



# Kac regularity and domination of quadratic forms

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## Abstract

A domain is called Kac regular for a quadratic form on  $L^2$  if every functions vanishing almost everywhere outside the domain can be approximated in form norm by functions with compact support in the domain. It is shown that this notion is stable under domination of quadratic forms. As applications measure perturbations of quasi-regular Dirichlet forms, Cheeger energies on metric measure spaces and Schrödinger operators on manifolds are studied. Along the way a characterization of the Sobolev space with Dirichlet boundary conditions on domains in infinitesimally Riemannian metric measure spaces is obtained.

**Keywords** Dirichlet forms · Kac regularity · Semigroup domination · Metric measure spaces · Cheeger energy

**Mathematics Subject Classification** 46E35 · 31E05 · 47A63

## 1 Introduction

Following Stroock, a domain  $\Omega \subset \mathbb{R}^n$  is called Kac regular if the first exit time of the Brownian motion from  $\Omega$  equals the first penetration time of the Brownian motion to  $\Omega^c$ . In analytic terms, Kac regularity has been proven by Herbst and Zhao [9] to be equivalent to the property that every  $u \in W^{1,2}(\mathbb{R}^n)$  with  $u = 0$  a.e. on  $\Omega^c$  can be approximated in  $W^{1,2}$ -norm by elements of  $C_c^\infty(\Omega)$ . As expected, this property very much depends on the regularity of the boundary of  $\Omega$ .

The above mentioned characterization of Kac regularity in terms of an approximation property allows to extend the definition of Kac regularity to open

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subsets of a sufficiently regular topological space with a Borel measure. The  $W^{1,2}$ -norm in the original definition is replaced by the norm induced by a closed quadratic form on (possibly vector-valued)  $L^2$ -functions, and the notion of Kac regularity depends on the choice of the quadratic form.

In this respect, it has been shown by Bei and Güneysu in [4] that an open subset of a Riemannian manifold  $X$  with its volume measure  $m$  is Kac regular for the canonical Dirichlet form on  $X$  induced by the scalar Laplace–Beltrami operator if and only if it is Kac regular for every reasonable semibounded quadratic form on  $L^2(X, m; E)$  which is induced by a covariant Schrödinger operator of the form  $\nabla^* \nabla + V$  on a Hermitian vector bundle  $E$  over  $X$ , where the potential  $V$  need not be semibounded. The aim of this article is to show that for positive perturbations  $V$  (allowing certain measure perturbations), this stability result can be understood in terms of domination of quadratic forms, thereby allowing much more general spaces than Riemannian manifolds.

In [4], the results are derived using deep techniques from stochastic analysis, in particular Feynman–Kac formulae and stochastic parallel transport. In contrast, our approach is purely functional analytical, drawing on the result of a joint article with Lenz and Schmidt [10], and works in the setting of (quasi-regular) Dirichlet forms so that it can readily be applied not only to quadratic forms on Riemannian manifolds, but also on (infinitesimally Riemannian) metric measure spaces. In this context we also give a characterization of the Sobolev space with Dirichlet boundary conditions on domains in infinitesimally Riemannian metric measure spaces, which might be of independent interest.

This article is structured as follows: In Sect. 1 we introduce the notion of Kac regularity with respect to quadratic forms on  $L^2$ -spaces and prove the abstract stability result under domination of quadratic forms (Theorem 1). In Sect. 2 we study Kac regular domains for quasi-regular Dirichlet forms, collect several equivalent definitions of the form domain with Dirichlet boundary conditions and prove that Theorem 1 implies the stability of Kac regularity under measure perturbations (Theorem 2). In Sect. 3 we apply the results of Sect. 2 to the Cheeger energy on infinitesimally Riemannian metric measure spaces (Theorems 3, 4). Finally, in Sect. 4 we show how the stability result of [4] fits into our framework (Theorem ).

## 2 Kac regular domains for quadratic forms

In this section, we introduce the concept of Kac regular domains for quadratic forms and prove an abstract stability theorem under domination. For simplicity's sake, all quadratic forms in this article are assumed to be non-negative. The generalization to the case of lower bounded forms is straightforward.

