High-Dimensional Expansion and Crossing Numbers of Simplicial Complexes

by

Pascal Frederik Wild

August, 2022

A thesis submitted to the
Graduate School
of the
Institute of Science and Technology Austria
in partial fulfillment of the requirements
for the degree of
Doctor of Philosophy

Committee in charge:

Mario de Bono, Chair
Uli Wagner
Herbert Edelsbrunner
Roy Meshulam

Institute of Science and Technology Austria

The thesis of Pascal Frederik Wild, titled *High-Dimensional Expansion and Crossing Numbers of Simplicial Complexes*, is approved by:

Supervisor: Uli Wagner, ISTA, Klosterneuburg, Austria

Signature: ____________________________________

Committee Member: Herbert Edelsbrunner, ISTA, Klosterneuburg, Austria

Signature: ____________________________________

Committee Member: Roy Meshulam, Technion – Israel Institute of Technology, Haifa, Israel

Signature: ____________________________________

Defense Chair: Mario de Bono, ISTA, Klosterneuburg, Austria

Signature: ____________________________________

Signed page is on file

ISTA Thesis, ISSN: 2663-337X

ISBN: 978-3-99078-021-3

I hereby declare that this thesis is my own work and that it does not contain other people's work without this being so stated; this thesis does not contain my previous work without this being stated, and the bibliography contains all the literature that I used in writing the dissertation.

I declare that this is a true copy of my thesis, including any final revisions, as approved by my thesis committee, and that this thesis has not been submitted for a higher degree to any other university or institution.

I certify that any republication of materials presented in this thesis has been approved by the relevant publishers and co-authors.

Signature: ______________________________

Pascal Frederik Wild
August, 2022

Signed page is on file

Abstract

In this dissertation we study coboundary expansion of simplicial complex with a view of giving geometric applications.

Our main novel tool is an equivariant version of Gromov's celebrated Topological Overlap Theorem. The equivariant topological overlap theorem leads to various geometric applications including a quantitative non-embeddability result for sufficiently thick buildings (which partially resolves a conjecture of Tancer and Vorwerk) and an improved lower bound on the pair-crossing number of (bounded degree) expander graphs. Additionally, we will give new proofs for several known lower bounds for geometric problems such as the number of Tverberg partitions or the crossing number of complete bipartite graphs.

For the aforementioned applications one is naturally lead to study expansion properties of joins of simplicial complexes. In the presence of a special certificate for expansion (as it is the case, e.g., for spherical buildings), the join of two expanders is an expander. On the flip-side, we report quite some evidence that coboundary expansion exhibits very non-product-like behaviour under taking joins. For instance, we exhibit infinite families of graphs $(G_n)_{n\in\mathbb{N}}$ and $(H_n)_{n\in\mathbb{N}}$ whose join $G_n * H_n$ has expansion of lower order than the product of the expansion constant of the graphs. Moreover, we show an upper bound of $(d+1)/2^d$ on the normalized coboundary expansion constants for the complete multipartite complex $[n]^{*(d+1)}$ (under a mild divisibility condition on n).

Via the probabilistic method the latter result extends to an upper bound of $(d+1)/2^d+\varepsilon$ on the coboundary expansion constant of the spherical building associated with $\mathrm{PGL}_{d+2}(\mathbb{F}_q)$ for any $\varepsilon > 0$ and sufficiently large $q = q(\varepsilon)$. This disproves a conjecture of Lubotzky, Meshulam and Mozes – in a rather strong sense.

By improving on existing lower bounds we make further progress towards closing the gap between the known lower and upper bounds on the coboundary expansion constants of $[n]^{*(d+1)}$. The best improvements we achieve using computer-aided proofs and flag algebras. The exact value even for the complete 3-partite 2-dimensional complex $[n]^{*3}$ remains unknown but we are happy to conjecture a precise value for every n.

In a loosely structured, last chapter of this thesis we collect further smaller observations related to expansion. We point out a link between discrete Morse theory and a technique for showing coboundary expansion, elaborate a bit on the hardness of computing coboundary expansion constants, propose a new criterion for coboundary expansion (in a very dense setting) and give one way of making the folklore result that expansion of links is a necessary condition for a simplicial complex to be an expander precise.

Acknowledgements

The past six years have been an extraordinary experience – equally challenging as rewarding, equally frustrating as fun. I have been fortunate that many wonderful people have accompanied me on this long journey.

First and for most, I would like to thank my supervisor Prof. Uli Wagner for generously sharing his ideas in countless long meetings. Uli's passion for and obsession with difficult problems have been a great source of inspiration.

I am indebted to my other two committee members Prof. Roy Meshulam and Prof. Herbert Edelsbrunner who carefully reviewed my thesis and served as very friendly examiners during my defense. Additionally, I had the opportunity to do a rotation project with Herbert but more importantly Herbert's witty words during various progress review meetings helped me to get a different, more relaxed perspective on being a researcher.

I am grateful to Prof. Ori Parzanchevski, who was an examiner during my qualifying exam, to Prof. Georgios Katsaros, who chaired my qualifying exam, and to Prof. Mario de Bono, the chair of my thesis defense.

I thank Prof. Jan Maas for his guidance during my rotation project and Prof. Beatriz Viscoso for her support as my mentor.

I would like to thank the current and former members of the Wagner group – Alan Arroyo, Sergey Avvakumov, Marek Filakovský, Peter Franek, Rado Fulek, Kristóf Huszár, Grisha Ivanov, Vojta Kaluza, Zuzka Masárová, Jakub Opršal, Pavel Paták, Zuzka Patáková, Elizaveta Streltsova, Gianluca Tosinato and Stephan Zhechev – for making the daily life of a young researcher enjoyable. I am particularly grateful to Kristóf who not only helped me with my first steps in the world of high-dimensional expanders but with whom I spent many "after work hours" discussing mathematics and much more. Some of these after hours were joined by Stephan.

I would like to extend my thank you to the broader IST(A) community: to my fellow PhD students of the cohort 2016 who made (especially the first year of) my PhD studies a wonderful time and among whom Kasia Kuzmicz-Kowalska, Laura Burnett and Michele Nardin deserve to be mentioned by name, to the "crypto people" for many lunches and some of them being great office mates, to the administrative stuff, including Astrid Bonventre-Darthé, Vlad Cozac, Christine Francois-Rennhofer, Christine Ostermann, the team of the Graduate School Office and the IT support, for their high-quality support on multiple occasions, to my fellow runners for 80+ runs from Vienna to ISTA (or back) over the years, and to all other members of ISTA I encountered during my journey.

I acknowledge the received funding from the European Union's Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie Grant Agreement No. 665385. Additionally, I appreciate that I have been part of the Austrian scholarship fund Pro Scientia during three years of my PhD studies.

I owe thanks to my former flatmates Maria at Barawitzkagasse and Caro, Felix B., Felix K., Simon and Levin at Rudolfinergasse for making Vienna indeed one of the most liveable cities in the world and for giving me many moments to escape the scientific bubble. I am indebted to all my other friends in Vienna, especially those who I met through the Pro Scientia network.

My friends in Switzerland, notably Patrick, Mat, Andreas, Giuliano and JJ, always warmly welcomed me on my trips there.

I am incredibly grateful to my partner Katharina-Maria Schön for her patience and love during the past years. Sharing moments of fascination, insight, obsession and frustration, while working on our dissertation projects, gave me the strength to finish this thesis. Her passion for languages, literature and teaching enriches my life on a daily basis.

Katharina introduced me to her family at the very beginning of a pandemic. Her parents, Martha and Mathias, and her siblings, Sebastian, Severin, Benedikt and Anastasia, generously hosted me during various lockdowns (and more). During these weeks some of the main results in this thesis were discovered and a new drinking game was invented.

Last but not least, I would like to thank my parents, Gudrun and Paul, my sister, Katrin, and my brother, Dominik, for their constant support and unconditional love. They always encouraged me to follow my own path.

About the Author

Pascal Wild did his undergraduate studies at the Eidgenössische Technische Hochschule Zürich (ETH Zurich) in Switzerland where he obtained a BSc in Mathematics and a MSc in Mathematics in 2013 and 2015, respectively. Before starting his PhD studies at IST Austria in September 2016 he worked for 9 months as a tutor at the African Institute for Mathematical Sciences in Muizenberg, Cape Town, South Africa.

Pascal joined the research group of Uli Wagner in May 2017. His main research interests are situated in discrete and computational topology and high-dimensional combinatorics, mostly in the rapidly developing area of high-dimensional expanders.

Beside working on mathematical problems, Pascal loves to spend his time playing the piano or running.

List of Publications

This thesis is partially based on the following publication which is currently under submission.

- Uli Wagner and Pascal Wild. Coboundary expansion, equivariant overlap, and crossing numbers of simplicial complexes. submitted to the *Israel Journal of Mathematics*, March 2022

Table of Contents

List of Figures

Chapter 1

Introduction

This thesis fits into the emerging research area of *high-dimensional expanders (HDXs)* which is a successful attempt to generalize the well-established theory of expander graphs to higher dimensions.[1]

Expander graphs are sparse but highly connected graphs. They can be defined in many different, (essentially) equivalent ways. Interestingly, for HDXs even the definition is not obvious at all. Many of the different characterizations of expander graphs have natural generalizations for higher dimensional simplicial or cellular complexes but it turns out that these generalizations are (usually) not equivalent anymore, often not even comparable.[2] Thus we end up with a whole array of genuinely different notions of HDXs, all with their own (potential) applications. There might even be further notions of HDXs to be discovered.

Despite its formative stage the theory of HDXs has already seen striking applications. One of them is a fully polynomial-time randomized approximation scheme for sampling and counting matroid basis which was given by Anari, Liu, Gharan and Vinzant in [7].[3] Another very recent breakthrough is the construction of locally testable codes with constant rate, constant distance and constant locality by Dinur, Evra, Livne, Lubotzky and Mozes in [31] and independently by Panteleev and Kalachev in [118] who additionally constructed asymptotically good quantum LDPC.[4]

All of these major applications solved long-standing open problems. The notion of HDXs relevant to these results are defined in terms of *spectral gaps* of combinatorial Laplacians (which were already introduced by Eckmann in the 1940s [38]) and related operators. A

[1] We refer the reader to [83, 93, 63] and [94] for two books and two surveys on expander graphs. [95] is a survey article on HDXs by Lubotzky presented as a plenary talk at the International Congress of Mathematics 2018.

[2] As an example, the papers [130] and [56] show non-comparability for the so-called coboundary expansion constant (which we introduce later in this introduction and more thoroughly in Chapter 3) and the spectral gap of high-dimensional Laplacians (for a definition see Chapter 2).

[3] This work is based on a general theory of spectral properties of random walks on simplicial complexes developed in a series of papers including [74, 33, 30, 114, 75] and [4]. Furthermore, it fostered a whole line of work of analyzing Markov chains via HDXs. Going beyond matroids there have been applications to the analysis of the hardcore model, Ising model, planar monomer-dimer systems, sampling (edge) colorings and many more. See for instance [25, 43, 5, 1, 24, 8] and [16], to name a few.

[4] A connection between HDXs and computer science was noticed early on in [73] by interpreting coboundary expansion as a certain property to be testable. The relevance of HDXs to probabilistic checkable proofs and error correcting codes was revealed in various works such as [33, 32, 30] and [3].

key tool for the proofs are local-to-global arguments which try to boost (easier-to-analyze) local expansion properties, such as expansion of links in simplicial complexes, to global ones. This is a truly high-dimensional phenomenon whose study can be traced back to the work of Garland in the 1970s [51].

In this thesis, we focus on a more combinatorial notion of expansion - *coboundary expansion*. This notion of HDXs arose independently in the work of Linial, Meshulam and Wallach [89, 109] and of Gromov [54], generalizes the edge expansion constant of graphs and provides a quantitative measure for vanishing cohomology.

To define this notion, consider a *pure*[5] d-dimensional simplicial complex X. We write $X(k)$ for the set of k-simplices of X. Endow X with the weight function $w\colon X \to \mathbb{R}_{\geq 0}$ given by

$$\sigma \mapsto w(\sigma) = \frac{|\{\tau \in X(d) : \sigma \subseteq \tau\}|}{\binom{d+1}{|\sigma|}|X(d)|}.$$

These weights, often called *Garland weights*, induce a norm $\|\cdot\|$ on cochain groups $C^k(X;\mathbb{F}_2)$ with coefficients in the finite field $\mathbb{F}_2$ with two elements[6] by

$$\|c\| = \sum_{\sigma \in X(k), c(\sigma) \neq 0} w(\sigma)$$

which itself gives rise to a quotient norm $\|[\cdot]\|$ on $C^k(X;\mathbb{F}_2)/B^k(X;\mathbb{F}_2)$ given by

$$\|[c]\| = \min\{\|c+b\| : b \in B^k(X;\mathbb{F}_2)\}.$$

Definition (Coboundary expansion constants[7])**.** Let X be a d-dimensional simplicial complex. Let $0 \leq k \leq d-1$. The *k-th coboundary expansion constant $\eta_k(X)$ of X (with respect to $\|\cdot\|$-norm and $\mathbb{F}_2$-coefficients)* is defined as

$$\eta_k(X) := \min_{c \in C^k(X;\mathbb{F}_2) \setminus B^k(X;\mathbb{F}_2)} \frac{\|\delta c\|}{\|[c]\|}.$$

Note that $\eta_k(X) > 0$ if and only if $\tilde{H}^k(X;\mathbb{F}_2) = 0$. This is why we think of $\eta_k(X)$ as quantifying the vanishing of the k-th cohomology group of X with coefficients in $\mathbb{F}_2$.

Gromov established a remarkable link between coboundary expansion and the so-called *topological overlap property*. We say that a d-dimensional simplicial complex is *c-topologically overlapping* for some constant $c > 0$ if for every continuous map[8] $f\colon |X| \to \mathbb{R}^d$ there is $p \in \mathbb{R}^d$ such that $|\{\sigma \in X(d) : p \in f(\sigma)\}| \geq c|X(d)|$. Informally speaking, Gromov's celebrated Topological Overlap Theorem states that for any dimension d and vector of positive real numbers $\eta = (\eta_0, \ldots, \eta_{d-1})$ there is a constant c, depending solely on d and η, such that if X is a d-dimensional simplicial complex with $\eta_k(X) \geq \eta_k$ for all $0 \leq k \leq d-1$ then X is c-topologically overlapping.[9] Gromov then shows that the

[5]A simplicial complex is pure if every (inclusion) maximal simplex has the same size.

[6]For a definition (co)chain groups and simplicial (co)homology see Section 2.2 below.

[7]We will give a more general definition for coboundary expansion constants in Chapter 3 including some motivation.

[8]Here $|X|$ denotes the *geometric realization* or *polyhedron* of X.

[9]More formally, there is another technical condition requiring some local sparseness of X that we sweep under the rug here. Moreover, the strong condition of vanishing cohomology can be weakened to the condition that every coboundary has a small cofilling and that every non-trivial cocycle has large norm. We refer to [36] for more details and a concise, streamlined proof of Gromov's Topological Overlap Theorem as well as Section 2.1-2.5 in [54] for Gromov's original argument.

complete d-dimensional complex K_n^d on n vertices satisfies $\eta_k(K_n^d) \geq 1$ for all $n, d \in \mathbb{Z}_{>0}$ and $0 \leq k \leq d-1$. In particular, K_n^d is c_d-topologically overlapping for some $c_d > 0$. This is related to a classical problem in discrete geometry – the *point selection problem*. There one asks for the optimal constant c_d such that for every affine map $f\colon |K_n^d| \to \mathbb{R}^d$ there is a point $p \in \mathbb{R}^d$ with

$$|\{\sigma \in K_n^d(d) : p \in f(\sigma)\}| \geq c_d \binom{n}{d+1} + o(n^{d+1}) \text{ as } n \to +\infty.$$[10]

The fact that $c_d > 0$ was shown in [18] for $d = 2$ (showing that $c_2 = 2/9$) and in [12] for $d \geq 3$.

Surprisingly, Gromov's proof of the Topological Overlap Theorem does not only generalize results for the point selection problem from affine to continuous maps but his lower bounds on the overlap constant improved upon previously known estimates in the affine setting.

Linial–Meshulam [89] (for $d = 2$) and Meshulam–Wallach [109] (for $d \geq 3$) also showed that $\eta_k(K_n^d) \geq 1$ for all $k, n \in \mathbb{Z}_{>0}$ and $0 \leq k \leq d-1$. They used this for an intricate cocycle counting argument to determine the exact threshold for the vanishing of $\tilde{H}^{d-1}(X; \mathbb{F}_2)$ where X is sampled according to the so-called *Linial–Meshulam model* $X_d(n, p)$. $X \sim X_d(n, p)$ is obtained as follows: Start with a complete $(d-1)$-skeleton on a vertex set V of size n. For every subset $\sigma \subseteq V$ with $|\sigma| = d+1$ add σ as a d-simplex independently at random with probability p.[11]

The works of Gromov, Linial–Meshulam and Meshulam–Wallach indicate that the notion of coboundary expansion is well-suited for topological applications. A main theme of the present thesis continues this line of research and provides a general method for the study of quantitative non-embeddability problems, such as crossing numbers of graphs and simplicial complexes, from the perspective of coboundary expansion properties of configuration spaces naturally associated with these types of problems.

Mostly, we will focus on the *(pair) crossing number problem.* Given a d-dimensional simplicial complex X and a continuous map $f\colon |X| \to \mathbb{R}^{2d}$ we define the *independent pair crossing number* $\operatorname{ipcr}(f)$ *of* f as

$$\operatorname{ipcr}(f) := \frac{1}{2}|\{(\sigma, \tau) \in X(d) \times X(d) : \sigma \cap \tau = \emptyset, f(\sigma) \cap f(\tau) \neq \emptyset\}|,$$

i.e. as the number of pairs of disjoint d-simplices of X whose images under f intersect. The *independent pair crossing number* $\operatorname{ipcr}(X)$ *of* X is defined as

$$\operatorname{ipcr}(X) := \min\{\operatorname{ipcr}(f) : f\colon |X| \to \mathbb{R}^{2d} \text{ continuous}\}.$$

Clearly, $\operatorname{ipcr}(X) > 0$ implies that X is not embeddable to $\mathbb{R}^{2d}$.[12] We think of $\operatorname{ipcr}(X)$ as a quantitative measure of non-embeddability for which we would like to prove lower bounds.

[10] Usually, this is formulated in the equivalent form that for every set of n points in $\mathbb{R}^d$ there is a point $p \in \mathbb{R}^d$ which is contained in at least $c_d\binom{n}{d+1} + o(n^{d+1})$ of the simplices spanned by $d+1$ points in P.

[11] By now, a much more fine-grained understanding of (topological) properties of the Linial–Meshulam and related random models of simplicial complexes have been obtained. We refer to [70] and [17] for two surveys on the study of random simplicial complexes - the former might be slightly outdated in some places.

[12] The converse is true for graphs ($d = 1$) and for $d \geq 3$ due to the completeness of the so-called *van Kampen obstruction* [47].

The embeddability question is often studied through the framework of configuration spaces and test maps, which provides a powerful topological toolbox to the study of a variety of geometric and combinatorical problems (see [107] and [139] for excellent introductions to the topic). In a nutshell this framework works as follows: One starts by constructing a configuration space to encode all possible solutions/configurations. A test map from the configuration space to some test space allows to separate some distinguished configurations from the others by, e.g., mapping them to zero. Often the configuration and test space are endowed with a group action of a (finite) group G and the test map is equivariant with respect to these actions, i.e., it commutes with the group action. One usually seeks for topological properties of the configuration/test space pair that prohibit the existence of an equivariant map from the configuration to the test space avoiding zero and whence guarantee the existence of a configuration with a desired property.

For the problem of embedding a d-dimensional simplicial complex X to $\mathbb{R}^{2d}$ a suitable configuration space is the *deleted join* X^{*2}_{Δ} *of* X. The *join* $X*Y$ of the simplicial complexes X and Y is the simplicial complex whose simplices are joins $\sigma \otimes \tau$ of pairs of simplices $\sigma \in X$ and $\tau \in Y$. Thinking of X and Y as abstract simplicial complexes, i.e. as a downward closed set systems, $\sigma \otimes \tau$ is simply the disjoint union of the sets σ and τ. Then X^{*2}_{Δ} is the subcomplex of $X * X$ given by

$$X^{*2}_{\Delta} := \{\sigma \otimes \tau : \sigma, \tau \in X, \sigma \cap \tau = \emptyset\}.$$

The points in the geometric realization of X^{*2}_{Δ} can be described as formal convex combinations $x = tx_1 \oplus (1-t)x_2$, where $t \in [0,1]$ and $x_1, x_2 \in |X|$ are from disjoint simplices of X. Here we use the convention that $0x_1 \oplus 1x_2 = 0x_1' \oplus 1x_2$ and $1x_1 \oplus 0x_2 = 1x_1 \oplus x_2'$ for all $x_1, x_1', x_2, x_2' \in |X|$.

Note that $\nu \colon |X^{*2}_{\Delta}| \to |X^{*2}_{\Delta}|$ given by

$$tx \oplus (1-t)y \mapsto (1-t)y \oplus tx$$

is a simplicial automorphism with $\nu \circ \nu = \mathrm{id}$. The map ν induces a free action of the cyclic group of two elements $\mathbb{Z}/2$ on X^{*2}_{Δ} turning it into a free $\mathbb{Z}/2$-complex.[13]

Now, given a continuous map $f \colon |X| \to \mathbb{R}^{2d}$, we get an induced map $F \colon |X^{*2}_{\Delta}| \to \mathbb{R}^{2d+1}$ given by

$$tx \oplus (1-t)y \mapsto \begin{pmatrix} 1-2t \\ tf(x) - (1-t)f(y) \end{pmatrix}.$$

Note that $\mathbb{Z}/2$ also acts on $\mathbb{R}^{2d+1}$ by the antipodal map $a \colon \mathbb{R}^{2d+1} \to \mathbb{R}^{2d+1}$ mapping x to $-x$. F is *equivariant*, i.e. $F \circ \nu = a \circ F$. Moreover, we have $F(tx \oplus (1-t)y) = 0$ if and only if $t = 1/2$ and $f(x) = f(y)$. It follows that

$$\mathrm{ipcr}(f) = \frac{1}{2}|\{\sigma \otimes \tau \in X^{*2}_{\Delta}(2d+1) : 0 \in F(\sigma \otimes \tau)\}|.$$

Furthermore, the existence of an embedding $f \colon X \to \mathbb{R}^{2d}$ gives rise to an equivariant map $F \colon X^{*2}_{\Delta} \to_{\mathbb{Z}/2} \mathbb{R}^{2d+1}$ whose image avoids 0. A generalization of the classical Borsuk–Ulam theorem (see for instance [141]) to $\mathbb{Z}/2$-complex/spaces rules out the existence of an

[13]For more detailed definitions of these notions we refer to Section 2.4.

equivariant map $F\colon |Y| \to_{\mathbb{Z}/2} \mathbb{R}^d$ avoiding 0 if Y is a d-dimensional simplicial complex with a free $\mathbb{Z}/2$-action and $\tilde{H}^k(Y;\mathbb{F}_2) = 0$ for all $0 \leq k \leq d-1$.[14] Our first main result is a quantitative version of this result for coboundary expanders which can be seen as an analogue of Gromov's Topological Overlap Theorem in the setting of free $\mathbb{Z}/2$-complexes and equivariant maps.

Theorem 1.1 (Quantitative Borsuk–Ulam Theorem)**.** *Let $d \in \mathbb{Z}_{>0}$ and $\eta = (\eta_0, \dots, \eta_{d-1})$ be a vector of positive real numbers. Let Y be a d-dimensional free $\mathbb{Z}/2$-complex such that $\eta_k(Y) \geq \eta_k$ for all $0 \leq k \leq d-1$. Then for any equivariant map $F\colon |Y| \to_{\mathbb{Z}_2} \mathbb{R}^d$ we have*

$$|\{\sigma \in Y(d) : 0 \in F(\sigma)\}| \geq \frac{\prod_{i=0}^{d-1} \eta_i}{2^d} |Y(d)|.$$

The quantitative Borsuk–Ulam Theorem implies a (non-trivial) lower bound on $\operatorname{ipcr}(X)$ for a d-dimensional simplicial complex X whenever we can prove good lower bounds for the coboundary expansion constants of X^{*2}_{Δ}. In particular, this would show the non-embeddability of X to $\mathbb{R}^{2d}$. Thus, it is natural to ask for conditions on X which ensure that X^{*2}_{Δ} is a coboundary expander. More specifically, is it possible to bound the coboundary expansion constants of X^{*2}_{Δ} in terms of the coboundary expansion of X? We are very far from a satisfying answer to this question (and as we will see the answer might be quite delicate) but for *sufficiently thick*[15] *spherical buildings* (for a definition of these complexes see Defintion 5.7) we know how to prove expansion for their deleted join. This leads us to the following application of Theorem 1.1.

Theorem 1.2 (Quantitative non-embeddability for sufficiently thick spherical buildings)**.** *For every dimension $d \in \mathbb{N}$ there exists $\delta_d > 0$ and $\mu_d > 0$ such that for every d-dimensional δ_d-thick spherical building X we have*

$$\operatorname{ipcr}(X) \geq \mu_d \cdot \binom{|X(d)|}{2}.$$

Theorem 1.2 makes progress on a conjecture of Tancer and Vorwerk (see [134, Conjecture 8.1]) who conjectured that no d-dimensional 3-thick spherical building embeds to $\mathbb{R}^{2d}$. Thus, under the stronger assumption of sufficiently large thickness, we can show non-embeddability of d-dimensional spherical buildings to $\mathbb{R}^{2d}$ in a strong quantitative sense.

As another application of Theorem 1.1 we will prove that a bounded degree expander graph $G = (V, E)$ satisfies $\operatorname{ipcr}(G) = \Omega(|V|^2/\log|V|)$ (see Theorem 5.14 for a more precise statement clarifying the dependencies of the hidden constants on the edge expansion constant and maximum degree of G). This improves a previous lower bound due to Kolman and Matoušek [79, Theorem 2] by a $\log|V|$-factor.

Our proof of Theorem 1.1 is quite robust. In particular, it also works for different choices of weights instead of the Garland weights. Moreover, we will prove a generalization (see Theorem 4.1 below) to the setting where $\mathbb{Z}/p$, for some prime p, acts on X freely and on $\mathbb{R}^d$ by orthogonal linear transformations and freely on $\mathbb{R}^d \setminus \{0\}$. Such setups naturally arise in the study of *Tverberg-type problems* (see [107, 139, 15] for more details). In fact,

14 The Borsuk–Ulam Theorem is the case when Y is (an equivariant triangulation of) the d-dimensional sphere $\mathbb{S}^d$ with the antipodal action.

15 By definition, we call a d-dimensional simplicial complex X *δ-thick* for some $\delta \in \mathbb{Z}_{>0}$ if every $(d-1)$-simplex of X is contained in at least δ d-simplices.

as an application we will give a new proof of the lower bound of Vučić and Živaljević ([138, Theorem 1]) on the number of *Tverberg partitions* in the prime case. We will show:

Theorem 1.3. *Let p be a prime. Let $N = (d+1)(p-1)$. Let σ^N be an N-dimensional simplex. Then for every continuous map $f\colon |\sigma^N| \to \mathbb{R}^d$ the number of unordered p-tuples $\{F_1, \ldots, F_p\}$ of pairwise disjoint faces of σ^N with $\bigcap_{i=1}^p f(F_i) \neq \emptyset$ is at least*

$$\frac{1}{(p-1)!}\left(\frac{p}{2}\right)^{N/2}.$$

With Theorem 1.1 in mind (and ignoring the difference between the deleted join and the join) it is worthwhile to get a good understanding of coboundary expansion properties of the join $X * X$ of a simplicial complex with itself. Using Künneth formula (see, e.g., [61, Chapter V.]) we see that $\tilde{H}^k(X * X; \mathbb{F}_2) = 0$ for all $0 \leq k \leq 2d$ if $\tilde{H}^j(X; \mathbb{F}_2) = 0$ for all $0 \leq j \leq d-1$. One could ask for a quantitative version of this result. The proof of such a result might be quite delicate and a very strong form of it would not further generalize to arbitrary joins $X * Y$ of simplicial complexes X and Y:

Proposition 1.4. *There are positive constants C and η such that there are infinite families of regular graphs $(G_n)_{n\in\mathbb{N}}$ and $(H_n)_{n\in\mathbb{N}}$ with the property that for all $n \in \mathbb{N}$*

(i) $\eta_0(G_n) \geq \eta$,

(ii) $\eta_0(H_n) \geq C\frac{\log|G_n(1)|}{|G_n(1)|}$,

(iii) $\eta_2(G_n * H_n) \leq \frac{6}{|G_n(1)|}$.[16]

In particular,

$$\lim_{n\to+\infty} \frac{\eta_2(G_n * H_n)}{\eta_0(G_n)\eta_0(H_n)} = 0.$$

On the positive side, we are able to establish coboundary expansion for $X * X$ if the coboundary expansion of X comes with a special certificate which we call a *random abstract cone*. Such a certificate is, for instance, available for spherical buildings.

An initial motivation for proving a quantitative Borsuk–Ulam type theorem such as Theorem 1.1 was to have a new approach to tackle various old conjectures on crossing numbers of graphs. Arguably the most prominent (and oldest) of these conjectures is *Turán's brick factory problem* (see for instance [13] or [126, Ch. 1]) which asks to determine the crossing number $\mathrm{cr}(K_{m,n})$ of a complete bipartite graph $K_{m,n}$.[17] It is conjectured that $\mathrm{cr}(K_{m,n}) = Z_{m,n}$ where

$$Z_{m,n} := \left\lfloor\frac{m-1}{2}\right\rfloor\left\lfloor\frac{m}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n}{2}\right\rfloor.$$

[16] Here we think of G_n as a 1-dimensional simplicial complex. Hence, $|G_n(1)|$ is the number of edges of G_n.

[17] The crossing number $\mathrm{cr}(G)$ of a graph is the smallest number of edge crossings of any drawing of G in the plane. Clearly, $\mathrm{ipcr}(G) \leq \mathrm{cr}(G)$ for all graphs G and it is a major open conjecture that $\mathrm{ipcr}(G) = \mathrm{cr}(G)$ (see [117, 126]).

This is also known as *Zarankiewicz' conjecture.* A construction attributed to Zarankiewicz shows that $\mathrm{cr}(K_{m,n}) \le Z_{m,n}$. Even the asymptotics of $\mathrm{cr}(K_{m,n})$ for $m, n \to +\infty$ is not fully understood.

Note that $(K_{m,n})_{\Delta}^{*2} = [m]_{\Delta}^{*2} * [n]_{\Delta}^{*2}$.[18] Thus, for large m and n, $(K_{m,n})_{\Delta}^{*2}$ is roughly equal to the complete 4-partite 3-dimensional complex $\Lambda^3_{m,m,n,n} := [m] * [m] * [n] * [n]$, in the sense that they only differ by a negligable number of simplices. What is more, it is not too difficult to show that $\eta_k(\Lambda^3_{m,m,n,n}) = \eta_k((K_{m,n})_{\Delta}^{*2}) + o(1)$ as $m, n \to +\infty$ (cf. Proposition 5.2). In particular, Theorem 1.1 implies

$$\mathrm{cr}(K_{m,n}) \ge \mathrm{ipcr}(K_{m,n}) \ge \frac{\eta_0(\Lambda^3_{m,m,n,n})\eta_1(\Lambda^3_{m,m,n,n})\eta_2(\Lambda^3_{m,m,n,n})}{16} (1 - o(1))\, m^2 n^2$$

as $m, n \to +\infty$ which would prove an asymptotic version Zarakiewicz' conjecture if we could show that $\eta_k(\Lambda^3_{m,m,n,n}) \ge 1$ for all $k \in \{0, 1, 2\}$. Unfortunately, at least for $k = 2$, this fails to be true.

Theorem 1.5. *Let $d \in \mathbb{N}$ be a dimension, $n_0, n_1, \ldots, n_d \ge 2$ integers. Write $\Lambda^d_{n_0,n_1,\ldots,n_d} = [n_0] * \cdots * [n_d]$ for the complete $(d+1)$-partite complex with parts of size $n_0, n_1, \ldots, n_{d-1}$ and n_d. If 2^d divides n_i for all $0 \le i \le d$, then*

$$\eta_{d-1}(\Lambda^d_{n_0,n_1,\ldots,n_d}) \le \frac{d+1}{2^d}.$$

We will write Λ^d_n for $\Lambda^d_{n_0,\ldots,n_d}$ if $n_0 = n_1 = \cdots = n_d = n$. The proof of Theorem 1.5 generalizes to give us upper bounds on $\eta_k(\Lambda^d_n)$ for $k < d-1$ which are exponentially small in d for constant codimension $d - k$ (see Proposition 7.8). Additionally, we will make a precise conjecture for the value of $\eta_1(\Lambda^2_n)$ for all $n \in \mathbb{Z}_{>0}$.

A key ingredient for the constructive proof of Theorem 1.5 is an interesting family of d-coboundaries with some extra algebraic structure (related to the *sum complexes* studied in [88]). We can obtain such a family of d-coboundaries for any $(d+1)$-partite d-dimensional complex. This allows us, using the probablistic method, to show an upper bound on the $(d-1)$-th expansion constant of the spherical building $A_d(\mathbb{F}_q)$ associated with $\mathrm{GL}_{d+2}(\mathbb{F}_q)$ for sufficiently large q. More precisely, given a prime power q, $A_d(\mathbb{F}_q)$ is the simplicial complex with vertices the non-trivial, proper subspaces of $\mathbb{F}_q^{d+2}$, a $(d+2)$-dimensional vector space over the finite field $\mathbb{F}_q$ with q elements, and k-simplices corresponding to chains $\{0\} \ne U_0 \subsetneq U_1 \subsetneq \cdots \subsetneq U_k \subsetneq \mathbb{F}_q^{d+2}$ of subspaces. In particular, $A_1(\mathbb{F}_q)$ is the points vs. lines graph of the Desarguesian projective plane of order q for which it is known that $\eta_0(A_1(\mathbb{F}_q)) \ge 1 - \frac{2\sqrt{q}}{q+1}$ (see [93, Section 8.3]). It was conjectured by Lubotzky, Meshulam and Mozes [98, Conjecture 5.1] that this extends to higher dimensions in the sense that $\eta_{d-1}(A_d(\mathbb{F}_q)) = 1 + o(1)$ as $q \to +\infty$. We disprove this conjecture in a rather strong sense for all $d \ge 2$.

Theorem 1.6. *For any dimension d and $\varepsilon > 0$ there is $Q = Q(d, \varepsilon) \in \mathbb{Z}_{>0}$ such that for all prime powers $q \ge Q$ we have*

$$\eta_{d-1}(A_d(\mathbb{F}_q)) \le \frac{d+1}{2^d} + \varepsilon.$$

[18] For $n \in \mathbb{Z}_{>0}$ we write $[n]$ for the set $\{1, \ldots, n\}$ which we also think of as 0-dimensional simplicial complex of n discrete points.

We will further close the gap between the known lower and upper bounds for $\eta_{d-1}(\Lambda_n^d)$ by also improving upon previously known lower bounds. Due to a recursive bound on $\eta_{d-1}(\Lambda_n^d)$ in terms on $\eta_{d-2}(\Lambda_n^{d-1})$ we focus on improving the lower bound for $\eta_1(\Lambda_n^2)$. Using a computer-aided proof, we will show

Proposition 1.7. *For all $n \in \mathbb{Z}_{>0}$ we have $\eta_1(\Lambda_n^2) \geq 0.67159$.*

One should compare this to the previously known lower bound $\eta_1(\Lambda_n^2) \geq 3/5$ and the upper bound $\eta_1(\Lambda_n^2) \leq 3/4$ from Theorem 1.5 (if 4 divides n). The exact value of $\eta_1(\Lambda_n^2)$ remains elusive.

In an attempt to circumvent the obstacle of Λ_n^d not having optimal coboundary expansion 1 and thus preventing us from a direct application of the quantitative Borsuk–Ulam theorem to prove Zarankiewicz' conjecture (up to lower order terms), we study the coboundary expansion constant $\zeta_{d-1}(\Lambda_n^d)$ of Λ_n^d with respect to integer coefficients and (normalized) ℓ_2^2-norm (see Section 3.1 for a discussion of coboundary expansion constants with respect to various coefficients and size functions/norms). We will show that for sufficiently large n we have $\zeta_{d-1}(\Lambda_n^d) > \eta_{d-1}(\Lambda_n^d)$. We came short to show that $\zeta_{d-1}(\Lambda_n^d) = 1$ but working within the setting of integer coefficients and (weighted) ℓ_2^2-norm, we can recover Kleitman's 4/5 bound [78] on the crossing number of complete bipartite graphs (up to lower order terms). More precisely, we will give a new proof of the following result

Proposition 1.8.

$$\lim_{m,n\to+\infty} \frac{\mathrm{cr}(K_{m,n})}{Z_{m,n}} \geq \frac{4}{5}.$$ [19]

Part of the proof of Proposition 1.8 is to show that $\zeta_2(\Lambda^3_{m,m,n,n}) \geq 4/5$.[20] We conjecture that in fact $\zeta_2(\Lambda^3_{m,m,n,n}) = 1$ for all $m, n \in \mathbb{Z}_{>0}$ which would imply

$$\lim_{m,n\to+\infty} \frac{\mathrm{cr}(K_{m,n})}{Z_{m,n}} = 1$$

and thus prove an asymptotic version of Zarankiewicz' conjecture.

1.1 Structure of Thesis

The remaining chapters of this thesis are structured as follows:

The next chapter gives some basic definitions regarding simplicial complexes and cohomology, etc.

In Chapter 3 we give a careful introduction to the notion of coboundary expansion. Moreover, we introduce the random cofilling technique, which is arguably the only known

[19] To be a bit more precise here, for every fixed $n \in \mathbb{N}$, we consider the function $m \mapsto \varphi_n(m) := \frac{\mathrm{cr}(K_{m,n})}{Z_{m,n}}$. One can show that φ_n converges pointwise to a function $\varphi_\infty(m) = \lim_{n\to+\infty} \varphi(m)$. Furthermore, it is another easy exercise to show that $\lim_{m\to+\infty} \varphi_\infty(m)$ exists. We write $\lim_{m,n\to+\infty} \frac{\mathrm{cr}(K_{m,n})}{Z_{m,n}}$ for $\lim_{m\to+\infty} \varphi_\infty(m)$.

[20] Strictly speaking, we show something a bit weaker, namely a lower bound of 4/5 on the 2nd expansion constant of $\Lambda_{m,m,n,n}$ with respect to some weighted ℓ_2^2-norm and integer coefficients. See Section 5.5 for the full proof of Proposition 1.8.

technique to prove lower bounds on coboundary expansion constants. We illustrate this technique by some examples.

In Chapter 4 we show our quantitative version of the Borsuk–Ulam Theorem.

We use this theorem in Chapter 5 to harvest some geometric applications such as the quantitative non-embeddability result for spherical buildings, lower bounds on (pair) crossing number of complete bipartite graphs and on the number of Tverberg partitions.

Chapter 6 is devoted to the study of expansion properties of the join of two simplicial complexes in general.

In Chapter 7 we provide upper bounds on coboundary expansion constants of $(d+1)$-partite d-dimensional simplicial complexes. In particular, we prove the upper bounds as claimed in Theorem 1.5 and Theorem 1.6.

In Chapter 8 we focus on lower bounds on expansion constants of Λ_n^d. We will see that $\zeta_{d-1}(\Lambda_n^d) > \eta_{d-1}(\Lambda_n^d)$ for $d \geq 2$ and sufficiently large n. Moreover, we make further progress on closing the gap between the known lower and upper bounds for $\eta_{d-1}(\Lambda_n^d)$ by improving upon existing lower bounds.

Chapter 9 of this thesis is a loose collection of various further observations related to expansion.

We close with a brief summary and an outlook in Chapter 10.

Chapter 2

Preliminaries

This chapter gives some basic definitions such as the definition of simplicial complexes and simplicial cohomology. While everything in this chapter is completely standard, introducing the material in some detail allows us to fix some notation which we can use throughout the thesis. The expert reader might want to skip this chapter on a first read and return to it if necessary. Our discussion here is mainly based on the respective parts in [107] and [110].

2.1 Simplicial Complexes and More General Cell Complexes

An *abstract simplicial complex* X is a downward closed set system $X \subseteq 2^V$ for some vertex set V. That is, if $\sigma \in X$ and $\tau \subseteq \sigma$ then $\tau \in X$. In particular, if X is non-empty then the empty set $\emptyset$ is in X. Elements of X are called *simplices*. The *dimension* $\dim \sigma$ of a simplex $\sigma \in X$ is $\dim \sigma = |\sigma| - 1$ and we will call σ a k-simplex if $\dim \sigma = k$. As usual, we use vertices, edges, triangles and tetrahedra as synonyms for 0-,1-,2- and 3-simplices, respectively. We write $X(k)$ for the set of k-simplices and $X^{(k)} = \bigcup_{-1 \leq i \leq k} X(i)$ for *k-skeleton of X*. A simplicial complex X is *s-partite* for some $s \in \mathbb{Z}_{>0}$ if there is a labelling $\lambda\colon X(0) \to [s]$ of the vertices of X such that $|\sigma \cap \lambda^{-1}(\{i\})| \leq 1$ for all $\sigma \in X$ and $i \in [s]$. All simplicial complexes considered in this thesis are finite, i.e. they have finitely many simplices or equivalently $|X(0)| < \infty$. The *dimension* $\dim X$ of a simplicial complex X is the maximal dimension of a simplex in X. A simplicial complex X is *pure* if every simplex is contained in a simplex of dimension $\dim X$. A *simplicial map* $f\colon X \to Y$ between abstract simplicial complexes is a map $f\colon X(0) \to Y(0)$ such that $f(\sigma) \in Y$ for all $\sigma \in X$.

To be able to talk about continuous maps from an abstract simplicial complex to $\mathbb{R}^d$ (or some other topological space), we need the notion of a geometric realization or polyhedron of X. To this end, let us first define geometric simplicial complexes.

A *geometric simplex* σ in $\mathbb{R}^d$ is the convex hull of finitely many affinely independent points $A \subseteq \mathbb{R}^d$. Elements in A are the *vertices* of σ. The *dimension* $\dim \sigma$ of a simplex is $\dim \sigma = |A| - 1$ and we call σ a *k-simplex* if $\dim \sigma = k$. The convex hull of a subset of A is called a *face* of σ.

A *geometric simplicial complex* Δ is a family of geometric simplices in $\mathbb{R}^d$ such that

(i) Every face of a simplex in Δ is also a simplex in Δ.

(ii) If $\sigma, \tau \in \Delta$ then the intersection $\sigma \cap \tau$ is a face of σ and of τ.

The *polyhedron* $|\Delta|$ of Δ is the union of all geometric simplices in Δ considered as a topological space endowed with the subspace topology of $\mathbb{R}^d$. The *vertex set* $V(\Delta)$ of Δ is the union of all vertices of simplices in Δ.

Every geometric simplicial complex has an associated abstract simplicial complex $X(\Delta) \subseteq 2^{V(\Delta)}$ where $\sigma \subseteq V(\Delta)$ is in $X(\Delta)$ if σ is the vertex set of some simplex in Δ.

A *geometric realization* of an abstract simplicial complex X is a geometric simplicial complex Δ with $X = X(\Delta)$.

It turns out that the polyhedron of any two geometric realizations of X are homeomorphic. We write $|X|$ for this topological space and call it the *polyhedron* or *geometric realization* of X.

A simplicial map $f\colon X \to Y$ induces a continuous map $|f|\colon |X| \to |Y|$ which maps vertices in $|X|$ to vertices of $|Y|$ according to f and for a point $x = \sum_{i=0}^{k} \lambda_i v_i \in X$ with vertices $v_i \in X$ and $\lambda_i \geq 0$, $\sum_{i=0}^{k} \lambda_i = 1$ we have $|f|(x) = \sum_{i=0}^{k} \lambda_i f(v_i)$.

A *subdivision* Δ' of a geometric simplicial complex Δ is a geometric simplicial complex such that $|\Delta'|$ and $|\Delta|$ are homeomorphic, every simplex $\sigma \in \Delta'$ is contained in some simplex $\tau \in \Delta$ and every $\tau \in \Delta$ is the union of finitely many simplices in Δ'.

A *subdivision* X' of an abstract simplicial complex X is an abstract simplicial complex associated with a subdivision of a geometric realization of X.

Using the geometric realization we can talk about continuous maps $f\colon |X| \to \mathbb{R}^d$ from a simplicial complex X to $\mathbb{R}^d$.

A map $f\colon |X| \to \mathbb{R}^d$ is *simplexwise affine* if it restricts to an affine map on each simplex in X. More generally, a map $f\colon |X| \to \mathbb{R}^d$ is *piecewise linear (a PL map)* if there is a subdivision X' of X such that the map $f\colon |X'| \to \mathbb{R}^d$ is simplexwise affine.

We will almost always use simplicial complexes as a combinatorial description of topological spaces except in Chapter 4 where we also encounter CW complexes.

Write $\mathbb{B}^d$ for the closed d-dimensional unit ball in $\mathbb{R}^d$. A *d-cell* is a topological space homeomorphic to $\mathbb{B}^d$. An *open d-cell* is a topological space homeomorphic to the interior of $\mathbb{B}^d$.

A *(finite) CW-complex* is a topological space X together with a finite collection $(\sigma_\alpha, f_\alpha)_{\alpha \in \mathcal{A}}$ where

(i) $\sigma_\alpha \subseteq X$ is a d_α-dimensional open cell,

(ii) X is the disjoint union of the σ_α's, i.e. $X = \bigcup_{\alpha \in \mathcal{A}} \sigma_\alpha$ and $\sigma_\alpha \cap \sigma'_\alpha = \emptyset$ for all $\alpha, \alpha' \in \mathcal{A}$ with $\alpha \neq \alpha'$,

(iii) $f_\alpha\colon \mathbb{B}^{d_\alpha} \to X$ is a continuous map such that $f_\alpha|_{\mathrm{Int}(\mathbb{B}^{d_\alpha})}$ is a homeomorphism and such that f_α maps $\partial\mathbb{B}^{d_\alpha}$ to a union of open cells each of dimension less than d_α.

The maps f_α are called *attaching maps.* A CW complex is *regular* if every attaching map is a homeomorphism.

Note that a d-dimensional geometric simplex is a d-cell. Thus, by gluing geometric simplices together along their boundary, we see that the polyhedron $|\Delta|$ of a geometric simplicial complex Δ has the structure of a regular CW complex. Furthermore, if X is a regular CW complex, we can triangulate its cells to obtain a simplicial complex refining the CW complex structure. To be a bit more precise, here is a specific way to achieve this: Given a (finite) partially ordered set $(\mathcal{P}, \prec)$ the *order complex* $\Delta(\mathcal{P})$ of $\mathcal{P}$ is the simplicial complex with vertices the elements in $\mathcal{P}$ and k-simplices $\{v_0, \ldots, v_k\}$ corresponding to chains $v_0 \prec v_1 \prec \cdots \prec v_k$ in $\mathcal{P}$. Given a regular CW complex X with cell structure $(\sigma_\alpha, f_\alpha)_{\alpha \in \mathcal{A}}$ we get a partially ordered set $(\mathcal{P}(X), \subseteq)$ consisting of the closed cells σ_α ordered by inclusion. One can check that the regularity of X implies that the order complex $\Delta(\mathcal{P}(X))$, which is a generalized barycentric subdivision, is a simplicial complex with $|\Delta(\mathcal{P}(X))|$ homeomorphic to X (cf. [101, Ch. III, Theorem 1.7]).

2.2 Chains and Cochains, Simplicial Homology and Cohomology

Let X be a simplicial complex. Let $\sigma \in X$. Two orderings of the vertices of σ are equivalent if they differ by a permutation with even sign. This defines an equivalence relation on all possible orderings of the vertices of σ with precisely two equivalence classes if $\dim \sigma > 0$. An *orientation of* σ is a choice of such an equivalence class. An *oriented simplex* is a simplex σ together with an orientation. We write X_k for the set of oriented k-simplices of X. Given an oriented simplex σ we write $-\sigma$ for the oriented simplex with the opposite orientation which we understand to be equal to σ if $\dim \sigma \in \{-1, 0\}$.

It is convenient to fix an orientation for each k-simplex according to a linear ordering $<$ on the vertices of X. Given such an ordering and a k-simplex $\sigma = \{v_0, \ldots, v_k\}$ with $v_0 < v_1, \cdots < v_k$ we give σ the orientation represented by the ordering $(v_0, \ldots, v_k)$ and write $[v_0, \ldots, v_k]$ for the corresponding oriented simplex. Sometimes it will be convenient to ease the notation and write $v_0 v_1 \ldots v_k$ instead of $[v_0, \ldots, v_k]$.

Let $\sigma = \{v_0, \ldots, v_k\} \in X(k)$ with $v_0 < \cdots < v_k$. For $\tau \in X(k-1)$ define the *oriented incidence number* $[\sigma : \tau]$ by

$$[\sigma : \tau] = \begin{cases} (-1)^j & \text{if } \tau \subseteq \sigma, \tau = \sigma \setminus \{v_j\}, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mathbb{A}$ be an abelian group. Later on, we will mostly work with $\mathbb{A} = \mathbb{Z}$ or $\mathbb{A} = \mathbb{F}_p$, (the additive group of) the field with p elements for some prime (power) p. Let $-1 \leq k \leq \dim X$. The *kth chain group of X with coefficients in* $\mathbb{A}$ is the abelian group $C_k(X; \mathbb{A})$ of all formal sums $c = \sum_{\sigma \in X_k} a_\sigma \sigma$ modulo the relation that $\sigma + (-\sigma) = 0$. Equivalently, it is the abelian group of all formal sums $c = \sum_{\sigma \in X(k)} a_\sigma \sigma$ where we fixed an orientation for every k-simplex σ. Elements in $C_k(X; \mathbb{A})$ are called *k-chains.*

There is a *boundary map* $\partial_k \colon C_k(X; \mathbb{A}) \to C_{k-1}(X; \mathbb{A})$ which is determined by

$$\partial_k(a_\sigma[v_0, \ldots, v_k]) = \sum_{i=0}^{k} ((-1)^i a_\sigma)[v_0, \ldots, \widehat{v_i}, \ldots, v_k],$$

where $\widehat{v_i}$ indicates that the vertex v_i is omitted. One checks that $\partial_k \circ \partial_{k+1} = 0$ for all $0 \le k \le d-1$. In particular, the image of ∂_{k+1} is contained in the kernel of ∂_k, i.e. $\mathrm{Im}(\partial_{k+1}) \subseteq \ker(\partial_k)$.

We call $Z_k(X;\mathbb{A}) := \ker \partial_k$ the *group of k-cycles* and $B_k(X;\mathbb{A}) := \mathrm{Im}\partial_{k+1}$ the *group of k-boundaries*. In particular, $B_k(X;\mathbb{A})$ is a subgroup of $Z_k(X;\mathbb{A})$ and we can define the *k-th homology group $\tilde{H}_k(X;\mathbb{A})$ of X with coefficients in* $\mathbb{A}$ as the quotient $\tilde{H}_k(X;\mathbb{A}) := Z_k(X;\mathbb{A})/B_k(X;\mathbb{A})$.

For an oriented simplex σ and $a \in \mathbb{A}$ define the *elementary cochain* $a\mathbb{1}_\sigma$ as the function $a\mathbb{1}_\sigma \colon X_k \to \mathbb{A}$ with $a\mathbb{1}_\sigma(\sigma) = a$, $a\mathbb{1}_\sigma(-\sigma) = -a$ and $a\mathbb{1}_\sigma(\tau) = 0$ for $\tau \in X_k \setminus \{\sigma, -\sigma\}$. Then we define the *k-th cochain group* $C^k(X;\mathbb{A})$ of X with coefficients in $\mathbb{A}$ as the abelian group of all formal sums $\sum_{\sigma\in X(k)} a_\sigma \mathbb{1}_\sigma$ with $a_\sigma \in \mathbb{A}$ where we assume that we fixed an orientation for every k-simplex.

There is a *coboundary map* $\delta_k \colon C^k(X;\mathbb{A}) \to C^{k+1}(X;\mathbb{A})$ given on elementary cochains $a\mathbb{1}_\sigma$ by

$$\delta_k(a\mathbb{1}_\sigma) = \sum_{\tau\in X(k+1)} [\tau : \sigma] a\mathbb{1}_\tau.$$

$B^k(X;\mathbb{A}) := \mathrm{Im}\delta_{k-1}$ is the *group of k-coboundaries of X* and $Z^k(X;\mathbb{A}) := \ker \delta_k$ is the *group of k-cocycles of X*. One checks that $\delta_k \circ \delta_{k-1} = 0$ and, hence, we can define the *k-th cohomology group $\tilde{H}^k(X;\mathbb{A})$ of X with coefficients in* $\mathbb{A}$ as the quotient $\tilde{H}^k(X;\mathbb{A}) := Z^k(X;\mathbb{A})/B^k(X;\mathbb{A})$.

Assume now that $\mathbb{A}$ is a ring with 1. Then by evaluating cochains on chains, we get a pairing $\langle\cdot,\cdot\rangle \colon C^k(X;\mathbb{A})\times C_k(X;\mathbb{A}) \to \mathbb{A}$. More precisely, for a cochain $\varphi = \sum_{\sigma\in X(k)} a_\sigma \mathbb{1}_\sigma \in C^k(X;\mathbb{A})$ and a chain $\psi = \sum_{\sigma\in X(k)} b_\sigma \sigma \in C_k(X;\mathbb{A})$ we define $\langle\varphi,\psi\rangle := \sum_{\sigma\in X(k)} a_\sigma b_\sigma$.

We have, essentially by definition, that $\langle\delta\varphi,\psi\rangle = \langle\varphi,\partial\psi\rangle$ for all $\varphi \in C^k(X;\mathbb{A})$ and $\psi \in C_{k+1}(X;\mathbb{A})$.

If $\mathbb{A} = \mathbb{F}$ is a field, this pairing gives us the following nice characterization of coboundaries.

Lemma 2.1. *Let $\mathbb{F}$ be a field. Let $c \in C^k(X;\mathbb{F})$. Then the following are equivalent:*

(i) $c \in B^k(X;\mathbb{F})$.

(ii) $\langle c, z\rangle = 0$ *for all cycle* $z \in Z_k(X;\mathbb{F})$.

(iii) $\langle c, z\rangle = 0$ *for* $z \in \mathcal{Z}$ *where* $\mathcal{Z} \subseteq Z_k(X;\mathbb{F})$ *is a generating set.*

The proof of Lemma 2.1 is an easy exercise in linear algebra which we leave to the reader.

Let X and Y be simplicial complexes. Let $f\colon X \to Y$ be a simplicial map. Let $\mathbb{A}$ be an abelian group. f induces a map $f^*\colon C^k(Y;\mathbb{A}) \to C^k(X;\mathbb{A})$ given by

$$(f^*c)([v_0,\dots,v_k]) = \begin{cases} c([f(v_0),\dots,f(v_k)]) & \text{if } \{f(v_0),\dots,f(v_k)\} \in Y(k), \\ 0 & \text{otherwise.} \end{cases}$$

Here we think of cochains as functions on oriented simplices. Note that $\delta f^*c = f^*\delta c$ for all $c \in C^k(Y;\mathbb{A})$. Applying this to the inclusion map $i\colon X \to Y$ of a subcomplex $X \subseteq Y$, we see that the restriction of a coboundary to a subcomplex is a coboundary.

Similarly, we get an induced map $f_*\colon C_k(X;\mathbb{A}) \to C_k(Y;\mathbb{A})$ which for $[v_0,\dots,v_k] \in X_k$ and $a \in \mathbb{A}$ is given by $f_*(a[v_0,\dots,v_k]) = a[f(v_0),\dots,f(v_k)]$ which we understand to be equal to 0 if $\{f(v_0),\dots,f(v_k)\} \notin Y(k)$.

2.3 Links and Localization

Let X be a simplicial complex. Let $\sigma \in X$. The *link* X_σ *of* X *at* σ is the simplicial complex $X_\sigma = \{\tau \in X : \sigma \cap \tau = \emptyset, \tau \cup \sigma \in X\}$. We think of X_σ as a local view of X around σ. Note that $X_\emptyset = X$.

Assume we fixed an ordering $<$ of the vertices. Let $\sigma = [v_0, \dots, v_k] \in X_k$ with $v_0 < \cdots < v_k$. Given an oriented simplex $\tau = [u_0, \dots, u_l] \in (X_\sigma)_l$ we let $\sigma \cup \tau \in X_{k+l}$ be the oriented simplex $[v_0, \dots, v_k, u_0, \dots, u_l]$. With this notation we can define the *localization* c_σ *of* $c \in C^k(X; \mathbb{A})$ as the cochain $c_\sigma \in C^{k-|\sigma|}(X_\sigma; \mathbb{A})$ given by $c_\sigma(\tau) = c(\sigma \cup \tau)$ for all oriented simplices $\tau \in (X_\sigma)_{k-|\sigma|}$.

2.4 G-Spaces and G-Complexes

Let G be a finite group with identity element e. A *group action of* G or G*-action* on a topological space X is a family $\Phi = (\varphi_g)_{g \in G}$ of homeomorphisms $\varphi_g \colon X \to X$ such that $\varphi_e = \mathrm{id}$ and $\varphi_g \circ \varphi_h = \varphi_{gh}$ for all $g, h \in G$. (X, Φ) is called a G*-space*.

Let (X, Φ) and (Y, Ψ) be G-spaces. A map $f \colon X \to Y$ is called G*-equivariant* or a G*-map* or (if the group G is understood) *equivariant* if $f \circ \varphi_g = \psi_g \circ f$ for all $g \in G$. We write $f \colon X \to_G Y$ to indicate that f is a G-equivariant map.

A G-space (X, Φ) is *free* if $\varphi_g(x) \neq x$ for all $x \in X, g \in G \setminus \{e\}$. For $x \in X$ the set $Gx = \{\varphi_g(x) : g \in G\}$ is called the *orbit of* x under the action of G. A G-action is *fixed-point free* if $|Gx| \geq 2$ for all $x \in X$.

A *simplicial* G*-complex* (X, Φ) is a simplicial complex X together with a family $\Phi = (\varphi_g)_{g \in G}$ of simplicial maps such that $(|X|, (|\varphi_g|)_{g \in G})$ is a G-space. The notion of a free or fixed-point free G-spaces naturally extends to simplicial G-complexes. For an abelian group $\mathbb{A}$ the maps $(\varphi_g)_* \colon C_k(X; \mathbb{A}) \to C_k(X; \mathbb{A})$ and $(\varphi_g)^* \colon C^k(X; \mathbb{A}) \to C^k(X; \mathbb{A})$ give rise to an induced G-action on (co)chains. We will write $g.c$ and $h.\tilde{c}$ instead of $(\varphi_g)_* c$ and $(\varphi_h)^* \tilde{c}$.

In this thesis we will only consider group actions by the cyclic group $\mathbb{Z}/p$ of order p for some prime p. In this case the group action of $\mathbb{Z}/p$ on X is determined by the action $\nu \colon X \to X$ of a generator of $\mathbb{Z}/p$. We will sometimes write (X, ν) in this case.

2.5 Expansion for Graphs and the Cheeger Inequality

Let $G = (V, E)$ be a (simple, undirected) graph on n vertices. For $v \in V$ we write $\deg(v)$ for its *vertex degree*, i.e. the number of edges incident to v. Write $A = (A_{u,v})_{u,v \in V} \in \mathbb{R}^{V \times V}$ for the adjacency matrix of G. That is $A_{u,v} = 1$ if $uv \in E$ and 0 otherwise. Let $D \in \mathbb{R}^{V \times V}$ be the diagonal matrix with diagonal entries $D_{v,v} = \deg(v)$. The *normalized Laplacian* $\mathcal{L}$ *of* G is defined as $\mathcal{L}(G) := I - D^{-1/2} A D^{-1/2}$.[1] Note that $\mathcal{L}(G)$ is symmetric and positive semi-definite. Thus, $\mathcal{L}(G)$ has real eigenvalues $0 \leq \lambda_1(G) \leq \lambda_2(G) \leq \cdots \leq \lambda_n(G)$. Moreover, the vector $v_0 = D^{1/2} \mathbb{1}$ satisfies $\mathcal{L}(G) v_0 = 0$. Hence, $\lambda_1(G) = 0$. It is not difficult

[1]Strictly speaking, this is not well-defined if G has isolated vertices, i.e. vertices with $\deg(v) = 0$. In that case, we define $D^{-1/2}_{v,v} = 0$ for all vertices v with $\deg(v) = 0$.

to see that the multiplicity of 0 as an eigenvalue is the number of connected components of G. Thus, $\lambda_2(G) > 0$ if and only if G is connected. Hence, $\lambda_2(G)$, sometimes called the *spectral gap of* G, is a measure of the connectivity of G.

A more combinatorial way of measuring the connectivity is in terms of the *(normalized) Cheeger constant* $\eta_0(G)$, which is sometimes also called *(normalized) edge expansion constant.* To define $\eta_0(G)$ let us introduce the volume $\mathrm{vol}(S)$ of a set of vertices $S \subseteq V$ as $\mathrm{vol}(S) := \frac{1}{2|E|}\sum_{v\in S}\deg(v)$. For $S \in V$ let $E(S, V\setminus S) = \{e \in E : |S\cap e| = |(V\setminus S)\cap e| = 1\}$ be the set of edges in the cut induced by S. Then, we define

$$\eta_0(G) := \min_{\emptyset\neq S\subsetneq V} \frac{|E(S, V\setminus S)|}{|E|\min\{\mathrm{vol}(S), \mathrm{vol}(V\setminus S)\}}.$$

Note that $\eta_0(G) > 0$ if and only if G is connected.

We remark that in the literature the Cheeger constant is often defined as

$$h_0(G) := \min_{\emptyset\neq S\subsetneq V} \frac{|E(S, V\setminus S)|}{\min\{|S|, |V\setminus S|\}}$$

and mostly studied for d-regular graphs. For d-regular graphs we have

$$h_0(G) = \frac{d}{2}\eta_0(G).$$

Define a family of graphs $(G_n)_{n\in\mathbb{N}}$ to be a family of *edge expander graphs* if there is $\eta > 0$ such that $\eta_0(G_n) \geq \eta$ for all $n \in \mathbb{N}$. Similarly, say that $(G_n)_{n\in\mathbb{N}}$ is a family of *spectral expander graphs* if there is $\lambda > 0$ such that $\lambda_2(G_n) \geq \lambda$ for all $n \in \mathbb{N}$.

It turns out that a family of graphs is a family of edge expander graphs if and only if it is a family of spectral expander graphs. This follows from the so-called discrete Cheeger inequality due to Dodziuk [34] and independently due to Alon and Milman [6]. The Cheeger inequality for graphs translates an analogous result of Cheeger [23] for Laplacians defined on Riemannian manifolds to a discrete setting.

Theorem 2.2 (Discrete Cheeger inequality, see, e.g., [26] or [63, Section 4.4])**.** *Let* $G = (V, E)$ *be a graph. Then*

$$\lambda_2(G) \leq \eta_0(G) \leq \sqrt{8\lambda_2(G)}.$$

In view of the discrete Cheeger inequality, we have two equivalent ways of describing families of expander graphs: in terms of the spectral gap of the Laplacian and in terms of the edge expansion constant. While computing the edge expansion constant is known to be NP-hard (see [71, Theorem 2]), $\lambda_2(G)$ can be efficiently computed (up to a prescribed error). This is why even the inequality $\lambda_2(G) \leq \eta_0(G)$, which is considered as the *easy part* of Theorem 2.2, is relevant in practice. Indeed, many known construction of infinite families of constant-degree expander graphs (see, e.g., [50, 92, 105, 106, 124]) establish the expansion properties of the constructed graphs by analyzing the eigenvalues of their Laplacians.

Expander graphs have been extensively studied since their existence was shown by Barzdin and Kolmogorov [80] and Pinsker [119] around 1970. The theory of expander graphs gave rise to a deep interplay between (pure) mathematics and (theoretical) computer science.

We do not touch any further on this theory here and refer the reader to the surveys and books [63, 83, 94, 93]. Here is a word of warning regarding higher dimensions: Both the spectral gap of combinatorial Laplacians and the edge expansion constant of graphs have a natural analogue for higher-dimensional simplicial complexes, but there is no analogue of the Cheeger inequality in higher dimensions. There is even a fairly simple reason why one should not expect such an inequality in dimension ≥ 2. The spectral gap of Laplacians for X can be thought of as a quantitative measure for vanishing cohomology $\tilde{H}^k(X;\mathbb{R})$, while the analogue of the edge expansion constant measures the vanishing of $\tilde{H}^k(X;\mathbb{F}_2)$. While, by the universal coefficient theorem (see, e.g., [59, Section 3.A]), $\tilde{H}^k(X;\mathbb{F}_2) = 0$ implies that $\tilde{H}^k(X;\mathbb{R}) = 0$, the converse does not hold (consider, e.g., a triangulation of the projective plane $\mathbb{R}P^2$). This rules out an analogue of the easy part $\lambda_2(G) \leq \eta_0(G)$ of the Cheeger inequality. In fact, one can rule out such an analogue even if one assumes that $\tilde{H}^k(X;\mathbb{F}_2) = 0$. For the other part of the Cheeger inequality one can construct an infinite family of simplicial complexes such that the spectral gap of the higher-dimensional Laplacian goes to zero much faster than the analogue of the edge expansion constant. We refer to [56, 130] for a detailed discussion.

We discuss the generalization of the edge expansion constant in length in Section 3.1. Higher-dimensional Laplacians have lead to various combinatorial applications. We will not use higher-dimensional Laplacians for any of the results in this thesis but for some remarks it will be helpful to refer to them. This is why, we briefly define them here and refer to [38, 64, 37, 82, 49, 51] and references therein for more background and some applications.

Let X be a d-dimensional simplicial complex. Assume that X is endowed with a weight function $w\colon X \to \mathbb{R}_{\geq 0}$. We can define a weighted inner product $\langle \cdot, \cdot \rangle_w$ on cochains $C^k(X;\mathbb{R})$ by

$$\langle f, g \rangle_w = \sum_{\sigma \in X_k} w(\sigma) f(\sigma) g(\sigma).$$

Write $\delta_k^*\colon C^{k+1}(X;\mathbb{R}) \to C^k(X;\mathbb{R})$ for the adjoint of the coboundary map δ_k. That is δ_k^* is defined through the relation $\langle \delta_k f, g \rangle_w = \langle f, \delta_k^* g \rangle_w$ for all $f \in C^k(X;\mathbb{R})$ and $g \in C^{k+1}(X;\mathbb{R})$. Then, we can define the *kth up-Laplacian* $\mathcal{L}_k^{up}$ *of* X by $\mathcal{L}_k^{up} := \delta_k^* \delta_k$. By definition $\mathcal{L}_k^{up}$ is self-adjoint (with respect to $\langle \cdot, \cdot \rangle_w$) and positive semidefinite. Moreover, $B^k(X;\mathbb{R}) \subseteq \ker \mathcal{L}_k^{up} = Z^k(X;\mathbb{R})$ and every $f \in B^k(X;\mathbb{R})$ is a *trivial eigenvector* of $\mathcal{L}_k^{up}$. All other *non-trivial* eigenvalues are coming from the restriction of $\mathcal{L}_k^{up}$ to $B^k(X;\mathbb{R})^\perp$, the orthogonal complement of $B^k(X;\mathbb{R})$ with respect to $\langle \cdot, \cdot \rangle_w$. Write $\lambda^{(k)}(X)$ for the smallest non-trivial eigenvalue of $\mathcal{L}_k^{up}$, i.e. by the variational characterization of eigenvectors we have

$$\lambda^{(k)}(X) = \min_{f \in B^k(X;\mathbb{R})^\perp, f \neq 0} \frac{\langle f, \mathcal{L}_k^{up} f \rangle_w}{\langle f, f \rangle_w}.$$

Note that $\lambda^{(k)}(X) > 0$ if and only if $\tilde{H}^k(X;\mathbb{R}) = 0$.

It is not difficult to check that for a graph $G = (V, E)$ (thought of as 1-dimensional simplicial complex), we have that $\mathcal{L}(G) = \mathcal{L}_0^{up}$ if we choose $w\colon G \to \mathbb{R}_{\geq 0}$ to be given by $w(v) = \deg(v)$ for $v \in V$ and $w(e) = 1$ for $e \in E$.

Chapter 3

Basics on Coboundary Expansion

This chapter serves two purposes:

(i) introducing the main notion used in this thesis: *coboundary expansion*,

(ii) elaborating on the random cofilling technique[1], which is arguably the only known technique to establish coboundary expansion.

3.1 Normed Cochain Groups and Coboundary Expansion

As mentioned in the introduction the notion of *coboundary expansion* was introduced (without calling it coboundary expansion) in the inspiring works by Gromov in [54] and independently by Linial–Meshulam in [89] as well as Meshulam–Wallach in [109]. The term coboundary expansion was later coined in [35]. Aforementioned applications of coboundary expansion are deep and, at first, it is not clear at all why coboundary expansion is useful to tackle these problems.

In retrospect, coboundary expansion can be seen as a natural generalization of edge expansion of graphs. Although historically incorrect, we use this point of view as an *a posteriori* motivation for the definition of coboundary expansion.

Recall from the introduction that given a (simple, undirected) graph $G = (V, E)$ and $S \subseteq V$ we write

$$E(S, V \setminus S) = \{e \in E : |e \cap S| = |e \cap (V \setminus S)| = 1\}$$

for the cut induced by S. Note that G is connected if and only if $|E(S, V \setminus S)| > 0$ for all $\emptyset \neq S \subsetneq V$. In fact, given $\emptyset \neq S \subsetneq V$, $|E(S, V \setminus S)|$ is the number of edges that have to

[1]This averaging technique was introduced by Gromov in [54] to show coboundary expansion for the complete complex, complete multipartite complexes and spherical buildings among others. Linial, Meshulam and Wallach gave essentially the same argument to establish coboundary expansion for the complete complex in their early work on coboundary expansion (see [89] and [109]). Gromov (as well as Guth in the survey [57] on Gromov's work on waist inequalities) remarks that the technique goes back to Federer's and Fleming's work in geometric measure theory (see [42] and [41]). The random cofilling technique was further elaborated on in [35, 98, 85, 76].

be removed from G such that S and $V \setminus S$ belong to two different connected components. Moreover, if $S \subseteq V$ is small, we should not expect too many edges going out of S. Thus, the Cheeger constant

$$h_0(G) = \min_{\emptyset \neq S \subsetneq V} \frac{|E(S, V \setminus S)|}{\min\{|S|, |V \setminus S|\}}$$

is a natural way to quantify the (robustness of) connectedness G.

There is an interpretation of $h_0(G)$ in terms of cohomology groups, which leads to the generalization to higher dimensions. For this purpose we think of G as a 1-dimensional simplicial complex. Then subsets $S \subseteq V$ are in one-to-one correspondence with cochains $\mathbb{1}_S \in C^0(G; \mathbb{F}_2)$ where $\mathbb{1}_S$ is the characteristic function of S, i.e. for $v \in V$

$$\mathbb{1}_S(v) := \begin{cases} 1 & \text{if } v \in S, \\ 0 & \text{otherwise.} \end{cases}$$

Similarly, subsets $F \subseteq E$ are in one-to-one correspondence with cochains $\mathbb{1}_F \in C^1(G; \mathbb{F}_2)$ by thinking of their characteristic functions $\mathbb{1}_F$ as $\mathbb{F}_2$-valued. Note that

$$\delta \mathbb{1}_S = \mathbb{1}_{E(S, V \setminus S)}$$

for all $S \subseteq V$.

Write $|\cdot|$ for the Hamming norm on $C^k(G; \mathbb{F}_2)$. That is $|c| := |\{\sigma \in G(k) : c(\sigma) \neq 0\}|$ for $c \in C^k(G; \mathbb{F}_2)$, $-1 \leq k \leq 1$.

Note that $B^0(G; \mathbb{F}_2) = \{0, \mathbb{1}\}$ is the space of constant functions. Hence, $\mathbb{1}_S$ and $\mathbb{1}_{V \setminus S}$ differ by a coboundary.

For $c \in C^0(G; \mathbb{F}_2)$ write $[c]$ for its equivalence class in $C^0(G; \mathbb{F}_2)/B^0(G; \mathbb{F}_2)$. Write

$$|[c]| = \min_{b \in B^0(G; \mathbb{F}_2)} |c + b|$$

for the quotient norm on $C^0(G; \mathbb{F}_2)/B^0(G; \mathbb{F}_2)$ induced by $|\cdot|$. The following table summarizes the correspondence of various terms in graph theoretical language and in cohomological language:

graph language	cohomological language
$S \subseteq V$	$\mathbb{1}_S \in C^0(G; \mathbb{F}_2)$
$F \subseteq E$	$\mathbb{1}_F \in C^1(G; \mathbb{F}_2)$
$E(S, V \setminus S)$	$\delta \mathbb{1}_S$
$\{\emptyset, V\}$	$B^0(G; \mathbb{F}_2)$
$\min\{\|S\|, \|V \setminus S\|\}$	$\|[\mathbb{1}_S]\|$.

We observe that

$$h_0(G) = \min_{c \in C^0(G; \mathbb{F}_2) \setminus B^0(G; \mathbb{F}_2)} \frac{|\delta c|}{|[c]|}.$$

With this point of view, we can think of $h_0(G)$ as quantifying the vanishing of $\tilde{H}^0(G; \mathbb{F}_2)$. Indeed, $h_0(G) > 0$ if and only if G is connected if and only if $\tilde{H}^0(G; \mathbb{F}_2) = 0$.

Moreover, this equivalent definition of $h_0(G)$ is very flexible. We can replace

- G by a d-dimensional simplicial complex X,
- $\mathbb{F}_2$ by some abelian group $\mathbb{A}$,
- $C^0(G;\mathbb{F}_2) \setminus B^0(G;\mathbb{F}_2)$ by $C^k(X;\mathbb{A}) \setminus B^k(X;\mathbb{A})$,
- $|\cdot|$ by some different way of measuring the size of cochains.

This flexibility allows us to define a notion which quantifies the vanishing of $\tilde{H}^k(X;\mathbb{A})$.

To make this more precise, we start with the notion of a size function on cochains.

Definition 3.1 (Size function on cochains)**.** Let X be a d-dimensional simplicial complex. Let $\mathbb{A}$ be an abelian group. Let $0 \leq k \leq d$. A *size function* $|\cdot|$ *on* $C^k(X;\mathbb{A})$ is a function $|\cdot|\colon C^k(X;\mathbb{A}) \to \mathbb{R}_{\geq 0}$ of the form

$$|c| = \sum_{\sigma \in X(k)} w(\sigma)|c(\sigma)|_{\mathbb{A}},$$

where $w\colon X(k) \to \mathbb{R}_{\geq 0}$ are non-negative weights on $X(k)$ and $|\cdot|_{\mathbb{A}}\colon \mathbb{A} \to \mathbb{R}_{\geq 0}$ is a non-negative function on $\mathbb{A}$ with $|0|_{\mathbb{A}} = 0$ and $|a|_{\mathbb{A}} = |-a|_{\mathbb{A}}$ for all $a \in \mathbb{A}$.
A size function $|\cdot|$ is *coboundary separating* if $|c| > 0$ for all $c \in C^k(X;\mathbb{A}) \setminus B^k(X;\mathbb{A})$.
A size function $|\cdot|$ is *positive on coboundaries* if $|b| = 0$ for $b \in B^k(X;\mathbb{A})$ implies $b = 0$.
A size function $|\cdot|$ *satisfies the triangle inequality* if for all $c, c' \in C^k(X;\mathbb{A})$ we have

$$|c + c'| \leq |c| + |c'|.$$

A size function $|\cdot|$ is a *norm* if it satisfies the triangle inequality and for $c \in C^k(X;\mathbb{A})$ we have $|c| = 0$ if and only if $c = 0$.

The reader might be confused at this point why we do not simply define a size function to be a norm (according to our definition). In fact, any norm is automatically coboundary separating and positive on coboundaries. But, as we will see later on, this would be too restrictive for some of the applications. We will encounter various size functions which do not satisfy the triangle inequality, are not coboundary separating or positive on coboundaries. It feels convenient to introduce some additional technicalities at this point in order to fit all examples of size functions and variants of coboundary expansion used throughout this thesis into one single definition. We hope that the examples at the end of this section already help to give an idea why size functions which are not norms could be useful.

We are ready to give the main definition in this thesis.

Definition 3.2 (Coboundary expansion constants)**.** Let X be a d-dimensional simplicial complex. Let $0 \leq k \leq d-1$. Let $\mathbb{A}$ be an abelian group. Let $|\cdot|$ be a size function on $C^k(X;\mathbb{A})$ and $C^{k+1}(X;\mathbb{A})$. For $c \in C^k(X;\mathbb{A})$ let $|[c]| := \min_{b \in B^k(X;\mathbb{A})} |c - b|$. Let $\eta \geq 0$. We say X is *η-coboundary expanding with respect to $\mathbb{A}$-coefficients and* $|\cdot|$ if

$$|\delta c| \geq \eta |[c]|$$

holds for all $c \in C^k(X;\mathbb{A})$.

We define the *kth coboundary expansion constant* $\eta_k^{|\cdot|}(X;\mathbb{A})$ *of X with respect to $\mathbb{A}$-coefficients and* $|\cdot|$ by

$$\eta_k^{|\cdot|}(X;\mathbb{A}) := \sup\{\eta \geq 0 : X \text{ is } \eta\text{-expanding with respect to } \mathbb{A}\text{-coefficients and } |\cdot|\}.$$

We should remark that many authors use the term 'coboundary expansion' exclusively for the case $\mathbb{A} = \mathbb{F}_2$. We prefer to take a more general approach here emphasizing the flexibility of the notion. As we will see later on that changing the size function and coefficients even within the same problem can lead to additional insights.

Before giving examples with various commonly used choices for $\mathbb{A}$ and $|\cdot|$, let us show that $\eta_k^{|\cdot|}(X;\mathbb{A})$ quantifies the vanishing of $\tilde{H}^k(X;\mathbb{A})$ (at least when $|\cdot|$ is a norm).

Lemma 3.3. *Let X be a d-dimensional simplicial complex. Let $0 \leq k \leq d$. Let $\mathbb{A}$ be an abelian group and $|\cdot|$ a size function on $C^j(X;\mathbb{A})$ for $j \in \{k, k+1\}$.*

(i) If $|\cdot|$ is coboundary separating on $C^k(X;\mathbb{A})$, then $\eta_k^{|\cdot|}(X;\mathbb{A}) > 0$ implies $\tilde{H}^k(X;\mathbb{A}) = 0$.

(ii) If $|\cdot|$ is positive on coboundaries on $C^{k+1}(X;\mathbb{A})$, then $\tilde{H}^k(X;\mathbb{A}) = 0$ implies $\eta_k^{|\cdot|}(X;\mathbb{A}) > 0$.

Proof. For (i) let $c \in Z^k(X;\mathbb{A})$ be a cocycle. We have

$$0 = |\delta c| \geq \eta_k^{|\cdot|}(X;\mathbb{A})|[c]|.$$

Since we assume that $\eta_k^{|\cdot|}(X;\mathbb{A}) > 0$ we must have $|[c]| = 0$. But $|\cdot|$ is coboundary separating, hence $c \in B^k(X;\mathbb{A})$. Thus, $Z^k(X;\mathbb{A}) = B^k(X;\mathbb{A})$ and $\tilde{H}^k(X;\mathbb{A}) = 0$.

For (ii) note that if $\eta_k^{|\cdot|}(X;\mathbb{A}) = 0$, then there is $c \in C^k(X;\mathbb{A}) \setminus B^k(X;\mathbb{A})$ such that $|\delta c| = 0$ while $|[c]| > 0$. Since $|\cdot|$ is positive on coboundaries, we get $\delta c = 0$. But then $c \in Z^k(X;\mathbb{A}) \setminus B^k(X;\mathbb{A})$ showing that $\tilde{H}^k(X;\mathbb{A}) \neq 0$. □

The following examples (also see Figure 3.1) show that, in general, we cannot (fully) remove the additional assumption in (i) and (ii) in the previous lemma.

