathcal{A} \subset \mathcal{L}^0(\mu)$, let $[\mathcal{A}]_\mu$ denote the family of the corresponding μ -classes.

Let now $F := \prod_{\zeta \in Z} L^2(\mu_\zeta)$. The *diagonal embedding* of $\mathcal{L}^2(\mu)$ into F , regarded up to μ_ζ -classes, is the map $\delta: f \mapsto (\zeta \mapsto \delta(f)_\zeta)$, where

$$\delta(f)_\zeta := \begin{cases} [f]_{\mu_\zeta} & \text{if } f \in \mathcal{L}^2(\mu_\zeta), \\ \mathbf{0}_{L^2(\mu_\zeta)} & \text{otherwise.} \end{cases} \tag{2.7}$$

Since $f \in L^2(\mu)$, we have $\delta(f)_\zeta = [f]_{\mu_\zeta}$ for ν -a.e. $\zeta \in Z$ by (2.6), and δ therefore is well defined as linear morphism mapping μ -classes to H -classes (see Proposition 2.15). Now, assume that

$$\mathcal{A} \text{ is a linear subspace of } \mathcal{L}^2(\mu), \text{ and } [\mathcal{A}]_\mu \text{ is dense in } L^2(\mu). \tag{2.8}$$

Since $[\mathcal{A}]_\mu$ is dense in $L^2(\mu)$ and the latter is separable, then there exists a countable family $\mathcal{U} \subset \mathcal{A}$ so that $[\mathcal{U}]_{\mu_\zeta}$ is total in $L^2(\mu_\zeta)$ for ν -a.e. $\zeta \in Z$. Thus for every \mathcal{A} as in (2.8) there exists a unique space of ν -measurable vector fields $S = S_{\mathcal{A}}$ containing $\delta(\mathcal{A})$, generated by $\delta(\mathcal{A})$ in the sense of Proposition 2.6. We denote by H the corresponding direct integral of Hilbert spaces

$$H := \int_Z^{\oplus} L^2(\mu_\zeta) \, d\nu(\zeta). \tag{2.9}$$

Remark 2.14. (cf. [8, Sect. 7.2, p. 84]) The direct integral H constructed in (2.9) is a *Banach lattice* (e.g., [6, 354A(b)]) for the order

$$h \geq \mathbf{0}_H \iff h_\zeta \geq \mathbf{0}_{L^2(\mu_\zeta)} \text{ for } \nu\text{-a.e. } \zeta \in Z.$$

Proposition 2.15. ([3, Prop. 2.25]) *Let (X, \mathcal{X}, μ) be σ -finite standard, (Z, \mathcal{Z}, ν) be σ -finite countably generated, and $(\mu_\zeta)_{\zeta \in Z}$ be a pseudo-disintegration of μ over ν . Then, the morphism*

$$\iota: L^2(\mu) \longrightarrow H := \int_Z^{\oplus} L^2(\mu_\zeta) \, d\nu(\zeta), \quad [f]_\mu \longmapsto [\delta(f)]_H \tag{2.10}$$

- (i) is well defined, linear, and an isometry of Hilbert spaces, additionally unitary if $(\mu_\zeta)_{\zeta \in Z}$ is separated;
- (ii) is a Riesz homomorphism (e.g., [6, 351H]). In particular,
 - for each $f \in L^2(\mu)$, it holds that $(\iota[f]_\mu)_\zeta \geq \mathbf{0}_{L^2(\mu_\zeta)}$ for v -a.e. $\zeta \in Z$ if and only if $f \geq 0$ μ -a.e.;
 - for each $f, g \in L^2(\mu)$, it holds that $(\iota[f \wedge g]_\mu)_\zeta = (\iota[f]_\mu)_\zeta \wedge (\iota[g]_\mu)_\zeta$ for v -a.e. $\zeta \in Z$.

2.6. Dirichlet forms

We recall a standard setting for the theory of Dirichlet forms, following [15].

Assumption 2.16. The quadruple $(X, \mathfrak{T}, \mathcal{X}, \mu)$ is so that (X, \mathfrak{T}) is a metrizable Luzin space with Borel σ -algebra \mathcal{X} and $\hat{\mu}$ is a Radon measure on $(X, \mathfrak{T}, \mathcal{X}^\mu)$ with full support.

By [6, 415D(iii), 424G] any space (X, \mathcal{X}, μ) satisfying Assumption 2.16 is σ -finite standard. The *support* of a (μ) -measurable function $f: X \rightarrow \mathbb{R}$ (possibly defined only on a μ -conegligible set) is defined as the support of the measure $|f| \cdot \mu$. Every such f has a support, independent of the μ -representative of f , cf. [15, p. 148].

A closed positive semi-definite quadratic form $(Q, \mathcal{D}(Q))$ on $L^2(\mu)$ is a (*symmetric*) *Dirichlet form* if

$$f \in \mathcal{D}(Q) \implies f^+ \wedge \mathbf{1} \in \mathcal{D}(Q) \quad \text{and} \quad Q(f^+ \wedge \mathbf{1}) \leq Q(f). \tag{2.11}$$

We shall denote Dirichlet forms by $(E, \mathcal{D}(E))$. A Dirichlet form $(E, \mathcal{D}(E))$ is *regular* if (X, \mathfrak{T}) is (additionally) locally compact and $\mathcal{D}(E) \cap C_0(X)$ is both $E_1^{1/2}$ -dense in $\mathcal{D}(E)$ and uniformly dense in $C_0(X)$.

Finally, we are interested in the notion of invariant sets of a Dirichlet form $(E, \mathcal{D}(E))$ on $L^2(\mu)$. We say that $A \subset X$ is *E-invariant* if it is μ -measurable and any of the following equivalent conditions holds.

- $T_t(\mathbf{1}_A f) = \mathbf{1}_A T_t f$ μ -a.e. for any $f \in L^2(\mu)$ and $t > 0$;
- $T_t(\mathbf{1}_A f) = 0$ μ -a.e. on A^c for any $f \in L^2(\mu)$ and $t > 0$;
- $\mathbf{1}_A f \in \mathcal{D}(E)$ for any $f \in \mathcal{D}(E)$ and

$$E(f, g) = E(\mathbf{1}_A f, \mathbf{1}_A g) + E(\mathbf{1}_{A^c} f, \mathbf{1}_{A^c} g), \quad f, g \in \mathcal{D}(E). \tag{2.12}$$

The form $(E, \mathcal{D}(E))$ is *irreducible* if every invariant set is either negligible or conegligible.

2.7. Direct integrals of Dirichlet forms

Let (X, \mathcal{X}, μ) be σ -finite standard, (Z, \mathcal{Z}, ν) be σ -finite countably generated, and $(\mu_\zeta)_{\zeta \in Z}$ be a pseudo-disintegration of μ over ν . Further let $\zeta \mapsto (Q_\zeta, \mathcal{D}(Q_\zeta))$ be a ν -measurable field of quadratic forms, each densely defined in $L^2(\mu_\zeta)$ with separable domain, and denote by $(Q, \mathcal{D}(Q))$ their direct integral in the sense of Definition 2.10.