Let  $X$  be a Lindelöf topological space (every open cover has a countable subcover),  $m$  a Borel measure on  $X$  and  $E \rightarrow X$  a Hermitian vector bundle (see, e.g., §2 of [13], where the term *Euclidean vector bundle* is used). In particular, each fiber  $E_x$  is a finite-dimensional Hilbert space with inner product  $\langle \cdot, \cdot \rangle_x$  and induced norm  $|\cdot|_x$ .

A (positive) *quadratic form* on a Hilbert space  $H$  is a map  $\alpha : H \rightarrow [0, \infty]$  satisfying

- $\alpha(u + v) + \alpha(u - v) = 2\alpha(u) + 2\alpha(v)$  for all  $u, v \in H$ ,
- $\alpha(\lambda u) = |\lambda|^2 \alpha(u)$  for all  $\lambda \in \mathbb{R}, u \in H$ .

Its domain is  $D(\alpha) = \{u \in H \mid \alpha(u) < \infty\}$ . Via polarization, every quadratic form induces a bilinear form on  $D(\alpha)$ , which shall be denoted by the same symbol.

A quadratic form  $\alpha$  is called *closed* if  $D(\alpha)$  is complete with respect to the form norm

$$\|\cdot\|_\alpha = \left( \|\cdot\|_{L^2}^2 + \alpha(\cdot) \right)^{\frac{1}{2}}.$$

Let  $\alpha$  be a closed quadratic form on  $L^2(X, m; E)$ . For an open subset  $\Omega$  of  $X$  denote by  $D(\alpha_\Omega)$  the  $\|\cdot\|_\alpha$ -closure of the set of all  $\Phi \in D(\alpha)$ , such that  $\text{supp } \Phi$  is compact and contained in  $\Omega$ . Here,  $\text{supp } \Phi$  is understood as the support of the measure  $|\Phi|m$ . Further let  $\alpha_\Omega$  be the restriction of  $\alpha$  to  $D(\alpha_\Omega)$ .

Moreover, let  $D(\tilde{\alpha}_\Omega) = \{\Phi \in D(\alpha) \mid \Phi = 0 \text{ a.e. on } \Omega^c\}$  and denote by  $\tilde{\alpha}_\Omega$  the restriction of  $\alpha$  to  $D(\tilde{\alpha}_\Omega)$ .

**Lemma 1** *The space  $D(\tilde{\alpha}_\Omega)$  is closed in  $D(\alpha)$ .*

**Proof** This follows immediately from the fact that every  $\|\cdot\|_\alpha$ -convergent sequences also converges in  $L^2$  and hence has an a.e. convergent subsequence.  $\square$

In the light of this lemma, one clearly has  $D(\alpha_\Omega) \subset D(\tilde{\alpha}_\Omega)$ . We will study domains where also the reverse inclusion holds.

**Definition 1** The set  $\Omega$  is called *Kac regular* for  $\alpha$  if  $D(\alpha_\Omega) = D(\tilde{\alpha}_\Omega)$ .

Now let us turn to domination of quadratic forms (see [12, 14]). A closed quadratic form  $b$  on  $L^2(X, m)$  satisfies the first Beurling–Deny criterion if  $u \in D(b)$  implies  $|u| \in D(b)$  and  $b(|u|) \leq b(u)$ .

**Lemma 2** *If  $b$  satisfies the first Beurling–Deny criterion and  $\Omega \subset X$  is open, then  $b_\Omega$  and  $\tilde{b}_\Omega$  also satisfy the first Beurling–Deny criterion.*

**Proof** In both cases it suffices to show that the domain is stable under taking absolute values. In the case of  $\tilde{b}_\Omega$  this is clear.