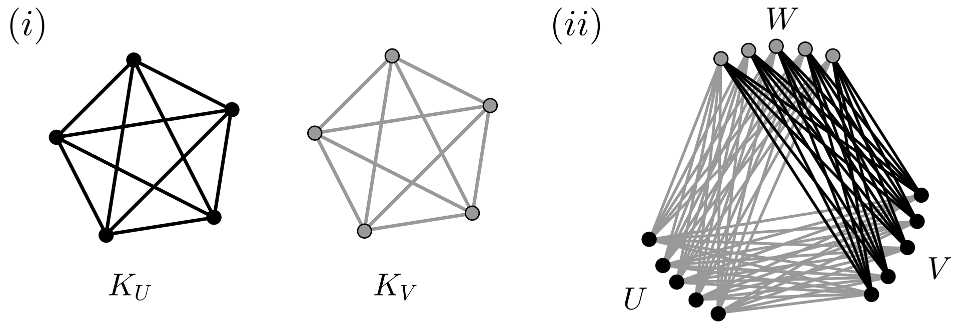


Figure 3.1: An illustration that in general $\eta_k^{|\cdot|}(X;\mathbb{A}) > 0$ is *not* equivalent to $\tilde{H}^k(X;\mathbb{A}) = 0$

Example 3.4. For both examples we let $\mathbb{A} = \mathbb{F}_2$ with the Hamming norm $|\cdot|_{\mathbb{A}}$. In (i) we have two disjoint copies of a complete graph. So, the 0th-cohomology does not vanish. But choosing weights to be constant 1 on K_U and 0 on K_V makes this graph an expander. In (ii) we have a complete tripartite graph which is connected and, hence, has vanishing 0th-cohomology. Choose the weights to be equal to 1 on the black vertices and edges and 0 on the gray vertices and edges. With respect to these weights the size of the coboundary of $\mathbb{1}_U$ is 0 but $|[\mathbb{1}_U]| > 0$. Hence, the expansion constant (with respect to this weighted Hamming norm) is 0.

3.1.1 Examples of Frequently Used Choices for Weights w and $|\cdot|_{\mathbb{A}}$ on $\mathbb{A}$

We will mostly work with the abelian group $\mathbb{A} = \mathbb{F}_2$, the finite field with two elements. Sometimes we will also consider $\mathbb{A} = \mathbb{F}_q$, the finite field with q elements for some prime power q, $\mathbb{A} = \mathbb{Z}$ or $\mathbb{A} = \mathbb{R}$.

We endow finite fields with the Hamming norm $|\cdot|_H$, i.e. $|x|_H = 1$ for $x \in \mathbb{F}_q \setminus \{0\}$ and $|0|_H = 0$. On $\mathbb{Z}$ and $\mathbb{R}$ we usually use the squared ℓ_2^2-norm $|\cdot|_2^2$, i.e. $|x|_2^2 = |x|^2$ for $x \in \mathbb{R}$.

In terms of weights w on a simplicial complex X we will mainly use three choices for w: the *Hamming weights* w_H, the *normalized Hamming weights* $\bar{w}_H$ and the *Garland weights* w_G.

The weight w_H is simply the constant 1 function, i.e. $w_H(\sigma) = 1$ for all $\sigma \in X$. The normalized Hamming weights are given by $\bar{w}_H(\sigma) = \frac{1}{|X(k)|}$ for $\sigma \in X(k)$.

Recall from the introduction that for a pure d-dimensional simplicial complex X, the Garland weights $w_G \colon X \to \mathbb{R}_{\geq 0}$ are given by

$$w_G(\sigma) := \frac{|\tau \in X(d) : \sigma \subseteq \tau|}{\binom{d+1}{|\sigma|}|X(d)|}$$

for $\sigma \in X$.

Note that $w_G(\sigma) \in [0, 1]$ for all $\sigma \in X$ and that $\sum_{\sigma \in X(k)} w_G(\sigma) = 1$ for all $-1 \leq k \leq d$. Thus, for each $-1 \leq k \leq d$, we can think of w_G as a probability distribution on $X(k)$. In fact, for $\sigma \in X(k)$, $w_G(\sigma)$ is the probability that the following random process ends up at σ: Sample $\tau \in X(d)$ uniformly at random. Then in each step remove a single vertex from τ uniformly at random until there are $k + 1$ vertices left.

The main advantage of working with Garland weights over (normalized) Hamming weights is that it allows to directly compare coboundary expansion constants of complexes of various sizes and with very different, non-constant degrees of simplices. In fact, in dimension $d \geq 2$ it is not obvious at all how to construct d-dimensional simplicial complexes such that every i-simplex is contained in the same number of j-simplices for all $0 \leq i < j \leq d$, let alone if one asks for additional expansion properties (see [22] for an attempt in this direction). It turns out that Garland weights often take care of potential issues arising from different degrees in an almost magical, automatic way.

For a more exotic choice of weights, which will be relevant for our result on the number of Tverberg partitions, consider the following weights on Λ_n^d: Write $\Lambda_n^d = U_0 * U_1 * \cdots * U_d$ with $U_0 = U_1 = \cdots = U_d = [n]$. For $\sigma \in \Lambda_n^d$ we call

$$I = \{i \in \{0, \ldots, d\} : U_i \cap \sigma \neq \emptyset\}$$

the *type of* σ. Now, define a weight function $w_- \colon \Lambda_n^d \to \mathbb{R}_{\geq 0}$ by

$$\sigma \mapsto w_-(\sigma) := \begin{cases} \frac{1}{n^{|\sigma|}} & \text{if } \sigma \text{ has type } \{0, 1, \ldots, |\sigma|\}, \\ 0 & \text{otherwise.} \end{cases}$$

In words, w_- gives weight $1/n^{k+1}$ to the k-simplex σ if it is contained in $U_0 * U_1 * \cdots * U_k \subseteq \Lambda_n^d$ and weight 0 otherwise.

It will be convenient to introduce special notation for coboundary expansion constants with respect to frequently used coefficients and size functions:

- We will write $\eta_k(X)$ for the kth coboundary expansion constant of X with respect to $\mathbb{F}_2$-coefficients and the size function induced by Garland weights w_G on X and the Hamming norm on $\mathbb{F}_2$.
- We will write $h_k(X)$ for the kth coboundary expansion constant of X with respect to $\mathbb{F}_2$-coefficients and the size function induced by the Hamming weights w_H on X and the Hamming norm on $\mathbb{F}_2$.
- We will write $\bar{h}_k(X)$ for the kth coboundary expansion constant of X with respect to $\mathbb{F}_2$-coefficients and the size function induced by the normalized Hamming weights w_H on X and the Hamming norm on $\mathbb{F}_2$.
- We will write $\zeta_k(X)$ for the kth coboundary expansion constant of X with respect to $\mathbb{Z}$-coefficients and the size function induced by Garland weights w_G on X and the ℓ_2^2-norm on $\mathbb{Z}$.

3.1.2 Further Remarks and Generalizations

Note that for the definition of the coboundary expansion constants $\eta_k^{|\cdot|}(X;\mathbb{A})$ we only need the structure of the cochain complex together with a way of measuring the size of cochains. As such, the definition of coboundary expansion constants immediately extends to (finite) cellular or polyhedral complexes. We will not make use of any such generalization except in Section 9.1, where we will consider the coboundary expansion constants $h_k(Q_d)$ with respect to $\mathbb{F}_2$-coefficients and Hamming weights of the d-dimensional hypercube Q_d thought of as a cubical complex.

Let $w\colon X \to \mathbb{R}_{>0}$ be a strictly positive weight function on a d-dimensional simplicial complex X. Let $0 \le k \le d-1$. Let $\mathcal{L}_k^{up}(X) = \delta_k^*\delta_k$ be the up-Laplacian where δ_k^* is the adjoint of δ_k with respect to the inner product $\langle\cdot,\cdot\rangle_w$ on $C^k(X;\mathbb{R})$ induced by w. Let $|\cdot|$ be the size function on $C^k(X;\mathbb{R})$ induced by the weights w and the squared ℓ_2-norm $|\cdot|_2^2$ on $\mathbb{R}$. Using the variational characterization of eigenvalues, it is not difficult to see that the smallest non-trivial eigenvalue $\lambda^{(k)}(X)$ of $\mathcal{L}_k^{up}(X)$ satisfies $\lambda^{(k)}(X) = \eta_k^{|\cdot|}(X;\mathbb{R})$.

3.1.3 Minimal Cochains and Cofillings

For future reference, we introduce the notion of *minimal cochains* and *minimal cofillings.*

Definition 3.5. Let X be a d-dimensional simplicial complex with the size function $|\cdot|$ induced by some weights $w\colon X \to \mathbb{R}_{\ge 0}$ and a weight function $|\cdot|_{\mathbb{A}}$ on the abelian group $\mathbb{A}$. Let $0 \le k \le d$. We say that $c \in C^k(X;\mathbb{A})$ is *minimal (with respect to* $|\cdot|$*)* if $|c| \le |c - \delta a|$ for all $a \in C^{k-1}(X;\mathbb{A})$.

Given $b \in B^k(X;\mathbb{A})$ we say that $c \in C^{k-1}(X;\mathbb{A})$ is a *cofilling* of b if $\delta c = b$. A cofilling c of $b \in B^k(X;\mathbb{A})$ is a *minimal cofilling* if $|c'| \ge |c|$ for any other cofilling c' of b.

We will use the following property of minimal cochains several times later on.

Lemma 3.6. *Let $X \subseteq Y$ be d-dimensional simplicial complexes with inclusion map $i\colon X \to Y$. Let $w\colon Y \to \mathbb{R}_{\ge 0}$ be a weight function. Endow X with weights obtained by restricing w to X. Let $\mathbb{A}$ be an abelian group with weight function $|\cdot|_{\mathbb{A}}$. Write $|\cdot|_X$ and $|\cdot|_Y$ for the induced size functions on cochains of X and Y, respectively. Let $0 \le k \le d$.*

Given $c \in C^k(X;\mathbb{A})$ write $\bar{c}$ for the extension by 0 of c to Y, i.e. $i^*\bar{c} = c$ and $\bar{c}(\sigma) = 0$ for all $\sigma \in Y(k) \setminus X(k)$. Assume c is minimal with respect to $|\cdot|_X$. Then $\bar{c}$ is minimal with respect to $|\cdot|_Y$.

Proof. Given minimal $c \in C^k(X;\mathbb{A})$ and $a \in C^k(Y;\mathbb{A})$ we compute

$$|\bar{c} + \delta a|_Y \geq |i^*(\bar{c} + \delta_Y a)|_X = |c + \delta_X i^* a|_X \geq |c|_X = |\bar{c}|_Y,$$

where we used the minimality of c for the second last step. □

3.2 The Random Cofilling Technique

In this section we introduce the random cofilling technique for showing coboundary expansion in an abstract setting. We will encounter such averaging arguments over and over again throughout this thesis. To help the reader get more acquainted with this type of arguments, we will illustrate the technique in proofs of the following two propositions.

Proposition 3.7. *Let $d \geq 1$ be a dimension. Let $n \geq d+1$. Then the complete d-dimensional complex K_n^d on n vertices is 1-coboundary expanding with respect to $\mathbb{Z}$-coefficients and ℓ_2^2-norm. More precisely, we have*

$$\zeta_{d-1}(K_n^d) \geq \frac{n}{n-d}.$$

Proposition 3.8. *Let $d \geq 1$ be a dimension. The 0th coboundary expansion constant $\eta_0(\Lambda_n^d)$ of the d-dimensional complete $(d+1)$-partite complex Λ_n^d satisfies*

$$\eta_0(\Lambda_n^d) \geq 1.$$

Proposition 3.7 seems new but the argument of Gromov [54] and Linial–Meshulam–Wallach [89, 109] for coboundary expansion ≥ 1 with respect to $\mathbb{F}_2$-coefficients and Garland weighted Hamming norm carries over to the setting with respect to $\mathbb{Z}$-coefficients and ℓ_2^2-norm without any difficulties.

Proposition 3.8 shows that the (normalized) edge expansion constant of the complete multipartite graph (with equally sized parts) is at least 1. This is a well-known fact and a simple exercise using explicit formulas for the size of a cut induced by a subset of vertices. We will give an alternative proof using the random abstract cofilling technique. This allows us to illustrate the technique in a simple setting. Moreover, having a random abstract cofilling as a certificate for $\eta_0(\Lambda_n^d) \geq 1$ will be useful for proving lower bounds on $\eta_k(\Lambda_n^d)$ for $k \geq 1$ later on (see Proposition 6.16).

Our discussion and notation for the abstract setting of the random cofilling technique loosely follows [85].

3.2.1 The Averaging Trick – Exemplified by a Proof of Proposition 3.7

Let X be a d-dimensional simplicial complex. Let $\mathbb{A}$ be an abelian group. Let $|\cdot|$ be a size function on $C^k(X;\mathbb{A})$ and $C^{k+1}(X;\mathbb{A})$. Given $c \in C^k(X;\mathbb{A})$ it is a priori not clear at all how to find $a \in C^{k-1}(X;\mathbb{A})$ such that $|c - \delta a|$ is small compared to $|\delta c|$.

Here is a simple yet powerful idea for a possible approach to this problem: Instead of exhibiting one single 'good' choice for $a \in C^{k-1}(X;\mathbb{A})$, one constructs a whole family $(a^{(s)})_{s\in S}$ of $(k-1)$-cochains and averages $|c-\delta a^{(s)}|$ over some distribution μ on S. The hope is that many complicated terms cancel out and the average $\mathbb{E}_{s\sim\mu}|c-\delta a^{(s)}|$ becomes easy (or at least easier) to analyze and to compare to $|\delta c|$. Since

$$\|[c]\| \leq \min_{s\in S} |c-\delta a^{(s)}| \leq \mathbb{E}_{s\sim\mu}|c-\delta a^{(s)}|,$$

we get $\eta_k^{|\cdot|}(X;\mathbb{A}) \geq 1/M$ for some $M > 0$, if for any $c \in C^k(X;\mathbb{A})$ we can construct a family $(a^{(s)})_{s\in S}$ of $(k-1)$-cochains with

$$\mathbb{E}_{s\sim\mu}|c-\delta a^{(s)}| \leq M|\delta c|.$$

In practice, it is sometimes easier to think about different cofillings of δc rather than trying to construct various choices for $a \in C^{k-1}(X;\mathbb{A})$ to make $c-\delta a$ small. In this case, the following easy lemma comes in handy:

Lemma 3.9. *Let X be a d-dimensional simplicial complex. Let $0 \leq k \leq d-1$. Let $\mathbb{A}$ be an abelian group and $|\cdot|$ a size function on $C^k(X;\mathbb{A})$ and $C^{k+1}(X;\mathbb{A})$. Let $\eta > 0$. Assume that $\tilde{H}^k(X;\mathbb{A}) = 0$ and that for every $b \in B^{k+1}(X;\mathbb{A})$ there is $c \in C^k(X;\mathbb{A})$ with $\delta c = b$ and*

$$|c| \leq \frac{1}{\eta}|b|.$$

Then $\eta_k^{|\cdot|}(X;\mathbb{A}) \geq \eta$.

Proof. Let $c \in C^k(X;\mathbb{A})$ be minimal. Let $b = \delta c$. By assumption there is $c' \in C^k(X;\mathbb{A})$ with $\delta c' = b$ and $|c'| \leq \frac{1}{\eta}|b|$. But then $\delta(c-c') = 0$ and since we assume that $\tilde{H}^k(X;\mathbb{A}) = 0$ we get $c-c' = \delta a$ for some $a \in C^{k-1}(X;\mathbb{A})$. We conclude that $\|[c]\| = \|[c']\| \leq \frac{1}{\eta}|\delta c|$, as desired. ☐

It is an easy consequence of Lemma 3.3 (i) that if $|\cdot|$ is coboundary separating, then the reverse implication in Lemma 3.9 is also true. Moreover, if $|\cdot|$ satisfies the triangle inequality, then one can show that $\eta_k^{|\cdot|}(X;\mathbb{A}) \geq \eta$ is equivalent to the fact that for every $b \in B^{k+1}(X;\mathbb{A})$ there is $c \in C^k(X;\mathbb{A})$ with

$$|c| \leq \frac{1}{\eta}|b|.$$

and that for every $c \in C^k(X;\mathbb{A})$ we have that $|\delta c| = 0$ implies $\|[c]\| = 0$.

It is convenient to introduce the following terminology.

Definition 3.10 (Cofilling inequality)**.** Let X be a d-dimensional simplicial complex. Let $0 \leq k \leq d-1$. Let $\mathbb{A}$ be an abelian group and $|\cdot|$ a size function on $C^k(X;\mathbb{A})$ and $C^{k+1}(X;\mathbb{A})$. We say that X *satisfies a cofilling inequality in dimension* $k+1$ *with cofilling constant* L if for every $b \in B^{k+1}(X;\mathbb{A})$ there is $c \in C^k(X;\mathbb{A})$ with $\delta c = b$ and such that

$$|c| \leq L|b|.$$

We illustrate the averaging trick by giving a proof of Proposition 3.7.

Proof of Proposition 3.7. K_n^d is the d-skeleton of the $(n-1)$-dimensional simplex σ^{n-1} on n vertices. Since σ^{n-1} is contractible and the $(d-1)$-th cohomology group only depends on the d-skeleton, we get that

$$\tilde{H}^{d-1}(K_n^d;\mathbb{Z}) \cong \tilde{H}^{d-1}(\sigma^{n-1};\mathbb{Z}) = 0.$$

Thus, according to Lemma 3.9, it suffices to show a cofilling inequality with cofilling constant $1 - d/n$. To this end, let $b = \delta c \in B^d(K_n^d;\mathbb{Z})$ for some $c \in C^{d-1}(K_n^d;\mathbb{Z})$. Given a vertex $v \in K_n^d(0)$ let $c^{(v)} \in C^{d-1}(K_n^d;\mathbb{Z})$ be given by

$$c^{(v)} := b_v,$$

where, by slight abuse of notation, we consider the localization b_v as a cochain on K_n^d instead of a cochain on the link $(K_n^d)_v$ by extending b_v by 0 on $K_n^d(d-1) \setminus (K_n^d)_v(d-1)$.

We claim that $\delta c^{(v)} = b$ for all $v \in K_n^d(0)$.

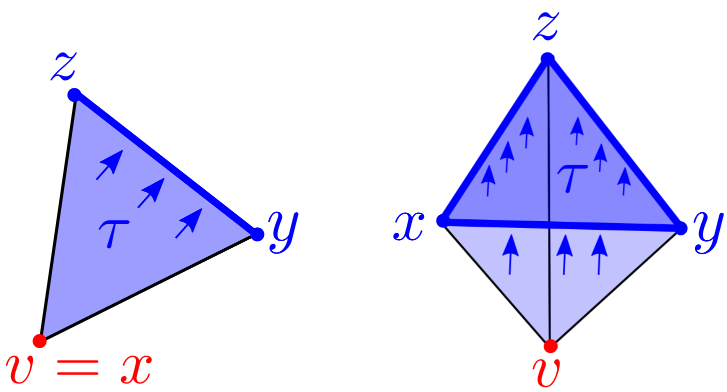


Figure 3.2: An illustration that $\delta c^{(v)} = b$ for $d = 2$: We distinguish two cases. On the left we consider a triangle $\tau = xyz$ with $v = x \in \tau$. By coning from v the value of b gets pushed to the opposite edge yz. On the right we assume that $\tau = xyz$ and $v \notin \tau$ we note that since b is a coboundary, b evaluates to zero on the boundary of the tetrahedron $vxyz$. Thus, $b(\tau) = b(vyz) - b(vxz) + b(vxy) = (\delta b_v)(xyz)$.

To see this (for an illustration for $d = 2$ see Figure 3.2), let $\sigma \in K_n^d(d)$. If $v \in \sigma$, then $\delta c^{(v)}(\sigma) = b(\sigma)$ is immediate by construction. If $v \notin \sigma$, then $\delta c^{(v)}(\sigma) = b(\sigma)$ follows from the fact that, as a coboundary, b evaluates to 0 on cycles which in particular means that

$$\langle b, \partial([v,\sigma])\rangle = 0.$$

Thus, we have found a family $(c^{(v)})_{v\in K_n^d(0)}$ of cofillings for b.

Let us compute the expected size of $c^{(v)}$. For this purpose, write $|\cdot|_2^2$ for the (unweighted) ℓ_2^2-norm on $C^k(K_n^d;\mathbb{Z})$, i.e.

$$|a|_2^2 = \sum_{\sigma\in K_n^d(k)} |a(\sigma)|^2$$

for any $a \in C^k(K_n^d;\mathbb{Z})$. We get by double counting (every d-simplex has $d+1$ vertices):

$$|[c]|_2^2 \le \frac{1}{n} \sum_{v\in K_n^d(0)} |c^{(v)}|_2^2 = \frac{1}{n} \sum_{v\in K_n^d(0)} |b_v|_2^2 = \frac{d+1}{n}|b|_2^2.$$

Using that Garland weights are uniform on K_n^d, we deduce that

$$\zeta_{d-1}(K_n^d) \geq \frac{n}{d+1} \frac{|K_n^d(d-1)|}{|K_n^d(d)|} = \frac{n}{d+1} \frac{\binom{n}{d}}{\binom{n}{d+1}} = \frac{n}{n-d},$$

as desired. □

3.2.2 Random Abstract Cofilling

Sometimes, as for the complete complex, it is not too difficult to come up with an ad-hoc construction for a family $(c^{(s)})_{s\in S}$ of cofillings of a given coboundary b. Inspired by the discussions in [85, Section 2.2] and in [76, Section 2], we outline a more systematic approach for finding families of cofillings, which we call *random abstract cofilling*. This approach allows to establish coboundary expansion in the presence of small fillings of cycles and an automorphism group acting transitively on the top dimensional faces.

A key insight leading towards random abstract cofilling is that the vanishing of $\tilde{H}^k(X;\mathbb{A})$ is implied by the existence of a cochain homotopy between the identity and 0-map on $C^k(X;\mathbb{A})$.

Lemma 3.11. *Let X be a d-dimensional simplicial complex. Let $0 \leq k \leq d-1$. Let $\mathbb{A}$ be an abelian group. Assume there is a cochain homotopy between the identity and 0-map on $C^k(X;\mathbb{A})$, i.e. for $i \in \{k, k+1\}$ there are*

$$T_i \colon C^i(X;\mathbb{A}) \to C^{i-1}(X;\mathbb{A})$$

such that for all $c \in C^k(X;\mathbb{A})$ it holds that

$$c = \delta T_k c + T_{k+1}\delta c.$$

Then $\tilde{H}^k(X;\mathbb{A}) = 0$.

If $\mathbb{A}$ is a field or $\mathbb{A} = \mathbb{Z}$, the converse is also true. That is $\tilde{H}^k(X;\mathbb{A}) = 0$ implies the existence of a cochain homotopy between the identity and 0-map on $C^k(X;\mathbb{A})$.

Proof. First let $T_i \colon C^i(X;\mathbb{A}) \to C^{i-1}(X;\mathbb{A})$, $i \in \{k, k+1\}$, be a cochain homotopy between the identity and 0-map on $C^k(X;\mathbb{A})$. Let $c \in C^k(X;\mathbb{A})$ with $\delta c = 0$. Then by assumption we get

$$c = \delta T_k c + T_{k+1}\delta c = \delta T_k c,$$

i.e. c is a coboundary with cofilling $T_k c$. Hence, $Z^k(X;\mathbb{A}) = B^k(X;\mathbb{A})$ and $\tilde{H}^k(X;\mathbb{A}) = 0$.

Now assume that $\mathbb{A}$ is a field or $\mathbb{A} = \mathbb{Z}$ and that $\tilde{H}^k(X;\mathbb{A}) = 0$. Thus, the (co)chain complex

$$\cdots \to C^{k-1}(X;\mathbb{A}) \xrightarrow{\delta} C^k(X;\mathbb{A}) \xrightarrow{\delta} C^{k+1}(X;\mathbb{A}) \to \ldots$$

is exact at $C^k(X;\mathbb{A})$. This implies that there is a short exact sequence (SES)

$$0 \to B^k(X;\mathbb{A}) \xrightarrow{i} C^k(X;\mathbb{A}) \xrightarrow{\delta} B^{k+1}(X;\mathbb{A}) \to 0,$$

where $i \colon B^k(X;\mathbb{A}) \to C^k(X;\mathbb{A})$ denotes the inclusion map. By the assumption on $\mathbb{A}$ this SES is split[2] and there are homomorphisms

$$r \colon C^k(X;\mathbb{A}) \to B^k(X;\mathbb{A}) \text{ and } s \colon B^{k+1}(X;\mathbb{A}) \to C^k(X;\mathbb{A})$$

[2] If $\mathbb{A}$ is a field then we are considering a SES of vector spaces which is always split. If $\mathbb{A} = \mathbb{Z}$, we see that $B^{k+1}(X;\mathbb{A})$ is a free abelian group which implies that the SES is split.

such that $r \circ i = \mathrm{id}_{B^k(X;\mathbb{A})}$ and $\delta \circ s = \mathrm{id}_{B^{k+1}(X;\mathbb{A})}$.

By the first isomorphism theorem applied to $\delta\colon C^{k-1}(X;\mathbb{A}) \to C^k(X;\mathbb{A})$ we get an induced isomorphism

$$\bar{\delta}\colon C^{k-1}(X;\mathbb{A})/Z^{k-1}(X;\mathbb{A}) \to B^k(X;\mathbb{A}).$$

In particular, if $\mathbb{A} = \mathbb{Z}$ we get that $C^{k-1}(X;\mathbb{A})/Z^{k-1}(X;\mathbb{A})$ is a free abelian group and for $\mathbb{A} = \mathbb{Z}$ or a field the following SES is split

$$0 \to Z^{k-1}(X;\mathbb{A}) \xrightarrow{i} C^{k-1}(X;\mathbb{A}) \xrightarrow{\pi} C^{k-1}(X;\mathbb{A})/Z^{k-1}(X;\mathbb{A}) \to 0.$$

Here i denotes the inclusion map and π the quotient map. In particular, there is $t\colon C^{k-1}(X;\mathbb{A})/Z^{k-1}(X;\mathbb{A}) \to C^{k-1}(X;\mathbb{A})$ with $\pi \circ t = \mathrm{id}_{C^{k-1}(X;\mathbb{A})/Z^{k-1}(X;\mathbb{A})}$.

The choice of $\mathbb{A}$ allows to extend $s\colon B^{k+1}(X;\mathbb{A}) \to C^k(X;\mathbb{A})$ to a homomorphism

$$T_{k+1}\colon C^{k+1}(X;\mathbb{A}) \to C^k(X;\mathbb{A}).$$

Also, define $T_k\colon C^k(X;\mathbb{A}) \to C^{k-1}(X;\mathbb{A})$ by

$$T_k := t \circ \bar{\delta}^{-1} \circ r.$$

We claim that $c = \delta T_k c + T_{k+1}\delta c$ for any $c \in C^k(X;\mathbb{A})$. This is essentially by construction, since by properties of SESs we have

$$c = (i \circ r)(c) + (s \circ \delta)(c)$$

for all $c \in C^k(X;\mathbb{A})$ and that $\delta \circ t \circ \bar{\delta}^{-1} = i\colon B^k(X;\mathbb{A}) \to C^k(X;\mathbb{A})$. This finishes the proof. □

Usually we are interested in lower bounds on $\eta_k^{|\cdot|}(X;\mathbb{A})$ for all $0 \le k \le d-1$. Thus, we would like to find (families of) homomorphisms

$$T_j\colon C^j(X;\mathbb{A}) \to C^{j-1}(X;\mathbb{A})$$

for all $0 \le j \le d$ such that

$$c = \delta T_k c + T_{k+1}\delta c$$

for all $c \in C^k(X;\mathbb{A})$, $0 \le k \le d-1$.

Often it easier to construct homotopies between the identity and 0-map on chain groups $C_k(X;\mathbb{A})$ and then dualize them to $C^k(X;\mathbb{A})$. More precisely, we would like to find homomorphisms $S_i\colon C_i(X;\mathbb{A}) \to C_{i+1}(X;\mathbb{A})$ for $-1 \le i \le d-1$ such that for all $c \in C_k(X;\mathbb{A})$, $0 \le k \le d-1$ we have

$$c = \partial S_k c + S_{k-1}\partial c.$$

Then, if $\mathbb{A}$ is a ring with 1, we can use the pairing between chains and cochains to define $T_k\colon C^k(X;\mathbb{A}) \to C^{k-1}(X;\mathbb{A})$ through the identity

$$\langle T_k c, a\rangle = \langle c, S_{k-1}a\rangle$$

for all $c \in C^k(X;\mathbb{A})$ and $a \in C_{k-1}(X;\mathbb{A})$.

The advantage of working on chain groups is that it allows for an inductive, bottom-up construction as follows:

Assume $\tilde{H}_k(X;\mathbb{A}) = 0$ for all $-1 \leq k \leq d-1$. Define $S_{-1}\colon C_{-1}(X;\mathbb{A}) \to C_0(X;\mathbb{A})$ by setting $S_{-1}\emptyset$ to be any chain c in $C_0(X;\mathbb{A})$ with $\partial c = \emptyset$. Such a chain exists since $\tilde{H}_{-1}(X;\mathbb{A}) = 0$ implies that X is non-empty and we could choose $c = v$ for some vertex $v \in X(0)$.

Next assume that by induction $S_i\colon C_i(X;\mathbb{A}) \to C_{i+1}(X;\mathbb{A})$ is already constructed for $-1 \leq i \leq k-1$ such that

$$\partial S_i c + S_{i-1}\partial c = c$$

for all $c \in C_i(X;\mathbb{A})$.[3]

Given $\sigma \in X(k)$ we would like to define $S_k\sigma \in C_{k+1}(X;\mathbb{A})$ such that

$$\partial S_k\sigma = \sigma - S_{k-1}\partial\sigma.$$

But

$$\partial(\sigma - S_{k-1}\partial\sigma) = \partial\sigma - \partial S_{k-1}\partial\sigma = \partial\sigma - (\partial\sigma - S_{k-2}\partial(\partial\sigma)) = 0,$$

so $\sigma - S_{k-1}\partial\sigma \in Z_k(X;\mathbb{A})$. Since we assume that $\tilde{H}_k(X;\mathbb{A}) = 0$ there is $S_k\sigma \in C_{k+1}(X;\mathbb{A})$ with $\partial S_k\sigma = \sigma - S_{k-1}\partial\sigma$, as desired.

Let $T_k\colon C^k(X;\mathbb{A}) \to C^{k-1}(X;\mathbb{A})$ and $T_{k+1}\colon C^{k+1}(X;\mathbb{A}) \to C^k(X;\mathbb{A})$ with

$$c = \delta T_k c + T_{k+1}\delta c$$

for all $c \in C^k(X;\mathbb{A})$. Then for any $c \in C^k(X;\mathbb{A})$, $T_{k+1}\delta c = c - \delta T_k c$ is a cofilling of the coboundary $\delta c \in B^{k+1}(X;\mathbb{A})$. A priori it is not clear how to choose T_{k+1} and T_k such $|T_{k+1}\delta c|$ is small compared to $|\delta c|$. What is more, there is no good reason why there should be a single choice for T_{k+1} and T_k that simultaneously works well for *all* $c \in C^k(X;\mathbb{A})$. But again, we can try to use the averaging trick and consider families $(T_k^{(\omega)}, T_{k+1}^{(\omega)})_{\omega\in\Omega}$ of cochain homotopies between the identity and 0-map on $C^k(X;\mathbb{A})$ parametrized by some probability space $(\Omega, \mathcal{B}, \mu)$. As we already observed, we get for all $c \in C^k(X;\mathbb{A})$ that

$$\|[c]\| \leq \mathbb{E}_{\omega\sim\mu}|T_{k+1}^{(\omega)}\delta c|.$$

The hope is that this expected value is easier to analyze and to compare to $|\delta c|$.

Let us introduce some terminology.

Definition 3.12 ((Random) abstract cone and (random) abstract cofilling)**.** Let X be a d-dimensional simplicial complex. Let $\mathbb{A}$ be an abelian group.

An *abstract cone* $(S_k)_{-2\leq k\leq d-1}$ *for* X is a family of homomorphisms

$$S_k\colon C_k(X;\mathbb{A}) \to C_{k+1}(X;\mathbb{A})$$

such that

$$c = \partial S_k c + S_{k-1}\partial c$$

[3] Note that we can make sense of this equation even for $i = -1$. Indeed, $C_{-2}(X;\mathbb{A})$ is 0 and, hence, there is a unique homomorphism $S_{-2}\colon C_{-2}(X;\mathbb{A}) \to C_{-1}(X;\mathbb{A})$. Therefore, for $i = -1$, the equation reduces to $\partial S_{-1}c = c$ for all $c \in C_{-1}(X;\mathbb{A})$ which holds by the choice of $S_{-1}\emptyset$.

for all $-1 \le k \le d-1$ and $c \in C_k(X;\mathbb{A})$.

An *abstract cofilling* $(T_k)_{-1\le k\le d}$ *for* X is a family of homomorphisms

$$T_k\colon C^k(X;\mathbb{A}) \to C^{k-1}(X;\mathbb{A})$$

such that

$$c = \delta T_k c + T_{k+1}\delta c$$

for all $-1 \le k \le d-1$ and $c \in C^k(X;\mathbb{A})$.

If $\mathbb{A}$ is a ring with 1, we say that an abstract cofilling $(T_k)_{-1\le k\le d}$ for X is *dual* to the abstract cone $(S_k)_{-2\le k\le d-1}$ for X if T_k is the dual map of S_{k-1} with respect to the pairing $\langle\cdot,\cdot\rangle$ between chain and cochains for all $-1 \le k \le d-1$.

Given a probability space $(\Omega,\mathcal{B},\mu)$ a *random abstract cone* $\mathcal{S}$ *for* X *parametrized by* $(\Omega,\mathcal{B},\mu)$ is a collection $\mathcal{S} = (S_k^{(\omega)})_{\omega\in\Omega,-2\le k\le d-1}$ of homomorphisms such that for every $\omega\in\Omega$ the family $(S_k^{(\omega)})_{-2\le k\le d-1}$ is an abstract cone for X.

Analogically, a *random abstract cofilling* $\mathcal{T}$ *for* X *parametrized by* $(\Omega,\mathcal{B},\mu)$ is a collection $\mathcal{T} = (T_k^{(\omega)})_{\omega\in\Omega,-1\le k\le d}$ of homomorphisms such that for every $\omega\in\Omega$ the family $(T_k^{(\omega)})_{-1\le k\le d}$ is an abstract cofilling for X.

We say that a random abstract cofilling $\mathcal{T} = (T_k^{(\omega)})_{\omega\in\Omega,-1\le k\le d}$ is *dual* to the abstract random cone $\mathcal{S} = (S_k^{(\omega)})_{-2\le k\le d-1}$ if for every $\omega\in\Omega$ the abstract cofilling $(T_k^{(\omega)})_{-1\le k\le d}$ is dual to the abstract cone $(S_k^{(\omega)})_{-2\le k\le d-1}$.

In the presence of a group of automorphisms G acting simplicially on X, we can use the induced action of G on chains to define an action of G on abstract cones. This allows us to turn a single abstract cone into a random abstract cone for X parametrized by $(G,\mathcal{B},\mu)$ for some measure μ on G. More precisely, given an abstract cone $(S_k)_{-2\le k\le d-1}$ for X and $g\in G$ we define another abstract cone $(g.S_k)_{-2\le k\le d-1}$ by

$$\begin{aligned} g.S_k\colon C_k(X;\mathbb{A}) &\to C_{k+1}(X;\mathbb{A}) \\ c &\mapsto (g.S_k)(c) := g.(S_k(g^{-1}.c)). \end{aligned}$$

Since G acts simplicially, the action of G on chain groups commutes with taking boundaries. It follows that $(g.S_k)_{-2\le k\le d-1}$ is indeed an abstract cone. Moreover, after short contemplation, we see that $(gh).S_k = g.(h.S_k)$ for all $g,h\in G$, i.e. that we get a group action of G on the set of abstract cones.

We close this subsection by showing a fairly generic lower bound on coboundary expansion constants in terms of properties of a random cofilling under some mild assumption on $\mathbb{A}$ and the size function $|\cdot|$. The reader should compare this result to [85, Theorem 2.5] and [76, Theorem 30].

Proposition 3.13. *Let X be a d-dimensional simplicial complex. Let $\mathbb{A}$ be a ring with 1. Let $0 \le k \le d-1$. Let $(\Omega,\mathcal{B},\mu)$ be a finite probability space. Let $|\cdot|$ be a size function on $C^k(X;\mathbb{A})$ induced by a weight function $|\cdot|_{\mathbb{A}}\colon \mathbb{A}\to\mathbb{R}_{\ge 0}$ with $|0|_{\mathbb{A}} = 0$, $|1|_{\mathbb{A}} = 1$, $|a+a'|_{\mathbb{A}} \le |a|_{\mathbb{A}} + |a'|_{\mathbb{A}}$ and $|a\cdot a'|_{\mathbb{A}} \le |a|_{\mathbb{A}}\cdot|a'|_{\mathbb{A}}$ for all $a,a'\in\mathbb{A}$ and strictly positive weights $w\colon X\to\mathbb{R}_{>0}$ on X.*

Let $\mathcal{S} = (S_k^{(\omega)})$ be a random abstract cone for X parametrized by $(\Omega,\mathcal{B},\mu)$ with dual random abstract cofilling $\mathcal{T} = (T_k^{(\omega)})$.

For $\tau \in X(k+1)$ let

$$\lambda(\tau) := \frac{1}{w(\tau)} \mathbb{E}_{\omega \sim \mu} |T_{k+1}^{(\omega)} \mathbb{1}_\tau| = \frac{1}{w(\tau)} \sum_{\omega \in \Omega} \sum_{\sigma \in X(k)} \mu(\omega) w(\sigma) |\langle \mathbb{1}_\tau, S_k^{(\omega)} \sigma \rangle|_{\mathbb{A}}.$$

Then

$$\eta_k^{|\cdot|}(X;\mathbb{A}) \geq \frac{1}{\max_{\tau \in X(k+1)} \lambda(\tau)}.$$

Moreover, if G is a group of automorphisms which acts simplicially on X and transitively on $X(d)$ and $w\colon X \to \mathbb{R}_{\geq 0}$ are the Garland weights, then for any abstract cone $(S_k)_{-2 \leq k \leq d-1}$ for X we have

$$\eta_k^{|\cdot|}(X;\mathbb{A}) \geq \frac{1}{\binom{d+1}{k+2} \mathrm{Size}_k((S_j)_{-2 \leq j \leq d-1})},$$

where

$$\mathrm{Size}_k((S_j)_{-2 \leq j \leq d-1}) := \max_{\sigma \in X(k)} \sum_{\tau \in X(k+1)} |\langle \mathbb{1}_\tau, S_k \sigma \rangle|_{\mathbb{A}}.$$

Before we dive into the proof of this proposition, let us shed some light on the complicated looking quantity $\lambda(\tau)$. Specializing to $w(\sigma) = \frac{1}{|X(\dim \sigma)|}$ for $\sigma \in X$, $\mathbb{A} = \mathbb{F}_2$, $|\cdot|_{\mathbb{A}}$ the Hamming norm and μ the uniform distribution on Ω, we get

$$\lambda(\tau) = \frac{|X(k+1)|}{|\Omega||X(k)|} |\{(\omega, \sigma) \in \Omega \times X(k) : \tau \in \mathrm{supp}(S_k^{(\omega)} \sigma)\}|.$$

Now, recall that $S_k^{(\omega)} \sigma$ is a filling of $\sigma + S_{k-1}^{(\omega)} \partial \sigma$. Thus, in order to get $\lambda(\tau)$ small for all $\tau \in X(k+1)$, we would need many small, well-distributed cycles inside of X.

Proof of Proposition 3.13. Let $c \in C^k(X;\mathbb{A})$. For $\omega \in \Omega$ let $c^{(\omega)} := T_{k+1}^{(\omega)} \delta c$. Since $\mathcal{T}$ is a random abstract cofilling, we have that for all $\omega \in \Omega$

$$c = c^{(\omega)} + \delta T_k^{(\omega)} c$$

and, hence, $[c] = [c^{(\omega)}] \in C^k(X;\mathbb{A})/B^k(X;\mathbb{A})$.

We estimate using the properties of $|\cdot|_{\mathbb{A}}$ that

$$\begin{aligned}
|[c]| &\leq \mathbb{E}_{\omega \sim \mu} |c^{(\omega)}| \\
&= \sum_{\omega \in \Omega} \sum_{\sigma \in X(k)} \mu(\omega) w(\sigma) |\langle T_{k+1}^{(\omega)} \delta c, \sigma \rangle|_{\mathbb{A}} \\
&= \sum_{\omega \in \Omega} \sum_{\sigma \in X(k)} \mu(\omega) w(\sigma) |\langle \delta c, S_k^{(\omega)} \sigma \rangle|_{\mathbb{A}} \\
&\leq \sum_{\omega \in \Omega} \sum_{\sigma \in X(k)} \mu(\omega) w(\sigma) \sum_{\tau \in X(k+1)} |\delta c(\tau)|_{\mathbb{A}} \cdot |\langle \mathbb{1}_\tau, S_k^{(\omega)} \sigma \rangle|_{\mathbb{A}} \\
&= \sum_{\tau \in X(k+1)} w(\tau) |\delta c(\tau)|_{\mathbb{A}} \lambda(\tau) \\
&\leq |\delta c| \max_{\tau \in X(k+1)} \lambda(\tau),
\end{aligned}$$

as desired.

For the second part we let $\Omega = G$ with the uniform distribution μ and $S_k^{(g)} = g.S_k$ for $g \in G$, $-2 \leq k \leq d-1$. According to the first part it remains to show that for this choice of random abstract cone/cofilling we have for all $\tau \in X(k+1)$ that

$$\lambda(\tau) \leq \binom{d+1}{k+2} \mathrm{Size}_k((S_j)_{-2 \leq j \leq d-1}).$$

To this end, let $\tau \in X(k+1)$. Write $G_\tau := \{g \in G : g.\tau = \tau\}$ for the stabilizer of τ under the action of G. Let R be a set of representatives of the cosets G/G_τ. Since we assume that G acts transitively on $X(d)$, we get

$$\bigcup_{g \in R} \{g.\sigma : \sigma \in X(d), \tau \subseteq \sigma\} = X(d).$$

Indeed, fix $\rho_0 \in X(d)$ with $\tau \subseteq \rho_0$. Since G acts transitively on $X(d)$, for any given $\rho \in X(d)$ there is $g \in G$ such that $g.\rho_0 = \rho$. Let $r \in R$ be the representative of gG_τ. So $r = gh$ for some $h \in G_\tau$. Let $\rho_0' := h^{-1}.\rho_0$. Then $\rho_0' \in X(d)$ with $\tau \subseteq \rho_0'$ and

$$r.\rho_0' = (gh).(h^{-1}.\rho_0) = g.\rho_0 = \rho.$$

It follows that

$$|X(d)| \leq \frac{|G|}{|G_\tau|} |\{\sigma \in X(d) : \tau \subseteq \sigma\}|.$$

By the definition of Garland weights this implies

$$|G_\tau| \leq w(\tau) \binom{d+1}{k+2} |G|.$$

Further note that for $g \in G$, $\tau \in X(k+1)$ and $\sigma \in X(k)$ we have

$$\begin{aligned}
&\tau \in \mathrm{supp}((g.S_k)\sigma) \text{ if and only if} \\
&\tau \in \mathrm{supp}(g.(S_k(g^{-1}.\sigma))) \text{ if and only if} \\
&g^{-1}.\tau \in \mathrm{supp}(S_k(g^{-1}.\sigma)).
\end{aligned}$$

Moreover, the Garland weights are invariant under the action of G, i.e. $w(\sigma) = w(g.\sigma)$ for all $g \in G$ and $\sigma \in X$.

With all these we estimate

$$\begin{aligned}
\lambda(\tau) &= \frac{1}{w(\tau)} \frac{1}{|G|} \sum_{g \in G} \sum_{\sigma \in X(k)} w(\sigma) |\langle \mathbb{1}_\tau, S_k^{(g)} \sigma \rangle|_{\mathbb{A}} \\
&= \frac{1}{w(\tau)} \frac{1}{|G|} \sum_{g \in G} \sum_{\sigma \in X(k)} w(\sigma) |\langle \mathbb{1}_{g^{-1}.\tau}, S_k(g^{-1}.\sigma) \rangle|_{\mathbb{A}} \\
&= \frac{1}{w(\tau)} \frac{1}{|G|} \sum_{g \in G} \sum_{\sigma \in X(k)} w(\sigma) |\langle \mathbb{1}_{g^{-1}.\tau}, S_k(\sigma) \rangle|_{\mathbb{A}} \\
&\leq \frac{1}{w(\tau)} \frac{1}{|G|} \sum_{g \in G} \sum_{\sigma \in X(k)} w(\sigma) \sum_{\rho \in X(k+1)} |\langle \mathbb{1}_\rho, S_k \sigma \rangle|_{\mathbb{A}} |\langle \mathbb{1}_{g^{-1}.\tau}, \rho \rangle|_{\mathbb{A}} \\
&\leq \frac{1}{w(\tau)} \frac{|G_\tau|}{|G|} \sum_{\sigma \in X(k)} w(\sigma) \sum_{\rho \in X(k+1)} |\langle \mathbb{1}_\rho, S_k \sigma \rangle|_{\mathbb{A}} \\
&\leq \binom{d+1}{k+2} \mathrm{Size}_k((S_j)_{-2 \leq j \leq d-1}) \sum_{\sigma \in X(k)} w(\sigma) \\
&= \binom{d+1}{k+2} \mathrm{Size}_k((S_j)_{-2 \leq j \leq d-1}),
\end{aligned}$$

which finishes the proof of the second part of the proposition. ☐

It is important to observe that, in principle, one can relax the assumptions in Proposition 3.13. First, notice that to define $c^{(\omega)}$ as $T_{k+1}^{(\omega)}\delta c$ we only need to know the value of $T_{k+1}^{(\omega)}$ on $B^{k+1}(X;\mathbb{A})$. Moreover, since we assume that

$$c = \delta T_k^{(\omega)} c + T_{k+1}^{(\omega)}\delta c,$$

$T_{k+1}^{(\omega)}\delta c$ is already determined on $B^{k+1}(X;\mathbb{A})$ once we fix $T_k^{(\omega)}$. Of course, trying to use the generic bound involving $\lambda(\tau)$, we would need to know $T_{k+1}^{(\omega)}$ on all of $C^{k+1}(X;\mathbb{A})$ in order to be able to understand its dual map $S_k^{(\omega)}$. But it could very well be that there are other means to analyze the expected value $\mathbb{E}_{\omega\sim\mu}|c - \delta T_k^{(\omega)}|$. We will encounter such a situation, for instance, when we show a lower bound on $\eta_k(G * G)$ for the join of an expander graph G with itself.

What is more, in order to define $c^{(\omega)}$ as $c - \delta T_k^{(\omega)} c$, we do not need $(T_k^{(\omega)}, T_{k+1}^{(\omega)})$ to form a cochain homotopy between the identity and 0-map on $C^k(X;\mathbb{A})$. In fact, we do not even need that $T_k^{(\omega)}\colon C^k(X;\mathbb{A}) \to C^k(X;\mathbb{A})$ is a homomorphism - any map would work. But then, it becomes less clear how to analyze $\mathbb{E}_{\omega\sim\mu}|c^{(\omega)}|$. Also, in view of Lemma 3.11 it seems reasonable to work with cochain homotopies between the identity and 0-map on $C^k(X;\mathbb{A})$ since they witness the vanishing of $\tilde{H}^k(X;\mathbb{A})$, a property we would like to quantify by giving a lower bound on $\eta_k^{|\cdot|}(X;\mathbb{A})$. Working on chain groups is especially appealing since we can construct chain homotopies inductively in a bottom-up fashion. This becomes particularly useful if there is an apparent family of small cycles and a group of automorphisms acting transitively on the top dimensional faces. Sometimes such a family of small cycles is witnessed by a nested family of (small) subcomplexes with vanishing cohomology as the following lemma shows:

Lemma 3.14. *Let X be a d-dimensional simplicial complex. Let $\mathbb{A}$ be an abelian group. Let $(B_\tau)_{\tau\in X^{(d-1)}}$ be a family of subcomplexes of X such that*

(i) $\tau \in B_\tau$ for all $\tau \in X$,

(ii) $B_\tau \subseteq B_{\tau'}$ whenever $\tau, \tau' \in X$ with $\tau \subseteq \tau'$, and

(iii) $\tilde{H}_j(B_\tau;\mathbb{A}) = 0$ for all $\tau \in X$ and $-1 \le j \le \dim\tau$.

Then there is an abstract cone $\mathcal{S} = (S_k)_{-2\le k\le d-1}$ for X with $\operatorname{supp}(S_k\sigma) \subseteq B_\sigma$ for all $\sigma \in X(k)$ and $-1 \le k \le d-1$.

Proof. We construct the abstract cone $\mathcal{S}$ by induction on k. For S_{-1} we note that by assumption $\tilde{H}_{-1}(B_\emptyset;\mathbb{A}) = 0$, hence $B_\emptyset$ is non-empty and we can define $S_{-1}\emptyset := v$ for some vertex $v \in B_\emptyset$.

Now assume that $S_j\colon C_j(X;\mathbb{A}) \to C_{j+1}(X;\mathbb{A})$ has already been constructed for $-1 \le j \le k-1$ such that $\operatorname{supp}(S_j\sigma) \subseteq B_\sigma$ for all $\sigma \in X(j)$ and such that $S_{j-1}\partial c + \partial S_j c = c$ for all $c \in C_j(X;\mathbb{A})$. Let $\sigma \in X(k)$. Note that by the induction hypothesis $\partial(\sigma - S_{k-1}\partial\sigma) = 0$. Using property (ii) of the family $(B_\tau)_{\tau\in X^{(d-1)}}$ of subcomplexes, we see that $\operatorname{supp}(\sigma - S_{k-1}\partial\sigma) \subseteq B_\sigma$, i.e. we can think of $\sigma - S_{k-1}\partial\sigma$ as a cycle in B_σ. But by assumption $\tilde{H}_k(B_\sigma;\mathbb{A}) = 0$. Hence, there is $c \in C_{k+1}(B_\sigma;\mathbb{A})$ with $\partial c = \sigma - S_{k-1}\partial\sigma$. Define $S_k\sigma := \bar{c}$

to be the extension of c by 0 to X. We have $S_{k-1}\partial\sigma + \partial S_k\sigma = \sigma$ and $\operatorname{supp}(S_k\sigma) \subseteq B_\sigma$ by construction. This finishes the proof. □

A family of subcomplexes as in Lemma 3.14 is at the heart of the definition of so-called *building-like complexes* in [98]. In particular, if X is a spherical building (see Definition 5.7 below) then it is not difficult to exhibit a family of subcomplexes as in Lemma 3.14 using the apartments in X.

3.2.3 A Lower Bound on $\eta_0(\Lambda_n^d)$ - Proof of Proposition 3.8

Write $\Lambda_n^d = U_0 * \cdots * U_d$ with $U_0 = \cdots = U_d = [n]$. We would like to exhibit many chain maps $S_k \colon C_k(\Lambda_n^d; \mathbb{F}_2) \to C_{k+1}(\Lambda_n^d; \mathbb{F}_2)$, $k \in \{-1, 0\}$ such that

$$\partial S_0 c + S_{-1}\partial c = c$$

for all $c \in C_0(\Lambda_n^d; \mathbb{F}_2)$. As discussed above, we could first define S_{-1} by $S_{-1}\emptyset := u$ for some vertex $u \in \Lambda_n^d(0)$. Then given $u' \in \Lambda_n^d(0)$ we have to define $S_0 u'$ to be a filling of $u + u'$. If u and u' are from different parts (i.e. $u \in U_i, u' \in U_j$ for some $i \neq j$), then we can set $S_0 u' = uu'$. If $u, u' \in U_i$, we could choose a third vertex $v \in U_j$ for some $j \neq i$ and set $S_0 u' = uv + u'v$.

Note that defining S_0 in this way, it depends on the ordered pair (u, v) of two vertices from two different parts in Λ_n^d. As we will shortly see, averaging over all possible ordered pairs will give the desired bound on $\eta_0(\Lambda_n^d)$.

To see this, we introduce the following notation. Let

$$\Omega := \bigsqcup_{0 \leq i,j \leq d, i \neq j} U_i \times U_j$$

and write μ for the uniform distribution on Ω. Let $u \in U_i, v \in U_j, i \neq j$. Let $\omega = (u, v) \in \Omega$ and define

$$\begin{aligned} S_{-1}^{(\omega)} \colon C_{-1}(\Lambda_n^d; \mathbb{F}_2) &\to C_0(\Lambda_n^d; \mathbb{F}_2) \\ \emptyset &\mapsto u \end{aligned}$$

and

$$\begin{aligned} S_0^{(\omega)} \colon C_0(\Lambda_n^d; \mathbb{F}_2) &\to C_1(\Lambda_n^d; \mathbb{F}_2) \\ u' &\mapsto \begin{cases} uu' & \text{if } u' \notin U_i, \\ uv + u'v & \text{if } u' \in U_i. \end{cases} \end{aligned}$$

By our discussion above, $\mathcal{S} := (S_{-1}^{(\omega)}, S_0^{(\omega)})_{\omega \in \Omega}$ is a random abstract cone in dimension 0 for Λ_n^d. Write $\mathcal{T} = (T_0^{(\omega)}, T_1^{(\omega)})_{\omega \in \Omega}$ for the dual random abstract cofilling.

For the analysis of $\mathbb{E}_{\omega \sim \mu} |T_1^{(\omega)} \delta c|$ for some $c \in C^0(\Lambda_n^d; \mathbb{F}_2)$ we will make use of the following negative type inequality.

Lemma 3.15. *Let $x_1, \ldots, x_k, y_1, \ldots, y_k \in \mathbb{R}$. Then*

$$\sum_{1 \leq i < j \leq k} (x_j - x_i)^2 + \sum_{1 \leq i < j \leq k} (y_j - y_i)^2 \leq \sum_{1 \leq i,j \leq k} (y_j - x_i)^2.$$

Consequently, if we write $\Lambda_n^1 = U * V$ with $U = \{u_1, \ldots, u_n\}$ and $V = \{v_1, \ldots, v_n\}$, then for all $c \in C^0(\Lambda_n^1; \mathbb{F}_2)$ we have

$$\sum_{uu' \in \binom{U}{2}} |c(u) + c(u')| + \sum_{vv' \in \binom{V}{2}} |c(v) + c(v')| \leq \sum_{u \in U, v \in V} |c(u) + c(v)|.$$

Proof. The second part follows from the first by setting $k = n$, $x_i = c(u_i) \in \{0, 1\} \subseteq \mathbb{R}$ and $y_i = c(v_i) \in \{0, 1\} \subseteq \mathbb{R}$, $1 \leq i \leq n$.

For the first part, a straightforward computation gives that

$$\sum_{1 \leq i,j \leq k} (y_j - x_i)^2 - \sum_{1 \leq i < j \leq k} (x_j - x_i)^2 - \sum_{1 \leq i < j \leq k} (y_j - y_i)^2 = \left(\sum_{i=1}^{k} x_i - \sum_{j=1}^{k} y_j \right)^2 \geq 0,$$

as desired. □

Now, given $\omega = (u, v) \in \Omega$ with $u \in U_i, v \in U_j$, $i \neq j$ and $c \in C^0(\Lambda_n^2; \mathbb{F}_2)$ we have

$$(T_1^{(\omega)} \delta c)(u') = \langle \delta c, S_0^{(\omega)} u' \rangle = \begin{cases} \delta c(uu') & \text{if } u' \notin U_i, \\ c(u) + c(u') & \text{if } u' \in U_i. \end{cases}$$

In particular, $T_1^{(\omega)} \delta c$ is independent of v. Using this, we compute for $c \in C^0(\Lambda_n^d; \mathbb{F}_2)$ that

$$\begin{aligned} |[c]| &\leq \mathbb{E}_{\omega \sim \mu} |T_1^{(\omega)} \delta c| \\ &= \frac{2}{(d+1)n} |\delta c| + \frac{2}{(d+1)n} \sum_{i=0}^{d} \sum_{uu' \in \binom{U_i}{2}} |c(u) + c(u')| \\ &= \frac{2}{(d+1)n} |\delta c| + \frac{2}{(d+1)n} \frac{1}{d} \sum_{0 \leq i < j \leq d} \left(\sum_{uu' \in \binom{U_i}{2}} |c(u) + c(u')| + \sum_{uu' \in \binom{U_j}{2}} |c(u) + c(u')| \right). \end{aligned}$$

Applying the negative-type inequality from Lemma 3.15 to the second term, we deduce

$$\begin{aligned} |[c]| &\leq \frac{2}{(d+1)n} |\delta c| + \frac{2}{d(d+1)n} \sum_{0 \leq i < j \leq d} \sum_{u \in U_i, u' \in U_j} |c(u) + c(u')| \\ &= \frac{2}{(d+1)n} \left(1 + \frac{1}{d} \right) |\delta c| \\ &= \frac{2}{dn} |\delta c|. \end{aligned}$$

Rearranging and normalizing finishes the proof of Proposition 3.8.

Chapter 4

A Quantitative Borsuk–Ulam Theorem

In this chapter, we prove the quantitative version of the Borsuk–Ulam Theorem (Theorem 1.1). In fact, we will prove the following more general result.

Theorem 4.1. *Let p be a prime. Let $G = \mathbb{Z}/p$. Let X be a d-dimensional simplicial complex with a free G-action. Fix a G-action on $\mathbb{R}^d$ by orthogonal maps which is free on $\mathbb{R}^d \setminus \{0\}$. Assume that there are positive constants $\eta_k > 0$ such that $\eta_k^{\|\cdot\|}(X; \mathbb{F}_p) \geq \eta_k > 0$ for all $0 \leq k \leq d-1$ where $\|\cdot\|$ is a size function induced by the Hamming norm on $\mathbb{F}_p$ and some $\mathbb{Z}/p$-invariant weight function w on X. Assume that $\tilde{H}^k(X; \mathbb{F}_p) = 0$ for all $0 \leq k \leq d-1$.*[1] *Then for every equivariant continuous map $F\colon |X| \to_G \mathbb{R}^d$ we have*

$$\|\{\sigma \in X(d) : 0 \in F(\sigma)\}\| \geq \|\mathbb{1}_{X(0)}\| \frac{1}{2^{d/2} p^{d/2}} \prod_{k=0}^{d-1} \eta_k.$$

The case $G = \mathbb{Z}/2$ already appeared in [140] where we also mentioned its generalization Theorem 4.1. The results in this chapter are joint work with Uli Wagner.

Our proof of Theorem 4.1 combines the idea of using approximation by piecewise-linear maps in general position and algebraic intersection numbers, as in the streamlined proof of Gromov's topological overlap theorem in [36], together with the idea of using a special $\mathbb{Z}/p$-invariant cell structure on spheres. For $\mathbb{Z}/2$ this cell structure is given by the hemispheres and was used in Walker's proof of the Borsuk–Ulam theorem for $\mathbb{Z}/2$-spaces in [141].

4.1 A Special $\mathbb{Z}/p$-invariant Cell Structure for $\mathbb{S}^d$

Given a map $F\colon |X| \to \mathbb{R}^d$ from a finite simplicial complex X to $\mathbb{R}^d$, using compactness, we can replace $\mathbb{R}^d$ by a closed ball $\mathbb{B}^d = B(0, R)$ of radius R centered at the origin such that the image $F(|X|)$ is contained in the interior of $\mathbb{B}^d$. Below we will make use of special triangulations of $\mathbb{B}^d$ which, restricted to the boundary sphere $\mathbb{S}^{d-1} = \partial\mathbb{B}^d$, refine a special $\mathbb{Z}/p$-invariant regular CW-complex structure on $\mathbb{S}^{d-1}$. For $\mathbb{Z}/2$ this structure is quite

[1] Recall that by Lemma 3.3 (i) $\tilde{H}^k(X; \mathbb{F}_p) = 0$ follows automatically from $\eta_k^{\|\cdot\|}(X; \mathbb{F}_p) > 0$ if $\|\cdot\|$ is coboundary separating on $C^k(X; \mathbb{F}_p)$.

apparent and given by the hemisphere. For $\mathbb{Z}/p$ our discussion is a slight generalization of [131, Section V.5].[2]

To define such a structure, write ν for a generator of $\mathbb{Z}/p$ and, by abuse of notation, we also write $\nu\colon \mathbb{R}^d \to \mathbb{R}^d$ for the orthogonal transformation of the action of ν on $\mathbb{R}^d$. It is not difficult to see that $\mathbb{Z}/p$ acts freely on $\mathbb{S}^{d-1} \subseteq \mathbb{R}^d$ if and only if ν has no fixed point (cf. [107, Observation 6.1.3]), i.e. if $\nu(x) \neq x$ for all $x \in \mathbb{S}^{d-1}$.

Now, if $p = 2$, then $\nu \circ \nu = \mathrm{id}$. In particular, all eigenvalues of ν are in $\{-1, 1\}$. But, since we assume that ν acts freely on $\mathbb{R}^d \setminus \{0\}$, all eigenvalues of ν are equal to -1. Thus, there is a orthogonal basis of $\mathbb{R}^d$ such that with respect to this basis ν is given as the antipodal map $x \mapsto -x$. It follows that without loss of generality we can assume that ν is the usual antipodal map on $\mathbb{R}^d$. For such ν a $\mathbb{Z}/2$-invariant, regular CW structure on $\mathbb{S}^{d-1}$ is given by the hemisphere. More precisely, such a structure has two cells in each dimension and is inductively obtained by decomposing a k-dimensional sphere into a $(k-1)$-dimensional equitorial sphere with two k-dimensional cells (upper and lower hemisphere) attached. We illustrate this cell structure for $\mathbb{S}^d$, $d \in \{0, 1, 2\}$, in Figure 4.1.

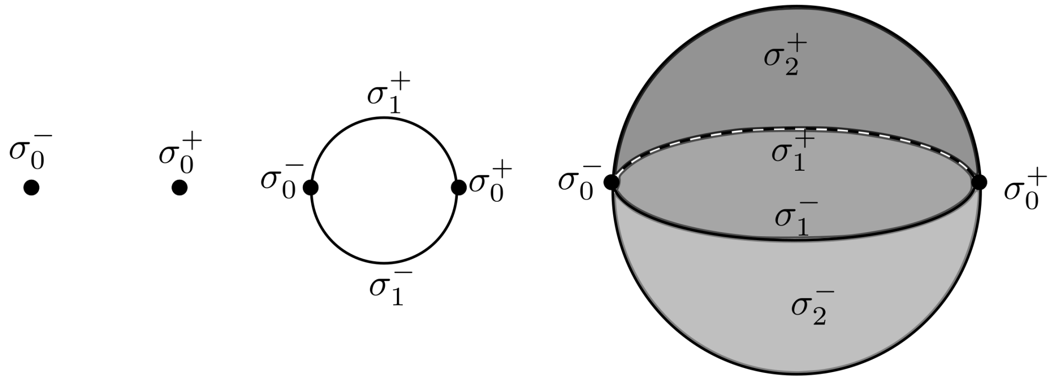


Figure 4.1: A $\mathbb{Z}/2$-invariant cell structure for $\mathbb{S}^d$, $d \in \{0, 1, 2\}$ with 2 cells in each dimension. For $d = 1$ we attach two semicircle σ_1^- and σ_1^+ to the two points σ_0^- and σ_0^+. For $d = 2$ we start with the cell structure for $\mathbb{S}^1$ and attach two hemispheres σ_2^- and σ_2^+ along this $\mathbb{S}^1$.

We can be a bit more explicit. Consider the unit sphere $\mathbb{S}^d \subseteq \mathbb{R}^{d+1}$. For $0 \leq k \leq d$ let $\mathbb{S}^k \subseteq \mathbb{S}^d$ be given by

$$\mathbb{S}^k = \{(x_0, \ldots, x_d) \in \mathbb{S}^d : x_j = 0 \text{ for } j > k\}.$$

For $0 \leq k \leq d$ let

$$\tau^{(k)} := \{(x_0, \ldots, x_d) \in \mathbb{S}^d : x_k \geq 0, x_j = 0 \text{ for } j > k\}.$$

Note that $f_k\colon \mathbb{B}^k \to \tau^{(k)}$ given by

$$x = (x_0, \ldots, x_{k-1}) \mapsto (x_0, \ldots, x_1, \ldots, x_{k-1}, \sqrt{1 - |x|^2}, 0, \ldots, 0)$$

is a homeomorphism. Note that $\tau^{(k)} \cap \nu\tau^{(k)} = \mathbb{S}^{k-1}$ for all $1 \leq k \leq d$. It follows that the cells $\{\tau^{(k)} : 0 \leq k \leq d\} \cup \{\nu\tau^{(k)} : 0 \leq k \leq d\}$ together with the attaching maps f_k

[2] We thank Ian Leary and especially Neil Strickland for their answers to our question on MathOverflow [65] regarding a special $\mathbb{Z}/p$-invariant cell structure on $\mathbb{S}^d$. They greatly helped us to clarify how to describe such a structure and pointed us to Steenrod's lectures [131].

and $\nu \circ f_k$, $0 \leq k \leq d$ form a $\mathbb{Z}/2$-invariant regular CW-structure on $\mathbb{S}^d$. Thinking of the cells as chains in $C_k(\mathbb{S}^d; \mathbb{F}_2)$ we have $\partial e_k = e_{k-1} + \nu e_{k-1}$ for all $1 \leq k \leq d$. Moreover, $e_d + \nu e_d = \mathbb{1}$.

To generalize this to free $\mathbb{Z}/p$-actions on $\mathbb{S}^d$ for a prime $p > 2$, we first note that in this case, we must have that d is odd. This follows for instance from the fact that if a finite group G acts freely on $\mathbb{S}^d$ then $|G|$ must divide the Euler characteristic which for even dimensional spheres $\mathbb{S}^{2m}$ is 2.

So, let $d = 2m + 1$, $m \in \mathbb{Z}_{\geq 0}$, be odd and think of $\mathbb{S}^d \subseteq \mathbb{C}^{m+1}$. By the representation theory of $\mathbb{Z}/p$ (see, e.g., [128, Chapter V]) we can without loss of generality assume that the action of the generator ν is given by $\nu.(z_0, \ldots, z_m) = (\lambda_0 z_0, \ldots, \lambda_m z_m)$ where λ_j is a primitive p-th root of unity, i.e. $\lambda_j = e^{\frac{2\pi s_j i}{p}}$ for some $s_j \in \{1, \ldots, p-1\}$.

There is a nested sequence of odd dimensional spheres $\mathbb{S}^1 \subseteq \mathbb{S}^3 \subseteq \cdots \subseteq \mathbb{S}^{2m+1}$ given by $\mathbb{S}^{2k+1} = \{(z_0, \ldots, z_m) \in \mathbb{S}^{2m+1} : z_j = 0 \text{ for } j > k\}$. Note that each of these spheres is invariant under the $\mathbb{Z}/p$-action. We can construct an equivariant cell decomposition of $\mathbb{S}^{2m+1}$ with p cells in each dimension inductively by constructing cell decomposition of $\mathbb{S}^{2k+1}$ for all $0 \leq k \leq m$. To pass from a decomposition of $\mathbb{S}^{2k-1}$ to $\mathbb{S}^{2k+1}$ we pick a $2k$-dimensional cell $\tau^{(2k)}$ in $\mathbb{S}^{2k+1}$ whose boundary is $\mathbb{S}^{2k-1}$ such that the complement of the orbit of $\tau^{(2k)}$ under $\mathbb{Z}/p$ is the disjoint union of p $(2k+1)$-dimensional (open) cells. Let us make this more precise. To this end, define for $0 \leq k \leq m$ the sets

$$\tau^{(2k)} := \{(z_0, \ldots, z_m) \in \mathbb{S}^{2m+1} : z_j = 0 \text{ if } j > k, z_k \in [0, 1]\}$$

and

$$\tau^{(2k+1)} := \{(z_0, \ldots, z_m) \in \mathbb{S}^{2m+1} : z_j = 0 \text{ if } j > k, z_k = re^{i\theta} \text{ with } r \in [0, 1], \theta \in [0, 2\pi/p]\}.$$

Note that $\tau^{(0)} = \{(1, 0, \ldots, 0)\} \subseteq \mathbb{S}^{2m+1}$. For $k \geq 1$ we have:

Claim 4.2. *$\tau^{(2k)}$ is homeomorphic to $\mathbb{B}^{2k}$ whose boundary is $\mathbb{S}^{2k-1}$.*

Proof. Thinking of $\mathbb{B}^{2k}$ as the unit ball in $\mathbb{C}^k$ we see that the map $f_{2k} \colon \mathbb{B}^{2k} \to \tau^{(2k)}$ given by $z \mapsto (z, \sqrt{1 - |z|^2}, 0, \ldots, 0)$ is a homeomorphism. The second part follows from the fact that $f_{2k}(z) = (z, 0, \ldots, 0)$ for $z \in \mathbb{B}^{2k}$ with $|z| = 1$. □

As expected we have for $k \geq 0$ that

Claim 4.3. *$\tau^{(2k+1)}$ is homeomorphic to $\mathbb{B}^{2k+1}$. Moreover, the boundary sphere of $\tau^{(2k+1)}$ is the union of $\tau^{(2k)}$ and $\nu^j \tau^{(2k)}$ where j is such that $\lambda_k^j = e^{\frac{2\pi i}{p}}$.*

Proof. Again we think of $\mathbb{B}^{2k}$ as the unit ball in $\mathbb{C}^k$. We think of $\mathbb{B}^{2k+1}$ as the unit ball in $\mathbb{C}^k \oplus \mathbb{R}$, i.e. $\mathbb{B}^{2k+1} = \{(z, t) \in \mathbb{C}^k \oplus \mathbb{R} : |z|^2 + t^2 \leq 1\}$. To see that $\tau^{(2k+1)}$ is homeomorphic to $\mathbb{B}^{2k+1}$ we will exhibit two surjective continuous maps $g_1 \colon \mathbb{B}^{2k} \times [0, 1] \to \tau^{(2k+1)}$ and $g_2 \colon \mathbb{B}^{2k} \times [0, 1] \to \mathbb{B}^{2k+1}$ such that $g_1(z, t) = g_1(z', t')$ if and only if $g_2(z, t) = g_2(z', t')$ for all $(z, t), (z', t') \in \mathbb{B}^{2k} \times [0, 1]$. From this it follows that the quotient space $\mathbb{B}^{2k} \times [0, 1]/\sim$ where $(z, t) \sim (z', t')$ if $g_1(z, t) = g_1(z', t')$ is both homeomorphic to $\tau^{(2k+1)}$ and $\mathbb{B}^{2k+1}$ and that there is a homeomorphism $f_{2k+1} \colon \mathbb{B}^{2k+1} \to \tau^{(2k+1)}$ such that $g_1 = f_{2k+1} \circ g_2$.

Define g_1 by $g_1(z,t) := (z, \sqrt{1-|z|^2}e^{\frac{2\pi t i}{p}})$ and g_2 by $g_2(z,t) := (z, \sqrt{1-|z|^2}(2t-1))$. One readily checks that g_1 and g_2 are indeed onto and that $g_1(z,t) = g_1(z',t')$ if and only if $z = z'$ and $t = t'$ or $|z| = 1$ if and only if $g_2(z,t) = g_2(z',t')$.

For the second part first note that $|g_2(z,t)| = 1$ if and only if $|z| = 1$ or $t \in \{0,1\}$. From this the second part follows from the definition of g_1. □

Consider the cells $C = \{\nu^i \tau^{(j)} : 0 \le i \le p-1, 0 \le j \le 2m+1\}$. Note that for each $0 \le k \le m$ the cells $\{\nu^i \tau^{(2k+1)} : 0 \le i \le p-1\}$ have pairwise disjoint interior and that $\mathbb{S}^{2k+1} = \bigcup_{i=0}^{p-1} \nu^i \tau^{(2k+1)}$. It follows that the cells C together with the attaching maps $\nu^i \circ f_j$, $0 \le i \le p-1, 0 \le j \le 2m+1$, define a $\mathbb{Z}/p$-invariant regular CW complex structure on $\mathbb{S}^{2m+1}$. Here f_j is defined as in the proofs of Claim 4.2 and Claim 4.3, respectively. Thinking of $\tau^{(j)}$ as a chain in $C_j(\mathbb{S}^{2m+1}; \mathbb{F}_p)$ we see that $\partial \tau^{(2k)} = \sum_{i=0}^{p-1} \nu^i \tau^{(2k-1)}$ for all $1 \le k \le m$ and that $\partial \tau^{(2k+1)} = \nu^{j_k} \tau^{(2k)} - \tau^{(2k)}$ for all $0 \le k \le m$, where $1 \le j_k \le p-1$ is such that $\lambda_k^{j_k} = e^{\frac{2\pi i}{p}}$.

4.2 Approximation by a Piecewise-Linear Map

The first step in the proof of Theorem 4.1 is a (fairly standard) limiting argument which allows us to replace arbitrary continuous maps by piecewise-linear maps in general position. First note that by compactness we can assume that $F(|X|)$ is contained in the interior of the closed ball $\mathbb{B}^d = B(0,R)$ for some sufficiently large R. We endow the boundary sphere $\mathbb{S}^d$ of $\mathbb{B}^d$ with an equivariant cell structure as defined in the previous section. Write $\tau_j^{(k)} = \nu^j \tau^{(k)}$, $0 \le j \le p-1, 0 \le k \le d-1$ for the p cells in each dimension k in this decomposition. We get an induced cell structure on $\mathbb{B}^d$ by adding the origin as an additional 0-cell and cone every cell $\tau_j^{(k)}$ with 0. That is, we add all cells of the form $\sigma_j^{(k+1)} := 0 * \tau_j^k$ for $0 \le k \le d-1$ and $0 \le j \le p-1$. We call a triangulation T of $\mathbb{B}^d$ *good* if it is invariant under the $\mathbb{Z}/p$-action and if it refines the cell structure on $\mathbb{B}^d$ given by the cells

$$\{0\} \cup \{\tau_j^{(k)} : 0 \le j \le p-1, 0 \le k \le d-1\} \cup \{\sigma_j^{(k)} : 0 \le j \le p-1, 1 \le k \le d\}.$$

Note that since the cell structure of $\mathbb{S}^d$ we start with is regular, the induced cell structure on $\mathbb{B}^d$ is regular as well. Thus, there is always a refinement which is a ($\mathbb{Z}/p$-invariant) triangulation. Below we will only work with good triangulations of $\mathbb{B}^d$.

Let Y be a simplicial complex. Recall that a map $f \colon |Y| \to \mathbb{R}^d$ is *piecewise-linear (PL)* if there is a subdivision Y' of Y such that the restriction of f to every simplex of Y' is an affine map.

Two affine spaces $A_1, A_2 \subseteq \mathbb{R}^d$ are *in general position* if

$$\dim(A_1 \cap A_2) = \max\{-1, \dim(A_1) + \dim(A_2) - d\}.$$

Note this amounts to say that the $d - \dim(A_1)$ equations defining A_1 are independent of the $d - \dim(A_2)$ equations defining A_2. We stress that if A_1, A_2 are in general position with $\max\{-1, \dim(A_1) + \dim(A_2) - d\} = -1$ then $A_1 \cap A_2 = \emptyset$. We say that a set of points $S \subseteq \mathbb{R}^d$ is *in general position* if for any two disjoint subsets $S_1, S_2 \subseteq S$ the affine hulls $\mathrm{aff}(S_1)$ and $\mathrm{aff}(S_2)$ are in general position. A simplexwise affine map $f \colon Y \to \mathbb{R}^d$ is *in general position* if it is injective on the vertices of Y and $\{f(v) : v \in Y(0)\} \subseteq \mathbb{R}^d$ is in general position.

Let T be a triangulation of $\mathbb{B}^d = B(0, R)$. Let $f\colon |Y| \to \mathbb{R}^d$ be a piecewise-linear map which is simplexwise affine on the subdivision Y' and for which $f(|Y|)$ is contained in the interior of $\mathbb{B}^d$. We say that f is *in general position with respect to T* if f as a simplexwise affine map $f\colon Y' \to \mathbb{R}^d$ is in general position and if for all $\sigma \in Y'$ and $\tau \in T$ we have $\dim(\mathrm{Aff}(f(\sigma) \cap \tau)) \le \max\{-1, \dim\sigma + \dim\tau - d\}$.

In order to avoid any confusion we will write $\mathrm{dist}(a, b)$ for the Euclidean distance of two points $a, b \in \mathbb{R}^d$.

With all these we have

Lemma 4.4. *Let X be a d-dimensional simplicial complex with a free $\mathbb{Z}/p$-action. Let $f\colon |X| \to_{\mathbb{Z}/p} \mathbb{R}^d$ be an equivariant, continuous map such that $f(|X|)$ is contained in the interior of $\mathbb{B}^d = B(0, R)$. Let T be a good triangulation of $\mathbb{B}^d$. Then for any $\varepsilon > 0$ there is an equivariant PL-map $g\colon |X| \to_{\mathbb{Z}/p} \mathbb{R}^d$ which is in general position with respect to T and such that $\mathrm{dist}(f(x), g(x)) \le \varepsilon$ for all $x \in |X|$.*

Proof. It follows from an equivariant version of the classical simplicial approximation theorem (see [19, I, Exercise 6])) that there is an equivariant PL-map $\tilde{g}\colon |X| \to_{\mathbb{Z}/p} \mathbb{R}^d$ such that $\mathrm{dist}(\tilde{g}(x), f(x)) \le \varepsilon/2$ for all $x \in |X|$. The map $\tilde{g}$ might not be in general position with respect to T yet. In order to fix this, let X' be a subdivision of X on which $\tilde{g}$ is simplexwise affine. Since X is a free $\mathbb{Z}/p$-complex, the vertex set $X'(0)$ of X' decomposes into a partion $X'(0) = V_0 \sqcup V_1 \sqcup \cdots \sqcup V_{p-1}$ such that each V_i contains precisely one vertex from each $\mathbb{Z}/p$-orbit. We can assume that $V_i = \nu^i V_0$ for all $0 \le i \le p-1$ where ν is (the action of) a generator of $\mathbb{Z}/p$. For each $v \in V_0$ pick a vector ε_v in $\mathbb{B}^d_{\varepsilon/2}(0) = \{x \in \mathbb{R}^d : \mathrm{dist}(x, 0) \le \varepsilon/2\}$ uniformly at random. Let $g\colon |X'| \to_{\mathbb{Z}/p} \mathbb{R}^d$ be the simplexwise affine map given by $g(\nu^j v) = \tilde{g}(\nu^j v) + \nu^j \varepsilon_v$ for $v \in V_0$ and $0 \le j \le p-1$. Since $\mathbb{Z}/p$ acts freely, we can assume (after passing to the barycentric subdivision of X', cf. [19, III, Proposition 1.1]) that $\sigma \cap \nu^j\sigma = \emptyset$ for all $\sigma \in X'$ and $0 \le j \le p-1$. This implies that with probability 1, the map g is in general position with respect to T. By construction we have $\mathrm{dist}(g(v), \tilde{g}(v)) \le \varepsilon/2$ for all $v \in X'(0)$. Moreover, using that both g and $\tilde{g}$ are simplexwise affine on X', the triangle inequality gives that for $x \in X$

$$\mathrm{dist}(f(x), g(x)) \le \mathrm{dist}(f(x), \tilde{g}(x)) + \mathrm{dist}(\tilde{g}(x), g(x)) \le \varepsilon/2 + \varepsilon/2 = \varepsilon,$$

as desired. □

Combining Lemma 4.4 with the following lemma, we can reduce the proof of Theorem 4.1 to the case of PL maps which are in general position with respect to a good triangulation T.

Lemma 4.5. *Let X be a d-dimensional free $\mathbb{Z}/p$-complex. Let $F\colon |X| \to_{\mathbb{Z}/p} \mathbb{R}^d$ be an equivariant map. Assume that $F_n\colon |X| \to_{\mathbb{Z}/p} \mathbb{R}^d$ is a sequence of equivariant maps converging uniformly to F, i.e. $\lim_{n\to+\infty} \sup_{x\in|X|} \mathrm{dist}(F(x), F_n(x)) = 0$. Let $\|\cdot\|$ be a size function $C^d(X; \mathbb{F}_p)$ induced by the Hamming norm on $\mathbb{F}_p$ and some weight function w on X. Let $S = \{\sigma \in X(d) : 0 \in F(\sigma)\}$ and $S_n = \{\sigma \in X(d) : 0 \in F_n(\sigma)\}$. If there is $\mu > 0$ such that $\|\mathbb{1}_{S_n}\| \ge \mu$ for all $n \in \mathbb{Z}_{>0}$ then $\|\mathbb{1}_S\| \ge \mu$.*

Proof. By compactness the infimum in

$$\rho := \inf\{\mathrm{dist}(F(x), 0) : \sigma \in X(d) \setminus S, x \in \sigma\}$$

is attained and $\rho > 0$. There is $n \in \mathbb{Z}_{>0}$ such that $\operatorname{dist}(F_n(x), F(x)) < \rho$ for all $x \in |X|$. Assume $\sigma \in S_n$ and let $x \in \sigma$ with $F_n(x) = 0$. We get that $\operatorname{dist}(F(x), 0) = \operatorname{dist}(F(x), F_n(x)) < \rho$. By choice of ρ this implies that $\sigma \in S$. We conclude that $S_n \subseteq S$ and the lemma follows from the monotonicity of $\| \cdot \|$. □

4.3 Algebraic Intersection Numbers

The advantage of working with PL maps in general position with respect to a good triangulation of $\mathbb{B}^d$ is that it allows to define algebraic intersection numbers. Here we only define these intersection numbers, which can be seen as a special case of Lefschetz intersection theory, and state some basic results we will need. We refer the reader to [103, Section 2.2] and references therein for a detailed review on intersection numbers.