Definition 2.17. ([3, Dfn. 2.26]) We say that $(Q, \mathcal{D}(Q))$ is *compatible* with the pseudo-disintegration $(\mu_\zeta)_{\zeta \in Z}$ if the space S_Q underlying $\zeta \mapsto \mathcal{D}(Q_\zeta)_1$ is of the form $S_{\mathcal{A}}$ for some \mathcal{A} as in (2.8) and additionally satisfying $\mathcal{A} \subset \mathcal{D}(Q)$.

Proposition 2.18. ([3, Prop. 2.29]) Let (X, \mathcal{X}, μ) be σ -finite standard, (Z, \mathcal{Z}, ν) be σ -finite countably generated, and $(\mu_\zeta)_{\zeta \in Z}$ be a separated pseudo-disintegration of μ over ν . Further let $(E, \mathcal{D}(E))$ be a direct integral of quadratic forms $\zeta \mapsto (E_\zeta, \mathcal{D}(E_\zeta))$ compatible with $(\mu_\zeta)_{\zeta \in Z}$. Then, $(E, \mathcal{D}(E))$ is a Dirichlet form on $L^2(\mu)$ if and only if $(E_\zeta, \mathcal{D}(E_\zeta))$ is so on $L^2(\mu_\zeta)$ for ν -a.e. $\zeta \in Z$.

Proposition 2.18 motivates the following definition.

Definition 2.19. A quadratic form $(E, \mathcal{D}(E))$ on $L^2(\mu)$ is a *direct integral of Dirichlet forms* $\zeta \mapsto (E_\zeta, \mathcal{D}(E_\zeta))$ if it is a direct integral of quadratic forms $\zeta \mapsto (E_\zeta, \mathcal{D}(E_\zeta))$, for each ζ the form $(E_\zeta, \mathcal{D}(E_\zeta))$ is a Dirichlet form on $L^2(\mu_\zeta)$, and the direct integral is additionally compatible with the *separated* pseudo-disintegration $(\mu_\zeta)_\zeta$ in the sense of Definition 2.17.

We refer to [3] for further comments on direct integrals of Dirichlet forms and related notions.

3. Order isomorphisms and intertwining operators

In this section, we first recall the notion of order isomorphisms between L^2 -spaces together with their main representation theorem, which shows that they are weighted composition operators. In the second part, we study intertwining operators and give a first result connecting intertwining operators and direct integral decompositions (Proposition 3.7).

Definition 3.1. (Order-preserving operators, order isomorphisms) For $i = 1, 2$ let $(X^i, \mathcal{X}^i, \mu^i)$ be σ -finite countably generated and countably separated. A linear operator $U: L^2(\mu^1) \rightarrow L^2(\mu^2)$ is *order-preserving* (also: *positivity-preserving*) if $f \geq 0$ implies $Uf \geq 0$ for each $f \in L^2(\mu^1)$. An *order isomorphism* is an invertible order-preserving linear operator with order-preserving inverse.

The structure of order isomorphisms between L^2 -spaces is characterized by the following Banach–Lamperti-type theorem.

Proposition 3.2. (Order isomorphism as weighted composition operator) *If $U: L^2(X^1, \mu^1) \rightarrow L^2(X^2, \mu^2)$ is an order isomorphism, then there exist a measurable map $h: X^2 \rightarrow (0, \infty)$ and an $\mathcal{X}^2/\mathcal{X}^1$ -measurable almost isomorphism $\tau: X^2 \rightarrow X^1$ such that*

$$Uf = h \cdot (f \circ \tau), \quad f \in L^2(X^1, \mu^1).$$

The maps h and τ are unique up to equality almost everywhere.

Proof. See [16, Thm. 5.1] for existence in the case of finite measures μ^i . For the general case argue as follows. Since $(X^i, \mathcal{X}^i, \mu^i)$ is σ -finite standard, there exists a function f_i satisfying $f_i > 0$ μ^i -a.e., and $\|f_i\|_{L^2(\mu^i)} = 1$. Thus, the multiplication operator $M_i := M_{f_i^{-1/2}} : L^2(\mu^i) \rightarrow L^2(f_i \mu^i)$ is an order isomorphism with (order) inverse $M_i^{-1} = M_{f_i^{1/2}}$, and therefore the operator $U' := M_2 \circ U \circ M_1^{-1} : L^2(f_1 \mu^1) \rightarrow L^2(f_2 \mu^2)$ is an order isomorphism as well. Applying the assertion for U' yields maps h' and τ' . Finally, letting $h := f_2^{-1/2} \circ \tau' \cdot f_1^{1/2} \cdot h'$ and $\tau := \tau'$ yields the assertion in the σ -finite case.

For uniqueness, one can similarly reduce to the case of finite measure. Then $h = U1$ is determined uniquely up to equality a.e. Moreover, if $\tau_1, \tau_2 : X^2 \rightarrow X^1$ are $\mathcal{X}^2/\mathcal{X}^1$ -measurable maps such that $h \cdot (f \circ \tau_1) = h \cdot (f \circ \tau_2)$ for all $f \in L^2(X^1, \mu^1)$, then $\mu^1(\tau_1^{-1}(B) \cap \tau_2^{-1}(B)) = 0$ for all $B \in \mathcal{X}^2$. From [6, 343F] it follows that $\tau_1 = \tau_2$ μ^2 -a.e. □

The maps h and τ associated with an order isomorphism U according to the previous proposition are the main players in the present article. We call h the *scaling* and τ the *transformation* associated with U .

Lemma 3.3. (Adjoint of an order isomorphism, [13, Lem. 1.2]) *Let $U : L^2(X^1, \mu^1) \rightarrow L^2(X^2, \mu^2)$ be an order isomorphism with associated scaling h and transformation τ . Then $\tau_{\#}\mu^2$ and μ^1 are mutually absolutely continuous. Moreover, the adjoint of U is given by*

$$U^* : L^2(X^2, \mu^2) \rightarrow L^2(X^1, \mu^1), \quad U^*g = \frac{d(\tau_{\#}\mu^2)}{d\mu^1}(hg) \circ \tau^{-1}, \quad g \in L^2(\mu^2).$$

Definition 3.4. (Intertwining operators) For $i = 1, 2$ let $V^{(i)}$ be a Banach space and $(A^{(i)}, \mathcal{D}(A^{(i)}))$ be a (possibly unbounded) densely defined operator on $V^{(i)}$. An invertible bounded linear operator $U : V^{(1)} \rightarrow V^{(2)}$ is said to *intertwine* $A^{(1)}$ and $A^{(2)}$ if

$$U\mathcal{D}(A^{(1)}) = \mathcal{D}(A^{(2)}) \quad \text{and} \quad UA^{(1)}f = A^{(2)}Uf, \quad f \in \mathcal{D}(A^{(1)}).$$

Two families of operators $(A_t^{(i)})_{t \in T}, i = 1, 2$, are *intertwined* by U if U intertwines $A_t^{(1)}, A_t^{(2)}$ for all $t \in T$.

The following is not difficult to show.

Lemma 3.5. ([11, Prop. A.1]) *Two strongly continuous contraction semigroups on Banach spaces are intertwined by U if and only if so are their generators.*

The next result is a variation of [13, Cor. 2.5], replacing the irreducibility assumption on the semigroups with the unitariness of the intertwining operator U . A proof is analogous, and therefore it is omitted.