To see this for  $b_\Omega$ , let  $u \in D(b_\Omega)$  and  $(u_n)$  a sequence in  $D(b)$  with  $\text{supp } u_n$  compact and contained in  $\Omega$  and  $u_n \rightarrow u$  in form norm. Then,  $|u_n|$  still has compact support contained in  $\Omega$ ,  $|u_n| \rightarrow |u|$  in  $L^2(X, m)$ , and  $\| |u_n| \|_b \leq \|u_n\|_b$ . Thus,  $(|u_n|)$  has a subsequence converging weakly to  $|u|$  in form norm. By the Banach–Saks theorem there is a sequence of finite convex combinations of elements of  $(|u_n|)$  that converges to  $|u|$  w.r.t.  $\|\cdot\|_b$ . Hence,  $|u| \in D(b_\Omega)$ .  $\square$

The closed quadratic form  $\mathfrak{a}$  on  $L^2(X, m; E)$  is said to be dominated by  $\mathfrak{b}$  if  $\mathfrak{b}$  satisfies the first Beurling–Deny criterion and for all  $\Phi \in D(\mathfrak{a})$  and  $v \in D(\mathfrak{b})$  with  $0 \leq v \leq |\Phi|$  one has  $|\Phi| \in D(\mathfrak{b})$ ,  $v \operatorname{sgn} \Phi \in D(\mathfrak{a})$  and

$$\mathfrak{b}(v, |\Phi|) \leq \operatorname{Rea}(\Phi, v \operatorname{sgn} \Phi).$$

Here  $\operatorname{sgn} \Phi(x) = \Phi(x)/|\Phi(x)|_x$  whenever  $\Phi(x) \neq 0$  (the value in the case  $\Phi(x) = 0$  is obviously irrelevant for the preceding definition).

Let  $A$  and  $B$  be the positive self-adjoint operators associated with  $\mathfrak{a}$  and  $\mathfrak{b}$ , respectively. Domination of quadratic forms has the following nice characterization in terms of the associated semigroups ([12], Theorem 4.1): The form  $\mathfrak{a}$  is dominated by  $\mathfrak{b}$  if and only if

$$|e^{-tA} \Phi| \leq e^{-tB} |\Phi|$$

for all  $t \geq 0$  and  $\Phi \in L^2(X, m; E)$ .

We will later use the following two facts: A closed quadratic form  $\mathfrak{b}$  that satisfies the first Beurling–Deny criterion is dominated by itself (see [14], Proposition 3.2) and the domain  $D(\mathfrak{b})$  is a lattice, that is,  $u, v \in D(\mathfrak{b})$  implies  $u \wedge v, u \vee v \in D(\mathfrak{b})$ . The latter is immediate from the formulas

$$\begin{aligned} u \wedge v &= \frac{1}{2}(u + v - |u - v|), \\ u \vee v &= \frac{1}{2}(u + v + |u - v|). \end{aligned}$$

**Lemma 3** *Let  $\mathfrak{a}$  be a closed quadratic form on  $L^2(X, m; E)$ ,  $\mathfrak{b}$  a closed quadratic form on  $L^2(X, m)$  and  $\Omega \subset X$  open. If  $\mathfrak{a}$  is dominated by  $\mathfrak{b}$ , then  $\mathfrak{a}_\Omega$  is dominated by  $\mathfrak{b}_\Omega$  and  $\tilde{\mathfrak{a}}_\Omega$  is dominated by  $\tilde{\mathfrak{b}}_\Omega$ .*

**Proof** If  $\Phi \in D(\tilde{\mathfrak{b}}_\Omega)$  and  $v \in D(\tilde{\mathfrak{a}}_\Omega)$  with  $0 \leq v \leq |\Phi|$ , then  $v \operatorname{sgn} \Phi \in D(\mathfrak{a})$  and  $|\Phi| \in D(\mathfrak{b})$ , since  $\mathfrak{a}$  is dominated by  $\mathfrak{b}$ . Moreover,  $|\Phi|$  and  $v \operatorname{sgn} \Phi$  vanish a.e. on  $\Omega^c$  by definition of  $D(\tilde{\mathfrak{a}}_\Omega)$  and  $D(\tilde{\mathfrak{b}}_\Omega)$ . Thus,  $|\Phi| \in D(\tilde{\mathfrak{b}}_\Omega)$  and  $v \operatorname{sgn} \Phi \in D(\tilde{\mathfrak{a}}_\Omega)$ . The inequality between  $\tilde{\mathfrak{a}}_\Omega$  and  $\tilde{\mathfrak{b}}_\Omega$  follows directly by restriction. The proof for  $\mathfrak{a}_\Omega, \mathfrak{b}_\Omega$  is analogous.  $\square$