Given a (geometric) k-simplex $\sigma \subseteq \mathbb{R}^d$ specifying an orientation for σ amounts to choosing an ordered basis of the linear space $L(\sigma)$ parallel to the k-dimensional affine space spanned by σ. If $\sigma = \operatorname{conv}(v_0, \dots, v_k)$ and the vertices are ordered as $v_0 < \dots < v_k$ we will always choose $(v_1 - v_0, \dots, v_k - v_0)$ as an ordered bases for $L(\sigma)$ and write $[v_0, \dots, v_k]$ for the oriented simplex σ with this orientation. Let $\sigma, \tau \subseteq \mathbb{R}^d$ be two oriented simplices in general position by which we mean that the vertices of σ and τ are pairwise distinct and that the union of the vertices is a set of points in $\mathbb{R}^d$ in general position as defined in the previous section. If $\dim \sigma + \dim \tau = d$ we have that $\sigma \cap \tau$ is empty or a single point. Let $(b_1, \dots, b_{\dim \sigma})$ and $(\tilde{b}_1, \dots, \tilde{b}_{\dim \tau})$ be ordered basis of $L(\sigma)$ and $L(\tau)$ corresponding to the choosen orientation of σ and τ. By general position $B = (b_1, \dots, b_{\dim \sigma}, \tilde{b}_1, \dots, \tilde{b}_{\dim \tau})$ is an ordered basis of $\mathbb{R}^d$. If $\sigma \cap \tau = \emptyset$, we define the *intersection number* $\sigma \bullet \tau$ *of* σ *and* τ to be 0. If $\sigma \cap \tau \neq \emptyset$ we define $\sigma \bullet \tau$ to be the sign of $\det B$, where we think of B as a matrix with the basis vectors as columns. That is, $\sigma \bullet \tau \in \{-1, 1\}$ and the sign depends on whether B has the same or opposite orientation as $\mathbb{R}^d$ (with the orientation determined by the standard basis vectors $(e_1, \dots, e_d)$).

We can extend this to k-dimensional PL chains in $\mathbb{R}^d$ which are formal linear combinations $c = \sum_{j \in J} a_j \sigma_j$ for some finite index set J and oriented k-simplices σ_j with $a_j \in \mathbb{Z}$. Consider a k-dimensional PL chain $c = \sum_{i \in I} a_i \sigma_i$ and a $(d-k)$-dimensional PL chain $c' = \sum_{j \in J} b_j \tau_j$ such that σ_i and τ_j are in general position for all $i \in I$ and $j \in J$. Then we define the intersection number $c \bullet c'$ of c and c' by $c \bullet c' := \sum_{i \in I, j \in J} a_i b_j (\sigma_i \bullet \tau_j) \in \mathbb{Z}$.

Given a PL map $F\colon |X| \to_{\mathbb{Z}/p} \mathbb{R}^d$ in general position with respect to a good trinagulation T of $\mathbb{B}^d$, we can define an *intersection homomorphism* $F^{\pitchfork}\colon C_k(T; \mathbb{Z}) \to C^{d-k}(X; \mathbb{Z})$ as follows: Let X' be a subdivision of X such that $F\colon |X'| \to_{\mathbb{Z}/p} \mathbb{R}^d$ is simplexwise affine. Fix an ordering of the vertices of X'. By general position we have that F is injective on $X'(0)$ and thus, the ordering on $X'(0)$ induces an ordering on $F(X'(0))$. Given an oriented k-simplex $\tau = [v_0, \dots, v_k]$, i.e. with $v_0 < v_1 < \dots < v_k$ according to the chosen ordering $<$ on $X'(0)$, we give $F(\tau)$ there orientation of $[F(v_0), \dots, F(v_k)]$. Any oriented simplex $\sigma \in X_k$ is the formal sum $\sigma = \tau_1 + \dots + \tau_l$ of some oriented simplices $\tau_i \in X'_k$. We get a k-dimensional PL chain $F_\sharp(\sigma) = \sum_{i=1}^{l} F(\tau_i)$ in $\mathbb{R}^d$. For $c \in C^k(T; \mathbb{Z})$ we can use this to define $F^{\pitchfork}(c) \in C^{d-k}(X; \mathbb{Z})$ to be given by $F^{\pitchfork}(c)(\sigma) := (-1)^k c \bullet F_\sharp(\sigma)$ for all $\sigma \in X_{d-k}$.

For any prime p we can consider $F^{\pitchfork}$ as a homomorphism $F^{\pitchfork}\colon C_k(T; \mathbb{F}_p) \to C^{d-k}(X; \mathbb{F}_p)$ by reducing $\bmod p$. In particular, for $p = 2$ the definition of $F^{\pitchfork}$ greatly simplifies since we do not have to take care of orientations. In that case we have by general position for $\tau \in T(k)$ and $\sigma \in X(d-k)$ that $\sigma \cap F^{-1}(\tau)$ is a set of finitely many points and we can define $\tau \bullet F(\sigma) := |\sigma \cap F^{-1}(\tau)| \mod 2$. This extends to the intersection homomoprhism $F^{\pitchfork}\colon C_k(T;\mathbb{F}_2) \to C^{d-k}(X;\mathbb{F}_2)$ as

$$F^{\pitchfork}(c)(\sigma) = \sum_{\tau \in T(k), c(\tau)=1} |\sigma \cap F^{-1}(\tau)| \mod 2$$

for $c \in C_k(T;\mathbb{F}_2)$ and $\sigma \in X(d-k)$. We will need the following property of $F^{\pitchfork}$ which is a consequence of Lemma 28 in [103].

Lemma 4.6. *For all $0 \le k \le d$ and $c \in C_k(X;\mathbb{Z})$ we have $\delta F^{\pitchfork}(c) = F^{\pitchfork}(\partial c)$.*

4.4 Pagodas and the Proof of Theorem 4.1

From our discussion in the previous sections, we see that it suffices to prove Theorem 4.1 for PL maps $F\colon |X| \to_{\mathbb{Z}/p} \mathbb{R}^d$ such that $F(|X|)$ is contained in the interior of $\mathbb{B}^d = B(0,R)$ and such that F is in general position with respect to a good triangulation T of $\mathbb{B}^d$.

Fix such a map $F\colon |X| \to_{\mathbb{Z}/p} \mathbb{R}^d$. Consider (co)chain groups with respect to $\mathbb{F}_p$-coefficients. In particular, we think of $F^{\pitchfork}$ as a homomorphism $F^{\pitchfork}\colon C_k(T;\mathbb{F}_p) \to C^{d-k}(X;\mathbb{F}_p)$. Note that

$$\{\sigma \in X(d) : F^{\pitchfork}(0)(\sigma) \neq 0\} \subseteq \{\sigma \in X(d) : 0 \in F(\sigma)\}.$$

Hence, by monotonicity of $\|\cdot\|$, it suffices to give a lower bound on $\|F^{\pitchfork}(0)\|$.

As before write $\nu\colon |X| \to |X|$ for the action of a generator of $\mathbb{Z}/p$. We also write ν for the induced action on (co)chains

We need the notion of a pagoda. For the definition of a pagoda we distinguish the case $p=2$ and $p \ge 3$, the former being somewhat a bit easier.

If $p=2$, a *pagoda for F* is a sequence of cochains $(b^{(d)}, a^{(d-1)}, b^{(d-1)}, \dots, a^{(0)}, b^{(0)})$ such that

(i) $b^{(d)} = F^{\pitchfork}(0)$ (where 0 is the vertex in T corresponding to the origin in $\mathbb{B}^d$),

(ii) $b^{(k)}, a^{(k)} \in C^k(X;\mathbb{F}_2)$ for all $0 \le k \le d-1$,

(iii) $b^{(k)} = a^{(k)} + \nu a^{(k)}$ for all $0 \le k \le d-1$, and

(iv) $b^{(k)} = \delta a^{(k-1)}$ for all $1 \le k \le d$.

For a prime $p \ge 3$ the definition of a pagoda for F is slightly more involved. Recall that in this case $d = 2(m+1)$ must be even and we can assume that ν acts on $\mathbb{R}^d \cong \mathbb{C}^{m+1}$ by $\nu.(z_0,\dots,z_m) = (\lambda_0 z_0, \dots, \lambda_m z_m)$ for some primitive pth roots of unity $\lambda_j = e^{2\pi n_j i/p}$ with $0 < n_j \le p-1$, $0 \le j \le m$.

Let $s = \sum_{i=0}^{p-1} \nu^i$ which we think of as an element of the group ring $\mathbb{F}_p[\mathbb{Z}/p]$ acting on (co)chain groups. For $0 \le j \le m$ let k_j such that $\lambda_j^{k_j} = e^{2\pi i/p}$ and let $t_j = \nu^{k_j} - \mathrm{id} \in \mathbb{F}_p[\mathbb{Z}/p]$.

For $p \ge 3$, a *pagoda for F* is a sequence of chains $(b^{(d)}, a^{(d-1)}, b^{(d-1)}, \dots, a^{(0)}, b^{(0)})$ such that

(i) $b^{(d)} = F^{\pitchfork}(0)$,

(ii) $b^{(k)}, a^{(k)} \in C^k(X;\mathbb{F}_2)$ for all $0 \leq k \leq d-1$,

(iii) $b^{(k)} = sa^{(k)}$ if $0 \leq k \leq d-1$ is even and $b^{(k)} = t_l a^{(k)}$ if $k = 2l+1$ is odd for some $0 \leq l \leq m$, and

(iv) $b^{(k)} = \delta a^{(k-1)}$ for all $1 \leq k \leq d$.

Note that we recover the definition of a pagoda for F for $p = 2$ if we set $s = t_l = \mathrm{id} + \nu$ for all l. To simplify our arguments below we will only prove the case $p \geq 3$. The case $p = 2$ can be proven similarly by setting $s = t_l = \mathrm{id} + \nu$. In fact, the proof $p = 2$ simplifies a bit, since we can ignore signs and we do not have to distinguish between odd and even dimensional cochains. We refer the reader to [140] where only the simpler case $p = 2$ is discussed.

We can always pullback the special cell decomposition of a good triangulation using $F^{\pitchfork}$ to construct a pagoda. We illustrate such a pagoda for a $\mathbb{Z}/2$-equivariant PL map from the octahedron Λ_2^2 to $\mathbb{B}^2$ in Figure 4.2. More generally, we have:

Lemma 4.7. *There exists a pagoda for F with $b^{(0)} = \mathbb{1}_{X(0)} \in B^0(X;\mathbb{F}_p)$.*

Proof. By abuse of notation, we write $\tau_j^{(k)}$ and $\sigma_j^{(l)}$ for the chains in T refining the corresponding cells in the special cell decomposition of $\mathbb{B}^d$.

Define $b^{(d)} := F^{\pitchfork}(0)$ and for $0 \leq k \leq d-1$ define $a^{(k)} := (-1)^k F^{\pitchfork}(\sigma_0^{(d-k)})$ and

$$b^{(k)} := \begin{cases} t_l a^{(k)} & \text{if } k = 2l+1 \text{ is odd} \\ sa^{(k)} & \text{if } k \text{ is even.} \end{cases}$$

Since the image of F is contained in the interior of T, we have $F^{\pitchfork}(\tau_j^{(k)}) = 0$ for all $0 \leq k \leq d-1$ and $0 \leq j \leq p-1$.

We compute using Lemma 4.6 that

$$\delta a^{(d-1)} = (-1)^{d-1}\delta F^{\pitchfork}(\sigma_0^{(1)}) = -F^{\pitchfork}(\partial\sigma_0^{(1)}) = F^{\pitchfork}(0) - F^{\pitchfork}(\tau_0^{(0)}) = F^{\pitchfork}(0) = b^{(d)}.$$

Similarly, for $1 \leq k \leq d-1$ we get

$$\begin{aligned} \delta a^{(k-1)} &= (-1)^{k-1}\delta F^{\pitchfork}(\sigma_0^{(d-k+1)}) \\ &= (-1)^{k-1}(F^{\pitchfork}(\partial\sigma_0^{(d-k+1)}) \\ &= (-1)^{k-1}(F^{\pitchfork}(\tau_0^{(d-k)}) - F^{\pitchfork}(0 * \partial\tau_0^{(d-k)}) \\ &= (-1)^k F^{\pitchfork}(0 * \partial\tau_0^{(d-k)}). \end{aligned}$$

If k is even, the right handside is equal to

$$(-1)^k F^{\pitchfork}(0 * s \cdot \tau_0^{(d-k-1)}) = (-1)^k s \cdot F^{\pitchfork}(\sigma_0^{(d-k)}) = s \cdot a^{(k)} = b^{(k)}$$

and if $k = 2l+1$ is odd, it is equal to

$$(-1)^k F^{\pitchfork}(0 * t_l \cdot \tau_0^{(d-k-1)}) = (-1)^k t_l \cdot F^{\pitchfork}(\sigma_0^{(d-k)}) = t_l \cdot a^{(k)} = b^{(k)}.$$

Finally, $b^{(0)} = s \cdot a^{(0)} = s \cdot F^{\pitchfork}(\sigma_0^{(d)}) = \mathbb{1}_{X(0)}$ since every vertex of X gets mapped to the interior of a unique d-simplex in T. □

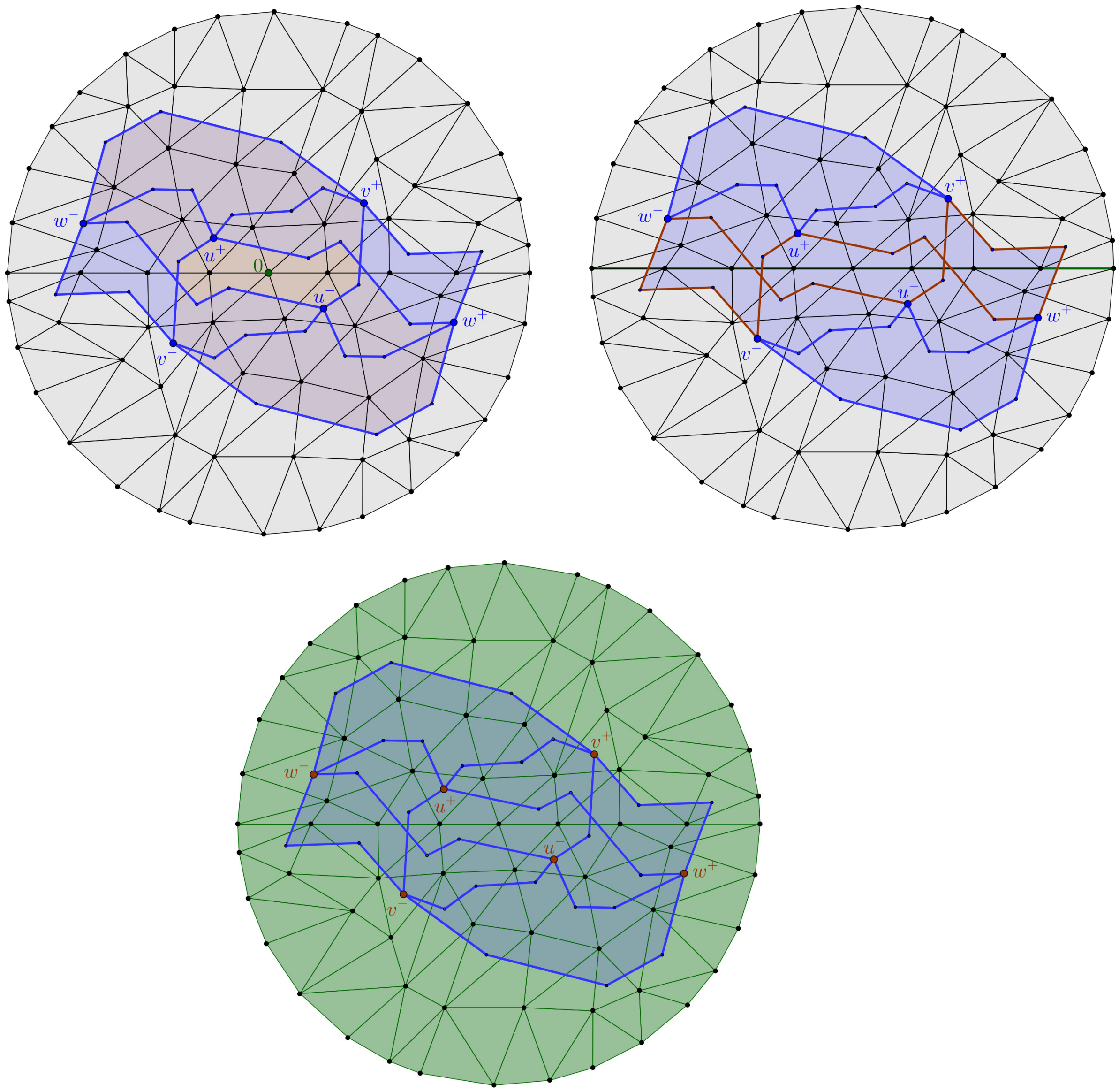


Figure 4.2: We illustrate the pagoda for an equivariant PL map $F\colon X \to_{\mathbb{Z}/2} \mathbb{B}^d$ in general position with respect to a good triangulation T where $X = \Lambda_2^2$ is an octahedron. In blue we show the image of X under F. At the top left, we have that $F^{\pitchfork}(0)$ are the two triangles $u^+v^-w^+$ and $u^-v^+w^-$ marked in red. At the top right, we depict $b^{(1)} = F^{\pitchfork}(\tau_1^-) + F^{\pitchfork}(\tau_1^+)$ in red and the chain $\tau_1^- + \tau_1^+$ in green. We see that the support of $b^{(1)}$ consists of all edges in $F(X)$ that the green line intersects an odd number of times. Finally, at the bottom we have $b^{(0)} = \mathbb{1}_{X(0)} = F^{\pitchfork}(\mathbb{1}_{T(2)})$.

Lemma 4.8. *Every pagoda for F satisfies $b^{(0)} = \mathbb{1}_{X(0)}$.*

Proof. Let $(b_*^{(d)}, a_*^{(d-1)}, \ldots, a_*^{(0)}, b_*^{(0)})$ be the pagoda constructed in the proof of Lemma 4.7. Let $(b^{(d)}, a^{(d-1)}, \ldots, a^{(0)}, b^{(0)})$ be another pagoda for F.

We will argue by induction on $0 \leq k \leq d$ that $b^{(k)} - b_*^{(k)}$ is the coboundary of a $(k-1)$-cochain of the form $s \cdot c$ for some $c \in C^{k-1}(X; \mathbb{F}_p)$ if k is even and of the form $t_l \cdot c$ with $c \in C^{k-1}(X; \mathbb{F}_p)$ if $k = 2l+1$ is odd. From this we easily conclude the lemma since $s \cdot c = 0$ for all $c \in C^{-1}(X; \mathbb{F}_p)$ and, hence, we must have $b^{(0)} - b_*^{(0)} = 0$ and $b^{(0)} = b_*^{(0)} = \mathbb{1}_{X(0)}$.

First note that $\delta(a^{(d-1)} - a_*^{(d-1)}) = b^{(d)} - b^{(d)} = 0$. Since $\tilde{H}^{d-1}(X; \mathbb{F}_p) = 0$ there is $c^{(d-2)} \in C^{d-2}(X; \mathbb{F}_p)$ with $\delta c^{(d-2)} = a^{(d-1)} - a_*^{(d-1)}$. It follows that

$$b^{(d-1)} - b_*^{(d-1)} = t_m a^{(d-1)} - t_m a_*^{(d-1)} = \delta(t_m \cdot c^{(d-2)}).$$

For the inductive step let $0 \leq k \leq d-2$. First assume that $k = 2l$ is even and that by the induction hypothesis $b^{(k+1)} - b_*^{(k+1)} = \delta(t_l \cdot c^{(k)})$. It follows that $a^{(k)} - a_*^{(k)} - t_l \cdot c^{(k)}$ is a cocycle and since $\tilde{H}^k(X; \mathbb{F}_p) = 0$ it is also a coboundary. Thus, there is $c^{(k-1)} \in C^{k-1}(X; \mathbb{F}_p)$ with $\delta c^{(k-1)} = a^{(k)} - a_*^{(k)} - t_l c^{(k)}$. We conclude

$$b^{(k)} - b_*^{(k)} = s \cdot (a^{(k)} - a_*^{(k)}) = s \cdot (\delta c^{(k-1)} + t_l c^{(k)}) = \delta(s \cdot c^{(k-1)}),$$

where we used that $s \cdot t_l = 0$.

Similarly, if $k = 2l+1$ is odd, we can assume by induction that $b^{(k+1)} - b_*^{(k+1)} = \delta(s \cdot c^{(k)})$ for some $c^{(k)} \in C^k(X; \mathbb{F}_p)$. Then $a^{(k)} - a_*^{(k)} - s \cdot c^{(k)}$ is a cocycle and again since $\tilde{H}^k(X; \mathbb{F}_p) = 0$ there is some $c^{(k-1)} \in C^{k-1}(X; \mathbb{F}_p)$ with $\delta c^{(k-1)} = a^{(k)} - a_*^{(k)} - sc^{(k)}$. We conclude

$$b^{(k)} - b_*^{(k)} = t_l \cdot (a^{(k)} - a_*^{(k)}) = t_l(\delta c^{(k-1)} + s \cdot c^{(k)}) = \delta(t_l \cdot c^{(k-1)})$$

where we used that $t_l \cdot s = 0$. This finishes the proof. □

We are ready to wrap-up the proof of Theorem 4.1 by inductively constructing a pagoda for F choosing minimal cofillings along the way. Coboundary expansion then guarantees that $F^{\pitchfork}(0)$ is large. For the details:

Proof of Theorem 4.1. We will define $a^{(k)} \in C^k(X; \mathbb{F}_p)$ inductively, $b^{(k)}$ is then determined by condition (iii) in the definition of a pagoda.

To start with, recall that $b^{(d)} = F^{\pitchfork}(0)$ is a coboundary and we choose $a^{(d-1)} \in C^{d-1}(X; \mathbb{F}_p)$ to be a minimal cofilling of $b^{(d)}$.

Let $0 \leq k \leq d-1$ and assume $a^{(k)}$ is already constructed. If $k = 2l+1$ is odd, we set $b^{(k)} = t_l \cdot a^{(k)}$ and for $k = 2l$ even, we set $b^{(k)} = s \cdot a^{(k)}$. We claim that $b^{(k)}$ is a cocycle. Indeed, if $k = d-1$ we have

$$\delta b^{(d-1)} = t_l \cdot \delta a^{(d-1)} = t_l \cdot b^{(d)} = 0,$$

since $\nu b^{(d)} = b^{(d)}$. If $k = 2l+1 < d-1$ is odd we have

$$\delta b^{(k)} = t_l \cdot \delta a^{(k)} = t_l \cdot b^{(k+1)} = t_l \cdot s \cdot a^{(k+1)} = 0,$$

since $t_l \cdot s = 0$. If $k = 2l$ is even, we have

$$\delta b^{(k)} = s \cdot \delta a^{(k)} = s \cdot b^{(k+1)} = s \cdot t_l \cdot a^{(k+1)} = 0,$$

since $s \cdot t_l = 0$. Since $\tilde{H}^k(X; \mathbb{F}_p) = 0$, $b^{(k)}$ is a coboundary and we choose $a^{(k-1)}$ to be a minimal cofilling of $b^{(k)}$.

By construction we have $\|b^{(k)}\| \geq \eta_{k-1}^{\|\cdot\|}(X; \mathbb{F}_p)\|a^{(k-1)}\|$ for all $1 \leq k \leq d$. By the $\mathbb{Z}/p$-invariance of $\|\cdot\|$ and the triangle inequality we have $\|b^{(k)}\| \leq 2\|a^{(k)}\|$ if $0 \leq k \leq d-1$ is odd and $\|b^{(k)}\| \leq p\|a^{(k)}\|$ if $0 \leq k \leq d-1$ is even. Moreover, by Lemma 4.8 we have $b^{(0)} = \mathbb{1}_{X(0)}$. Combining all these we estimate

$$\|\mathbb{1}_{X(0)}\| = \|b^{(0)}\| \leq p\|a^{(0)}\| \leq \frac{p}{\eta_0^{\|\cdot\|}(X; \mathbb{F}_p)}\|b^{(1)}\| \leq \cdots \leq \frac{2^{d/2}p^{d/2}}{\prod_{i=0}^{d-1} \eta_i^{\|\cdot\|}(X; \mathbb{F}_p)}\|b^{(d)}\|,$$

as desired. □

4.5 Remarks on the Assumptions of Theorem 4.1

Being a coboundary expander is a very strong condition. It would be interesting to weaken this assumption for the quantitative Borsuk–Ulam theorem. In fact, for Gromov's topological overlap theorem we do not need that X is a coboundary expander. It suffices for X to be a so-called *cosystolic expander*. A d-dimensional simplicial complex X is a (η, θ)-cosystolic expander if for every $0 \leq k \leq d-1$ and $\beta \in B^{k+1}(X; \mathbb{F}_2)$ there is $\alpha \in C^k(X; \mathbb{F}_2)$ with $\delta\alpha = \beta$ and $\|\alpha\| \leq \frac{1}{\eta}\|\beta\|$ and if for all $z \in Z^k(X; \mathbb{F}_2) \setminus B^k(X; \mathbb{F}_2)$ we have $\|z\| \geq \theta$.

The following example shows that cosystolic expansion is not suitable to give a quantitative Borsuk–Ulam theorem.

Example 4.9. Let G be a connected graph. Let $X = G \sqcup G$ be the disjoint union of two copies of G. We turn X into a free $\mathbb{Z}/2$-complex by considering the $\mathbb{Z}/2$-action which interchanges the two copies of G in X. Note that X is a $(\eta_0(G), 1/2)$-cosystolic expander. Indeed, every connected component of X has expansion $\eta_0(G)$ and contains half of the vertices. But the map $f\colon |X| \to_{\mathbb{Z}/2} \mathbb{R}$ which maps one copy of G in X to $+1$ and the other copy of G to -1 is an equivariant map with $0 \notin f(X)$.

On the other hand, if $G = (V, E)$ is a graph with a free $\mathbb{Z}/2$-action ν such that every subset $S \subset V$ of vertices with precisely one vertex of each ν-orbit is expanding then $|E \cap f^{-1}(0)|$ has to be large for every equivariant map $f\colon G \to_{\mathbb{Z}/2} \mathbb{R}$. This is a much weaker condition than G to be an expander graph. It would be interesting to find such weaker conditions in higher dimensions too.

Chapter 5

Applications of Equivariant Overlap Theorem

In this chapter, we harvest some applications of our quantitative Borsuk–Ulam theorem. The quantitative non-embeddability result for sufficiently spherical building already appears in [140]. Some of the other applications have been mentioned there but here we work out the details for the first time. The results in this chapter are joint work with Uli Wagner.

5.1 Expansion of Join Versus Expansion of Deleted Join

For the applications of the equivariant topological overlap theorem to quantitative non-embeddability results of simplicial complexes it will often be more convenient/easier to establish lower bounds on the coboundary expansion constants of the join X^{*2} and then relate the coboundary expansion constants of the deleted join X^{*2}_{Δ} to the ones of X^{*2}. The purpose of this section is to make such a relationship precise in a general context such that we can use it later on in a blackbox fashion.

In fact, sometimes giving lower bounds on the coboundary expansion constants of the deleted join is more than we need. If we know (by other means) that $\tilde{H}^k(X^{*2}_{\Delta};\mathbb{F}_2) = 0$ for all $0 \leq k \leq 2\dim X$, we can bound $\mathrm{ipcr}(X)$ more directly in terms of the coboundary expansion constants of the join X^{*2} and an error term (see Lemma 5.1 in Section 5.1.1). If we do not know that X^{*2}_{Δ} has vanishing cohomology, we would still expect (at least for large complexes) that X^{*2}_{Δ} and X^{*2} have comparable coboundary expansion constants. We will give one way of making this precise and quantitative in Section 5.1.2.

5.1.1 $\tilde{H}^k(X^{*2}_{\Delta};\mathbb{F}_2) = 0$ and Sum of Squared Degree Error Term

Lemma 5.1. *Let Y be a d-dimensional simplicial complex with a $\mathbb{Z}/2$-action $\nu\colon Y \to Y$. Let $Y_0 \subseteq Y$ be an invariant (i.e. $\nu(Y_0) \subseteq Y_0$) d-dimensional subcomplex such that the restriction of ν to Y_0 turns Y_0 into a free $\mathbb{Z}/2$-complex. Assume $Y_0(0) = Y(0)$ and $\tilde{H}^k(Y_0;\mathbb{F}_2) = 0$ for all $0 \leq k \leq d-1$. Let $w\colon Y \to \mathbb{R}_{\geq 0}$ be a weight function on Y with induced weighted Hamming norm $\|\cdot\|$. Let $w^*\colon Y_0 \to \mathbb{R}_{\geq 0}$ be the restriction of w to X inducing the weighted Hamming norm $\|\cdot\|_*$. Then for every equivariant map*

$F\colon |Y_0| \to_{\mathbb{Z}/2} \mathbb{R}^d$ *we have*

$$\|\{\sigma \in Y_0(d) : 0 \in F(\sigma)\}\|_* \geq \frac{\|\mathbb{1}_{Y(0)}\|}{2^d}\left(\prod_{i=0}^{d-1} \eta_i^{\|\cdot\|}(Y;\mathbb{F}_2)\right) - \sum_{k=0}^{d-1} \frac{\|\mathbb{1}_{\Delta(d-k)}\|}{2^k} \prod_{i=0}^{k-1} \eta_{d-1-i}^{\|\cdot\|}(Y;\mathbb{F}_2)$$

where $\Delta(j) := \{\sigma \in Y(j) \setminus Y_0(j) : \tau \subseteq \sigma$ *for some* $\tau \in Y_0(j-1)\}$.

Proof. As in the proof of the equivariant overlap theorem (Theorem 4.1) it suffices to give a lower bound on the norm of $b^{(d)} = F^{\pitchfork}(0) \in B^d(Y_0;\mathbb{F}_2)$ for any PL-map $F\colon |Y_0| \to_{\mathbb{Z}/2} \mathbb{R}^d$ which is in general position with respect to a sufficiently fine triangulation of a ball containing the image of $|Y_0|$ under F.

In fact, it suffices to construct a pagoda $(b^{(d)}, a^{(d-1)}, b^{(d-1)}, \ldots, a^{(0)}, b^{(0)})$ for F such that $\|b^{(k)}\|_* \geq \eta_{k-1}^{\|\cdot\|}(Y;\mathbb{F}_2)\|a^{(k-1)}\| - \|\mathbb{1}_{\Delta(k)}\|$ for all $1 \leq k \leq d$.

We can construct such a pagoda recursively as follows: Assume that $b^{(k)} \in B^k(Y_0;\mathbb{F}_2)$ has been constructed. Then $b^{(k)} = \delta c^{(k-1)}$ for some $c^{(k-1)} \in C^{k-1}(Y_0;\mathbb{F}_2)$. Let $\tilde{b}^{(k)} := \delta\bar{c}^{(k-1)} \in B^k(Y;\mathbb{F}_2)$, where $\bar{c}^{(k-1)}$ is the extension by 0 of $c^{(k-1)}$ to Y. Let $\tilde{a}^{(k-1)}$ be a minimal cofilling of $\tilde{b}^{(k)}$. In particular, $\|\tilde{b}^{(k)}\| \geq \eta_{k-1}^{\|\cdot\|}(Y;\mathbb{F}_2)\|\tilde{a}^{(k-1)}\|$. Note that every k-simplex in the support of $\tilde{b}^{(k)}$ which is not in Y_0 must have a $(k-1)$-face which is in Y_0. Thus, $\|\tilde{b}^{(k)}\| \leq \|\mathbb{1}_\Delta(k)\| + \|b^{(k)}\|_*$.

Now let $a^{(k-1)} \in C^{k-1}(Y_0;\mathbb{F}_2)$ be the restriction of $\tilde{a}^{(k-1)}$ to Y_0. Then $\delta a^{(k-1)} = b^{(k)}$ and combining the two inequalities above, we obtain

$$\|b^{(k)}\|_* \geq \eta_{k-1}^{\|\cdot\|}(Y;\mathbb{F}_2)\|a^{(k-1)}\| - \|\mathbb{1}_{\Delta(k)}\|,$$

as desired. ☐

5.1.2 Quantitative Relationship of Expansion Constants of Join and Deleted Join

The purpose of this section is to establish a simple quantitative but rather general relationship between expansion constants of the join and the deleted join. In specific settings, one can most likely obtain better bounds by carrying out estimates in a more situation-taylored way but our result here will suffice for our purpose and hopefully keeps the technicalities at a reasonable level.

Proposition 5.2. *Let X be a d-dimensional simplicial complex. Let $J = X^{*2}$ and $J_\Delta = X^{*2}_\Delta$. Let $w\colon J \to \mathbb{R}_{\geq 0}$ be a weight function. Let $|\cdot|$ be the induced weighted Hamming norm on $C^k(J;\mathbb{F}_2)$, $0 \leq k \leq 2d$. Write w_* for the restriction of w to J_Δ and $|\cdot|_*$ for the induced weighted Hamming norm on $C^k(J_\Delta;\mathbb{F}_2)$. Let $0 \leq k \leq 2d$ and assume that there is $\alpha > 0$ such that for all $\sigma \in J_\Delta(k)$ we have*

$$\sum_{\substack{\tau \in J(k+1)\setminus J_\Delta(k+1) \\ \sigma \subseteq \tau}} w(\tau) \leq \alpha w_*(\sigma).$$

Then $\eta_k^{|\cdot|_*}(J_\Delta;\mathbb{F}_2) \geq \eta_k^{|\cdot|}(J;\mathbb{F}_2) - \alpha$.

Proof. For $j \in \{k, k+1\}$, let $\Delta(j) = J(j) \setminus J_\Delta(j)$.

Let $c \in C^k(J_\Delta; \mathbb{F}_2)$ be minimal. Write $\bar{c} \in C^k(J; \mathbb{F}_2)$ for the extension by 0 of c to J, i.e. $\bar{c}(\sigma) = c(\sigma)$ for $\sigma \in J_\Delta(k)$ and $\bar{c}(\sigma) = 0$ for $\sigma \in \Delta(k)$. By Lemma 3.6 $\bar{c}$ is minimal. It follows that $|\delta\bar{c}| \geq \eta_k^{|\cdot|}(J; \mathbb{F}_2)|c|_*$.

We have $|\delta\bar{c}| = |\delta c|_* + |(\delta\bar{c})_{|\Delta(k+1)}|$. Note that for every $\tau \in \Delta(k+1)$ with $\delta\bar{c}(\tau) = 1$ there must be $\sigma \in J_\Delta(k)$ with $c(\sigma) = 1$. It follows that

$$|(\delta\bar{c})_{|\Delta(k+1)}| \leq \sum_{\sigma \in J_\Delta(k), c(\sigma)=1} \sum_{\tau \in \Delta(k+1), \sigma \subseteq \tau} w(\tau) \leq \sum_{\sigma \in J_\Delta(k), c(\sigma)=1} \alpha w_*(\sigma) = \alpha|c|_*.$$

We conclude

$$\eta_k^{|\cdot|}(J; \mathbb{F}_2)|c|_* \leq |\delta\bar{c}| \leq |\delta c|_* + \alpha|c|_*,$$

which shows $\eta_k^{|\cdot|_*}(J_\Delta; \mathbb{F}_2) \geq \eta_k^{|\cdot|}(J; \mathbb{F}_2) - \alpha$, as desired. □

We will see that for the Garland weights $w = w_G$, we can choose α in the previous lemma arbitarily small provided that X is sufficiently thick.

Definition 5.3. Let X be a d-dimensional simplicial complex. The *thickness* $\delta(X)$ *of* X is defined by

$$\min_{\sigma \in X(d-1)} |X_\sigma(0)|.$$

We say that X is δ*-thick* for some $\delta > 0$ if $\delta(X) \geq \delta$.

For the remaining part of this section, we write w for the Garland weights on a simplicial complex X and w_σ for the Garland weights on the link X_σ at $\sigma \in X$.

By definition, if X is δ-thick, then for all $\sigma \in X(d-1)$ and $v \in X_\sigma(0)$, we have

$$w_\sigma(v) = \frac{1}{|X_\sigma(0)|} \leq \frac{1}{\delta}.$$

For our estimates below, we will need such a bound for all $-1 \leq k \leq d-1$, $\sigma \in X(k)$ and $v \in X_\sigma(0)$. Fortunately, δ-thickness implies such bounds, as the following lemma shows.

Lemma 5.4. *Let X be a d-dimensional, δ-thick simplicial complex. Then*

(i) for any $\sigma \in X(k), -1 \leq k \leq d-1$ the link X_σ is a $(d - |\sigma|)$-dimensional, δ-thick simplicial complex.

(ii) for every $v \in X(0)$ we have $w(v) \leq \frac{1}{\delta}$.

(iii) for every $\sigma \in X(k)$, $-1 \leq k \leq d-1$ and $v \in X_\sigma(0)$ we have $w_\sigma(v) \leq \frac{1}{\delta}$.

Proof. For (i) we simply observe that for $\tau \in X_\sigma(d - |\sigma| - 1)$ we have

$$|(X_\sigma)_\tau(0)| = |X_{\sigma\cup\tau}(0)| \geq \delta,$$

since X is δ-thick. For (ii) we first note that since X is δ-thick

$$\delta|X(d-1)| \leq \sum_{\sigma \in X(d-1)} |X_\sigma(0)| = (d+1)|X(d)|.$$

Then for $v \in X(0)$ we compute

$$w(v) = \frac{|X_v(d-1)|}{(d+1)|X(d)|} \leq \frac{|X_v(d-1)|}{\delta |X(d-1)|} \leq \frac{1}{\delta},$$

where we used $X_v(d-1) \subseteq X(d-1)$ for the last inequality.

(iii) follows from combining (i) and (ii). □

Let us give some intuition on the condition that $w_\sigma(u) \leq \varepsilon$ for some $\varepsilon > 0$. To this end, we first note that for $\sigma \in X(k)$, $u \in X_\sigma(0)$ we have

$$w(\sigma) = \frac{1}{k+2} \sum_{v \in X_\sigma(0)} w(\sigma \sqcup v) \text{ and } w_\sigma(u) = \frac{w(\sigma \sqcup u)}{(k+2)w(\sigma)}.$$

Thus, the condition $w_\sigma(u) \leq \varepsilon$ for some $\varepsilon > 0$ is equivalent to $\frac{1}{k+2} w(\sigma \sqcup u) \leq \varepsilon w(\sigma)$. This is to say, that every $(k+1)$-simplex containing σ contributes only a small fraction to the weight of σ.

The following consequence of δ-thickness shows that working with Garland weights, we can apply Proposition 5.2 to δ-thick complexes with $\alpha = \frac{1}{\delta}(k+1)(k+2)$.

Lemma 5.5. *Let X be a d-dimensional simplicial complex which is δ-thick for some $\delta > 0$. Then for all $0 \leq k \leq 2d$ and $\tau \in X^{*2}_\Delta(k)$ we have*

$$\sum_{\sigma \in X^{*2}(k+1) \setminus X^{*2}_\Delta(k+1), \tau \subseteq \sigma} w(\sigma) \leq \frac{1}{\delta}(k+1)(k+2) w_*(\tau).$$

For the proof of Lemma 5.5 we need the following identities.

Claim 5.6. *Let X be a d-dimensional simplicial complex. For $-1 \leq i, j \leq d$ let*

$$c_{i,j} = \frac{\binom{d+1}{i+1}\binom{d+1}{j+1}}{\binom{2d+2}{i+j+2}}.$$

Then:

(i) For all $\sigma, \tau \in X$ we have for $\sigma \otimes \tau \in X^{*2}$ that

$$w(\sigma \otimes \tau) = c_{|\sigma|-1, |\tau|-1} w(\sigma) w(\tau).$$

(ii) For all $-1 \leq i, j \leq d$ we have

$$c_{i,j} = c_{j,i}.$$

(iii) For all $-1 \leq i \leq d$, $0 \leq j \leq d$ we have

$$\frac{c_{i,j}}{c_{i,j-1}} = \frac{d+1-j}{j+1} \frac{i+j+2}{2d-i-j+1}.$$

Also, if $\sigma \in X$ and $v \in X_\sigma(0)$ we have

$$w(\sigma \sqcup v) = w_\sigma(v)(|\sigma|+1).$$

The proof of this claim is a straightforward computation which we omit. We turn to the proof of Lemma 5.5.

Proof of Lemma 5.5. Let $\Delta(k) = X^{*2}(k) \setminus X^{*2}_{\Delta}(k)$. Let $\tau = \tau' \otimes \tau'' \in X^{*2}_{\Delta}(k)$ with $\tau', \tau'' \in X, \tau' \cap \tau'' = \emptyset$. It will be convenient to extend the weight function w to arbitrary subsets of $X^{*2}(0)$ and set $w(s) = 0$ for $s \subseteq X^{*2}(0)$ if $s \notin X^{*2}$. Similarly, for $u \in X(0)$ we interpret $w_\sigma(u)$ as 0 if u is not a vertex of X_σ. Write $\tau' = \{v_0, \ldots v_l\}$ (we allow $l = -1$ if $\tau' = \emptyset$) and $\tau'' = \{v_{l+1}, \ldots, v_k\}$. Using the identities in Claim 5.6 and Lemma 5.4 (iii), we compute

$$\begin{aligned}
\sum_{\sigma \in \Delta(k+1), \tau \subseteq \sigma} w(\sigma) &= \sum_{i=0}^{l} w(\tau' \otimes (\tau'' \cup v_i)) + \sum_{i=l+1}^{k} w((\tau' \cup v_i) \otimes \tau'') \\
&= \sum_{i=0}^{l} c_{l,k-l} w(\tau') w_{\tau''}(v_i) w(\tau'')(|\tau''| + 1) \\
&+ \sum_{i=l+1}^{k} c_{l+1,k-l-1} w_{\tau'}(v_i) w(\tau')(|\tau'| + 1) w(\tau'') \\
&\leq \frac{1}{\delta} \left((|\tau''| + 1) c_{l,k-l}(l+1) + (|\tau'| + 1) c_{l+1,k-l-1}(k-l) \right) w(\tau') w(\tau'') \\
&= \frac{1}{\delta} \frac{(l+1)(k-l+1) c_{l,k-l} + (l+2)(k-l) c_{l+1,k-l-1}}{c_{l,k-l-1}} w_*(\tau) \\
&= \frac{1}{\delta} \left(\frac{(l+1)(k+2)(d-k+l+1) + (k+2)(d-l)(k-l)}{2d-k+1} \right) w_*(\tau) \\
&\leq \frac{1}{\delta} \frac{k+2}{2d-k+1} \left((k+1)(d-k+l+1+(d-l) \right) w_*(\tau) \\
&= \frac{1}{\delta}(k+2)(k+1) w_*(\tau)
\end{aligned}$$

This finishes the proof. ☐

5.2 Quantitative Non-Embeddability of Spherical Buildings

We give a very brief introduction to spherical buildings. Buildings are highly symmetric (combinatorial) structures that have been extensively studied since their introduction by Jacques Tits in the 1960s. We will only need very few basic facts and refer the interested reader to the books [2], [52] or [135].

We start with the definition of a (spherical) building.

Definition 5.7 (Building)**.** A *d-dimensional (thick) building* X is a d-dimensional simplicial complex X for which there is a family $\mathcal{A}$ of subcomplexes, called *apartments*, such that

(i) X is pure and every $\sigma \in X(d-1)$ is contained in at least three d-simplices.

(ii) Any two simplices of X are contained in a common apartment $A \in \mathcal{A}$.

(iii) Any $(d-1)$-simplex in an apartment A is incident to precisely two d-simplices of A.

(iv) For any two d-simplices σ, σ' in an apartment A there is a sequence of d-simplices $\sigma_0, \dots, \sigma_n \in A$ such that $\sigma = \sigma_0, \sigma' = \sigma_n$ and $|\sigma_i \cap \sigma_{i+1}| = d$ for all $0 \le i \le n-1$.

(v) If $\sigma, \tau \in X$ are contained in apartments $A, A' \in \mathcal{A}$ then there is a simplicial isomorphism $\phi\colon A \to A'$ which fixes σ and τ pointwise.

A building is called *spherical* if every apartment is finite.

It turns out that for a given building X there is a Coxeter system (W, S) such that every apartment A is isomorphic to the Coxeter complex associated with (W, S).[1] In particular, every apartment of X has the same number of d-simplices, namely $|A(d)| = |W|$. We will denote this number by $w_d(X)$ and call it the *width of* X. Elaborating on the work of Gromov in [54] the following lower bound on the coboundary expansion constants of spherical buildings was shown in [98].

Theorem 5.8 (Expansion spherical buildings (Corollary 3.6 in [98]))**.** *Let X be a d-dimensional spherical building. Then for any $0 \le k \le d-1$ we have*

$$\eta_k(X) \ge \frac{1}{\binom{d+1}{k+2}^2 w_d(X)}.$$

It is not hard to see that the join X^{*2} of a d-dimensional spherical building X with itself is a $(2d+1)$-dimensional spherical building with width $w_{2d+1}(X^{*2}) = w_d(X)^2$. Indeed, if $\mathcal{A}$ is an apartment structure on X then the family of subcomplexes

$$\mathcal{A}^{*2} = \{A * A' \subseteq X^{*2} : A, A' \in \mathcal{A}\}$$

of X^{*2} forms an apartment structure on X^{*2}. We immediately deduce

Corollary 5.9 (Expansion join spherical buildings)**.** *Let X be a d-dimensional spherical building. Then for all $0 \le k \le 2d$ we have*

$$\eta_k(X^{*2}) \ge \frac{1}{\binom{2d+2}{k+2}^2 w_d(X)^2}.$$

We are ready to prove the following slightly refined version of Theorem 1.2 from the introduction.

Theorem 5.10 (Quantitative non-embeddability spherical buildings)**.** *Let X be a d-dimensional building such that $\delta(X) > (k+2)(k+1)\binom{2d+2}{k+2}^2 w_d(X)^2$ for all $0 \le k \le 2d$. Then*

$$\operatorname{pcr}(X) \ge \left(\frac{1}{2^{2d+1}} \prod_{k=0}^{2d} \left(\frac{1}{\binom{2d+2}{k+2}^2 w_d(X)^2} - (k+2)(k+1)\frac{1}{\delta(X)} \right) \right) \binom{|X(d)|}{2}.$$

[1]It is not important here what these are exactly. Let us just mention that a Coxeter system (W, S) is a group W with a generating set S satisfying special types of relations. The associated Coxeter complex (W, S) is a triangulation of a $(|S|-1)$-dimensional sphere if W is finite and reflects the group structure of W geometrically.

Proof. We apply the quantitative Borsuk–Ulam theorem (Theorem 4.1) to X_Δ^{*2} where we use the norm on cochains obtained by restricting the Garland weights on X^{*2} to X_Δ^{*2}. Then the result follows by plugging-in the bounds from Corollary 5.9, Lemma 5.5 and Proposition 5.2. □

We remark that there is some constant w_d such that $w_d(X) \leq w_d$ for all d-dimensional spherical buildings. Thus, if one wished, one could make the assumption on the thickness of X in the previous theorem not to depend on $w_d(X)$.

5.3 Number of Tverberg Partitions

A classical result in discrete geometry is Tverberg's theorem which says that any set of $(d+1)(r-1)+1$ points in $\mathbb{R}^d$ can be partitioned into r pairwise disjoint subsets with intersecting convex hulls. We will call any such partition a *Tverberg partition.* A topological version, which implies Tverberg's theorem, holds when r is a prime power.

Theorem 5.11 (Topological Tverberg theorem)**.** *Let p be a prime, $k \in \mathbb{Z}_{>0}$. Let $r = p^k$. Let $d \in \mathbb{Z}_{>0}$. Let $N = (d+1)(r-1)$. Then for every continuous map $f\colon |\sigma^N| \to \mathbb{R}^d$. there are r pairwise disjoint faces $F_1, \ldots, F_p \in \sigma^N$ such that $\bigcap_{i=1}^r f(F_i) \neq \emptyset$.*

Theorem 5.11 was first proven by Bárány, Shlosman and Szűcs in 1981 [11] for r prime and by Özaydin in 1987 in the unpublished manuscript [115] for the prime power case. The topological Tverberg theorem fails to hold when r is *not* a prime power. First counterexamples were constructed by Frick [48] heavily relying on the machinery introduced by Mabillard and Wagner [102, 103] and using an observation which was independently observed by Gromov in [54, p.445]. See [15] for a general survey on the topological Tverberg story.

We call faces $\{F_1, \ldots, F_p\}$ as in the conclusion of the topological Tverberg theorem a Tverberg partition.

While the Tverberg theorem guarantees the existence of at least one Tverberg partition, it is natural to ask whether there is a lower bound on the number of Tverberg partitions. A long-standing conjecture due to Sierksma [129] states that for a set of $(r-1)(d+1)+1$ points in general position in $\mathbb{R}^d$ there are at least $((r-1)!)^d$ Tverberg partitions.[2]

Using our equivariant topological overlap theorem (Theorem 4.1) we can recover the following lower bound on the number of Tverberg partititions in the topological setting due to Vućic and Živaljević (see [138, Theorem 1] or [107, Theorem 6.5.1] as well as [60, Theorem 2] for an extension to the prime power case).

Theorem 5.12. *Let p be a prime. Let $N = (d+1)(p-1)$. Let σ^N be an N-dimensional simplex. Then for every continuous map $f\colon |\sigma^N| \to \mathbb{R}^d$ the number of unordered p-tuples $\{F_1, \ldots, F_p\}$ of pairwise disjoint faces of σ^N with $\bigcap_{i=1}^p f(F_i) \neq \emptyset$ is at least*

$$\frac{1}{(p-1)!}\left(\frac{p}{2}\right)^{N/2}.$$

[2] This number is an upper bound attained by the configuration of $(d+1)$ clusters of $r-1$ points around the vertices of a d-simplex σ in $\mathbb{R}^d$ and a point at the barycenter of σ.

For the proof of Theorem 5.12 we encode Tverberg partitions into the configuration space/test map paradigm. Once we establish coboundary expansion of the configuration space, we can apply the equivariant topological overlap theorem to deduce the claimed lower bound on the number of Tverberg partitions.

A natural candidate for the configuration space is the p-fold deleted join $X = (\sigma^N)^{*p}_{\Delta(2)}$ of σ^N. X is the subcomplex of the p-fold join $(\sigma^N)^{*p}$ consisting of all simplices $\sigma_1 \otimes \sigma_2 \otimes \dots \sigma_p$ with $\sigma_1, \dots, \sigma_p \in \sigma^N$ pairwise disjoint. Note that ordered partitions $(F_1, \dots, F_p)$ of $[N+1]$ are in 1-to-1 correspondence with maximal simplices of X making X a suitable configuration space. Moreover, every continuous map $f\colon |\sigma^N| \to \mathbb{R}^d$ induces a map $f^{*p}\colon |X| \to \mathbb{R}^{p(d+1)}$ given by

$$t_1 x_1 \oplus \cdots \oplus t_p x_p \mapsto f^{*p}(t_1 x_1 \oplus \cdots \oplus t_p x_p) := (t_1, f(x_1), t_2, f(x_2), \dots, t_p, f(x_p)).$$

Here and from now on, we will think of $\mathbb{R}^{p(d+1)} = \mathbb{R}^{d+1} \oplus \dots \mathbb{R}^{d+1}$ as a direct sum of p copies of $\mathbb{R}^{d+1}$.

Now, (ordered) Tverberg partitions correspond to maximal simplices of X whose image under f^{*p} intersects the thin diagonal $\mathcal{D} := \{(x, \dots, x) \in \mathbb{R}^{p(d+1)} : x \in \mathbb{R}^{d+1}\}$. Thus, if we denote by $\mathcal{D}^\perp$ the orthogonal complement of $\mathcal{D}$ and by $\pi\colon \mathbb{R}^{p(d+1)} \to \mathcal{D}^\perp$ the orthogonal projection onto $\mathcal{D}^\perp$, we get a test map $F\colon |X| \to \mathcal{D}^\perp$, $F = \pi \circ f^{*p}$, such that maximal simplices of X containing 0 in their image are in 1-to-1 correspondence with ordered Tverberg partitions (of f).

$\mathbb{Z}/p$ acts on X and on $\mathbb{R}^{p(d+1)}$ by cyclically shifting coordinates. More precisely, if ν is a generator of $\mathbb{Z}/p$, ν acts on X by

$$t_1 x_1 \oplus \dots t_p x_p \mapsto t_2 x_2 \oplus \dots t_p x_p \oplus t_1 x_1$$

and on $\mathbb{R}^{p(d+1)}$ by

$$(x_1, \dots, x_p) \mapsto (x_2, \dots, x_p, x_1).$$

The action $\mathbb{R}^{p(d+1)}$ restricts to an action (by orthogonal linear maps) on the $(p-1)(d+1)$-dimensional space $\mathcal{D}^\perp$ which is free on $\mathcal{D}^\perp \setminus \{0\}$. The action on X is free. Moreover, $F\colon |X| \to \mathcal{D}^\perp$ is $\mathbb{Z}/p$-equivariant.

Thus, we are precisely in the setting where we could apply the equivariant overlap theorem provided X is a coboundary expander with respect to $\mathbb{F}_p$-coefficients and a $\mathbb{Z}/p$-invariant size function. This is precisely what we will establish in the remaining part of this section.

To this end, we first observe that

$$X = (\sigma^N)^{*p}_\Delta = (\{\cdot\}^{*(N+1)})^{*p}_\Delta = (\{\cdot\}^{*p}_\Delta)^{*(N+1)} = [p]^{*(N+1)}$$

is the complete $(N+1)$-partite N-dimensional complex with parts of size p. Writing X as $[p]^{*(N+1)}$ the $\mathbb{Z}/p$-action on X is given by a cyclic shift on each copy of $[p]$.

Write $X = U_0 * \cdots * U_N$ with $U_i = [p]$. Consider the weight function $w\colon X \to \mathbb{R}_{\geq 0}$ given by

$$\sigma \mapsto w(\sigma) = \begin{cases} \frac{1}{p^{|\sigma|}} & \text{if } \sigma \subseteq U_0 * \cdots * U_{\dim \sigma} \subseteq X \\ 0 & \text{otherwise.} \end{cases}$$

Write $\|\cdot\|$ for the induced weighted Hamming norm on cochain groups of X with respect to $\mathbb{F}_p$-coefficients.

With the notation above we have

Lemma 5.13. *For all $0 \le k \le N$ we have $\eta_k^{\|\cdot\|}(X;\mathbb{F}_p) \ge 1$.*

Before we prove this lemma, let us first show how it helps us to prove Theorem 5.12.

Proof of Theorem 5.12. Write $S_f(d,p)$ for the number of (unordered) Tverberg partitions of $f\colon \sigma^N \to \mathbb{R}^d$. With the notation introduced above we have that $\|\cdot\|$ is the normalized Hamming on $X(N)$. It follows that

$$S_f(d,p) \ge \frac{|X(N)|}{p!}\|\{\sigma \in X(N) : 0 \in F(\sigma)\}\| = \frac{p^N}{(p-1)!}\|\{\sigma \in X(N) : 0 \in F(\sigma)\}\|.$$

Plugging the inequality $\eta_k^{\|\cdot\|}(X;\mathbb{F}_p) \ge 1$ into the lower bound of the equivariant overlap theorem (Theorem 4.1), we get

$$\|\{\sigma \in X(N) : 0 \in F(\sigma)\}\| \ge \|\mathbb{1}_{X(0)}\|\frac{1}{2^{\lceil (N+1)/2\rceil}p^{\lfloor (N+1)/2\rfloor}} = \frac{1}{2^{\lceil (N+1)/2\rceil}p^{\lfloor (N+1)/2\rfloor}}.$$

Combining these altogether gives

$$S_f(d,p) \ge \frac{1}{(p-1)!}\left(\frac{p}{2}\right)^{\lceil N/2\rceil},$$

as desired. □

It remains to prove Lemma 5.13.

Proof of Lemma 5.13. We use a random cofilling argument. Given $0 \le k \le N$ and $\beta \in B^{k+1}(X;\mathbb{F}_p)$ and $u \in U_{k+1}$ we will construct $\alpha^{(u)} \in C^k(X;\mathbb{F}_p)$ with $\delta\alpha^{(u)} = \beta$ such that

$$\frac{1}{n}\sum_{u\in U_{k+1}} \|\alpha^{(u)}\| = \|\beta\|.$$

Since $\tilde{H}^k(X;\mathbb{F}_p) = 0$ for all $0 \le k \le N$ this would finish the proof according to Lemma 3.9.

To define $\alpha^{(u)} \in C^k(X;\mathbb{F}_p)$ note that $X_u = X_{u'} \cong [p]^{*N}$ for all $u, u' \in U_i$, $0 \le i \le N+1$. Let $u, u' \in U_i$ with $u \ne u'$. For an oriented simplex $\tau = [v_0,\dots,v_k]$ we write $\tau \setminus v_i$ for the oriented simplex $(-1)^i[v_0,\dots,\widehat{v_i},\dots,v_k]$, where $\widehat{v_i}$ indicates that the vertex v_i is omitted.

Since localizing along a cycle commutes with taking coboundaries, the localization $\beta_u - \beta_{u'}$ is a coboundary, i.e. $\beta_u - \beta_{u'} \in B^k(X_u;\mathbb{F}_p)$. Let $\alpha^{(u,u')} \in C^{k-1}(X_u;\mathbb{F}_p)$ be a cofilling of $\beta_{u'} - \beta_u$. Now, define $\alpha^{(u)} \in C^k(X;\mathbb{F}_p)$ by

$$a^{(u)}(\sigma) = \begin{cases} 0 & \text{if } u \in \sigma, \\ b_u(\sigma) & \text{if } \sigma \cap U_i = \emptyset, \\ a^{(u,u')}(\sigma \setminus u') & \text{if } \sigma \cap U_i = \{u'\}, u \ne u'. \end{cases}$$

It is straightforward to check that $\alpha^{(u)}$ is indeed a cofilling of β, i.e. $\delta\alpha^{(u)} = \beta$ (cf. Section 8.1).

To finish the proof we estimate

$$\begin{aligned}\|[\alpha^{(u)}]\| &\leq \frac{1}{p}\sum_{u\in U_{k+1}}\|\alpha^{(u)}\|\\ &= \frac{1}{p}\sum_{u\in U_{k+1}}\sum_{\sigma\in U_0*\dots*U_k}\frac{1}{p^{k+1}}|\alpha^{(u)}(\sigma)|\\ &= \frac{1}{p^{k+2}}\sum_{u\in U_{k+1}}\sum_{\sigma\in U_0*\dots*U_k}|\beta(\sigma\sqcup u)|\\ &= \|\beta\|.\end{aligned}$$

□

5.4 Pair-Crossing Number of Bounded Degree Expander Graphs

A classical result on crossing numbers (see, e.g., [116]) asserts that for any graph $G = (V, E)$

$$\mathrm{cr}(G) \geq \Omega(\mathrm{b}(G)^2) - O(\mathrm{ssqd}(G)).$$

Here $\mathrm{b}(G)$ denotes the *bisection width of* G which is the smallest number $|E(S, V \setminus S)|$ for all subsets $S \subseteq V$ with $\min\{|S|, |V \setminus S|\} \geq \frac{1}{3}|V|$ and $\mathrm{ssqd}(G) = \sum_{v\in V}\deg(v)^2$ is the sum of squared vertex degrees. Note that $\mathrm{b}(G) \geq \frac{h_0(G)}{3}|V|$. In particular, for a bounded degree expander graph $\mathrm{cr}(G) = \Omega(h_0(G)^2|V|^2)$. The usual proof starts with an optimal drawing of G in the plane and replaces every crossing with a new vertex of degree 4. Then a seperator theorem is applied to the resulting planar graph. This approach fails terribly for the pair-crossing number since there is almost no control about the total number of crossings. For the pair-crossing number the best lower bound in the literature, we were able to find, is due to Kolman and Matoušek ([79, Theorem 2]) who show that $\mathrm{pcr}(G) \geq \Omega\left(\frac{\mathrm{b}(G)^2}{\log(|V|)^2}\right) - O(\mathrm{ssqd}(G))$. Using the quantitative Borsuk–Ulam theorem, we could get rid of the factor $\log|V|^2$-factor if we could show a constant lower bound on $\eta_k(G^{*2}_\Delta)$ for all $0 \leq k \leq 2$. Unfortunately, we only know how to obtain such a bound for $k = 0$ and $k = 1$ which allows us to remove one of the $\log|V|$-factors and to show the following result.

Theorem 5.14. *Let $G = (V, E)$ be a connected graph such that $\tilde{H}^k(G^{*2}_\Delta; \mathbb{F}_2) = 0$ for $0 \leq k \leq 2$. Let Δ be the maximum vertex degree of a vertex in G. Then*

$$\mathrm{ipcr}(G) \geq \Omega\left(\frac{h_0(G)^3}{\Delta^3}\frac{|E|^2}{\log|V|}\right) - O(\mathrm{ssqd}(G)).$$

The reader might object that our lower bound on $\mathrm{ipcr}(G)$ in Theorem 5.14 requires much stronger assumptions than the bound on $\mathrm{pcr}(G)$ due to Kolman and Matoušek or the classical bound on $\mathrm{cr}(G)$. We would like to remark that assuming that G is connected is not a severe restriction. Indeed, we have the following lemma:

Lemma (Lemma 5 in [79])**.** *Let G be a graph on n vertices with bisection with* $\mathrm{b}(G)$*. Then G contains a subgraph on at least $\frac{2}{3}n$ vertices with edge expansion constant $h_0(G)$ at least $\frac{\mathrm{b}(G)}{n}$.*

The assumption that $\tilde{H}^k(G^{*2}_\Delta; \mathbb{F}_2) = 0$ for $0 \leq k \leq 2$, which we impose in order to be able to apply the equivariant topological overlap theorem as well as Lemma 5.1, seems more restrictive and harder to check. In Section 5.4.2 below we will give sufficient (geometric) conditions on G which guarantee $\tilde{H}^k(G^{*2}_\Delta; \mathbb{F}_2) = 0$ for $0 \leq k \leq 2$. In particular, G^{*2}_Δ has vanishing cohomology for sufficiently good expander graphs.

5.4.1 Proof of Theorem 5.14

The strategy of the proof of Theorem 5.14 is as follows: We first establish lower bounds on the coboundary expansion constants of G^{*2} with respect to $\mathbb{F}_2$-coefficients and a suitably weighted Hamming norm. Then, we will make use of Lemma 5.1 to get lower bounds on $\mathrm{ipcr}(G)$. Since G^{*2} contains a complete bipartite graph the bound on $\eta_0^{|\cdot|}(G^{*2}; \mathbb{F}_2)$ will be straightforward. For lower bounds $\eta_k^{|\cdot|}(G^{*2}; \mathbb{F}_2)$, $k \in \{1, 2\}$, we will use a random cofilling argument. For $k = 1$ this is fairly straightforward, for $k = 2$ we will make use of the notion of a low congestion embedding of K_n to G, which was already used in the proof of the lower bound on $\mathrm{pcr}(G)$ by Kolman and Matoušek.

Let us fix some notation. Let $X = G^{*2}$ and $n = |V|$.

It will be convenient to distinguish the two copies of G in X and write $X = G_L * G_R$ with $G_L = (V_L, E_L)$ and $G_R = (V_R, E_R)$ being two distinguished copies of $G = (V, E)$. In particular, $X(0) = V_L \sqcup V_R$.

We endow X with the weight function $w\colon X \to \mathbb{R}_{\geq 0}$ which is equal to $1/|X(k)|$ on $X(k)$ for $k \in \{0, 2, 3\}$ and given by

$$w(e) = \begin{cases} \frac{1}{n^2} & \text{if } e = x \otimes y \text{ for } x \in V_L, y \in V_R \\ 0 & \text{otherwise} \end{cases}$$

on $X(2)$. Write $\|\cdot\|$ for the induced weighted Hamming norm on $C^k(X; \mathbb{F}_2)$, i.e. $\|\cdot\|$ is the normalized Hamming norm on $C^k(X; \mathbb{F}_2)$ for $k \in \{0, 2, 3\}$ and $\|c\| = \frac{1}{n^2} \sum_{x \in V_L, y \in V_R} c(x \otimes y)$ for $c \in C^1(X; \mathbb{F}_2)$. We write $|\cdot|$ for the (unnormalized) Hamming norm.

For the rest of this section, we consider coboundary expansion of X with respect to $\mathbb{F}_2$-coefficients and size function $\|\cdot\|$.

A lower bound on $\eta_0^{\|\cdot\|}(X; \mathbb{F}_2)$

Lemma 5.15. $\eta_0^{\|\cdot\|}(X; \mathbb{F}_2) \geq 1$.

Proof. By the choice of weights $w\colon X \to \mathbb{R}_{\geq 0}$, $\eta_0^{\|\cdot\|}(X; \mathbb{F}_2)$ is simply the normalized edge expansion constant of a complete bipartite graph $K_{n,n}$. It is well-known (and a special case of Proposition 3.8) that this constant is at least 1. □

A lower bound on $\eta_1^{\|\cdot\|}(X; \mathbb{F}_2)$

Lemma 5.16. $\eta_1^{\|\cdot\|}(X; \mathbb{F}_2) \geq \frac{h_0(G)}{2\Delta}$.

Proof. For $(u, v) \in V_L \times V_R$ define $S^{(u,v)}\colon C_0(X; \mathbb{F}_2) \to C_1(X; \mathbb{F}_2)$ by

$$x \mapsto \begin{cases} u \otimes x & \text{if } x \in V_R \\ (u + x) \otimes v & \text{if} x \in V_L. \end{cases}$$

Let $T^{(u,v)}\colon C^1(X;\mathbb{F}_2) \to C^0(X;\mathbb{F}_2)$ be the dual map of $S^{(u,v)}$.

Now, given $c \in C^1(X;\mathbb{F}_2)$ and $(u,v) \in V_L \times V_R$ let $c^{(u,v)} := c + \delta T^{(u,v)} c$. By interchanging the role of G_L and G_R we can assume that $\|(\delta c)_{|V_L * E_R}\| \le \frac{1}{2}\|\delta c\|$. Given $x \in V_L, y \in V_R$ we have

$$\begin{aligned} c^{(u,v)}(x \otimes y) &= c(x \otimes y) + \langle \delta T^{(u,v)} c, x \otimes y \rangle \\ &= c(x \otimes y) + \langle c, S^{(u,v)} x + S^{(u,v)} y \rangle \\ &= c(x \otimes y) + c(u \otimes v) + c(x \otimes v) + c(u \otimes y) \\ &= \delta_{K_{V_R}}(c_x + c_u)(vy), \end{aligned}$$

which we understand as 0 if $v = y$. Thus, by averaging over $(u,v) \in V_L \times V_R$,

$$\|[c]\| \le \frac{1}{n^2} \sum_{(u,v) \in V_L \times V_R} \|c^{(u,v)}\| = \frac{4}{n^4} \sum_{uu' \in \binom{V_L}{2}} \sum_{vv' \in \binom{V_R}{2}} |\delta_{K_{V_R}}(c_u + c_{u'})(vv')|.$$

Expansion of G implies that $|\delta_{K_{V_R}} a| \le \frac{n}{h_0(G)} |\delta_{G_R} a|$ for all $a \in C^0(G_R;\mathbb{F}_2)$. Hence,

$$\|[c]\| \le \frac{4}{n^3 h_0(G)} \sum_{uu' \in \binom{V_L}{2}} \sum_{e \in E_R} |\delta_{G_R}(c_u + c_{u'})(e)|.$$

Note that $\delta_{G_R}(c_u + c_{u'})(e) = \delta c(u \otimes e) + \delta c(u' \otimes e)$. Combining this with the triangle inequality, we get

$$\|[c]\| \le \frac{4(n-1)}{n^3 h_0(G)} |X(2)| \|(\delta c)_{|V_L * E_R}\|.$$

Using the assumption $\|(\delta c)_{|V_L * E_R}\| \le \frac{1}{2}\|\delta c\|$ and that $|X(2)| = 2|V||E| \le \Delta n^2$ we conclude

$$\|[c]\| \le \frac{2\Delta}{h_0(G)} \|\delta c\|,$$

as desired. □

A lower bound on $\eta_2^{\|\cdot\|}(X;\mathbb{F}_2)$ Our lower bound on $\eta_2^{\|\cdot\|}(X;\mathbb{F}_2)$ depends on the *congestion of an embedding of K_n to G*. Given graphs G and H, an *embedding of H to G* is a pair (f,φ) where $f\colon V(H) \to V(G)$ is an injective map and φ maps edges $e = uv \in E(H)$ to a path $\varphi(e)$ connecting $f(u)$ and $f(v)$. The *congestion* $\mathrm{cong}(f,\varphi)$ *of an embedding* (f,φ) is

$$\mathrm{cong}(f,\varphi) := \max_{e \in E(G)} |\{k \in E(H) : e \in \varphi(k)\}|,$$

i.e. the maximum number of paths passing through an edge e of G. We write $\mathrm{cong}(H;G)$ for the minimum congestion of all embeddings (f,φ) of H to G. If G is a graph on n vertices, we write $\mathrm{cong}(G)$ instead of $\mathrm{cong}(K_n;G)$. We will show that

Lemma 5.17. $\eta_2^{\|\cdot\|}(X;\mathbb{F}_2) \ge \frac{|V|}{|E|}\left(\frac{1}{h_0(G)} + \left(\frac{2\Delta}{h_0(G)} + 1\right)\frac{\mathrm{cong}(G)}{|V|}\right)^{-1} = \Omega\left(\frac{h_0(G)^2}{\Delta^2 \log|V|}\right).$

For the proof of Lemma 5.17 we need the following result

Theorem 5.18 (Theorem 4 in [79])**.** *Let G be a connected graph on n vertices. Then*

$$\mathrm{cong}(G) = O(h_0(G)^{-1} n \log n).$$

Proof of Lemma 5.17. Note that an embedding of K_n to G amounts to choosing a path $\gamma^{(x,y)}$ connecting x with y for any pair of vertices $x, y \in V$. Let us fix such a collection $(\gamma^{(x,y)})_{xy \in \binom{V}{2}}$ of paths coming from a low congestion embedding of K_n to G, i.e. such that every edge of G appears in at most $O(h_0(G)^{-1} n \log n)$ of the paths.

We will also think of $\gamma^{(x,y)}$ as a 1-chain. Moreover, we extend the collection of paths by $(\gamma^{(x,x)})_{x \in V}$ which we interpret as the empty path or zero chain in $C_1(G; \mathbb{F}_2)$.

With these notations we can construct a random abstract cone parametrized by $V_L \times V_R$. For $(u, v) \in V_L \times V_R$ define $S^{(u,v)} \colon C_1(X; \mathbb{F}_2) \to C_2(X; \mathbb{F}_2)$ by

$$\{x, y\} \mapsto \begin{cases} u \otimes xy & \text{if } xy \in G_R, \\ xy \otimes v & \text{if } xy \in G_L, \\ \gamma^{(u,x)} \otimes (v + y) & \text{if } x \in V_L, y \in V_R. \end{cases}$$

Let $T^{(u,v)} \colon C^2(X; \mathbb{F}_2) \to C^1(X; \mathbb{F}_2)$ be the dual of $S^{(u,v)}$. Given $c \in C^2(X; \mathbb{F}_2)$ let $c^{(u,v)} := c + \delta T^{(u,v)} c$.

For $\tau = xy \otimes z \in E_L \otimes V_R$ we have

$$\begin{aligned} c^{(u,v)}(\tau) &= c(\tau) + \langle c, S^{(u,v)}(xy + y \otimes z + x \otimes z) \rangle \\ &= \langle c, (\gamma^{(u,x)} + \gamma^{(u,y)} + xy) \otimes (v + z) \rangle \\ &= \delta_{K_{V_R}} \left(c_{\gamma^{(u,x)} + \gamma^{(u,y)} + xy} \right) (vz). \end{aligned}$$

For $\tau = z \otimes xy \in V_L \otimes E_R$ we compute

$$\begin{aligned} c^{(u,v)}(\tau) &= c(\tau) + \langle c, S^{(u,v)}(xy + z \otimes y + z \otimes x) \rangle \\ &= c(z \otimes xy) + c(u \otimes xy) + \langle c, \gamma^{(u,z)} \otimes (v + y) \rangle + \langle c, \gamma^{(u,z)} \otimes (v + x) \rangle \\ &= \langle \delta c, \gamma^{(u,z)} \otimes xy \rangle. \end{aligned}$$

Using these, we estimate

$$\begin{aligned} \|[c]\| &\leq \frac{1}{n^2} \sum_{(u,v) \in V_L \times V_R} \|c^{(u,v)}\| \\ &= \frac{1}{n^2} \sum_{(u,v) \in V_L \times V_R} \sum_{xy \in E_L} \sum_{z \in V_R} \frac{1}{2|E||V|} |\delta_{K_{V_R}} \left(c_{\gamma^{(u,x)} + \gamma^{(u,y)} + xy} \right) (vz)| \\ &+ \frac{1}{n^2} \sum_{(u,v) \in V_L \times V_R} \sum_{z \in V_L} \sum_{xy \in E_R} \frac{1}{2|E||V|} |\langle \delta c, \gamma^{(u,z)} \otimes xy \rangle|. \end{aligned}$$

Using the triangle inequality we see that the second summand is at most

$$\frac{\mathrm{cong}(G)}{n|E||V|} |\delta c| = \frac{|E| \,\mathrm{cong}(G)}{n|V|} \|\delta c\|.$$

For the first summand we use expansion of G_R to bound it by

$$\frac{1}{n|E||V| h_0(G)} \sum_{u \in V_L} \sum_{xy \in E_L} \sum_{vv' \in E_R} |\langle \delta c, (\gamma^{(u,x)} + \gamma^{(u,y)} + xy) \otimes vv' \rangle|.$$

Using the triangle inequality this can in turn be upper bound by

$$\frac{1}{|E||V|h_0(G)}|\delta c| + \frac{1}{n|E||V|h_0(G)} \sum_{u\in V_L} \sum_{\in V_L} \deg(x) \sum_{vv'\in E_R} |\langle \delta c, \gamma^{(u,x)} \otimes vv' \rangle|$$
$$\leq \frac{|E|}{|V|h_0(G)}\|\delta c\| + \frac{2\Delta|E|\operatorname{cong}(G)}{n|V|h_0(G)}\|\delta c\|.$$

Overall we get

$$\begin{aligned}
\eta_2^{\|\cdot\|}(X;\mathbb{F}_2) &\geq \frac{|V|}{|E|}\left(\frac{1}{h_0(G)} + \left(\frac{2\Delta}{h_0(G)}+1\right)\frac{\operatorname{cong}(G)}{|V|}\right)^{-1} \\
&= \Omega\left(\frac{2}{\Delta}\left(\frac{1}{h_0(G)} + \left(\frac{2\Delta}{h_0(G)}+1\right)\frac{\log|V|}{h_0(G)}\right)^{-1}\right) \\
&= \Omega\left(\frac{h_0(G)^2}{\Delta^2\log|V|}\right),
\end{aligned}$$

where we used that $|E| \leq \frac{\Delta|V|}{2}$ and $\operatorname{cong}(G) = O(h_0(G)^{-1} n \log n)$ for the second step. □

Putting everything together We would like to apply Lemma 5.1 with $Y = G^{*2}$ and $Y_0 = G^{*2}_\Delta$ with the weighted Hamming norm $\|\cdot\|$ as defined above. Plugging in the lower bounds on $\eta_k^{\|\cdot\|}(X;\mathbb{F}_2)$ from Lemma 5.15, Lemma 5.16 and Lemma 5.17 to the bound in Lemma 5.1, we see that the following lemma would finish the proof of Theorem 5.14.

Lemma 5.19. *For every* $1 \leq k \leq 3$ *we have*

$$|E|^2\|\mathbb{1}_{\Delta(k)}\| = O(\operatorname{ssqd}(G)).$$

where $\Delta(k) = \{\sigma \in X(k) \setminus G^{*2}_\Delta(k) : \tau \subseteq \sigma$ *for some* $\tau \in G^{*2}_\Delta(k-1)\}$.

Proof. We first note that by Cauchy–Schwarz inequality

$$|E|^2 = \left(\frac{1}{2}\sum_{v\in V}\deg(v)\right)^2 \leq \frac{1}{4}|V|\operatorname{ssqd}(G).$$

For $k = 1$ we observe that there are $|V|$ edges in X which are not in G^{*2}_Δ. It follows

$$|E|^2\|\mathbb{1}_{\Delta(1)}\| = \frac{|E|^2}{|V|} = O(\operatorname{ssqd}(G)),$$

where we used that $|E|^2 = O(|V|\operatorname{ssqd}(G))$, as shown above.

For $k = 2$ note that triangles in $\Delta(2)$ are of the form $e \otimes x$ or $x \otimes e$ for some edge $e \in E$ and $x \in e$. It follows that

$$|E|^2\|\mathbb{1}_{\Delta(1)}\| = \frac{4|E|^3}{2|E||V|} = \frac{2|E|^2}{|V|} = O(\operatorname{ssqd}(G)),$$

where we again used that $|E|^2 = O(|V|\operatorname{ssqd}(G))$.

Finally for $k = 3$, we note that every $\sigma \in \Delta(3)$ is of the form $\sigma = e \otimes e'$ for some $e, e' \in E$ with $|e \cap e'| = 1$. It follows that

$$|E|^2\|\mathbb{1}_{\Delta(3)}\| = |\mathbb{1}_{\Delta(3)}| \leq \sum_{v\in V}\deg(v)(\deg(v)-1) = O(\operatorname{ssqd}(G)),$$

which finishes the proof of the lemma and, hence, the proof of Theorem 5.14. □

5.4.2 Sufficient Conditions for $\tilde{H}^k(G^{*2}_\Delta;\mathbb{F}_2) = 0$

Let $G = (V, E)$ be a connected graph. We will give sufficient conditions for $\tilde{H}^k(G^{*2}_\Delta;\mathbb{F}_2) = 0$ for all $0 \le k \le 2$. For $k = 0$ we have the following lemma whose proof is straightforward and we omit.

Lemma 5.20. *Let G be a connected graph. Then $\tilde{H}^0(G^{*2}_\Delta;\mathbb{F}_2) = 0$ if and only if G is not a single vertex.*

For $k = 1$ and $k = 2$ the following simple but general observation will be useful.

Lemma 5.21. *Let $X \subseteq Y$ be simplicial complexes with inclusion map $i\colon X \to Y$. Let $\mathbb{A}$ be an abelian group. If $\tilde{H}^k(Y;\mathbb{A}) = 0$ and every k-cocycle in X can be extended to a k-cocycle in Y (i.e. for all $z \in Z^k(X;\mathbb{A})$ there is $\tilde{z} \in Z^k(Y;\mathbb{A})$ with $i^*\tilde{z} = z$) then $\tilde{H}^k(X;\mathbb{A}) = 0$.*

Proof. Let $z \in Z^k(X;\mathbb{A})$. By assumption there is $\tilde{z} \in C^k(Y;\mathbb{A})$ with $i^*\tilde{z} = z$. $\tilde{H}^k(Y;\mathbb{A}) = 0$ implies that $\tilde{z} = \delta\tilde{a}$ for some $\tilde{a} \in C^{k-1}(Y;\mathbb{A})$. Let $a := i^*\tilde{a} \in C^{k-1}(X;\mathbb{A})$. Then $\delta a = \delta i^*\tilde{a} = i^*\delta\tilde{a} = i^*\tilde{z} = z$. This shows that $z \in B^k(X;\mathbb{A})$, hence $Z^k(X;\mathbb{A}) = B^k(X;\mathbb{A})$ and $\tilde{H}^k(X;\mathbb{A}) = 0$, as desired. □

The following notation will be convenient: Write $G^{*2} = G_L * G_R$ with $G_L = (V_L, E_L)$ and $G_R = (V_R, E_R)$ being two distinguished copies of G. Write $\nu\colon G^{*2} \to G^{*2}$ for the $\mathbb{Z}/2$-action on G^{*2}. Given $\sigma \in G^{*2}$ write $\bar{\sigma}$ for the image of σ under ν. For $x, y \in V$ write $x \sim y$ if $xy \in E$ and $N_G(x) = \{y \in V : y \sim x\}$ for the set of neighbours of x in G.

Lemma 5.22. *Let $G = (V, E)$ be a connected graph with at least two vertices. If there is $v_0 \in V$ with $\deg(v_0) \ge 3$, then $\tilde{H}^1(G^{*2}_\Delta;\mathbb{F}_2) = 0$.*

Proof. According to Lemma 5.21 it suffices to show that every $z \in Z^1(G^{*2}_\Delta;\mathbb{F}_2)$ can be extended to a cocycle $\tilde{z}$ in G^{*2}. So, given $x \in V_L$, we would like to define $\tilde{z}(x \otimes \bar{x})$ such that $\delta\tilde{z}(\tau) = 0$ for all $\tau \in G^{*2}(2) \setminus G^{*2}_\Delta(2)$.