Proposition 3.6. For $i = 1, 2$ let $(X^i, \mathcal{X}^i, \mu^i)$ be σ -finite standard, $(T_t^{(i)})_{t \geq 0}$ be a sub-Markovian semigroup on $L^2(\mu^i)$ with corresponding Dirichlet form $(E^i, \mathcal{D}(E^i))$, and $U : L^2(\mu^1) \rightarrow L^2(\mu^2)$ be a unitary order isomorphism intertwining $(T_t^{(1)})_{t \geq 0}$ and $(T_t^{(2)})_{t \geq 0}$. Then,

$$U \mathcal{D}(E^1) = \mathcal{D}(E^2) \quad \text{and} \quad E^2(Uf, Ug) = E^1(f, g), \quad f, g \in \mathcal{D}(E^1).$$

Proposition 3.7. For $i = 1, 2$ let $(X^i, \mathcal{X}^i, \mu^i)$ be σ -finite standard, (Z, \mathcal{Z}, ν) be σ -finite countably generated, and $(\mu_\zeta^i)_{\zeta \in Z}$ be a separated pseudo-disintegration of μ^i over ν . Further let $\zeta \mapsto U_\zeta$ be a ν -measurable field of bounded operators $U_\zeta : L^2(\mu_\zeta^1) \rightarrow L^2(\mu_\zeta^2)$ in the sense of Definition 2.9, and set

$$U := \int_Z^\oplus U_\zeta \, d\nu(\zeta).$$

Then, U is a bounded operator $U : L^2(\mu^1) \rightarrow L^2(\mu^2)$ and

(i) $U : L^2(\mu^1) \rightarrow L^2(\mu^2)$ is unitary if and only if U_ζ is so for ν -a.e. $\zeta \in Z$, in which case

$$U^{-1} = \int_Z^\oplus U_\zeta^{-1} \, d\nu(\zeta).$$

(ii) $U : L^2(\mu^1) \rightarrow L^2(\mu^2)$ is order-preserving if and only if U_ζ is so for ν -a.e. $\zeta \in Z$.

For $i = 1, 2$ further let $\zeta \mapsto Q_\zeta^i$ be a ν -measurable field of closed quadratic forms $(Q_\zeta^i, \mathcal{D}(Q_\zeta^i))$ on $L^2(\mu_\zeta^i)$ in the sense of Definition 2.10, each with semigroup $(T_{\zeta,t}^{(i)})_{t \geq 0}$, $\zeta \in Z$, and set

$$T_t^{(i)} := \int_Z^\oplus T_{\zeta,t}^{(i)} \, d\nu(\zeta), \quad t > 0.$$

Then, for each fixed $t > 0$, the operator $T_t^{(i)}$ satisfies $T_t^{(i)} \in \mathcal{B}(L^2(\mu^i))$ and

(iii) $U : L^2(\mu^1) \rightarrow L^2(\mu^2)$ intertwines $T_t^{(1)}$ and $T_t^{(2)}$ if and only if U_ζ intertwines $T_{\zeta,t}^{(1)}$ and $T_{\zeta,t}^{(2)}$ for ν -a.e. $\zeta \in Z$.

Proof. By definition of U , we have

$$U : \int_Z^\oplus L^2(\mu_\zeta^1) \, d\nu(\zeta) \longrightarrow \int_Z^\oplus L^2(\mu_\zeta^2) \, d\nu(\zeta).$$

Let $i = 1, 2$. Since the disintegration $(\mu_\zeta^i)_{\zeta \in Z}$ is separated, by Proposition 2.15 (i) there exists a unitary order isomorphism $\iota^{(i)}$ satisfying (2.10). Since $\iota^{(i)}$ is as well an order isomorphism by Proposition 2.15 (ii), in the rest of the proof we may identify U with $\iota^{(2)} \circ U \circ (\iota^{(1)})^{-1} : L^2(\mu^1) \rightarrow L^2(\mu^2)$.

(i) is [5, Sect. II.2.3, Example, p. 182].

(ii) Both implications are a consequence of the definition of the order structure on the direct integral of $\zeta \mapsto L^2(\mu_\zeta)$, together with Proposition 2.15 (i).

(iii) Let $t > 0$ be fixed. Assume first that U_ζ intertwines $T_{\zeta,t}^{(1)}, T_{\zeta,t}^{(2)}$. By [5, §II.2.3, Prop. 3, p. 182],

$$\begin{aligned} U \circ T_t^{(1)} &= \int_Z^\oplus U_\zeta \, d\nu(\zeta) \circ \int_Z^\oplus T_{\zeta,t}^{(1)} \, d\nu(\zeta) = \int_Z^\oplus U_\zeta \circ T_{\zeta,t}^{(1)} \, d\nu(\zeta) \\ &= \int_Z^\oplus T_{\zeta,t}^{(2)} \circ U_\zeta \, d\nu(\zeta) = T_t^{(2)} \circ U. \end{aligned}$$

Vice versa assume that U intertwines $T_t^{(1)}$ and $T_t^{(2)}$. By Proposition 2.12 (iii) and [5, §II.2.3, Prop. 3, p. 182],

$$\int_Z^\oplus U_\zeta \circ T_{\zeta,t}^{(1)} \, d\nu(\zeta) = U \circ T_t^{(1)} = T_t^{(2)} \circ U = \int_Z^\oplus T_{\zeta,t}^{(2)} \circ U_\zeta \, d\nu(\zeta),$$

hence, by a further application of [5, §II.2.3, Prop. 3, p. 182], and by [5, §II.2.3, Prop. 2, p. 181],

$$\begin{aligned} 0 &= \|U \circ T_t^{(1)} - T_t^{(2)} \circ U\|_{\text{op}} = \left\| \int_Z^\oplus (U_\zeta \circ T_{\zeta,t}^{(1)} - T_{\zeta,t}^{(2)} \circ U_\zeta) \, d\nu(\zeta) \right\|_{\text{op}} \\ &= v\text{-esssup}_\zeta \|U_\zeta \circ T_{\zeta,t}^{(1)} - T_{\zeta,t}^{(2)} \circ U_\zeta\|_{\text{op}}, \end{aligned}$$

which concludes the proof. □

4. Intertwining of ergodic decompositions

In this section, we describe the structure of order isomorphisms intertwining Dirichlet forms that are not necessarily irreducible. Informally, one might expect that one can simply decompose the Dirichlet forms into their “connected components” and apply the known result for irreducible Dirichlet forms. However, it is technically nontrivial to make this notion of “connected components” rigorous for abstract Dirichlet forms.

We rely here on the notion of ergodic decompositions introduced in [3]. We will first show that these ergodic decompositions are essentially unique (Theorem 4.4) and that intertwining order isomorphisms carry ergodic decompositions to ergodic decompositions (Theorem 4.5). As a consequence, whenever the Dirichlet forms in question admit ergodic decompositions, intertwining order isomorphisms act componentwise as anticipated by the informal discussion above (Corollary 4.6).

4.1. Ergodic decompositions

Let us first introduce a rigorous definition for the ergodic decomposition of a regular Dirichlet form.