**Theorem 1** *Let  $\mathfrak{a}$  be a closed quadratic form on  $L^2(X, m; E)$ ,  $\mathfrak{b}$  a closed quadratic form on  $L^2(X, m)$  and  $\Omega \subset X$  open. If  $\mathfrak{a}$  is dominated by  $\mathfrak{b}$  and  $\Omega$  is Kac regular for  $\mathfrak{b}$ , then it is Kac regular for  $\mathfrak{a}$ .*

**Proof** By Lemma 3, the form  $\tilde{\mathfrak{a}}_\Omega$  is dominated by  $\tilde{\mathfrak{b}}_\Omega$ . Let  $D_\mathfrak{a}$  and  $D_\mathfrak{b}$  be the set of all elements of  $D(\mathfrak{a})$  and  $D(\mathfrak{b})$ , respectively, with compact support in  $\Omega$ . By assumption,  $D_\mathfrak{b}$  is dense in  $D(\tilde{\mathfrak{b}}_\Omega)$ .

In general,  $D(\tilde{\mathfrak{b}}_\Omega)$  will of course not be densely defined in  $L^2(X, m)$ . However, since  $D(\tilde{\mathfrak{b}}_\Omega)$  is a lattice, its closure in  $L^2(X, m)$  is order isometric to an  $L^2$ -space ([15], Corollary 2.4), so we can view  $\tilde{\mathfrak{a}}_\Omega$  and  $\tilde{\mathfrak{b}}_\Omega$  as densely defined forms in the closures of their respective domains. Then, Theorem 2.3 from [10] (see also

Corollary 2.4) is applicable and implies that  $D_\alpha$  is dense in  $D(\tilde{\alpha}_\Omega)$ . Hence,  $\Omega$  is Kac regular for  $\alpha$ . □

### 3 Quasi-regular Dirichlet forms

In this section, we give a characterization of  $D(\mathcal{E}_\Omega)$  that is better suited for applications in the case when  $\mathcal{E}$  is a quasi-regular Dirichlet form and discuss an application of Theorem 1 to measure perturbations of Dirichlet forms.

The definition of quasi-regular Dirichlet forms along with all necessary properties can be found in [11]. At this point let us just mention that every Dirichlet form satisfies the first Beurling–Deny criterion and is thus dominated by itself.

Further let us recall some standard terminology. Let  $X$  be a Hausdorff space,  $m$  a  $\sigma$ -finite Borel measure on  $X$  of full support, and  $\mathcal{E}$  a quasi-regular Dirichlet form on  $L^2(X, m)$ . An ascending sequence  $(F_k)$  of closed subsets of  $X$  is called *nest* if  $\bigcup_k D(\mathcal{E})_{F_k}$  is dense in  $\mathcal{E}$ . A Borel set  $B \subset X$  is called *exceptional* if  $B \subset \bigcap_k F_k^c$  for some nest  $(F_k)$ . A property is said to hold *quasi-everywhere*, abbreviated *q.e.*, if it holds outside an exceptional set. A function  $u$  on  $X$  is called *quasi-continuous* if there exists a nest  $(F_k)$ , such that  $u|_{F_k}$  is continuous for all  $k \in \mathbb{N}$ . Every element  $u$  of the domain of a quasi-regular Dirichlet form has a quasi-continuous  $m$ -version, uniquely determined up to equality q. e. We will denote it by  $\tilde{u}$ .

For a Borel set  $B \subset X$  let

$$D(\mathcal{E})_B = \{u \in D(\mathcal{E}) \mid \tilde{u} = 0 \text{ q.e. on } B^c\}.$$

Alternatively,  $D(\mathcal{E})_B$  can be characterized as the closure of all functions with compact support in  $B$ , as the following lemma ([17], Lemma 2.7) shows.