Given $x \in V_L$ let $u \sim x$ be a neighbour of x and define $\tilde{z}(x \otimes \bar{x}) := z(xu) + z(u \otimes \bar{x})$. Note that it suffices to check that $\tilde{z}$ is independent of the choice of neighbour u of x and that $z(ux) + z(x \otimes \bar{v}) + z(\bar{v}\bar{x}) + z(u \otimes \bar{x}) = 0$ for all $u \sim x, v \sim x$, $x \in V_L$. This amounts to show that $\langle z, a\rangle = 0$ for all cycles $a \in Z_k(G^{*2}_\Delta;\mathbb{F}_2)$ of the form $a = ux + xv + v \otimes \bar{x} + v \otimes \bar{x}$ and $a = ux + x \otimes \bar{v} + \bar{v}\bar{x} + u \otimes \bar{x}$ for any $ux, vx \in E_L$. For this, it suffices to show that each such cycle is a boundary in G^{*2}_Δ.

It is probably more instructive to look at Figure 5.1 instead of trying to digest the formulas that follow for the sake of completness.

First assume that a is of the form $a = ux + xv + v \otimes \bar{x} + u \otimes \bar{x}$. If $\deg(x) \ge 3$ we can pick $w \sim x$ with $w \ne u, w \ne v$. Then $c := ux \otimes \bar{w} + xv \otimes \bar{w} + v \otimes \bar{x}\bar{w} + u \otimes \bar{x}\bar{w} \in C_2(G^{*2}_\Delta;\mathbb{F}_2)$ is a filling of a. If $\deg(x) = 2$, let $\gamma = (y_0, y_1, \ldots y_l, y_{l+1})$ a sequence of vertices such that $(y_0, \ldots, y_l)$ is a path in G from v to v_0 with $\deg(v_0) \ge 3$, $y_{l+1} \sim v_0, y_{l+1} \ne y_{l-1}$. By interchanging the roles of u and v, if necessary, we can assume that $x \notin \{y_1, \ldots, y_l\}$. We can argue by induction on l. By adding $\partial(xu \otimes \bar{v} + u \otimes \bar{x}\bar{v} + y_1 v \otimes \bar{x} + y_1 \otimes \bar{v}x)$ to a we reduce to a path of length $l - 1$ if $l > 0$ and to the previous situation where $\deg x \ge 3$ if $l = 0$.

To see that $a = ux + x \otimes \bar{v} + \bar{v}\bar{x} + u \otimes \bar{x}$ is a boundary in G^{*2}_{Δ} for any $ux, vx \in E_L$, we distinguish two cases: If $u \neq v$ we have $\partial_X(ux \otimes \bar{v} + u \otimes \bar{v}\bar{x}) = z$. If $u = v$, at least one of x and u must have another neighbour. We can assume that $w \sim x, x \neq u$. By adding $\partial(wx \otimes \bar{u} + w * \bar{x}\bar{u})$ to a we reduce to a situation we already dealt with. □

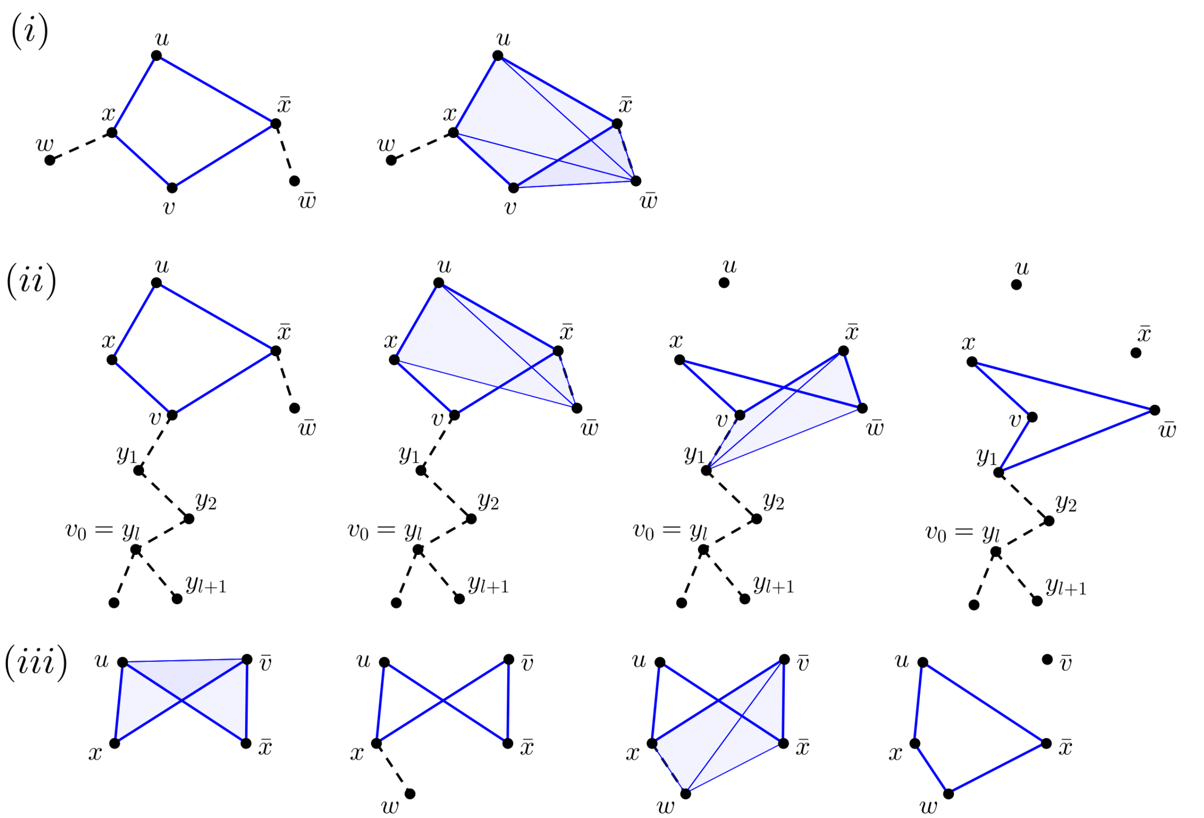


Figure 5.1: A cycle $vx + v \otimes \bar{x} + u \otimes \bar{x} + ux$ as in (i) is a boundary in the deleted join if x has a third neighbour w. A filling is obtained by coning over $\bar{w}$. To see that in (ii) the cycle $vx + v\bar{x} + \bar{x}u + ux$ is a boundary in the deleted join, we add the boundary of $u \otimes \bar{x}\bar{v} + ux \otimes \bar{v}$ leaving us to find a filling of the cycle $xv + v \otimes \bar{x} + \bar{x}\bar{v} + x \otimes \bar{v}$. By adding the boundary of $vy_1 \otimes \bar{x} + y_1 \otimes \bar{x}\bar{v}$ we then reduce to the cycle $xv + vy_1 + y_1 \otimes \bar{v} + x \otimes \bar{v}$. By going along a path from v to a vertex v_0 with degree at least 3 we reduce end up in a situation as in (i). In (iii) we easily fill the cycle $ux + x\bar{v} + \bar{v}\bar{x} + \bar{x}u$ by $ux \otimes \bar{v} + u \otimes \bar{v}\bar{x}$ if $u \neq v$ or reduce to a situation as in (ii) by adding a boundary $wx \otimes \bar{u} + w * \bar{x}\bar{u}$.

Let us remark that it is not difficult to show that if G is a conneceted graph which is not a single vertex then $\tilde{H}^1(G^{*2}_{\Delta}; \mathbb{F}_2) = 0$ implies that G has a vertex of degree at least 3, i.e. that the converse of Lemma 5.22 is true in this case.

Similar to the case $k = 1$ and under the mild assumption that G has minimum vertex degree at least 3, we can deduce $\tilde{H}^2(G^{*2}_{\Delta}; \mathbb{F}_2) = 0$ by showing that a certain (small) set of (short) cycles in G^{*2}_{Δ} are boundaries. We have:

Lemma 5.23. *Let $G = (V, E)$ be a connected graph with minimum vertex degree at least 3. Then $\tilde{H}^2(G^{*2}_{\Delta}; \mathbb{F}_2) = 0$ if and only if the following two types of 2-cycles are boundaries in G^{*2}_{Δ}:*

(i) $\partial_{G^{*2}}((ax + xb) * (\bar{c}\bar{x} + \bar{x}\bar{d}))$ for every $x \in V$ and pairwise distinct $a, b, c, d \in N_G(x)$ and

(ii) $\partial_{G^{*2}}(ay*(\bar{y}\bar{c}+\bar{y}\bar{x})+yx*(\bar{c}\bar{y}+\bar{y}\bar{x}+\bar{x}\bar{d})+bx*(\bar{y}\bar{x}+\bar{x}\bar{d}))$ *for all* $xy \in E$, $a, c \in N_G(y)\setminus\{x\}$, $a \neq c$, $b, d \in N_G(x) \setminus \{y\}$, $b \neq d$.

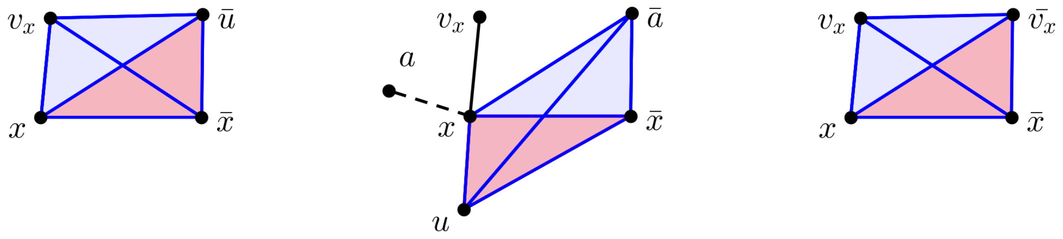


Figure 5.2: Assume that c can be extended to a cocycle $\tilde{c}$ in G^{*2}. By adding a coboundary of some edges of the form $x\bar{x}$ we can assume that $\tilde{c}(v_x x * \bar{x}) = 0$ for some fixed neighbour v_x of x. This determines the value of $\tilde{c}(x \otimes \bar{x}\bar{u})$ if $u \neq v_x$. But this forces the value of $\tilde{c}(ux \otimes \bar{x})$ for a neighbour u of x, $u \neq v_x$. Finally, this also determines $\tilde{c}(x \otimes \bar{x}\bar{v}_x)$.

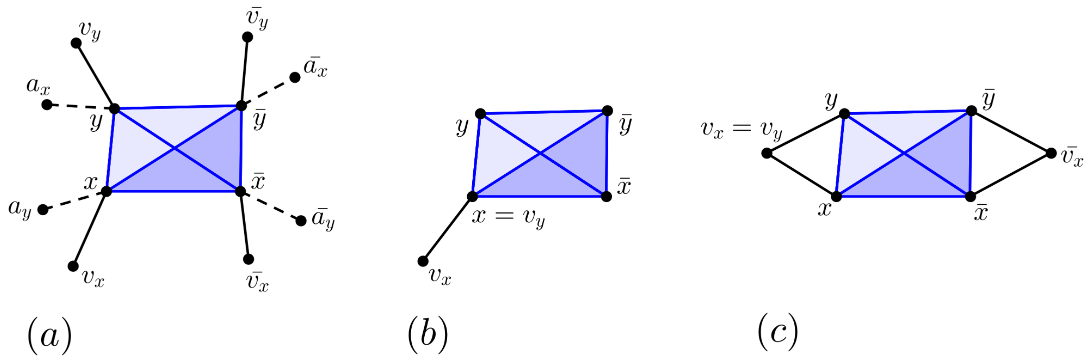


Figure 5.3: Three cases to be distinguished.

Proof of Lemma 5.23. If $\tilde{H}^2(G^{*2}_\Delta; \mathbb{F}_2) = 0$ then every 2-cycle is a boundary in G^{*2}_Δ. For the converse direction, we show that if all the 2-cycles as in (i) and (ii) are boundaries in G^{*2}_Δ, then every $c \in Z^2(G^{*2}_\Delta; \mathbb{F}_2)$ can be extended to a cocycle in G^{*2} (which will finish the proof by Lemma 5.21).

To this end, let $c \in Z^2(G^{*2}_\Delta; \mathbb{F}_2)$ and for every $x \in V$ let $v_x \in N_G(x)$ be a fixed neighbour of x in G. We can assume that if $xy \in E$ then $x \neq v_y$ or $y \neq v_x$. Define $\tilde{c} \in C^2(G^{*2}; \mathbb{F}_2)$ with $\mathrm{supp}(\tilde{c}) \subseteq G^{*2}(2) \setminus G^{*2}_\Delta(2)$ as follows (see also Figure 5.2):

(i) For every $x \in V$ define $\tilde{c}(xv_x * \bar{x}) = 0$

(ii) For $u \in N_G(x), u \neq v_x$ define $\tilde{c}(x \otimes \bar{x}\bar{u}) = c(xv_x \otimes \bar{u}) + c(v_x \otimes \bar{u}\bar{x})$.

(iii) For $u \in N_G(x), u \neq v_x$, pick $a \in N_G(x)$, $a \notin \{v_x, u\}$ and define

$$\tilde{c}(ux \otimes \bar{x}) = \tilde{c}(x \otimes \bar{x}\bar{a}) + c(xu \otimes \bar{a}) + c(u \otimes \bar{x}\bar{a}).$$

(iv) For $x \in V$ define $\tilde{c}(x \otimes \bar{x}\bar{v}_x) = \tilde{c}(v_x x \otimes \bar{v}_x) + \tilde{c}(xv_x \otimes \bar{x}) + \tilde{c}(v_x \otimes \bar{v}_x\bar{x})$.

It remains to check that $\tilde{c}$ is well-defined, i.e. that in (iii) $\tilde{c}(ux \otimes \bar{x})$ does not depend on the choice of a and that for every $xy \in E$ with $y \neq v_x$ we have

$$\tilde{c}(x \otimes \bar{x}\bar{y}) + \tilde{c}(y \otimes \bar{x}\bar{y}) + \tilde{c}(xy \otimes \bar{x}) + \tilde{c}(xy \otimes \bar{y}) = 0.$$

For any other $\sigma \in G^{*2}(3) \setminus G^{*2}_{\Delta}(3)$ we have $\delta_{G^{*2}}\tilde{c}(\sigma) = 0$ by construction.

To see that in (iii) the value of $\tilde{c}(ux \otimes \bar{x})$ does not depend on the choice of a let $x \in V$ and $u, v_x, a, a' \in N_K(x)$ pairwise distinct neighbours. We compute

$$\begin{aligned}
\tilde{c}(x \otimes \bar{x}\bar{a}) + c(xu \otimes \bar{a}) + c(u \otimes \bar{x}\bar{a}) + \tilde{c}(x \otimes \bar{x}\bar{a}') + c(xu \otimes \bar{a}') + c(u \otimes \bar{x}\bar{a}')\\
= c(xv_x \otimes \bar{a}) + c(v_x \otimes \bar{a}\bar{x}) + c(xu \otimes \bar{a}) + c(u \otimes \bar{x}\bar{a})\\
+ c(xv_x \otimes \bar{a}') + c(v_x \otimes \bar{a}'\bar{x}) + c(xu \otimes \bar{a}') + c(u \otimes \bar{x}\bar{a}')\\
= \langle c, \partial_{G^{*2}}(ux + v_x x) \otimes (\bar{a}\bar{x} + \bar{a}'\bar{x})\rangle.
\end{aligned}$$

But by assumption $\partial_{G^{*2}}(ux + v_x x) \otimes (\bar{a}\bar{x} + \bar{a}'\bar{x}) = \partial_{G^{*2}_{\Delta}} b$ for some $b \in C^3(G^{*2}_{\Delta}; \mathbb{F}_2)$. But then

$$\langle c, \partial_{G^{*2}_{\Delta}} b\rangle = \langle \delta_X c, b\rangle = 0,$$

as desired.

It remains to check that for every $xy \in E$ with $y \neq v_x$ we have

$$\tilde{c}(x\bar{x} \otimes \bar{y}) + \tilde{c}(y\bar{x} \otimes \bar{y}) + \tilde{c}(xy \otimes \bar{x}) + \tilde{c}(xy \otimes \bar{y}) = 0.$$

To this end, fix $xy \in E$ with $y \neq v_x$. We distinguish three cases as depicted in Figure 5.3. In (a) we assume that $v_y \notin \{x, v_x\}$. Let $a_x \in N_G(y) \setminus \{v_y, x\}, a_y \in N_G(x) \setminus \{v_x, y\}$. We compute

$$\begin{aligned}
&\tilde{c}(x\bar{x} \otimes \bar{y}) + \tilde{c}(y \otimes \bar{x}\bar{y}) + \tilde{c}(xy \otimes \bar{x}) + \tilde{c}(xy \otimes \bar{y})\\
&= c(xv_x \otimes \bar{y}) + c(v_x \otimes \bar{x}\bar{y}) + c(v_y y \otimes \bar{x}) + c(v_y \otimes \bar{x}\bar{y})\\
&+ \tilde{c}(x \otimes \bar{x}\bar{a}_y) + c(xy \otimes \bar{a}_y) + c(y \otimes \bar{x}\bar{a}_y)\\
&+ \tilde{c}(y \otimes \bar{y}\bar{a}_x) + c(xy \otimes \bar{a}_x) + c(x \otimes \bar{y}\bar{a}_x)\\
&= c(xv_x \otimes \bar{y}) + c(v_x \otimes \bar{x}\bar{y}) + c(v_y y \otimes \bar{x}) + c(v_y \otimes \bar{x}\bar{y})\\
&+ c(xv_x \otimes \bar{a}_y) + c(v_x \otimes \bar{x}\bar{a}_y) + c(xy \otimes \bar{a}_y) + c(y \otimes \bar{x}\bar{a}_y)\\
&+ c(yv_y \otimes \bar{a}_x) + c(v_y \otimes \bar{y}\bar{a}_x) + c(xy \otimes \bar{a}_x) + c(x\bar{y}\bar{a}_x)\\
&= \langle c, \partial_{G^{*2}}(v_y y \otimes (\bar{y}\bar{a}_x + \bar{y}\bar{x}) + xy \otimes (\bar{y}\bar{a}_x + \bar{x}\bar{y} + \bar{a}_y\bar{x}) + v_x x \otimes (\bar{x}\bar{y} + \bar{x}\bar{a}_y)\rangle\\
&= 0,
\end{aligned}$$

since $\partial_{G^{*2}}\left(v_y y \otimes (\bar{y}\bar{a}_x + \bar{y}\bar{x}) + xy \otimes (\bar{y}\bar{a}_x + \bar{x}\bar{y} + \bar{a}_y\bar{x}) + v_x x \otimes (\bar{x}\bar{y} + \bar{x}\bar{a}_y)\right)$ is a boundary in G^{*2}_{Δ} by assumption.

In (b) we have $x = v_y$ and thus $\tilde{c}(x\bar{x} \otimes \bar{y}) + \tilde{c}(y \otimes \bar{x}\bar{y}) + \tilde{c}(xy \otimes \bar{x}) + \tilde{c}(xy \otimes \bar{y}) = 0$ by construction.

For (c) we see that eventhough $v_x = v_y$ the same computation as in (a) goes through. $\square$

Under the following conditions on G the assumption of the previous lemma hold:

Lemma 5.24. *Let $G = (V, E)$ be a connected graph.*

(i) *Let $x \in V$, $a, b, c, d \in N_G(x)$ pairwise distinct. Assume there are vertex disjoint paths γ_{ab} connecting a with b and γ_{cd} connecting c with d which do not pass through x. Then $\partial_{G^{*2}}((ax + xb) * (\bar{c}\bar{x} + \bar{x}\bar{d}))$ is a boundary in G^{*2}_{Δ}.*

(ii) *Let $xy \in E$, $a \neq c \in N_G(y) \setminus \{x\}$, $b \neq d \in N_K(x) \setminus \{y\}$. Assume there are vertex disjoint paths γ_{ab} connecting a with b and γ_{cd} connecting c with d which do not pass through x and y, then $\partial_{G^{*2}}(ay * (\bar{y}\bar{c} + \bar{y}\bar{x} + \bar{y}\bar{x}) + yx * (\bar{c}\bar{y} + \bar{x}\bar{d}) + bx * (\bar{y}\bar{x} + \bar{x}\bar{d}))$ is a boundary in G^{*2}_{Δ}.*

Proof of Lemma 5.24. For (i) let $\alpha = (\gamma_{ab} + ax + xb) * (\bar{\gamma_{cd}} + \bar{c}\bar{x} + \bar{x}\bar{d})$. Notice that α is the join of two cycles and, hence, $\partial_{G^{*2}}\alpha = 0$. We deduce that

$$\partial_{G^{*2}}((ax + xb) * (\bar{c}\bar{x} + \bar{x}\bar{d})) = \partial_{G^{*2}}(\gamma_{ab} * (\bar{c}\bar{x} + \bar{x}\bar{d} + \bar{\gamma_{cd}}) + (ax + xb) * \bar{\gamma_{cd}}).$$

Since γ_{ab} and γ_{cd} are vertex disjoint and do not pass through x, we have $\gamma_{ab} * (\bar{c}\bar{x} + \bar{x}\bar{d} + \bar{\gamma_{cd}}) + (ax + xb) * \bar{\gamma_{cd}} \in C_3(G^{*2}_{\Delta}; \mathbb{F}_2)$, showing that $(ax + xb) * (\bar{c}\bar{x} + \bar{x}\bar{d}) \in B_2(G^{*2}_{\Delta}; \mathbb{F}_2)$, as desired.

The argument for (ii) is similar and omitted. □

The conditions on G in Lemma 5.24 are related to 2-linkedness of a graph:

Definition 5.25. Let $k > 0$ be a positive integer. A graph G is *k-linked* if G has at least $2k$ vertices and for every sequence $(s_1, \ldots, s_k, t_1, \ldots, t_k)$ of $2k$ pairwise distinct vertices there are k vertex-disjoint paths $\gamma_1, \ldots, \gamma_k$ such that γ_i connects s_i with t_i.

In this language, we see that if G is a graph such that $G \setminus e$ is 2-linked for every edge e then G satisfies the conditions of Lemma 5.24. 2-linkedness is related to vertex connectivity. Recall that a graph G is *k-vertex-connected* if it has more than k vertices and remains connected after removing fewer than k vertices. Jung shows in [69]

Theorem 5.26 (Satz 2 in [69])**.** *Let $G = (V, E)$ be a 4-vertex-connected (in the sense of vertex-connectivity) graph. Then G is 2-linked if and only if G is non-planar or maximal planar (i.e. if $3|V| - 6 = |E|$). In particular, if G is 6-connected then G is 2-linked.*

As a consequence of Jung's result it is not difficult to show that:

Corollary 5.27. *Let G be a 6-vertex-connected, d-regular graph on n vertices. Then $\tilde{H}^2(G^{*2}_{\Delta}) = 0$.*

We close this section with the remark that vertex connectivity can be related to expansion properties of graphs. Here is a fairly old result in this direction due to Fiedler:

Theorem (4.1 in [44])**.** *Let G be a graph which is not a complete graph. Let $\lambda_2(G)$ the second smallest eigenvalue of its Laplacian. If $\lambda_2(G) \geq k$ for some $k \in \mathbb{Z}_{>0}$ then G is k-connected.*

5.4.3 A different approach without assuming $\tilde{H}^k(G^{*2}_\Delta;\mathbb{F}_2)=0$

We briefly outline a different approach to relate coboundary expansion of G^{*2}_Δ to coboundary expansion of G^{*2} without a priori assuming that $\tilde{H}^k(G^{*2}_\Delta;\mathbb{F}_2)=0$ for all $k\in\{0,1,2\}$. We assume that we work with a weighted Hamming norm $\|\cdot\|$ on G^{*2}_Δ where the weights are obtained by restricting the Garland weights or normalized Hamming norm on G^{*2} to G^{*2}_Δ. Write δ_G for the minimum vertex degree of G. Then for $k=0$ and $k=1$, we can use Proposition 5.2 to get $\eta_k^{\|\cdot\|}(G^{*2}_\Delta;\mathbb{F}_2)\geq \eta_k^{\|\cdot\|}(G^{*2};\mathbb{F}_2)-\frac{C}{\delta_G}$ where $C>0$ is an absolute constant $C>0$.

For $k=2$ we can use the local-to-global criterion of Evra–Kaufman (see [40, Theorem 5]) or an ad-hoc argument to show expansion for small cochains in G^{*2} provided that G is a sufficiently good expander. That is to say, if $\eta_0(G)$ is sufficiently large, then there are constants $\mu,\eta>0$ (solely depending on the expansion of G) such that $\|\delta c\|\geq\eta\|[c]\|$ for all cochains $c\in C^2(G^{*2};\mathbb{F}_2)$ with $\|[c]\|\leq\mu$. Let $c\in C^2(G^{*2}_\Delta;\mathbb{F}_2)$ be minimal. As before write $\bar{c}\in C^2(G^{*2};\mathbb{F}_2)$ for the extension by 0 of c to G^{*2}. We distinguish two cases. If $\|c\|\leq\mu$, we get

$$\|\delta c\|\geq\left(\eta-\frac{C}{\delta_G}\right)\|c\|$$

for some constant $C>0$ and for $\|c\|\geq\mu$ we get

$$\|\delta c\|\geq\left(\frac{C'}{\log|V|}-\frac{\operatorname{ssqd}(G)}{\mu|E|^2}\right)\|c\|.$$

Unfortunately, the proofs using local-to-global arguments for expansion of small 2-cochains in G^{*2}_Δ require quite strong expansion of G and result in very small constants μ and η. To be a bit more precise, let us state one of the results, we know how to prove.

Proposition. *Let $G=(V,E)$ be a connected d-regular graph. Let*

$$\phi(G)=\min_{\emptyset\neq S\subsetneq V}\frac{|E(S,V\setminus S)|}{\frac{d}{|V|}|S||V\setminus S|}.^{3}$$

Let $\delta\in(0,1)$ and assume that $\phi(G)\geq 1-\delta$. Let $\varepsilon,\tilde{\varepsilon}\in(0,1)$ and $0\leq\mu\leq 1/2$. Assume that $c\in C^2(G^{*2};\mathbb{F}_2)$ is minimal with $\|c\|\leq\mu$. Then

$$|\delta c|\geq d|c|\left(\frac{3}{5}(1-\delta)-1+\left(\frac{\varepsilon}{2}(1-\delta\frac{1}{5}-\left(\frac{\delta}{2(1-\varepsilon)}+\frac{3\varepsilon\tilde{\varepsilon}}{(1-\varepsilon)^2}\right)\right)\left(2-\left(\delta+\frac{3\mu}{\tilde{\varepsilon}^2}\right)\right)\right).$$

In particular, for $\delta=1/1000$, $\varepsilon=9/10$, $\tilde{\varepsilon}=1/54000$ and $\mu=\frac{1}{972000000000}$ we get

$$|\delta c|\geq\frac{152209}{2000000}d|c|.$$

In particular, in order to get a positive lower bound on $\eta_2^{\|\cdot\|}(G^{*2}_\Delta;\mathbb{F}_2)$, we would need $\phi(G)$ quite close to 1 (and δ_G and $|E|$ to be sufficiently large).

[3] Note that for a d-regular graph G, $\eta_0(G)\geq\phi(G)$.

5.5 Crossing Number of $K_{m,n}$

As mentioned in the introduction, an initial motivation to prove Theorem 4.1 was to have a (new) tool to attack various old conjectures regarding crossing numbers of various families of graphs. Recall *Zarankiewicz' conjecture* claims that the crossing number $\mathrm{cr}(K_{m,n})$ of the complete bipartite graph $K_{m,n}$ (when we write $K_{m,n}$, we assume that $m \leq n$) is

$$\mathrm{cr}(K_{m,n}) = Z_{m,n},$$

where for $m, n \geq 1$,

$$Z_{m,n} := \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \sim \frac{m^2 n^2}{16}.$$

A classical construction due to Zarankiewicz (see Figure 5.4) shows that $\mathrm{cr}(K_{m,n}) \leq Z_{m,n}$. But even the asymptotics of $\mathrm{cr}(K_{m,n})$ as $m, n \to +\infty$ remains not fully understood. Note

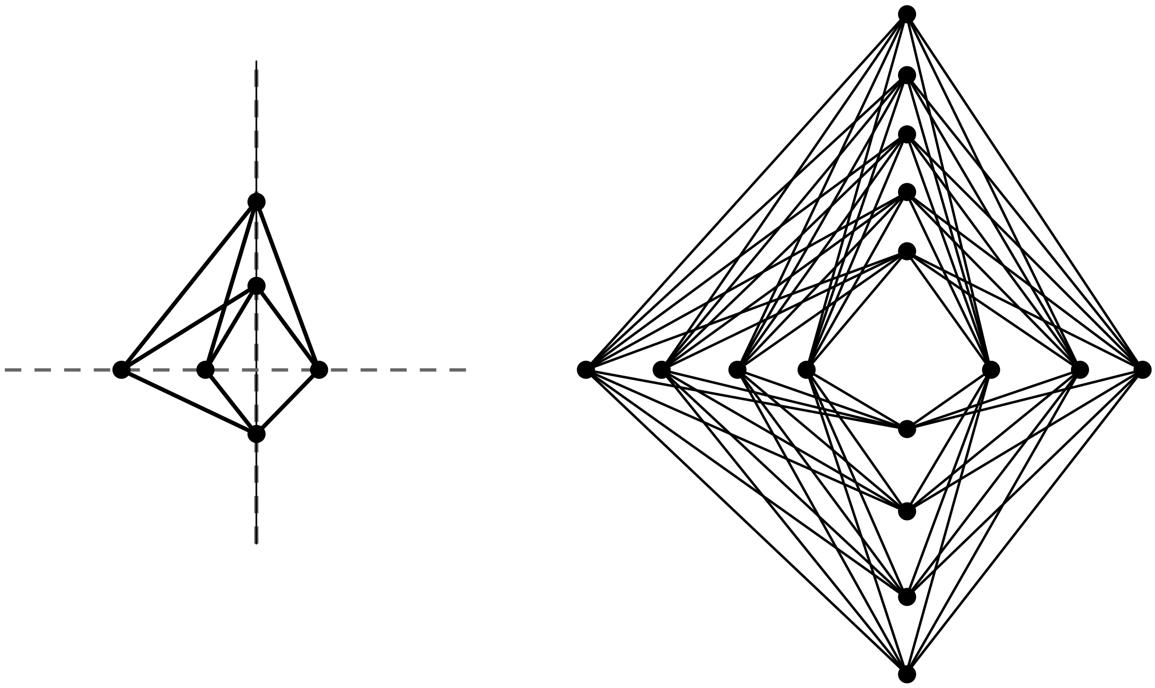


Figure 5.4: We show Zarakiewicz' construction for a drawing f of $K_{m,n}$ with $\mathrm{cr}(f) = Z_{m,n}$ for $(m, n) \in \{(3, 3), (7, 9)\}$. The m vertices of one part are placed on a horizontal axis, the n vertices of the other are placed on a vertical axis, such that roughly half of the vertices of each part end up on one side of axis. Then, the edges are drawn using straight segments.

that $(K_{m,n})^{*2}_{\Delta} = [m]^{*2}_{\Delta} * [n]^{*2}_{\Delta}$ which for large m and n we think of to be roughly equal to $\Lambda^3_{m,m,n,n} = [m]^{*2} * [n]^{*2}$. Theorem 4.1 (in combination with Proposition 5.2) would imply the asymptotic version of Zarankiewicz' conjecture

$$\mathrm{cr}(K_{m,n}) \geq \mathrm{ipcr}(K_{m,n}) \geq \frac{1}{16}(1 + o(1))m^2 n^2, \text{ as } m, n \to +\infty,$$

if we could show that $\eta_k(\Lambda^3_{m,m,n,n}) \geq 1$ for all $0 \leq k \leq 2$. But, unfortunately, as we will see in Theorem 7.7, this does not hold (at least for $k = 2$).

Due to the flexibility of the equivariant overlap theorem, we would still get the asymptotic version of Zarankiewicz' conjecture if we could show $\eta_k^{\|\cdot\|}(\Lambda^3_{m,m,n,n};\mathbb{F}_2) \geq 1$ with respect to some weighted Hamming norm $\|\cdot\|$ such that $\|\cdot\|$ is invariant under the $\mathbb{Z}/2$-action, is equal to the normalized Hamming norm on $\Lambda^3_{m,m,n,n}(3)$ and such that $\|\mathbb{1}_{\Lambda^3_{m,m,n,n}(0)}\| = 1$. Unfortunately, we do not know how to prove such a result.

But using an interesting combination of expansion with respect to $\mathbb{Z}$-coefficients and $\mathbb{F}_2$-coefficients we can show that:

Proposition 5.28. *We have for $m \leq n$ that*

$$\operatorname{cr}(K_{m,n}) \geq \frac{4}{5}\frac{1}{16}m^2n^2 - \frac{3}{10}mn^2 - 254016(mn^2 + m^2n^2 - 2mn).$$

In particular,

$$\lim_{m,n\to+\infty} \frac{\operatorname{cr}(K_{m,n})}{Z_{m,n}} \geq \frac{4}{5}.$$

It is important to note that the lower bound in Proposition 5.28 is on $\operatorname{cr}(K_{m,n})$ and not on $\operatorname{ipcr}(K_{m,n})$. In fact, a starting point of the proof is the observation that an optimal drawing $f\colon |K_{m,n}| \to \mathbb{R}^2$ of $K_{m,n}$ achieving $\operatorname{cr}(K_{m,n})$ can be assumed to have the following properties (see for instance [132, Section 1]):

(i) f is piecewise-linear in general position,

(ii) no two edges which share an endpoint have another common point except this endpoint, and

(iii) any two edges cross at most once.

We call a drawing satisfying (i)-(iii) a *good* drawing. It follows from properties (i)-(iii) that $b := F^{\pitchfork}(0) \in B^3({K_{m,n}}^{*2}_{\Delta};\mathbb{Z})$ is $\{-1,0,+\}$-valued where $F\colon {K_{m,n}}^{*2}_{\Delta} \to_{\mathbb{Z}/2} \mathbb{R}^3$ is the equivariant map induced by f. We have $b(e\otimes e') \in \{-1,+1\}$ if and only if the edges e and e' cross (for the drawing f). Moreover, the value of $b(e \otimes e')$ then depends on the 'sign' of the crossing of e and e'. In particular, for such f, we have $|b|_2^2 = 2\operatorname{cr}(K_{m,n})$ where $|\cdot|_2^2$ denotes the squared ℓ_2-norm. Then, we will use expansion with respect to $\mathbb{Z}$-coefficients in dimension 2 and expansion with respect to $\mathbb{F}_2$-coefficients to deduce the lower bound as claimed.

Before we add the details to this outline, let us compare the bound in Proposition 5.28 to existing bounds in the literature. An old bound from 1970 due to Kleitman [78] is that

$$\operatorname{cr}(K_{m,n}) \geq \frac{4}{5}Z_{m,n}$$

for all $m \geq 5$. This was the state-of-the-art until in 2003 Nahas (see [111]) gave a tiny improvement on Kleitman's bound showing that for sufficiently large m and n

$$\operatorname{cr}(K_{m,n}) \geq)\frac{1}{5}m(m-1)\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor + 9.9\times 10^{-6}m^2n^2.$$

Using semidefinite programming techniques de Klerk et al. were able to improve the asymptotic bound in a series of works [28, 29] to

$$\lim_{n\to+\infty}\frac{\operatorname{cr}(K_{m,n})}{Z_{m,n}} \geq 0.8594\frac{m}{m-1},$$

whenever $m \geq 9$. In a recent preprint [20] Brosch and Polak show that the constant 0.8594 can be improved to 0.8878 if $m \geq 13$. Furthermore, it was already announced in 2013 [113] by Norin and Zwols that the constant could be further improved to 0.905 using flag algebra techniques. As far as we know this work has not been published (yet).

In view of these results we see that Proposition 5.28 recovers Kleitman's bound asymptotically.

5.5.1 Setting-up the Stage

For the proof of Proposition 5.28 we will need the following notation. Assume $3 \leq m \leq n$ are fixed. Let $X := (K_{m,n})^{*2}_{\Delta}$ and $Y = K^{*2}_{m,n}$. We will write $Y = A * B * C * D$ with $A = C = [m]$ and $B = D = [n]$. We will establish expansion properties for Y and then use the fact that $|Y(k) \setminus X(k)|/|Y(k)| = o(1)$ for $m, n \to +\infty$ for all $k \in \{0, 1, 2, 3\}$ to get good bounds for the expansion constants of X as well. In order to pass from Y to X we will make use of the fact that $\tilde{H}^k(X; \mathbb{Z}) = 0$ for all $0 \leq k \leq 2$. Indeed, note that $X = (K_{m,n})^{*2}_{\Delta} \cong [m]^{*2}_{\Delta} * [n]^{*2}_{\Delta}$ is the join of two connected graphs (here we need that $n \geq m \geq 3$), hence the vanishing of the cohomology groups of X follows from the Künneth theorem.

We will consider expansion of Y with respect to integer coefficients in dimension 2 to 3 and with respect to $\mathbb{F}_2$-coefficients in dimension 0 to 1 and 1 to 2. We will also make use of two different weight functions $w_1, w_2 \colon Y \to \mathbb{R}_{\geq 0}$ given as follows:

- $w_1(\sigma) = w_2(\sigma) = \frac{1}{m^2 n^2}$ for $\sigma \in Y(3)$.
- $w_1(\tau) = \frac{1}{2m^2 n}$ for $\tau \in Y(2)$ with $\tau \in A * B * C \sqcup A * C * D$ and 0 otherwise while $w_2(\tau) = \frac{1}{2mn^2}$ for $\tau \in Y(2)$ with $\tau \in A * B * D \sqcup B * C * D$ and 0 otherwise.
- $w_1(e) = \frac{1}{m^2}$ for $e \in Y(1)$ with $e \in A * C$ and 0 otherwise while $w_2(e) = \frac{1}{n^2}$ for $e \in Y(1)$ with $e \subseteq B * D$ and 0 otherwise.
- $w_1(x) = \frac{1}{2m}$ for $x \in A \sqcup C$ and 0 elsewhere while $w_2(x) = \frac{1}{2n}$ for $x \in B \sqcup D$ and 0 elsewhere.

We endow X with the weights obtained by restricting the weights w_i on Y to X. We will denote them by w_i as well. We write $\| \cdot \|_i$, $i \in \{1, 2\}$, for the induced weighted Hamming norm on cochain groups $C^k(Y; \mathbb{F}_2)$ (or $C^k(X; \mathbb{F}_2)$) and $| \cdot |^2_i$, $i \in \{1, 2\}$, for the size function on $C^k(Y; \mathbb{Z})$ (or $C^k(X; \mathbb{Z})$) induced by the weights w_i and the ℓ^2_2-norm on $\mathbb{Z}$. Note that the weights w_i are invariant under the $\mathbb{Z}/2$-action on X (or Y) hence the induced norms are $\mathbb{Z}/2$-invariant as well.

Having all these notations at hand, we will state a couple of lemmata in the next subsection which establish the required expansion properties of Y and will help us to pass from Y to X. Then, we will first see how the lemmata help to prove Proposition 5.28 before we close the section with the proofs of the lemmata and some remarks.

5.5.2 A Bunch of Lemmata

We are able to show the following expansion properties for Y:

Lemma 5.29. *For $i \in \{1, 2\}$ we have $\eta_0^{\| \cdot \|_i}(Y; \mathbb{F}_2) \geq 1$.*

Lemma 5.30. *For $i \in \{1,2\}$ we have $\eta_1^{\|\cdot\|_i}(Y;\mathbb{F}_2) \geq 1$.*

Lemma 5.31. *For any $f \in C^2(Y;\mathbb{Z})$ we have $|\delta f|_1^2 \geq \frac{4}{5}\min\{|[f]|_1^2, |[f]|_2^2\}$.*

To relate expansion properties of X with those of Y the following two lemmata will be useful:

Lemma 5.32. *For $k \in \{1,2,3\}$ let $\Delta(k) = \{\sigma \in Y(k) :$ there exists $\tau \in X(k-1), \tau \subseteq \sigma\}$. Then,*

(i) $\|\mathbb{1}_{\Delta(1)}\|_1 \leq 1/m$ *and* $\|\mathbb{1}_{\Delta(1)}\|_2 \leq 1/n$.

(ii) $\|\mathbb{1}_{\Delta(2)}\|_1 \leq 1/m$ *and* $\|\mathbb{1}_{\Delta(2)}\|_2 \leq 1/n$.

(iii) $|\mathbb{1}_{\Delta(3)}|_1^2 \leq \frac{m+n-2}{mn}$.

Lemma 5.33. *Let $b \in B^3(X;\mathbb{Z})$ with $b(\sigma) \in \{-1,0,1\}$ for all $\sigma \in X(3)$. Then there is $\tilde{b} \in B^3(Y;\mathbb{Z})$ such that $|\tilde{b}|_1^2 \leq |b|_1^2 + 254016\frac{m+n-2}{mn}$ and such that the restriction of $\tilde{b}$ to X is equal to b.*

5.5.3 Proof of Proposition 5.28 Assuming the Lemmata

We show how the lemmata can be put together to prove Proposition 5.28.

Proof of Proposition 5.28. Let $f\colon |K_{m,n}| \to \mathbb{R}^2$ be a *good* drawing of $K_{m,n}$ achieving $\mathrm{cr}(K_{m,n})$. As above write $Y = K_{m,n}^{*2}$ for the join and $X = (K_{m,n})^{*2}_{\Delta}$ for the deleted join of $K_{m,n}$ with itself. Let $i\colon X \to Y$ be the inclusion map. Write $F\colon X \to_{\mathbb{Z}/2} \mathbb{R}^3$ for the induced equivariant map. Let $b^{(3)} := F^{\pitchfork}(0) \in B^3(X;\mathbb{Z})$. As discussed above $b^{(3)}$ is $\{-1,0,1\}$-valued with $2\,\mathrm{cr}(K_{m,n}) = |b^{(3)}|_1^2$. Let $\bar{b}^{(3)} \in B^3(X;\mathbb{F}_2)$ the reduction of $b^{(3)}$ modulo 2. Note that $\|\bar{b}^{(3)}\|_1 = |b^{(3)}|_1^2$.

Next we construct a pagoda for $\bar{b}^{(3)}$. To this end, let $\tilde{b}^{(3)} \in B^3(X;\mathbb{Z})$ such that $\tilde{b}^{(3)}$ restricts to $b^{(3)}$ on X and such that $|\tilde{b}^{(3)}|_1^2 \leq |b^{(3)}|_1^2 + 254016\frac{m+n-2}{mn}$. According to Lemma 5.33 we can always find such a $\tilde{b}^{(3)}$.

By Lemma 5.31 there is $\tilde{a}^{(2)} \in B^2(Y;\mathbb{Z})$ with $\delta\tilde{a}^{(2)} = \tilde{b}^{(3)}$ and such that $|\tilde{b}^{(3)}|_1^2 \geq 4/5\min\{|\tilde{a}^{(2)}|_1^2, |\tilde{a}^{(2)}|_2^2\}$. Upon interchanging the roles of m and n we can assume that $\min\{|\tilde{a}^{(2)}|_1^2, |\tilde{a}^{(2)}|_2^2\} = |\tilde{a}^{(2)}|_1^2$.

Let $\bar{a}^{(2)} = i^*\tilde{a}^{(2)} \mod 2 \in C^2(X;\mathbb{F}_2)$. Note that $|\tilde{a}^{(2)}|_1^2 \geq \|\bar{a}^{(2)}\|_1$.

Let $b^{(2)} = \bar{a}^{(2)} + \nu\bar{a}^{(2)}$, where ν is the $\mathbb{Z}/2$-action on X. As in the proof of the equivariant overlap theorem we have $\delta b^{(2)} = 0$ and since $\tilde{H}^2(X;\mathbb{F}_2) = 0$ there is $c^{(1)} \in C^1(X;\mathbb{F}_2)$ with $\delta c^{(1)} = b^{(2)}$.

Let $\tilde{b}^{(2)} = \delta\bar{c}^{(1)}$ where $\bar{c}^{(1)}$ denotes the extension by 0 of $c^{(1)}$ to $C^1(Y;\mathbb{F}_2)$. By Lemma 5.32 (ii) we have $\|\tilde{b}^{(2)}\|_1 \leq \|b^{(2)}\|_1 + 1/m$.

According to Lemma 5.30 there is a cofilling $\tilde{a}^{(1)} \in C^1(Y;\mathbb{F}_2)$ of $\tilde{b}^{(2)}$ such that $\|\tilde{b}^{(2)}\|_1 \geq \|\tilde{a}^{(1)}\|$.

Let $a^{(1)} = i^*\tilde{a}^{(1)} \in C^1(X;\mathbb{F}_2)$ and $b^{(1)} = a^{(1)} + \nu a^{(1)}$. Then $\delta b^{(1)} = 0$ and since $\tilde{H}^1(X;\mathbb{F}_2) = 0$ we find $c^{(0)} \in C^0(X;\mathbb{F}_2)$ with $\delta c^{(0)} = b^{(1)}$. Let $\tilde{b}^{(1)} = \delta\bar{c}^{(0)}$. By Lemma 5.32 (i) we have $\|\tilde{b}^{(1)}\|_1 \leq \|b^{(1)}\| + 1/m$.

Lemma 5.29 implies that there is a cofilling $\tilde{a}^{(0)}$ of $\tilde{b}^{(0)}$ with $\|\tilde{b}^{(0)}\|_1 \geq \|\tilde{a}^{(0)}\|_1$. Let $a^{(0)} = i^*\tilde{a}^{(0)}$ and $b^{(0)} = a^{(0)} + \nu a^{(0)}$.

We have constructed a pagoda $(\bar{b}^{(3)}, \bar{a}^{(2)}, b^{(2)}, a^{(1)}, b^{(1)}, a^{(0)}, b^{(0)})$ for $\bar{b}^{(3)}$. In particular, by Lemma 4.8, we must have $b^{(0)} = \mathbb{1}_{X(0)}$.

Putting all the estimates together, we conclude

$$\frac{2\operatorname{cr}(K_{m,n})}{m^2n^2} + 254016\frac{m+n-2}{mn} \geq |b^{(3)}|_1^2 \geq= \frac{1}{10} - \frac{3}{5m},$$

finishing the proof after rearranging. □

5.5.4 Proofs of the Lemmata

Proof of Lemma 5.29. We simply note that by definition of the weights w_i we have $\eta_0^{\|\cdot\|_1}(Y;\mathbb{F}_2) = \eta_0(K_{m,m})$ and $\eta_0^{\|\cdot\|_2}(Y;\mathbb{F}_2) = \eta_0(K_{n,n})$. It is a well-known fact (and a special case of Proposition 3.8) that $\eta_0(K_{n,n}) \geq 1$ for all $n \in \mathbb{Z}_{>0}$. □

Proof of Lemma 5.30. For the proof we will never use that $m \leq n$. Thus, upon interchanging the roles of m and n, it suffices to consider the norm $\|\cdot\|_1$. We will use a random cofilling argument. To this end, let $\gamma \in C^1(Y;\mathbb{F}_2)$ be minimal. Let $\beta := \delta\gamma \in B^2(Y;\mathbb{F}_2)$. Interchanging the roles of B and D we can assume that

$$\frac{1}{m^2n}\sum_{a\in A,b\in B,c\in C}|\beta(abc)| \leq \frac{1}{2}\|\beta\|_1 \tag{5.1}$$

Fix $a_0 \in A$. For $b \in B$ define $S^{(b)}\colon C_0(Y;\mathbb{F}_2) \to C_1(Y;\mathbb{F}_2)$ by

$$S^{(b)}x := \begin{cases} bx & \text{if } x \in A \sqcup C \sqcup D \\ a_0b + a_0x & \text{if } x \in B. \end{cases}$$

Let $T^{(b)}\colon C^1(Y;\mathbb{F}_2) \to C^0(Y;\mathbb{F}_2)$ be the dual map of $S^{(b)}$. For $b \in B$ let $\gamma^{(b)} := \gamma + \delta T^{(b)}\gamma$.

We compute for $a \in A$ and $c \in C$ that

$$\begin{aligned}\gamma^{(b)}(ac) &= \gamma(ac) + \langle \delta T^{(b)}\gamma, ac\rangle \\ &= \gamma(ac) + \langle \gamma, S^{(b)}c + S^{(b)}a\rangle \\ &= \gamma(ac) + \gamma(bc) + \gamma(ab) \\ &= \beta(abc).\end{aligned}$$

If follows that

$$\begin{aligned}\|\gamma\|_1 &\leq \min_{b\in B}\|\gamma^{(b)}\|_1 \\ &\leq \frac{1}{n}\sum_{b\in B}\|\gamma^{(b)}\|_1 \\ &= \frac{1}{m^2n}\sum_{b\in B}\sum_{a\in A,c\in C}|\gamma^{(b)}(ac)| \\ &= \frac{1}{m^2n}\sum_{a\in A,b\in B,c\in C}|\beta(abc)| \\ &\leq \|\beta\|_1,\end{aligned}$$

where we used the assumption (5.1) for the last inequality. □

For the proof of Lemma 5.31 we will need the following inequality.

Claim 5.34. *Let $\beta \in B^3(Y;\mathbb{Z})$. Then*

$$\frac{1}{m}\sum_{xx'\in\binom{A}{2}\sqcup\binom{C}{2}}|\beta_{x'}-\beta_x|^2+\frac{1}{n}\sum_{xx'\in\binom{B}{2}\sqcup\binom{D}{2}}|\beta_{x'}-\beta_x|^2\leq 3m^2n^2|\beta|_1^2.$$

Proof. This is a special case of Lemma 8.4 below. □

Proof of Lemma 5.31. We use a random cofilling argument. Let $\beta \in B^3(Y;\mathbb{Z})$. As before we will not use the assumption $m \leq n$. Thus, upon changing m with n, we can assume by Claim 5.34 that

$$\frac{1}{m}\left(\sum_{aa'\in\binom{A}{2}}|\beta_{a'}-\beta_a|^2+\sum_{cc'\in\binom{C}{2}}|\beta_{c'}-\beta_c|^2\right)\leq\frac{3}{2}m^2n^2|\beta|_1^2. \tag{5.2}$$

Given $aa' \in \binom{A}{2}$ and $c \in C$ let $\gamma^{(a,a',c)}$ be a cofilling of $\beta_{a'}-\beta_a \in B^2(B*C*D;\mathbb{Z})$ such that

$$\gamma^{(a,a',c)}(bd)=\beta(abcd)-\beta(a'bcd)$$

for all $b\in B, d\in D$. Such a cofilling always exists. Indeed, we could choose $\gamma^{(a,a',c)} = \gamma-\delta\bar{\gamma}_c$ for any cofilling γ of $\beta_{a'}-\beta_a$.

Similarly, for $cc' \in \binom{C}{2}$ and $a\in A$ let $\gamma^{(c,c',a)}$ be a cofilling of $\beta_{c'}-\beta_c \in B^2(A*B*D;\mathbb{Z})$ such that

$$\gamma^{(c,c',a)}(bd)=\beta(abcd)-\beta(abc'd)$$

for all $b\in B, d\in D$.

Now let $(a,c,\varepsilon)\in A\times C\times\{-,+\}$. If $\varepsilon=-$ define $\gamma^{(a,c,\varepsilon)}\in C^2(Y;\mathbb{Z})$ by

$$\gamma^{(a,c,-)}(\tau)=\begin{cases}\beta(a\tau) & \text{if } \tau\in B*C*D\\ \gamma^{(a,a',c)}(xy) & \text{if } \tau=a'xy\in A*(B*C\sqcup B*D\sqcup C*D).\end{cases}$$

Similarly, if $\varepsilon=+$ define $\gamma^{(a,c,\varepsilon)}\in C^2(X;\mathbb{Z})$ by

$$\gamma^{(a,c,+)}(\tau)=\begin{cases}\beta(a'b'cd') & \text{if } \tau=a'b'd'\in A*B*D\\ \gamma^{(c,c',a)}(xy) & \text{if } \tau=xyc'\in A*B*C,\\ \gamma^{(c,c',a)}(xy) & \text{if } \tau=xc'y\in A*C*D \text{ or } \tau=xc'y\in B*C*D.\end{cases}$$

It is straightforward to check that $\delta\gamma^{(a,c,\varepsilon)}=\beta$ for all $a\in A, c\in C, \varepsilon\in\{-,+\}$.

Averaging over all choices of $a \in A, c \in C, \varepsilon \in \{-,+\}$ gives

$$\begin{aligned}
\min_{a\in A, c\in C, \varepsilon\in\{-,+\}} |\gamma^{(a,c,\varepsilon)}|_2^2 &\leq \frac{1}{2m^2} \sum_{a\in A, c\in C} \sum_{b'\in B, c'\in C, d'\in D} \frac{1}{2mn^2} |\gamma^{(a,c,+)}(b'c'd')|^2 \\
&+ \frac{1}{2m^2} \sum_{a\in A, c\in C} \sum_{a'\in A, b'\in B, d'\in D} \frac{1}{2mn^2} |\gamma^{(a,c,+)}(a'b'd')|^2 \\
&+ \frac{1}{2m^2} \sum_{a\in A, c\in C} \sum_{b'\in B, c'\in C, d'\in D} \frac{1}{2mn^2} |\gamma^{(a,c,-)}(b'c'd')|^2 \\
&+ \frac{1}{2m^2} \sum_{a\in A, c\in C} \sum_{a'\in A, b'\in B, d'\in D} \frac{1}{2mn^2} |\gamma^{(a,c,-)}(a'b'd')|^2 \\
&= \frac{1}{2}|b|_1^2 + \frac{1}{2m^3n^2} \sum_{aa'\in\binom{A}{2}} |\beta_{a'} - \beta_a|^2 + \frac{1}{2m^3n^2} \sum_{cc'\in\binom{C}{2}} |\beta_{c'} - \beta_c|^2 \\
&\leq \frac{1}{2}|b|_1^2 + \frac{3}{4}|b|_1^2 \\
&= \frac{5}{4}|b|_1^2.
\end{aligned}$$

Here we used (5.1) for the last inequality. This finishes the proof.

□

Proof of Lemma 5.32. The inequalities easily follows from the defnition of $\|\cdot\|_i$ and $|\cdot|_i^2$ observing that

$$\begin{aligned}
\Delta(1) &= \{\{a, \nu a\} : a \in A\} \cup \{\{b, \nu b\} : b \in B\}, \\
\Delta(2) &= \{\{a, b, \nu a\} : a \in A, b \in B\} \cup \{\{a, b, \nu b\} : a \in A, b \in B\} \\
&\quad \cup \{\{\nu c, c, d\} : c \in C, d \in D\} \cup \{\{\nu d, c, d\} : c \in D, d \in D\}, \text{ and} \\
\Delta(3) &= \{\{a, b, \nu a', \nu b\} : a, a' \in A, a \neq a', b \in B\} \\
&\quad \cup \{\{a, b, \nu a, \nu b'\} : a \in A, b, b' \in B, b \neq b'\}.
\end{aligned}$$

□

Proof of Lemma 5.33. Fix pairwise distinct vertices $a_0, a_1, a_2 \in A$ and $b_0, b_1, b_2 \in B$. Given $\sigma \in X$ let

$$A_\sigma := (A \cap \sigma) \cup (A \cap \nu\sigma) \cup \{a_0, a_1, a_2\} \text{ and } B_\sigma = (B \cap \sigma) \cup (B \cap \nu\sigma) \cup \{b_0, b_1, b_2\}.$$

Let $\Sigma_\sigma := (A_\sigma * B_\sigma)^{*2}_\Delta \subseteq X$. Note that $\Sigma_\sigma \cong (K_{|A_\sigma|,|B_\sigma|})^{*2}_\Delta$ satisfies $\tilde{H}^k(\Sigma_\sigma; \mathbb{Z}) = 0$ for all $0 \leq k \leq 2$, since $\min\{|A|_\sigma, |B|_\sigma\} \geq 3$.

Also if $\tau \subseteq \sigma \in X$ we clearly have $\Sigma_\tau \subseteq \Sigma_\sigma$. Thus, we can apply Lemma 3.14 to get an abstract cone $(S_k)_{-2\leq k\leq 2}$ for X such that $\operatorname{supp}(S_k\sigma) \subseteq B_\sigma$ for all $\sigma \in X(k)$, $-1 \leq k \leq 2$. Let $T_3\colon C^3(X;\mathbb{Z}) \to C^2(X;\mathbb{Z})$ be the dual map of S_2.

Now, let $\beta \in B^3(X;\mathbb{Z})$ with $\beta(\sigma) \in \{-1, 0, +1\}$ for all $\sigma \in X(3)$. Since $(S_k)_{-2\leq k\leq 2}$ is an abstract cone, $\gamma := T_3\beta \in C^2(X;\mathbb{Z})$ is a cofilling of β. Moreover, for $\tau \in X(2)$ we have

$$|\gamma(\tau)| = |\langle T_3\beta, \tau\rangle| = |\langle \beta, S_2\tau\rangle| \leq |B_\tau(3)|,$$

where we used that β only takes values in $\{-1,0,1\}$. Note that $|B_\tau(3)| \leq 252$ for all $\tau \in X(2)$. Let $\bar{\gamma} \in C^2(Y;\mathbb{Z})$ be the extension by 0 of γ to Y and let $\tilde{b} := \delta\bar{\gamma} \in B^3(Y;\mathbb{Z})$.

Let $\Delta(3) = \{\sigma \in Y(3) : \text{there is } \tau \in X(2), \tau \subseteq \sigma\}$. Note that if $\sigma \in \Delta(3)$ then σ has precisely two of its boundary triangles in X. It follows that for such σ we have

$$|\tilde{\beta}(\sigma)|^2 \leq 4|B_\tau(3)|^2 \leq 254016.$$

We conclude that

$$\begin{aligned} |\tilde{b}|_1^1 &= |\beta|_1^2 + |\tilde{b}_{|\Delta(3)}|_1^2 \\ &\leq |\beta|_1^2 + 254016|\mathbb{1}_{\Delta(3)}|_1^2 \\ &\leq |\beta|_1^2 + 254016\frac{m+n-2}{mn}, \end{aligned}$$

where we used Lemma 5.32 (iii) for the last inequality. □

5.5.5 Further Discussion and Remarks

For $n \in \mathbb{Z}_{>0}$ let $K_{n,n,n}$ be the complete tripartite graph with equally sized parts of size n. It was shown in [53] that $\mathrm{cr}(K_{n,n,n}) \leq A_n$ where

$$A_n := 3\left(\left(\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor\right)^2 + \left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n^2}{2}\right\rfloor\right).$$

It is conjectured that $\mathrm{cr}(K_{n,n,n}) = A_n$. Using similar arguments as for $K_{m,n}$ we were able to show that[4]

$$\lim_{n\to+\infty}\frac{\mathrm{cr}(K_{n,n,n})}{A_n} \geq \frac{2}{3},$$

recovering a bound proven in [53, Theorem 1.2]. Since we were not able to improve upon existing bounds in the literature and the argument for $K_{n,n,n}$ is significantly more technical than for $K_{m,n}$, we refrain from giving a proof here.

Note that any improvement on the constant $4/5$ in Lemma 5.31 would immediately lead to an improvement on the bound in Proposition 5.28. We are happy to conjecture that

Conjecture 5.35. *Let $m,n \in \mathbb{Z}_{>0}$.*

- *(Weak form) For all $c \in C^2(K^{*2}_{m,n};\mathbb{Z})$ we have $|\delta c|_1^2 \geq \min\{|[c]|_1^2, |[c]|_2^2\}$.*
- *(Strong form) $\zeta_2(\Lambda^3_{m,m,n,n}) \geq 1$.*

Note that both forms of Conjecture 5.35 imply an asymptotic version of Zarankiewicz' conjecture, namely that $\lim_{m,n\to+\infty}\frac{\mathrm{cr}(K_{m,n})}{Z_{m,n}} = 1$.

[4]To show the existence of the limit is a not too difficult exercise.

Chapter 6

Expansion of Joins

In this chapter, we give a general discussion on expansion of joins.

On the one hand, we give examples showing that under taking joins coboundary expansion (with respect to $\mathbb{F}_2$-coefficients and Garland weighted Hamming norm) does not behave as well as one might naively expect.

More precisely, in Section 6.1 we exhibit two infinite families $(G_n)_{n\in\mathbb{N}}$ and $(H_n)_{n\in\mathbb{N}}$ of connected graphs for which $\eta_2(G_n * H_n)$ is of lower order than $\eta_0(G_n)\eta_0(H_n)$ as $n \to +\infty$.

As another example illustrating the difficulty of analyzing expansion of properties of joins, we show in Section 6.2 that minimality of cochains is not always preserved under taking joins.

Contrasting these negative results, we give a join construction for random abstract cones in Section 6.3. This allows us to establish coboundary expansion for $X * Y$ if the coboundary expansion of X and Y is certified by a random abstract cone. We illustrate the construction by proving a lower bound on $\eta_k(\Lambda_n^d)$.

6.1 Non Product-Like Behaviour for Expansion Constants under Taking Joins

The goal of this section is to show

Proposition 6.1. *There are positive constants C and η such that there are infinite families of regular graphs $(G_n)_{n\in\mathbb{N}}$ and $(H_n)_{n\in\mathbb{N}}$ with the property that for all $n \in \mathbb{N}$*

(i) $\eta_0(G_n) \geq \eta$,

(ii) $\eta_0(H_n) \geq C\frac{\log|G_n(1)|}{|G_n(1)|}$,

(iii) $\eta_2(G_n * H_n) \leq \frac{6}{|G_n(1)|}$.

In particular,

$$\lim_{n\to+\infty} \frac{\eta_2(G_n * H_n)}{\eta_0(G_n)\eta_0(H_n)} = 0.$$

Let us first describe the two families $(G_n)_{n\in\mathbb{N}}$ and $(H_n)_{n\in\mathbb{N}}$ we will use for the proof of Proposition 6.1.

We choose $(G_n = (V_n, E_n))_{n\in\mathbb{N}}$ to be any infinite family of d-regular graphs for a sufficiently large but fixed d such that $\eta_0(G_n) \geq \eta$ for some $\eta > 0$ and all $n \in \mathbb{N}$ and such that the girth[1] $\mathrm{g}(G_n)$ of G_n is at least $c \log |V_n|$ for some $c > 0$ and all $n \in \mathbb{N}$. Such families of expander graphs are known to exist. For instance we could work with the Ramanujan graphs constructed by Luboztky, Phillips and Sarnak.

Theorem 6.2. *Let $p \equiv 1 \mod 4$ be a prime. Then there is an infinite family of $(p+1)$-regular graphs $(G_n = (V_n, E_n))_{n\in\mathbb{N}}$ such that*

(i) $\eta_0(G_n) \geq 1 - \frac{2\sqrt{p}}{p+1}$ *for all $n \in \mathbb{N}$ and*

(ii) $\mathrm{g}(G_n) > \log_p(|V_n|)$ *for all $n \in \mathbb{N}$.*

Proof. This is an immediate consequence of Theorem 3.4 and Theorem 4.1 in [92] combined with the Cheeger inequality (see Theorem 2.2). □

Assume we have fixed a family $(G_n = (V_n, E_n))_{n\in\mathbb{N}}$ with the desired properties as listed above. Let $\mathbb{A}_n := C^1(G_n; \mathbb{F}_2)/B^1(G_n; \mathbb{F}_2)$ and

$$H_n := \mathrm{Cay}(\mathbb{A}_n, \{[\mathbb{1}_e] : e \in E_n\})$$

be the Cayley graph of $\mathbb{A}_n$ with generating set $\{[\mathbb{1}_e] : e \in E_n\}$. In other words, H_n is the graph with vertex set $\mathbb{A}_n$ and edges $\{[c], [c']\}$ whenever $[c + c'] = [\mathbb{1}_e] \in \mathbb{A}_n$ for some edge $e \in E_n$.

Now, Proposition 6.1 will be an immediate consequence of the following two lemmata.

Lemma 6.3. *Let $(G_n = (V_n, E_n))_{n\in\mathbb{N}}$ and H_n as described above. Then for sufficiently large n, we have*

$$\eta_2(G_n * H_n) \leq \frac{6}{|E_n|}.$$

Lemma 6.4. *Let $(G_n = (V_n, E_n))_{n\in\mathbb{N}}$ and H_n as described above. Then there is a constant $s > 0$, such that for sufficiently large n we have*

$$\eta_0(H_n) \geq s \frac{\log |E_n|}{|E_n|}.$$

We start with the proof of Lemma 6.3. To this end, let $X_n = G_n * H_n$ and write $X_n(0) = V(G_n) \sqcup V(H_n)$. For $v \in V(H_n) = \mathbb{A}_n$ let $a^{(v)} \in C^1(G_n; \mathbb{F}_2)$ be a minimal representative of $v \in \mathbb{A}_n$. For $e = \{u, v\} \in E(H_n)$ let $a^{(e)} \in C^0(G; \mathbb{F}_2)$ be minimal such that $a^{(u)} + a^{(v)} + \delta a^{(e)} \in C^1(G_n; \mathbb{F}_2)$ is minimal. Here we consider minimality with respect to Garland weighted Hamming norm which we denote by $\|\cdot\|$. Furthermore, we will write $|\cdot|$ (unnormalized) Hamming norm on cochain groups.

Define $c^{(n)} \in C^2(X_n; \mathbb{F}_2)$ by

$$c^{(n)} := \sum_{v \in V(H_n)} a^{(v)} \otimes \mathbb{1}_v + \sum_{e \in E(H_n)} a^{(e)} \otimes \mathbb{1}_e.$$

We claim that

[1]The girth $\mathrm{g}(H)$ of a graph H is the length of a smallest cycle in H.

Claim 6.5. $\|\delta c^{(n)}\| = \frac{1}{|E_n|}$ *for all* $n \in \mathbb{N}$.

and

Claim 6.6. $\|[c^{(n)}]\| \geq 1/6$ *for sufficiently large* n *provided that* $(G_n)_{n\in\mathbb{N}}$ *is a family of* d*-regular graphs for sufficiently large* d.

Note that these two claims immediately imply Lemma 6.3.

Proof of Claim 6.5. We compute

$$\begin{aligned}
\|\delta c^{(n)}\| &= \frac{1}{|E(G_n)|}\frac{1}{|E(H_n)|}\sum_{f=xy\in E(G_n)}\sum_{e=uv\in E(H_n)}|c_u^{(n)}(f)+c_v^{(n)}(f)+c_e(u)+c_e(v)| \\
&= \frac{1}{|E(G_n)|}\frac{1}{|E(H_n)|}\sum_{uv\in E(H_n)}|a^{(u)}+a^{(v)}+\delta a^{(uv)}| \\
&= \frac{1}{|E(G_n)|}\frac{1}{|E(H_n)|}\sum_{uv\in E(H_n)}\|[\mathbb{1}_{uv}]\| \\
&= \frac{1}{|E(G_n)|}.
\end{aligned}$$

Here we used the definition of $c^{(n)}$ and that $\mathbb{1}_{uv}$ is a minimal representative of $[\mathbb{1}_{uv}] \in \mathbb{A}_n$ since a single edge cannot form a coboundary/cut in G_n due to its expansion properties. □

The proof of Claim 6.6 requires a bit more work. First let $s \in C^1(X_n;\mathbb{F}_2)$ such that $\tilde{c}^{(n)} := c^{(n)} + \delta s$ is minimal. Write s' for the restriction of s to $C^1(G_n;\mathbb{F}_2) \subseteq C^1(X_n;\mathbb{F}_2)$. We get that

$$\begin{aligned}
\|[c^{(n)}]\| &= \|\tilde{c}^{(n)}\| \\
&\geq \frac{1}{2|E(G_n)||V(H_n)|}\sum_{xy\in E(G_n)}\sum_{v\in V(H_n)}|\tilde{c}_v^{(n)}(xy)| \\
&= \frac{1}{2|E(G_n)||V(H_n)|}\sum_{xy\in E(G_n)}\sum_{v\in V(H_n)}|c_v^{(n)}(xy)+s'(xy)+\delta s_v(xy)| \\
&\geq \frac{1}{2|V(H_n)|}\sum_{v\in V(H_n)}\|[a^{(v)}+s']\| \\
&= \frac{1}{2|V(H_n)|}\sum_{v\in V(H_n)}\|[a^{(v)}]\|.
\end{aligned}$$

Thus, it remains to show that on average a minimal 1-cochain $c \in C^1(G_n;\mathbb{F}_2)$ contains a constant fraction of the edges of G_n.

This can be fairly easily shown using a probabilistic argument. Indeed, by using a Chernoff bound, it was shown in [85, Claim 5.2] that

Claim 6.7. *Consider the probability space of* 1*-cochains* c *in* $C^1(G_n;\mathbb{F}_2)$ *of the form*

$$c = \sum_{e\in E(G_n)} X_e \mathbb{1}_e$$

where the X_e *are independent* $\{0,1\}$*-valued random variables with* $\mathbb{P}(X_e = 0) = 1/2$ *and* $\mathbb{P}(X_e = 1) = 1/2$*. Then*

$$\mathbb{P}\left(\|[c]\| < 1/2\left(1 - 20\sqrt{\frac{2}{d}}\right)\right) < 0.8^{|V_n|}.$$

Using Claim 6.7 we can finish the proof of Claim 6.6

Proof of Claim 6.6. Claim 6.7 implies that there are at least $2^{|E_n|}(1 - 0.8^{|V_n|})$ cochains $c \in C^1(G; \mathbb{F}_2)$ with

$$\|[c]\| \geq \frac{1}{2}\left(1 - 20\sqrt{\frac{2}{d}}\right)$$

These cochains give rise to at least $2^{|E_n|-|V_n|+1}(1 - 0.8^{|V_n|})$ different equivalence classes in $\mathbb{A}_n$. It follows $c^{(n)} \in C^2(X_n; \mathbb{F}_2)$ satisfies

$$\begin{aligned}\|[c^{(n)}]\| &\geq \frac{1}{2|V(H_n)|}\sum_{v \in V(H_n)} \|[a^{(v)}]\| \\ &\geq \frac{1}{4}\left(1 - 20\sqrt{\frac{2}{d}}\right)(1 - 0.8^{|V_n|}) \geq \frac{1}{6},\end{aligned}$$

for sufficiently large d and n. □

For the proof of the lower bound on $\eta_0(H_n)$ we use the Cheeger inequality and the fact that the eigenvalues and eigenvectors of the normalized Laplacian of Cayley graphs of abelian groups can be described in terms of the characters of the group.

Given a group $\mathbb{A}$ and a symmetric[2], generating[3] set $S \subseteq \mathbb{A}$ the *Cayley graph* $\mathrm{Cay}(\mathbb{A}, S)$ *of* $\mathbb{A}$ *with generating set* S is the graph with vertex set $\mathbb{A}$ and edges $\{a, a'\}$ whenever $aa'^{-1} \in S$.

A *character* χ *of* $\mathbb{A}$ is a group homomoprhism $\chi\colon \mathbb{A} \to \mathbb{C}^\times$ from $\mathbb{A}$ to the multiplicative group of complex numbers.

Interestingly, the eigenvectors of the normalized Laplacian of the Cayley graph of an abelian group are precisely given by the characters and, hence, independent of the generating set. More precisely, we have

Proposition 6.8 (see [90] or [10, Corollary 3.2])**.** *Let* $\mathbb{A}$ *be an abelian group and* $S \subseteq \mathbb{A}$ *a symmetric, generating set. Let* $\Gamma = \mathrm{Cay}(\mathbb{A}, S)$ *be the Cayley graph of* $\mathbb{A}$ *with generating set* S*. Let* $\chi\colon \mathbb{A} \to \mathbb{C}^\times$ *be a character. Then* χ *is an eigenvector of the normalized Laplacian of* Γ *with eigenvalue*

$$1 - \frac{1}{|S|}\sum_{s \in S} \chi(s).$$

The characters of $\mathbb{F}_2^k$ are easy to describe.

[2] A subset $S \subseteq \mathbb{A}$ is *symmetric* if $s \in S$ if and only if $s^{-1} \in S$.

[3] A subset $S \subseteq \mathbb{A}$ is *generating* if every element $a \in \mathbb{A}$ can be written as a finite product of elements in S.

Lemma 6.9 (see, e.g.,[128, Chapter V])**.** *Let $\mathbb{A} = \mathbb{F}_2^k$ for some positive integer k. Then for every $x \in \mathbb{A}$ the function $\chi_x \colon \mathbb{A} \to \mathbb{C}^\times$ given by*

$$y \mapsto \chi_x(y) := (-1)^{\langle x,y\rangle}$$

where

$$\langle x, y\rangle = \sum_{i=1}^{k} x_i y_i$$

is a character of $\mathbb{A}$.

We are ready to prove Lemma 6.4:

Proof of Lemma 6.4. Recall that $\mathbb{A}_n = C^1(G_n;\mathbb{F}_2)/B^1(G_n;\mathbb{F}_2)$. Thus, characters of $\mathbb{A}_n$ are in one-to-one correspondence with characters of $C^1(G_n;\mathbb{F}_2)$ which contain $B^1(G_n;\mathbb{F}_2)$ in their kernel. By Lemma 6.9 these are precisely the characters χ_x for some $x \in C^1(G_n;\mathbb{F}_2)$ for which $\langle x, b\rangle = 0$ for all $b \in B^1(G_n;\mathbb{F}_2)$. But by Lemma 2.1 these are precisely the cycles $Z_1(G_n;\mathbb{F}_2)$.

Note that given $x \in \mathbb{A}_n$ and $e \in E_n$ we have $\chi_x(\mathbb{1}_e) = 1$ if $x(e) = 0$ and $\chi_x(\mathbb{1}_e) = -1$ if $x(e) = 1$. It follows that the eigenvalue of the normalized Laplacian of H_n corresponding to the character χ_x for $x \in Z_1(G_n;\mathbb{F}_2)$ is given by

$$1 - \frac{1}{|E_n|}\sum_{e\in E_n} \chi_x(\mathbb{1}_e) = 1 - \frac{1}{|E_n|}(-|x| + |E_n| - |x|) = \frac{2|x|}{|E_n|}.$$

Since we have chosen G_n to have logarithmic girth ($\mathrm{g}(G_n) \geq C\log|V_n|$ for some constant $C > 0$), we have that $|x| \geq C\log|V_n|$ for all $x \in Z_1(G_n;\mathbb{F}_2)$, $x \neq 0$. Thus, every non-trivial eigenvalue of the normalized Laplacian of G_n is at least $2C\log|V_n|/|E_n|$. An application of the Cheeger inequality (Theorem 2.2) finishes the proof. □

Proof of Proposition 6.1. Proposition 6.1 immediately follows by combining Lemma 6.3 and Lemma 6.4. □

The above example is very unbalanced in the sense that we consider joins $G * H$ where H is exponentially larger than G. It would be interesting to construct examples for which G and H are of comparable size or even examples with $G = H$. Furthermore, Proposition 6.1 rules out the existence of an universal constant $C > 0$ such that $\eta_2(G*H) \geq C\eta_0(G)\eta_0(H)$ for any two (connected) graphs G and H. But, for instance, we have not ruled out the possibility that $\eta_2(G * H) \geq C(\eta_0(G)\eta_0(H))^2$ for some constant $C > 0$.

6.2 Join of Minimal Cochains Not Necessarily Minimal

Throughout this section we consider cochains with respect to $\mathbb{F}_2$-coefficients. We endow cochain groups with the Hamming norm $|\cdot|$.

Let X and Y be simplicial complexes. For cochains $c \in C^i(X;\mathbb{F}_2)$ and $c' \in C^j(Y;\mathbb{F}_2)$ we write $c\otimes c'$ for their *join* which is the cochain in $C^{i+j+1}(X*Y;\mathbb{F}_2)$ given by $(c\otimes c')(\sigma\otimes\tau) = c(\sigma)c'(\tau)$ for $\sigma \in X(j), \tau \in Y(j)$ and $(c \otimes c')(\rho) = 0$ for any other $(i + j + 1)$-simplex in $X * Y$. Given minimal cochains $c \in C^i(X;\mathbb{F}_2)$, $c' \in C^j(Y;\mathbb{F}_2)$, it is natural to think that the join $c \otimes c' \in C^{i+j+1}(X * Y;\mathbb{F}_2)$ is minimal. This is not the case in general.

Proposition 6.10. *For every $m \in \mathbb{Z}_{>0}$ there is $c \in C^1(K_{5m};\mathbb{F}_2)$ such that $c \otimes c \in C^3(K_{5m}^{*2};\mathbb{F}_2)$ satisfies $\|[c \otimes c]\| < \|[c]\|^2$.*

For the proof of Proposition 6.10 we will first show the case $m = 1$ and then use a blow-up construction to deduce the general case $m \geq 2$. For $m = 1$ we have the following lemma:

Lemma 6.11. *Let $\mathbb{1} \in C^1(K_5;\mathbb{F}_2)$ be the all-one cochain, i.e. the cochain with $\mathbb{1}(e) = 1$ for all $e \in K_5(1)$. Then $\|[\mathbb{1}]\| = 4$ while $\mathbb{1} \otimes \mathbb{1} \in C^3(K_5^{*2};\mathbb{F}_2)$ satsifies*

$$\|[\mathbb{1} \otimes \mathbb{1}]\| = 14 < 16 = \|[\mathbb{1}]\|^2.$$

Proof. Note that every cochain in $[\mathbb{1}] \in C^1(K_5;\mathbb{F}_2)/B^1(K_5;\mathbb{F}_2)$ is the complement of a cut from which we easily see that $\|[\mathbb{1}]\| = 4$.

Next we will define a cochain $c \in C^3(K_5^{*2};\mathbb{F}_2)$ with $|c| = 14$ and $c \in [\mathbb{1} \otimes \mathbb{1}]$. To this end, we let $U = \{u_0, \dots, u_4\}$ and $V = \{v_0, \dots, v_4\}$. Write K_U and K_V for the complete graph with vertex set U and V, respectively. Let $X = K_U * K_V \cong K_5^{*2}$. Now define $c \in C^3(X;\mathbb{F}_2)$ to be the cochain with support

$$\begin{aligned}\operatorname{supp}(c) = \{&u_0u_1 \otimes v_1v_2, u_0u_2 \otimes v_0v_1, u_0u_2 \otimes v_3v_4, u_0u_3 \otimes v_0v_4, u_0u_3 \otimes v_1v_3, u_0u_4 \otimes v_2v_4,\\ &u_1u_2 \otimes v_0v_2, u_1u_3 \otimes v_2v_3, u_1u_4 \otimes v_0v_3, u_1u_4 \otimes v_1v_4, u_2u_3 \otimes v_0v_3, u_2u_3 \otimes v_1v_4,\\ &u_2u_4 \otimes v_2v_3, u_3u_4 \otimes v_0v_2\}.\end{aligned}$$

We have $|c| = 14$. Hence, it remains to show that $c \in [\mathbb{1} \otimes \mathbb{1}]$, i.e. $c + \mathbb{1} = \delta a$ for some $a \in C^2(X;\mathbb{F}_2)$ (here $\mathbb{1}$ is the constant 1 cochain in $C^3(X;\mathbb{F}_2)$). By Lemma 2.1 this amounts to show that $\langle c + \mathbb{1}, z\rangle = 0$ for a generating set of cycles $z \in Z_3(X;\mathbb{F}_2)$. Note that $Z_3(X;\mathbb{F}_2) = Z_1(K_U;\mathbb{F}_2) \otimes Z_1(K_V;\mathbb{F}_2)$ and that the space of cycles $Z_1(K_n;\mathbb{F}_2)$ of a complete graph K_n is generated by cycles of length 3. Thus, it suffices to show that $\langle c + \mathbb{1}, z \otimes z'\rangle = 0$ for cycles $z \in Z_1(K_U;\mathbb{F}_2), z' \in Z_1(K_V;\mathbb{F}_2)$ of length 3. Localizing, this is equivalent to $\langle c_z + \mathbb{1}, z'\rangle = 0$ for all cycles $z \in Z_1(K_U;\mathbb{F}_2), z' \in Z_1(K_V;\mathbb{F}_2)$ of length 3. This amounts to show that c_z is the complement of a cut/coboundary in K_V for every cycle $z \in Z_1(K_U;\mathbb{F}_2)$ of length 3. In Figure 6.1 we depict the support of c_z in blue for every such cycle z. We see that c_z is indeed the complement of a cut. ☐

As mentioned for $k \geq 2$ we use a blow-up construction. Given a simplicial complex X and $m \in \mathbb{Z}_{>0}$ we write $\widehat{X}^{(m)}$ for the simplicial complex with vertex set $X(0) \times [m]$ and k-simplices $\{(v_0, i_0), \dots, (v_k, i_k)\}$ for $\{v_0, \dots, v_k\} \in X(k)$ and $i_0, \dots, i_k \in [m]$. Note that there is a projection $\pi\colon \widehat{X}^{(m)} \to X$ given by $\{(v_0, i_0), \dots, (v_k, i_k)\} \mapsto \{v_0, \dots, v_k\}$. Given $c \in C^k(X;\mathbb{F}_2)$ we let $\widehat{c}^{(m)} := \pi^* c$. Note that $\widehat{K_n}^{(m)} \cong (\Lambda_m^{n-1})^{(1)} \subseteq K_{mn}$.