Definition 4.1. Let $(X, \mathfrak{T}, \mathcal{X}, \mu)$ be a locally compact Polish Radon-measure space, and $(E, \mathcal{D}(E))$ be a regular Dirichlet form on $L^2(\mu)$. We say that $(E, \mathcal{D}(E))$ admits an ergodic decomposition $\zeta \mapsto (E_\zeta, \mathcal{D}(E_\zeta))$ indexed by (Z, \mathcal{Z}, ν) if there exist

- (i) a separable countably generated probability space (Z, \mathcal{Z}, ν) and a measurable map $s : (X, \mathcal{X}) \rightarrow (Z, \mathcal{Z})$;
- (ii) a pseudo-disintegration $(\mu_\zeta)_{\zeta \in Z}$ of μ over ν , strongly consistent with s such that if $s^{-1}(\zeta)$ is endowed with the subspace topology and the trace σ -algebra inherited by $(X, \mathfrak{T}, \mathcal{X}^\mu)$, then $(s^{-1}(\zeta), \hat{\mu}_\zeta)$ is a Radon-measure space for ν -a.e. $\zeta \in Z$;
- (iii) a ν -measurable field $\zeta \mapsto (E_\zeta, \mathcal{D}(E_\zeta))$ of regular irreducible Dirichlet forms $(E_\zeta, \mathcal{D}(E_\zeta))$ on $L^2(\mu_\zeta)$;

such that

$$L^2(\mu) = \int_Z^\oplus L^2(\mu_\zeta) \, d\nu(\zeta) \quad \text{and} \quad E = \int_Z^\oplus E_\zeta \, d\nu(\zeta)$$

as a direct integral of Dirichlet forms in the sense of Definition 2.19.

Remark 4.2. Since the pseudo-disintegration $(\mu_\zeta)_{\zeta \in Z}$ in Definition 4.1(ii) is s -separated, the map s is implicitly always assumed to be surjective.

Remark 4.3. The existence of ergodic decompositions has been shown:

- (a) in [3] under the additional assumptions that either μ be a finite measure, or that the form $(E, \mathcal{D}(E))$ be strongly local and admitting a carré du champ operator;
- (b) by K. Kuwae in [12] under the additional assumption that the transition probabilities of the Markov process properly associated to the Dirichlet form be absolutely continuous w.r.t. the reference measure μ .

It is one further result of [12] that, under the same assumption as in (b) above, the set Z indexing the ergodic decomposition is in fact *at most countable*, which greatly simplifies the discussion. This type of ergodic decomposition however rules out some interesting examples, in particular from infinite-dimensional analysis (see [3, §3.4] and references therein).

Let us also point out that, again in the case of (b) above, our main result (i.e., the extension of the results in [13] to the non-irreducible case) has also been obtained by L. Li and H. Lin in [14, Thm. 4.6].

Ergodic decompositions of Dirichlet forms are *essentially projectively unique*, in the sense of the next theorem, expanding the scope of all the projective uniqueness results in [3].

Theorem 4.4. *Let (X, \mathcal{X}, μ) be a locally compact Polish Radon-measure space, and $(E, \mathcal{D}(E))$ be a regular Dirichlet form on $L^2(\mu)$. Further assume that $(E, \mathcal{D}(E))$ admits*

- an ergodic decomposition $\zeta \mapsto (E_\zeta^1, \mathcal{D}(E_\zeta^1))$ indexed by $(Z^1, \mathcal{Z}^1, \nu^1)$;
- an ergodic decomposition $\eta \mapsto (E_\eta^2, \mathcal{D}(E_\eta^2))$ indexed by $(Z^2, \mathcal{Z}^2, \nu^2)$.

Then, there exists an almost isomorphism $\varrho: (Z^1, \mathcal{Z}^1, \mathcal{N}^{v^1}) \rightarrow (Z^2, \mathcal{Z}^2, \mathcal{N}^{v^2})$ and a \mathcal{Z}^2 -measurable function $h: Z^2 \rightarrow (0, \infty)$ so that $\varrho_{\#}v^1 \sim v^2$ and $E_{\zeta}^1 = h(\varrho(\zeta))E_{\varrho(\zeta)}^2$ for v^1 -a.e. $\zeta \in Z^1$.

Proof. Let \mathcal{X}_0 be the family of μ -measurable E -invariant subsets of X , and note that \mathcal{X}_0 is a σ -subalgebra of \mathcal{X}^μ , e.g., [7, Lem. 1.6.1, p. 53]. Let μ_0 be the restriction of $\hat{\mu}$ to (X, \mathcal{X}_0) . By our Assumption 2.16, \mathcal{X} is countably generated, thus \mathcal{X}_0 is μ_0 -essentially countably generated by $\mathcal{X}^* := \mathcal{X} \cap \mathcal{X}_0$.

Further let \mathcal{C} be a special standard core for $(E, \mathcal{D}(E))$ witnessing the compatibility of the direct integral representation, i.e., so that $\mathcal{C} = \mathcal{A}$ as in Definition 2.17. Note that for every $u \in \mathcal{C}$ we may choose $u_{\zeta} \equiv u$ as a μ_{ζ} -representative of the diagonal embedding $\delta(u)_{\zeta}$.

Step 1: Invariant sets. For simplicity of notation, in this step let Z denote either Z^1 or Z^2 , and analogously for all the other symbols.

Claim I: For every $B \in \mathcal{Z}$ the set $A := s^{-1}(B)$ is E -invariant. Since $(\mu_{\zeta})_{\zeta \in Z}$ is s -separated, $(X_{\zeta})_{\zeta \in Z}$, with $X_{\zeta} := s^{-1}(\zeta)$, is a separating family in the sense of Definition 2.13. The multiplication operator $M_{X_{\zeta}} := M_{\mathbf{1}_{X_{\zeta}}}: L^2(\mu_{\zeta}) \rightarrow L^2(\mu_{\zeta})$ satisfies $M_{X_{\zeta}} = \text{id}_{L^2(\mu_{\zeta})}$, hence $T_{\zeta,t}M_{X_{\zeta}} = M_{X_{\zeta}}T_{\zeta,t}$. As a consequence,

$$\begin{aligned} M_A \circ T_t &= \int_B^{\oplus} M_{X_{\zeta}} \, d\nu(\zeta) \circ \int_Z^{\oplus} T_{\zeta,t} \, d\nu(\zeta) = \int_Z^{\oplus} M_{X_{\zeta}} \circ T_{\zeta,t} \, d\nu(\zeta) \\ &= \int_Z^{\oplus} T_{\zeta,t} \circ M_{X_{\zeta}} \, d\nu(\zeta) = \int_Z^{\oplus} T_{\zeta,t} \, d\nu(\zeta) \circ \int_B^{\oplus} M_{X_{\zeta}} \, d\nu(\zeta) \\ &= T_t \circ M_A, \end{aligned}$$

that is, $A := s^{-1}(B)$ is E -invariant.