**Lemma 4** *For every Borel set  $B \subset X$  there exists an ascending sequence  $(F_k)$  of compact subsets of  $B$ , such that  $\bigcup_k D(\mathcal{E})_{F_k}$  is  $\|\cdot\|_{\mathcal{E}}$ -dense in  $D(\mathcal{E})_B$ .*

**Corollary 1** *If  $\Omega \subset X$  is open, then  $D(\mathcal{E}_\Omega) = D(\mathcal{E})_\Omega$ .*

Note that in this situation,  $\mathcal{E}_\Omega$  is again a quasi-regular Dirichlet form ([17], Lemma 2.12).

In many cases it is more customary to define the Sobolev space with Dirichlet boundary conditions not as the closure of *all* compactly supported functions in the domain, but as the closure of a certain subset (for example continuous functions or smooth ones). The following definition gives a general setup for these situations.

**Definition 2** A subalgebra  $\mathcal{C}$  of  $D(\mathcal{E}) \cap C_b(X)$  is called *generalized special standard core* if

- $\mathcal{C}$  is dense in  $D(\mathcal{E})$  with respect to the form norm,
- for every  $\epsilon > 0$  there exists an increasing 1-Lipschitz function  $C_\epsilon : \mathbb{R} \rightarrow \mathbb{R}$ , such that  $-\epsilon \leq C_\epsilon \leq 1 + \epsilon$ ,  $C(t) = t$  for  $t \in [0, 1]$  and  $C_\epsilon \circ u \in \mathcal{C}$  for all  $u \in \mathcal{C}$ ,
- for every compact  $K \subset X$  and every open  $G \subset X$  with  $K \subset G$  there exists  $u \in \mathcal{C}$ , such that  $u \geq 0$ ,  $u|_K = 1$  and  $\text{supp } u \subset G$ .

As the name suggests, this concept generalizes special standard cores in the regular setting (see [6], Sect. 1.1). In particular, if  $\mathcal{E}$  is a regular Dirichlet form, then  $D(\mathcal{E}) \cap C_c(X)$  is a special standard core. More examples will be discussed in the following sections.

The crucial property of generalized special standard cores for our purpose is that their density in the form domain can be localized in the sense of the following lemma (cf. Lemma 2.3.4 in [6] in the case of special standard cores).

**Lemma 5** *If  $\mathcal{C}$  is a generalized special standard core, then  $\{u \in \mathcal{C} \mid \text{supp } u \subset \Omega\}$  is  $\|\cdot\|_{\mathcal{E}}$ -dense in  $D(\mathcal{E}_\Omega)$ .*

**Proof** We have to show that every function  $u \in D(\mathcal{E}_\Omega)$  can be approximated in  $(D(\mathcal{E}), \langle \cdot, \cdot \rangle_{\mathcal{E}})$  by elements from  $\mathcal{C}$  with support in  $\Omega$ . By Lemma 4 we can assume that  $u$  has compact support in  $\Omega$ , and by a standard approximation argument we may further assume that  $0 \leq u \leq 1$ .

Denote the support of  $u$  by  $K$ . By the definition of generalized special standard core, there exists  $w \in \mathcal{C}$ , such that  $w \geq 0$ ,  $w|_K = 1$  and  $\text{supp } w \subset \Omega$  and a sequence  $(v_n)$  in  $\mathcal{C}$ , such that  $v_n \rightarrow u$  w.r.t.  $\|\cdot\|_{\mathcal{E}}$ . Moreover, let  $C_\epsilon$  be a bounded 1-Lipschitz function as in Definition 2 for some  $\epsilon > 0$ .

Let  $u_n = (C_\epsilon \circ v_n) \cdot w$ . By definition of a generalized special standard core, we have  $u_n \in \mathcal{C}$ , and clearly  $\text{supp } u_n \subset \Omega$ . Furthermore

$$\begin{aligned} \int_X |(C_\epsilon \circ v_n) \cdot w - u|^2 dm &= \int_X |(C_\epsilon \circ v_n) \cdot w - (C_\epsilon \circ u) \cdot w|^2 dm \\ &\leq \|w\|_\infty \int_X |v_n - u|^2 dm \\ &\rightarrow 0, \end{aligned}$$

and

$$\mathcal{E}((C_\epsilon \circ v_n) \cdot w)^{1/2} \leq \|C_\epsilon\|_\infty \mathcal{E}(w)^{1/2} + \|w\|_\infty \mathcal{E}(v_n)^{1/2}.$$