Given simplicial complexes $X \subseteq Y$ and $c \in C^k(X;\mathbb{F}_2)$ we write $\bar{c}^Y$ for the extension of c to Y by 0. Usually Y is understood from the context and we will write $\bar{c}$ instead of $\bar{c}^Y$. The following fact was shown in [84, Theorem 6.3]:

Lemma 6.12. *Let $m, n \in \mathbb{Z}_{>0}$. Let $X := \widehat{K_n}^{(m)} \subseteq Y := K_{mn}$. If $c \in C^1(K_n;\mathbb{F}_2)$ is minimal then $\overline{\widehat{c}^{(m)}} \in C^1(Y;\mathbb{F}_2)$ is minimal.*

With these preparations we can finish the proof of Proposition 6.10.

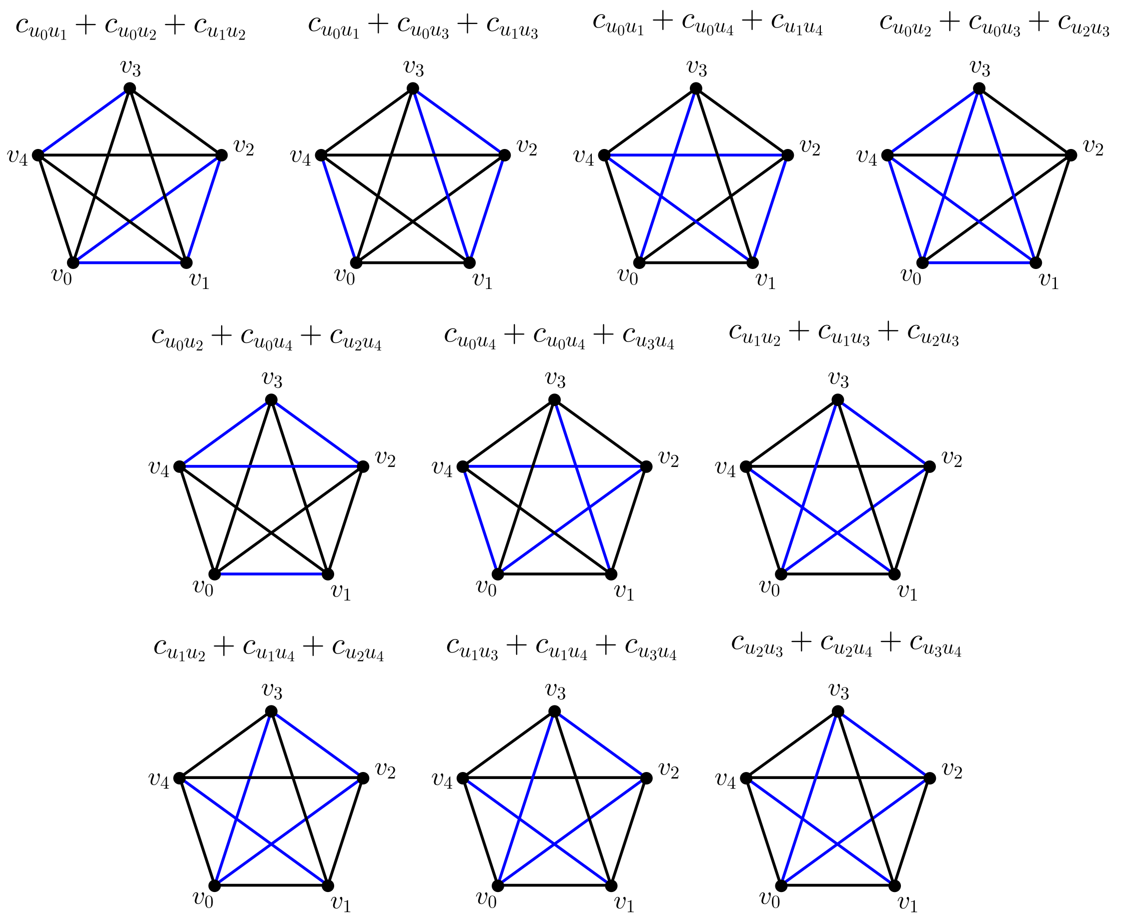


Figure 6.1: For every triple of pairwise distinct vertices $x, y, z \in U$ we show the support of $c_{xy} + c_{xz} + c_{yz}$ in blue. We see that all these cochains are complements of cuts.

Proof of Proposition 6.10. Fix $m \in \mathbb{Z}_{>0}$. Let $X = \widehat{K_5 * K_5}^{(m)} \cong \widehat{K_5}^{(m)} * \widehat{K_5}^{(m)}$ which we think of as a subcomplex of $Y = K_{5m} * K_{5m}$. Let $c \in C^1(K_5; \mathbb{F}_2)$ be a minimal representative of $[\mathbb{1}]$. By Lemma 6.12 $c_m := \overline{\widehat{c}^{(m)}} \in C^1(K_{5m}; \mathbb{F}_2)$ is minimal. Note that $|c_m| = m^2|c|$. Let $a \in C^3(K_5^{*2}; \mathbb{F}_2)$ such that $\gamma := c \otimes c + \delta a$ is minimal. Note that $\delta_Y \overline{\widehat{a}^{(m)}} = \overline{\delta_X \widehat{a}^{(m)}}$. Indeed, every $\sigma \in Y(3) \setminus X(3)$ has all, two or none of its boundary triangles in X. If it has two of its boundary triangle in X, $\widehat{a}^{(m)}$ has the same value on both of them. It follows that $\overline{\widehat{\gamma}^{(m)}} = c_m \otimes c_m + \delta_Y \overline{\widehat{a}^{(m)}}$. We conclude

$$|[c_m \otimes c_m]| \leq |\overline{\widehat{\gamma}^{(m)}}| = m^4|\gamma| < m^4|c|^2 = |c_m|^2,$$

as desired. □

It would be interesting to strengthen the above construction by giving an affirmative answer to the following question: Are there infinite families of simplicial complexes $(X_n)_{n\in\mathbb{N}}, (Y_n)_{n\in\mathbb{N}}$ and cochains $c_n \in C^i(X_n; \mathbb{F}_2)$, $c'_n \in C^j(Y_n; \mathbb{F}_2)$ for which

$$\lim_{n\to+\infty} \frac{|[c_n \otimes c'_n]|}{|[c_n]||[c'_n]|} = 0?$$

6.3 Joining Random Abstract Cones

In this section we give a join construction for random abstract cones. We will illustrate the construction by proving a lower bound on $\eta_k(\Lambda_n^d)$.

6.3.1 Joining Abstract Cones

Let X be a d_X-dimensional, Y a d_Y-dimensional simplicial complex. Let $J = X * Y$ be the join of X and Y. Let $\mathbb{A}$ be an abelian group. Let $(S_k^X)_{-2\leq k\leq d_X-1}$ be an abstract cone for X and $(S_k^Y)_{-2\leq k\leq d_Y-1}$ an abstract cone for Y. For $-2 \leq k \leq d_X + d_Y$ define $S_k^J\colon C_k(J;\mathbb{A}) \to C_{k+1}(J;\mathbb{A})$ by

$$\sigma \otimes \tau \mapsto \begin{cases} S^X_{\dim\sigma}\sigma \otimes \tau, & \text{if } \dim\sigma < d_X, \\ (-1)^{|\sigma|}(\sigma - S^X_{\dim\sigma-1}\partial\sigma) \otimes S^Y_{\dim\tau}\tau & \text{if } \dim\sigma = d_X. \end{cases}$$

We call $(S_k^J)_{-2\leq k\leq d_X+d_Y}$ the *join of* $(S_k^X)_{-2\leq k\leq d_X-1}$ *and* $(S_k^Y)_{-2\leq k\leq d_X-1}$. As the reader might have expected, we have

Lemma 6.13. $(S_k^J)_{-2\leq k\leq d_X+d_Y}$ *is an abstract cone for* J.

Proof. The proof is a straightforward computation. Notice that it is enough to show that for basic simplices $\sigma\otimes\tau \in J(k)$ with $\sigma \in X, \tau \in Y$ we have

$$\partial S_k^J(\sigma\otimes\tau) + S_{k-1}^J\partial(\sigma\otimes\tau) = \sigma\otimes\tau.$$

Throughout the proof we will frequently use that for all $\sigma \in X, \tau \in Y$,

$$\partial(\sigma\otimes\tau) = \partial\sigma\otimes\tau + (-1)^{|\sigma|}\sigma\otimes\partial\tau.$$

Let $\sigma \in X(i)$ for some $-1 \leq i \leq d_X$ and $\tau \in Y(j)$ for some $-1 \leq j \leq d_Y$. Let $k = i + j + 1 = \dim(\sigma\otimes\tau)$. We distinguish two cases. First assume that $i = \dim\sigma < d_X$. Then we compute

$$\begin{aligned} \partial S_k^J(\sigma\otimes\tau) + S_{k-1}^J\partial(\sigma\otimes\tau) &= \partial(S_i^X\sigma\otimes\tau) + S_{k-1}^J(\partial\sigma\otimes\tau + (-1)^{|\sigma|}\sigma\otimes\partial\tau) \\ &= \partial S_i^X\sigma\otimes\tau + (-1)^{|\sigma|+1}S_i^X\sigma\otimes\partial\tau \\ &+ S_{i-1}^X\partial\sigma\otimes\tau + (-1)^{|\sigma|}S_i^X\sigma\otimes\partial\tau \\ &= (\partial S_i^X\sigma + S_{i-1}^X\partial\sigma)\otimes\tau \\ &= \sigma\otimes\tau, \end{aligned}$$

where we used that $(S_k^X)_{-2\leq k\leq d_X-1}$ is an abstract cone for the last equality.

Now assume that $\dim\sigma = d_X$. We have

$$\begin{aligned} \partial S_k^J(\sigma\otimes\tau) &= \partial\left((-1)^{|\sigma|}(\sigma - S_{i-1}^X\partial\sigma)\otimes S_j^Y\tau\right) \\ &= (-1)^{|\sigma|}\partial\sigma\otimes S_j^Y\tau + \sigma\otimes\partial S_j^Y\tau \\ &+ (-1)^{|\sigma|+1}\partial S_{i-1}^X\partial\sigma\otimes S_j^Y\tau - S_{j-1}^X\partial\sigma\otimes\partial S_j^Y\tau \\ &= (\sigma - S_{i-1}^X\partial\sigma)\otimes\partial S_j^Y\tau, \end{aligned}$$

where we used that $\partial S_{i-1}^X\partial\sigma = \partial\sigma$ since $(S_k^X)_{-2\leq k\leq d_X-1}$ is an abstract cone for X.

Furthermore,

$$\begin{aligned} S^J_{k-1}\partial(\sigma \otimes \tau) &= S^J_{k-1}(\partial\sigma \otimes \tau) + (-1)^{|\sigma|} S^J_{k-1}(\sigma \otimes \partial\tau) \\ &= S^X_{i-1}\partial\sigma \otimes \tau + (\sigma - S^X_{i-1}\partial\sigma) \otimes S^Y_{j-1}\partial\tau. \end{aligned}$$

Combinining these two, we deduce

$$\begin{aligned} \partial S^J_k(\sigma \otimes \tau) + S^J_{k-1}\partial(\sigma \otimes \tau) &= \left(\sigma - S^X_{i-1}\partial\sigma\right) \otimes \left(\partial S^Y_j \tau + S^Y_{j-1}\partial\tau\right) + S^X_{i-1}\partial\sigma \otimes \tau \\ &= (\sigma - S^X_{i-1}\partial\sigma) \otimes \tau + S^X_{i-1}\partial\sigma \otimes \tau \\ &= \sigma \otimes \tau, \end{aligned}$$

where we used that $(S^Y_k)_{-2 \le k \le d_Y - 1}$ is an abstract cone for Y. □

6.3.2 Analysis for Join of Random Abstract Cones

Using the join construction for abstract cones, we can join random abstract cones for two simplicial complexes X and Y to obtain a random abstract cone for the join $X * Y$. Using Proposition 3.13, this gives a way to bound the coboundary expansion constants of $X * Y$ in terms of the lower bounds of the coboundary expansion constants of X and Y obtained through the random abstract cones.

For completeness, we give the technical details: Let X and Y be simplicial complexes of dimension d_X and d_Y, respectively. Let $w_X \colon X \to \mathbb{R}_{>0}$ and $w_Y \colon Y \to \mathbb{R}_{>0}$ be positive weight functions. Furthermore, we assume that there is a constant $C_X > 0$ such that for all $\sigma \in X(d_X - 1)$ we have

$$\sum_{\tau \in X(d_X), \sigma \subseteq \tau} w_X(\tau) \le C_X w_X(\sigma).$$

Note that if w_X are the Garland weights, we can choose $C_X = d_X + 1$ and the inequality becomes an equality.

Let $J = X * Y$ be the join of X and Y. Let $d_J = \dim J$. We endow J with the weight function w_J given by $w_J(\sigma \otimes \tau) = c_{i,j} w_X(\sigma) w_Y(\tau)$ for all $\sigma \in X(i)$ and $\tau \in Y(j)$ where for $-1 \le i \le d_X$ and $-1 \le j \le d_Y$ we let

$$c_{i,j} := \frac{\binom{d_X+1}{i+1}\binom{d_Y+1}{j+1}}{\binom{d_X+d_Y+2}{i+j+2}}.$$

Note that w_J are the Garland weights if w_X and w_Y are the Garland weights. This motivates the somewhat cumbersome normalizing factor $c_{i,j}$.

Let Z be a simplicial complex endowed with a weight function $w_Z \colon Z \to \mathbb{R}_{\ge 0}$. Let $f \colon Z(j) \to \mathbb{R}$ be a function. Eventhough w_Z might not induce a probability measure on $Z(j)$, it will be convenient to write $\mathbb{E}_{\sigma \in Z(j)} f(\sigma)$ for $\sum_{\sigma \in Z(j)} w_Z(\sigma) f(\sigma)$.

Let $\mathbb{A}$ be a ring with 1 endowed with a norm $|\cdot|_{\mathbb{A}}$. We write $|\cdot|$ for the size function on cochain groups induced by $|\cdot|_{\mathbb{A}}$ and a fixed weight function on a simplicial complex.

Let $(\Omega_X, \mathcal{B}_X, \mu_X)$ and $(\Omega_Y, \mathcal{B}_Y, \mu_Y)$ be two finite probability spaces such that there are random abstract cones $\mathcal{S}_X = (S^{(\omega)}_{X,k})_{-2 \le k \le d_X - 1, \omega \in \Omega_X}$ for X and $\mathcal{S}_Y = (S^{(\omega)}_{Y,k})_{-2 \le k \le d_Y - 1, \omega \in \Omega_Y}$ for Y.

Write $(\Omega_J, \mathcal{B}_J, \mu_J)$ for the product probability space of $(\Omega_X, \mathcal{B}_X, \mu_X)$ and $(\Omega_Y, \mathcal{B}_Y, \mu_Y)$. For $\omega = (\omega', \omega'') \in \Omega_J = \Omega_X \times \Omega_Y$ the join of the abstract cones $(S^{(\omega')}_{X,k})_{-2\le k\le d_X-1}$ and $(S^{(\omega'')}_{Y,k})_{-2\le k\le d_Y-1}$ defines an abstract cone $(S^{(\omega)}_{J,k})_{-2\le k\le d_J-1}$ for J. Write $\mathcal{S}_J = (S^{(\omega)}_{J,k})_{-2\le k\le d_J-1,\omega\in\Omega_J}$ for the resulting random abstract cone.

Write $\mathcal{T}_X = (T^{(\omega)}_{X,k})_{-1\le k\le d_X,\omega\in\Omega_X}, \mathcal{T}_Y = (T^{(\omega)}_{Y,k})_{-1\le k\le d_Y,\omega\in\Omega_Y}$ and $\mathcal{T}_J = (T^{(\omega)}_{J,k})_{-1\le k\le d_J,\omega\in\Omega_J}$ for random abstract cofillings dual to $\mathcal{S}_X$, $\mathcal{S}_Y$ and $\mathcal{S}_J$, respectively.

For $-1 \le i \le d_X - 1$ let

$$\lambda_i(X) := \max_{\tau\in X(i+1)} \frac{1}{w_X(\tau)} \mathbb{E}_{\omega\sim\Omega_X} |T^{(\omega)}_{X,i+1} \mathbb{1}_\tau|$$

and for $-1 \le j \le d_Y - 1$ let

$$\lambda_j(Y) := \max_{\tau\in Y(j+1)} \frac{1}{w_Y(\tau)} \mathbb{E}_{\omega\sim\Omega_Y} |T^{(\omega)}_{Y,j+1} \mathbb{1}_\tau|.$$

We extend this definition and set $\lambda_{-2}(X) := 0, \lambda_{-2}(Y) := 0$. For $0 \le i \le d_X, 0 \le j \le d_Y$ let

$$\lambda_{i,j}(J) = \begin{cases} \frac{(i+1)(d_X+d_Y-i-j+1)}{(d_X-i+1)(i+j+2)}\lambda_{i-1}(X) & \text{if } i < d_X \\ \frac{(d_X+1)(d_Y-j+1)}{d_X+j+2}\lambda_{i-1}(X) + \frac{j+1}{d_X+j+2}(\lambda_{j-1}(Y) + C_X\lambda_{i-1}(X)\lambda_{j-1}(Y)) & \text{if } i = d_X \end{cases}.$$

We have:

Proposition 6.14. *With the assumptions and notation above, we have for* $0 \le k \le d_J - 1$ *that*

$$\eta_k(J) \ge \frac{1}{\lambda_k(J)},$$

where

$$\lambda_k(J) := \max_{i+j=k} \lambda_{i,j}(J).$$

Proof. By Proposition 3.13 it suffices to show that for all $\sigma\otimes\tau \in J(k)$, $0 \le k \le d_J - 1$, we have

$$\lambda(\sigma\otimes\tau) := \frac{1}{w_J(\sigma\otimes\tau)} \mathbb{E}_{\omega\sim\mu_J} |T^{(\omega)}_{J,k+1} \mathbb{1}_{\sigma\otimes\tau}| \le \lambda_k(J).$$

To this end, fix $0 \le k \le d_J - 1$ and $\sigma\otimes\tau \in J(k)$ with $\dim\sigma = i, \dim\tau = j$ $(i+j = k-1)$. We distinguish the two cases (i) $\dim\sigma < d_X$ and (ii) $\dim\sigma = d_X$.

For (i) we first note that for $\rho \in J(k)$

$$\langle T^{(\omega)}_{J,k+1}\mathbb{1}_{\sigma\otimes\tau}, \rho\rangle = \langle \mathbb{1}_{\sigma\otimes\tau}, S^{(\omega)}_{J,k}\rho\rangle$$

can only be non-zero if $\rho = \sigma'\otimes\tau'$ with $\sigma' \in X(i-1)$. It follows that

$$\begin{aligned}
\lambda(\sigma\otimes\tau) &= \frac{1}{w_J(\sigma\otimes\tau)} \mathbb{E}_{\omega\sim\mu_j} \mathbb{E}_{\sigma\sim J(k)} |\langle \mathbb{1}_{\sigma\otimes\tau}, S^{(\omega)}_{J,k+1}\mathbb{1}_\rho\rangle|_{\mathbb{A}} \\
&= \frac{c_{i-1,j}}{c_{i,j} w_X(\sigma) w_Y(\tau)} \mathbb{E}_{(\omega',\omega'')\sim\mu_J} \mathbb{E}_{\sigma'\sim X(i-1)} \mathbb{E}_{\tau'\sim Y(j)} |\langle \mathbb{1}_\sigma, S^{(\omega')}_{X,i-1}\sigma'\rangle \cdot \langle \mathbb{1}_\tau, \tau'\rangle|_{\mathbb{A}} \\
&\le \frac{c_{i-1,j}}{c_{i,j} w_X(\sigma)} \mathbb{E}_{\omega'\sim\mu_X} \mathbb{E}_{\sigma'\sim X(i-1)} |\langle \mathbb{1}_\sigma, S^{(\omega')}_{X,i-1}\sigma'\rangle|_{\mathbb{A}} \\
&\le \frac{c_{i-1,j}}{c_{i,j}} \lambda_{i-1}(X) \\
&= \frac{(i+1)(d_J-i-j)}{(d_X-i+1)(i+j+2)} \lambda_{i-1}(X).
\end{aligned}$$

For the second case (ii), $\dim\sigma = d_X$, we note that $\rho = \sigma' \otimes \tau' \in J(k)$ can only be in the support of $T^{(\omega)}_{J,k+1}\mathbb{1}_{\sigma\otimes\tau}$ if $\dim\sigma' = i-1$ and $\dim\tau' = j$ or if $\dim\sigma' = i$ and $\dim\tau' = j-1$. By the previous estimate, those simplices ρ with $\dim\sigma' = i-1$ contribute at most

$$\frac{(d_X+1)(d_Y-j+1)}{d_X+j+2}\lambda_{d_X-1}(X)$$

to $\lambda(\sigma\otimes\tau)$.

For the second type of simplices ρ (those with $\dim\sigma' = d_X$) we estimate using the triangle inequality of $|\cdot|_{\mathbb{A}}$ that

$$\begin{aligned}
&\frac{c_{i,j-1}}{w_J(\sigma\otimes\tau)}\mathbb{E}_{\substack{\sigma'\sim X(d_X)\\ \tau'\sim Y(j-1)}}\mathbb{E}_{(\omega',\omega'')\sim\mu_J}|\langle\mathbb{1}_{\sigma\otimes\tau},(-1)^{d_X+1}(\sigma'-S^{(\omega')}_{X,d_X-1}\partial\sigma')\otimes S^{(\omega'')}_{Y,j-1}\tau'\rangle|_{\mathbb{A}}\\
&\le\frac{c_{i,j-1}}{c_{i,j}w_X(\sigma)w_Y(\tau)}\mathbb{E}_{\substack{\sigma'\sim X(d_X)\\ \tau'\sim Y(j-1)}}\mathbb{E}_{(\omega',\omega'')\sim\mu_J}|\langle\mathbb{1}_{\sigma\otimes\tau},\sigma'\otimes S^{(\omega'')}_{Y,j-1}\tau'\rangle|_{\mathbb{A}}\\
&+\frac{c_{i,j-1}}{c_{i,j}w_X(\sigma)w_Y(\tau)}\mathbb{E}_{\substack{\sigma'\sim X(d_X)\\ \tau'\sim Y(j-1)}}\sum_{\alpha\in\partial\sigma'}\mathbb{E}_{(\omega',\omega'')\sim\mu_J}|\langle\mathbb{1}_{\sigma},S^{(\omega')}_{X,d_X-1}\alpha\rangle|_{\mathbb{A}}\cdot|\langle\mathbb{1}_{\tau},S^{(\omega'')}_{Y,j-1}\tau'\rangle|_{\mathbb{A}}.
\end{aligned}$$

The first summand is at most

$$\frac{c_{i,j-1}}{c_{i,j}}\lambda_{j-1}(Y) = \frac{j+1}{d_X+j+2}\lambda_{j-1}(Y),$$

which follows from the estimates for the case (i) interchanging the roles of X and Y.

Using that for $\alpha\in X(d_X-1)$ we have

$$\sum_{\beta\in X(d_X),\alpha\subseteq\beta} w_X(\beta)\le C_X w_X(\alpha),$$

we see that the second summand is at most

$$\begin{aligned}
&\frac{c_{i,j-1}}{c_{i,j}}\left(\frac{C_X}{w_X(\sigma)}\mathbb{E}_{\substack{\alpha\sim X(d_X-1)\\ \omega'\sim\mu_X}}|\langle\mathbb{1}_\sigma,S^{(\omega')}_{X,d_X-1}\alpha\rangle|_{\mathbb{A}}\right)\left(\frac{1}{w_Y(\tau)}\mathbb{E}_{\substack{\tau'\sim Y(j-1)\\ \omega''\sim\mu_Y}}|\langle\mathbb{1}_\tau,S^{(\omega'')}_{Y,j-1}\tau'\rangle|_{\mathbb{A}}\right)\\
&\le\frac{c_{i,j-1}}{c_{i,j}}C_X\lambda_{i-1}(X)\lambda_{j-1}(Y)\\
&=\frac{j+1}{d_X+j+2}\lambda_{i-1}(X)\lambda_{j-1}(Y).
\end{aligned}$$

In total, we conclude that

$$\lambda(\sigma\otimes\tau)\le\frac{(d_X+1)(d_Y-j+1)}{d_X+j+2}\lambda_{d_X-1}(X)+\frac{j+1}{d_X+j+2}\left(\lambda_{j-1}(Y)+\lambda_{i-1}(X)\lambda_{j-1}(Y)\right).$$

Comparing the obtained upper bounds on $\lambda(\sigma\otimes\tau)$ with the definition of $\lambda_k(J)$ and $\lambda_{i,j}(J)$, we see that this finishes the proof. □

We close this section by elaborating on how the second part of Proposition 3.13 is useful for the join construction as well. This part of Proposition 3.13 gives a lower bound on coboundary expansion constants in terms of the size of an abstract cone under the additional assumption that there is a group of automorphisms acting transitively on the top-dimensional faces.

Recall that given a d_Z-dimensional simplicial complex Z and an abstract cone $\mathcal{S}_Z = (S_k)_{-2\leq k\leq d_Z-1}$ for Z, the size $\mathrm{size}_k(\mathcal{S}_Z)$, $-1 \leq k \leq d_Z - 1$, is defined as

$$\mathrm{size}_k(\mathcal{S}_Z) = \max_{\sigma\in Z(k)} \sum_{\tau\in X(k+1)} |\langle \mathbb{1}_\tau, S_k\sigma\rangle|_{\mathbb{A}}.$$

Now, let X and Y be simplicial complexes of dimension d_X and d_Y, respectively. Assume that G is a group acting by automorphisms on X, H is a group acting by automorphisms on Y. Then, $G \times H$ acts on $J = X * Y$ by

$$\sigma \otimes \tau \mapsto (g,h).(\sigma\otimes\tau) := (g.\sigma) \otimes (h.\tau)$$

for every $\sigma\otimes\tau \in J, (g,h) \in G\times H$. Note that if the action of G is transitive on $X(d_X)$ and the action of H is transitive on $Y(d_Y)$, then $G \times H$ acts transtively on $J(\dim J)$.

Thus, we could use the second part of Proposition 3.13 to give a lower bound on coboundary expansion constants of J. Fortunately, given an abstract cone $\mathcal{S}_X = (S_k^X)_{-2\leq k\leq d_X-1}$ for X and an abstract cone $\mathcal{S}_Y = (S_k^Y)_{-2\leq k\leq d_Y-1}$ for Y, it is straightforward to give an upper bound on the size of the join $\mathcal{S}_J = (S_k^J)_{-2,\leq k\leq d_X+d_Y}$ of $\mathcal{S}_X$ and $\mathcal{S}_Y$ in terms of the size of $\mathcal{S}_X$ and $\mathcal{S}_Y$. We have

Lemma 6.15. *Let X be a d_X-dimensional simplicial complex, Y be a d_Y-dimensional simplicial complex. Let $J = X * Y$. Let $\mathcal{S}_X = (S_k^X)_{-2\leq k\leq d_X-1}$ be an abstract cone for X and $\mathcal{S}_Y = (S_k^Y)_{-2\leq k\leq d_Y-1}$ be an abstract cone for Y. Let $\mathcal{S}_J = (S_k^J)_{-2,\leq k\leq d_X+d_Y}$ be the join of $\mathcal{S}_X$ and $\mathcal{S}_Y$. Let $-1 \leq k \leq d_X + d_Y$. If $k < d_X$ then*

$$\mathrm{Size}_k(\mathcal{S}_J) \leq \max_{-1\leq i\leq k} \mathrm{Size}_i(\mathcal{S}_X).$$

If $k \geq d_X$ then

$$\mathrm{Size}_k(\mathcal{S}_J) \leq \max\{\max_{k-d_Y-1\leq i\leq d_X-1} \mathrm{Size}_i(\mathcal{S}_X), \mathrm{Size}_{k-1-d_X}(\mathcal{S}_Y)(1+(d_X+1))\mathrm{Size}_{d_X-1}(\mathcal{S}_X)\}.$$

Proof. The proof is straightforward from the definitions. □

6.3.3 A Lower Bound on $\eta_k(\Lambda_n^d)$ via Joining Abstract Cones

Using the join construction of random abstract cones we will show that

Proposition 6.16. *Let $n \in \mathbb{N}$. Let $d \geq 1$ be a dimension. Let $0 \leq k \leq d-1$. Then*

$$\eta_k(\Lambda_n^d) \geq \frac{k+2}{\left((d-k)\sum_{i=0}^{k-1} 2^i \left(\frac{n-1}{n}\right)^i \frac{\binom{k+2}{i+1}}{\binom{d+1}{i+1}} + 2^{k+1}\left(\frac{n-1}{n}\right)^k \frac{\binom{k+2}{k}}{\binom{d+1}{k}}\right)}.$$

In particular,

$$\eta_0(\Lambda_n^d) \geq 1$$

and

$$\eta_{d-1}(\Lambda_n^d) \geq \frac{d+1}{3\cdot 2^{d-1}-1}$$

for all $d \geq 1$.

A slightly weaker lower bound as stated in Proposition 6.16 appeared in [98, Theorem 3.3] and for the special case $k = d - 1$ also in [35, Proposition 5.7]. It turns out that both arguments can lead to the bound stated here by a bit more careful analysis. Both arguments use the averaging trick. In [35] an ad-hoc construction of many different cofillings is given and combined with an induction on d for fixed codimension $d - k$. The argument in [98] uses a random abstract cone which can be seen as an iterated join of abstract cones $(S_{-1}^{(u)})_{u\in[n]}$ on $[n]$ given by $S_{-1}^{(u)}\emptyset = u$ for each copy of $[n]$ in $\Lambda_n^d = [n]^{*(d+1)}$.

We use a blend of the two arguments: The random abstract cone obtained as a join of abstract cones on $[n]$ together with an induction on d for fixed codimension $d - k$.

We start with the following lemma.

Lemma 6.17. *Let $d \geq 2$ be a dimension. Let $1 \leq k \leq d-1$. Assume there is a probability space $(\Omega, \mathcal{B}, \mu)$ and a random abstract cone $(S_{k-1}^{(\omega)}, S_{k-2}^{(\omega)})_{\omega\in\Omega}$ in dimension $k-1$ for Λ_n^{d-1} with dual random abstract cofilling $(T_k^{(\omega)}, T_{k-1}^{(\omega)})$ such that for all $c \in C^{k-1}(\Lambda_n^{d-1};\mathbb{F}_2)$ we have*

$$\mathbb{E}_{\omega\sim\mu}|T_k^{(\omega)}\delta c| \leq L_{k-1,d-1}(n)|\delta c|$$

for some positive constant $L_{k-1,d-1}(n) > 0$.

Then there is a probability space $(\tilde{\Omega}, \tilde{\mathcal{B}}, \tilde{\mu})$ and a random abstract cone $(S_k^{(\omega)}, S_{k-1}^{(\omega)})_{\omega\in\tilde{\Omega}}$ in dimension k for Λ_n^d with dual random abstract cofilling $(T_{k+1}^{(\omega)}, T_k^{(\omega)})$ such that for all $c \in C^k(\Lambda_n^d;\mathbb{F}_2)$ we have

$$\mathbb{E}_{\omega\sim\tilde{\mu}}|T_{k+1}^{(\omega)}\delta c| \leq L_{k,d}(n)|\delta c|$$

where

$$L_{k,d}(n) = \frac{1}{n}\frac{k+2}{d+1}(1 + 2(n-1)L_{k-1,d-1}(n)).$$

Proof. Write Λ_n^d as $\Lambda_n^d = U_0 * \cdots * U_d$ with $U_i = [n]$ for $0 \leq i \leq d$. Let $U = U_0$ and $Y = U_1 * \cdots * U_d$.

We think of the random abstract cone $(S_{k-1}^{(\omega)}, S_{k-2}^{(\omega)})_{\omega\in\Omega}$ in dimension $k-1$ for Λ_n^{d-1} as being defined on Y.

For $u \in U$ let $S_{-1}^{(u)} \colon C_{-1}(U;\mathbb{F}_2) \to C_0(U;\mathbb{F}_2)$ be given by $\emptyset \mapsto S_{-1}^{(u)}\emptyset := u$. Endow U with the uniform distribution ν.

Note that the data $(S_{-1}^{(u)})_{u\in U}$ and $(S_{k-1}^{(\omega)}, S_{k-2}^{(\omega)})_{\omega\in\Omega}$ suffices to use the join construction for abstract cones to define a random abstract cone in dimension k for Λ_n^d. More precisely, let $\tilde{\Omega} = U \times \Omega$ endowed with the product measure $\tilde{\mu} = \nu \otimes \mu$. Let $\tilde{\omega} = (u, \omega) \in \tilde{\Omega}$. For $j \in \{k-1, k\}$ let

$$\begin{aligned} S_j^{(\tilde{\omega})} \colon C_j(\Lambda_n^d;\mathbb{F}_2) &\to C_{j+1}(\Lambda_n^d;\mathbb{F}_2) \\ \sigma &\mapsto \begin{cases} u \otimes \sigma, & \text{if } \sigma \cap U = \emptyset, \\ (u + u') \otimes S_{j-1}^{(\omega)}(\sigma \setminus \{u'\}) & \text{if } \sigma \cap U = \{u'\}. \end{cases} \end{aligned}$$

By Lemma 6.13 $(S_k^{(\tilde{\omega})}, S_{k-1}^{(\tilde{\omega})})_{\tilde{\omega}\in\tilde{\Omega}}$ is a random abstract cone in dimension k for Λ_n^d. Write $(T_{k+1}^{(\tilde{\omega})}, T_k^{(\tilde{\omega})})_{\tilde{\omega}\in\tilde{\Omega}}$ for the dual random abstract cofilling.

Given $c \in C^k(\Lambda_n^d; \mathbb{F}_2)$, let $b = \delta c$. We estimate

$$\begin{aligned}
\mathbb{E}_{\tilde\omega\sim\tilde\mu}|T_{k+1}^{(\tilde\omega)}b| &= \frac{1}{n}\sum_{u\in U}\sum_{\omega\in\Omega}\mu(\{\omega\})\sum_{\sigma\in\Lambda_n^d(k)}|\langle T_{k+1}^{(\tilde\omega)}b,\sigma\rangle| \\
&= \frac{1}{n}\sum_{u\in U}\sum_{\omega\in\Omega}\mu(\{\omega\})\sum_{\sigma\in\Lambda_n^d(k)}|\langle b, S_k^{(\tilde\omega)}\sigma\rangle| \\
&= \frac{1}{n}\sum_{u\in U}\sum_{\omega\in\Omega}\mu(\{\omega\})\sum_{\sigma\in\Lambda_n^d(k),\sigma\cap U=\emptyset}|\langle b, u\otimes\sigma\rangle| \\
&+ \frac{1}{n}\sum_{u\in U}\sum_{\omega\in\Omega}\mu(\{\omega\})\sum_{u'\in U}\sum_{\sigma'\in Y(k-2)}|\langle b, (u+u')\otimes S_{k-2}^{(\omega)}\sigma'\rangle| \\
&= \frac{1}{n}|b_{|_{C^0(U;\mathbb{F}_2)\otimes C^{k-1}(Y;\mathbb{F}_2)}}| + \frac{2}{n}\sum_{uu'\in\binom{U}{2}}\mathbb{E}_{\omega\sim\mu}|T_{k-1}^{(\omega)}(b_u+b_{u'})| \\
&\le \frac{1}{n}|b_{|_{C^0(U;\mathbb{F}_2)\otimes C^{k-1}(Y;\mathbb{F}_2)}}| + \frac{2}{n}\sum_{uu'\in\binom{U}{2}}L_{k-1,d-1}(n)|b_u+b_{u'}| \\
&\le \frac{1}{n}(1+2(n-1)L_{k-1,d-1}(n))|b_{|_{C^0(U;\mathbb{F}_2)\otimes C^{k-1}(Y;\mathbb{F}_2)}}|.
\end{aligned}$$

Note that (by double counting)

$$\sum_{i=0}^{d}\sum_{u\in U_i}|b_u| = (k+2)|b|,$$

Thus, if we additionally average over the choice (U,Y) with $U = U_i$, $Y = (\Lambda_n^d)_u$ for some $u \in U_i$ over $i \in \{0,\dots,d\}$, we obtain the bound as claimed. □

We are ready to give a proof of Proposition 6.16.

Proof of Proposition 6.16. Note that the Garland weights on Λ_n^d are uniform. Thus,

$$\eta_k(\Lambda_n^d) = \frac{|\Lambda_n^d(k)|}{|\Lambda_n^d(k+1)|}h_k(\Lambda_n^d) = \frac{1}{n}\frac{k+2}{d-k}h_k(\Lambda_n^d).$$

For integers k, d with $0 \le k \le d-1$ recursively define $L_{k,d}(n)$ by

$$L_{0,d}(n) := \frac{2}{dn}\text{for all } d \ge 1$$

and

$$L_{k,d}(n) := \frac{1}{n}\frac{k+2}{d+1}(1+2(n-1)L_{k-1,d-1}(n)).$$

Combining Lemma 6.17 with Proposition 3.8 we deduce that

$$h_k(\Lambda_n^d) \ge \frac{1}{L_{k,d}(n)}$$

or equivalently

$$\eta_k(\Lambda_n^d) \ge \frac{1}{nL_{k,d}(n)}\frac{k+2}{d-k}.$$

Solving the recursion for $L_{k,d}(n)$ leads to the lower bound on $\eta_k(\Lambda_n^d)$ as claimed in the proposition. We omit this step here and leave it to the reader as a straightforward exercise. □

Chapter 7

Upper Bounds on Expansion Constants of Partite Complexes

In this chapter, we prove various upper bounds on coboundary expansion constants of $(d+1)$-partite d-dimensional complexes. At the heart of the proofs is an explicit construction (which we give in the next section) of exponentially many d-coboundaries in $\Lambda^d_{n_1,\dots,n_d}$ with some additional algebraic structure (closely related to sum complexes as studied in [88]). Using a probabilistic argument we make use of these coboundaries to prove Theorem 1.6 in Section 7.2. In Section 7.3 we prove Theorem 1.5 as well as more refined upper bounds on $\eta_1(\Lambda^2_n)$. Furthermore, using a product construction, we can also get upper bounds on $\eta_k(\Lambda^d_n)$ for $0 < k < d-1$ which for constant codimension $d-k$ are exponentially small in d (see Proposition 7.8). Part of the results in this chapter are already discussed in [140].

7.1 A Wealth of Coboundaries

The following proposition provides us with a wealth of coboundaries.

Proposition 7.1. *Let $d \in \mathbb{Z}_{>0}$ be a dimension. Let $n_0, n_1, \dots, n_d \in \mathbb{Z}$ with $n_i \geq 2$ for all $0 \leq i \leq d$. Let $X = \Lambda^d_{n_0,\dots,n_d}$. Given $\varphi\colon X(0) \to \mathbb{F}_2^d$ define $c^\varphi \in C^d(X;\mathbb{F}_2)$ by*

$$c^\varphi(\{v_0,\dots,v_d\}) := \begin{cases} 1 & \text{if } \sum_{i=0}^d \varphi(v_i) = 0 \in \mathbb{F}_2^d, \\ 0 & \text{otherwise .} \end{cases}$$

Then c^φ is a coboundary, i.e. $c^\varphi \in B^d(X;\mathbb{F}_2)$.

For the proof of Proposition 7.1 we will make use of the characterization of coboundaries given in Lemma 2.1. In view of this lemma, the following basis for $Z_d(\Lambda^d_{n_0,\dots,n_d};\mathbb{F}_2)$ will be useful.

Lemma 7.2. *Let $\Lambda^d_{n_0,\dots,n_d} = U_0 * U_1 * \cdots * U_d$ with $|U_i| = n_i$. Given pairwise distinct vertices $u_i^+, u_i^- \in U_i$, $0 \leq i \leq d$, let $\lozenge^d((u_i^+,u_i^-)_{0\leq i\leq d}) := \{u_0^+,u_0^-\} * \cdots * \{u_d^+,u_d^-\}$ be the octahedral sphere spanned by the vertices $u_0^+,u_0^-,\dots,u_d^+,u_d^-$. We will think of $\lozenge^d((u_i^+,u_i^-)_{0\leq i\leq d})(d)$ as a chain in $C_d(\Lambda^d_{n_0,\dots,n_d};\mathbb{F}_2)$. Then for any fixed $u_i^+ \in U_i, 0 \leq i \leq d$ the set*

$$\{\lozenge^d((u_i^+,u_i^-)_{0\leq i\leq d})(d) \in C_d(\Lambda^d_{n_0,\dots,n_d};\mathbb{F}_2) : u_i^- \in U_i \setminus \{u_i^+\}, 0 \leq i \leq d\}$$

is a basis for $Z_d(\Lambda^d_{n_0,\dots,n_d};\mathbb{F}_2)$.

Proof. Fix $u_i^+ \in U_i, 0 \le i \le d$ and let

$$\mathcal{Z} = \{\Diamond^d((u_i^+, u_i^-)_{0\le i\le d})(d) \in C_d(\Lambda^d_{n_0,\dots,n_d};\mathbb{F}_2) : u_i^- \in U_i \setminus \{u_i^+\}, 0 \le i \le d\}.$$

Clearly, every $z \in \mathcal{Z}$ is a cycle.

Note that for any choice of $u_i^- \in U_i \setminus \{u_i^+\}, 0 \le i \le d$, there is precisely one $z \in \mathcal{Z}$ which contains $\{u_0^-, \dots, u_d^-\}$ in its support. This implies that the cycles in $\mathcal{Z}$ are linearly independent.

Note that $|\mathcal{Z}| = \prod_{i=0}^d (n_i - 1)$.

On the other hand, since $\tilde{H}_k(\Lambda^d_{n_0,\dots,n_d};\mathbb{F}_2) = 0$ for all $-1 \le k \le d-1$, we get by the rank–nullity theorem that

$$\begin{aligned}\dim Z_d(\Lambda^d_{n_0,\dots,n_d};\mathbb{F}_2) &= \sum_{i=0}^{d+1}(-1)^i \dim C_{d-i}(\Lambda^d_{n_0,\dots,n_d};\mathbb{F}_2)\\ &= \sum_{i=0}^{d+1}(-1)^i \sum_{0\le i_0<\dots<i_{d-i}\le d}\prod_{l=0}^{d-i} n_{i_l}\\ &= \prod_{i=0}^{d}(n_i-1).\end{aligned}$$

Thus, $\mathcal{Z}$ generates all of $Z_d(\Lambda^d_{n_0,\dots,n_d};\mathbb{F}_2)$. □

We are ready to prove Proposition 7.1.

Proof of Proposition 7.1. Write $X = U_0 * \dots * U_d$ with $U_i = [n_i]$. By Lemma 2.1 and Lemma 7.2 it suffices to check that for every collection of pairs $\{u_i^+, u_i^-\} \in \binom{U_i}{2}, 0 \le i \le d$, the crosspolytope $\Diamond^d = \{u_0^+, u_0^-\} * \dots * \{u_d^+, u_d^-\}$ contains an even number of d-simplices from c^φ.

So let us fix a choice of pairs $\{u_i^+, u_i^-\} \in \binom{U_i}{2}$, $0 \le i \le d$, and consider the corresponding crosspolytope $\Diamond^d = \{u_0^+, u_0^-\} * \dots * \{u_d^+, u_d^-\}$. First we reduce to the case when

$$\varphi(u_0^+) = \varphi(u_1^+) = \dots = \varphi(u_d^+) = 0.$$

If $\Diamond^d$ does not contain a d-simplex from c^φ, we are done. Otherwise we can assume (after relabeling the vertices in $\Diamond^d$) that

$$\sum_{i=0}^{d} \varphi(u_i^+) = 0.$$

Now consider $\tilde{\varphi}\colon X(0) \to \mathbb{F}_2^d$ given by

$$\tilde{\varphi}(u_i) = \varphi(u_i) + \varphi(u_i^+)$$

for any $u_i \in U_i$, $0 \le i \le d$. Since $\sum_{i=0}^d \varphi(u_i^+) = 0$ we have $c^{\tilde{\varphi}} = c^\varphi$. Moreover $\tilde{\varphi}(u_i^+) = 0$ for all $0 \le i \le d$ by construction. So we are left with the case when $\varphi(u_i^+) = 0$ for all

$0 \le i \le d$. In this case, there is a one-to-one correspondence between d-simplices in $\Diamond^d$ from c^φ and vectors $(\alpha_0, \dots, \alpha_d) \in \mathbb{F}_2^{d+1}$ for which

$$\sum_{i=0}^{d} \alpha_i \varphi(u_i^-) = 0.$$

The number of such vectors equals $2^{\dim \ker A}$ where $A \in \mathbb{F}_2^{d\times(d+1)}$ is the matrix with columns $\varphi(u_0^-), \dots, \varphi(u_d^-)$. Note that $\dim \ker A \ge 1$ (we consider linear dependencies of $d+1$ vectors in the d-dimensional vector space $\mathbb{F}_2^d$), hence $2^{\dim \ker A}$ is even. This finishes the proof. □

7.2 Upper Bound for Spherical Building $A_d(\mathbb{F}_q)$

In this subsection we prove Theorem 1.6 which we restate here for easier reference.

Theorem. *For any dimension d and $\varepsilon > 0$ there is $Q = Q(d, \varepsilon) \in \mathbb{Z}_{>0}$ such that for all prime powers $q \ge Q$ we have*

$$\eta_{d-1}(A_d(\mathbb{F}_q)) \le \frac{d+1}{2^d} + \varepsilon.$$

Recall that for a prime power q and a dimension d, $A_d(\mathbb{F}_q)$ is the d-dimensional simplicial complex with vertex set the non-trivial, proper subspaces of $\mathbb{F}_q^{d+2}$ and k-simplices corresponding to chains of subspaces $\{0\} \ne U_0 \subsetneq U_1 \subsetneq \cdots \subsetneq U_k \subsetneq \mathbb{F}_q^{d+2}$.

Let us start by collecting a few very basic combinatorial properties of $A_d(\mathbb{F}_q)$. Note that every $(d-1)$-simplex of $A_d(\mathbb{F}_q)$ is contained in precisely $(1+q)$ of the d-simplices of $A_d(\mathbb{F}_q)$ ($(1+q)$ is the number of 1-dimensional subspaces of a 2-dimensional vector space over $\mathbb{F}_q$). In particular,

$$|A_d(\mathbb{F}_q)(d-1)| = \frac{d+1}{q+1} |A_d(\mathbb{F}_q)(d)|.$$

On the other $|A_d(\mathbb{F}_q)(d)| = [d+2]_q!$ where for $k \ge 1$ we let $[k]_q! = [k]_q \cdot [k-1]_q \cdot \dots \cdot [1]_q$ with $[j]_q = \sum_{i=0}^{j-1} q^i$. It follows that $|A_d(\mathbb{F}_q)(d-1)|$ is a polynomial in q with leading term $(d+1)q^{\frac{d(d+3)}{2}}$. Hence, for sufficiently large q ($q \ge (d+2)!$ suffices) we have

$$|A_d(\mathbb{F}_q)(d-1)| \le 2(d+1)q^{\frac{d(d+3)}{2}}.$$

Clearly, the map $\lambda\colon A_d(\mathbb{F}_q)(0) \to \{1, 2, \dots, d+1\}$ given by $U \mapsto \lambda(U) := \dim(U)$ is a labeling of the vertices of $A_d(\mathbb{F}_q)$ showing that $A_d(\mathbb{F}_q)$ is $(d+1)$-partite. This gives rise to an embedding $\iota\colon A_d(\mathbb{F}_q) \to \Lambda^d_{n_0,\dots,n_d}$ where $n_k = \frac{[d+2]_q!}{[k+1]_q![d+1-k]_q!}$ is the number of $k+1$ dimensional subspaces of $\mathbb{F}_q^{d+2}$.

Outline of proof of Theorem 1.6 We first observe that since the restriction of a coboundary to a subcomplex is a coboundary, Proposition 7.1 also provides a wealth of coboundaries in $A_d(\mathbb{F}_q)$.

Corollary 7.3. *Let* $\varphi\colon A_d(\mathbb{F}_q)(0) \to \mathbb{F}_2^d$*. Let* $c^\varphi \in C^d(A_d(\mathbb{F}_q);\mathbb{F}_2)$ *be given by*

$$c^\varphi(\{u_0,\dots,u_d\}) = \begin{cases} 1 & \text{if } \sum_{i=0}^d \varphi(u_i) = 0 \in \mathbb{F}_2^d, \\ 0 & \text{otherwise }. \end{cases}$$

Then c^φ *is a coboundary, i.e.* $c^\varphi \in B^d(A_d(\mathbb{F}_q);\mathbb{F}_2)$*.*

Now the idea is to pick φ uniformly at random and consider c^φ. That is for every vertex $v \in A_d(\mathbb{F}_q)(0)$ we choose $\varphi(v) \in \mathbb{F}_2^d$ independently and uniformly at random. It will turn out that as $q \to +\infty$, with positive probability, there is some coboundary $b = c^\varphi$ for which every $(d-1)$-simplex in $A_d(\mathbb{F}_q)$ is contained in at most $\frac{q+1}{2^d} + o(q)$ d-simplices of b. Writing $|\cdot|$ for the Hamming norm of cochains, we see that every cofilling c of b must satisfy

$$\left(\frac{q+1}{2^d} + o(q)\right)|c| \geq |b|$$

giving us a cochain $c \in C^{d-1}(A_d(\mathbb{F}_q);\mathbb{F}_2)$ for which

$$\frac{|\delta c|}{|[c]|} \leq \left(\frac{q+1}{2^d} + o(q)\right).$$

Normalizing we get

$$\eta_{d-1}(A_d(\mathbb{F}_q)) \leq \frac{d+1}{q+1}\left(\frac{q+1}{2^d} + o(q)\right) = \frac{d+1}{2^d} + o(1)$$

as $q \to +\infty$.

Proof of Theorem 1.6 We add some more details to the proof outline above. To this end, let $(\Omega, \mathcal{B}, \mathbb{P})$ be the probability space with $\Omega = (\mathbb{F}_2^d)^{A_d(\mathbb{F}_q)(0)}$, i.e. Ω is the set of maps $\varphi\colon A_d(\mathbb{F}_q)(0) \to \mathbb{F}_2^d$, $\mathcal{B} = 2^\Omega$ and $\mathbb{P}$ the uniform distribution. For $\omega \in \Omega$ we let $b(\omega) := c^\omega \in B^d(A_d(\mathbb{F}_q);\mathbb{F}_2)$ as defined in Corollary 7.3. For $\tau \in A_d(\mathbb{F}_q)(d)$ let $b^{(\tau)}\colon \Omega \to \mathbb{R}$ be given by

$$b^{(\tau)}(\omega) := \begin{cases} 1 & \text{if } b(\omega)(\tau) = 1, \\ 0 & \text{otherwise }. \end{cases}$$

For $\sigma \in A_d(\mathbb{F}_q))(d-1)$ let $d^{(\sigma)}\colon \Omega \to \mathbb{R}$ be given by

$$d^{(\sigma)}(\omega) := \sum_{\tau \in A_d(\mathbb{F}_q)(d), \sigma \subseteq \tau} b^{(\tau)}(\omega),$$

i.e. $d^{(\sigma)}(\omega)$ is the number of d-simplices incident to τ which are contained in $b(\omega)$. We have

Lemma 7.4. *(i)* $\mathbb{P}(b^{(\tau)} = 1) = \mathbb{E}[b^{(\tau)}] = \frac{1}{2^d}$ *for all* $\tau \in A_d(\mathbb{F}_q)(d)$*.*

(ii) $\mathbb{E}[d^{(\sigma)}] = \frac{q+1}{2^d}$ *for all* $\sigma \in A_d(\mathbb{F}_q)(d-1)$*.*

Proof. (i) follows from the fact that for any fixed $a_0, a_1, \dots, a_{d-1} \in \mathbb{F}_2^d$ the equation $a_0 + a_1 + \cdots + a_{d-1} + x = 0$ has precisely one solution for x. (ii) then follows from (i) by linearity of expectation using that every $(d-1)$-simplex of $A_d(\mathbb{F}_q)$ is contained in exactly $q+1$ d-simplices. □

The following observation is crucial as it will allow us to use Hoeffding's inequality for $d^{(\sigma)}$.

Lemma 7.5. *Fix $\sigma \in A_d(\mathbb{F}_q)(d-1)$. Let $\tau_1, \dots, \tau_{q+1}$ be the $q+1$ d-simplices incident to σ. Then the random variables $b^{(\tau_1)}, \dots, b^{(\tau_{q+1})}$ are independent.*

Proof. Let $\sigma = \{v_0, \dots v_{d-1}\}$. When randomly picking $\varphi\colon A_d(\mathbb{F}_q)(0) \to \mathbb{F}_2^d$ we can think that the values of φ on the vertices of σ have already been picked. Then the value of $b^{(\tau_i)}$ solely depends on the choice of φ on the remaining vertex $v \in \tau_i \setminus \sigma$. These choices are independent. □

Recall Hoeffding's inequality

Theorem 7.6 (Hoeffding's inequality, [62, Theorem 1])**.** *Let $X_1, \dots, X_n$ be $\{0,1\}$-valued independent identically distributed (i.i.d.) random variables with $p = \mathbb{E}X_i$. Then for any $t \geq 0$ we have*

$$\mathbb{P}\left(\sum_{i=1}^{n} X_i \geq (p+t)n\right) \leq e^{-2t^2 n}.$$

By Lemma 7.4 (i) and Lemma 7.5 $d^{(\sigma)}$ is a sum of $\{0,1\}$-valued i.i.d. random variables with success probability $p = \frac{1}{2^d}$. Thus we can apply Hoeffding's inequality to $d^{(\sigma)}$ with

$$n = q+1, p = \frac{1}{2^d}, t = \sqrt{\frac{(d(d+3)+2)\log q}{4(q+1)}}$$

and combine it with a union bound over all $(d-1)$-simplices $\sigma \in A_d(\mathbb{F}_q)(d-1)$ to get (for $q \geq (d+2)!$) that

$$\begin{aligned}\mathbb{P}\left(\exists \sigma \in A_d(\mathbb{F}_q)(d-1) \text{ with } d^{(\sigma)} \geq \left(\frac{1}{2^d} + t\right)(q+1)\right) &\leq |A_d(\mathbb{F}_q)(d-1)| e^{-2t^2(q+1)} \\ &\leq 2(d+1) q^{\frac{d(d+3)}{2}} e^{-\left(\frac{d(d+3)}{2}+1\right)\log q} \\ &= \frac{2(d+1)}{q}.\end{aligned}$$

For the last inequality we used that $|A_d(\mathbb{F}_q)(d-1)| \leq 2(d+1)q^{\frac{d(d+3)}{2}}$ whenever $q \geq (d+2)!$. In particular, for $q \geq (d+2)!$ there is some $\omega \in \Omega$ such that for all $\sigma \in A_d(\mathbb{F}_q)(d-1)$ it holds that

$$d^{(\sigma)}(\omega) \leq \frac{q+1}{2^d} + (q+1)\sqrt{\frac{(d(d+3)+2)\log q}{4(q+1)}} = \frac{q+1}{2^d} + \frac{1}{2}\sqrt{(d(d+3)+2)(q+1)\log q}.$$

As we noticed earlier, this implies that every $c \in C^{d-1}(A_d(\mathbb{F}_q); \mathbb{F}_2)$ with $\delta c = b(\omega)$ must satisfy

$$\left(\frac{q+1}{2^d} + \frac{1}{2}\sqrt{(d(d+3)+2)(q+1)\log q}\right)|c| \geq |b(\omega)|.$$

It follows that

$$\eta_{d-1}(A_d(\mathbb{F}_q)) \leq \frac{d+1}{2^d} + \frac{(d+1)\sqrt{(d(d+3)+2)(q+1)\log q}}{2(q+1)}.$$

Since

$$\lim_{q\to+\infty} \frac{(d+1)\sqrt{(d(d+3)+2)(q+1)\log q}}{2(q+1)} = 0$$

this finishes the proof of Theorem 1.6.

Some remarks regarding Theorem 1.6. We conclude this section with two remarks regarding our upper bound on $\eta_{d-1}(A_d(\mathbb{F}_q))$.

(i) The proof of Theorem 1.6 carries over to arbitrary infinite families $(X_n)_{n\in\mathbb{N}}$ of $(d+1)$-partite d-dimensional simplicial complex for which every $(d-1)$-simplex of X_n is contained in (roughly) $q(n)$ d-simplices for a sequence of positive integers $q(n)$ which grows to infinity sufficiently fast.

(ii) The simplicial complexes $A_d(\mathbb{F}_q)$ show up as the (vertex) links of so-called Ramanujan complexes [100]. As an application of the necessity of expansion of links, we can use Theorem 1.6 to get an upper bound on the top-dimensional cofilling constants of Ramanujan complexes (see Section 9.4 below).

7.3 Upper Bound for Complete Multipartite Complexes

7.3.1 Upper Bound on $\eta_{d-1}(\Lambda^d_{n_0,n_1,\ldots,n_d})$

Let $d \in \mathbb{N}$ be a dimension, $n_0, n_1, \ldots, n_d \geq 2$ integers. We will write $\Lambda^d_{n_0,n_1,\ldots,n_d}$ as $\Lambda^d_{n_0,n_1,\ldots,n_d} = V_0 * V_1 * \cdots * V_d$ with $V_i = [n_i]$, $0 \leq i \leq d$. We write $|\cdot|$ for the Hamming norm on cochains and $\|\cdot\|$ for the Garland weighted Hamming norm on cochains. In this subsection we prove the following slightly refined version of Theorem 1.5.

Theorem 7.7. *If 2^d divides n_i for all $0 \leq i \leq d$, then*

$$\eta_{d-1}(\Lambda^d_{n_0,n_1,\ldots,n_d}) \leq \frac{d+1}{2^d}.$$

Moreover, let $\varepsilon > 0$. If $\min\{n_0, \ldots, n_d\} \geq 2^d + \frac{d+1}{\varepsilon}$, then

$$\eta_{d-1}(\Lambda^d_{n_0,n_1,\ldots,n_d}) \leq \frac{d+1}{2^d} + \varepsilon.$$

Proof. Let $X = \Lambda^d_{n_0,n_1,\ldots,n_d}$. Write $n_i = l_i 2^d + r_i$ with $0 \leq r_i < 2^d$, $l_i \in \mathbb{Z}_{\geq 0}$, $0 \leq i \leq d$. Partition $V_i = \bigsqcup_{j=1}^{2^d} V_{ij}$ as equally as possible, i.e. such that $||V_{ij}| - |V_{ij'}|| \leq 1$ for all $j, j' \in \{1, \ldots, 2^d\}$. Let $\psi\colon [2^d] \to \mathbb{F}_2^d$ be a bijection. Define $\varphi\colon X(0) \to \mathbb{F}_2^d$ by $\varphi(v) := \psi(j)$ for $v \in V_{ij}$, $1 \leq j \leq 2^d, 0 \leq i \leq d$. Let $b = c^\varphi \in B^d(X;\mathbb{F}_2)$ as defined in Proposition 7.1.

Given $\sigma \in X(d-1)$ there is a unique $i \in \{0, 1, \ldots, d\}$ for which $\sigma \cap V_i = \emptyset$. We call this i the type of σ.

First assume that $r_i = 0$ for all $0 \leq i \leq d$, i.e. that 2^d divides n_i for all $0 \leq i \leq d$. Consider $c \in C^{d-1}(X;\mathbb{F}_2)$ with $\delta c = b$. Decompose $c = \sum_{i=0}^d c^{(i)}$ where the support of $c^{(i)}$

is given by the $(d-1)$-simplices of type i in the support of c. Since 2^d divides n_i, every $(d-1)$-simplex of type i is contained in exactly l_i d-simplices in the support of b. Hence,

$$\sum_{i=0}^{d} l_i |c^{(i)}| \geq |b|.$$

Note that a $(d-1)$-simplex σ of type i has Garland weight $w(\sigma) = \frac{n_i}{(d+1)|X(d)|}$. It follows that

$$\begin{aligned}\|c\| &= \sum_{i=0}^{d} \frac{n_i}{(d+1)|X(d)|} |c^{(i)}| \\ &= \frac{2^d}{(d+1)|X(d)|} \sum_{i=0}^{d} l_i |c^{(i)}| \\ &\geq \frac{2^d}{(d+1)|X(d)|} |b| \\ &= \frac{2^d}{d+1} \|b\|.\end{aligned}$$

This shows that

$$\eta_{d-1}(\Lambda^d_{n_0,\ldots,n_d}) \leq \frac{d+1}{2^d},$$

whenever 2^d divides all n_i, $0 \leq i \leq d$.

If not all the n_i's are divisible by 2^d, we still have that every $(d-1)$-simplex σ of type i is contained in at most $l_i + 1$ d-simplices from b. Thus, every cofilling $c \in C^{d-1}(X; \mathbb{F}_2)$ of b must satisfy

$$\sum_{i=0}^{d} (l_i + 1) |c^{(i)}| \geq |b|,$$

where again we decompose $c = \sum_{i=0}^{d} c^{(i)}$ according to the type of $(d-1)$-simplices. Note that

$$\frac{n_i}{l_i + 1} = \frac{2^d l_i + r_i}{l_i + 1} \geq 2^d \frac{l_i}{l_i + 1}.$$

Let $l_{\min} = \min_{0 \leq i \leq d} \frac{l_i}{l_i+1}$. Then

$$\begin{aligned}\|c\| &= \frac{1}{(d+1)|X(d)|} \sum_{i=0}^{d} n_i |c^{(i)}| \\ &\geq \frac{1}{(d+1)|X(d)|} \sum_{i=0}^{d} 2^d \frac{l_i}{l_i + 1} (l_i + 1) |c^{(i)}| \\ &\geq \frac{2^d l_{\min}}{(d+1)|X(d)|} \sum_{i=0}^{d} (l_i + 1) |c^{(i)}| \\ &\geq \frac{2^d l_{\min}}{d+1} \|b\|.\end{aligned}$$

Now assume that $\min_{0 \leq i \leq d} n_i \geq 2^d + \frac{d+1}{\varepsilon}$. Then for any $0 \leq i \leq d$ we have

$$l_i = \left\lfloor \frac{n_i}{2^d} \right\rfloor \geq \frac{n_i}{2^d} - 1 \geq \frac{d+1}{2^d \varepsilon}.$$

It follows that

$$l_{\min} \geq 1 - \frac{1}{1 + \frac{d+1}{2^d \varepsilon}}.$$

Therefore

$$\eta_{d-1}(\Lambda^d_{n_0,\ldots,n_d}) \leq \frac{d+1}{2^d} \frac{1}{l_{\min}} \leq \frac{d+1}{2^d} \frac{1 + \frac{d+1}{2^d\varepsilon}}{\frac{d+1}{2^d\varepsilon}} = \frac{d+1}{2^d} + \varepsilon,$$

as desired. □

7.3.2 Upper Bound on $\eta_k(\Lambda^d_n)$

By taking products, we can extend the construction yielding the upper bound on $\eta_{d-1}(\Lambda^d_n)$ to give the following upper bound on $\eta_k(\Lambda^d_n)$.

Proposition 7.8. *Let $l, n \in \mathbb{Z}_{>0}$, $d_1, \ldots, d_l \in \mathbb{Z}_{\geq 0}$. Let $d = (\sum_{i=1}^l d_i) + l - 1$ and $k = (\sum_{i=1}^l d_i) - 1$. If $2^{\max_{1 \leq i \leq l} d_i}$ divides n then*

$$\eta_k(\Lambda^d_n) \leq \frac{k+2}{d-k} \sum_{i=1}^{l} \frac{1}{2^{d_i}}.$$

In particular, for every $d \in \mathbb{Z}_{\geq 0}$ and $0 \leq k < d-1$ we have

$$\eta_k(\Lambda^d_n) \leq \frac{k+2}{2^{\lfloor (k+1)/(d-k) \rfloor}}$$

whenever n is divisible by $2^{\lceil (k+1)/(d-k) \rceil}$.

We remark that it is not difficult to further extend the above results to an upper bound

$$\eta_k(\Lambda^d_{n_0,\ldots,n_d}) \leq \frac{k+2}{d-k} \sum_{i=1}^{l} \frac{1}{2^{d_i}}$$

provided that 2^{d_i} divides n_{k_i+s} for $0 \leq s \leq d_i$ where $k_i = \sum_{t=1}^{i-1}(d_t + 1)$. Moreover, for every $\varepsilon > 0$ there is $n_\varepsilon \in \mathbb{Z}_{>0}$ such that if $\min\{n_0, \ldots, n_d\} \geq n_\varepsilon$ then

$$\eta_k(\Lambda^d_{n_0,\ldots,n_d}) \leq \frac{k+2}{d-k} \sum_{i=1}^{l} \frac{1}{2^{d_i}} + \varepsilon.$$

The proof of these slight extensions do not really need any new idea but require some additional technicalities which we prefer to omit.

Proof of Proposition 7.8. For $1 \leq i \leq l$ write n as $n = 2^{d_i} r_i$ for some $r_i \in \mathbb{Z}_{>0}$. Let $b^{(i)} \in B^{d_i}(\Lambda^{d_i}_n; \mathbb{F}_2)$ be a coboundary as constructed in the proof of Theorem 7.7 witnessing that $\eta_{d_i-1}(\Lambda^{d_i}_n) \leq (d_i+1)/2^{d_i}$. Let $c^{(i)} \in C^{d_i-1}(\Lambda^{d_i}_n; \mathbb{F}_2)$ be a minimal cofilling of $b^{(i)}$.

Think of Λ^d_n as $\Lambda^d_n = \Lambda^{d_1}_n * \Lambda^{d_2}_n * \cdots * \Lambda^{d_l}_n$. With this decomposition we consider $c = c^{(1)} \otimes c^{(2)} \otimes \ldots \otimes c^{(l)} \in C^k(\Lambda^d_n; \mathbb{F}_2)$. Let $b = \delta c$. Recall that $b^{(i)}$ has the property that every $(d_i - 1)$-simplex in $\Lambda^{d_i}_n$ is contained in at most r_i simplices of the support of $b^{(i)}$. Consequently, every k-simplex $\sigma \in \Lambda^d_n$ is contained in at most

$$\sum_{i=1}^{l} r_i = n \sum_{i=1}^{l} \frac{1}{2^{d_i}}$$

simplices in the support of b. It follows that

$$|b| \leq n \left(\sum_{i=1}^{l} \frac{1}{2^{d_i}} \right) \|[c]\|.$$

Since $|\Lambda_n^d(j)| = \binom{d+1}{j+1} n^{j+1}$ for all $0 \leq j \leq d$, we easily deduce by normalizing that

$$\eta_k(\Lambda_n^d) \leq \frac{|\Lambda_n^d(k)|}{|\Lambda_n^d(k+1)|} n \sum_{i=1}^{l} \frac{1}{2^{d_i}} = \frac{k+2}{d-k} \sum_{i=1}^{l} \frac{1}{2^{d_i}},$$

as desired. The second part follows from the first by writing

$$k+1 = s(d-k) + r$$

with integers $s \geq 0$ and $0 \leq r < d-k$ and setting $l = d-k$, $d_i = s+1$ for $1 \leq i \leq r$ and $d_i = s$ for $r < i \leq d-k$. This gives

$$\eta_k(\Lambda_n^d) \leq \frac{k+2}{d-k} \left(\frac{r}{2^{s+1}} + \frac{d-k-r}{2^s} \right) \leq \frac{k+2}{2^{\lfloor (k+1)/(d-k) \rfloor}}$$

and finishes the proof. □

7.3.3 Refined Upper Bound for Λ_n^2

For Λ_n^2 we have the following refined upper bound on $\eta_1(\Lambda_n^2)$ which we conjecture to be the exact value.

Proposition 7.9. *Let $n \in \mathbb{Z}_{>0}$.*

- *If $n \equiv 0 \mod 4$, then*
$$\eta_1(\Lambda_n^2) \leq \frac{3}{4}.$$
- *If $n \equiv 1 \mod 4$, then*
$$\eta_1(\Lambda_n^2) \leq \frac{3n^3+9}{4n^3-3n^2+3n}.$$
- *If $n \equiv 2 \mod 4$ and $n \neq 2$,*[1] *then*
$$\eta_1(\Lambda_n^2) \leq \frac{3n^3+24}{4n^3-2n^2+4n}.$$
- *If $n \equiv 3 \mod 4$, then*
$$\eta_1(\Lambda_n^2) \leq \frac{3n^3+3}{4n^3-3n^2+n}.$$

We remark that for $1 \leq n \leq 5$ the exact values of $\eta_1(\Lambda_n^2)$ are

$$\eta_1(\Lambda_1^2) = 3, \eta_1(\Lambda_2^2) = 1, \eta_1(\Lambda_3^2) = 1, \eta_1(\Lambda_4^2) = 3/4 \text{ and } \eta_1(\Lambda_5^2) = 48/55,$$

[1] It is known that $\eta_1([2]^{*3}) = 1$ (see [35, Proposition 5.5]).

matching the upper bounds of the proposition.[2]

For the proof of Proposition 7.9 we will consider $b = c^{\varphi}$ for a specific choice of $\varphi\colon \Lambda_n^2(0) \to \mathbb{F}_2^2$, for which $|\varphi^{-1}(x) - \varphi^{-1}(y)| \leq 1$ for all $x, y \in \mathbb{F}_2^2$. We will explicitly write down a minimal cofilling of such a b which gives the desired upper bound. The minimality of these cofillings we check with the help of a computer. The following lemma, which might be of independent interest, allows us to reduce the number of cases to a feasible amount.

Lemma 7.10 (Minimality for product-like cochains)**.** *Write $\Lambda_n^2 = U_0 * U_1 * U_2$ with $U_0 = U_1 = U_2 = [n]$. For $i \in \{0,1,2\}$ let $U_i = \bigsqcup_{s=1}^{l_i} U_s^{(i)}$ be a partition of U_i. Let $c \in C^1(\Lambda_n^2; \mathbb{F}_2)$ such that the restriction $c|_{U_s^{(i)} * U_t^{(j)}}$ is constant for all $0 \leq i < j \leq 2$ and $1 \leq s \leq l_i$, $1 \leq t \leq l_j$. Then the following are equivalent:*

(i) c is minimal, i.e. $|c + \delta a| \geq |c|$ for all $a \in C^0(\Lambda_n^2; \mathbb{F}_2)$.

(ii) For all $S \subseteq \Lambda_n^2(0)$, $|\mathrm{supp}(c) \cap \mathrm{supp}(\delta \mathbb{1}_S)| \leq \frac{|\mathrm{supp}(\delta \mathbb{1}_S)|}{2}$.

(iii) For all $S \subseteq X(0)$ with $S \cap U_s^{(i)} \in \{\emptyset, U_s^{(i)}\}$ for all $0 \leq i \leq 2$, $1 \leq s \leq l_i$,

$$|\mathrm{supp}(c) \cap \mathrm{supp}(\delta \mathbb{1}_S)| \leq \frac{|\mathrm{supp}(\delta \mathbb{1}_S)|}{2}.$$

The proof of this lemma is inspired by the proof of Theorem 6.3 in [84] where a similar result for the complete 2-dimensional complex is proven.

Proof. To ease notation let $X = \Lambda_n^2$. We will write δS instead of $\delta \mathbb{1}_S$ for $S \subseteq X(0)$ as well as $c \cap \delta S$ instead of $\mathrm{supp}(c) \cap \mathrm{supp}(\delta \mathbb{1}_S)$.

The equivalence of (i) and (ii) holds for any $c \in C^1(X; \mathbb{F}_2)$ and easily follows from the observation that $|c + \delta a| = |c| + |\delta a| - 2|c \cap \delta a|$.

(ii) cleary implies (iii). For the converse implication we argue that given $c \in C^1(X; \mathbb{F}_2)$ with $|c \cap \delta S| > \frac{|\delta S|}{2}$ for some $S \subseteq X(0)$, there is also $\tilde{S} \subseteq X(0)$ with $\tilde{S} \cap U_s^{(i)} \in \{\emptyset, U_s^{(i)}\}$ for all $0 \leq i \leq 2, 1 \leq s \leq l_i$ and $|c \cap \delta \tilde{S}| > \frac{|\delta \tilde{S}|}{2}$. Assume, by contradiction, this is not the case for some $c \in C^1(X; \mathbb{F}_2)$. Then there is some $S \subseteq X(0)$ with $|c \cap \delta S| > \frac{|\delta S|}{2}$ such that the condition $S \cap U_s^{(i)} \in \{\emptyset, U_s^{(i)}\}$ is violated for the fewest number of $0 \leq i \leq 2$ and $1 \leq s \leq l_i$ among all $S' \subseteq X(0)$ with $|c \cap \delta S'| > \frac{|\delta S'|}{2}$. After relabeling we can without loss of generality assume that $\emptyset \neq S \cap U_1^{(0)} \subsetneq U_1^{(0)}$. Let $A = S \cap U_1^{(0)}$ and $B = U_1^{(0)} \setminus A$. Let $S^- = S \setminus A$ and $S^+ = S \sqcup B$. Note that S^- and S^+ violate fewer of the conditions on the intersection with $U_s^{(i)}$ than S. Thus, showing that $|\delta S^- \cap c| > \frac{1}{2}|\delta S^-|$ or $|\delta S^+ \cap c| > \frac{1}{2}|\delta S^+|$ would contradict the choice of S and finish the proof of the lemma.

To see that this is indeed the case, fix $u \in U_1^{(0)}$ and let

$$\beta = |\{s \in S \cap (U_1 \sqcup U_2) : c(us) = 1\}|$$

and

$$\gamma = |\{s \in (X(0) \setminus S) \cap (U_1 \sqcup U_2) : c(us) = 1\}|.$$

[2]For $n = 1$ and $n = 3$ these bounds can be checked by hand. The case $n = 2$ is part of [35, Proposition 5.5]. For $n = 4$ the random cofilling argument (carrying lower order terms in n along) as in Proposition 8.6 yields a lower bound of 3/4 matching the upper bound of Theorem 7.7. For $n = 5$ we run an exhaustive search on a computer for a stronger upper bound after reducing to a feasible amount of cases.

Note that

$$|\delta S^- \cap c| = |\delta S \cap c| - (\gamma - \beta)|A|, \text{ and}$$
$$|\delta S^+ \cap c| = |\delta S \cap c| + (\gamma - \beta)|B|.$$

This implies

$$|\delta S \cap c| = (1-\lambda)|\delta S^- \cap c| + \lambda|\delta S^+ \cap c|$$

with $\lambda = \frac{|A|}{|A|+|B|}$. Now let $s_i = |S \cap U_i|$ for $i \in \{0,1,2\}$. Define the function $\varphi\colon \mathbb{R} \to \mathbb{R}$ by

$$s \mapsto \varphi(s) := s(2n - s_1 - s_2) + s_1(2n - s - s_2) + s_2(2n - s - s_1).$$

Note that φ is affine and hence concave. Moreover,

$$|\delta S| = \varphi(s_0), |\delta S^-| = \varphi(s_0 - |A|) \text{ and } |\delta S^+| = \varphi(s_0 + |B|).$$

Now assume, by contradiction, that both $|\delta S^- \cap c| \leq \frac{1}{2}|\delta S^-|$ and $|\delta S^+ \cap c| \leq \frac{1}{2}|\delta S^+|$. Then

$$\begin{aligned}
|\delta S \cap c| &= (1-\lambda)|\delta S^- \cap c| + \lambda|\delta S^+ \cap c| \\
&\leq (1-\lambda)\frac{|\delta S^-|}{2} + \lambda\frac{|\delta S^+|}{2} \\
&= \frac{1}{2}\left((1-\lambda)\varphi(s_0 - |A|) + \lambda\varphi(s_0 + |B|)\right) \\
&\leq \frac{1}{2}\varphi((1-\lambda)(s_0 - |A|) + \lambda(s_0 + |B|)) \\
&= \frac{1}{2}\varphi\left(s_0 - \frac{|A| \cdot |B|}{|A| + |B|} + \frac{|B| \cdot |A|}{|A| + |B|}\right) \\
&= \frac{1}{2}|\delta S|,
\end{aligned}$$

where we used concavity of φ for the second inequality. We obtained a contradiction to our assumption that $|\delta S \cap c| > \frac{1}{2}|\delta S|$. This finishes the proof. □

Another ingredient for the proof of Proposition 7.9 is a particular cofilling of the coboundary $b \in B^2(\Lambda_4^2; \mathbb{F}_2)$ showing that $\eta_1(\Lambda_4^2) \leq 3/4$.

Lemma 7.11. *Let $\Lambda_4^2 = U * V * W$ with $U = \{u_0, u_1, u_2, u_3\}, V = \{v_0, v_1, v_2, v_3\}$ and $W = \{w_0, w_1, w_2, w_3\}$. Let $\psi\colon \{0,1,2,3\} \to \mathbb{F}_2^2$ be given by*

$$\psi(0) = (0,0), \psi(1) = (1,0), \psi(2) = (0,1) \text{ and } \psi(3) = (1,1).$$

Let $b \in B^2(\Lambda_4^2; \mathbb{F}_2)$ be given by[3]

$$b(\{u_i, v_j, w_k\}) = \begin{cases} 1 & \text{if } \psi(i) + \psi(j) + \psi(k) = 0 \in \mathbb{F}_2^2 \\ 0 & \text{otherwise.} \end{cases}$$

Let $c \in C^1(\Lambda_4^2; \mathbb{F}_2)$ be given by

$$\begin{aligned}
\operatorname{supp} c = \{&u_0v_0, u_0v_3, u_0w_1, u_0v_2, u_1v_0, u_1v_2, u_1w_0, u_1w_2, \\
&u_2v_0, u_2v_1, u_2w_0, u_2w_1, v_0w_3, v_1w_2, v_2w_1, v_3w_0\}.
\end{aligned}$$

Then $\delta c = b$.

[3] Note that b is indeed a coboundary since $b = c^\varphi$ for $\varphi\colon \Lambda_4^2(0) \to \mathbb{F}_2^2$ given by $\varphi(x_i) = \psi(i)$ for $x \in \{u, v, w\}$ and c^φ as in Proposition 7.1.

Proof. Instead of tediously checking that $\delta c = b$ let us describe how one would come up with c in the first place. To this end, we first note that $c_{|V*W} = b_{u_3}$. Thus, if we define $\tilde{c} \in C^1(\Lambda_4^2; \mathbb{F}_2)$ to be equal to b_{u_3} on $V * W$ and to be 0 on edges containing u_3, we already achieved that $\delta\tilde{c}(\tau) = b$ for all triangles τ containing u_3. In order to extend $\tilde{c}$ to a cofilling of b we would need to satisfy

$$b(u_i v_j w_k) \overset{?!}{=} \delta\tilde{c}(u_i v_j w_k) = b(u_3 v_j w_k) + \tilde{c}(u_i w_k) + \tilde{c}(u_i v_j),$$

or equivalently that $(\delta\tilde{c}_{u_i})(v_j w_k) = (b_{u_i} + b_{u_3})(v_j w_k)$. This amounts to choose $\tilde{c}_{u_i}$ as a cofilling of $b_{u_i} + b_{u_3} \in B^1(V * W; \mathbb{F}_2)$.

In Figure 7.1 we depict $b_{u_i} + b_{u_3}$ and c_{u_i} for $i \in \{0,1,2\}$. We see that c_{u_i} is indeed a cofilling of $b_{u_i} + b_{u_3}$ for $i \in \{0,1,2\}$ and conclude $\delta c = b$, as desired. □

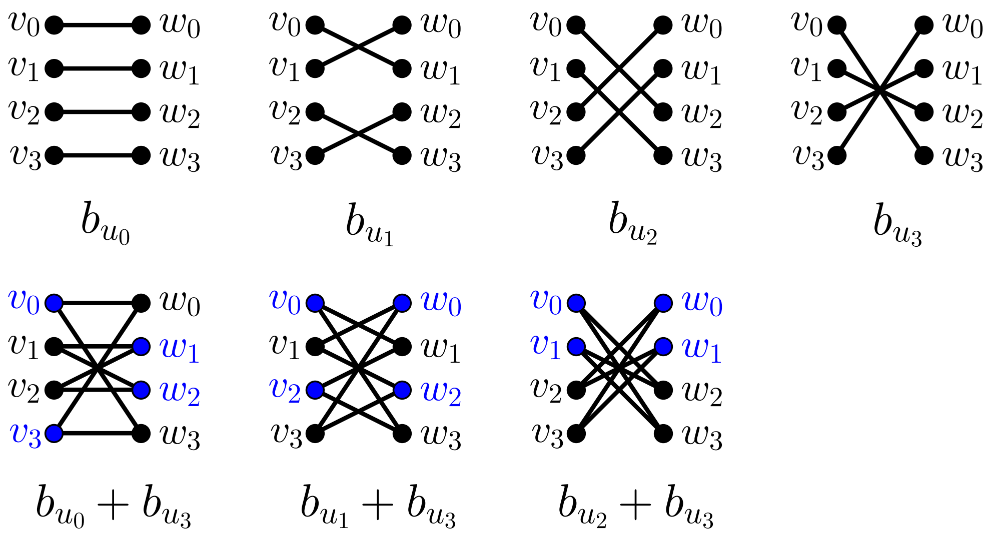


Figure 7.1: The top row shows b_{u_i} for $i \in \{0,1,2,3\}$. The bottom row shows $b_{u_i} + b_{u_3}$ for $i \in \{0,1,2\}$. We marked the vertices in the support of c_{u_i} in blue. We note that for each $i \in \{0,1,2\}$ the vertices in c_{u_i} induce the cut $b_{u_i} + b_{u_3}$

We are ready to give a proof of Proposition 7.9.