Claim II: for every E -invariant $A \in \mathcal{X}_0$ there exists $B \in \mathcal{Z}$ with $\mu_0(A \Delta s^{-1}(B)) = 0$. It suffices to show the statement for $A \in \mathcal{X}^*$. Since (Z, \mathcal{Z}) is separable countably generated, it is countably separated, hence by [10, Prop. 12.1(iii)] there exists a separable metrizable topology \mathfrak{T}_Z on Z so that \mathcal{Z} is the Borel σ -algebra generated by \mathfrak{T}_Z . As a consequence, $s: X \rightarrow Z$ is Borel, and therefore $s(A)$ is analytic by [6, 423G(b)], thus \mathcal{Z} -universally measurable by [6, 434D(c)], hence finally ν -measurable. In particular, there exist $B_0 \subset s(A) \subset B_1$ with $B_0, B_1 \in \mathcal{Z}$ and $\nu(B_1 \setminus B_0) = 0$. By the previous claim, we have that $A_0 := s^{-1}(B_0), A_1 := s^{-1}(B_1) \in \mathcal{X}_0$. The conclusion follows since $A_0 \subset A \subset A_1$ by definition, and since $\mu_0(A_1 \setminus A_0) = 0$ combining (2.5) with $\nu(B_1 \setminus B_0) = 0$.

Claim III: $s^*\mathcal{Z} := \{s^{-1}(B) : B \in \mathcal{Z}\}$ is μ_0 -essentially countably generating \mathcal{X}_0 . It suffices to combine the previous two claims with the fact that \mathcal{Z} is countably generated by assumption.

Claim IV: The measure space (Z, \mathcal{Z}, ν) is perfect. We shall use Lemma 2.2 in numerous instances, without explicit mention. As a Radon space $(X, \mathfrak{T}, \mathcal{X}^\mu, \mu)$ is perfect.

Since it is σ -finite, there exists a probability measure λ' equivalent to $\hat{\mu}$ on \mathcal{X}^μ , and the space $(X, \mathcal{X}^{\lambda'}, \lambda')$ is perfect as well. Let $\lambda := \lambda'|_{s^*\mathcal{Z}}$ and $\sigma := s_{\#}\lambda$, and note that the restricted measure space $(X, s^*\mathcal{Z}, \lambda)$ is perfect. Thus (Z, \mathcal{Z}, σ) is perfect, too.

Now, let $B \in \mathcal{Z}$ with $\nu B = 0$. Then, $s^{-1}(B)$ is μ -negligible by (2.5). Since $\lambda \sim \mu$ on $s^*\mathcal{Z}$, the set $s^{-1}(B)$ is as well λ -negligible, hence B is as well σ -negligible, which shows that $\sigma \ll \nu$. Vice versa let $B \in \mathcal{Z}$ with $\sigma B = 0$. Then, $s^{-1}(B)$ is λ -negligible, and therefore μ -negligible. Since μ_ζ is non-zero for every $\zeta \in Z$ by definition of pseudo-disintegration, we conclude again from (2.5) that B is as well ν -negligible, which shows $\nu \ll \sigma$. Thus $\nu \sim \sigma$.

Finally, since both ν and σ are finite measures, ν has a density w.r.t. σ by the Radon–Nikodym Theorem, and therefore (Z, \mathcal{Z}, ν) is perfect since so is (Z, \mathcal{Z}, σ) .

Step 2: Almost isomorphism. Throughout this step, let $i = 1, 2$. By construction, the measure algebra $(\mathfrak{Z}_i, \bar{\sigma}_i)$ of $(Z, \mathcal{Z}^i, \sigma_i)$ is isomorphic to the measure algebra of $(X, s_i^*\mathcal{Z}^i, \lambda^i)$. Furthermore, by *Step 1: Claim III* the σ -algebra \mathcal{Z}^i is μ_0 -essentially countably generating the algebra \mathcal{X}_0 of E -invariant sets. Thus, \mathcal{Z}^i is as well $\lambda_0 := \lambda'|_{\mathcal{X}^*}$ -essentially countably generating \mathcal{X}_0 , and therefore

$$(X, (s_1^*\mathcal{Z}^1)^{\lambda_1}, \hat{\lambda}_1) = (X, (\mathcal{X}^*)^{\lambda_0}, \hat{\lambda}_0) = (X, (s_2^*\mathcal{Z}^2)^{\lambda_2}, \hat{\lambda}_2).$$

As a consequence, since the measure algebra of a measure space coincides with that of its completion, the measure algebras $(\mathfrak{Z}^1, \bar{\sigma}^1)$ and $(\mathfrak{Z}^2, \bar{\sigma}^2)$ are isomorphic. By all of the above, the spaces $(Z^i, (\mathcal{Z}^i)^{\sigma_i}, \hat{\sigma}_i)$ are countably separated perfect complete probability spaces, with isomorphic measure algebras. Therefore, the corresponding measure spaces $(Z^i, \mathcal{Z}^i, \sigma^i)$ are almost isomorphic by [6, 344C], via some almost isomorphism $\varrho: Z^1 \rightarrow Z^2$ satisfying $\varrho_{\#}\sigma^1 = \sigma^2$. Since $\sigma^i \sim \nu^i$ by the proof of *Step 1 Claim IV*, we conclude that $\varrho_{\#}\nu^1 \sim \nu^2$.

Step 3: Forms identifications. Since $\varrho_{\#}\nu^1 \sim \nu^2$, there exists a measurable $h: (Z^2, \mathcal{Z}^2) \rightarrow (0, \infty)$ satisfying $h \cdot \varrho_{\#}\nu^1 = \nu^2$. Now

$$\begin{aligned} \mu(A \cap (s^2)^{-1}(B^2)) &= \int_{B^2} \mu_\eta^2 A \, d\nu^2(\eta) \\ &= \int_{\varrho^{-1}(B^2)} \mu_{\varrho(\zeta)}^2 A \cdot h(\varrho(\zeta))^{-1} \, d\nu^1(\zeta) \end{aligned}, \quad A \in \mathcal{X}, B^2 \in \mathcal{Z}^2,$$
(4.1)

which shows that $(h(\varrho(\zeta))^{-1} \cdot \mu_{\varrho(\zeta)}^2)_{\zeta \in Z^1}$ is a pseudo-disintegration of μ over ν^1 , strongly consistent with $\varrho^{-1} \circ s^2$.

Furthermore, exchanging the roles of ν^1 and ν^2 in *Step 2* we have that $\varrho_{\#}\nu^1 \sim \nu^2$ and $\varrho_{\#}^{-1}\nu^2 \sim \nu^1$. Thus, $f: Z^2 \rightarrow \mathbb{R}$ is ν^2 -measurable if and only if $f \circ \varrho: Z^1 \rightarrow \mathbb{R}$ is ν^1 -measurable. As a consequence, the function $h \circ \varrho: Z^1 \rightarrow (0, \infty)$ is ν^1 -measurable and, since $\eta \mapsto (E_\eta^2, \mathcal{D}(E_\eta^2))$ is a ν^2 -measurable field of quadratic forms, for every $u, v \in \mathcal{C}$ the map

$$\zeta \mapsto E_{\varrho(\zeta), 1}^2(u, v) = E_{\varrho(\zeta)}^2(u_{\varrho(\zeta)}, v_{\varrho(\zeta)}) + \langle u_{\varrho(\zeta)} \mid v_{\varrho(\zeta)} \rangle_{L^2(\mu_{\varrho(\zeta)}^2)}$$

is ν^1 -measurable, which shows that $\zeta \mapsto (h(\rho(\zeta))E_{\varrho(\zeta)}^2, \mathcal{D}(E_{\varrho(\zeta)}^2))$ is a ν^1 -measurable field of quadratic forms on $L^2(\mu_{\varrho(\zeta)}^2)$. The corresponding direct integral form $(\tilde{E}, \mathcal{D}(\tilde{E}))$ is compatible with the disintegration in (4.1) in the sense of Definition 2.17 with underlying space $S_{\mathcal{C}}$, since \mathcal{C} is a core for