Thus  $(u_n)$  has a subsequence converging weakly in  $(D(\mathcal{E}), \langle \cdot, \cdot \rangle_{\mathcal{E}})$  to  $u$ . By the Banach–Saks theorem there is a sequence of finite convex combinations of elements of  $(u_n)$  that converges to  $u$  w.r.t.  $\|\cdot\|_{\mathcal{E}}$ .  $\square$

Next, we study a class of perturbations of quasi-regular Dirichlet forms that are dominated by the original form.

A positive Borel measure  $\mu$  on  $X$  is called *smooth* if  $\mu(B) = 0$  for every exceptional set  $B$  and there exists a nest  $(F_k)$  of compact sets, such that  $\mu(F_k) < \infty$  for all  $k \in \mathbb{N}$ .

If  $\mu$  is a smooth measure, define the quadratic form  $\mathcal{E}^\mu$  by

$$D(\mathcal{E}^\mu) = \{u \in D(\mathcal{E}) \mid \tilde{u} \in L^2(X, \mu)\}, \quad \mathcal{E}^\mu(u) = \mathcal{E}(u) + \int_X \tilde{u}^2 d\mu.$$

The form  $\mathcal{E}^\mu$  is again a quasi-regular Dirichlet form ([17], Proposition 2.3).

**Lemma 6** *The form  $\mathcal{E}^\mu$  is dominated by  $\mathcal{E}$ .*

**Proof** Let  $u \in D(\mathcal{E}^\mu)$  and  $v \in D(\mathcal{E})$  with  $0 \leq v \leq |u|$  a.e. Then  $|u| \in D(\mathcal{E})$  and  $v \operatorname{sgn} u \in D(\mathcal{E})$ , since  $\mathcal{E}$  is dominated by itself. Furthermore

$$\int_X |v \widetilde{\operatorname{sgn} u}|^2 d\mu = \int_X |\tilde{v}|^2 d\mu \leq \int_X |\tilde{u}|^2 d\mu < \infty.$$

Thus  $v \operatorname{sgn} u \in D(\mathcal{E}^\mu)$ .

Finally

$$\mathcal{E}^\mu(u, v \operatorname{sgn} u) = \mathcal{E}(u, v \operatorname{sgn} u) + \int v |\tilde{u}| d\mu \geq \mathcal{E}(u, v \operatorname{sgn} u) \geq \mathcal{E}(|u|, v)$$

since  $\mathcal{E}$  is dominated by itself. □

Now, we can combine Lemma 6 with Theorem 1 to obtain stability of Kac regularity under measure perturbations.

**Theorem 2** *If an open set  $\Omega \subset X$  is Kac regular for  $\mathcal{E}$ , then it is Kac regular for  $\mathcal{E}^\mu$  for every smooth measure  $\mu$ .*

Even in the case of the standard Dirichlet energy on Euclidean space, this theorem provides new examples, as [4] only studies the case of absolutely continuous measures  $\mu$ , whereas in the present framework also singular measures can be treated without additional effort.

**Example 1** Let

$$\mathcal{E} : W^1(\mathbb{R}^d) \longrightarrow [0, \infty), \mathcal{E}(u) = \int_{\mathbb{R}^d} |\nabla u|^2 dx.$$

The Hausdorff measure  $\mu$  on a  $(d - 1)$ -dimensional hyperplane is a smooth measure (see [6], Exercise 2.2.1). Thus, every domain that is Kac regular for  $\mathcal{E}$  is also Kac regular for the perturbed form  $\mathcal{E}^\mu$ .

### 4 Metric measure spaces

In this section, we apply the results of the previous section to the Cheeger energy on infinitesimally Riemannian metric measure spaces.