Proof of Proposition 7.9. Write $n = 4k + l$ with $l \in \{0,1,2,3\}$. If $l = 0$, we already know from Theorem 7.7 that $\eta_1(\Lambda_n^2) \leq \frac{3}{4}$. So we can assume that $l \in \{1,2,3\}$.

Write $\Lambda_n^2 = U * V * W$ with $U = V = W = [n]$. Partition $U = \bigsqcup_{i=0}^3 U_i$, $V = \bigsqcup_{i=0}^3 V_i$ and $W = \bigsqcup_{i=0}^3 W_i$ such that $|U_i| = |V_i| = k$ for $0 \leq i \leq 3 - l$, $|U_i| = |V_i| = k+1$ for $3 - l < i \leq 3$ and $1 \leq l \leq 3$ and $|W_0| = k+1$, $|W_1| = |W_2| = |W_3| = k$ if $l = 1$, $|W_0| = |W_1| = k+1$, $|W_2| = |W_3| = k$ if $l = 2$ and $|W_0| = |W_2| = |W_3| = k+1$, $|W_1| = k$ if $l = 3$.

Label the vertices of Λ_4^2 as in Lemma 7.11. Let $f\colon \Lambda_n^2(0) \to \Lambda_4^2$ be such that $f(u) = u_i$ for all $u \in U_i$, $f(v) = v_i$ for all $v \in V_i$ and $f(w) = w_i$ for all $w \in W_i$, $0 \leq i \leq 3$. Let $f\colon \Lambda_n^2 \to \Lambda_4^2$ be the induced simplicial map. Let $b_0 \in B^2(\Lambda_4^2; \mathbb{F}_2)$ and $c_0 \in C^1(\Lambda_4^2; \mathbb{F}_2)$ be the cochains as considered in Lemma 7.11. Let $b := f^* b_0$ and $c := f^* c_0$. Clearly, $\delta c = b \in B^2(\Lambda_n^2; \mathbb{F}_2)$.

Using that

$$\begin{aligned}|b| = |U_0| \cdot (|V_0| \cdot |W_0| + |V_1| \cdot |W_1| + |V_2| \cdot |W_2| + |V_3| \cdot |W_3|) \\ + |U_1| \cdot (|V_0| \cdot |W_1| + |V_1| \cdot |W_0| + |V_2| \cdot |W_3| + |V_3| \cdot |W_2|) \\ + |U_2| \cdot (|V_0| \cdot |W_2| + |V_1| \cdot |W_3| + |V_2| \cdot |W_0| + |V_3| \cdot |W_1|) \\ + |U_3| \cdot (|V_0| \cdot |W_3| + |V_1| \cdot |W_2| + |V_2| \cdot |W_1| + |V_3| \cdot |W_0|)\end{aligned}$$

and

$$\begin{aligned}|c| = |U_0| \cdot (|V_0| + |V_3| + |W_1| + |W_2|) \\ + |U_1| \cdot (|V_0| + |V_2| + |W_0| + |W_2|) \\ + |U_2| \cdot (|V_0| + |V_1| + |W_0| + |W_1|) \\ + |V_0| \cdot |W_3| + |V_1| \cdot |W_2| + |V_2| \cdot |W_1| + |V_3| \cdot |W_0|\end{aligned}$$

one can check that $\frac{3}{n}\frac{|b|}{|c|}$ would give the upper bound on $\eta_1(\Lambda_n^2)$ as claimed in the statement of the proposition. Thus, it remains to show that c is minimal.

By construction c has product-like structure. Therefore, by Lemma 7.10, it suffices to show that for all $S \subseteq \Lambda_n^2(0)$ with $S \cap X_j \in \{\emptyset, X_j\}$ for all $0 \leq j \leq 3$ and $X \in \{U, V, W\}$, it holds that $|c \cap \delta S| \leq \frac{|\delta S|}{2}$.

This amounts to show that $|c \cap \delta f^* S| \leq \frac{|\delta f^* S|}{2}$ for all $S \subseteq \Lambda_4^2(0)$.

For fixed $l \in \{1, 2, 3\}$ and fixed $S \subseteq \Lambda_4^2(0)$ we can think of $|\delta f^* S| - 2|c \cap \delta f^* S|$ as a polynomial $p_{S,l}(k)$ in k (recall that k is such that $n = 4k + l$). Note that $p_{S,l}$ has degree at most 2 and is thus determined by the values at three different k's. Since $\delta f^* S = f^* \delta S$ for each l, we only have to consider 2^{11} choices for $S \subseteq \Lambda_4^2(0)$ and check whether $p_{S,l}(k)$ is non-negative for all $k \in \mathbb{Z}_{\geq 0}$. It turns out that all $p_{S,l}$ have non-negative coefficients, from which $p_{S,l}(k) \geq 0$ for all $k \in \mathbb{R}_{\geq 0}$ immediately follows. This can be readily verified by using a short computer script.[4] □

7.3.4 Blow-Up Construction for Λ_n^2

In the proof of the upper bounds on $\eta_1(\Lambda_n^2)$ there is a blow-up construction lurking in the background which is worth elaborating on. In fact, the blow-up construction is independent of coefficients and will also be helpful to prove our upper bound on $\zeta_1(\Lambda_n^2)$ in the next subsection.

Given a simplicial complex X on vertex set $V = X(0)$ and $t \in \mathbb{Z}_{>0}$ we define the *t-fold blow-up $X^{\{t\}}$ of X* to be the simplicial complex $X^{\{t\}}$ with vertex set $V \times [t]$ such that $\{(v_0, i_0), \ldots, (v_k, i_k)\}$ is a k-simplex of $X^{\{t\}}$ if and only if $\{v_0, \ldots, v_k\} \in X(k)$. Note that there is a projection map $\pi : X^{\{t\}} \to X$ which maps a k-simplex $\{(v_0, i_0), \ldots, (v_k, i_k)\}$ to $\{v_0, \ldots v_k\}$. This allows to pullback a cochain $c \in C^k(X; \mathbb{A})$ (for some abelian group $\mathbb{A}$) to a cochain $c^{\{t\}} = \pi^* c \in C^k(X^{\{t\}}; \mathbb{A})$. We will call $c^{\{t\}}$ the *blow-up of c*.

Note that for $n, t \in \mathbb{Z}_{>0}$ we have $(\Lambda_n^d)^{\{t\}} \cong \Lambda_{tn}^d$. Moreover, in dimension $d = 2$, blow-up of cochains preserves minimality. More precisely:

Proposition 7.12. *(i) Let $c \in C^1(\Lambda_n^2; \mathbb{Z})$. Then c is minimal (with respect to ℓ_2^2-norm) if and only if $c^{\{t\}} \in C^1(\Lambda_{tn}^2; \mathbb{Z})$ is minimal.*

[4] Our Python code, which we used for this, will be made available via the library of IST Austria.

(ii) *Let* $c \in C^1(\Lambda_n^2; \mathbb{F}_2)$. *Then* c *is minimal (with respect to Hamming norm) if and only if* $c^{\{t\}} \in C^1(\Lambda_{tn}^2; \mathbb{F}_2)$ *is minimal.*

The following corollary is immediate but still worth stating separately. It says that the blow-up construction allows to transfer an upper bound on the coboundary expansion constant of Λ_n^2 to arbitrarily large complexes.

Corollary 7.13. *For every* $k, n \in \mathbb{Z}_{>0}$ *we have* $\eta_1(\Lambda_{kn}^2) \leq \eta_1(\Lambda_n^2)$ *and* $\zeta_1(\Lambda_{kn}^2) \leq \zeta_1(\Lambda_n^2)$.

Blow-Up and Minimality over $\mathbb{Z}$

We show part (i) of Proposition 7.12.

Let $X = \Lambda_n^2$ and $\widehat{X} = X^{\{t\}} = \Lambda_{tn}^2$. Write $\widehat{X}(0) = X \times [t]$.

First assume that $c \in C^1(X; \mathbb{Z})$ is not minimal. Then there is $a \in C^0(X; \mathbb{Z})$ with $|c - \delta a|^2 < |c|^2$. But then

$$|c^{\{t\}}|^2 = t^2|c|^2 > t^2|c - \delta a|^2 = |(c - \delta a)^{\{t\}}|^2 = |c^{\{t\}} - \delta a^{\{t\}}|^2,$$

showing that the blow-up $c^{\{t\}} \in C^1(\widehat{X}; \mathbb{Z})$ of c is not minimal.

For the converse we need the following lemma:

Lemma 7.14. *Let* $c \in C^1(\Lambda_n^2; \mathbb{Z})$. *Then the following are equivalent*

(i) c *is minimal with respect to the* ℓ_2^2*-norm* $|\cdot|^2$.

(ii) $|\langle c, \delta a\rangle| \leq \frac{1}{2}|\delta a|^2$ *for all* $a \in C^0(\Lambda_n^2; \mathbb{Z})$.

(iii) $|\langle c, \delta a\rangle| \leq \frac{1}{2}|\delta a|^2$ *for all* $a \in C^0(\Lambda_n^2; \mathbb{Z})$ *with* $a(u) \in \{0, 1\}$ *for all* $u \in \Lambda_n^2(0)$.

Proof. The equivalence of (i) and (ii) follows from expanding the inequality $|c+\delta a|^2 \geq |c|^2$, which holds for all $a \in C^0(\Lambda_n^2; \mathbb{Z})$ if c is minimal, in terms of inner products.

Clearly (ii) implies (iii). For the reverse implication let $a \in C^0(\Lambda_n^2; \mathbb{Z})$ be arbitrary. Decompose $a = a_+ + a_-$ where $a_+(u) = \max\{0, a(u)\}$ and $a_-(u) = -\min\{0, a(u)\}$ for all $u \in \Lambda_n^2(0)$. Since $(x - y)^2 \geq x^2 + y^2$ for real numbers $x, y \in \mathbb{R}$ with $xy \geq 0$ we have

$$|\delta a(e)|^2 \geq |\delta a_+(e)|^2 + |\delta a_-(e)|^2$$

for all $e \in \Lambda_n^2(1)$.

Let $l_+ = \max_{u \in \Lambda_n^2(0)} a_+(u)$ and $l_- = \max_{u \in \Lambda_n^2(0)} a_-(u)$ For $\varepsilon \in \{-, +\}$ and $1 \leq i \leq l_\varepsilon$ let $a_\varepsilon^{(i)} \in C^0(\Lambda_n^2; \mathbb{Z})$ be given by

$$a_\varepsilon^{(i)}(u) = \begin{cases} 1 & \text{if } a_\varepsilon(u) \geq i, \\ 0 & \text{otherwise .} \end{cases}$$

We have $a = \sum_{i=1}^{l_+} a_+^{(i)} - \sum_{j=1}^{l_-} a_-^{(j)}$ by construction. We estimate assuming (iii)

$$\begin{aligned}
|\langle c, \delta a\rangle| &= |\langle c, \delta \left(\sum_{i=1}^{l_+} a_+^{(i)} - \sum_{j=1}^{l_-} a_-^{(j)}\right)\rangle| \\
&\leq \sum_{i=1}^{l_+} |\langle c, \delta a_+^{(i)}\rangle| + \sum_{j=1}^{l_-} |\langle c, \delta a_-^{(j)}\rangle| \\
&\leq \frac{1}{2}\sum_{i=1}^{l_+} |\delta a_+^{(i)}|^2 + \frac{1}{2}\sum_{j=1}^{l_-} |\delta a_-^{(j)}|^2 \\
&= \frac{1}{2}\sum_{xy\in\Lambda_n^2(1)} \left(|a_+(x) - a_+(y)| + |a_-(x) - a_-(y)|\right) \\
&\leq \frac{1}{2}\left(|\delta a_+|^2 + |\delta a_-|^2\right) \\
&\leq \frac{1}{2}|\delta a|^2,
\end{aligned}$$

where we used the triangle inequality for the first inequality, (iii) for the second, that $|x| \leq x^2$ for all $x \in \mathbb{Z}$ for the third, and $|\delta a_+(e)|^2 + |\delta a_-(e)|^2 \leq |\delta a(e)|^2$ for all $e \in \Lambda_n^2(1)$ for the last inequality. □

Write X as $X = V_0 * V_1 * V_2$ with $V_0 = V_1 = V_2 = [n]$. Let $c \in C^1(X;\mathbb{Z})$ be a minimal cochain. Let $\widehat{c} \in C^1(\widehat{X};\mathbb{Z})$ the blow-up of c. Assume that $\widehat{c}$ is not minimal. By Lemma 7.14 there is $S \subseteq \widehat{X}(0)$ such that

$$|\langle c, \delta \mathbb{1}_S\rangle| > \frac{|\delta \mathbb{1}_S|^2}{2}.$$

We will argue that we can choose S to be of the form $S = S' \times [t]$ for some $S' \subseteq X(0)$. This will give us the desired contradiction, since for S of such form the minimality of c implies that

$$|\langle \widehat{c}, \delta \mathbb{1}_S\rangle| = t^2|\langle c, \delta \mathbb{1}_{S'}\rangle| \leq t^2\frac{|\delta \mathbb{1}_{S'}|^2}{2} = \frac{|\delta \mathbb{1}_S|^2}{2}.$$

So, let $a \in C^0(\widehat{X};\mathbb{Z})$ with $a(u) \in \{0,1\}$ for all $u \in \widehat{X}$ such that $|\langle \widehat{c}, \delta a\rangle| > \frac{|\delta a|^2}{2}$. Assume there is a vertex v (which without loss of generality we can assume to be in V_0) such that

$$\emptyset \neq (\{v\} \times [t]) \cap \operatorname{supp}(a) \subsetneq \{v\} \times [t].$$

Let $A = \{v\} \times [t] \cap \operatorname{supp}(a)$, $B = \{v\} \times [t] \setminus A$, $a^- = a - \mathbb{1}_A$ and $a^+ = a + \mathbb{1}_B$. Note that $\langle \widehat{c}, \delta \mathbb{1}_A\rangle = |A|\langle c, \delta \mathbb{1}_v\rangle$ while $\langle \widehat{c}, \delta \mathbb{1}_B\rangle = |B|\langle c, \delta \mathbb{1}_v\rangle$. Let $\alpha = |A|\langle c, \delta_X \mathbb{1}_v\rangle$, $\beta = |B|\langle c, \delta_X \mathbb{1}_v\rangle$, and

$$\lambda = \frac{\alpha}{\alpha + \beta} = \frac{|A|}{|A| + |B|}.$$

Note that $\lambda \in [0,1]$. We have that

$$(1-\lambda)\langle \widehat{c}, \delta a^-\rangle + \lambda\langle \widehat{c}, \delta a^+\rangle = \langle \widehat{c}, \delta a\rangle.$$

Assume that $|\langle \widehat{c}, \delta a^-\rangle| \leq \frac{|\delta a^-|^2}{2}$ and $|\langle \widehat{c}, \delta a^+\rangle| \leq \frac{|\delta a^+|^2}{2}$. Note that for $\{0,1\}$-valued cochains $a \in C^0(X;\mathbb{Z})$ $|\delta a|^2$ only depends on $a_i = |\operatorname{supp}(a) \cap V_i|$ for $i \in \{0,1,2\}$. Moreover $|\delta a|^2$ is an affine function $a_0 \mapsto \varphi(a_0)$ in a_0 if a_1 and a_2 are fixed. We deduce that

$$\begin{aligned}
|\langle \widehat{c}, \delta a\rangle| &\le (1-\lambda)|\langle \widehat{c}, \delta a^-\rangle| + \lambda |\langle \widehat{c}, \delta a^+\rangle| \\
&\le (1-\lambda)\frac{|\delta a^-|^2}{2} + \lambda \frac{|\delta a^+|^2}{2} \\
&= \frac{1}{2}(1-\lambda)\varphi(a_0 - |A|) + \frac{1}{2}\lambda\varphi(a_0+|B|) \\
&= \frac{1}{2}\frac{|B|}{|A|+|B|}\varphi(a_0-|A|) + \frac{1}{2}\frac{|A|}{|A|+|B|}\varphi(a_0+|B|) \\
&\le \frac{1}{2}\varphi\left(\frac{|B|}{|A|+|B|}(a_0-|A|) + \frac{|A|}{|A|+|B|}(a_0+|B|)\right) \\
&= \frac{1}{2}\varphi(a_0) \\
&= \frac{|\delta a|^2}{2}.
\end{aligned}$$

For the last inequality we used that, as an affine function, φ is concave.

We obtained a contradiction to our assumption that $|\langle \widehat{c}, \delta a\rangle| > \frac{|\delta a|^2}{2}$. Thus, we must have $|\langle \widehat{c}, \delta a^-\rangle| > \frac{|\delta a^-|^2}{2}$ or $|\langle \widehat{c}, \delta a^+\rangle| > \frac{|\delta a^+|^2}{2}$. But a^- and a^+ are both $\{0,1\}$-valued 0-cochains for which there are fewer vertices v with

$$\emptyset \neq (\{v\} \times [t]) \cap \operatorname{supp}(a) \subsetneq \{v\} \times [t]$$

than for a.

Proceeding by induction, we can obtain a cochain a' for which $|\langle \widehat{c}, \delta a'\rangle| > \frac{|\delta a'|^2}{2}$ such that a' is of the form $a' = \mathbb{1}_S$ for some $S = S' \times [t]$ with $S \subseteq X(0)$.

Blow-Up and Minimality over $\mathbb{F}_2$

For part (ii) of Proposition 7.12 we observe that the blow-up $c^{\{t\}}$ of a cochain $c \in C^1(\Lambda_n^2; \mathbb{F}_2)$ is product-like (in the sense of Lemma 7.10). Since

$$t^2|c+\delta a| = |(c+\delta a)^{\{t\}}| = |c^{\{t\}} + \delta a^{\{t\}}|$$

part(ii) is a special case of Lemma 7.10.

7.3.5 Upper Bound on $\zeta_{d-1}(\Lambda_n^d)$

We have the following upper bound on $\zeta_{d-1}(\Lambda_n^d)$.

Proposition 7.15. *(i) For all $n \ge 2$ we have $\zeta_1(\Lambda_n^2) \le 1$.*

(ii) If $d \ge 3$ and $d+1$ divides n then $\zeta_{d-1}(\Lambda_n^d) \le 1$.

Proof. For (i) we distinguish two cases depending on the parity of n. First assume that $n = 2k$ is even. Using the blow-up construction (Corollary 7.13) it suffices to consider the case $n = 2$. Write Λ_2^2 as $\Lambda_2^2 = \{u_0, u_1\} * \{v_0, v_1\} * \{w_0, w_1\}$. Let $f \in C^1(\Lambda_2^2; \mathbb{Z})$ be given by $f([u_0, v_0]) = 1$, $f([v_0, w_0]) = f([u_1, w_0]) = -1$ and $f(e) = 0$ for all other (oriented) edges.

Note that $\delta f = \mathbb{1}_{[u_0,v_0,w_1]} + \mathbb{1}_{[u_1,v_1,w_0]}$. Moreover, f is minimal. Indeed, there is no cofilling of δf with fewer than three edges in its support since the triangles in the support of δf

are antipodal triangles in the octahedron Λ_2^2 (see Figure 7.2 for an illustration of f and δf). It follows that

$$\zeta_1(\Lambda_2^2) \leq \frac{3}{2}\frac{|\delta f|^2}{|f|^2} = 1,$$

as desired.

Next we assume that $n = 2k+1$ is odd with $k \in \mathbb{Z}_{>0}$. Write Λ_n^2 as $\Lambda_n^2 = U_0 * U_1 * U_2$ with $U_i = \{u_{-k}^{(i)}, u_{-k+1}^{(i)}, \ldots, u_{-1}^{(i)}, u_0^{(i)}, u_1^{(i)}, \ldots, u_k^{(i)}\}$, $i \in \{0,1,2\}$.

Define $a \in C^1(\Lambda_n^2; \mathbb{Z})$ by

$$a([u_s^{(i)}, u_t^{(j)}]) = \begin{cases} 1 & \text{if } s > 0, t > 0 \text{ and } t \geq k+s+1, \\ -1 & \text{if } s > 0, t < 0 \text{ and } s \geq k+t+1, \\ 0 & \text{otherwise.} \end{cases}$$

for all $s, t \in \{-k, -k+1, \ldots, -1, 0, 1, \ldots, k\}$ and $0 \leq i < j \leq 2$. See Figure 7.2 for a drawing of a for $n = 5$.

We claim that a is minimal. By Lemma 7.14 it suffices to check that

$$|\langle a, \delta \mathbb{1}_S \rangle| \leq \frac{|\delta \mathbb{1}_S|^2}{2}$$

for all $S \subseteq \Lambda_n^2(0)$. Recall that $\langle a, \delta \mathbb{1}_S \rangle = \langle \partial a, \mathbb{1}_S \rangle$. Observe that $b := \partial a$ is given by $\partial a(u_j^{(i)}) = 2j$ for all $-k \leq j \leq k$ and $0 \leq i \leq 2$.

Fix $S \subseteq \Lambda_n^2(0)$. Note that $|\delta \mathbb{1}_S|^2$ does only depend on $s_i := |S \cap U_i|$, $0 \leq i \leq 2$. In fact $|\delta \mathbb{1}_S|^2 = 2n(s_0+s_1+s_2) - 2(s_0 s_1 + s_0 s_2 + s_1 s_2)$. Now, for $i \in \{0,1,2\}$ let $\tilde{s}_i = \min\{s_i, k+1\}$ and note that

$$\begin{aligned} |\langle \partial a, \mathbb{1}_S \rangle| &\leq \sum_{i=}^{2} \sum_{l=0}^{\tilde{s}_i - 1} 2(k-l) - \sum_{i=0}^{2} \sum_{l=1}^{s_i - \tilde{s}_i} 2l \\ &= n(\tilde{s}_0 + \tilde{s}_1 + \tilde{s}_2) - \sum_{i=0}^{2} \tilde{s}_i^2 - \sum_{i=0}^{2} t_i(t_i+1), \end{aligned}$$

where $t_i = \min\{0, s_i - \tilde{s}_i\}$.

We compute that

$$n\tilde{s}_i - \tilde{s}_i^2 - (s_i - \tilde{s}_i)(s_i - \tilde{s}_i + 1) = ns_i - s_i^2 - (n+1)(s_i - \tilde{s}_i) + 2\tilde{s}_i(s_i - \tilde{s}_i) \leq ns_i - s_i^2,$$

since $\tilde{s}_i \leq s_i$ and $\tilde{s}_i \leq k+1 = \frac{n+1}{2}$. It follows that

$$|\langle \partial a, \mathbb{1}_S \rangle| \leq n(s_0+s_1+s_2) - (s_0^2+s_1^2+s_2^2) \leq n(s_0+s_1+s_2) - (s_0 s_1 + s_0 s_2 + s_1 s_2) = \frac{|\delta \mathbb{1}_S|^2}{2},$$

where we used that $xy + xz + yz \leq x^2 + y^2 + z^2$ for all $x, y, z \in \mathbb{R}$. This finishes the proof of the minimality of a.

We compute

$$|a|^2 = |\operatorname{supp}(a)| = 6\sum_{i=1}^{k} i = 6\frac{k(k+1)}{2} = 3\frac{n-1}{2}\frac{n+1}{2} = \frac{3(n^2-1)}{4}.$$

To find the value of $|\delta a|^2$ we first note that the edges in the support of a form a triangle-free graph. Also, by the choice of signs $\delta a(\tau) = 0$ for any triangle τ for which two of its boundary edges are in the support of a. Thus δa takes values in $\{-1, 0, 1\}$ and $|\delta a|^2$ is the number of triangles in Λ_n^2 that have exactly one boundary edge in the support of a. We conclude

$$|\delta a|^2 = 6 \sum_{i=1}^{k} \sum_{j=1}^{i} (n - i - (k - j + 1)) = \frac{n(n^2 - 1)}{4},$$

where the last equality follows by some straightforward computation.

Overall we conclude that

$$\zeta_1(\Lambda_n^2) \leq \frac{3}{n} \frac{|\delta a|^2}{|a|^2} = \frac{3}{n} \frac{n(n^2-1)}{4} \frac{4}{3(n^2-1)} = 1.$$

This finishes the proof of part (i). For (ii) we use the construction in the proof of Claim 3.4 in [98] where it was shown that $\eta_{d-1}(\Lambda_n^d) \leq 1$ whenever $d+1$ divides n. We assume that $d+1$ divides n. Write Λ_n^d as $\Lambda_n^d = V_1 * \cdots * V_{d+1}$. For $1 \leq i \leq d+1$ partition V_i into $V_i = V_{i,1} \sqcup V_{i,2} \sqcup \cdots \sqcup V_{i,d+1}$ such that $|V_{i,j}| = |V_{i,j'}|$ for all $1 \leq j, j' \leq d+1$. Consider $f \in C^{d-1}(\Lambda_n^d; \mathbb{Z})$ be given by

$$f = \sum_{1 \leq i_1 < \cdots < i_d \leq d+1} \sum_{\pi \in S_d} \operatorname{sgn}(\pi) \mathbb{1}_{V_{i_1,\pi(1)} \times \cdots \times V_{i_d,\pi(d)}},$$

where $\operatorname{sgn}(\pi)$ denotes the sign of the permutation π. Note that δf is $\{-1, 0, 1\}$-valued and $\sigma = [v_0, \ldots, v_d]$, with $v_i \in V_i$, is in the support of δf if and only if there is a permutation $\pi \in S_{d+1}$ such that $v_i \in V_{i,\pi(i)}$ for all $1 \leq i \leq d+1$. It follows that $|\delta f|^2 = \left(\frac{n}{d+1}\right)^{d+1} (d+1)!$. Also, note that every $(d-1)$-simplex in Λ_n^d is contained in $\frac{n}{d+1}$ d-simplices of the support of δf. Thus, we must have $|[f]|^2 \geq \left(\frac{d+1}{n}\right) |\delta f|^2 = |f|^2$. It follows that f is minimal and

$$\zeta_{d-1}(\Lambda_n^d) \leq \frac{d+1}{n} \frac{|\delta f|^2}{|f|^2} = 1,$$

as desired. □

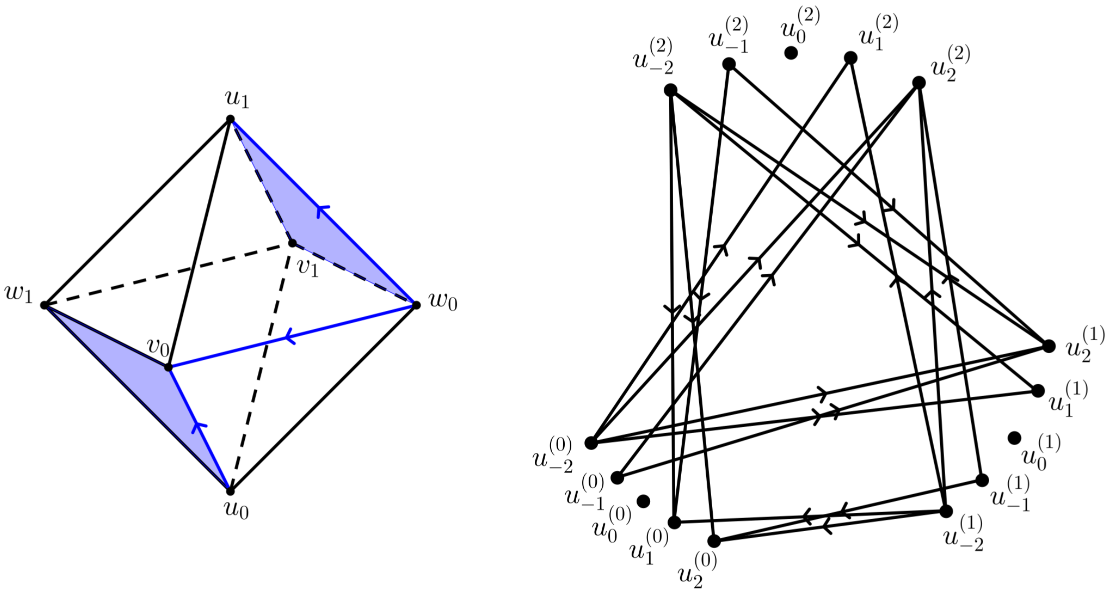


Figure 7.2: On the left we illustrate an example of $f \in C^1(\Lambda_2^2; \mathbb{Z})$ showing $\zeta_1(\Lambda_2^2) \leq 1$. The blue triangles are the triangles in the support of δf. The blue edges are in the support of f. The arrow on the edge indicates the sign of its value in $\{-1, 1\}$. On the right we illustrate the example of a cochain $a \in C^1(\Lambda_5^2; \mathbb{Z})$ showing $\zeta_1(\Lambda_5^2) \leq 1$. Marked edges indicate that an edge is in the support of a and the arrow shows its sign.

Chapter 8

Lower Bounds on Expansion Constants of $\Lambda^d_{n_0,\ldots,n_d}$

In this chapter, we give recursive lower bounds on $\zeta_{d-1}(\Lambda^d_{n_0,\ldots,n_d})$ and $\eta_{d-1}(\Lambda^d_{n_0,\ldots,n_d})$ in terms of $\zeta_{d-2}(\Lambda^{d-1}_{n_1,\ldots,n_d})$ and $\eta_{d-2}(\Lambda^{d-1}_{n_1,\ldots,n_d})$, respectively. We will see that for $d > 1$ (and sufficiently large n_i), $\zeta_{d-1}(\Lambda^d_{n_0,\ldots,n_d}) > \eta_{d-1}(\Lambda^d_{n_0,\ldots,n_d})$.

The proof of the recursive lower bound for $\eta_{d-1}(\Lambda^d_{n_0,\ldots,n_d})$ allows to recover the bound on $\eta_{d-1}(\Lambda^d_n)$ proven in Proposition 6.16 and is a minor recast of said argument. In fact, the bound is already contained in the proof of Proposition 5.7 in [35].

Furthermore, any improvement on a lower bound $\eta_1(\Lambda^2_n)$ would automatically give improved lower bounds on $\eta_{d-1}(\Lambda^d_n)$ for any $d \geq 1$ as well. We will discuss various approaches leading to such improvements in Section 8.4.

To the best of our knowledge, all results in this chapter, except the recursive lower bound for $\eta_{d-1}(\Lambda^d_{n_0,\ldots,n_d})$, are new.

Throughout this chapter, we will write $\Lambda^d_{n_0,\ldots,n_d}$ as $\Lambda^d_{n_0,\ldots,n_d} = U_0 * U_1 * \cdots * U_d$ with $U_i = [n_i]$ for positive integers n_i and $0 \leq i \leq d$.

8.1 Cofilling for $b \in B^d(\Lambda^d_{n_0,\ldots,n_d}; \mathbb{A})$ by Coning from a Vertex

As for K^d_n there is a fairly natural ad-hoc way of defining a cofilling for $b \in B^d(\Lambda^d_{n_0,\ldots,n_d}; \mathbb{A})$ by 'coning' from a vertex. For better illustration, let us first consider the case $d = 2$ and $n_0 = n_1 = n_2 = n$. Write $\Lambda^2_n = U * V * W$ with $U = V = W = [n]$. Given $b \in B^2(\Lambda^2_n; \mathbb{A})$ and $u \in U$ we could attempt to construct a cofilling $a^{(u)} \in C^1(\Lambda^2_n; \mathbb{A})$ of b by defining $a^{(u)}$ to be equal to b_u on $V * W$ and $a^{(u)}(e) = 0$ if $u \in e$. This already achieves that $\delta a^{(u)}(\tau) = b(\tau)$ for all triangles τ containing u. In order to correct the remaining values, we note that we would like to satisfy

$$b(u'v'w) \overset{?!}{=} \delta a^{(u)}(u'v'w') = b(uv'w') - a^{(u)}(u'w') + a^{(u)}(u'v') = b(uv'w') - \delta a^{(u)}_{u'}(v'w').$$

In other words, we would need to choose $a^{(u)}_{u'}$ to be a cofilling of $b_u - b_{u'}$. Fortunately, $b_u - b_{u'}$ is indeed a coboundary in $(\Lambda^2_n)_u$.

Let us generalize this to arbitary dimension d and parts of not necessarily equal size. To this end, let $X := \Lambda^d_{n_0,\dots,n_d}$. For an oriented simplex $\tau = [v_0,\dots,v_k]$ we write $\tau \setminus v_i$ for the oriented simplex $(-1)^i[v_0,\dots,\widehat{v_i},\dots,v_k]$, where $\widehat{v_i}$ indicates that the vertex v_i is omitted. Now, let $\mathbb{A}$ an abelian group and let $b \in B^d(X;\mathbb{A})$. Since the localization of a coboundary along a cycle is a coboundary, $b_u - b_{u'}$ is a $(d-1)$-coboundary in X_u for all $u, u' \in U_i$, $0 \le i \le d$.

For $0 \le i \le d$ and $uu' \in \binom{U_i}{2}$ let $a^{(u,u')} \in C^{d-2}(X_u;\mathbb{A})$ be a cofilling of $b_u - b_{u'} \in B^{d-1}(X_u;\mathbb{A})$.

For $u \in U_i$, $0 \le i \le d$, let $a^{(u)} \in C^{d-1}(X;\mathbb{A})$ be given by

$$a^{(u)}(\sigma) = \begin{cases} 0 & \text{if } u \in \sigma, \\ b_u(\sigma) & \text{if } \sigma \cap U_i = \emptyset, \\ a^{(u,u')}(\sigma \setminus u') & \text{if } \sigma \cap U_i = \{u'\}, u \neq u'. \end{cases}$$

We have:

Lemma 8.1. *With the notations above we have $\delta a^{(u)} = b$ for all $u \in U_i$, $0 \le i \le d$.*

Proof. Let $\sigma = [u_0,\dots,u_d]$ be an oriented d-simplex with $u_i \in U_i$. If $u \in \sigma$, then $\delta a^{(u)}(\sigma) = b(\sigma)$ is immediate. Otherwise, we compute, carefully keeping track of signs,

$$\begin{aligned}
\delta a^{(u)}(\sigma) &= \sum_{k=0}^{d} (-1)^k a^{(u)}([u_0,\dots,\widehat{u_k},\dots,u_d]) \\
&= \sum_{0\le k<i} (-1)^k a^{(u,u_i)}([u_0,\dots,\widehat{u_k},\dots,u_d] \setminus u_i) + (-1)^i b_u([u_0,\dots,\widehat{u_i},\dots,u_d]) \\
&+ \sum_{k<i\le d} (-1)^k a^{(u,u_i)}([u_0,\dots,\widehat{u_k},\dots,u_d] \setminus u_i) \\
&= (-1)^i b_u([u_0,\dots,\widehat{u_i},\dots,u_d]) \\
&+ \sum_{0\le k<i} (-1)^{k+i-1} a^{(u,u_i)}([u_0,\dots,\widehat{u_k},\dots,\widehat{u_i},\dots,u_d]) \\
&+ \sum_{i<k\le d} (-1)^{k+i} a^{(u,u_i)}([u_0,\dots,\widehat{u_i},\dots,\widehat{u_k},\dots,u_d]) \\
&= (-1)^i b_u([u_0,\dots,\widehat{u_i},\dots,u_d]) + (-1)^{i-1}\delta a^{(u,u_i)}([u_0,\dots,\widehat{u_i},\dots u_d]) \\
&= (-1)^i b_u([u_0,\dots,\widehat{u_i},\dots,u_d]) + (-1)^{i-1} b_u([u_0,\dots,\widehat{u_i},\dots u_d]) \\
&- (-1)^{i-1} b_{u_i}([u_0,\dots,\widehat{u_i},\dots u_d]) \\
&= (-1)^i b_{u_i}([u_0,\dots,\widehat{u_i},\dots u_d])) \\
&= b(\sigma).
\end{aligned}$$

□

In what follows, we will call $a^{(u)}$ a *cone for b based at u*. Note that $a^{(u)}$ depends on the particular choice of cofillings $a^{(u,u')}$ of $b_u - b_{u'}$ for $u' \in U_i \setminus u$. Often we will make additional assumptions on these cofillings, e.g., that they are as small as possible with respect to some size function.

8.2 Recursive Lower Bound on $\zeta_{d-1}(\Lambda^d_{n_0,\dots,n_d})$

We show the following recursive lower bound on $\zeta_{d-1}(\Lambda^d_{n_0,\dots,n_d})$.

Proposition 8.2. *For $d \in \mathbb{Z}_{>0}$ let*

$$\zeta_{d-1} := \inf\{\zeta_{d-1}(\Lambda^d_{n_0,\dots,n_d}) : n_0,\dots,n_d \in \mathbb{Z}_{>0}\}.$$

Then

$$\zeta_{d-1} \geq \frac{d+1}{1+\frac{2d^2}{(d+1)\zeta_{d-2}}}.$$

In particular, since $\zeta_0(\Lambda^1_{m,n}) \geq 1$ for all $m,n \in \mathbb{Z}_{>0}$ we get for all $n_0,\dots,n_d \in \mathbb{Z}_{>0}$ that

$$\zeta_{d-1}(\Lambda^d_{n_0,\dots,n_d}) \geq \frac{(d+1)^2}{2^{d+2}-d-3}.$$

Comparing Proposition 8.2 with Proposition 7.7, we see that (for all n_i, $0 \leq i \leq d$, sufficiently large) $\Lambda^d_{n_0,\dots,n_d}$ has strictly better expansion with respect to integer coefficients and ℓ^2_2-norm than with respect to $\mathbb{F}_2$-coefficients and Hamming norm, i.e. that $\zeta_{d-1}(\Lambda^d_{n_0,\dots,n_d}) > \eta_{d-1}(\Lambda^d_{n_0,\dots,n_d})$ for all $d \geq 2$ and $\min_{0\leq i\leq d} n_i$ sufficiently large. It would be interesting to know whether the exponential decay in d is an artifact of the proof method or an actual structural property of $\Lambda^d_{n_0,\dots,n_d}$ which is also present in the setting with $\mathbb{Z}$-coefficients. Is there a lower bound on $\zeta_{d-1}(\Lambda^d_{n_0,\dots,n_d})$ which does not decay to 0 exponentially fast in d or which is even independent of d? So far, we have not excluded the possibility of $\eta_{d-1}(\Lambda^d_{n_0,\dots,n_d}) \geq 1$ which for $d=3$ would imply an asymptotic version of Zarankiewicz' conjecture on the crossing number of complete partite graphs (cf. Conjecture 5.35 in Section 5.5).

For the sake of completeness, let us establish the base case $\zeta_0(\Lambda^1_{m,n}) \geq 1$:

Lemma 8.3. *For all $m,n \in \mathbb{Z}_{>0}$ we have $\zeta_0(\Lambda^1_{m,n}) \geq 1$.*

Proof. Write $X := \Lambda^1_{m,n}$ as $\Lambda^1_{m,n} = U * V$ with $U = [m]$, $V = [n]$. Write $\langle\cdot,\cdot\rangle_w$ for the weighted inner product on cochains induced by Garland weights. That is, given $f,g \in C^k(X;\mathbb{Z})$ we have $\langle f,g\rangle_w = \sum_{\sigma\in X(k)} w(\sigma)f(\sigma)g(\sigma)$ where $w\colon X \to \mathbb{R}_{\geq 0}$ are the Garland weights. Write $\|\cdot\|^2$ for the induced ℓ^2_2-norm, i.e. $\|f\|^2 = \langle f,f\rangle_w$ for $f \in C^k(X;\mathbb{Z})$. Note that $f \in C^0(X;\mathbb{Z})$ is minimal (with respect to $\|\cdot\|^2$) if and only if $|\langle f,\mathbb{1}\rangle_w| \leq 1/2$. Moreover, since f is integer valued, we have $|\langle f,\mathbb{1}\rangle_w| \leq \|f\|^2$. We compute that

$$\begin{aligned}
\|\delta f\| &= \frac{1}{mn}\sum_{u\in U, v\in V}(f(v)-f(u))^2 \\
&= \frac{1}{n}\sum_{v\in V} f(v)^2 + \frac{1}{m}\sum_{u\in U} f(u)^2 - 8\left(\frac{1}{2m}\sum_{u\in U} f(u)\right)\left(\frac{1}{2n}\sum_{v\in V} f(v)\right) \\
&\geq 2\|f\|^2 - 2\langle f,\mathbb{1}\rangle_w^2 \\
&\geq 2\|f\|^2 - |\langle f,\mathbb{1}\rangle_w| \\
&\geq \|f\|^2,
\end{aligned}$$

where we used that $ab \leq (a+b)^2/4$ for all real numbers $a,b \in \mathbb{R}$ for the first inequality, minimality of f for the second, and $|\langle f,\mathbb{1}\rangle_w| \leq \|f\|^2$ for the last inequality. □

Another key ingredient for the proof of Proposition 8.2 is the following lemma:

Lemma 8.4. *Let $d \in \mathbb{Z}_{\geq 0}$. Let $X = \Lambda^d_{n_0,\dots,n_d}$. Then, for every $b \in B^d(X;\mathbb{Z})$ we have*

$$\sum_{i=0}^{d} \frac{1}{n_i} \sum_{uu' \in \binom{U_i}{2}} |b_{u'} - b_u|^2 \leq d|b|^2.$$

Before we give the somewhat technical proof of Lemma 8.4, let us first prove Proposition 8.2 using this lemma.

Proof of Proposition 8.2. Let $X = \Lambda^d_{n_0,\dots,n_d}$. Write $|\cdot|^2$ for the ℓ_2^2-norm and $\|\cdot\|^2$ for the Garland weighted ℓ_2^2-norm. Since $\tilde{H}^{d-1}(X;\mathbb{Z}) = 0$ it suffices, by Lemma 3.9, to prove a cofilling inequality. To this end, let $b \in B^d(X;\mathbb{Z})$. By Lemma 8.4 there is $i \in \{0,\dots,d\}$ with

$$\frac{1}{n_i} \sum_{uu' \in \binom{U_i}{2}} |b_{u'} - b_u|^2 \leq \frac{d}{d+1} |b|^2$$

or, equivalently,

$$\frac{1}{n_i} \sum_{uu' \in \binom{U_i}{2}} \|b_{u'} - b_u\|^2 \leq \frac{d}{d+1} n_i \|b\|^2.$$

For $u \in U_i$ let $a^{(u)}$ be a cone for b based at u such that $a^{(u,u')}$ in the definition of $a^{(u)}$ is a minimal cofilling of $b_u - b_{u'}$ with respect to the Garland weighted ℓ_2^2-norm $\|\cdot\|^2$ on X_u. In particular, we have

$$\|a^{(u,u')}\|^2 \leq \frac{1}{\zeta_{d-2}} \|b_u - b_{u'}\|^2.$$

Averaging over all $u \in U_i$, we estimate

$$\begin{aligned}
\min_{u \in U_i} \|a^{(u)}\|^2 &\leq \frac{1}{n_i} \sum_{u \in U_i} \|a^{(u)}\|^2 \\
&= \frac{1}{d+1} \|b\|^2 + \frac{2}{n_i^2} \frac{d}{d+1} \sum_{uu' \in \binom{U_i}{2}} \|a^{(u,u')}\|^2 \\
&\leq \frac{1}{d+1} \|b\|^2 + \frac{2d}{(d+1)\zeta_{d-2} n_i^2} \sum_{uu' \in \binom{U_i}{2}} \|b_u - b_{u'}\|^2 \\
&\leq \left(\frac{1}{d+1} + \frac{2}{\zeta_{d-2}} \left(\frac{d}{d+1} \right)^2 \right) \|b\|^2.
\end{aligned}$$

Since $b \in B^d(X;\mathbb{Z})$ was arbitrary, we conclude

$$\zeta_{d-1} \geq \frac{d+1}{1 + \frac{2d^2}{(d+1)\zeta_{d-2}}},$$

proving the first part. Solving this recurrence with $\zeta_0(\Lambda^1_{m,n}) \geq 1$ (which holds by Lemma 8.3) gives the second part. □

It remains to prove Lemma 8.4. For this, we need some preparation. Let $X = \Lambda^d_{n_0,\dots,n_d} = U_0 * \cdots * U_d$ and $Y = \Lambda^r_{m_0,\dots,m_r} = V_0 * \cdots * V_r$ with $V_i = [m_i]$, $0 \le i \le r$ for positive integers $m_0, \dots, m_r$ and $r \in \mathbb{Z}_{\ge 0}$. For $I \subseteq \{0, \dots, d\}$ we let $X_I := X_{\{u_i : i \in I\}}$ where $u_i \in U_i$ for $i \in I$. In other words, X_I is the link of X at some simplex $\sigma \in X$ with one vertex from each U_i with $i \in I$. For $I \subseteq \{0, \dots, d\}$ write $I^c := \{0, \dots, d\} \setminus I$ for the complement of I. For $f \in C^d(X; \mathbb{R})$ let $f_I \in C^{d-|I|}(X_I; \mathbb{R})$ be given by $f_I := \sum_{\sigma \in X_{I^c}(|I|-1)} f_\sigma$ (as usual f_σ denotes the localization of f at σ). Moreover, for $f \in C^r(Y; \mathbb{R})$ we write $Z(f)$ for

$$Z(f) := \sum_{i=0}^{r} \sum_{v_i v_i' \in \binom{V_i}{2}} |\langle f, z_{(v_i v_i')_{i \in \{0,\dots,r\}}} \rangle|^2,$$

where $z_{(v_i v_i')_{i \in \{0,\dots,r\}}} = \otimes_{i \in \{0,\dots,r\}} (v_i' - v_i) \in Z_r(Y; \mathbb{R})$ is the fundamental cycle of the octahedral sphere $\{v_0, v_0'\} * \cdots * \{v_r, v_r'\}$. We extend this definition to $r = -1$ in which case Y is the empty simplicial complex with the single (-1)-simplex $\emptyset$. So $f \in C^{-1}(\{\emptyset\}; \mathbb{R})$ is a constant function $f = \alpha \mathbb{1}_\emptyset$ and we define $Z(f) := \alpha^2$.

With all these notations we have

Claim 8.5. *Let $f \in C^d(X; \mathbb{R})$, $g \in C^{d-1}(X; \mathbb{R})$ then*

$$n_0 \cdot n_1 \cdot \dots \cdot n_d |f - \delta g|^2 = \sum_{I \subseteq \{0,\dots,d\}} Z((f - \delta g)_I).$$

Moreover, for $b \in B^d(X; \mathbb{R})$ we have

$$n_0 \cdot n_1 \cdot \dots \cdot n_d |b|^2 = \sum_{I \subseteq \{0,\dots d\}, I \ne \emptyset} Z(b_I).$$

Proof. We argue by induction on d. We will use the fact that for a 0-cochain $h \in C^0(K_U; \mathbb{R})$ on the complete graph K_U with n vertices we have

$$|\delta h|^2 = n|h|^2 - \left(\sum_{u \in U} h(u) \right)^2. \tag{8.1}$$

The base case $d = 0$ is essentially this identity. Indeed, for $f \in C^0([n_0]; \mathbb{R})$ and $g = \alpha \mathbb{1}_\emptyset \in C^{-1}([n_0]; \mathbb{R})$ we have

$$Z((f - \delta g)_\emptyset) = \sum_{uu' \in \binom{[n_0]}{2}} |(f - \delta g)(u') - (f - \delta g)(u)|^2$$

and

$$Z((f - \delta g)_{\{0\}}) = \left(\sum_{u \in [n_0]} (f - \delta g)(u) \right)^2.$$

For the inductive step, using (8.1), we compute for $f \in C^d(X; \mathbb{R})$ and $g \in C^{d-1}(X; \mathbb{R})$ that

$$\begin{aligned} |f - \delta g|^2 &= \sum_{i=0}^{d} \sum_{u_i \in U_i} \left(f([u_0, \dots, u_d]) - \sum_{j=0}^{d} (-1)^j g([u_0, \dots, \widehat{u_j}, \dots, u_d]) \right)^2 \\ &= \frac{1}{n_0} \sum_{u_0 u_0' \in \binom{U_0}{2}} |f_{u_0'} - f_{u_0} - (\delta g_{u_0'} - \delta g_{u_0})|^2 + \frac{1}{n_0} |(f - \delta g)_{\{0\}}|^2. \end{aligned} \tag{8.2}$$

Note that for $I \subseteq \{u_1, \dots, u_d\}$ we have

$$\sum_{u_0u_0' \in \binom{U_0}{2}} Z((f_{u_0'} - f_{u_0} - \delta(g_{u_0'} - g_{u_0}))_I) = Z((f - \delta g)_I)$$

and

$$Z\left(\left((f - \delta g)_{\{0\}}\right)_I\right) = Z\left((f - \delta g)_{I \cup \{0\}}\right).$$

Using these and applying the induction hypothesis in (8.2), we get

$$\begin{aligned} n_0 \cdot \dots \cdot n_d |f - \delta g|^2 &= \sum_{u_0u_0' \in \binom{U_0}{2}} \sum_{I \subseteq \{u_1,\dots,u_d\}} Z((f_{u_0'} - f_{u_0} - \delta(g_{u_0'} - g_{u_0}))_I) \\ &\quad + \sum_{I \subseteq \{u_1,\dots,u_d\}} Z\left(\left((f - \delta g)_{\{0\}}\right)_I\right) \\ &= \sum_{I \subseteq \{u_0,\dots,u_d\}, u_0 \notin I} Z((f - \delta g)_I) + \sum_{I \subseteq \{u_0,\dots,u_d\}, u_0 \in I} Z((f - \delta g)_I) \\ &= \sum_{I \subseteq \{u_0,\dots,u_d\}} Z((f - \delta g)_I), \end{aligned}$$

as desired.

The second part follows from the fact that for $b \in B^d(X;\mathbb{R})$ we have $\langle b, z\rangle = 0$ for all $z \in Z_d(X;\mathbb{R})$ and, hence, $Z(b_\emptyset) = 0$. □

We conclude this section with the proof of Lemma 8.4.

Proof of Lemma 8.4. For $b \in B^d(X;\mathbb{R})$ we compute using Claim 8.5 that

$$\begin{aligned} \sum_{i=0}^{d} \left(\prod_{j \in \{0,\dots,d\} \setminus \{i\}} n_i \right) \sum_{uu' \in \binom{U_i}{2}} |b_{u'} - b_u|^2 &= \sum_{i=0}^{d} \sum_{uu' \in \binom{U_i}{2}} \sum_{\substack{I \subseteq \{0,\dots d\} \setminus \{i\} \\ I \neq \emptyset}} Z((b_{u'} - b_u)_I) \\ &= \sum_{i=0}^{d} \sum_{\substack{I \subseteq \{0,\dots d\} \setminus \{i\} \\ I \neq \emptyset}} Z(b_I) \\ &= \sum_{\substack{I \subseteq \{0,\dots d\} \setminus \{i\} \\ I \neq \emptyset, |I| \leq d}} (d + 1 - |I|) Z(b_I) \\ &\leq d \sum_{\substack{I \subseteq \{0,\dots d\} \\ I \neq \emptyset}} Z(b_I) \\ &= d|b|^2 \left(\prod_{i=0}^{d} n_i \right). \end{aligned}$$

Dividing both sides by $\prod_{i=0}^{d} n_i$ finishes the proof. □

8.3 Recursive Lower Bound on $\eta_{d-1}(\Lambda^d_{n_0,\dots,n_d})$

For $\eta_{d-1}(\Lambda^d_{n_0,\dots,n_d})$ we have a similar recursive bound:

Proposition 8.6. *For $d \in \mathbb{Z}_{>0}$ let*

$$\eta_{d-1} := \inf\{\eta_{d-1}(\Lambda^d_{n_0,\dots,n_d}) : n_0, \dots, n_d \in \mathbb{Z}_{>0}\}.$$

Then, we have for $d \geq 2$

$$\eta_{d-1} \geq \frac{d+1}{1+\frac{2d}{\eta_{d-2}}}.$$

In particular,

$$\eta_{d-1} \geq \frac{d+1}{2^{d-2}\left(1+\frac{3}{\eta_1}\right)-1}.$$

and using that $\eta_0(\Lambda^1_{m,n}) \geq 1$ we get

$$\eta_{d-1} \geq \frac{d+1}{3 \cdot 2^{d-1} - 1}.$$

For the sake of completeness, let us first show the case $d = 0$ which asks to show that the complete bipartite graph has edge expansion at least 1 with respect to Garland weighted Hamming norm.

Lemma 8.7. *Let $m, n \in \mathbb{Z}_{>0}$. Then $\eta_0(\Lambda^1_{m,n}) \geq 1$.*

Proof. Write $X := \Lambda^1_{m,n}$ as $X = U * V$ with $U = [m]$ and $V = [n]$. Write $\|\cdot\|$ for the Garland weighted Hamming norm. Let $S \subseteq U, T \subseteq V$ and $c = \mathbb{1}_S + \mathbb{1}_T \in C^0(X; \mathbb{F}_2)$. Assume c is minimal with respect to $\|\cdot\|$, i.e. that $\|c\| \leq 1/2$. We compute

$$\begin{aligned}
\|\delta c\| &= \frac{1}{mn}\left(|S|(n-|T|) + |T|(m-|S|)\right) \\
&= 2\|c\| - 8\left(\frac{|S|}{2m}\right)\cdot\left(\frac{|T|}{2n}\right) \\
&\geq 2\|c\| - 2\left(\frac{|S|}{2m} + \frac{|T|}{2n}\right)^2 \\
&= 2\|c\| - 2\|c\|^2 \\
&\geq \|c\|,
\end{aligned}$$

where we used that $4ab \leq (a+b)^2$ for all real numbers $a, b \in \mathbb{R}$ for the first inequality and minimality of c for the second inequality. $\square$

Proof of Proposition 8.6. Let $X = \Lambda^d_{n_0,\dots,n_d}$. Write $|\cdot|$ for the Hamming norm and $\|\cdot\|$ for the Garland weighted Hamming norm. Since $\tilde{H}^{d-1}(X; \mathbb{F}_2) = 0$ it suffices, by Lemma 3.9, to prove a cofilling inequality. To this end, let $b \in B^d(X; \mathbb{F}_2)$. We proceed as in the proof of Proposition 8.2 except that for $\mathbb{F}_2$-coefficients we do not have Lemma 8.4 at our hands. Instead, we will use the inequality

$$\sum_{i=0}^{d} \frac{1}{n_i} \sum_{uu' \in \binom{U_i}{2}} |b_u + b_{u'}| \leq (d+1)|b|, \tag{8.3}$$

which holds for all $b \in B^d(X;\mathbb{F}_2)$ and follows by an application of the triangle inequality. In particular, (8.3) implies that there is $i \in \{0,\ldots,d\}$ such that

$$\frac{1}{n_i} \sum_{uu' \in \binom{U_i}{2}} |b_u + b_{u'}| \leq |b| \tag{8.4}$$

or equivalently, that

$$\frac{1}{n_i} \sum_{uu' \in \binom{U_i}{2}} \|b_u + b_{u'}\| \leq n_i \|b\|.$$

Now, for $u \in U_i$ let $a^{(u)} \in C^{d-1}(X;\mathbb{F}_2)$ be a cone for b based at u such that $a^{(u,u')}$ is a minimal cofilling (with respect to the Garland weighted Hamming norm $\|\cdot\|$) of $b_u + b_{u'} \in B^{d-1}(X_u;\mathbb{F}_2)$ for all $uu' \in \binom{U_i}{2}$. In particular,

$$\|a^{(u,u')}\| \leq \frac{1}{\eta_{d-2}} \|b_u + b_{u'}\|.$$

Averaging over $u \in U_i$, we estimate

$$\begin{aligned}
\min_{u \in U_i} \|a^{(u)}\| &\leq \frac{1}{n_i} \sum_{u \in U_i} \|a^{(u)}\| \\
&= \frac{1}{d+1}\|b\| + \frac{2}{n_i^2}\frac{d}{d+1} \sum_{uu' \in \binom{U_i}{2}} \|a^{(u,u')}\| \\
&\leq \frac{1}{d+1}\|b\| + \frac{2d}{(d+1)\eta_{d-2}n_i^2} \sum_{uu' \in \binom{U_i}{2}} \|b_u + b_{u'}\| \\
&\leq \frac{1}{d+1}\left(1 + \frac{2d}{\eta_{d-2}}\right)\|b\|,
\end{aligned}$$

where we used expansion of X_u for the first inequality and (8.4) for the second inequality. Since $b \in B^d(X;\mathbb{F}_2)$ was arbitrary, we get

$$\eta_{d-1} \geq \frac{d+1}{1 + \frac{2d}{\eta_{d-2}}}.$$

This shows the first part. The other parts easily follow from solving the recursion using $\eta_0(\Lambda^1_{m,n}) \geq 1$ (which holds by Lemma 8.7). □

8.4 Improved Lower Bound on $\eta_1(\Lambda^2_n)$

Note that Proposition 8.6 implies that $\eta_1(\Lambda^2_n) \geq 3/5$. In this section, we will improve upon this bound.

Without using a computer we are able to show the following lower bound:

Proposition 8.8. *For all $n \in \mathbb{Z}_{>0}$ we have $\eta_1(\Lambda^2_n) \geq 0.6358$.*

Relying on the computational power of a computer, we can further improve this to:

Proposition 8.9. *For all $n \in \mathbb{Z}_{>0}$ we have $\eta_1(\Lambda^2_n) \geq 0.67159$.*

The proof of Proposition 8.9 shows that for sufficiently small or sufficiently large cochains we do have normalized expansion at least 3/4. More precisely

Corollary 8.10. *Let $c \in C^1(\Lambda_n^2; \mathbb{F}_2)$ be minimal. Write $\|\cdot\|$ for the normalized Hamming norm. If $0 < \|c\| \leq 13/124$ or $\|c\| \geq 1/3$ then*

$$\frac{\|\delta c\|}{\|c\|} \geq 3/4.$$

A common feature of the proofs of Proposition 8.8 and Proposition 8.9 is that they deal with cochains of small and large norm separately. This dichotomy between small and large cochains has occurred before, e.g. in [72, 97, 96].

We will start our argument by improving upon the 3/5-bound for large cochains. We achieve this by a variation of the analysis of the random cofilling argument showing $\eta_1(\Lambda_n^2) \geq 3/5$.

Using ideas from [97] and [108], we then give a local-to-global argument to improve upon the 3/5-bound for small cochains.

Before combining these two into a proof of Proposition 8.8, it will be helpful to establish the existence of $\lim_{n\to+\infty} \eta_1(\Lambda_n^2)$.

Using a computer we can further improve upon the expansion of small cochains in two ways: (i) by replacing the application of the triangle inequality in the random cofilling argument giving the 3/5-lower bound with a stronger bound and (ii) by using a flag algebra approach inspired by [86], where flag algebras are used to show good expansion properties for small cochains in the complete 2-dimensional complex K_n^2.

Throughout this section the following notation will be useful. We write $|\cdot|$ for the Hamming norm of cochains on Λ_n^2 and $\|\cdot\|$ for the normalized Hamming norm. Moreover, for $0 \leq \alpha \leq 1/2$ we let

$$\eta(\alpha) := \liminf_{n\to+\infty} \min\{\|\delta c\| : c \in C^1(\Lambda_n^2; \mathbb{F}_2), \|[c]\| \geq \alpha\}.$$

8.4.1 Expansion of Large Cochains

We revisit the random cofilling argument. To this end, let $b \in B^2(\Lambda_n^2; \mathbb{F}_2)$, $b = \delta a$ for some $a \in C^1(\Lambda_n^2; \mathbb{F}_2)$. For $u \in U_0$ let $a^{(u)}$ be a cone for b based at u such that $a^{(u,u')} \in C^0((\Lambda_n^2)_u; \mathbb{F}_2)$ is a minimal cofilling (with respect to the Hamming norm) of $b_u + b_{u'} \in B^1((\Lambda_n^2)_u; \mathbb{F}_2)$ for all $u' \in U_0 \setminus \{u\}$.

Averaging over $u \in U_0$ we get as before

$$\begin{aligned} n|[a]| &\leq \sum_{u\in U_0} |a^{(u)}| \\ &= |b| + 2 \sum_{uu' \in \binom{U_0}{2}} |a^{(u,u')}| \\ &\leq |b| + \frac{4}{n} \sum_{uu' \in \binom{U_0}{2}} |b_u + b_{u'}|, \end{aligned}$$

where we used that $h_0(K_{n,n}) \geq \frac{n}{2}$ for the last step.

Now, instead of applying the triangle inequality to the last term, we can also rewrite it as

$$\sum_{uu' \in \binom{U_0}{2}} |b_u + b_{u'}| = \sum_{e \in U_1 * U_2} |b_e|(n - |b_e|).$$

Indeed, for $uu' \in \binom{U_0}{2}$, $|b_u + b_{u'}|$ counts the edges $e \in U_1 * U_2$ such that precisely one of the triangles $u \otimes e$ and $u' \otimes e$ is in the support of b. On the other hand $|b_e|(n - |b_e|)$, for $e \in U_1 * U_2$, counts the number of pairs of triangles sharing the edge e and with precisely one of the triangles in the support of b. Thus, the identity above holds by double counting.

Further note that, by the Cauchy–Schwarz inequality, we have

$$|b|^2 = \left(\sum_{e \in U_1 * U_2} |b_e| \right)^2 \le n^2 \sum_{e \in U_1 * U_2} |b_e|^2.$$

Using these, we get

$$\begin{aligned} \sum_{uu' \in \binom{U_0}{2}} |b_u + b_{u'}| &= \sum_{e \in U_1 * U_2} |b_e|(n - |b_e|) \\ &= n|b| - \sum_{e \in U_1 * U_2} |b_e|^2 \\ &\le n|b| - \frac{1}{n^2}|b|^2 \\ &= n(1 - \|b\|)|b|. \end{aligned}$$

We deduce that

$$\|b\| \ge \frac{3}{5 - 4\|b\|} \|[a]\|.$$

In particular, if $\|[a]\| \ge \alpha$ we get that

$$\|b\|(5 - 4\|b\|) \ge 3\alpha.$$

This implies that

$$-4\eta(\alpha)^2 + 5\eta(\alpha) - 3\alpha \ge 0.$$

Solving this for $\eta(\alpha)$ we conclude

Lemma 8.11. *For any $0 \le \alpha \le \frac{1}{2}$ we have*

$$\eta(\alpha) \ge \frac{5}{8} - \frac{1}{8}\sqrt{25 - 48\alpha}.$$

In particular, for $\alpha \ge 1/3$ we have $\eta(\alpha) \ge \frac{3}{4}\alpha$.

We find it interesting to observe that $1/3$ is precisely the density of the cochain which shows $\eta_1(\Lambda^2_n) \le 3/4$ for n divisible by 4 (cf. Theorem 7.7).

8.4.2 Expansion for Small Cochains – an Upper Bound First

Before we give lower bounds on $\eta(\alpha)$ for small α, it is worthwhile to give an upper bound on $\eta(\alpha)$ we can compare our lower bounds to. Interestingly, we will be able to match our upper bound with a lower bound on $\eta(\alpha)$ of the same order for $\alpha \to 0$. We should remark that our construction here is a simple extension of a construction for the complete 2-dimensional simplicial complex K_n^2 given in [108].

Lemma 8.12. *For $\alpha \in [0, 2/9]$ we have*

$$\eta(\alpha) \leq \frac{3}{4}\alpha\left(1+\sqrt{1-4\alpha}\right) = \frac{3}{2}\alpha - \frac{3}{2}\alpha^2 - \frac{3}{2}\alpha^3 + O(\alpha^4)$$

as $\alpha \to 0$.

Proof. Let $\sigma = \frac{1}{2}\left(1-\sqrt{1-4\alpha}\right)$. Since $\alpha \leq 2/9$ we get $\sigma \leq 1/3$. For $i \in \{0,1,2\}$ partition $U_i = U_i^{(0)} \sqcup U_i^{(1)} \sqcup U_i^{(2)}$ with $|U_i^{(0)}| = \sigma$ and $|U_i^{(1)}| = |U_i^{(2)}| = \frac{1-\sigma}{2}n$.[1] Let $c \in C^1(\Lambda_n^2; \mathbb{F}_2)$ be the cochain with support being all edges in $U_i^{(0)} * U_j^{(1)}$ for $i, j \in \{0,1,2\}$, $i \neq j$. Note that

$$\|c\| = \frac{1}{3n^2} \cdot 6\sigma\frac{1-\sigma}{2}n^2 = \sigma(1-\sigma) = \alpha$$

while

$$\|\delta c\| = \frac{1}{n^3}6\left(\frac{1-\sigma}{2}\right)^2\sigma n^3 = \frac{3}{2}(1-\sigma)^2\sigma n^3 = \frac{3}{2}(1-\sigma)\|c\| = \frac{3}{4}\alpha\left(1+\sqrt{1-4\alpha}\right).$$

This gives the desired bound if we can show that c is minimal. To this end, we observe that if $c' \in C^1(\Lambda_n^2; \mathbb{F}_2)$ satisfies $\delta c' = \delta c$ then every triangle in the support of δc must have at least one of its boundary edges in the support of c'. But every edge in Λ_n^2 is contained in at most $\frac{1-\sigma}{2}n$ triangles of δc. Thus, $|c'| \geq |\delta c|\frac{2n}{1-\sigma} = |c|$, showing minimality of c. □

8.4.3 A Local-to-Global Argument for Expansion of Small Cochains

Inspired by some arguments in [97, 108], we give a lower bound on $\eta(\alpha)$ for small α.

Let $c \in C^1(\Lambda_n^2; \mathbb{F}_2)$. For $i \in \{0,1,2,3\}$ write t_i for the number of triangles in Λ_n^2 with precisely i of their boundary edges in c. We have the following simple claim.

Claim 8.13. *(i)* $|\delta c| = t_1 + t_3$.

(ii) $n|c| = t_1 + 2t_2 + 3t_3$.

(iii) $\sum_{x \in \Lambda_n^2(0)} |\delta_{(\Lambda_n^2)_x} c_x| = 2t_1 + 2t_2$.

Proof. (i) is by definition of the coboundary map. (ii) follows from the fact that every edge is contained in precisely n triangles. For (iii) we note that an edge $e = yz \in (\Lambda_n^2)_x$ contributes to $|\delta_{(\Lambda_n^2)_x} c_x|$ if and only if precisely one of the edges xy and xz is in the support of c. Depending on the value of $c(yz)$ this means that the triangle xyz has 1 or 2 of its boundary edges in c. Moreover, every such triangle gets counted twice. □

[1] Strictly speaking we should take divisibility issues into account here. If necessary, we should approximate σ with a rational number q and then choose an infinite sequence of positive integers n_k such that σn_k and $\frac{1-\sigma}{2}n_k$ are both integers. This can be all worked out but for the sake of a simpler presentation we sweep these technicalities under the rug.

Subtracting part (ii) from part (iii) in the previous claim, we deduce

$$|\delta c| \geq \sum_{x\in\Lambda_n^2(0)} |\delta_{(\Lambda_n^2)_x} c_x| - n|c|.$$

Since every vertex link $(\Lambda_n^2)_x$ is a complete bipartite graph, we can decompose $c_x = c_x^L + c_x^R$ where the support of c_x^L and c_x^R are contained in different parts of $(\Lambda_n^2)_x$. With this, we can rewrite

$$\begin{aligned}
|\delta c| &\geq \sum_{x\in\Lambda_n^2(0)} |\delta_{(\Lambda_n^2)_x} c_x| - n|c| \\
&= \sum_{x\in\Lambda_n^2(0)} \Big(|c_x^L|(n-|c_x^R|) + |c_x^R|(n-|c_x^L|)\Big) - n|c| \\
&= n|c| - 2\sum_{x\in\Lambda_n^2(0)} |c_x^R||c_x^L| \\
&\geq n|c| - \frac{1}{2}\sum_{x\in\Lambda_n^2(0)} |c_x|^2,
\end{aligned}$$

where we used that $ab \leq (a+b)^2/4$ for all real numbers a, b for the last inequality.

Thinking of c as a graph, we see that $\sum_{x\in\Lambda_n^2(0)} |c_x|^2$ is the sum of its squared vertex degrees.

We would like to find strong upper bounds on $\sum_{x\in\Lambda_n^2(0)} |c_x|^2$ for small minimal cochains c. For the bound we will give below, we will not use the full strength of minimality but we will only use the fact that if c is minimal then $|c_x| \leq n$ for all $x \in \Lambda_n^2(0)$ (otherwise we would have $|c + \delta \mathbb{1}_x| < |c|$). Moreover, we will give an upper bound on $\sum_{v\in K_{3n}(0)} |c_v|^2$ for any cochain $c \in C^1(K_{3n}; \mathbb{F}_2)$ with $|c_v| \leq n$ for all $v \in K_{3n}(0)$. We use the same argument as in Lemma 10 in [108] with different parameters. For the sake of completeness we give the full argument here. It would be interesting to exploit more of the structure of minimal cochains and the fact that we are considering cochains in Λ_n^2 rather than in K_{3n}.

Lemma 8.14. *Let $G = (V, E)$ be a graph with $|V| = 3n$, $|E| = 3\alpha n^2$ for some $\alpha \in [0, 1/6]$ and such that $\deg(v) \leq n$ for all $v \in V$. Then, as $n \to +\infty$, we have*

$$\sum_{v\in V} \deg(v)^2 \leq (\sigma + \sigma^2 - \sigma^3 + o(1))n^3,$$

where $\sigma = 1 - \sqrt{1-6\alpha}$.

Proof. Given $G = (V, E)$ as in the assumption, we can turn G into a specific form using a sequence of transformations which do not change the number of edges and do not decrease the sum of squared degrees. The sum of squared degrees for graphs in this specific form will be easy to analyze.

To start with, we number the vertices $v_1, \dots, v_{3n}$ such that $d_1 \geq \dots \geq d_{3n}$ where we let $d_i := \deg(v_i)$. Note that if $d_i \geq d_j$ then $(d_i+1)^2 + (d_j-1)^2 > d_i^2 + d_j^2$. Thus, if we change our graph such that d_i increases by 1 and d_j decreases by 1 while all other vertex degrees remain fixed, we do not change the number of edges but increase the sum of squared degrees.

The ordering of the vertices induces an orientiation of the edges. If $e = \{v_i, v_j\}$ is an edge with $i < j$ then we call v_i the left end of e and v_j the right end of j.

We claim that we can transform G without changing its number of edges and without decreasing the sum of its squared degrees into a graph with the following three properties

(i) If k is such that $d_1 = d_2 = \cdots = d_k = n$ while $d_{k+1} < n$ then we can assume that all left ends of all edges are among $v_1, \ldots, v_{k+1}$.

(ii) $v_1, \ldots, v_k$ form a clique, i.e. $\{v_i, v_j\} \in E$ for all $1 \leq i < j \leq k$.

(iii) The right neighbours of each v_i, $1 \leq i \leq k$, form a contiguous interval $v_{i+1}, \ldots, v_{n+1}$.

If (i) does not hold, there is an edge $\{v_i, v_j\}$ with $i > k+1$. We can replace this edge with $\{v_{k+1}, v_j\}$ without decreasing the sum of squared degree. This possibly increases k but it also increases $\sum_{i=1}^{k+1} d_i$ which is a bounded function. So after finitely many steps we must satisfy (i).

Then since we assume $\alpha \leq 1/6 < 1/3$ we have $k < n$. Now suppose $1 \leq i < j \leq k$ with $\{v_i, v_j\} \notin E$. Since $d_i = d_j = n$, v_i and v_j are incident to at least two vertices among $v_{k+2}, \ldots, v_{3n}$. In particular, there are $l, m \geq k+2$, $l \neq m$, such that $\{v_i, v_l\} \in E$, $\{v_j, v_m\} \in E$. By (i) we have $\{v_l, v_m\} \notin E$. Thus, we can delete the edges $\{v_i, v_l\}$ and $\{v_j, v_m\}$ and add the edges $\{v_i, v_j\}$ and $\{v_l, v_m\}$. This does not change the sum of squared degrees or the number of edges but increases the number of edges on $\{v_1, \ldots, v_k\}$. Then transformations to achieve (i) do not affect the number of edges in $\{v_1, \ldots, v_k\}$. Thus, after finitely many steps we can achieve both (i) and (ii).

Finally, if v_i, $1 \leq i \leq k$, is connected to v_{l+1} but not to v_l for some $l > k$, then we can replace the edge $\{v_i, v_{l+1}\}$ with $\{v_i, v_l\}$. This also achieves (iii) after finitely many steps.

It remains to analyze the sum of squared degrees for graphs satisfying (i), (ii) and (iii). Note that such a graph is such that $v_1, \ldots, v_k$ is connected to the first n vertices and there are no other edges except possible some edges incident to v_{k+1}. Thus,

$$3n^2\alpha = |E| = kn - \binom{k}{2} + O(n).$$

Let $k = \sigma n$. Then, the above equation gives $\sigma = 1 - \sqrt{1 - 6\alpha} + o(1)$. Finally, we estimate

$$\sum_{v \in V} \deg(v)^2 \leq kn^2 + (n-k)k^2 + O(n^2) = \sigma n^3 + (1-\sigma)\sigma^2 n^3 + O(n^2) = (\sigma + \sigma^2 - \sigma^3 + o(1))n^3,$$

as desired. □

Plugging this into the bound

$$|\delta c| \geq n|c| - \frac{1}{2} \sum_{x \in \Lambda_n^2(0)} |c_x|^2$$

one easily deduces the following corollary.

Corollary 8.15. *Let $\alpha \in [0, 1/6]$. Let $\sigma = 1 - \sqrt{1 - 6\alpha}$. Then, for any $\varepsilon > 0$ there is some positive integer N such that for all $n \geq N$ and $c \in C^1(\Lambda_n^2; \mathbb{F}_2)$ minimal with $\|c\| \geq \alpha$, we have*

$$\|\delta c\| \geq 3\left(1 - \frac{1}{6\alpha}(\sigma + \sigma^2 - \sigma^3 + \varepsilon)\right)\|c\|.$$

Moreover,

$$\eta(\alpha) \geq 3\alpha - \frac{1}{2}(\sigma + \sigma^2 - \sigma^3) = \frac{1}{2}(1 - \sqrt{1-6\alpha})(1-6\alpha) = \frac{3}{2}\alpha - \frac{27}{4}\alpha^2 - \frac{27}{4}\alpha^3 + O(\alpha^4),$$

as $\alpha \to 0$.

8.4.4 Existence of $\lim_{n\to+\infty} \eta_1(\Lambda^2_n)$ and Relation to $\eta(\alpha)$

Before we give the proof of Proposition 8.8, it is worth to take a small detour and show that the limit $\lim_{n\to+\infty} \eta_1(\Lambda^2_n)$ exists. This might seem obvious but requires some argument. Moreover, we will clarify the relation of $\lim_{n\to+\infty} \eta_1(\Lambda^2_n)$ and $\eta(\alpha)$.

Regarding the existence of the limit $\lim_{n\to+\infty} \eta_1(\Lambda^2_n)$ we will show something slightly stronger:

Proposition 8.16. *Let* $0 < \eta < 1$. *Then the following are equivalent*

(i) There exists $n_0 \in \mathbb{Z}_{>0}$ *such that* $\eta_1(\Lambda^2_{n_0}) < \eta$.

(ii) $\limsup_{n\to+\infty} \eta_1(\Lambda^2_n) < \eta$.

(iii) $\liminf_{n\to+\infty} \eta_1(\Lambda^2_n) < \eta$.

In particular, the limit

$$\eta_\infty := \lim_{n\to+\infty} \eta_1(\Lambda^2_n)$$

exists and we have $\eta_1(\Lambda^2_n) \geq \eta_\infty$ *for all* n.

For the proof of Proposition 8.16 we will use Corollary 7.13 and the following lemma:

Lemma 8.17. *There is a constant* $C > 0$ *such that for sufficiently large* n *we have*

$$|\eta_1(\Lambda^2_{n+1}) - \eta_1(\Lambda^2_n)| \leq \frac{C}{n}.$$

Before proving Lemma 8.17, let us show how it helps to prove Proposition 8.16.

Proof of Proposition 8.16 assuming Lemma 8.17. $(ii) \Rightarrow (iii)$ and $(iii) \Rightarrow (i)$ are easy. For the implication $(i) \Rightarrow (ii)$ we assume that $\eta_1(\Lambda^2_{n_0}) < \eta$ for some $n_0 \in \mathbb{Z}_{>0}$ and $\eta \in (0,1)$. Let $\delta > 0$ such that $\eta_1(\Lambda^2_{n_0}) < \eta - \delta$. By Corollary 7.13 we get $\eta_1(\Lambda^2_{kn_0}) \leq \eta_1(\Lambda^2_{n_0}) \leq \eta - \delta$ for all positive integers k. Let $M > 0$ such that the conclusion of Lemma 8.17 holds for all $n \geq M$. Let $n \in \mathbb{Z}$ with $n \geq M + n_0$. Write $n = kn_0 + r$ for non-negative integers k and r with $0 \leq r \leq n_0 - 1$. Using Lemma 8.17 we get

$$\eta_1(\Lambda^2_n) \leq \eta_1(\Lambda^2_{kn_0}) + \frac{Cr}{kn_0} \leq \eta - \delta + \frac{2Cn_0}{n}.$$

This implies that $\eta_1(\Lambda^2_n) \leq \eta - \delta/2$ for all $n > \max\{M + n_0, \frac{4Cn_0}{\delta}\}$. In particular

$$\limsup_{n\to+\infty} \eta_1(\Lambda^2_n) < \eta,$$

as desired.

To see the second part, let $\eta := \liminf_{n\to+\infty} \eta_1(\Lambda_n^2)$. Then using the implication $(iii) \Rightarrow (i)$ we get that for any $\varepsilon > 0$ the inequality

$$\limsup_{n\to+\infty} \eta_1(\Lambda_n^2) < \liminf_{n\to+\infty} \eta_1(\Lambda_n^2) + \varepsilon$$

holds. By letting ε tend to 0 this implies that

$$\limsup_{n\to+\infty} \eta_1(\Lambda_n^2) \leq \liminf_{n\to+\infty} \eta_1(\Lambda_n^2)$$

and shows the existence of $\eta_\infty = \lim_{n\to+\infty} \eta_1(\Lambda_n^2)$. The fact that $\eta_1(\Lambda_n^2) \geq \eta_\infty$ for all $n \in \mathbb{Z}_{>0}$ now follows from the impliciation $(i) \Rightarrow (ii)$. □

It remains to show Lemma 8.17. We will make use of the following claim which asserts that the density of a minimal cochain achieving $\eta_1(\Lambda_n^2)$ is strictly bounded away from 0.

Claim 8.18. *There is $\mu > 0$ such that for all sufficiently large $n \in \mathbb{Z}_{>0}$ we have that if $c \in C^1(\Lambda_n^2; \mathbb{F}_2)$ is a minimal cochain with $\eta_1(\Lambda_n^2) = \frac{\|\delta c\|}{\|c\|}$ then $\|c\| \geq \mu$.*

Proof. This is an immediate consequence of Corollary 8.15 and Theorem 7.7. □

We are ready to give a proof of Lemma 8.17.