$$(h(\rho(\zeta))E_{\varrho(\zeta)}^2, \mathcal{D}(E_{\varrho(\zeta)}^2))$$

for ν^1 -a.e. $\zeta \in Z^1$. Finally, let us show that the form $(\tilde{E}, \mathcal{D}(\tilde{E}))$ coincides with $(E, \mathcal{D}(E))$. It suffices to note that, for all $u, v \in \mathcal{C}$ we have

$$\begin{aligned} \tilde{E}(u, v) &= \int_{Z^1} h(\varrho(\zeta))E_{\varrho(\zeta)}^2(u_{\varrho(\zeta)}, v_{\varrho(\zeta)}) \, d\nu^1(\zeta) \\ &= \int_{Z^2} h(\eta)E_{\eta}^2(u_{\eta}, v_{\eta}) \, d\varrho_{\#}\nu^1 = \int_{Z^2} E_{\eta}^2(u_{\eta}, v_{\eta}) \, d\nu^2(\eta) = E(u, v). \quad \square \end{aligned}$$

4.2. Intertwining

We are now ready to state our main result.

Theorem 4.5. *For $i = 1, 2$, let $(X^i, \mathfrak{A}^i, \mathcal{X}^i, \mu^i)$ be a locally compact Polish Radon-measure space, and $(E^i, \mathcal{D}(E^i))$ be a regular Dirichlet form on $L^2(\mu^i)$ with semigroup $(T_t^{(i)})_{t \geq 0}$. Suppose further that*

- (a) $(E^1, \mathcal{D}(E^1))$ admits ergodic decomposition $\zeta \mapsto (E_{\zeta}^1, \mathcal{D}(E_{\zeta}^1))$ indexed by (Z, \mathcal{Z}, ν) ;
- (b) there exists a unitary order isomorphism $U: L^2(\mu^1) \rightarrow L^2(\mu^2)$ intertwining $(T_t^{(1)})_{t \geq 0}, (T_t^{(2)})_{t \geq 0}$, represented by a scaling h and a transformation τ in the sense of Proposition 3.2.

Finally, set

$$\mu_{\zeta}^2 = \frac{d\mu^2}{d(\tau^{-1})_{\#}\mu^1}(\tau^{-1})_{\#}\mu_{\zeta}^1 \quad \text{for } \nu\text{-a.e. } \zeta \in Z. \tag{4.2}$$

Then,

- (i) $(\mu_{\zeta}^2)_{\zeta \in Z}$ is a separated pseudo-disintegration of μ^2 over (Z, \mathcal{Z}, ν) ;
- (ii) $U: L^2(\mu^1) \rightarrow L^2(\mu^2)$ is decomposable over (Z, \mathcal{Z}, ν) and represented by a ν -measurable field of order isomorphisms $U_{\zeta}: L^2(\mu_{\zeta}^1) \rightarrow L^2(\mu_{\zeta}^2)$.

For $\zeta \in Z$ and $t > 0$ further define an operator $T_{\zeta,t}^{(2)}: L^2(\mu_{\zeta}^2) \rightarrow L^2(\mu_{\zeta}^2)$ by

$$T_{\zeta,t}^{(2)} \circ U_{\zeta} = U_{\zeta} \circ T_{\zeta,t}^{(1)}. \tag{4.3}$$

Then,

- (iii) $(T_{\zeta,t}^{(2)})_{t \geq 0}$ is a sub-Markovian strongly continuous contraction semigroup on $L^2(\mu_{\zeta}^2)$ for ν -a.e. $\zeta \in Z$;

(iv) $(E^2, \mathcal{D}(E^2))$ admits an ergodic decomposition $\zeta \mapsto (E_\zeta^2, \mathcal{D}(E_\zeta^2))$ indexed by (Z, \mathcal{Z}, ν) , where the Dirichlet form $(E_\zeta^2, \mathcal{D}(E_\zeta^2))$ is the image form of $(E_\zeta^1, \mathcal{D}(E_\zeta^1))$ via U_ζ for ν -a.e. $\zeta \in Z$ and the associated semigroup is $(T_{\zeta,t}^{(2)})_{t \geq 0}$.

Proof. Up to (re-)defining τ , resp. τ^{-1} , on a μ^1 -, resp. μ^2 -, negligible set, we can assume without loss of generality that τ is defined everywhere. Note however that τ is neither surjective nor injective.

Step 1: Disintegrations. By Lemma 3.3, the definition of $(\mu_\zeta^2)_{\zeta \in Z}$ in (4.2) is well-posed. By definition of pseudo-disintegration, $\zeta \mapsto \mu_\zeta^1 f$ is ν -measurable for every \mathcal{X}^1 -measurable real-valued f on X^1 . As a consequence, the map

$$\zeta \mapsto \mu_\zeta^2 A = \int_A \frac{d\mu^2}{d(\tau^{-1})_\# \mu^1} d(\tau^{-1})_\# \mu_\zeta^1, \quad A \in \mathcal{X}^2,$$

is as well ν -measurable, by $\mathcal{X}^2/\mathcal{X}^1$ -measurability of τ . Furthermore, by (2.5) for $(\mu_\zeta^1)_{\zeta \in Z}$,

$$\begin{aligned} \int_Z \int_A \frac{d\mu^2}{d(\tau^{-1})_\# \mu^1} d(\tau^{-1})_\# \mu_\zeta^1 d\nu(\zeta) &= \int_Z \int_A \frac{d\mu^2}{d(\tau^{-1})_\# \mu^1} \circ \tau^{-1} d\mu_\zeta^1 d\nu(\zeta) \\ &= \int_A \frac{d\mu^2}{d(\tau^{-1})_\# \mu^1} \circ \tau^{-1} d\mu^1 \\ &= \int_A \frac{d\mu^2}{d(\tau^{-1})_\# \mu^1} d(\tau^{-1})_\# \mu^1 \\ &= \mu^2 A, \end{aligned}$$

that is, the family $(\mu_\zeta^2)_\zeta$ is a pseudo-disintegration of μ^2 over (Z, \mathcal{Z}, ν) . Since τ^{-1} is μ^1 -a.e. injective, it is as well μ_ζ^1 -a.e. injective for ν -a.e. $\zeta \in Z$. As a consequence, the pseudo-disintegration $(\mu_\zeta^2)_\zeta$ is as well separated, which concludes a proof of (i).

Step 2: Direct-integral representations of L^2 -spaces. Since $(\mu_\zeta^1)_{\zeta \in Z}$ is separated by assumption and since $(\mu_\zeta^2)_{\zeta \in Z}$ is separated by (i), Proposition 2.15 yields the direct-integral representations

$$L^2(\mu^i) = \int_Z^{S^i \oplus} L^2(\mu_\zeta^i) d\nu(\zeta), \quad i = 1, 2,$$

for every space of ν -measurable vector fields $S^i = S_{\mathcal{A}^i}$ induced by \mathcal{A}^i as in (2.8). In the following, we shall always choose $\mathcal{A}^1 := \mathcal{C}$ to be a special standard core for the (regular) form $(E^1, \mathcal{D}(E^1))$, and set $\mathcal{A}^2 := \{h \cdot f \circ \tau : f \in \mathcal{A}^1\}$. In light of Proposition 3.6 we have $\mathcal{A}^2 \subset \mathcal{D}(E^2)$. Having fixed the spaces S^i throughout the proof, we omit them from the notation.