Let  $(X, d)$  be a complete, separable metric space and  $m$  a  $\sigma$ -finite Borel measure of full support on  $X$  satisfying  $m(B_r(x)) < \infty$  for all  $x \in X, r > 0$ . Denote by  $\operatorname{Lip}_b(X, d)$  the space of all bounded Lipschitz functions on  $X$ . For  $f \in \operatorname{Lip}_b(X, d)$  the local Lipschitz constant is defined as

$$|Df|(x) = \limsup_{y \rightarrow x} \frac{|f(x) - f(y)|}{d(x, y)}.$$

The Cheeger energy  $\operatorname{Ch}$  is the  $L^2$ -lower semicontinuous relaxation of

$$\text{Lip}_b(X, d) \cap L^2(X, m) \longrightarrow [0, \infty], f \mapsto \frac{1}{2} \int_X |Df|^2 dm.$$

If  $\text{Ch}$  is a quadratic form, then  $(X, d, m)$  is called *infinitesimally Riemannian* (see, e.g., [2, 7]). In this case,  $\text{Ch}$  is a quasi-regular Dirichlet form ([18], Theorem 4.1—the theorem is formulated for RCD spaces, but the proof of quasi-regularity holds under the weaker assumptions stated above).

Let  $(X, d, m)$  be an infinitesimally Riemannian metric measure space and  $\text{Ch}$  the Cheeger energy. The space  $\text{Lip}_{\text{bs}}(X, d)$  of Lipschitz functions with bounded support is  $\|\cdot\|_{\text{Ch}}$ -dense in  $D(\text{Ch})$ , as follows for example from [1], Lemma 4.3, combined with a standard localization argument. From that fact it is easy to see that  $\text{Lip}_{\text{bs}}(X, d)$  is indeed a generalized special standard core for  $\text{Ch}$ .

The results from the last section (Corollary 1, Lemma 5) immediately give the following characterization of the first-order Sobolev space with Dirichlet boundary conditions on domains in infinitesimally Riemannian metric measure spaces.

**Theorem 3** *For an open subset  $\Omega$  of  $X$ , the following spaces all coincide:*

- The  $\|\cdot\|_{\text{Ch}}$ -closure of  $\{u \in D(\text{Ch}) \mid \text{supp } u \subset \Omega \text{ compact}\}$ .
- The  $\|\cdot\|_{\text{Ch}}$ -closure of  $\{u \in \text{Lip}_{\text{bs}}(X, d) \mid \text{supp } u \subset \Omega\}$ .
- The set  $\{u \in D(\text{Ch}) \mid \bar{u} = 0 \text{ q.e. on } \Omega^c\}$ .

Now, we turn to perturbations of the Cheeger energy by a positive potential.

For measurable  $V : X \longrightarrow [0, \infty]$  let

$$\text{Ch}^V : L^2(X, m) \longrightarrow [0, \infty], \text{Ch}^V(u) = \text{Ch}(u) + \int_X |u|^2 V dm.$$

For the following theorem recall that an  $\text{RCD}^*(K, N)$  space is an infinitesimally Hilbertian metric measure space with Ricci curvature bounded below by  $K$  and dimension bounded above by  $N$  (see [5], Definition 3.16 for a precise statement).

**Theorem 4** *If  $V \in L^1_{\text{loc}}(X, m)$  is non-negative and  $\Omega$  is Kac regular for  $\text{Ch}$ , so it is for  $\text{Ch}^V$ . Conversely, if  $(X, d, m)$  is an  $\text{RCD}^*(K, N)$  space for  $K > 0, N \in (2, \infty)$ ,  $V \in L^p(X, m)$  for  $p \in [N/2, \infty]$ , and  $\Omega$  is Kac regular for  $\text{Ch}^V$ , then it is also Kac regular for  $\text{Ch}$ .*

**Proof** Since  $\text{Ch}$  is quasi-regular, there is a nest  $(F_k)_k$  of compact sets. By assumption  $Vm(F_k) < \infty$  for all  $k \in \mathbb{N}$ . Of course,  $Vm$  does not charge measures of capacity zero. Hence,  $Vm$  is a smooth measure, and the first implication follows from Theorem 5.