Proof of Lemma 8.17. Fix an inclusion $i\colon \Lambda_n^2 \to \Lambda_{n+1}^2$. First let $c \in C^1(\Lambda_n^2; \mathbb{F}_2)$ be minimal with $\eta_1(\Lambda_n^2) = \frac{\|\delta c\|}{\|c\|}$. Let $\bar{c} \in C^1(\Lambda_{n+1}^2; \mathbb{F}_2)$ be the extension by 0 of c to Λ_{n+1}^2. By Lemma 3.6 $\bar{c}$ is minimal. Moreover, note that $|\delta\bar{c}| \leq |\delta c| + |c|$. Indeed, every triangle in the coboundary of $\bar{c}$ is in Λ_n^2 or it must contain a vertex from $\Lambda_{n+1}^2(0) \setminus \Lambda_n^2(0)$ and an edge from the support of c. But every edge in Λ_n^2 is contained in precisely one triangle with a vertex from $\Lambda_{n+1}^2(0) \setminus \Lambda_n^2(0)$. It follows that

$$|\delta c| \geq |\delta\bar{c}| - |c| \geq \left(\frac{n+1}{3}\eta_1(\Lambda_{n+1}^2) - 1\right)|c|.$$

Normalizing gives

$$\eta_1(\Lambda_n^2) \geq \frac{n+1}{n}\eta_1(\Lambda_{n+1}^2) - \frac{3}{n} \geq \eta_1(\Lambda_{n+1}^2) - \frac{3}{n},$$

or equivalently

$$\eta_1(\Lambda_n^2) - \eta_1(\Lambda_{n+1}^2) \geq -\frac{3}{n}.$$

For a reverse inequality let $c \in C^1(\Lambda_{n+1}^2; \mathbb{F}_2)$ be minimal with $\eta_1(\Lambda_{n+1}^2) = \frac{\|\delta c\|}{\|c\|}$. Let $\tilde{c} = i^*c \in C^1(\Lambda_n^2; \mathbb{F}_2)$ be the restriction of c to Λ_n^2. Let $a \in C^0(\Lambda_n^2; \mathbb{F}_2)$ such that $|[\tilde{c}]| = |\tilde{c} + \delta a|$. Let $\bar{a} \in C^0(\Lambda_{n+1}^2; \mathbb{F}_2)$ be the extension by 0 of a to Λ_{n+1}^2. By minimality of c we get

$$|c| \leq |c + \delta\bar{a}| = |\tilde{c} + \delta a| + |(c + \delta\bar{a})_{|\Lambda_{n+1}^2 \setminus \Lambda_n^2}| \leq |[\tilde{c}]| + 6(n+1),$$

where we used that $|\Lambda^2_{n+1}(1) \setminus \Lambda^2_n(1)| = 6n+3 \leq 6(n+1)$ for the last inequality. It follows that

$$\begin{aligned}\eta_1(\Lambda^2_n) &\leq \frac{3}{n}\frac{|\delta\tilde{c}|}{|[\tilde{c}]|}\\ &\leq \frac{3}{n}\frac{|\delta c|}{|[\tilde{c}]|}\\ &\leq \frac{n+1}{n}\eta_1(\Lambda^2_{n+1})\frac{|c|}{|[\tilde{c}]|}\\ &\leq \frac{n+1}{n}\eta_1(\Lambda^2_{n+1})\frac{|c|}{|c|-6(n+1)}.\end{aligned}$$

Claim 8.18 implies that $|c| \geq 3\mu(n+1)^2$ for some positive constant $\mu > 0$ if $n \geq N_0$ for some $N_0 \in \mathbb{Z}_{>0}$. Plugging this into above inequality, we get

$$\eta_1(\Lambda^2_n) \leq \frac{n+1}{n}\eta_1(\Lambda^2_{n+1})\frac{1}{1-\frac{2}{\mu(n+1)}} \leq \eta_1(\Lambda^2_{n+1}) + \left(1+\frac{4}{\mu}\right)\frac{1}{n},$$

provided that $n+1 > \max\{\frac{2}{\mu}, N_0\}$.

We have shown that for sufficiently large n

$$-\frac{3}{n} \leq \eta_1(\Lambda^2_n) - \eta_1(\Lambda^2_{n+1}) \leq \left(1+\frac{4}{\mu}\right)\frac{1}{n}.$$

This finishes the proof. □

After the existence of $\eta_\infty := \lim_{n\to+\infty}\eta_1(\Lambda^2_n)$ being established, we can observe a simple relationship between η_∞ and $\eta(\alpha)$.

Lemma 8.19. *Let $\lambda \in (0,1)$. Then $\eta_\infty \geq \lambda$ if and only if $\eta(\alpha) \geq \lambda\alpha$ for all $\alpha \in (0,1/2]$. Moreover,*

$$\eta_\infty = \inf_{0<\alpha\leq 1/2}\frac{\eta(\alpha)}{\alpha}.$$

Proof. First assume that $\eta(\alpha) \geq \lambda\alpha$ for all $\alpha > 0$. We will show that this implies $\eta_1(\Lambda^2_n) \geq \lambda$ for all positive integers n and, hence, $\eta_\infty \geq \lambda$ as well. Let $c \in C^1(\Lambda^2_n;\mathbb{F}_2)$ be minimal with $\frac{\|\delta c\|}{\|c\|} = \eta_1(\Lambda^2_n)$. Let $\alpha = \|c\|$. Via a blow-up construction as in Section 7.3.4 we get a sequence $c^{(k)} \in C^1(\Lambda^2_{nk};\mathbb{F}_2)$ of minimal cochains with $\|c^{(k)}\| = \alpha$ and $\|\delta c^{(k)}\| = \|\delta c\|$. We deduce that

$$\eta_1(\Lambda^2_n) = \frac{\|\delta c^{(k)}\|}{\|c^{(k)}\|} \geq \frac{\eta(\alpha)}{\alpha} \geq \lambda.$$

Conversely, if $\eta_\infty \geq \lambda$ then by Proposition 8.16 $\eta_1(\Lambda^2_n) \geq \lambda$ for all positive integers n. Now consider a sequence of minimal cochains $c^{(n_k)} \in C^1(\Lambda^2_{n_k};\mathbb{F}_2)$ such that $\|c^{(n_k)}\| \geq \alpha$ and such that $\lim_{k\to+\infty}\|\delta c^{(n_k)}\| = \eta(\alpha)$. We deduce

$$\eta(\alpha) = \lim_{k\to+\infty}\|\delta c^{(n_k)}\| \geq \lambda \liminf_{k\to+\infty}\|c^{(n_k)}\| \geq \lambda\alpha,$$

as desired.

The second part of the lemma follows easily from the first part. □

8.4.5 Proof of Proposition 8.8

Let us put the pieces above together to show Proposition 8.8.

Proof of Proposition 8.8. By Lemma 8.19 it suffices to show that $\eta(\alpha) \geq 0.6358\alpha$ for all $\alpha \in (0, 1/2]$. Consider the functions

$$\begin{aligned} f\colon (0,1/2] &\to \mathbb{R} \\ \alpha &\mapsto \frac{5-\sqrt{25-48\alpha}}{8\alpha} \end{aligned}$$

and

$$\begin{aligned} g\colon (0,1/6] &\to \mathbb{R} \\ \alpha &\mapsto \frac{1}{2\alpha}(1-\sqrt{1-6\alpha})(1-6\alpha). \end{aligned}$$

Lemma 8.11 and Corollary 8.15 imply that $\eta(\alpha)/\alpha \geq f(\alpha)$ for $\alpha \in (0, 1/2]$ and $\eta(\alpha)/\alpha \geq g(\alpha)$ for $\alpha \in (0, 1/6]$.

Note that f is monotonically increasing on $(0, 1/2]$, while g is monotonically decreasing on $(0, 1/6]$. Numerically solving the equation $f(\alpha) = g(\alpha)$ suggests a root at $\alpha \approx 0.1109$. Computing $f(0.1109)$ and $g(0.1109)$ and using the monotonicity of f and g, we get $g(\alpha) \geq 0.6358$ for all $0 < \alpha \leq 0.1109$ and $f(\alpha) \geq 0.6358$ for all $0.1109 \leq \alpha \leq 1/2$. This finishes the proof. □

8.4.6 An Improved Triangle Inequality

Revisiting the random cofilling argument showing $\eta_1(\Lambda_n^2) \geq 3/5$, it is natural to ask to which extent each estimate is tight. In particular, the inequality

$$\sum_{i=0}^{2} \sum_{uu' \in \binom{U_i}{2}} |b_u + b_{u'}| \leq 3(n-1)|b|$$

for $b \in B^2(\Lambda_n^2; \mathbb{F}_2)$ seems wasteful since it is merely an application of the triangle inequality and does not make use of any properties of coboundaries.

For $n = 5$ we were able to tighten above inequality. With the help of a computer we can show

Lemma 8.20. *For any $b \in B^2(\Lambda_5^2; \mathbb{F}_2)$ it holds that*

$$\sum_{i=0}^{2} \sum_{uu' \in \binom{U_i}{2}} |b_u + b_{u'}| \leq \frac{336}{31}|b|.$$

Note that an application of the triangle inequality would only give an upper bound of $12|b|$ on the right hand side. Since the restriction of a coboundary to a subcomplex is a coboundary, we can use Lemma 8.20 and averaging over subcomplexes $\Lambda_5^2 \subseteq \Lambda_n^2$ to get an improved triangle inequality for all $n \geq 5$.

Corollary 8.21. *Let $n \geq 5$. Then for any $b \in B^2(\Lambda^2_n; \mathbb{F}_2)$ it holds that*

$$\sum_{i=0}^{2} \sum_{uu' \in \binom{U_i}{2}} |b_u + b_{u'}| \leq \frac{84}{31}(n-1)|b|.$$

Moreover, $\eta_1(\Lambda^2_n) \geq 93/143$.

As mentioned the proof of Lemma 8.20 is computer-aided. We give the reduction to a feasible set of cases. Then we wrote a piece of C++ code that checks this remaining set of cases.[2]

Proof of Lemma 8.20. Fix $b \in B^2(\Lambda^2_5; \mathbb{F}_2)$. As in the proof of Lemma 8.11 we write

$$\begin{aligned}
\sum_{i=0}^{2} \sum_{uu' \in \binom{U_i}{2}} |b_u + b_{u'}| &= \sum_{i=0}^{2} \sum_{e \in \Lambda^2_5(1), e \cap U_i = \emptyset} |b_e|(5 - |b_e|) \\
&= 15|b| - \sum_{e \in \Lambda^2_5(1)} |b_e|^2 \\
&\leq \left(15 - \frac{3}{25}|b|\right)|b|.
\end{aligned}$$

If $|b| \geq 35$ this gives

$$\sum_{i=0}^{2} \sum_{uu' \in \binom{U_i}{2}} |b_u + b_{u'}| \leq \left(15 - \frac{3}{25} \cdot 35\right)|b| = 10.8|b| < \frac{336}{31}|b|.$$

For $|b| \leq 34$ a computer comes into play. $|b| \leq 34$ implies that there is a vertex $u \in \Lambda^2_5(0)$ for which $|b_u| \leq 6$. Also, $b_u + b_{u'} \in B^1(K_{5,5}; \mathbb{F}_2)$ for all $uu' \in \binom{U_i}{2}$. Thus, once we fix b_{u_0} with $|b_{u_0}| \leq 6$, there are only 512 choices for each of the remaining four b_u with $u \in U_0 \setminus \{u_0\}$. Thus, for each subgraph of $K_{5,5}$ (up to isomorphism) with at most 6 edges we have to test no more than $\binom{515}{4} = 2896986240 \approx 2.9 \cdot 10^9$ cases. This is feasible. In Figure 8.1 and Figure 8.2 we give for each subgraph G of $K_{5,5}$ with at most 6 edges the smallest value of α for which

$$\sum_{i=0}^{2} \sum_{uu' \in \binom{U_i}{2}} |b_u + b_{u'}| \leq \alpha|b|$$

for all $b \in B^2(\Lambda^2_5; \mathbb{F}_2)$ with the support of b_{u_0} being isomorphic to G and $|b_{u_0}| = \min_{u \in U_0} |b_u|$. These values have been computed using some C++ code. We see that all α satisfy $\alpha \leq \frac{336}{31}$ and thus this verifies the claimed bound. ☐

Proof of Corollary 8.21. Fix $b \in B^2(\Lambda^2_n; \mathbb{F}_2)$. Write $\mathcal{X}$ for the family of subcomplexes of Λ^2_n given by

$$\mathcal{X} = \{V_0 * V_1 * V_2 \subseteq \Lambda^2_n : V_i \subseteq U_i, |V_i| = 5\}.$$

Given $X \in \mathcal{X}$ and $i \in \{0,1,2\}$ let $X_i := X(0) \cap U_i$. Also, we write b^X for the restriction of b to X.

[2]This code will be made available via the library of IST Austria.

Recall that the expression $\sum_{i=0}^{2} \sum_{uu' \in \binom{U_i}{2}} |b_u + b_{u'}|$ counts the number of pairs of triangles sharing an edge and with precisely one of them in the support of b. Note that any pair of triangles sharing an edge is contained in $\binom{n-2}{3}\binom{n-1}{4}^2$ of the subcomplexes in $\mathcal{X}$. Similarly, any triangle in Λ_n^2 is in $\binom{n-1}{4}^4$ of the subcomplexes in $\mathcal{X}$. It follows that

$$\begin{aligned}
\sum_{i=0}^{2} \sum_{uu' \in \binom{U_i}{2}} |b_u + b_{u'}| &= \frac{1}{\binom{n-2}{3}\binom{n-1}{4}^2} \sum_{X \in \mathcal{X}} \sum_{i=0}^{2} \sum_{vv' \in \binom{X_i}{2}} |b_v^X + b_{v'}^X| \\
&\leq \frac{1}{\binom{n-2}{3}\binom{n-1}{4}^2} \frac{336}{31} \sum_{X \in \mathcal{X}} |b^X| \\
&= \frac{336}{31} \frac{\binom{n-1}{4}^3}{\binom{n-2}{3}\binom{n-1}{4}^2} |b| \\
&= \frac{84}{31}(n-1)|b|,
\end{aligned}$$

where we used Lemma 8.20 for the inequality. This proves the first part. The lower bound $\eta_1(\Lambda_n^2) \geq 93/143$ now follows by plugging this improved triangle inequality into the random cofilling argument giving the 3/5-bound. □

8.4.7 Expansion for Small Cochains Using Flag Algebras

In this subsection, we use *flag algebras* to show

Lemma 8.22. *For all $\alpha \in [0, 1/2]$ we have $\eta(\alpha) \geq \frac{31}{37}\alpha(1-\alpha)$.*

Flag algebras, which were introduce in Razborov's seminal paper [121], provide a framework to tackle problems in (asymptotic) extremal combinatorics. Its strength stems from providing a systematic way to generate bounds on parameters in extremal combinatorics by computer-assisted, semi-automated proofs.

A typical application of flag algebras is to bound densities of (induced) subgraphs in graphs not containing any subgraph isomorphic to a graph in a family of forbidden graphs. These are very classical problems in extremal graph theory going back to Mantel's theorem [104] stating that a triangle-free graph on n vertices has at most $n^2/4$ edges and its generalization due to Turán [136] saying that a K_{r+1}-free graph on n vertices has at most $\left(1 - \frac{1}{r}\right)\frac{n^2}{2}$ edges.

A much harder problem of this type is the *Erdős pentagon problem* which asks whether any graph on $5n$ vertices with no triangle contains at most n^5 pentagons (i.e. 5-cycles). This question was asked by Erdős in 1984 [39] and remained open until around 2012 an affirmative answer was given in [55] and independently in [58] heavily relying on flag algebras. Later in [87] a complete characterization of all extremal examples of triangle-free graphs on n vertices maximizing the number of 5-cycles for all n (not necessarily divisible by 5) was given.

Another striking application of flag algebras is the precise description of the (asymptotically) minimal possible density $g_r(\rho)$ of copies of K_r in a graph with given edge density

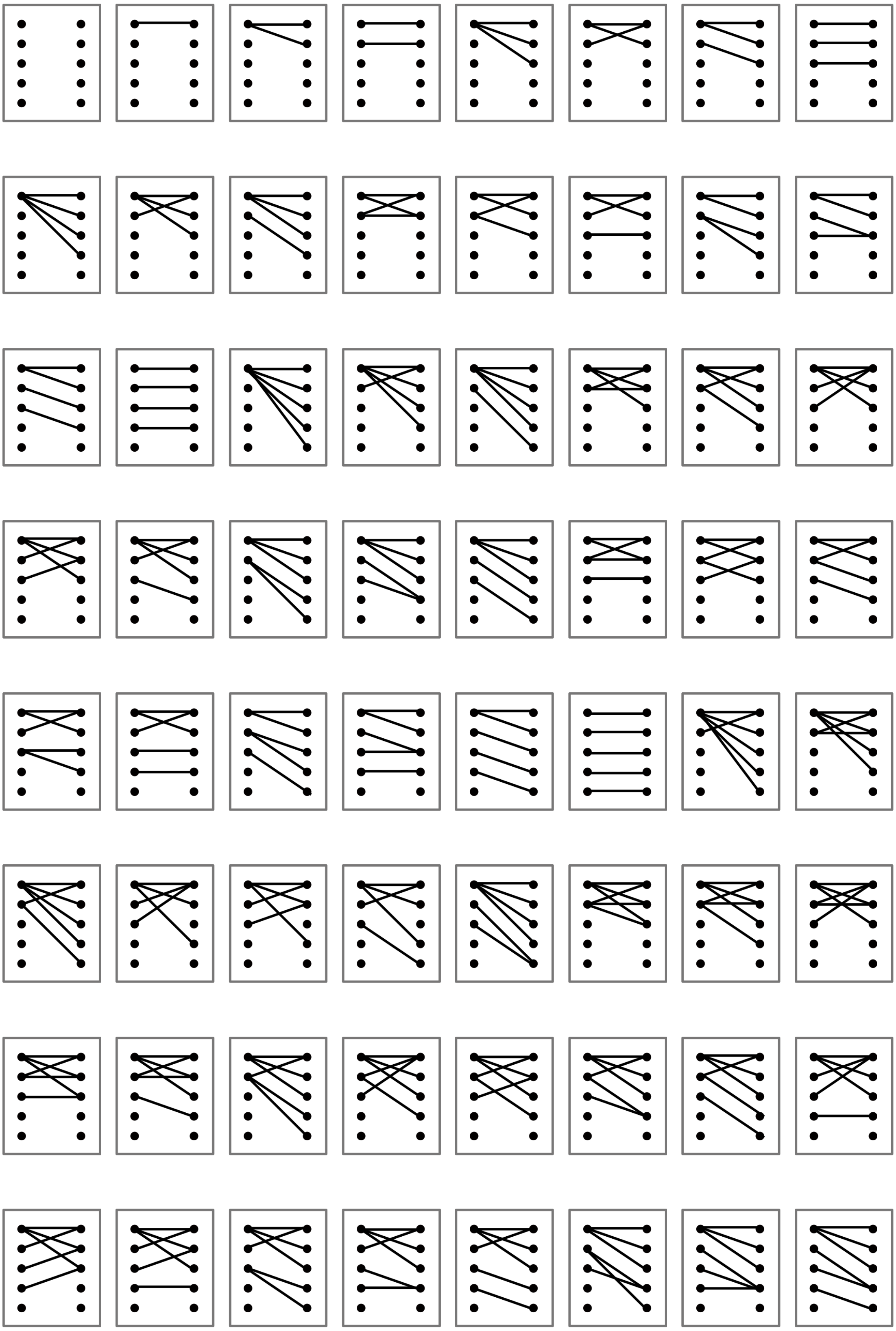


Figure 8.1: The figure shows subgraphs of $K_{5,5}$ with at most 6 edges (up to isomorphism). The value above each graph indicates the maximal ratio of $\sum_{i=0}^{2}\sum_{uu'\in\binom{U_i}{2}}|b_u+b_{u'}|/|b|$ over all $b\in B^2(\Lambda_5^2;\mathbb{F}_2)$ for which b_{u_0} is the given subgraph and $|b_{u_0}|=\min_{u\in U_0}|b_u|$.

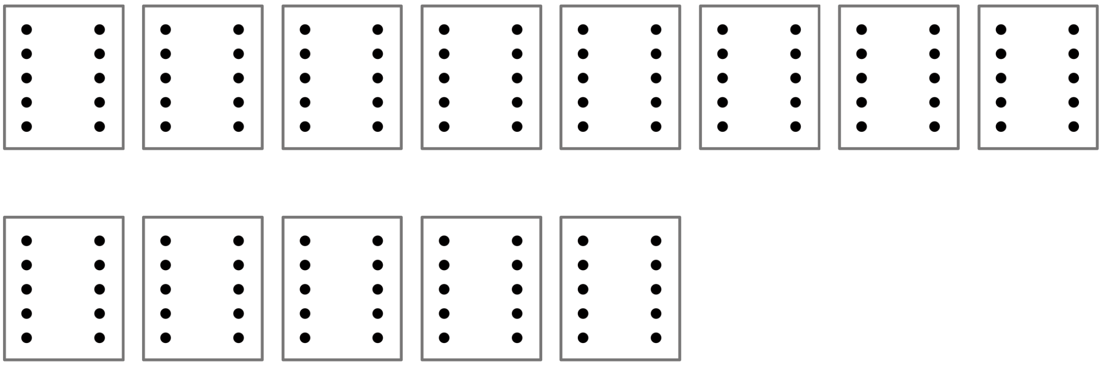
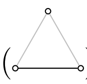
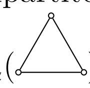


Figure 8.2: The figure shows the remaining subgraphs of $K_{5,5}$ with at most 6 edges (up to isomorphism). The value above each graph indicates the maximal ratio of $\sum_{i=0}^{2} \sum_{uu' \in \binom{U_i}{2}} |b_u + b_{u'}|/|b|$ over all $b \in B^2(\Lambda_5^2; \mathbb{F}_2)$ for which b_{u_0} is the given subgraph and $|b_{u_0}| = \min_{u \in U_0} |b_u|$.

$\rho \in [0,1]$ for all values of ρ. This is due to Razborov in [122] for $r = 3$, Nikiforov in [112] for $r = 4$ and Reiher in [123] for $r > 4$.

Flag algebras have also occurred in the context of coboundary expansion (with respect to $\mathbb{F}_2$-coefficients) in [86]. There, the authors use them to prove good expansion properties for small 1-cochains in the complete 2-dimensional complex K_n^2.[3] The goal of this subsection is to translate (some of) the ideas in [86] to the setting of Λ_n^2 leading to a proof of Lemma 8.22.

Beside the paper [86] the resources [121] and [27] helped us to gain some acquaintance with flag algebras. Our discussion below draws from all these sources. Currently, our arguments do not need the full machinery of the flag algebra toolbox and we will only introduce the concepts relevant for our proofs here.

The main idea is as follows: Let $\alpha \in (0, 1/2]$. Consider a family of minimal 1-cochains $c_k \in C^1(\Lambda_{n_k}^2; \mathbb{F}_2)$, for some sequence $(n_k)_{k \in \mathbb{N}}$ with $n_k \to +\infty$, with $\|c_k\| \geq \alpha$ and $\lim_{k \to +\infty} \|\delta c_k\| = \eta(\alpha)$. We will think of c_k as a subgraph of the 1-skeleton of $\Lambda_{n_k}^2$. It will be convenient to write $(G_k)_{k \in \mathbb{N}}$ instead of $(c_k)_{k \in \mathbb{N}}$. Given another (tripartite) graph H let $p(H; G_k)$ be the probability that a randomly choosen subgraph G_k with $|V(H)|$ vertices is isomorphic to H. By compactness there is a subsequence $(G_{k_l})_{l \in \mathbb{N}}$ such that the limit $\phi_\alpha(H) := \lim_{l \to +\infty} p(H; G_{k_l})$ exists for all (tripartite) graphs H. Note that $\phi_\alpha(H) \geq 0$ for all (tripartite) graphs. Intuitively speaking, we have $\phi_\alpha($$) \geq \alpha$ and $\eta(\alpha) = \phi_\alpha($$) + \phi_\alpha($$)$. We can extend ϕ_α linearly to all formal linear combinations of (tripartite) graphs. The values of ϕ_α for different graphs and their linear combinations are highly correlated. In fact, we will derive various inequalities of the form $0 \leq \phi_\alpha(F)$ for some linear combination of graphs F. Using linear programming, we will find linear combinations with non-negative coefficients of such inequalities leading to lower bounds on $\eta(\alpha)$ in terms on α.

Before we can write down such inequalities, we should take a bit more care introducing our formalism. For instance, it will be useful to consider tripartite graphs up to reordering the vertices in each part. That is, we consider the three parts of a tripartite graph as

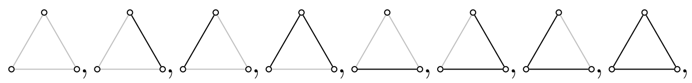
[3]This is relevant to the point selection problem mentioned in the introduction and lead to an improvement on c_3 as defined in the introduction.

distinguishable but the vertices within each part as indistinguishable. Such technicalities and the fact that the cofilling argument showing optimal expansion $\eta_1(K_n^2) \geq 1$ is much simpler than the cofilling argument showing $\eta_1(\Lambda_n^2) \geq 3/5$ give rise to more complicated formulas here compared to the arguments in [86]. The reader might find it helpful to have a look at [86] in parallel or prior to reading this section.

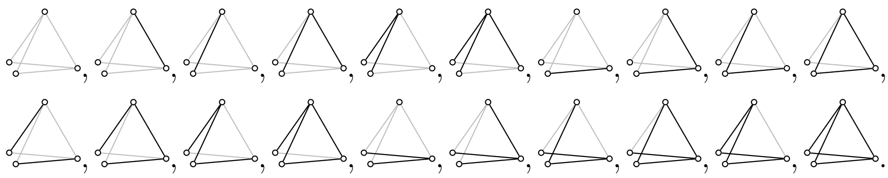
After all these remarks let us finally get our hands dirty: We write $\mathcal{F}$ for the set of 3-partite graphs up to reordering the vertices in each part.

Given $(l_1, l_2, l_3) \in \mathbb{Z}^3_{\geq 0}$ we write $\mathcal{F}_{(l_1,l_2,l_3)}$ for the set of 3-partite graphs with parts of sizes equal to l_1, l_2 and l_3, respectively. Notice that, since we consider the parts as distinguishable, we have that, for instance, $\mathcal{F}_{(1,2,1)} \neq \mathcal{F}_{(2,1,1)}$.

The next line shows the eight flags in $\mathcal{F}_{(1,1,1)}$:



and we have the following twenty flags in $\mathcal{F}_{(2,1,1)}$:



Given $F \in \mathcal{F}_{(l_1,l_2,l_3)}$ write $V(F) = V_1(F) \sqcup V_2(F) \sqcup V_3(F)$ for the vertex set of F with $|V_i(F)| = l_i$, $1 \leq i \leq 3$. Given $S \subseteq V(F)$ we write $F[S]$ for the subgraph of F induced by S.

We call $\sigma \in \mathcal{F}_{(k_1,k_2,k_3)}$ a *type of size* (k_1, k_2, k_3). We write $\emptyset$ for the unique type of size $(0, 0, 0)$.

Let σ be a type of size (k_1, k_2, k_3). Let $l_1, l_2, l_3 \in \mathbb{Z}$ with $l_i \geq k_i$ for $1 \leq i \leq 3$. Let $F \in \mathcal{F}_{(l_1,l_2,l_3)}$. An *embedding of σ to F* is a triple $\theta = (\theta_1, \theta_2, \theta_3)$ of maps $\theta_i : [k_i] \to V_i(F)$ such that $F[\bigsqcup_{i=1}^3 \theta_i([k_i])]$ is isomorphic to σ (where again we consider the parts as distinguishable). Given a type σ, $F \in \mathcal{F}$ and an embedding $\theta : \sigma \to F$ we call (F, θ) a *σ-flag of size* (l_1, l_2, l_3) where $l_i = |V_i(F)|$. Thus, a σ-flag is nothing else than a 3-partite graph containing a labelled copy of σ. It is natural to define that σ-flags (F, θ) and (F', θ') are *isomorphic* if for $1 \leq i \leq 3$ there are bijections $\rho_i : V_i(F) \to V_i(F')$ that induce a graph isomorphism between F and F' which is label preserving, i.e. $\theta'_i = \rho_i \circ \theta_i$. We write $\mathcal{F}^\sigma$ for the set of σ-flags. For $(l_1, l_2, l_3) \in \mathbb{Z}^3_{\geq 0}$ we let $\mathcal{F}^\sigma_{(l_1,l_2,l_3)} = \mathcal{F}^\sigma \cap \mathcal{F}_{(l_1,l_2,l_3)}$.

Let σ be a type of size (s_1, s_2, s_3). Let $F_1, \ldots, F_t, (G, \theta)$ be σ-flags, $F_i \in \mathcal{F}^\sigma_{(l_1^{(i)}, l_2^{(i)}, l_3^{(i)})}$, $1 \leq i \leq t$, and $G \in \mathcal{F}^\sigma_{(k_1,k_2,k_3)}$, say. We say that $F_1, \ldots, F_t$ *fit into* G if for all $1 \leq i \leq 3$ we have

$$k_i - s_i \geq \sum_{j=1}^{t} (l_i^{(j)} - s_i).$$

This allows us to define the key quantity $p(F_1, \ldots, F_t; G)$ as the probability that if we pick pairwise disjoint $U_i^{(1)}, \ldots U_i^{(t)} \subseteq V_i(G) \setminus \mathrm{Im}\theta_i$ of sizes $|U_i^{(j)}| = l_i^j - s_i$, $1 \leq j \leq t$, uniformly at random the induced σ-flag $G[\bigsqcup_{i=1}^3 (U_i^{(j)} \cup \mathrm{Im}\theta_i)]$ is isomorphic to F_j for all $1 \leq j \leq t$.

Notice that the sampling process in the definition of $p(F; G)$ can be replaced by a two step sampling process as follows: First sample a σ-flag H inside G and then inside H

sample F. This holds in general and leads to the following *chain rule* which can be proven by applying the total law of probability.

Lemma (Chain rule, see Lemma 2.3 in [121])**.** *Let σ be a type of size (s_1, s_2, s_3), $F_i \in \mathcal{F}^\sigma_{(l_1^{(i)}, l_2^{(i)}, l_3^{(i)})}$, $1 \le i \le t$, $G \in \mathcal{F}^\sigma_{(k_1,k_2,k_3)}$, $(\tilde{l}_1, \tilde{l}_2, \tilde{l}_3)$ with $\tilde{l}_i \le k_i$, $1 \le i \le 3$, such that for all $i \in \{1,2,3\}$ we have*

$$\tilde{l}_i - s_i \ge \sum_{j=1}^{s} (l_i^{(j)} - s_i) \text{ (i.e. } F_1, \dots, F_s \text{ fit into } (\tilde{l}_1, \tilde{l}_2, \tilde{l}_3) - \text{flags)}$$

and

$$k_i - s_i \ge (\tilde{l}_i - s_i) + \sum_{j=s+1}^{t} (l_i^{(j)} - s_i) \text{ (i.e. } \tilde{F}, F_{s+1}, \dots, F_t \text{ fit into } G \text{ for all } \tilde{F} \in \mathcal{F}^\sigma_{(\tilde{l}_1,\tilde{l}_2,\tilde{l}_3)}).$$

Then

$$p(F_1, \dots, F_t; G) = \sum_{\tilde{F} \in \mathcal{F}^\sigma_{(\tilde{l}_1,\tilde{l}_2,\tilde{l}_3)}} p(F_1, \dots, F_s; \tilde{F}) p(\tilde{F}, F_{s+1}, \dots, F_t; G).$$

In particular, if $s = t$ and

$$\tilde{l}_i - s_i \ge \sum_{j=1}^{t} (l_i^{(j)} - s_i)$$

for $i \in \{1,2,3\}$ then

$$p(F_1, \dots, F_t; G) = \sum_{\tilde{F} \in \mathcal{F}^\sigma_{(\tilde{l}_1,\tilde{l}_2,\tilde{l}_3)}} p(F_1, \dots, F_t; \tilde{F}) p(\tilde{F}; G).$$

In general it is not true that $p(F_1, F_2; G)$ and $p(F_1; G)p(F_2; G)$ are equal. But if $V_1(G), V_2(G)$ and $V_3(G)$ are all very large then sampling subsets of a fixed small size in $V_i(G)$ are likely to be disjoint. Thus, we expect $p(F_1, F_2; G)$ and $p(F_1; G)p(F_2; G)$ to be equal 'in the limit' when the size of G tends to infinity. For a more precise general statement we have

Lemma (Almost product, see Lemma 2.3 in [121])**.** *For $1 \le i \le t$ let $F_i \in \mathcal{F}^\sigma_{(l_1^{(i)}, l_2^{(i)}, l_3^{(i)})}$, $G \in \mathcal{F}^\sigma_{(k_1,k_2,k_3)}$. Assume that $F_1, \dots, F_t$ fit into G. Then*

$$|p(F_1, \dots, F_t; G) - \prod_{j=1}^{t} p(F_j; G)| \le \sum_{i=1}^{3} \frac{\left(\sum_{j=1}^{t} l_i^{(j)}\right)^2}{k_i}.$$

This suggest that there might be some limiting object where we end up with an actual product. This is what we will define now. Note that for a fixed σ-flag G we can think of $p(\cdot; G)$ as a function on $\mathcal{F}^\sigma$ (where we define $p(F; G) = 0$ if F does not fit into G). It seems natural to enrich the structure a bit and consider $\mathbb{R}\mathcal{F}^\sigma$ as the real linear space spanned by elements in $\mathcal{F}^\sigma$ for a type σ of size (s_1, s_2, s_3). We can extend $p(\cdot; G)$ to $\mathbb{R}\mathcal{F}^\sigma$ linearly. Let $\mathcal{K}^\sigma \subseteq \mathbb{R}\mathcal{F}^\sigma$ be the subspaces spanned by elements of the form

$$\tilde{F} - \sum_{F \in \mathcal{F}^\sigma_{(l_1,l_2,l_3)}} p(\tilde{F}, F) F$$

with $\tilde{F} \in \mathcal{F}^\sigma_{(k_1,k_2,k_3)}$ such that $s_i \le k_i \le l_i$ for all $i \in \{1,2,3\}$. Notice that by the chain rule $p(H; G) = 0$ for all $H \in \mathcal{K}^\sigma, G \in \mathcal{F}^\sigma$. Let us define $\mathcal{A}^\sigma = \mathbb{R}\mathcal{F}^\sigma / \mathcal{K}^\sigma$. We can endow

$\mathcal{A}^\sigma$ with a product which will turn it into a commutative associative algebra. To this end, let us first define a bilinear map $\cdot : \mathbb{R}\mathcal{F}^\sigma \times \mathbb{R}\mathcal{F}^\sigma \to \mathcal{A}^\sigma$. Given $F_1 \in \mathcal{F}^\sigma_{(l_1,l_2,l_3)}$ and $F_2 \in \mathcal{F}^\sigma_{(l'_1,l'_2,l'_3)}$ choose $(k_1,k_2,k_3) \in \mathbb{Z}^3_{>0}$ such that $k_i - s_i \geq (l_i - s_i) + (l'_i - s_i)$. Then define

$$F_1 \cdot F_2 = \sum_{F \in \mathcal{F}^\sigma_{(k_1,k_2,k_3)}} p(F_1, F_2; F) F$$

and extend it bilinearly to $\mathbb{R}\mathcal{F}^\sigma \times \mathbb{R}\mathcal{F}^\sigma$. We have the following important lemma.

Lemma (see Lemma 2.4 in [121]). *(i) $\cdot$ is well-defined, i.e. $F_1 \cdot F_2$ is independent of the choice of (k_1,k_2,k_3).*

(ii) For any $f \in \mathcal{K}^\sigma$ and $g \in \mathbb{R}\mathcal{F}^\sigma$ we have $f \cdot g \in \mathcal{K}^\sigma$. Moreover, $\cdot$ induces a symmetric bilinear map $\mathcal{A}^\sigma \times \mathcal{A}^\sigma \to \mathcal{A}^\sigma$.

(iii) $\cdot$ turns $\mathcal{A}^\sigma$ into a commutative associative algebra.

$\mathcal{A}^\sigma$ is called the *flag algebra (of type σ)*. We will write $\mathcal{A}^\sigma_{(l_1,l_2,l_3)}$ for the projection of the subspace $\mathcal{F}^\sigma_{(l_1,l_2,l_3)}$ in $\mathcal{F}^\sigma$ to $\mathcal{A}^\sigma$.

We can use the almost product behaviour of $p(\cdot, G)$ to construct homomorphisms $\mathcal{A}^\sigma \to \mathbb{R}$. To this end, we will say that a sequence of σ-flags $(G_k)_{k\in\mathbb{N}}$ of sizes $(l_1^{(k)}, l_2^{(k)}, l_3^{(k)})$ is *increasing* if the sizes $(l_i^{(k)})_{k\in\mathbb{N}}$ is a strictly increasing sequence of $i \in \{1,2,3\}$. A *convergent sequence of σ-flags* is an increasing sequence $(G_k)_{k\in\mathbb{N}}$ of σ-flags such that

$$\phi(F) = \lim_{k\to\infty} p(F, G_k)$$

exists for all $F \in \mathcal{F}^\sigma$. Notice that by compactness every increasing sequence of σ-flags contains a convergent subsequence. We will extend ϕ linearly to $\mathbb{R}\mathcal{F}^\sigma$. Notice that by the chain rule $\phi(K) = 0$ for all $K \in \mathcal{K}^\sigma$. Thus we obtain an induced map $\phi : \mathcal{A}^\sigma \to \mathbb{R}$. The almost product behaviour of $p(\cdot, G)$ implies that ϕ is an algebra homomorphism. In particular $\phi(f \cdot g) = \phi(f)\phi(g)$ for all $f, g \in \mathcal{A}^\sigma$. We call such ϕ a *limit functional.* Clearly $\phi(F) \geq 0$ for all σ-flags F. We say that $\psi \in \mathrm{Hom}(\mathcal{A}^\sigma, \mathbb{R})$ is *positive* if $\psi(F) \geq 0$ for all σ-flags F. We write $\mathrm{Hom}^+(\mathcal{A}^\sigma, \mathbb{R})$ for the set of positive homomorphisms. It turns out that the positive homomorphisms are precisely the limit functionals.

Theorem (Theorem 3.3 in [121]). *Every limit functional is a positive homomorphism. Conversely, every positive homomorphism is a limit functional for some convergent sequences of flags.*

Given $f \in \mathcal{A}^\sigma$ let us write $f \succeq_\sigma 0$ if $\phi(f) \geq 0$ for all $\phi \in \mathrm{Hom}^+(\mathcal{A}^\sigma, \mathbb{R})$. It will be convenient to extend this notation as follows. Given $\mathcal{G}^\sigma \subseteq \mathcal{F}^\sigma$ let us write $\Phi_{\mathcal{G}^\sigma} \subseteq \mathrm{Hom}^+(\mathcal{A}^\sigma, \mathbb{R})$ for the limit functionals that can be obtained from convergent sequences contained in $\mathcal{G}^\sigma$. We write $f \succeq_\sigma^{\mathcal{G}^\sigma} 0$ if $\phi(f) \geq 0$ for all $\phi \in \Phi_{\mathcal{G}^\sigma}$.

We can consider $c \in C^1(\Lambda^2_n; \mathbb{F}_2)$ as an element of $\mathcal{F}_{(n,n,n)} \subseteq \mathcal{F}$. With this identification and given $\alpha \in (0, 1/2]$, we write $\mathcal{G}_\alpha \subseteq \mathcal{F}$ for the set of minimal cochains with normalized Hamming norm at least α. Similarly, for a type σ we write $\mathcal{G}^\sigma_\alpha \subseteq \mathcal{F}^\sigma$ for the set of σ-flags corresponding to minimal cochains with norm at least α. Note that $\mathcal{G}^\emptyset_\alpha = \mathcal{G}_\alpha$.

We are interested in finding inequalities of the form $f \succeq_\emptyset^{\mathcal{G}_\alpha} 0$. It is often easier to find valid inequalities $f \succeq_\sigma 0$ for non-empty type σ. This will induce a valid inequality $[\![f]\!]_\sigma \succeq_\emptyset 0$

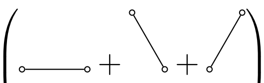
for some $[\![f]\!]_\sigma \in \mathcal{A}^\emptyset$ via the so-called *downward operator*, which is a linear operator $[\![\cdot]\!]_\sigma : \mathcal{A}^\sigma \to \mathcal{A}^\emptyset$. In order to define $[\![\cdot]\!]_\sigma$ first note that given a σ-flag (F, θ) we can easily obtain a $\emptyset$-flag $\downarrow F$ by simply forgetting about the embedding θ. For a σ-flag F with σ a type of size (s_1, s_2, s_3) let $q_\sigma(F)$ be the probability that injective maps $\theta_i : [s_i] \to V_i(F)$ picked uniformly at random induce a σ-flag $(\downarrow F, (\theta_1, \theta_2, \theta_3))$ which is isomorphic to F. Then define $[\![F]\!]_\sigma = q_\sigma(F) \downarrow F$. This can be extend to a linear map $\mathbb{R}\mathcal{F}^\sigma \to \mathcal{A}^\emptyset$. One can show that $[\![\mathcal{K}^\sigma]\!]_\sigma \subseteq \mathcal{K}^\emptyset$. So in fact we obtain a linear operator $[\![\cdot]\!]_\sigma : \mathcal{A}^\sigma \to \mathcal{A}^\emptyset$. A key property of $[\![\cdot]\!]_\sigma$ is that if $f \in \mathcal{A}^\sigma$ with $f \succeq_\sigma 0$ then $[\![f]\!]_\sigma \succeq_\emptyset 0$. Also, if $f \succeq_\sigma^{\mathcal{G}_\alpha^\sigma} 0$ then $[\![f]\!]_\sigma \succeq_\emptyset^{\mathcal{G}_\alpha} 0$.

Proof of $\eta(\alpha) \geq \frac{3}{5}\alpha$

Proposition 6.16 shows, as a special case, that there is a random abstract cone certifying that $\eta_1(\Lambda_n^2) \geq 3/5$. For illustrative purposes, we reformulate this random abstract cone argument into the language of flag algebras and show that $\eta(\alpha) \geq \frac{3}{5}\alpha$.

First note that by definition

$$\alpha \preceq_\emptyset^{\mathcal{G}_\alpha} \frac{1}{3}\Bigg(\quad \Bigg). \tag{8.5}$$



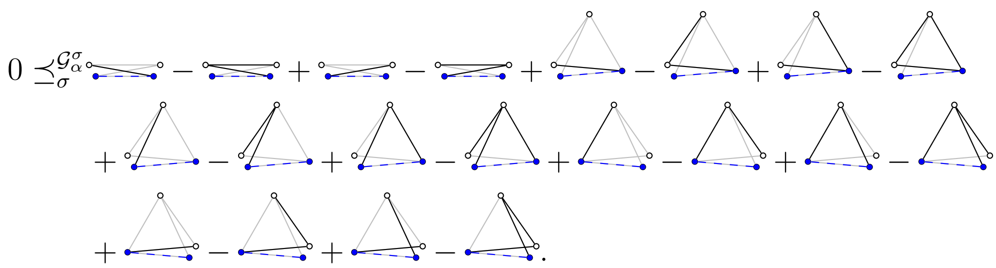
Next let us recall part of the random abstract cone argument. Write Λ_n^2 as $\Lambda_n^2 = U * V * W$ with $U = V = W = [n]$. Then spelling out the recursive construction in Proposition 6.16, we see that for $(u, v) \in U \times V$ the chain map $S_0^{(u,v)} \colon C_0(\Lambda_n^2; \mathbb{F}_2) \to C_1(\Lambda_n^2; \mathbb{F}_2)$ is given by

$$y \mapsto \begin{cases} uy & \text{if } y \in V \sqcup W \\ uv + yv & \text{if } y \in U. \end{cases}$$

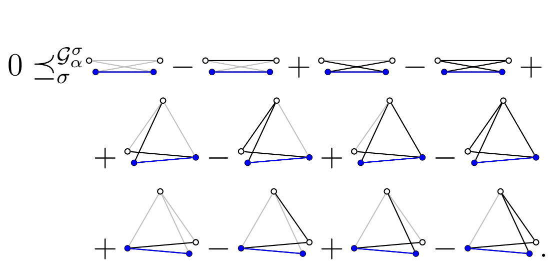
Write $T_1^{(u,v)}$ for the dual map of $S_0^{(u,v)}$. Let $c \in C^1(\Lambda_n^2; \mathbb{F}_2)$ and $\gamma^{(u,v)} = T_1^{(u,v)} c \in C^0(\Lambda_n^2; \mathbb{F}_2)$. By definition we have

$$\gamma^{(u,v)}(y) = \begin{cases} c(uy) & \text{if } y \in V \sqcup W \\ c(uv) + c(yv) & \text{if } y \in U \end{cases}.$$

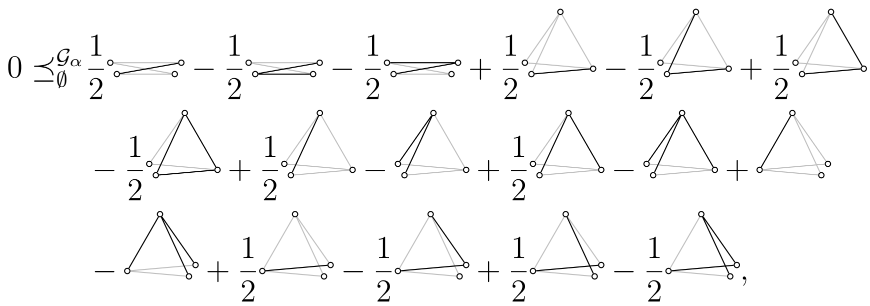
If we assume now that c is minimal, we get that $\|c + \delta\gamma^{(u,v)}\| \geq \|c\|$. If $c(uv) = 0$ this translates into the following inequality for the flag $\sigma =$



Similarly, if $c(uv) = 1$ this translates into an inequality for the flag $\sigma =$

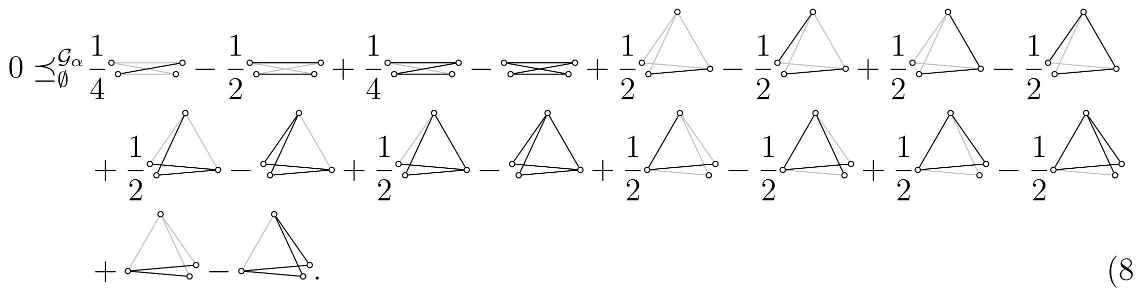


Applying the averaging operator to these inequalities gives



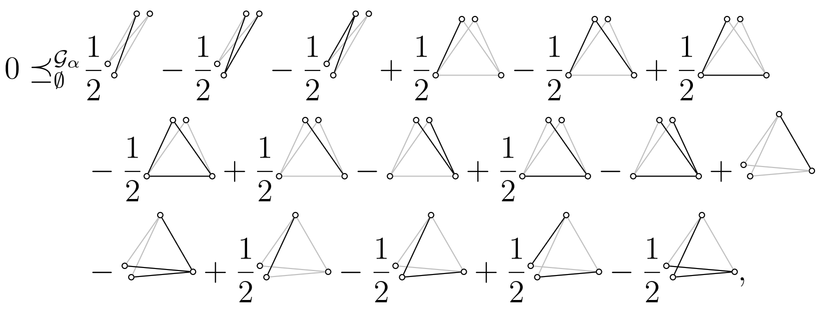
(8.6)

and



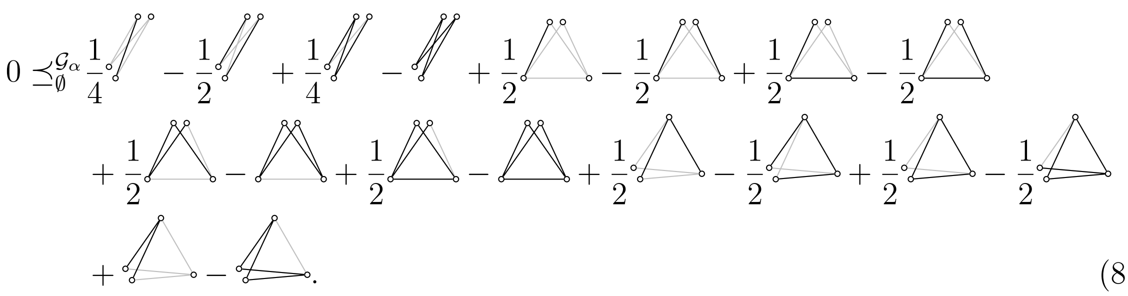
(8.7)

If we replace the role of $(u, v) \in U \times V$ by a pair $(w, u) \in W \times U$, we symmetrically get the following two inequalities:



(8.8)

and

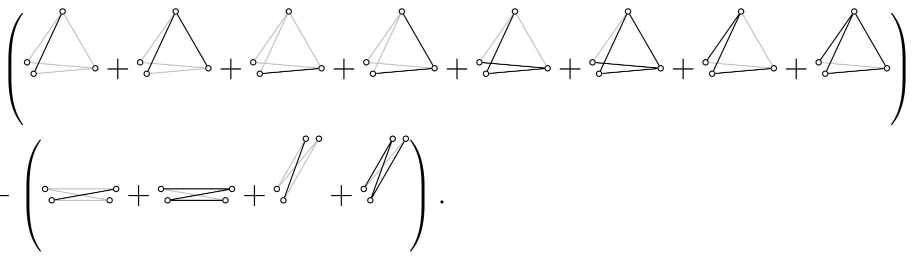


(8.9)

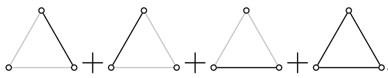
Recall that for the proof of the existence of a random abstract cone certificate for $\eta_0(\Lambda_n^d) \geq 1$ we used an inequality of negative type (Lemma 3.15). This enters into the proof of Proposition 6.16 as the base case of an induction. Spelling it out for Λ_n^2, it boils down to the inequality

$$\sum_{vv' \in \binom{V}{2}} |c(uv) + c(u'v) + c(u'v') + c(uv')| + \sum_{ww' \in \binom{W}{2}} |c(uw) + c(u'w) + c(u'w') + c(uw')|$$
$$\leq \sum_{v \in V, w \in W} |c(uv) + c(u'v) + c(u'w) + c(uw)|,$$

which holds for all $c \in C^1(\Lambda_n^2; \mathbb{F}_2)$ and $uu' \in \binom{U}{2}$. It is a consequence of Lemma 3.15 applied to the cochain $c_u + c_u' \in C^0(V * W; \mathbb{F}_2)$. In terms of flags this inequality translates into the following inequality

$0 \preceq_{\emptyset}^{\mathcal{G}_\alpha} 2$  (8.10)

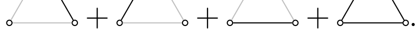
Now the idea is to find the largest $\lambda > 0$ for which there are $\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5 \in \mathbb{R}_{\geq 0}$ such that

$\lambda(8.5) + \lambda_1(8.6) + \lambda_2(8.7) + \lambda_3(8.8) + \lambda_4(8.12) + \lambda_5(8.10) \preceq_{\emptyset}^{\mathcal{G}_\alpha}$ 

Note that this a linear optimization problem. Any feasible solution λ would show that $\eta(\alpha) \geq \lambda\alpha$. To actually be able to feed this into a computer, we can represent all elements from $\mathcal{A}^\emptyset$ appearing in the inequalities above as linear combinations by flags in $\mathcal{F}_{(2,2,2)}$. This gives us vectors in $\mathbb{R}^{|\mathcal{F}_{(2,2,2)}|}$. Then, using that $\phi(F) \geq 0$ for all $F \in \mathcal{F}$ and limit functionals ϕ, we see that an inequality of the form $A \preceq_{\emptyset}^{\mathcal{G}_\alpha} A'$ holds for the elements from $\mathcal{A}^\emptyset$ represented by $A = \sum_{\sigma \in \mathcal{F}_{(2,2,2)}} \lambda_\sigma \sigma$ and $A' = \sum_{\sigma \in \mathcal{F}_{(2,2,2)}} \lambda'_\sigma$ if $\lambda_\sigma \leq \lambda'_\sigma$ for all $\sigma \in \mathcal{F}_{(2,2,2)}$.

We used a Python script[4] to check that if we sum up inequalities (8.5)-(8.10) with coefficients $3/5, 1/10, 1/10, 1/10, 1/10$ and $1/10$, we have

$$\frac{3}{5}\alpha \preceq_{\emptyset}^{\mathcal{G}_\alpha} \frac{3}{5}(8.5) + \frac{1}{10}((8.6) + (8.7) + (8.8) + (8.9) + (8.10))$$

$\preceq_{\emptyset}^{\mathcal{G}_\alpha}$ 

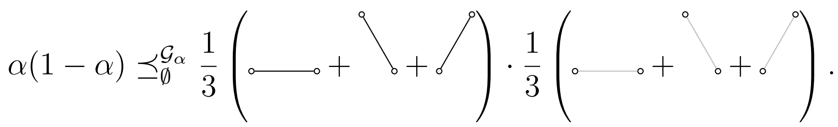
This shows that $\eta(\alpha) \geq \frac{3}{5}\alpha$, as desired.

Proof of Lemma 8.22

For the proof of Lemma 8.22 we will include some further inequalities into our linear optimization problem. For the first one, we notice that since $\alpha(1-\alpha)$ is an increasing

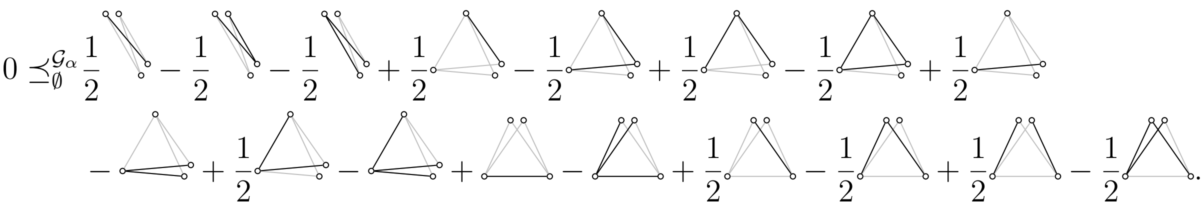
[4] This script will be made available via the library of IST Austria.

function of α on $(0, 1/2]$ we get that



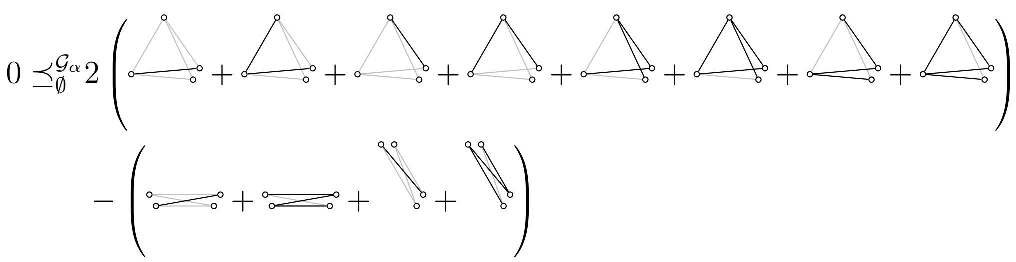
(8.11)

Replacing the role of $(u, v) \in (U, V)$ by $(v, w) \in (V, W)$ in inequality (8.6) we get by symmetry the following inequality



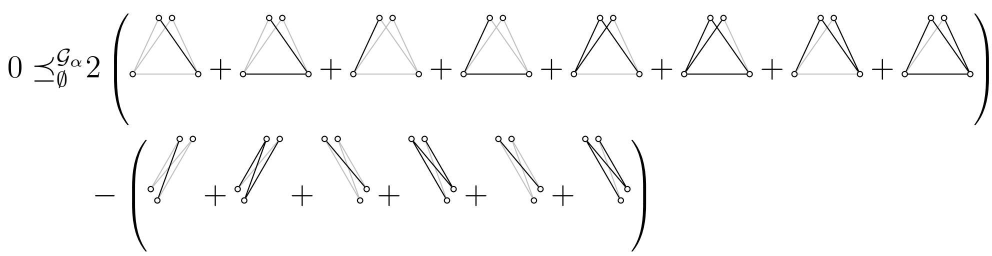
(8.12)

Similarly, by symmetry and interchanging the roles of U, V and W in (8.10) we get the following two inequalities



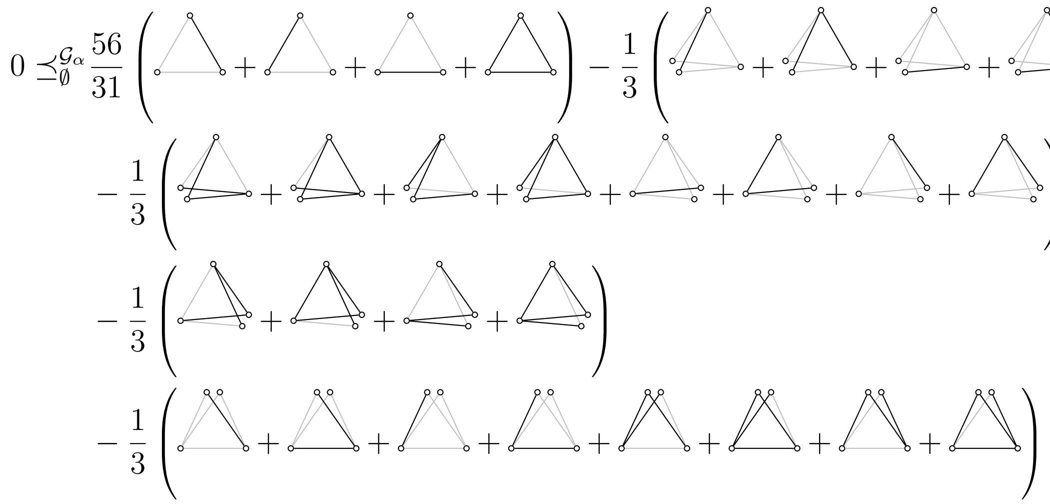
(8.13)

and



(8.14)

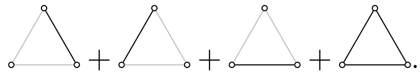
Finally, the improved triangle inequality from Corollary 8.21 translates into the following inequality of flags:



(8.15)

Using a Python script[5], we run a linear optimization problem which shows that if we take the linear combination of (8.11),(8.6),(8.8),(8.12),(8.10),(8.13),(8.14) and (8.15) with coefficients 31/37, 31/333, 31/333, 31/333, 217/5328, 217/5328, 217/5328 and 217/888, respectively, we get

$$
\begin{aligned}
\frac{31}{37}\alpha(1-\alpha) \preceq_{\emptyset}^{\mathcal{G}_\alpha} & \frac{31}{37}(8.11) + \frac{31}{333}\left((8.6) + (8.8) + (8.12)\right) \\
& + \frac{217}{5328}\left((8.10) + (8.13) + (8.14)\right) + \frac{217}{5328}(8.15) \\
& \preceq_{\emptyset}^{\mathcal{G}_\alpha}
\end{aligned}
$$



This shows $\eta(\alpha) \geq \frac{31}{37}\alpha(1-\alpha)$, as desired.

8.4.8 Proof of Proposition 8.9 and Corollary 8.10

We can put everything together and prove Proposition 8.9.

Proof of Proposition 8.9. By Lemma 8.19 it suffices to show that $\eta(\alpha) \geq 0.67159\alpha$ for all $\alpha \in (0, 1/2]$. Consider the functions

$$
\begin{aligned}
f\colon (0, 1/2] &\to \mathbb{R} \\
\alpha &\mapsto \frac{5 - \sqrt{25 - 48\alpha}}{8\alpha}
\end{aligned}
$$

and

$$
\begin{aligned}
h\colon (0, 1/2] &\to \mathbb{R} \\
\alpha &\mapsto \frac{31}{37}(1-\alpha).
\end{aligned}
$$

Lemma 8.11 and Lemma 8.22 imply that $\eta(\alpha)/\alpha \geq \max\{f(\alpha), h(\alpha)\}$ for all $\alpha \in (0, 1/2]$.

Note that f is monotonically increasing on $(0, 1/2]$, while h is monontonically decreasing on $(0, 1/2]$. A numerical solver suggested to us that the equation $f(\alpha) = h(\alpha)$ has a root for $\alpha \approx 0.19842$. One can now check that $f(\alpha) \geq 0.67159$ if $\alpha > 0.19841$ while $h(\alpha) \geq 0.67159$ if $\alpha < 0.19842$. This finishes the proof. □

Finally, we note that Corollary 8.10 follows from the fact $f(\alpha) \geq 3/4$ for $\alpha \geq 1/3$ and $h(\alpha) \geq 3/4$ for $\alpha \leq 13/124$, where f and h are the functions in the proof of Proposition 8.9.

8.4.9 Some Remarks

We close with some remarks how the approaches above might lead to further improvements on the lower bound on $\eta_1(\Lambda_n^2)$ but also indicate some limitations:

- The upper bound on the sum of squared degrees of a small minimal cochain $c \in C^1(\Lambda_n^2; \mathbb{F}_2)$ (Lemma 8.14) does not use the full strength of the minimality assumption but only the fact that $|c_x| \leq n$ for all $x \in \Lambda_n^2(0)$. Moreover, we do not

[5] Our Python code will be made available via the library of IST Austria.

exploit the fact that we are considering cochains in Λ^2_n rather than K_{3n}. Additionally, we could try to tighten the estimate

$$\sum_{v\in\Lambda^2_n(0)} |c^L_v||c^R_v| \leq \frac{1}{4}\sum_{v\in\Lambda^2_n(0)} |c_v|^2$$

by distinguishing 'left' and 'right' degrees of vertices.

- So far all inequalities for the flag algebra approach were obtained in a rather ad-hoc way, partially motivated by the random cofilling argument. There is a more systematic way of finding inequalities of the form $f \in \mathcal{A}^\sigma$ with $f \succeq_\sigma 0$ as follows: Fix some (not too large) size $(l_1, l_2, l_3) \in \mathbb{Z}^3_{\geq 0}$. Consider a positive semidefinite matrix $Q : \mathcal{F}^\sigma_{(l_1,l_2,l_3)} \times \mathcal{F}^\sigma_{(l_1,l_2,l_3)} \to \mathbb{R}$. Then the projection of

$$f = \sum_{F,G\in\mathcal{F}^\sigma_{(l_1,l_2,l_3)}} F \cdot Q(F,G) \cdot G$$

  to $\mathcal{A}^\sigma$ clearly satisfies $f \succeq_\sigma 0$. This opens the possibility to generate Cauchy–Schwarz/sum-of-squares type inequalities in an automated way and optimize parameters via semi-definite programming. While this could give us further inequalities within $\mathcal{F}^\sigma$, it is less clear how to generate such inequalities involving minimality, i.e. inequalities which only hold for limit functionals of convergent sequences in $\mathcal{G}_\alpha$.

- We have not considered expressions outside of $\mathcal{F}_{(2,2,2)}$ yet. But the upper bound example for small α can be viewed as a blow-up of an example in Λ^2_3 and the general upper bound $\eta_\infty \leq 3/4$ stems from blowing-up a cochain in Λ^2_4. This suggests that one should consider expressions inside $\mathcal{F}_{(3,3,3)}$ for small cochains or even in $\mathcal{F}_{(4,4,4)}$ for densities close to $1/3$ in order to fully capture the upper bound examples. Especially for $\mathcal{F}_{(4,4,4)}$ this might be computationally very challenging due to the size of $\mathcal{F}_{(4,4,4)}$.

Chapter 9

Miscellaneous

In this chapter, we collect a couple of losely connected further ponderings related to coboundary expansion. Namely,

- we point out a link between discrete Morse theory and abstract cones and use this to give a new proof that $h_k(Q_d) \geq 1$ for all $0 \leq k \leq d-1$ where Q_d denotes the d-dimensional hypercube viewed as a cubical complex,
- we provide a criterion for coboundary expansion (in a very dense regime) using intersection of links,
- we comment on the (computational) hardness of coboundary expansion constants (with respect to $\mathbb{F}_2$-coefficients),
- we give a thorough proof for the folklore fact that expansion of links is a necessary condition for expansion of the whole complex.

9.1 Abstract Cones via Discrete Morse Functions

In this section, we describe how given a discrete Morse function on a d-dimensional simplicial or cellular complex X without critical simplices in dimension $0 \leq k \leq d-1$, we can construct a cone for X. In particular, averaging over many different choices of such Morse functions one might get a lower bound on the coboundary expansion constants of X. We illustrate this by showing that $h_k(Q_d) \geq 1$ for all $0 \leq k \leq d-1$ where Q_d denotes the d-dimensional hypercube (as a cubical complex). This gives another, arguably more conceptional proof of Theorem 4.3 in [85].

9.1.1 Primer on Discrete Morse Theory

We give a very brief introduction to Forman's Discrete Morse Theory. We give the most basic definitions and state some results (without proofs) which we need. We refer the reader to [45] and [127] for a thorough treatment as well as [46] for a gentle user's guide introduction to the topic.

Definition 9.1 (Discrete Morse function)**.** Let X be a d-dimensional cellular complex. A function $f\colon X \to \mathbb{R}$ is a *discrete Morse function* if for all $\sigma \in X(k)$, $-1 \leq k \leq d$ we have

(i) $|\{\tau \in X(k+1) : \sigma \subseteq \tau, f(\tau) \leq f(\sigma)\}| \leq 1$, and

(ii) $|\{\rho \in X(k-1) : \rho \subseteq \sigma, f(\sigma) \leq f(\rho)\}| \leq 1$.

Given a discrete Morse function f, a simplex $\sigma \in X(k)$ is *critical* if

(i) $|\{\tau \in X(k+1) : \sigma \subseteq \tau, f(\tau) \leq f(\sigma)\}| = 0$ and

(ii) $|\{\rho \in X(k-1) : \rho \subseteq \sigma, f(\sigma) \leq f(\rho)\}| = 0$.

If $\sigma \in X$ is not critical, we call *non-critical.*

It is not difficult to see that for a non-critical simplex $\sigma \in X(k)$ of a Morse function f we cannot have both

$|\{\tau \in X(k+1) : \sigma \subseteq \tau, f(\tau) \leq f(\sigma)\}| = 1$ and $|\{\rho \in X(k-1) : \sigma \subseteq \tau, f(\sigma) \leq f(\rho)\}| = 1$.

Thus, every Morse function induces a partial on the cells of X such that $\sigma \in X(k)$ is matched with $\tau \in X(k+1)$ whenever $f(\sigma) \geq f(\tau)$ and such that the unmatched cells are precisely the critical cells. Using this, we can define the *gradient vector field V_f of f* as a map between chain groups $V_f \colon C_*(X;\mathbb{Z}) \to C_{*+1}(X;\mathbb{Z})$ by

$$\sigma \mapsto V_f(\sigma) = \begin{cases} -\langle \sigma, \partial\tau \rangle \tau & \text{if } \tau \in X(\dim\sigma + 1) \text{ with } f(\sigma) \geq f(\tau) \\ 0 & \text{otherwise.} \end{cases}$$

As in the smooth setting, the gradient vector field V_f contains all the relevant information.

More generally, we define a *discrete vector field V on X* as a partial matching of the Hasse diagram of X. In other words, V is a collection of pairs (σ, τ) such that $\sigma \subseteq \tau$ with $\dim\sigma = \dim\tau - 1$ and such that every cell appears in at most one pair. As for partial matchings coming from a discrete Morse function, we can also think of V as map between chain groups.

Given a discrete vector field V, we call a sequence of $(\sigma_0, \tau_0, \sigma_1, \tau_1, \ldots, \tau_r, \sigma_{r+1})$ a *V-path of length r* if there is k such that

(i) $\dim\sigma_i = k$ for all $0 \leq i \leq r+1$ and $\dim\tau_i = k+1$ for all $0 \leq i \leq r$,

(ii) $\sigma_i \subseteq \tau_i$ for all $0 \leq i \leq r$,

(iii) $\sigma_{i+1} \subseteq \tau_i$ and $\sigma_{i+1} \neq \sigma_i$ for all $0 \leq i \leq r$, and

(iv) $(\sigma_i, \tau_i) \in V$ for all $0 \leq i \leq r$.

A V-path is a *non-trivial, closed V-path* if $r \geq 0$ and $\sigma_0 = \sigma_{r+1}$. There is a nice criteria for a discrete vector field V to be the gradient vector field of a discrete Morse function.

Theorem 9.2 (Theorem 3.5 in [46])**.** *Let X be a cell complex. Let V be a discrete vector field on X. Then V is the gradient vector field V_f of a discrete Morse function f on X if and only if there is no non-trivial, closed V-path.*

Every gradient vector field V induces a *gradient flow* $\Phi\colon C_*(X;\mathbb{Z}) \to C_*(X;\mathbb{Z})$ given by $\Phi := \mathrm{id} + \partial \circ V + V \circ \partial$.

It turns out that Φ stabilizes, i.e. for sufficiently large N we have $\Phi^\infty := \Phi^N = \Phi^{N+1}$, where $\Phi^N = \Phi^{N-1} \circ \Phi$ with $\Phi^0 = \mathrm{id}$.

Write C_k^Φ for the Φ-invariant k-chains, i.e. for those $c \in C_k(X;\mathbb{Z})$ with $\Phi(c) = c$. Note that since $\partial \circ \Phi = \Phi \circ \partial$, ∂ maps Φ-invariant k-chains to Φ-invariant k-chains and C_*^Φ forms a sub(chain)complex of $C_*(X;\mathbb{Z})$, which is often called the *Morse complex of X associated with f*. Moreover, Φ^∞ induces a map $\Phi^\infty\colon C_*(X;\mathbb{Z}) \to C_*^\Phi$.

The Morse complex can also be described in terms of the critical simplices as follows: Let $\mathcal{M}_k \subseteq C_k(X;\mathbb{Z})$ be the span of critical simplices. Then Φ^∞ restricts to a map $\Phi^\infty\colon \mathcal{M}_k \to C_k^\Phi$. Let $\pi_{\mathcal{M}}\colon C_k(X;\mathbb{Z}) \to \mathcal{M}_k$ be given by

$$c = \sum_{\sigma\in X(k)} c_\sigma \sigma \mapsto \sum_{\sigma\in X(k),\sigma \text{ critical}} c_\sigma \sigma.$$

It turns out that restricting $\pi_{\mathcal{M}}$ to C_k^Φ gives an inverse to Φ^∞:

Theorem 9.3 (Theorem 8.2 in [45])**.** *$\Phi^\infty\colon \mathcal{M}_k \to C_k^\Phi$ is an isomorphism with inverse $\pi_{\mathcal{M}}\colon C_k^\Phi \to \mathcal{M}_k$.*

In particular, this shows that if f does not have any critical cells in dimension k, $\Phi^\infty\colon C_k(X;\mathbb{Z}) \to C_k(X;\mathbb{Z})$ is the zero map.

Moreover, it is not too difficult to show (see [45, Theorem 7.3]) that the map $L\colon C_*(X;\mathbb{Z}) \to C_{*+1}(X;\mathbb{Z})$ given by $L = -V(\mathrm{id} + \Phi + \cdots + \Phi^N)$ satisfies $\mathrm{id} - \iota \circ \Phi^\infty = \partial \circ L + L \circ \partial$, where $\iota\colon C_*^\Phi \to C_*(X;\mathbb{Z})$ denotes the inclusion map. This implies that $H_*(C_*^\Phi) \cong \tilde{H}_*(X;\mathbb{Z})$. It also shows that if f does not have any critical cells in dimension k, then (L_k, L_{k-1}) is an abstract cone in dimension k for X.

Now, we could try to exhibit many different Morse functions on X without critical simplices and try to average over the induced abstract cones. We illustrate this idea in the next subsection by proving a lower bound on the coboundary expansion constants for the hypercube thought of as a cell complex. We would like to point out that there are many different families of simplicial complexes for which the vanishing of their (co)homology groups can be proven by constructing Morse functions without critical simplices. This includes, among others, all shellable complexes. As elaborated above the random abstract cofilling technique to prove lower bounds on coboundary expansion constants only works well if there is a large collection of well-distributed cycles. Even if such a collection of cycles is not available, it might still be interesting to try to use Morse functions with a small number of critical simplices to simplify a given cell complex X to a complex X' with comparable expansion constants and then use a different argument to show expansion for X'.

9.1.2 Discrete Morse Matchings on Q_d

Let Q_d be the d-dimensional hypercube given as a (cubical) cell complex. Cells of dimension at least 0 in Q_d can be described as vectors $x = (x_1, \ldots, x_d) \in \{0, 1, *\}^d$ such that the number of x_i's with $x_i = *$ is the dimension of the cell x. We add the empty cell $\emptyset$ to Q_d. With this description of the cells and $k \geq 1$ the boundary operator $\partial\colon C_k(Q_d;\mathbb{F}_2) \to C_{k-1}(Q_d;\mathbb{F}_2)$ is given by

$$x \mapsto \partial x = \sum_{i\in[d],x_i=*} (x_1,\dots,x_{i-1},0,x_{i+1},\dots,x_d) + (x_1,\dots,x_{i-1},1,x_{i+1},\dots,x_d),$$

i.e. the occurrence of every $*$ in x is replaced by 1 or 0.

Fix a vertex $x^{(0)} \in Q_d(0)$ and a permutation $\pi \in S_d$. Define a matching $\mathcal{M}_{x^{(0)},\pi}$ on Q_d as follows: Match $x^{(0)}$ with the empty cell $\emptyset$. Given $x = (x_1,\dots,x_d) \in Q_d \setminus \{x^{(0)},\emptyset\}$ let $i = \min\{k : x_{\pi(k)} \neq x^{(0)}_{\pi(k)}\}$. Match x with $y = (y_1,\dots,y_d)$ where $y_{\pi(j)} = x_{\pi(j)}$ for $j \in [d], j \neq i$ and $y_{\pi(i)}$ being the unique element in $\{0,1,*\} \setminus \{x^{(0)}_{\pi(i)}, x_{\pi(i)}\}$. Write $V^{(x^{(0)},\pi)}$ for the discrete vector field induced by $\mathcal{M}_{x^{(0)},\pi}$. We have

Lemma 9.4. *For every $\pi \in S_d$ and $x^{(0)} \in Q_d(0)$ the vector field $V^{(x^{(0)},\pi)}$ is the gradient vector field of a discrete Morse function.*

Proof. We show that there is no non-trivial, closed $V^{(x^{(0)},\pi)}$-path. For given $x \in Q_d \setminus \emptyset$, we let $i(x) := \min\{k : x_{\pi(k)} \neq x^{(0)}_{\pi(k)}\}$ and $s(x) := \min\{k : x_{\pi(k)} = *\}$. Here, we understand $\min\emptyset$ to be equal to $+\infty$, so that $i(x), s(x) \in [d] \cup \{+\infty\}$. We extend the natural linear order on $[d]$ to $[d] \cup \{+\infty\}$ by saying that $i < +\infty$ for all $i \in [d]$. Write $\prec$ for the induced lexicographic ordering on $([d] \cup \{+\infty\}) \times ([d] \cup \{+\infty\})$, i.e. $(i,j) \prec (i',j')$ if $i < i'$ or $i = i'$ and $j < j'$. Now given a $V^{(x^{(0)},\pi)}$-path $(x_0, x_1, \dots, x_{r+1})$ we note that $(i(x_0), s(x_0)) \succ (i(x_1), s(x_1)) \succ \cdots \succ (i(x_{r+1}), s(x_{r+1}))$ which implies that $(x_0, x_1, \dots, x_{r+1})$ cannot be a non-trivial, closed path. $\square$

Note that $V^{(x^{(0)},\pi)}$ does not have any critical simplices and, hence, induces an abstract cone $(S_k^{(x^{(0)},\pi)})_{-2\leq k\leq d-1}$. Endow $\Omega := Q_d(0) \times S_d$ with the uniform distribution such that we get a random abstract cone $(S_k^{(\omega)})_{\omega\in\Omega,-2\leq k\leq d-1}$ parametrized by Ω. We claim that this random abstract cone is a certificate for (unnormalized) expansion 1 for Q_d.

Proposition 9.5. *For all $d \geq 1$ and $0 \leq k \leq d-1$ we have $h_k(Q_d) \geq 1$.*

Proof. Fix a dimension $d \geq 1$ and $0 \leq k \leq d-1$. Let $c \in C^k(Q_d;\mathbb{F}_2)$. As in the proof of Proposition 3.13 we have

$$|[c]| \leq \sum_{\tau\in Q_d(k+1)} |\delta c(\tau)|\lambda(\tau)$$

where $\lambda(\tau) = \frac{1}{d!2^d} \sum_{\pi\in S_d, x^{(0)}\in Q_d(0)} \sum_{\sigma\in Q_d(k)} |\langle \mathbb{1}_\tau, S_k^{(x^{(0)},\pi)}\sigma\rangle|$.

By symmetry $\lambda := \lambda(\tau)$ is independent of $\tau \in Q_d(k+1)$. Hence,

$$\lambda|Q_d(k+1)| = \frac{1}{d!2^d} \sum_{\pi\in S_d, x^{(0)}\in Q_d(0)} \sum_{\sigma\in Q_d(k)} |S_k^{(x^{(0)},\pi)}\sigma|.$$

By symmetry $\Theta_k := \sum_{\sigma\in Q_d(k)} |S_k^{(x^{(0)},\pi)}\sigma|$ is independent of $(x^{(0)},\pi) \in \Omega$. It follows that

$$h_k(Q_d) \geq \frac{|Q_d(k+1)|}{\Theta_k}.$$

It remains to compute $\Theta_k = \sum_{\sigma \in Q_d(k)} |S_k^{(x^{(0)},\pi)}\sigma|$ for some $\pi \in S_d$ and $x^{(0)} \in Q_d(0)$. Let us choose $\pi := \mathrm{id}$ and $x^{(0)} := (0, \dots, 0)$ the zero vector. To ease notation, write $V := V^{(x^{(0)},\pi)}$ and $S_k := S_k^{(x^{(0)},\pi)}$. Let $x = (x_1, \dots, x_d) \in Q_d(k)$. Let

$$e(x) := \min\{i \in [d] : x_i = 1\} \text{ and } s(x) := \min\{i \in [d] : x_i = *\}.$$

We set $\min \emptyset = +\infty$ with the relation $l < +\infty$ for all $l \in [d]$.

Note that $V(x) = 0$ if $s(x) < e(x)$ and $V(x) = (x_1, \dots, x_{e(x)-1}, *, x_{e(x)+1}, \dots, x_d)$ if $e(x) < s(x)$. Using this, we easily compute the discrete gradient flow $\Phi = \mathrm{id} + \partial V + V\partial$ to be given by

$$\Phi(x) = (x_1, \dots, x_{e(x)-1}, 0, x_{e(x)+1}, \dots x_d)$$

if $e(x) < s(x)$. If $x \in Q_d$ with $s(x) < e(x)$ let $j = \min\{l \in [d] : l > s(x), x_l \neq 0\}$. Then

$$\Phi(x) = \begin{cases} 0 & \text{if } x_j = * \\ (x_1, \dots, x_{s(x)-1}, 0, \dots, x_{j-1}, *, x_{j+1}, \dots, x_d) & \text{if } x_j = 1. \end{cases}$$

From this, we easily compute that $|S_k x| = |\{j \in [d] : j < s(x), x_j = 1\}|$. We conclude that

$$\Theta_0 = \sum_{x \in Q_d(0)} |S_0 x| = \sum_{i=0}^{d} i \binom{d}{i} = d2^{d-1} = |Q_d(1)|$$

and for $k \geq 1$ that

$$\Theta_k = \sum_{x \in Q_d(k)} |S_k x| = \sum_{i=1}^{d-k+1} \binom{d-i}{k-1} 2^{d-i-k+1} \sum_{s=0}^{i-1} s \binom{i-1}{s} = 2^{d-k-1} \binom{d}{k+1} = |Q_d(k+1)|.$$

Plugging these into $h_k(Q_d) \geq \frac{|Q_d(k+1)|}{\Theta_k}$ we get $h_k(Q_d) \geq 1$, as desired. □

9.2 Double-Link Criterion for Coboundary Expansion

In this section, we present a simple criterion, which we call the *double-link criterion*, for coboundary expansion by considering intersections of vertex links. In its current version, the criterion requires very dense complexes and, hence, has very limited applications.

Our starting point for finding our double-link criterion was to come up with a simple way to distinguish two different random models for 2-dimensional simplicial complexes with complete 1-skeleton: complexes according to the Linial–Meshulam model $X^2(n, 1/2)$ (see [89, 109]) and a random construction due to Gundert and Wagner which we will denote by $Y^2_{GW}(n, 1/2)$ (see [56]).

$X \sim X^2(n, 1/2)$ is obtained from a complete 1-skeleton on n vertices by adding each possible triangle independently with probability 1/2.

$Y \sim Y^2_{GW}(n, 1/2)$ is obtained as follows: Pick $c \in C^1(K_n^2; \mathbb{F}_2)$ to be a random cochain with density 1/2, i.e. every edge of the underlying graph of K_n^2 is picked independently with probability 1/2 to be in the support of c. We can think of c to be a random Erdős–Rényi graph $c \sim G(n, 1/2)$. Define Y_c to be the simplicial complex on vertex set $[n]$ with complete 1-skeleton and triangles $K_n^2(2) \setminus \mathrm{supp}(\delta_{K_n^2} c)$ and set $Y = Y_c$.

It is not too difficult to see that for every vertex $u \in X(0)$ or $v \in Y(0)$ the vertex link X_u or Y_v is a random Erdős–Rényi $X_u, Y_v \sim G(n-1, 1/2)$. In particular, with high probability, all vertex links are good expander. Garland's method then implies that the up-Laplacian of both X and Y has a good spectral gap. On the other hand, while $X \sim X^2(n, 1/2)$ is a good coboundary expander (constant expansion with high probability with respect to normalized Hamming norm), we have that $\delta_{Y_c} c = 0$ by construction while $\|[c]\| \geq \frac{1}{2} - \varepsilon$ with high probability for any $\varepsilon > 0$. In particular, $\tilde{H}^1(Y; \mathbb{F}_2) \neq 0$. This example also shows that it is not sufficient to consider (vertex) links in order to find a criterion for coboundary expansion. We refer the reader to [56] for a complete discussion of all these results.

