A Cautionary Tale: Burning the Medial Axis Is Unstable

Erin Chambers
Saint Louis University, MO, USA

Christopher Fillmore
IST Austria, Klosterneuburg, Austria

Elizabeth Stephenson
IST Austria, Klosterneuburg, Austria

Mathijs Wintraecken
IST Austria, Klosterneuburg, Austria

Abstract

The medial axis of a set consists of the points in the ambient space without a unique closest point on the original set. Since its introduction, the medial axis has been used extensively in many applications as a method of computing a topologically equivalent skeleton. Unfortunately, one limiting factor in the use of the medial axis of a smooth manifold is that it is not necessarily topologically stable under small perturbations of the manifold. To counter these instabilities various prunings of the medial axis have been proposed. Here, we examine one type of pruning, called burning. Because of the good experimental results, it was hoped that the burning method of simplifying the medial axis would be stable. In this work we show a simple example that dashes such hopes based on Bing's house with two rooms, demonstrating an isotopy of a shape where the medial axis goes from collapsible to non-collapsible.

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1 Introduction

The medial axis $\mathrm{ax}(\mathcal{S})$ of a closed set $\mathcal{S} \subset \mathbb{R}^d$ is the set of points in $\mathbb{R}^d$ for which the closest point in $\mathcal{S}$ is not unique. We note that although Federer [22] already studied the (complement of the) medial axis, the name was coined later by Blum [8]. The medial axis is used in many applications as a method of computing a topologically equivalent skeleton. The medial axis also has deep connections to singularity theory [2, 9, 17, 28, 29, 34, 35, 38].

Unfortunately, one limiting factor in the use of the medial axis is its (topological) instable under small perturbations [3]. Here small is understood to be small with respect to the Hausdorff distance. See Figure 1 for a standard example of such an instability. However the medial axis does capture the homotopy type [26, 36]. The radius function and medial axis together suffice to reconstruct the original set (under reasonable assumptions) [14, 15, 16, 18].

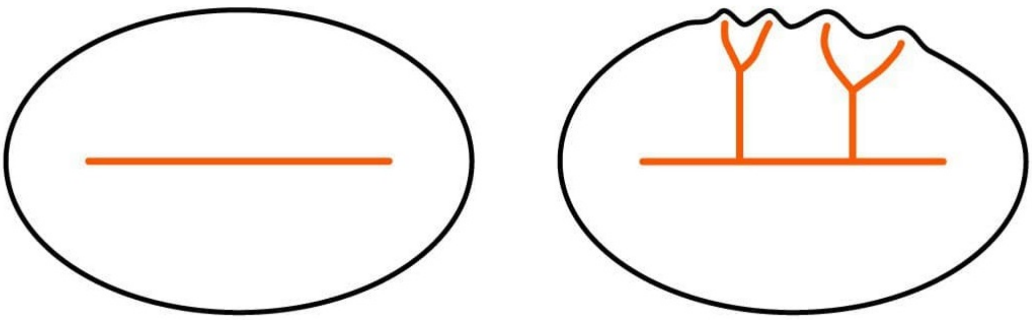

Figure 1 Small perturbations (with respect to the Hausdorff distance) can lead to large perturbations of the medial axis.