For the converse implication it is sufficient to show that  $D(\text{Ch}^V) = D(\text{Ch})$ . Let  $q$  be the dual exponent of  $p$  and  $u \in D(\text{Ch})$ . Since  $q \in [1, \frac{N}{N-2}]$ , the Sobolev embedding theorem ([3], Proposition 6.2.3, see also [16], Proposition 3.3) implies  $u \in L^{2q}(X, m)$ . Thus



$$\int_X V|u|^2 \, dm \leq \|V\|_p \|u\|_{2q}^{1/2} < \infty,$$

that is,  $u \in D(\text{Ch})$ . □

More examples in the spirit of [4] can be constructed using the first-order differential calculus for the Cheeger energy developed in [7]. There is an  $L^2$ -normed  $L^\infty$ -module  $L^2(TX)$ , the tangent module, and a linear map  $\nabla : D(\text{Ch}) \rightarrow L^2(TX)$  satisfying the Leibniz rule such that the Cheeger energy can be written as

$$\text{Ch}(u) = \int_X |\nabla u|^2 \, dm.$$

If  $b \in L^2(TX)$  with  $|b| \in L^\infty(X, m)$ , then the form

$$\text{Ch}^b : D(\text{Ch}) \rightarrow [0, \infty), \text{Ch}^b(u) = \int_X |(\nabla - ib)u|^2 \, dm$$

is dominated by Ch and thus Kac regularity for Ch implies Kac regularity for  $\text{Ch}^b$ . With the calculus rules proven in [7], the proof can be carried out in essentially the same way as the one of Proposition 3.7 from [10]. We do not repeat it here as it is quite lengthy.

### 5 Schrödinger operators on Riemannian manifold

In this section, Kac regularity for quadratic forms generated by Schrödinger operators on vector bundles over manifolds is examined.

Let  $(M, g)$  be a Riemannian manifold and

$$\mathcal{E} : W_0^{1,2}(M) \rightarrow [0, \infty], u \mapsto \int_M |\nabla u|^2 \, d\text{vol}_g.$$

The space  $C_c^\infty(M)$  is obviously a special standard core for  $\mathcal{E}$  and so  $D(\mathcal{E}_\Omega)$  coincides with  $W_0^{1,2}(\Omega)$  for  $\Omega \subset M$  open.

Further let  $E \rightarrow M$  be a Hermitian vector bundle with metric covariant derivative  $\nabla$  and let  $V : M \rightarrow \text{End}(E)$  be a measurable section with  $V(x) \geq 0$  for a.e.  $x \in M$  and  $|V| \in L^1_{\text{loc}}(M)$ . In particular,  $V(x)$  is hermitian for a.e.  $x \in M$ .

Denote by  $\Gamma_{C_c^\infty}(M; E)$  the smooth section of  $E$  with compact support and let  $\mathcal{E}^{\nabla, V}$  be the closure of

$$\begin{aligned} \Gamma_{C_c^\infty}(M; E) &\rightarrow [0, \infty), \\ \Phi &\mapsto \int_M (|\nabla \Phi(x)|_x^2 + \langle V(x)\Phi(x), \Phi(x) \rangle_x) \, d\text{vol}_g(x). \end{aligned}$$

From the proof of [8], Proposition 2.2, together with the semigroup characterization of domination ([12], Theorem 4.1) it follows that  $\mathcal{E}^{\nabla, 0}$  is dominated by  $\mathcal{E}$  (alternatively, one could also adapt the proof of [10], Proposition 3.7, to the case of Dirichlet boundary conditions). Given this result, it is not hard to see (with an

argument along the lines of Lemma 6) that  $\mathcal{E}^{\nabla, V}$  is dominated by  $\mathcal{E}$  as well. Thus, we can apply Theorem 1 to obtain the stability of Kac regularity in this setting.

**Theorem 5** *If  $\Omega$  is Kac regular for  $\mathcal{E}$ , then it is also Kac regular for  $\mathcal{E}^{\nabla, V}$ .*

A standard approximation argument shows that  $D(\mathcal{E}_\Omega^{\nabla, V})$  is the closure of  $\Gamma_{C_c^\infty}(\Omega; E)$ . Therefore, the previous theorem recovers partially Theorem 2.13 a) of [4] with the restriction to *non-negative* potentials.

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