As a consequence, if $U : L^2(\mu^1) \rightarrow L^2(\mu^2)$ is decomposable, and represented by a ν -measurable field of operators $\zeta \mapsto U_\zeta$, then $U_\zeta : L^2(\mu_\zeta^1) \rightarrow L^2(\mu_\zeta^2)$ for ν -a.e. $\zeta \in Z$.

Step 3: Decomposability of U . For every $A \in (\mathcal{X}^1)^{\mu^1}$, and every $f \in L^2(\mu^1)$,

$$\begin{aligned} UM_A f &= h \cdot \tau^*(\mathbf{1}_A f) = h \cdot \tau^* \mathbf{1}_A \cdot \tau^* f \\ &= \tau^* \mathbf{1}_A \cdot h \cdot \tau^* f = \tau^* \mathbf{1}_A \cdot Uf = M_{\tau^{-1}(A)} Uf. \end{aligned} \tag{4.4}$$

Now, let A be E^1 -invariant. Combining (4.4) and the definition of invariant set yields

$$\begin{aligned} M_{\tau^{-1}(A)} \circ T_t^{(2)} \circ U &= M_{\tau^{-1}(A)} \circ U \circ T_t^{(1)} \\ &= U \circ M_A \circ T_t^{(1)} \\ &= U \circ T_t^{(1)} \circ M_A \\ &= T_t^{(2)} \circ U \circ M_A \\ &= T_t^{(2)} \circ M_{\tau^{-1}(A)} \circ U, \end{aligned}$$

whence, pre-composing with U^{-1} on both sides yields that $\tau^{-1}(A)$ is E^2 -invariant.

Arguing as in *Step 3* in the proof of [3, Thm. 3.4], it follows that U commutes with all the operators in $\mathcal{B}(L^2(\mu^1))$ diagonalizable over Z in the sense of [5, §II.2.4, Dfn. 3, p. 185]. By the characterization of decomposable operators via diagonalizable operators [5, §II.2.5, Thm. 1, p. 187], the operator U is decomposable, and represented by a ν -measurable field of bounded operators $\zeta \mapsto U_\zeta : L^2(\mu_\zeta^1) \rightarrow L^2(\mu_\zeta^2)$. For ν -a.e. $\zeta \in Z$, the operator U_ζ is additionally an order isomorphism, by the forward implication in Proposition 3.7. This concludes the proof of (ii).

Step 4: Image forms. Since U_ζ is unitary and $(T_{\zeta,t}^{(1)})_{t \geq 0}$ is a strongly continuous contraction semigroup on $L^2(\mu_\zeta^1)$ for ν -a.e. $\zeta \in Z$, the same holds for $(T_{\zeta,t}^{(2)})_{t \geq 0}$. For $t > 0$, let

$$\tilde{T}_t^{(2)} := \int_Z^{\oplus} T_{\zeta,t}^{(2)} d\nu(\zeta).$$

By *Step 2* above, $\tilde{T}_t^{(2)}$ is a bounded operator in $\mathcal{B}(L^2(\mu^2))$. By the decomposability of U shown in *Step 4* and by definition (4.3) of $T_{\zeta,t}^{(2)}$, we may apply the reverse implication in Proposition 3.7(iii) to obtain

$$\tilde{T}_t^{(2)} \circ U = U \circ T_t^{(1)}, \quad t > 0,$$

and conclude that $\tilde{T}_t^{(2)} = T_t^{(2)}$ for each $t > 0$, since U is unitary.

For ν -a.e. $\zeta \in Z$, let $(E_\zeta^2, \mathcal{D}(E_\zeta^2))$ be the quadratic form on $L^2(\mu_\zeta^2)$ corresponding to $(T_{\zeta,t}^{(2)})_{t \geq 0}$.

Step 5: Direct-integral representation. The direct integral representation of $(E^2, \mathcal{D}(E^2))$ as a direct integral of Dirichlet forms follows from that of $(T_t^{(2)})_{t \geq 0}$ by Proposition 2.12, provided we show

- (a) the compatibility of $(E^2, \mathcal{D}(E^2))$ with the representation by $\zeta \mapsto E_\zeta^2$ in the sense of Definition 2.17; and that
- (b) $(E_\zeta^2, \mathcal{D}(E_\zeta^2))$ is a Dirichlet form for ν -a.e. $\zeta \in Z$.

In light of Step 2, in order to show (a) it suffices to show that \mathcal{A}^2 is contained in $\mathcal{D}(E^2)$, which follows from Proposition 3.6. In order to show (b), note that, since $(T_t^{(2)})_{t \geq 0}$ is sub-Markovian, it follows by Proposition 2.18 that $(T_{\zeta,t}^{(2)})_{t \geq 0}$ is sub-Markovian for ν -a.e. $\zeta \in Z$. □

Corollary 4.6. *For $i = 1, 2$ let $(X^i, \mathfrak{T}^i, \mathcal{X}^i, \mu^i)$ be a locally compact Polish Radon-measure space, and $(E^i, \mathcal{D}(E^i))$ be a regular Dirichlet form on $L^2(\mu^i)$ with semigroup $(T_t^{(i)})_{t \geq 0}$, each admitting ergodic decomposition $\zeta \mapsto (E_\zeta^i, \mathcal{D}(E_\zeta^i))$ indexed by $(Z^i, \mathcal{Z}^i, \nu^i)$. Suppose further that there exists a unitary order isomorphism $U : L^2(\mu^1) \rightarrow L^2(\mu^2)$ intertwining $(T_t^{(1)})_{t \geq 0}$, $(T_t^{(2)})_{t \geq 0}$, represented by a scaling h and a transformation τ in the sense of Proposition 3.2. Then, there exists an almost isomorphism $\varrho : (Z^1, \mathcal{Z}^1, \nu^1) \rightarrow (Z^2, \mathcal{Z}^2, \nu^2)$ such that*

- (i) $U : L^2(\mu^1) \rightarrow L^2(\mu^2)$ is decomposable over $(Z^1, \mathcal{Z}^1, \nu^1)$ and represented by a ν^1 -measurable field of order isomorphisms $U_\zeta : L^2(\mu_\zeta^1) \rightarrow L^2(\mu_{\varrho(\zeta)}^2)$;
- (ii) $U^{-1} : L^2(\mu^2) \rightarrow L^2(\mu^1)$ is decomposable over $(Z^2, \mathcal{Z}^2, \nu^2)$. Furthermore, the field of order isomorphisms $(U_\zeta)^{-1} : L^2(\mu_{\varrho(\zeta)}^2) \rightarrow L^2(\mu_\zeta^1)$ is ν^2 -measurable, and it represents U^{-1} ;
- (iii) for ν^1 -a.e. $\zeta \in Z^1$, the order isomorphism U_ζ intertwines the semigroup $(T_{\zeta,t}^{(1)})_{t \geq 0}$ associated to $(E_\zeta^1, \mathcal{D}(E_\zeta^1))$ with the semigroup $(T_{\varrho(\zeta),t}^{(2)})_{t \geq 0}$ associated to $(E_{\varrho(\zeta)}^2, \mathcal{D}(E_{\varrho(\zeta)}^2))$.