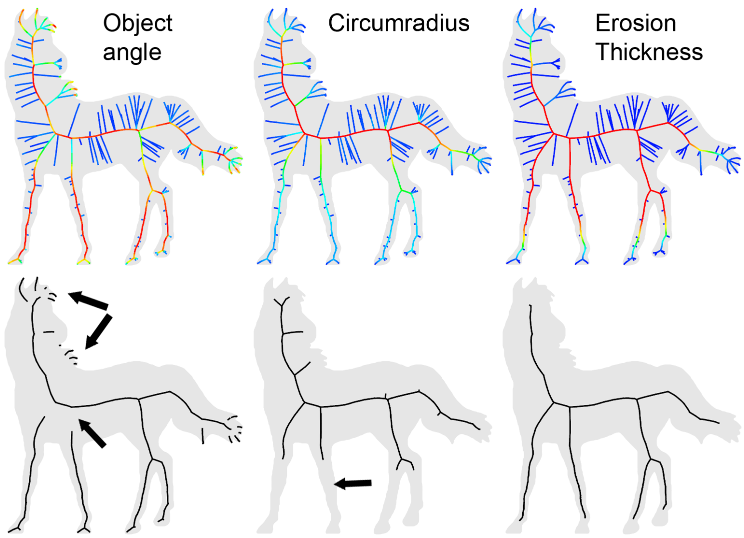

Figure 2 Various pruning methods, from left to right: Object angles [4, 21], radius of the set of closest points [12] (the λ-medial axis, also used in our computation), and a burning method proposed in [37], with various undesirable features indicated. The value of the object angle, radius of of the set of closest points, and burning time is indicated in colour on top. Reproduced from [37].

If we restrict ourselves to a smaller class of spaces and perturbations, stability results are available: Chazal and Soufflet [11] proved that the medial axis is stable with respect to the Hausdorff distance under ambient diffeomorphisms, if we assume that the set of positive reach is a C^2 manifold and the distortion is a C^2 diffeomorphism of $\mathbb{R}^b$.

Significant effort has gone into the simplification (pruning) of the medial axis. This was motivated by applications in graphics (where it is used as a skeleton, see the surveys [30, 33]), data reduction, shape recognition, and learning (see for example [5, 10, 19, 25, 27, 31, 32, 37]).

Many prunings of the medial axis have been proposed in many different settings [1, 4, 7, 12, 20, 21, 24, 27, 32]. See Figure 2 for an illustration of some commonly used ones and their pitfalls: the object angle, which is of historic importance in the community but can disconnect the medial axis, the λ-medial axis, which is used to compute a close approximation of the medial axis but which can truncate "thin" regions undesirably, and the burning method which we consider in this work.

2 Burning Bing's house

The simplification which we focus on for this work is the burning of the medial axis [37], which generalizes Blum's original "grassfire" analogy for the medial axis. The burning of the medial axis removes the extremities of the medial axis by "starting a fire" at the boundary of the medial axis which stops if the fire hits an obstacle, as illustrated in Figure 3.

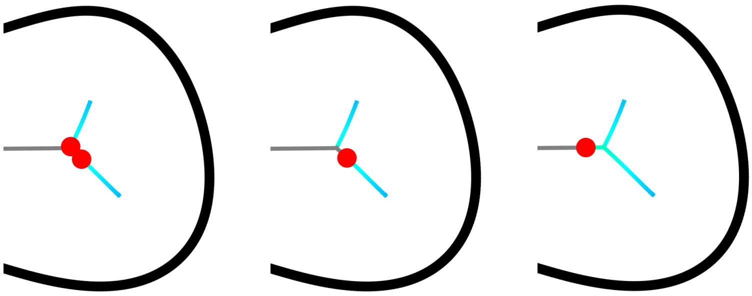

Figure 3 The fire front progression on a the medial axis (grey) of a curve (black). As the fire front (indicated by the red dot) hits an unburned junction, it stops. If the junction is already burned (with the colour indicating the burn time) the fire continues.

Because of the good experimental results, it was conjectured that the burning method of simplification of the medial axis would be stable [27], i.e. no discontinuous jumps. In this work we show that this is not the case. The counter example is based on the standard deformation retract from the closed ball to Bing's house with two rooms [6], which is a contractible but not collapsible two dimensional simplicial complex, see Figure 6. Bing's house is not collapsible, as there is no boundary.

Before we go into the main statement we consider a deformation of Bing's house which makes it collapsible. This deformation will be mirrored in the medial axis in our construction, this deformation is depicted in Figure 5, see Figure 4 for the nomenclature. In this construction we cut a flap open so that the room no longer completely runs around the corridor. This cutting exposes an edge of one of the walls of the corridor and path that goes from the edge to the bottom room. We can use this edge to collapse along the path into the bottom room, then the room, and from this the rest of Bing's house.