It is natural to ask for a structure within $Y \sim Y^2_{GW}(n, 1/2)$ which detects that Y is not a coboundary expander. It turns out that by looking at intersections of two different vertex links, we can distinguish $X \sim X^2(n, 1/2)$ from $Y \sim Y^2_{GW}(n, 1/2)$. Indeed, fix $X \sim X^2(n, 1/2)$ and $Y \sim Y^2_{GW}(n, 1/2)$. For $u, v \in X(0), u \neq v$ it is easy to see that $X_u \cap X_v \sim G(n-2, 1/4)$ is a random Erdős-Rényi graph on $n-2$ vertices with density $1/4$.

The situation for $u, v \in Y(0), u \neq v$ is very different. Let $c \in C^1(Y; \mathbb{F}_2)$ be the 1-cochain defining Y. Given a vertex x in $Y_u \cap Y_v$ we call $c(ux) + c(vx) \in \mathbb{F}_2$ its type. Notice that there can only be an edge between x and y in $Y_u \cap Y_v$ if x and y have the same type. In that case there will be an edge with probability $1/2$ (there is exactly one remaining choice for the value $c(xy)$ such that both vxy and uxy are in $Y(2)$). It follows that $Y_u \cap Y_v$ is a disjoint union of two Erdős-Rényi graphs with density $1/2$, one on the set of type 0 vertices and one on the set of type 1 vertices. Since types of vertices in $Y_u \cap Y_v$ are independent and take only two distinct values with probability $1/2$ we expect these two graphs to be of roughly equal size. Notice that $Y_u \cap Y_v$ is not connected (see Figure 9.1). We can turn this observation into a quantitative statement. For this, it is important to note that given a d-dimensional simplicial complex X, $u, v \in X(0)$, $u \neq v$, $c \in C^{d-1}(X; \mathbb{F}_2)$ and a $(d-1)$-simplex $\sigma \in X_u \cap X_v$ with $\delta_{X_u \cap X_v}(c_u + c_v)(\sigma) = 1$, then precisely one of the d-simplices $\sigma \sqcup \{u\}$ and $\sigma \sqcup \{v\}$ is in the support of δc.

Now, assume for a moment that X has a complete $(d-1)$-skeleton, i.e. $X \subseteq Y$ with $X^{(d-1)} = Y^{(d-1)}$ and $Y = K_n^d$ for some n, and that $\delta_{X_u \cap X_v}(c_u + c_v)(\sigma)$ is large compared to $\delta_{Y_u \cap Y_v}(c_u + c_v)$. Then it would suffice to give a lower bound on the number of pairs of d-simplices (σ, σ') in Y with $\sigma \cap \sigma' \in Y(d-1)$ and precisely one of σ and σ' in the support of $\delta_Y c$. The number of such pairs can be interpreted as the cut induced by $\delta_Y c$ thought of as a subset of vertices in the graph $\Gamma_Y = (V(\Gamma_Y), E(\Gamma_Y))$ with vertex set $V(\Gamma_Y) = Y(d)$ and edges $\{\sigma, \sigma'\}$ if $\sigma \cap \sigma' \in Y(d-1)$. Thus, expansion properties of Γ_Y and Y would finalize this argument and deduce expansion for X. For $Y = K_n^d$ the graph Γ_Y is well-known in the literature as the *Johnson graph* $J(n, d+1)$. The eigenvalues of $J(n, d+1)$ are fully understood (see, e.g., [21, Chapter 4]) and using Cheeger's inequality (Theorem 2.2) one can show that $h_0(\Gamma_Y) \geq n/2$. Also, $h_{d-1}(K_n^2) \geq n/(d+1)$.

There is nothing special about $Y = K_n^d$ in the above argument. As long as $Y^{(d-1)} = X^{(d-1)}$, Y and Γ_Y are expander and $\delta_{X_u \cap X_v}(c_u + c_v)(\sigma)$ can be compared to $\delta_{Y_u \cap Y_v}(c_u + c_v)$, the argument goes through. This suggests to introduce the following terminology: Let $X \subseteq Y$ be d-simplicial complexes defined on the same vertex set $V = X(0) = Y(0)$. Given

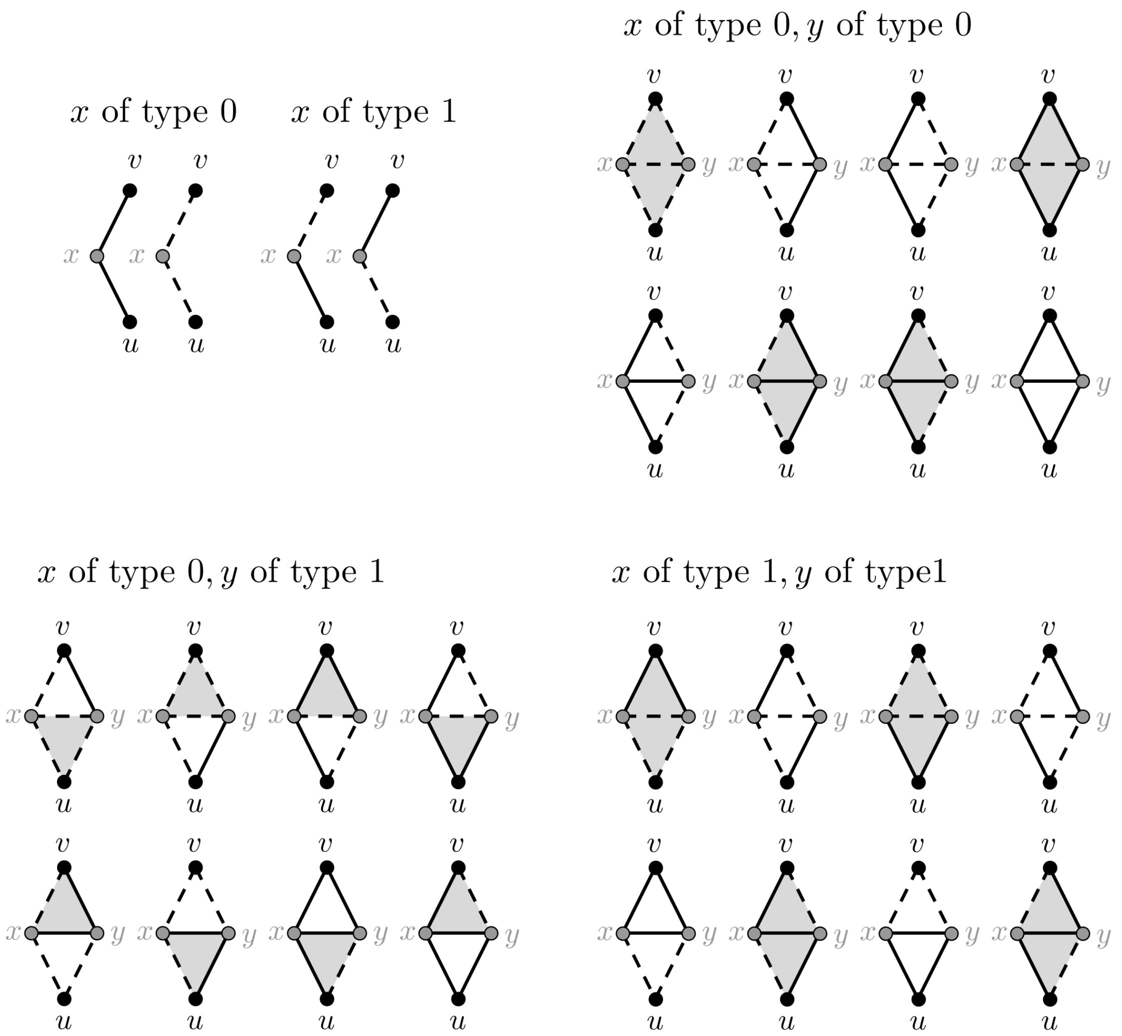


Figure 9.1: An illustration for $Y_u \cap Y_v$ in $Y \sim Y^2_{GW}(n, 1/2)$: Edges in the support of c defining Y are marked black, all other edges are dashed. Triangles in Y are filled in. We distinguish two types of vertices $x \in Y_u \cap Y_v$ depending on the value $c(ux) + c(vx)$. We see that an edge xy cannot end up in $Y_u \cap Y_v$ unless x and y have the same type. In this case, xy will be in $Y_u \cap Y_v$ with probability $1/2$.

$c \in C^{d-1}(X; \mathbb{F}_2)$ write $\bar{c} \in C^{d-1}(Y; \mathbb{F}_2)$ for the extension by 0 of c to Y. For $uv \in \binom{V}{2}$ let

$$\eta_{uv}(X; Y) := \min_{c \in C^{d-2}(X_u \cap X_v; \mathbb{F}_2)} \frac{|\delta_{X_u \cap X_v} c|}{|\delta_{Y_u \cap Y_v} \bar{c}|},$$

where we define the quotient on the right hand side to be $+\infty$ whenever $\delta_{Y_u \cap Y_v} \bar{c} = 0$. Let

$$\eta_{\mathrm{DL}}(X; Y) := \min_{uv \in \binom{V}{2}} \eta_{uv}(X; Y)$$

which we call the *double-link expansion of X with respect to Y*. We write $D_X := \max_{\sigma \in X(d-1)} |X_\sigma(0)|$ for the largest degree of a $(d-1)$-simplex in X.

With all these preparations we are ready to show:

Proposition 9.6. *Let $X \subseteq Y$ be two d-dimensional simplicial complexes on the same vertex set V. Assume that X and Y have the same codimension 1 skeleton, i.e. $X^{(d-1)} =$*

$Y^{(d-1)}$. *Then*

$$h_{d-1}(X) \geq \frac{1}{(d+1)D_X} \frac{|Y(d)|}{|Y(d-1)|} \min\{1/2, \bar{h}_{d-1}(Y)\} \eta_{\mathrm{DL}}(X;Y) h_0(\Gamma_Y).$$

Proof. Let $c \in C^{d-1}(X;\mathbb{F}_2)$ be minimal (with respect to the Hamming norm $|\cdot|$). From our discussion above, we get that

$$\sum_{uv \in \binom{V}{2}} |\delta_{X_u \cap X_v}(c_u + c_v)|$$

counts the number of pairs $\{\sigma, \sigma'\} \in \binom{X(d)}{2}$ with precisely one of σ and σ' in the support of δc. Note that every $\sigma \in \operatorname{supp}(\delta c)$ gets counted at most $D_X(d+1)$ times. Using this, the definition of double-link expansion and the fact that $X^{(d-1)} = Y^{(d-1)}$ we estimate

$$\begin{aligned}
(d+1)D_X|\delta_X c| &\geq \sum_{\{u,v\} \in \binom{V}{2}} |\delta_{X_u \cap X_v}(c_u + c_v)| \\
&\geq \eta_{\mathrm{DL}}(X;Y) \sum_{\{u,v\} \in \binom{V}{2}} |\delta_{Y_u \cap Y_v}(\overline{c_u + c_v})| \\
&= \eta_{\mathrm{DL}}(X;Y) |E_{\Gamma_Y}(\operatorname{supp}(\delta_Y \bar{c}), V(\Gamma_Y) \setminus \operatorname{supp}(\delta_Y \bar{c}))| \\
&\geq \eta_{\mathrm{DL}}(X;Y) h_0(\Gamma_Y) \min\{|\delta_Y \bar{c}|, |\mathbb{1} + \delta_Y \bar{c}|\}.
\end{aligned}$$

We distinguish two cases. If $|\delta_Y \bar{c}| > \frac{1}{2}|Y(d)|$, we get

$$\begin{aligned}
|\delta_X c| &\geq \frac{1}{(d+1)D_X} \eta_{\mathrm{DL}}(X;Y) h_0(\Gamma_Y) \frac{|Y(d)|}{2} \\
&\geq \frac{1}{(d+1)D_X} \eta_{\mathrm{DL}}(X;Y) h_0(\Gamma_Y) \frac{|Y(d)|}{2|Y(d-1)|} |c|.
\end{aligned}$$

If $|\delta_Y \bar{c}| \leq \frac{1}{2}|Y(d)|$ we use expansion of Y to deduce

$$\begin{aligned}
|\delta_X c| &\geq \frac{1}{(d+1)D_X} \eta_{\mathrm{DL}}(X;Y) \bar{h}_{d-1}(Y) \frac{|Y(d)|}{|Y(d-1)|} |[\bar{c}]| \\
&= \frac{1}{(d+1)D_X} \eta_{\mathrm{DL}}(X;Y) \bar{h}_{d-1}(Y) \frac{|Y(d)|}{|Y(d-1)|} |c|,
\end{aligned}$$

where we used Lemma 3.6 for the last equality. Combining the two cases gives the lower bound on $h_{d-1}(X)$ as claimed and finishes the proof. □

9.3 Hardness of Computing Coboundary Expansion Constants

Computing $h_0(G)$ for a given graph $G = (V, E)$ is a computationally hard problem, known to be NP-hard (see, for instance, [71, Theorem 2]). Since eigenvalues can be computed efficiently (up to a priori fixed accuracy), the Cheeger inequality (Theorem 2.2) yields an approximation algorithm for $h_0(G)$. Unfortunately, the approximation ratio guaranteed by the Cheeger inequality is fairly poor, especially for small values of $h_0(G)$. Using semidefinite programming approaches a $O(\sqrt{\log |V|})$-approximation algorithm was given in [9]. This is the best currently known approximation algorithm and the precise approximation ratio for $h_0(G)$ one could achieve remains unknown. In fact, a variant of computing $h_0(G)$, called the *gap-small-set expansion problem*[1], is intimately related to the *unique games conjecture*, one of the major open problems in computational complexity theory (see [77] and [120] for starting points into this topic).

Intuitively speaking, one should expect that computing $h_{d-1}(X)$ for a d-dimensional simplicial complex X with $d \geq 2$ should be at least as hard as computing the edge expansion constant $h_0(G)$ of graphs. In fact, we are not aware of any known efficient approximation algorithm for $h_{d-1}(X)$ with a guaranteed, non-trivial approximation ratio. As mentioned earlier, the Cheeger inequality fails in dimension $d \geq 2$ (see [130, 56]). Furthermore, it is known (see [66]) that even deciding whether a given cochain $c \in C^1(K_n; \mathbb{F}_2)$ is minimal (with respect to the Hamming norm) is NP-complete.

As a tiny step towards a better understanding of the complexity of computing $h_{d-1}(X)$, we give a simple construction which turns a given graph G and a dimension d into a d-dimensional simplicial complex X such that $h_{d-1}(X)$ is within a factor d of $h_0(G)$. More precisely, we have:

Proposition 9.7. *Let $G = (V, E)$ be a connected graph and $d \geq 2$. Let $\varepsilon > 0$. Let $N \in \mathbb{Z}_{>0}$ with $N \geq |V|/\varepsilon$. Let $X = K_N^{d-2} * G$ be the join of G with a complete $(d-2)$-dimensional complex on N vertices. Then*

$$\frac{1}{d}(1-\varepsilon)h_0(G) \leq h_{d-1}(X) \leq h_0(G).$$

Proof. For the lower bound we use a random cofilling argument. Using Künneth formula for the cohomology groups of joins (see [61, Chapter V.]) we have that $\tilde{H}^{d-1}(X; \mathbb{F}_2) = 0$. Hence, according to Lemma 3.9, it suffices to show a cofilling inequality. To this end, let $b \in B^d(X; \mathbb{F}_2)$ with $b = \delta a$ for some $a \in C^{d-1}(X; \mathbb{F}_2)$. Write U for the vertices of the copy of K_N^{d-2} in X. For $s \in U$ define $a^{(s)} \in C^{d-1}(X; \mathbb{F}_2) = C^{d-2}(K_N^{d-2}; \mathbb{F}_2) \otimes C^0(G; \mathbb{F}_2) \oplus C^{d-3}(K_N^{d-2}; \mathbb{F}_2) \otimes C^1(G; \mathbb{F}_2)$ by

$$a^{(s)}(\sigma \otimes \tau) = \begin{cases} 0 & \text{if } s \in \sigma \\ b((v \sqcup \sigma) \otimes \tau) & \text{if } \sigma \in K_N^{d-2}(d-3), \tau \in E, s \notin \sigma \\ \alpha^{(s,\sigma)}(\tau), & \text{if } \sigma \in K_N^{d-2}(d-2), \tau \in V, s \notin \sigma, \end{cases}$$

where $\alpha^{(s,\sigma)}$ is a minimal cofilling of

$$\sum_{\tau \subseteq s \sqcup \sigma, |\tau| = d-1} b_\tau \in B^1(G; \mathbb{F}_2).$$

$\sum_{\tau \subseteq s \sqcup \sigma, |\tau| = d-1} b_\tau$ is indeed a coboundary since it is the localization of the coboundary b along the cycle $\partial(s \sqcup \sigma) \in Z_{d-2}(K_N^{d-2}; \mathbb{F}_2)$.

Note that $\delta a^{(s)} = b$ for all $s \in U$, essentially by construction. We estimate

$$\begin{aligned} |[a]| &\leq \frac{1}{N} \sum_{s \in U} |a^{(s)}| \\ &= \frac{d-1}{N}|b| + \frac{1}{N} \sum_{s \in U, \sigma \in \binom{[N] \setminus \{s\}}{d-2}} |a^{(s,\sigma)}|. \end{aligned}$$

[1]The gap-small-set expansion problem asks to distinguish whether given a d-regular graph $G = (V, E)$ and constants $\delta, \eta > 0$ there is $S \subseteq V$ with $|S| = \delta|V|$ and $|E(S, V \setminus S)| \leq \eta d|S|$ or whether $|E(S, V \setminus S)| \geq (1-\eta)d|S|$ for all $S \subseteq V$ with $|S| = \delta|V|$.

Using the expansion of G and the triangle inequality we get

$$\begin{aligned}
||[a]|| &\le \frac{d-1}{N}|b| + \frac{d}{Nh_0(G)} \sum_{\tau \in \binom{[N]}{d}} |b_{\partial\tau}| \\
&\le \frac{d-1}{N}|b| + \frac{d(N-d+1)}{Nh_0(G)} \sum_{\sigma \in K_N^{d-2}(d-2)} |b_\sigma| \\
&= \left(\frac{d-1}{N} + \frac{d(N-d+1)}{Nh_0(G)}\right)|b|.
\end{aligned}$$

Thus,

$$h_{d-1}(K_N^{d-2}) \ge h_0(G)\frac{N}{h_0(G)(d-1)+d(N-d+1)} \ge \frac{h_0(G)}{d}(1-\varepsilon),$$

where we used the assumption $N \ge |V|/\varepsilon$ for the last inequality.

To see the upper bound, fix $S \subseteq V$ achieving $h_0(G)$, i.e. $0 < |S| \le |V|/2$ and $|E(S, V \setminus S| = h_0(G)|S|$. Fix $\sigma \in K_N^{d-2}(d-2)$ and define $c := \mathbb{1}_\sigma \otimes \mathbb{1}_S \in C^{d-1}(X;\mathbb{F}_2)$. We claim that c is minimal, which would finish the proof since then

$$h_{d-1}(X) \le \frac{|\delta c|}{||[c]||} = \frac{|\delta c|}{|c|} = \frac{|\mathbb{1}_\sigma \otimes \delta_G \mathbb{1}_S|}{|S|} = h_0(G).$$

To see that c is minimal let $\varphi\colon S \to V \setminus S$ be an injective function. Fix a vertex $v \in K_N^{d-2}(0) \setminus \sigma$ (we can assume that $N \ge d$). Consider the family of cycles $(z^{(s)})_{s\in S} \subseteq Z_{d-1}(X;\mathbb{F}_2)$ given by $z^{(s)} = (\partial(\sigma \sqcup v)) \otimes (s + \varphi(s))$. Note that the cycles $z^{(s)}$ have pairwise disjoint support. Moreover, for any $s \in S$ and $a \in C^{d-2}(X;\mathbb{F}_2)$ we have

$$\langle c + \delta a, z^{(s)}\rangle = \langle c, z^{(s)}\rangle \ne 0.$$

This implies $||[c]|| \ge |S| = |c|$, as desired. □

9.4 Expansion of Links Is Necessary

Local-to-global argument have been used to show expansion for small cochains with respect to $\mathbb{F}_2$-coefficients (and normalized or Garland weighted Hamming norm) at various places in the literature (see, for instance, [97, 96, 72] and especially Evra–Kaufman's local-to-global criterion [40, Theorem 5]). Common to all these argument is that they exploit good expansion properties of the links combined with excellent expansion of various graphs and the fact that the cochains under consideration have small norm.

Having these results in mind, it seems natural to think that expansion of the links is actually a necessary condition for expansion of the whole complex. We believe that this is a folklore result known within the community of HDXs but we could not find any formal argument in the literature. In this section, we would like to fill-in this gap and show the following:

Proposition 9.8. *Let X be a d-dimensional simplicial complex. Let $0 \le k \le d-2$. Let $v \in X(0)$ and $\rho_v^{(k)} := \max_{\sigma \in X_v(k)} w_\sigma(v)$ (here we write w_σ for the Garland weights on X_σ). Assume that $\eta_k(X) \ge 2(k+2)\rho_v^{(k)}$. Then*

$$\eta_k(X_v) \ge \frac{k+2}{k+3}\eta_{k+1}(X).$$

Recall from Lemma 5.4 that we always have $w_\sigma(v) \leq 1/\delta(X)$ where $\delta(X)$ denotes the thickness of X. Thus, the assumption $\eta_k(X) \geq 2(k+2)\rho_v$ is fairly mild.

Working with the Hamming norm, we can also show a result in terms of cofilling constants. To state such a result, let us introduce the following notation: Given a d-dimensional simplicial complex X and $1 \leq k \leq d$ let

$$L_k(X) := \max_{a \in C^{k-1}(X;\mathbb{F}_2) \setminus Z^{k-1}(X;\mathbb{F}_2)} \frac{\min_{z \in Z^{k-1}(X;\mathbb{F}_2)} |a+z|}{|\delta a|}$$

where $|\cdot|$ is the Hamming norm. In other words, $L_k(X)$ is the smallest number L such that for all $b \in B^k(X;\mathbb{F}_2)$ there is $a \in C^{k-1}(X;\mathbb{F}_2)$ with $\delta a = b$ and $|a| \leq L|b|$. We have

Proposition 9.9. *Let X be a d-dimensional simplicial complex and $1 \leq k \leq d-1$. Let $v \in X(0)$. If $L_k(X_v) \leq 1$ then $L_k(X_v) \leq L_{k+1}(X)$.*

Before we give the proofs of Proposition 9.8 and Proposition 9.9, let us mention two consequences: Using Proposition 9.8 we see that (at the cost of potentially worse bounds) for an application of Evra–Kaufman's local-to-global criterion [40, Theorem 5], it suffices to give a lower bound on $\eta_k(X_\sigma)$ for all $\sigma \in X(0)$ rather than all $\sigma \in X$.

The upper bound on $\eta_{d-2}(A_{d-1}(\mathbb{F}_q))$ in Theorem 1.6 translates into the lower bound

$$L_{d-1}(A_{d-1}(\mathbb{F}_q)) \geq \frac{2^{d-1}}{q+1}(1-\varepsilon)$$

for any $\varepsilon > 0$ and sufficiently large $q \geq Q(\varepsilon)$. On the other hand there is a positive constant $\eta_{d-1} > 0$ independent of q such that $\eta_{d-2}(A_{d-1}(\mathbb{F}_q)) \geq \eta_{d-1}$ (see, e.g., [98, Corollary 3.6]). Equivalently, we have $L_{d-1}(A_{d-1}(\mathbb{F}_q)) \leq \frac{d+1}{\eta_{d-1}(q+1)}$. Now $A_{d-1}(\mathbb{F}_q)$ shows up as the vertex links of d-dimensional Ramanujan complexes[2] and, hence, for large enough q, we can use Proposition 9.9 with the lower bound on $L_{d-1}(A_{d-1}(\mathbb{F}_q))$ to deduce lower bounds on the cofilling constants of Ramanujan complexes as well.[3]

9.4.1 Proof of Proposition 9.8

The idea of the proof of Proposition 9.8 is very simple: Given a minimal k-cochain $c \in C^k(X_v;\mathbb{F}_2)$ with $\|\delta c\| = \eta_k(X_v)\|c\|$, the lift $\tilde{c} = I^v c \in C^{k+1}(X;\mathbb{F}_2)$, given by $\tilde{c}(\sigma) = 0$ if $v \notin \sigma$ and $\tilde{c}(\sigma) = c(\sigma \setminus \{v\})$ if $v \in \sigma$, seems a natural candidate for an upper bound on $\eta_{k+1}(X)$ in terms of $\eta_k(X_v)$. Indeed, using basic properties of the Garland weights one readily computes that

$$\|\delta\tilde{c}\| = \|I^v \delta_{X_v} c\| = (k+3)w(v)\|\delta_{X_v} c\|_v = (k+3)w(v)\eta_k(X_v)\|c\|_v = \frac{k+3}{k+2}\eta_k(X_v)\|\tilde{c}\|.$$

[2] It is not important here what these complexes are exactly. Let us just mention that they are remarkable families of explicitly constructed simplicial complexes of bounded degree generalizing the construction of Ramanujan graphs due to Lubotzky, Philipps and Sarnak [92] to higher dimensions [99, 100]. In particular, they have – in a very precise sense – optimal spectral expansion properties. Furthermore, they gave rise to infinite families of simplicial complexes of bounded degree with the topological overlap property [72, 40] and played a guiding role in [31]. They are essentially the only known family of simplicial complexes of bounded degree exhibiting such strong expansion properties and are thus extremely relevant for applications.

[3] Since d-dimensional Ramanujan complexes do not necessarily have vanishing $(d-1)$th cohomology, it is important to work with cofilling constants and Proposition 9.9 rather than coboundary expansion constants and Proposition 9.8 here.

Here and throughout the proof of Proposition 9.8 we write $\|\cdot\|$ for the Garland weighted Hamming norm on X and $\|\cdot\|_\sigma$ for the Garland weighted Hamming norm on the link X_σ of X at σ for some $\sigma \in X$.

Note that the computation above would finish the proof, if we could show that the lift $\tilde{c} = I^v c$ is minimal in X. This is where the assumption on $\eta_k(X)$ enters:

Lemma 9.10. *Let X be a d-dimensional simplicial complex. Let $v \in X(0)$. Let $0 \leq k \leq d-2$ and assume that $\eta_k(X) \geq 2(k+2)\rho_v^{(k)}$, where $\rho^{(k)} = \max_{\sigma \in X_v(k)} w_\sigma(v)$. Let $c \in C^k(X_v;\mathbb{F}_2)$. Then $I^v c \in C^{k+1}(X;\mathbb{F}_2)$ is minimal.*

For the proof of this lemma some notation will be useful: For $v \in X(0)$ we write $X \setminus v$ for the simplicial complex obtained from X by removing all simplices containing v. We endow $X \setminus v$ with the weights obtained by restricting the Garland weights of X to $X \setminus v$ and $\|\cdot\|$ for the induced weighted Hamming norm. We write $i_{X_v} : X_v \to X$ for the inclusion map.

In preparation for the proof of Lemma 9.10 we need the following two claims:

Claim 9.11. *Given $c \in C^k(X \setminus v;\mathbb{F}_2)$ minimal, we have*

$$\|\delta c\| \geq \left(\eta_k(X) - (k+2)\rho_v^{(k)}\right)\|c\|$$

where $\rho_v^{(k)} = \max_{\sigma \in X_v(k)} w_\sigma(v)$

Claim 9.12. *Let $b \in B^k(X;\mathbb{F}_2)$. Then there is $a \in C^{k-1}(X;\mathbb{F}_2)$ with $\delta a = b$ and $\operatorname{supp}(a) \subseteq X \setminus v$.*

Proof of Lemma 9.10 assuming Claim 9.11 and Claim 9.12. Let $c \in C^k(X_v;\mathbb{F}_2)$ be minimal. Let $\tilde{c} = I^v c \in C^{k+1}(X;\mathbb{F}_2)$. We would like to show that $\|\tilde{c} + \delta a\| \geq \|\tilde{c}\|$ for all $a \in C^k(X;\mathbb{F}_2)$. By Claim 9.12 we can assume that $\operatorname{supp}(a) \subseteq X \setminus v$. Write a' for the restriction of a to $X \setminus v$. Let $s \in C^{k-1}(X \setminus v;\mathbb{F}_2)$ such that $a' + \delta s$ is minimal. We compute using Claim 9.11, the triangle inequality, that $\|I^v u\| = w(v)(k+2)\|u\|$ for any $u \in C^k(X_v;\mathbb{F}_2)$ and minimality of c:

$$\begin{aligned}
\|\tilde{c} + \delta a\| &= \|\delta_{X\setminus v} a'\| + \|I^v(c + i^*_{X_v} a)\| \\
&\geq \left(\eta_k(X) - (k+2)\rho_v^{(k)}\right)\|a' + \delta s\| + \|I^v(c + i^*_{X_v} a)\| \\
&= \left(\eta_k(X) - (k+2)\rho_v^{(k)}\right)\|a' + \delta s\| + w(v)(k+2)\|c + i^*_{X_v} a\|_v \\
&\geq \left(\eta_k(X) - (k+2)\rho_v^{(k)}\right)\|a' + \delta s\| \\
&\quad + w(v)(k+2)\left(\|c + i^*_{X_v}\delta s\|_v - \|i^*_{X_v}(a + \delta s)\|_v\right) \\
&\geq \left(\eta_k(X) - (k+2)\rho_v^{(k)}\right)\|a' + \delta s\| + w(v)(k+2)\left(\|c\|_v - \|i^*_{X_v}(a + \delta s)\|_v\right) \\
&= \|\tilde{c}\| + \left(\eta_k(X) - (k+2)\rho_v^{(k)}\right)\|a' + \delta s\| - \|I^v(i^*_{X_v}(a + \delta s))\|.
\end{aligned}$$

It remains to give an upper bound on $\|I^v(i^*_{X_v}(a + \delta s))\|$. For this we simply compute

$$\begin{aligned}
\|I^v(i^*_{X_v}(a + \delta s))\| &= \sum_{\sigma \in X_v(k), a(\sigma)+\delta s(\sigma)=1} w(\sigma \cup v) \\
&= (k+2) \sum_{\sigma \in X_v(k), a(\sigma)+\delta s(\sigma)=1} w_\sigma(v) w(\sigma) \\
&\leq (k+2)\rho_v^{(k)} \sum_{\sigma \in X_v(k), a(\sigma)+\delta s(\sigma)=1} w(\sigma) \\
&\leq (k+2)\rho_v^{(k)}\|a' + \delta s\|.
\end{aligned}$$

Plugging this into above estimates we conclude

$$\|\tilde{c}+\delta a\| \geq \|\tilde{c}\| + \left(\eta_k(X) - 2(k+2)\rho_v^{(k)}\right)\|a'+\delta s\| \geq \|\tilde{c}\|,$$

since we assume that $\eta_k(X) \geq 2(k+2)\rho_v^{(k)}$. □

It remains to prove the two claims.

Proof of Claim 9.11. Let $\Delta := \{\sigma \in X(k+1) : \sigma \notin X \setminus v\}$. Write $\bar{c}$ for the extension of c by 0 to X. By Lemma 3.6 $\bar{c}$ is minimal in X. Using expansion of X we get

$$\|\delta_{X\setminus v} c\| = \|\delta_X \bar{c}\| - \|(\delta_X \bar{c})_{|\Delta}\| \geq \eta_k(X)\|c\| - \|(\delta_X \bar{c})_{|\Delta}\|.$$

Now, observe that

$$\begin{aligned}\|(\delta_X \bar{c})_{|\Delta}\| &= \sum_{\sigma\in X_v(k), c(\sigma)=1} w(\sigma\cup v)\\ &= (k+2)\sum_{\sigma\in X_v(k), c(\sigma)=1} w_\sigma(v)w(\sigma)\\ &\leq (k+2)\rho_v^{(k)}\sum_{\sigma\in X_v(k), c(\sigma)=1} w(\sigma)\\ &\leq (k+2)\rho_v^{(k)}\|c\|.\end{aligned}$$

Plugging this into the previous estimate, we get

$$\|\delta_{X\setminus v} c\| \geq \left(\eta_k(X) - (k+2)\rho_v^{(k)}\right)\|c\|,$$

as desired. □

Proof of Claim 9.12. Given $a \in C^k(X;\mathbb{F}_2)$ we simply note that the support of $a + \delta_X \overline{I_v a}$ is contained in $X \setminus v$. □

9.4.2 Proof of Proposition 9.9

The proof of Proposition 9.9 is somewhat simpler than the proof of Proposition 9.8.

Proof of Proposition 9.9. Let $b \in B^k(X_v;\mathbb{F}_2)$ with cofilling $a \in C^{k-1}(X_v;\mathbb{F}_2)$ such that

$$|a| = \min_{z\in Z^{k-1}(X_v;\mathbb{F}_2)} |a+z| = L_k(X_v)|b|.$$

Let $\tilde{b} = I^v b = \delta I^v a$ be the lift of a to X. We claim that $|I^v a| = \min_{z\in Z^k(X;\mathbb{F}_2)} |I^v a + z|$. This will finish the proof since then

$$L_k(X_v)|b| = |a| = |I^v a| \leq L_{k+1}(X)|\tilde{b}| = L_{k+1}(X)|b|.$$

We finish the proof by showing that $|I^v a| = \min_{z\in Z^k(X;\mathbb{F}_2)} |I^v a + z|$. To this end, let $z \in Z^k(X;\mathbb{F}_2)$ and note that if $\sigma \in \operatorname{supp}(I^v a + z)$ then either $v \notin \sigma$ and $\sigma \in \operatorname{supp}(z_{|X\setminus v})$ or $v \in \sigma$ and $\sigma \setminus \{v\} \in \operatorname{supp}(I_v(I^v a + z))$.

Note that $\delta_{X_v} I_v z = z_{|X_v}$. Indeed, we compute for $\sigma \in X_v(k)$ that

$$0 = \delta_X z(\sigma \sqcup v) = z(\sigma) + \sum_{u \in \sigma} z(\sigma \setminus \{u\} \sqcup \{v\}) = z(\sigma) + \sum_{u \in \sigma} I_v z(\sigma \setminus \{u\}) = z(\sigma) + \delta I_v z(\sigma).$$

Now let $\tilde{z} \in Z^{k-1}(X_v; \mathbb{F}_2)$ such that $|I_v z + \tilde{z}| = \min_{z' \in Z^{k-1}(X_v;\mathbb{F}_2)} |I_v z + z'|$.

Using these observations we estimate

$$\begin{aligned}
|I^v a + z| &= |z_{|X \setminus v}| + |I_v(I^v a + z)| \\
&\geq |z_{|X_v}| + |a + I_v z| \\
&= |\delta_{X_v} I_v z| + |a + I_v z| \\
&\geq \frac{1}{L_k(X_v)} |I_v z + \tilde{z}| + |a + \tilde{z}| - |I_v z + \tilde{z}| \\
&\geq \left(\frac{1}{L_k(X_v)} - 1 \right) |I_v z + \tilde{z}| + |a| \\
&\geq |a| \\
&= |I^v a|,
\end{aligned}$$

where we used the assumption $L_k(X_v) \leq 1$ for the last inequality. □

Chapter 10

Conclusion

In this dissertation we presented an equivariant version of Gromov's Topological Overlap Theorem as a general tool for studying quantitative non-embeddability problems through the lens of HDXs and gave various applications (Chapter 5). We believe that the results presented here are only the tip of an iceberg and that there are many further geometric and topological applications that could come out of a symbiosis of HDXs with the configuration space/test map framework. For this, it would be desirable to generalize Theorem 4.1 by, e.g., weakening the strong assumption of vanishing cohomology or extending it to different, not necessarily free group actions. For instance, a quantitative version of the following theorem due to Volovikov would be interesting as it allows to give a proof of the topological Tverberg theorem as well as a generalization of Theorem 5.12 for the prime power case (see [137] and [60, Theorem 2], respectively).

Proposition (Lemma in [137]). *Let p be prime. Let $G = (\mathbb{Z}/p)^n$ be the product of n copies of the cyclic group $\mathbb{Z}/p$. Assume G acts on the spaces X and Y without fixed points. If Y is a k-dimensional cohomology sphere over $\mathbb{Z}/p$ and $\tilde{H}^j(X;\mathbb{Z}/p) = 0$ for all $0 \leq i \leq k-1$ then there is no G-equivariant map from X to Y.*

There are various other topological results which have been used within the configuration space/test map framework. It would be interesting to attempt to prove quantitative versions of them which might even lead to new notions of HDXs. In particular, it would be nice to develop the theory well enough to give an affirmative answer to the question whether $\mathrm{ipcr}(X) \geq c|X(d)|^2$ for some constant $c > 0$ where X is a d-dimensional Ramanujan complex or, more generally, a compact quotient of an affine Bruhat–Tits building and in particular, show that these complexes do not embed into $\mathbb{R}^{2d}$. This question was explicitly ask in [54, p. 447] and [95, Section 3].

But even Theorem 4.1 itself might have further applications if one could show sufficiently good expansion properties of certain simplicial complexes. In particular, it could lead to first lower bounds on the number of Tverberg partitions for colorful Tverberg-type problems which do not follow from the so-called *constraint method* [14] and, hence, for which Theorem 5.12 is not available to imply quantitative bounds in a blackbox fashion. One such result in this direction is:

Theorem (Theorem 3.1 in [67], Theorem 2.1 in [68]). *Let p be prime. Let $r = p^s$ for some $s \in \mathbb{Z}_{>0}$. Let $d \geq 1$ be a positive integer. Let $N \geq (r-1)(d+2)$ and $rk + s \geq (r-1)d$ for integers $k \geq 0$ and $0 \leq s < r$. Let σ^N be the N-dimensional simplex on $N+1$ vertices.*

Then for every continuous map $f\colon |\sigma^N| \to \mathbb{R}^d$ there are r pairwise disjoint faces $\sigma_1, \dots, \sigma_r$ of σ^N such that $\bigcap_{i=1}^r f(\sigma_i) \neq \emptyset$ and such that $\dim \sigma_i \leq k+1$ for $1 \leq i \leq s$ and $\dim \sigma_i \leq k$ for $s < i \leq r$.

The proof of this theorem shows vanishing cohomology up to the relevant dimension (by exhibiting a discrete Morse function without critical simplices of small dimension) for a suitable configuration space associated with the problem and applies Volovikov's lemma above. Thus, at least for the prime case $r = p$, Theorem 4.1 would give a quantitative lower bound on the number of r-tuples as in the conclusion of the theorem, provided that we could show good enough expansion properties for the configuration space in question. Unfortunately, the lower bounds on the expansion constants we were able to show, do not lead to any non-trivial lower bound.

Furthermore, it would be interesting to know whether $\zeta_2(\Lambda^3_{m,m,n,n}) \geq 1$ holds for all $m, n \in \mathbb{Z}_{>0}$. As discussed in Section 5.5, this would imply an asymptotic version Zarankiewicz' conjecture. More generally, we conjecture that

Conjecture 10.1. *Let $d, n_0, \dots, n_d \in \mathbb{Z}_{>0}$, $d \geq 2$. Then*

(i) $\eta_{d-1}(\Lambda^d_{n_0,\dots,n_d}) \geq \frac{d+1}{2^d}$.

(ii) $\zeta_{d-1}(\Lambda^d_{n_0,\dots,n_d}) \geq 1$.

For $d = 2$, we conjecture that the upper bound on $\eta_1(\Lambda^2_n)$ given in Proposition 7.9 is the true value for all $n \in \mathbb{Z}_{>0}$.

It is somewhat surprising that the exact value of $\eta_{d-1}(X)$ remains unknown even for the most basic families of d-dimensional simplicial complexes such as d-dimensional complete $(d+1)$-partite complexes Λ^d_n. But even for the complete complex K^d_n the precise value of $\eta_{d-1}(K^d_n)$ is not known for all n. It is known though that $\eta_{d-1}(K^d_n) \geq \frac{n}{n-d}$ for all n with equality if $d+1$ divides n (see for instance [109, Section 2]). From this it is not too hard to deduce that $\eta_{d-1}(K^d_n) \leq 1 + o(1)$ as $n \to +\infty$. Furthermore, for $d = 2$ it was shown in [84, Theorem 4.2] that $\eta_1(K^2_n) = \frac{n}{n-2}$ if n is not a power of 2. The situation is even less understood if one asks for a more fine-grained understanding of $\eta_{d-1}(K^d_n)$ in terms of the *cofilling profile*

$$\eta_{d-1}(\alpha) := \liminf_{n \to +\infty} \min\{\|\delta c\| : c \in C^{d-1}(K^d_n; \mathbb{F}_2), \|[c]\| \geq \alpha\}$$

for $\alpha \in (0, 1/2]$ [108, 86]. This is relevant to the point selection problem mentioned in the introduction.

We think that there are extremly interesting combinatorial questions surrounding the problem of finding the exact value of $\eta_k(X)$. In particular, getting a better understanding of the structure of minimal cochains seems crucial to make progress on any of these problems. In view of the fact that checking minimality of cochains is NP-hard even for $c \in C^1(K_n; \mathbb{F}_2)$ this is likely to be a difficult task.

One might wonder whether there are tools in (extremal) combinatorics already available for tackling such problems. For instance, for showing $\eta_1(\Lambda^2_n) \geq 3/4$ we know by Corollary 8.10 that it suffices to consider minimal cochains $c \in C^1(\Lambda^2_n; \mathbb{F}_2)$ with $13/124 < \|c\| < 1/3$. In particular, we can work in the setting of large dense graphs where Szemerédi's regularity lemma (usually in combination with some counting lemma) has proven to be a powerful tool (see, for instance, Szemerédi's original paper [133], the two surveys [125, 81] or Chapter 9 in [91]). Is it possible to show that $\eta_1(\Lambda_n^2) \geq 3/4$ or at least an improved lower bound using some sort of regularity/counting lemma argument?

Furthermore, we would like to repeat the quest for a better understanding of the expansion constants $\eta_k(X * Y)$ of the join of two simplicial complexes X and Y, especially in the case $X = Y$. In view of Proposition 6.1, solving this problem might be quite subtle but nevertheless we would like to ask whether, given a d-dimensional simplicial complex X, it is possible to bound the coboundary expansion constants $\eta_k(X^{*2})$, $0 \leq k \leq 2d$, in terms of the expansion constants $\eta_j(X)$, $0 \leq j \leq d-1$? A positive answer even for $d = 1$ and $X = G$ a bounded degree expander graph on n vertices would be interesting since it would give that $\mathrm{ipcr}(G) = \Omega(n^2)$.

We have seen that $\zeta_{d-1}(\Lambda_n^d) > \eta_{d-1}(\Lambda_n^d)$ for all $d \geq 2$ and sufficiently large n. Thus, at least for the iterated join of a discrete set of n points coboundary expansion behaves better with respect to integer coefficients and ℓ_2^2-norm than with respect to $\mathbb{F}_2$-coefficients and Garland weighted Hamming norm. It would be interesting to know whether this is a more general phenomenon meaning whether under taking joins $J = X * Y$ the constants $\zeta_k(J)$ behave better than $\eta_k(J)$. On the one hand, additional tools such as eigenvalues and eigenspaces of Laplacians and some discrete lattices naturally show up in the study of expansion constants with respect to integer coefficients and ℓ_2^2-norm. On the other hand, we cannot expect a Cheeger inequality in dimension at least 2 since even qualitatively vanishing cohomology with respect to $\mathbb{R}$-coefficients does not imply vanishing integer cohomology.

Bibliography

[1] Dorna Abdolazimi, Kuikui Liu, and Shayan Oveis Gharan. A Matrix Trickle-Down Theorem on Simplicial Complexes and Applications to Sampling Colorings. In *62nd Annual IEEE Symposium on Foundations of Computer Science–FOCS 2021*, pages 161–172. IEEE Computer Soc., Los Alamitos, CA, 2022. doi:10.1109/FOCS52979.2021.00024.

[2] Peter Abramenko and Kenneth S. Brown. *Buildings. Theory and applications.*, volume 248 of *Graduate Texts in Mathematics*. Springer, New York, 2008. doi:10.1007/978-0-387-78835-7, MR:2439729, Zbl:1214.20033.

[3] Vedat Levi Alev, Fernando Granha Jeronimo, Dylan Quintana, Shashank Srivastava, and Madhur Tulsiani. List decoding of direct sum codes. In *Proceedings of the 2020 ACM-SIAM Symposium on Discrete Algorithms*, pages 1412–1425. SIAM, Philadelphia, PA, 2020. MR:4141267.

[4] Vedat Levi Alev and Lap Chi Lau. Improved analysis of higher order random walks and applications. In *STOC '20—Proceedings of the 52nd Annual ACM SIGACT Symposium on Theory of Computing*, pages 1198–1211. ACM, New York, [2020] ©2020. doi:10.1145/3357713.3384317, MR:4141833.

[5] Yeganeh Alimohammadi, Nima Anari, Kirankumar Shiragur, and Thuy-Duong Vuong. Fractionally Log-Concave and Sector-Stable Polynomials: Counting Planar Matchings and More. In *STOC'21—Proceedings of the 53rd Annual ACM SIGACT Symposium on Theory of Computing*, page 433–446. ACM, New York, 2021. URL: https://doi.org/10.1145/3406325.3451123, doi:10.1145/3406325.3451123.

[6] N. Alon and V. D. Milman. λ_1, isoperimetric inequalities for graphs, and superconcentrators. *J. Comb. Theory Ser. B*, 38(1):73–88, 1985. doi:10.1016/0095-8956(85)90092-9, MR:782626, Zbl:0549.05051.

[7] Nima Anari, Kuikui Liu, Shayan Oveis Gharan, and Cynthia Vinzant. Log-concave polynomials II: High-dimensional walks and an FPRAS for counting bases of a matroid. In *STOC'19—Proceedings of the 51st Annual ACM SIGACT Symposium on Theory of Computing*, pages 1–12. ACM, New York, 2019. doi:10.1145/3313276.3316385, MR:4003314, Zbl:1433.68606.

[8] Nima Anari, Kuikui Liu, and Shayan Oveis Gharan. Spectral independence in high-dimensional expanders and applications to the hardcore model. In *2020 IEEE 61st Annual Symposium on Foundations of Computer Science*, pages 1319–1330. IEEE Computer Soc., Los Alamitos, CA, [2020] ©2020. doi:10.1109/FOCS46700.2020.00125, MR:4232133.

[9] Sanjeev Arora, Satish Rao, and Umesh Vazirani. Expander flows, geometric embeddings and graph partitioning. *J. ACM*, 56(2):Art. 5, 37, 2009. doi:10.1145/1502793.1502794, MR:2535878, Zbl:1325.68255.

[10] László Babai. Spectra of Cayley graphs. *J. Combin. Theory Ser. B*, 27(2):180–189, 1979. doi:10.1016/0095-8956(79)90079-0, MR:546860, Zbl:0338.05110.

[11] I. Bárány, S. B. Shlosman, and A. Szűcs. On a topological generalization of a theorem of Tverberg. *J. Lond. Math. Soc., II. Ser.*, 23(1):158–164, 1981. doi:10.1112/jlms/s2-23.1.158, MR:602247, Zbl:0453.55003.

[12] Imre Bárány. A generalization of Carathéodory's theorem. *Discrete Math.*, 40(2-3):141–152, 1982. doi:10.1016/0012-365X(82)90115-7, MR:676720, Zbl:0492.52005.

[13] Lowell Beineke and Robin Wilson. The early history of the brick factory problem. *Math. Intell.*, 32(2):41–48, 2010. doi:10.1007/s00283-009-9120-4, MR:2657999, Zbl:1226.05093.

[14] Pavle V. M. Blagojević, Florian Frick, and Günter M. Ziegler. Tverberg plus constraints. *Bull. Lond. Math. Soc.*, 46(5):953–967, 2014. doi:10.1112/blms/bdu049, MR:3262197, Zbl:1305.52021.

[15] Pavle V. M. Blagojević and Günter M. Ziegler. Beyond the Borsuk-Ulam theorem: the topological Tverberg story. In *A journey through discrete mathematics. A tribute to Jiří Matoušek*, pages 273–341. Springer, Cham, 2017. MR:3726602, Zbl:1470.52007.

[16] Antonio Blanca, Pietro Caputo, Zongchen Chen, Daniel Parisi, Daniel Štefankovič, and Eric Vigoda. On mixing of markov chains: Coupling, spectral independence, and entropy factorization*. In *Proceedings of the 2022 ACM-SIAM Symposium on Discrete Algorithms (SODA)*, pages 3670–3692. [Society for Industrial and Applied Mathematics (SIAM)], Philadelphia, PA, 2022.

[17] Omer Bobrowski and Dmitri Krioukov. Random simplicial complexes: Models and phenomena. 2021. 40 pages. arXiv:2105.12914.

[18] Endre Boros and Zoltán Füredi. The number of triangles covering the center of an n-set. *Geom. Dedicata*, 17(1):69–77, 1984. doi:10.1007/BF00181519, MR:771183, Zbl:0595.52002.

[19] Glen E. Bredon. *Introduction to compact transformation groups*. Pure and Applied Mathematics, Vol. 46. Academic Press, New York-London, 1972. MR:0413144, Zbl:0246.57017.

[20] Daniel Brosch and Sven Polak. New lower bounds on crossing numbers of $k_{m,n}$ from permutation modules and semidefnite programming. 2022. 15 pages, 3 figures, 3 tables. arXiv:2206.02755.

[21] Andries E. Brouwer, Arjeh M. Cohen, and Arnold Neumaier. *Distance-regular graphs*, volume 18 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1989. doi:10.1007/978-3-642-74341-2, MR:1002568, Zbl:0747.05073.

[22] Michael Chapman, Nati Linial, and Yuval Peled. Expander graphs—both local and global. *Combinatorica*, 40(4):473–509, 2020. doi:10.1007/s00493-019-4127-8, MR:4150879, Zbl:1474.05237.

[23] Jeff Cheeger. A lower bound for the smallest eigenvalue of the Laplacian. In *Problems in analysis (Papers dedicated to Salomon Bochner, 1969)*, pages 195–199. 1970. MR:0402831, Zbl:0212.44903.

[24] Zongchen Chen, Andreas Galanis, Daniel Štefankovič, and Eric Vigoda. Rapid mixing for colorings via spectral independence. In *Proceedings of the 2021 ACM-SIAM Symposium on Discrete Algorithms (SODA)*, pages 1548–1557. [Society for Industrial and Applied Mathematics (SIAM)], Philadelphia, PA, 2021. doi:10.1137/1.9781611976465.94, MR:4262527.

[25] Zongchen Chen, Kuikui Liu, and Eric Vigoda. Optimal mixing of Glauber dynamics: Entropy factorization via high-dimensional expansion. In *STOC'21—Proceedings of the 53rd Annual ACM SIGACT Symposium on Theory of Computing*, pages 1537–1550, 2021. doi:10.1145/3406325.3451035.

[26] Fan Chung. Four proofs for the Cheeger inequality and graph partition algorithms. In *Fourth International Congress of Chinese Mathematicians. Proceedings of the ICCM '07, Hangzhou, China, December 17–22, 2007*.

[27] Marcel K. de Carli Silva, Fernando Mário de Oliveira Filho, and Cristiane Maria Sato. Flag algebras: a first glance. *Nieuw Arch. Wiskd. (5)*, 17(3):193–199, 2016. MR:3643687, Zbl:1364.05090.

[28] E. de Klerk, J. Maharry, D. V. Pasechnik, R. B. Richter, and G. Salazar. Improved bounds for the crossing numbers of $k_{m,n}$ and k_n. *SIAM J. Discrete Math.*, 20(1):189–202, 2006. doi:10.1137/S0895480104442741, MR:2257255, Zbl:1111.05029.

[29] Etienne de Klerk, Dmitrii V. Pasechnik, and Alexander Schrijver. Reduction of symmetric semidefinite programs using the regular $*$-representation. *Math. Program.*, 109(2-3, Ser. B):613–624, 2007. doi:10.1007/s10107-006-0039-7, MR:2296566, Zbl:1200.90136.

[30] Yotam Dikstein, Irit Dinur, Yuval Filmus, and Prahladh Harsha. Boolean function analysis on high-dimensional expanders. In *Approximation, randomization, and combinatorial optimization. Algorithms and techniques*, volume 116 of *LIPIcs. Leibniz Int. Proc. Inform.*, pages Art. No. 38, 20. Schloss Dagstuhl. Leibniz-Zent. Inform., Wadern, 2018. MR:3857276.

[31] Irit Dinur, Shai Evra, Ron Livne, Alexander Lubotzky, and Shahar Mozes. Locally Testable Codes with constant rate, distance, and locality. 2021. 37 pages. arXiv:2111.04808.

[32] Irit Dinur, Yuval Filmus, Prahladh Harsha, and Madhur Tulsiani. Explicit SoS lower bounds from high-dimensional expanders. In *12th Innovations in Theoretical Computer Science Conference*, volume 185 of *LIPIcs. Leibniz Int. Proc. Inform.*, pages Art. No. 38, 16. Schloss Dagstuhl. Leibniz-Zent. Inform., Wadern, 2021. MR:4214282.

[33] Irit Dinur and Tali Kaufman. High dimensional expanders imply agreement expanders. In *58th Annual IEEE Symposium on Foundations of Computer Science—FOCS 2017*, pages 974–985. IEEE Computer Soc., Los Alamitos, CA, 2017. doi:10.1109/FOCS.2017.94, MR:3734297.

[34] Jozef Dodziuk. Difference equations, isoperimetric inequality and transience of certain random walks. *Trans. Am. Math. Soc.*, 284(2):787–794, 1984. doi:10.2307/1999107, MR:743744, Zbl:0512.39001.

[35] Dominic Dotterrer and Matthew Kahle. Coboundary expanders. *J. Topol. Anal.*, 4(4):499–514, 2012. doi:10.1142/S1793525312500197, MR:3021774, Zbl:1259.05151.

[36] Dominic Dotterrer, Tali Kaufman, and Uli Wagner. On expansion and topological overlap. *Geom. Dedicata*, 195:307–317, 2018. doi:10.1007/s10711-017-0291-4, MR:3820508, Zbl:1393.05315.

[37] Art M. Duval, Caroline J. Klivans, and Jeremy L. Martin. Simplicial matrix-tree theorems. *Trans. Am. Math. Soc.*, 361(11):6073–6114, 2009. doi:10.1090/S0002-9947-09-04898-3, MR:2529925, Zbl:1207.05227.

[38] Beno Eckmann. Harmonische Funktionen und Randwertaufgaben in einem Komplex. *Comment. Math. Helv.*, 17:240–255, 1945. doi:10.1007/BF02566245, MR:13318, Zbl:0061.41106.

[39] Paul Erdős. On some problems in graph theory, combinatorial analysis and combinatorial number theory. Graph theory and combinatorics, Proc. Conf. Hon. P. Erdős, Cambridge 1983, 1-17 (1984)., 1984. Zbl:0546.05002.

[40] Shai Evra and Tali Kaufman. Bounded degree cosystolic expanders of every dimension. In *STOC'16—Proceedings of the 48th Annual ACM SIGACT Symposium on Theory of Computing*, pages 36–48. ACM, New York, 2016. doi:10.1145/2897518.2897543, MR:3536553, Zbl:1376.05095.

[41] Herbert Federer. *Geometric measure theory*, volume 153. Springer-Verlag New York, Inc., New York, 1969. Zbl:0176.00801.

[42] Herbert Federer and Wendell H. Fleming. Normal and integral currents. *Ann. of Math. (2)*, 72:458–520, 1960. doi:10.2307/1970227, MR:123260, Zbl:0187.31301.

[43] Weiming Feng, Heng Guo, Yitong Yin, and Chihao Zhang. Rapid mixing from spectral independence beyond the Boolean Domain. In *Proceedings of the 2021 ACM-SIAM Symposium on Discrete Algorithms (SODA)*, pages 1558–1577. [Society for Industrial and Applied Mathematics (SIAM)], Philadelphia, PA, 2021. doi:10.1137/1.9781611976465.95, MR:4262528.

[44] Miroslav Fiedler. Algebraic connectivity of graphs. *Czechoslovak Math. J.*, 23(98):298–305, 1973. MR:318007, Zbl:0265.05119.

[45] Robin Forman. Morse theory for cell complexes. *Adv. Math.*, 134(1):90–145, 1998. doi:10.1006/aima.1997.1650, MR:1612391, Zbl:0896.57023.

[46] Robin Forman. A user's guide to discrete Morse theory. *Sém. Lothar. Combin.*, 48:Art. B48c, 35, 2002. MR:1939695, Zbl:1048.57015.

[47] Michael H. Freedman, Vyacheslav S. Krushkal, and Peter Teichner. Van Kampen's embedding obstruction is incomplete for 2-complexes in $\mathbb{R}^4$. *Math. Res. Lett.*, 1(2):167–176, 1994. doi:10.4310/MRL.1994.v1.n2.a4, MR:1266755, Zbl:0847.57005.

[48] Florian Frick. Counterexamples to the topological Tverberg conjecture. *Oberwolfach Reports*, 12:318–322, 2015.

[49] J. Friedman. Computing Betti numbers via combinatorial Laplacians. *Algorithmica*, 21(4):331–346, 1998. doi:10.1007/PL00009218, MR:1622290, Zbl:0911.57021.

[50] Ofer Gabber and Zvi Galil. Explicit constructions of linear size superconcentrators. In *20th Annual Symposium on Foundations of Computer Science (San Juan, Puerto Rico, 1979)*, pages 364–370. IEEE, New York, 1979. MR:598118.

[51] Howard Garland. p-adic curvature and the cohomology of discrete subgroups of p-adic groups. *Ann. of Math. (2)*, 97:375–423, 1973. doi:10.2307/1970829, MR:320180, Zbl:0262.22010.

[52] Paul Garrett. *Buildings and classical groups*. Chapman & Hall, London, 1997. doi:10.1007/978-94-011-5340-9, MR:1449872, Zbl:0933.20019.

[53] Ellen Gethner, Leslie Hogben, Bernard Lidický, Florian Pfender, Amanda Ruiz, and Michael Young. On crossing numbers of complete tripartite and balanced complete multipartite graphs. *J. Graph Theory*, 84(4):552–565, 2017. doi:10.1002/jgt.22041, MR:3623394, Zbl:1359.05027.

[54] Mikhail Gromov. Singularities, expanders and topology of maps. Part 2: From combinatorics to topology via algebraic isoperimetry. *Geom. Funct. Anal.*, 20(2):416–526, 2010. doi:10.1007/s00039-010-0073-8, MR:2671284, Zbl:1251.05039.

[55] Andrzej Grzesik. On the maximum number of five-cycles in a triangle-free graph. *J. Combin. Theory Ser. B*, 102(5):1061–1066, 2012. doi:10.1016/j.jctb.2012.04.001, MR:2959390, Zbl:1252.05093.

[56] Anna Gundert and Uli Wagner. On eigenvalues of random complexes. *Israel J. Math.*, 216(2):545–582, 2016. doi:10.1007/s11856-016-1419-1, MR:3557457, Zbl:1353.05079.

[57] Larry Guth. The waist inequality in Gromov's work. *The Abel Prize*, 2012:181–195, 2008.

[58] Hamed Hatami, Jan Hladký, Daniel Kráľ, Serguei Norine, and Alexander Razborov. On the number of pentagons in triangle-free graphs. *J. Combin. Theory Ser. A*, 120(3):722–732, 2013. doi:10.1016/j.jcta.2012.12.008, MR:3007147, Zbl:1259.05087.

[59] Allen Hatcher. *Algebraic topology*. Cambridge University Press, Cambridge, 2002. MR:1867354, Zbl:1044.55001.

[60] Stephan Hell. On the number of Tverberg partitions in the prime power case. *European J. Combin.*, 28(1):347–355, 2007. doi:10.1016/j.ejc.2005.06.005, MR:2261824, Zbl:1121.05007.

[61] P. John Hilton and Urs Stammbach. *A course in homological algebra*, volume 4 of *Graduate Texts in Mathematics*. New York, NY: Springer, second edition, 1997. doi:10.1007/978-1-4419-8566-8, MR:1438546, Zbl:0863.18001.

[62] Wassily Hoeffding. Probability inequalities for sums of bounded random variables. *J. Amer. Statist. Assoc.*, 58:13–30, 1963. doi:10.2307/2282952, MR:144363, Zbl:0127.10602.

[63] Shlomo Hoory, Nathan Linial, and Avi Wigderson. Expander graphs and their applications. *Bull. Amer. Math. Soc. (N.S.)*, 43(4):439–561, 2006. doi:10.1090/S0273-0979-06-01126-8, MR:2247919, Zbl:1147.68608.

[64] Danijela Horak and Jürgen Jost. Spectra of combinatorial Laplace operators on simplicial complexes. *Adv. Math.*, 244:303–336, 2013. doi:10.1016/j.aim.2013.05.007, MR:3077874, Zbl:1290.05103.

[65] Neil Strickland (https://mathoverflow.net/users/10366/neil strickland). Special cell decomposition for spheres with free $\mathbb{Z}/p\mathbb{Z}$-action by orthogonal transformations? MathOverflow. (version: 2022-05-26). URL: https://mathoverflow.net/q/423355.

[66] Vít Jelínek, Eva Jelínková, and Jan Kratochvíl. On the hardness of switching to a small number of edges. In *Computing and combinatorics. 22nd international conference, COCOON 2016, Ho Chi Minh City, Vietnam, August 2–4, 2016. Proceedings*, volume 9797 of *Lecture Notes in Comput. Sci.*, pages 159–170. Springer, [Cham], 2016. doi:10.1007/978-3-319-42634-1_13, MR:3557674, Zbl:1477.68232.

[67] Duško Jojić, Gaiane Panina, and Rade Živaljević. A Tverberg type theorem for collectively unavoidable complexes. *Israel J. Math.*, 241(1):17–36, 2021. doi:10.1007/s11856-021-2087-3, MR:4242144, Zbl:1462.05363.

[68] Duško Jojić, Siniša T. Vrećica, and Rade T. Živaljević. Symmetric multiple chessboard complexes and a new theorem of Tverberg type. *J. Algebr. Comb.*, 46(1):15–31, 2017. doi:10.1007/s10801-017-0743-9, MR:3666410, Zbl:1368.05163.

[69] H. A. Jung. Eine Verallgemeinerung des n-fachen Zusammenhangs für Graphen. *Math. Ann.*, 187:95–103, 1970. doi:10.1007/BF01350174, MR:268067, Zbl:0184.27601.

[70] Matthew Kahle. Topology of random simplicial complexes: a survey. In *Algebraic topology: applications and new directions*, volume 620 of *Contemp. Math.*, pages 201–221. Amer. Math. Soc., Providence, RI, 2014. doi:10.1090/conm/620/12367, MR:3290093, Zbl:1333.05324.

[71] Volker Kaibel. On the expansion of graphs of 0/1-polytopes. In *The sharpest cut. The impact of Manfred Padberg and his work. Papers from the workshop in honor of Manfred Padberg's 60th birthday, Berlin, Germany, October 11–13, 2001.*, MPS/SIAM Ser. Optim., pages 199–216. SIAM, Philadelphia, PA, 2004. MR:2077562, Zbl:1079.52008.

[72] Tali Kaufman, David Kazhdan, and Alexander Lubotzky. Isoperimetric inequalities for Ramanujan complexes and topological expanders. *Geom. Funct. Anal.*, 26(1):250–287, 2016. doi:10.1007/s00039-016-0362-y, MR:3494490, Zbl:1339.05075.

[73] Tali Kaufman and Alexander Lubotzky. High dimensional expanders and property testing. In *ITCS'14—Proceedings of the 2014 Conference on Innovations in Theoretical Computer Science*, pages 501–506. ACM, New York, 2014. doi:10.1145/2554797.2554842, MR:3359501, Zbl:1365.68462.

[74] Tali Kaufman and David Mass. High dimensional random walks and colorful expansion. In *8th Innovations in Theoretical Computer Science Conference*, volume 67 of *LIPIcs. Leibniz Int. Proc. Inform.*, pages Art. No. 4, 27. Schloss Dagstuhl. Leibniz-Zent. Inform., Wadern, 2017. doi:10.4230/LIPIcs.ITCS.2017.4, MR:3754928, Zbl:1402.05197.

[75] Tali Kaufman and Izhar Oppenheim. High order random walks: beyond spectral gap. *Combinatorica*, 40(2):245–281, 2020. doi:10.1007/s00493-019-3847-0, MR:4085988, Zbl:1463.05543.

[76] Tali Kaufman and Izhar Oppenheim. Coboundary and cosystolic expansion from strong symmetry. In *48th International Colloquium on Automata, Languages, and Programming*, volume 198 of *LIPIcs. Leibniz Int. Proc. Inform.*, pages Art. No. 84, 16. Schloss Dagstuhl. Leibniz-Zent. Inform., Wadern, 2021. MR:4288914.

[77] Subhash Khot. On the power of unique 2-prover 1-round games. In *Proceedings of the Thirty-Fourth Annual ACM Symposium on Theory of Computing*, pages 767–775. ACM, New York, 2002. doi:10.1145/509907.510017, MR:2121525, Zbl:1192.68367.

[78] Daniel J. Kleitman. The crossing number of $k_{5,n}$. *J. Combinatorial Theory*, 9:315–323, 1970. doi:10.1016/S0021-9800(70)80087-4, MR:280403, Zbl:0205.54401.

[79] Petr Kolman and Jiří Matoušek. Crossing number, pair-crossing number, and expansion. *J. Combin. Theory Ser. B*, 92(1):99–113, 2004. doi:10.1016/j.jctb.2003.09.002, MR:2078496, Zbl:1051.05038.

[80] A. N. Kolmogorov and Ya. M. Barzdin'. Über die Realisierung von Netzen im dreidimensionalen Raum. *Probl. Kibern.*, 19:261–268, 1967. Zbl:0183.01903.

[81] J. Komlós and M. Simonovits. Szemerédi's regularity lemma and its applications in graph theory. In *Combinatorics, Paul Erdős is eighty, Vol. 2 (Keszthely, 1993)*, volume 2 of *Bolyai Soc. Math. Stud.*, pages 295–352. János Bolyai Math. Soc., Budapest, 1996. MR:1395865, Zbl:0851.05065.

[82] W. Kook, V. Reiner, and D. Stanton. Combinatorial Laplacians of matroid complexes. *J. Amer. Math. Soc.*, 13(1):129–148, 2000. doi:10.1090/S0894-0347-99-00316-1, MR:1697094, Zbl:0940.05021.

[83] Emmanuel Kowalski. *An introduction to expander graphs*, volume 26 of *Cours Spécialisés [Specialized Courses]*. Société Mathématique de France, Paris, 2019. MR:3931316, Zbl:1421.05002.

[84] Dmitry N. Kozlov. The first Cheeger constant of a simplex. *Graphs Combin.*, 33(6):1543–1564, 2017. doi:10.1007/s00373-017-1853-9, MR:3735719, Zbl:1380.05204.

[85] Dmitry N. Kozlov and Roy Meshulam. Quantitative aspects of acyclicity. *Res. Math. Sci.*, 6(4):Paper No. 33, 32, 2019. doi:10.1007/s40687-019-0195-z, MR:4011565, Zbl:1437.55024.

[86] Daniel Kráľ, Lukáš Mach, and Jean-Sébastien Sereni. A new lower bound based on Gromov's method of selecting heavily covered points. *Discrete Comput. Geom.*, 48(2):487–498, 2012. doi:10.1007/s00454-012-9419-3, MR:2946458, Zbl:1262.05151.

[87] Bernard Lidický and Florian Pfender. Pentagons in triangle-free graphs. *European J. Combin.*, 74:85–89, 2018. doi:10.1016/j.ejc.2018.07.008, MR:3846228, Zbl:1394.05061.

[88] N. Linial, R. Meshulam, and M. Rosenthal. Sum complexes—a new family of hypertrees. *Discrete Comput. Geom.*, 44(3):622–636, 2010. doi:10.1007/s00454-010-9252-5, MR:2679058, Zbl:1213.05038.

[89] Nathan Linial and Roy Meshulam. Homological connectivity of random 2-complexes. *Combinatorica*, 26(4):475–487, 2006. doi:10.1007/s00493-006-0027-9, MR:2260850, Zbl:1121.55013.

[90] L. Lovász. Spectra of graphs with transitive groups. *Period. Math. Hungar.*, 6(2):191–195, 1975. doi:10.1007/BF02018821, MR:398886, Zbl:0395.05057.

[91] László Lovász. volume 60 of *Colloq. Publ., Am. Math. Soc.* Providence, RI: American Mathematical Society (AMS), 2012. doi:10.1090/coll/060, MR:3012035, Zbl:1292.05001.

[92] A. Lubotzky, R. Phillips, and P. Sarnak. Ramanujan graphs. *Combinatorica*, 8(3):261–277, 1988. doi:10.1007/BF02126799, MR:963118.

[93] Alexander Lubotzky. *Discrete groups, expanding graphs and invariant measures.* Modern Birkhäuser Classics. Birkhäuser Verlag, Basel, 2010. With an appendix by Jonathan D. Rogawski, Reprint of the 1994 edition. doi:10.1007/978-3-0346-0332-4, MR:2569682, Zbl:1183.22001.

[94] Alexander Lubotzky. Expander graphs in pure and applied mathematics. *Bull. Amer. Math. Soc. (N.S.)*, 49(1):113–162, 2012. doi:10.1090/S0273-0979-2011-01359-3, MR:2869010, Zbl:1232.05194.

[95] Alexander Lubotzky. High dimensional expanders. In *Proceedings of the International Congress of Mathematicians—Rio de Janeiro 2018. Vol. I. Plenary lectures*, pages 705–730. World Sci. Publ., Hackensack, NJ, 2018. MR:3966743, Zbl:1448.05125.

[96] Alexander Lubotzky, Zur Luria, and Ron Rosenthal. Random Steiner systems and bounded degree coboundary expanders of every dimension. *Discrete Comput. Geom.*, 62(4):813–831, 2019. doi:10.1007/s00454-018-9991-2, MR:4027617, Zbl:1426.05183.

[97] Alexander Lubotzky and Roy Meshulam. Random Latin squares and 2-dimensional expanders. *Adv. Math.*, 272:743–760, 2015. doi:10.1016/j.aim.2014.12.015, MR:3303247, Zbl:1306.05020.

[98] Alexander Lubotzky, Roy Meshulam, and Shahar Mozes. Expansion of building-like complexes. *Groups Geom. Dyn.*, 10(1):155–175, 2016. doi:10.4171/GGD/346, MR:3460334, Zbl:1405.55019.

[99] Alexander Lubotzky, Beth Samuels, and Uzi Vishne. Explicit constructions of Ramanujan complexes of type $\tilde{A}_d$. *European J. Combin.*, 26(6):965–993, 2005. doi:10.1016/j.ejc.2004.06.007, MR:2143204, Zbl:1135.05038.

[100] Alexander Lubotzky, Beth Samuels, and Uzi Vishne. Ramanujan complexes of type $\tilde{A}_d$. *Israel J. Math.*, 149:267–299, 2005. Probability in mathematics. doi:10.1007/BF02772543, MR:2191217, Zbl:1087.05036.

[101] Albert T. Lundell and Stephen Weingram. *The topology of CW complexes.* The University Series in Higher Mathematics. Van Nostrand Reinhold Co., New York, 1969. doi:10.1007/978-1-4684-6254-8, MR:3822092, Zbl:0207.21704.

[102] Isaac Mabillard and Uli Wagner. Eliminating Tverberg points, I. An analogue of the Whitney trick. In *Proceedings of the 30th annual symposium on computational geometry, SoCG '14, Kyoto, Japan, June 8–11, 2014*, pages 171–180. New York, NY: Association for Computing Machinery (ACM), 2014. doi:10.1145/2582112.2582134, MR:3382296, Zbl:1397.57042.

[103] Isaac Mabillard and Uli Wagner. Eliminating higher-multiplicity intersections, I. A Whitney trick for Tverberg-type problems. 2015. 46 pages, 14 figures. arXiv:1508.02349.

[104] W. Mantel. Vraagstuk XXVIII. *Wiskundige Opgaven*, 10:60–61, 1907.

[105] G. A. Margulis. Explicit constructions of expanders. *Problemy Peredači Informacii*, 9(4):71–80, 1973. MR:0484767, Zbl:0312.22011.

[106] G. A. Margulis. Explicit group-theoretic constructions of combinatorial schemes and their applications in the construction of expanders and concentrators. *Problemy Peredachi Informatsii*, 24(1):51–60, 1988. MR:939574, Zbl:0708.05030.

[107] Jiří Matoušek. *Using the Borsuk-Ulam theorem. Lectures on Topological methods in Combinatorics and Geometry.* Universitext. Springer-Verlag, Berlin, 2008. Written in cooperation with Anders Björner and Günter M. Ziegler, 2nd, corrected printing. doi:10.1007/978-3-540-76649-0, MR:1988723, Zbl:1234.05002.

[108] Jiří Matoušek and Uli Wagner. On Gromov's method of selecting heavily covered points. *Discrete Comput. Geom.*, 52(1):1–33, 2014. doi:10.1007/s00454-014-9584-7, MR:3231028, Zbl:1298.51027.

[109] Roy Meshulam and Nathan Wallach. Homological connectivity of random k-dimensional complexes. *Random Structures Algorithms*, 34(3):408–417, 2009. doi:10.1002/rsa.20238, MR:2504405, Zbl:1177.55011.

[110] James R. Munkres. *Elements of algebraic topology.* Addison-Wesley Publishing Company, Menlo Park, CA, 1984. MR:755006, Zbl:0673.55001.

[111] Nagi H. Nahas. On the crossing number of $k_{m,n}$. *Electron. J. Combin.*, 10:Note 8, 6, 2003. MR:2014536, Zbl:1023.05039.

[112] V. Nikiforov. The number of cliques in graphs of given order and size. *Trans. Amer. Math. Soc.*, 363(3):1599–1618, 2011. doi:10.1090/S0002-9947-2010-05189-X, MR:2737279, Zbl:1231.05129.

[113] Sergey Norin. Turan brickyard problem and flag algebras. Presentation at the BIRS Workshop on geometric and topological graph theory (13w5091), October 1, (2013)., 10 2013. https://www.birs.ca/events/2013/5-day-workshops/13w5091/videos/watch/201310011538-Norin.html - accessed May 10, 2022.

[114] Izhar Oppenheim. Local spectral expansion approach to high dimensional expanders Part I: Descent of spectral gaps. *Discrete Comput. Geom.*, 59(2):293–330, 2018. doi:10.1007/s00454-017-9948-x, MR:3755725, Zbl:1383.05312.

[115] Murad Özaydin. Equivariant maps for the symmetric group. Preprint, 17 pages, http://digital.library.wisc.edu/1793/63829, 1987.

[116] J. Pach, F. Shahrokhi, and M. Szegedy. Applications of the crossing number. *Algorithmica*, 16(1):111–117, 1996. doi:10.1007/s004539900039, MR:1394496, Zbl:0851.68088.

[117] János Pach and Géza Tóth. Which crossing number is it anyway? *J. Combin. Theory Ser. B*, 80(2):225–246, 2000. doi:10.1006/jctb.2000.1978, MR:1794693, Zbl:1023.05042.

[118] Pavel Panteleev and Gleb Kalachev. Asymptotically Good Quantum and Locally Testable Classical LDPC Codes. 2021. 51 pages. arXiv:2111.

[119] Mark S Pinsker. On the complexity of a concentrator. In *7th International Telegraffic Conference*, volume 4, pages 1–318, 1973.

[120] Prasad Raghavendra and David Steurer. Graph expansion and the unique games conjecture. In *STOC'10—Proceedings of the 2010 ACM International Symposium on Theory of Computing*, pages 755–764. ACM, New York, 2010. MR:2743325.

[121] Alexander A. Razborov. Flag algebras. *J. Symb. Log.*, 72(4):1239–1282, 2007. doi:10.2178/jsl/1203350785, MR:2371204, Zbl:1146.03013.

[122] Alexander A. Razborov. On the minimal density of triangles in graphs. *Combin. Probab. Comput.*, 17(4):603–618, 2008. doi:10.1017/S0963548308009085, MR:2433944, Zbl:1170.05036.

[123] Christian Reiher. The clique density theorem. *Ann. of Math. (2)*, 184(3):683–707, 2016. doi:10.4007/annals.2016.184.3.1, MR:3549620, Zbl:1348.05103.

[124] Omer Reingold, Salil Vadhan, and Avi Wigderson. Entropy waves, the zig-zag graph product, and new constant-degree expanders. *Ann. Math. (2)*, 155(1):157–187, 2002. doi:10.2307/3062153, Zbl:1008.05101.

[125] Vojtěch Rödl and Mathias Schacht. Regularity lemmas for graphs. In *Fete of combinatorics and computer science. Selected papers of the conference held in Keszthely, Hungary, August 11–15, 2008 dedicated to László Lovász on the occasion of his 60th birthday*, volume 20 of *Bolyai Soc. Math. Stud.*, pages 287–325. János Bolyai Math. Soc., Budapest, 2010. doi:10.1007/978-3-642-13580-4_11, MR:2798368, Zbl:1218.05147.

[126] Marcus Schaefer. *Crossing numbers of graphs.* Discrete Mathematics and its Applications (Boca Raton). CRC Press, Boca Raton, FL, 2018. doi:10.1201/9781315152394, MR:3751397, Zbl:1388.05005.

[127] Nicholas A. Scoville. *Discrete Morse theory*, volume 90 of *Stud. Math. Libr.* Providence, RI: American Mathematical Society (AMS), 2019. doi:10.1090/stml/090, MR:3970274, Zbl:1433.58002.

[128] Jean-Pierre Serre. *Linear representations of finite groups.* Graduate Texts in Mathematics, Vol. 42. Springer-Verlag, New York-Heidelberg, 1977. Translated from the second French edition by Leonard L. Scott. MR:0450380, Zbl:0355.20006.

[129] Gerard Sierksma. Convexity without linearity; the Dutch cheese problem. Mimeographed notes, University of Groningen, 1979.

[130] John Steenbergen, Caroline Klivans, and Sayan Mukherjee. A Cheeger-type inequality on simplicial complexes. *Adv. in Appl. Math.*, 56:56–77, 2014. doi:10.1016/j.aam.2014.01.002, MR:3194207, Zbl:1305.55010.

[131] N. E. Steenrod. *Cohomology operations. Lectures. Written and revised by D.B.A. Epstein.*, volume 50 of *Ann. Math. Stud.* Princeton University Press, Princeton, N.J., 1962. doi:10.1515/9781400881673, MR:0145525, Zbl:0102.38104.

[132] László A. Székely. A successful concept for measuring non-planarity of graphs: The crossing number. volume 276, pages 331–352. 2004. 6th International Conference on Graph Theory. doi:10.1016/S0012-365X(03)00317-0, MR:2046646, Zbl:1035.05034.

[133] Endre Szemerédi. Regular partitions of graphs. In *Problèmes combinatoires et théorie des graphes (Colloq. Internat. CNRS, Univ. Orsay, Orsay, 1976)*, volume 260 of *Colloq. Internat. CNRS*, pages 399–401. CNRS, Paris, 1978. MR:540024, Zbl:0413.05055.

[134] Martin Tancer and Kathrin Vorwerk. Non-embeddability of geometric lattices and buildings. *Discrete Comput. Geom.*, 51(4):779–801, 2014. doi:10.1007/s00454-014-9591-8, MR:3216663, Zbl:1415.51012.

[135] Jacques Tits. *Buildings of spherical type and finite BN-pairs*. Lecture Notes in Mathematics, Vol. 386. Springer-Verlag, Berlin-New York, 1974. MR:0470099, Zbl:0295.20047.

[136] Paul Turán. Eine Extremalaufgabe aus der Graphentheorie. *Mat. Fiz. Lapok*, 48:436–452, 1941. MR:18405, Zbl:0026.26903.

[137] A. Yu. Volovikov. On a topological generalization of the Tverberg theorem. *Math. Notes*, 59(3):324–326, 1996. doi:10.1007/BF02308547, MR:1399973, Zbl:0879.55004.

[138] Aleksandar Vučić and Rade T. Živaljević. Note on a conjecture of Sierksma. *Discrete Comput. Geom.*, 9(4):339–349, 1993. doi:10.1007/BF02189327, MR:1206796, Zbl:0784.52007.

[139] Rade T. Živaljević. Topological methods. In *Handbook of discrete and computational geometry*, CRC Press Ser. Discrete Math. Appl., pages 209–224. CRC, Boca Raton, FL, 1997. MR:1730167, Zbl:0955.52500.

[140] Uli Wagner and Pascal Wild. Coboundary expansion, equivariant overlap, and crossing numbers of simplicial complexes. submitted to the *Israel Journal of Mathematics*, March 2022.

[141] James W. Walker. A homology version of the Borsuk-Ulam theorem. *Amer. Math. Monthly*, 90(7):466–468, 1983. doi:10.2307/2975728, MR:711647, Zbl:0542.55002.