Example 4.7. Theorem 4.5 and Corollary 4.6 provide an alternative proof for the construction of the ergodic decomposition of the Dirichlet–Ferguson diffusion [4] on the space of probability measures over a closed Riemannian manifold, obtained by ad hoc techniques in [4, Thm. 6.15(ii)]. The ergodic decomposition of the associated form consists of uncountably many components, and therefore the result does not follow from the aforementioned work [14] (cf. Remark 4.3 above).

Theorem 4.5 has the following natural converse, a proof of which is a straightforward application of Proposition 3.7

Proposition 4.8. *Let $(X, \mathfrak{T}, \mathcal{X}, \mu)$ be a locally compact Polish Radon-measure space, (Z, \mathcal{Z}, ν) be a separable countably generated probability space, and $s : (X, \mathcal{X}^\mu) \rightarrow (Z, \mathcal{Z})$ be a measurable map. For $i = 1, 2$ further let $(\mu_\zeta^i)_{\zeta \in Z}$ be an s -separated pseudo-disintegration of μ over (Z, \mathcal{Z}, ν) , and $\zeta \mapsto (E_\zeta^i, \mathcal{D}(E_\zeta^i))$ be a ν -measurable field of Dirichlet forms compatible with the disintegration. Further assume that there*

exists a ν -measurable field $\zeta \mapsto U_\zeta$ of unitary order isomorphisms $U_\zeta : L^2(\mu_\zeta) \rightarrow L^2(\mu_\zeta)$ intertwining the semigroups of $(E_\zeta^i, \mathcal{D}(E_\zeta^i))$ as in (4.3).

Then, the direct integral forms $E^i := \int_Z^{\oplus} E_\zeta^i \, d\nu(\zeta)$ are intertwined by the unitary order isomorphism $U^i := \int_Z^{\oplus} U_\zeta \, d\nu(\zeta)$.

By the transfer method for quasi-regular Dirichlet spaces, e.g., [2], it is possible to extend all the previous results to the quasi-regular case. The Definition 4.1 of ergodic decomposition is readily adapted by letting $(X, \mathfrak{T}, \mathcal{X}, \mu)$ be satisfying Assumption 2.16 (as opposed to: locally compact Polish). We only spell out the adaptation of Theorem 4.5. The easy adaptation of Proposition 4.4 and Corollary 4.6 is left to the reader.

Corollary 4.9. *For $i = 1, 2$, let $(X^i, \mathfrak{T}^i, \mathcal{X}^i, \mu^i)$ be satisfying Assumption 2.16, and $(E^i, \mathcal{D}(E^i))$ be a quasi-regular Dirichlet form on $L^2(\mu^i)$ with semigroup $(T_t^{(i)})_{t \geq 0}$. Let all other assumptions of Theorem 4.5 hold. Then, all the conclusions of that Theorem hold as well.*

Remark 4.10. (On quasi-homeomorphisms) It is shown in [13, Thm. 3.11] that the transformation τ associated to any order isomorphism U intertwining two irreducible regular Dirichlet forms has a version $\tilde{\tau}$ which is additionally a quasi-homeomorphism. This result has been extended by Li and Lin in [14] to the case of up-to-countable ergodic decompositions as in Remark 4.3(b). Whereas there is a reasonable expectation for this result to hold—at least under the additional assumption that U be unitary, as in Theorem 4.5—a proof seems currently beyond reach.

Acknowledgements

The authors are grateful to an anonymous reviewer for their careful reading of the manuscript and useful suggestions.

Funding Information Open access funding provided by Austrian Science Fund (FWF).

Data Availability Statement The manuscript has no associated data.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

Open Access. This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article's Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article's Creative Commons licence and your intended use

is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit <http://creativecommons.org/licenses/by/4.0/>.

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

REFERENCES

- [1] Arendt, W., Biegert, M., ter Elst, A.F.M.: Diffusion determines the manifold. *J. reine angew. Math.* **667**, 1–25 (2012). <https://doi.org/10.1515/crelle.2011.131>
- [2] Chen, Z.-Q., Ma, Z.-M., Röckner, M.: Quasi-homeomorphisms of Dirichlet forms. *Nagoya Math. J.* **136**, 1–15 (1994). <https://doi.org/10.1017/S0027763000024934>
- [3] Dello Schiavo, L.: Ergodic Decomposition of Dirichlet Forms via Direct Integrals and Applications. *Potential Anal.* (2021). <https://doi.org/10.1007/s11118-021-09951-y>
- [4] Dello Schiavo, L.: The Dirichlet–Ferguson Diffusion on the Space of Probability Measures over a Closed Riemannian Manifold. *Ann. Probab.* **50**(2), 591–648 (2022). <https://doi.org/10.1214/21-AOP1541>
- [5] Dixmier, J.: *Von Neumann Algebras*. North-Holland (1981)
- [6] Fremlin, D. H.: *Measure Theory – Volume I - IV, V Part I and II*. Torres Fremlin (ed.) (2000–2008)
- [7] Fukushima, M., Oshima, Y., Takeda, M.: Dirichlet forms and symmetric Markov processes, *De Gruyter Studies in Mathematics*, vol. 19, extended edn. de Gruyter (2011)
- [8] Haydon, R., Levy, M., Raynaud, Y.: Randomly normed spaces. *Travaux en cours*. Hermann (1991)
- [9] Kac, M.: Can One Hear the Shape of a Drum? *Amer. Math. Month.* **73**(4), 1–23 (1966)
- [10] Kechris, A. S.: *Classical Descriptive Set Theory*, Graduate Texts in Mathematics, vol. 156. Springer-Verlag, New York (1995)
- [11] Keller, M., Lenz, D., Schmidt, M., Wirth, M.: Diffusion determines the recurrent graph. *Adv. Math.* **269**, 364–398 (2015). <https://doi.org/10.1016/j.aim.2014.10.003>
- [12] Kuwae, K.: Irreducible decomposition for Markov processes. *Stoch. Proc. Appl.* **140**, 339–356 (2021). <https://doi.org/10.1016/j.spa.2021.06.012>
- [13] Lenz, D., Schmidt, M., Wirth, M.: Geometric Properties of Dirichlet Forms under Order Isomorphism (2018) [arXiv:1801.08326](https://arxiv.org/abs/1801.08326)
- [14] Li, L., Lin, H.: On Order Isomorphisms Intertwining Semigroups for Dirichlet Forms (2022) [arXiv:2204.02975](https://arxiv.org/abs/2204.02975)
- [15] Ma, Z.-M., Röckner, M.: *Introduction to the Theory of (Non-Symmetric) Dirichlet Forms*. Springer, Berlin (1992)
- [16] Weis, L.: On the Representation of Order Continuous Operators by Random Measures. *Trans. Amer. Math. Soc.* **285**(2), 535–563 (1984). <https://doi.org/10.2307/1999450>

*Lorenzo Dello Schiavo and Melchior Wirth
Institute of Science and Technology Austria
Klosterneuburg
Austria
E-mail: lorenzo.delloschiavo@ist.ac.at*

*Melchior Wirth
E-mail: melchior.wirth@ist.ac.at*

Accepted: 24 November 2022