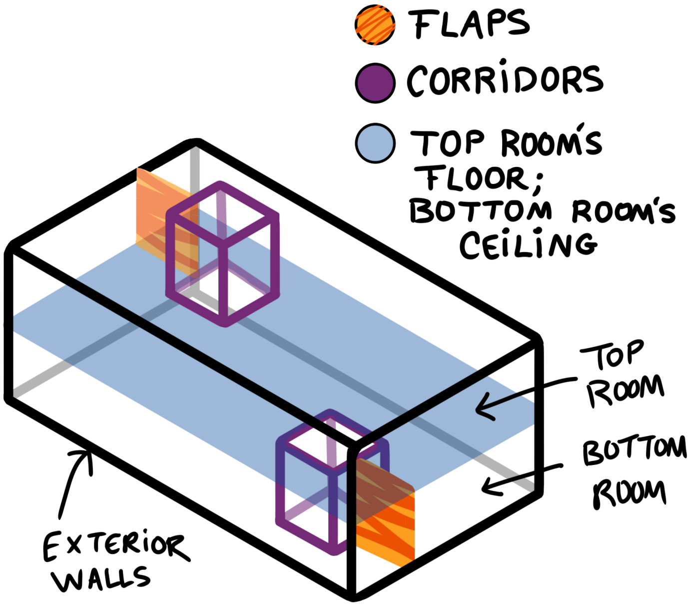

Figure 4 The various parts of Bing's house indicated.

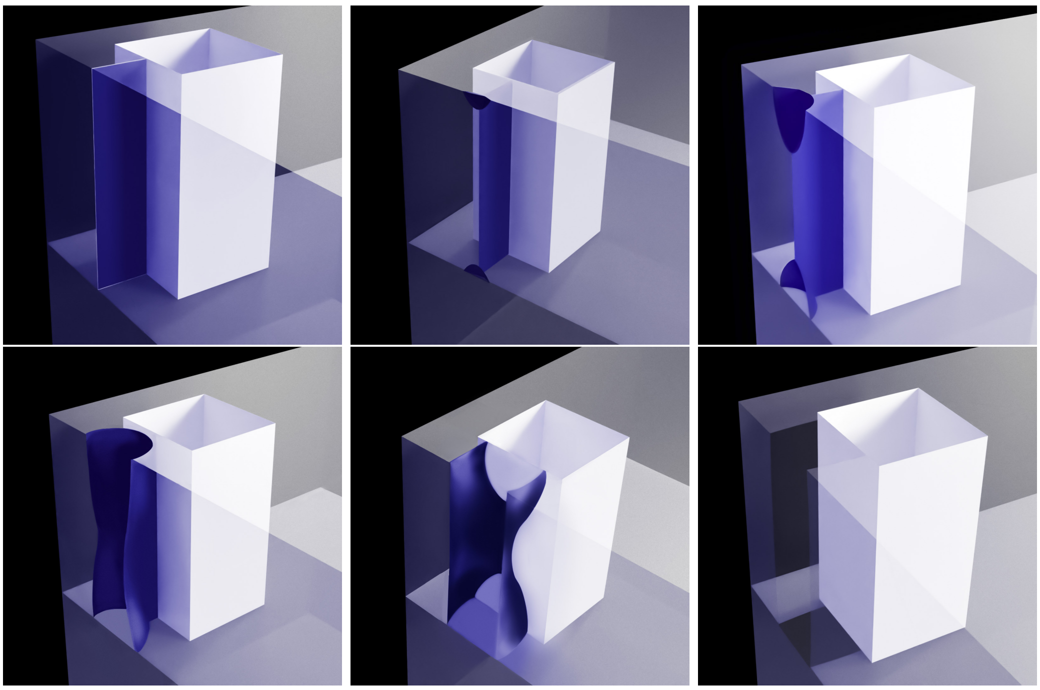

Figure 5 This deformation, which cuts a flap open, makes Bing's house collapsible.

The precise result is the following:

▶ **Theorem 1.** *There exists a smooth ambient isotopy $H_t : [0,1] \times \mathbb{S}^2 \to \mathbb{R}^3$ such that:*

- *The medial axis* $\mathrm{ax}(H_0(\mathbb{S}^2))$ *is collapsible/burn to a single point.*
- *The medial axis* $\mathrm{ax}(H_1(\mathbb{S}^2))$ *is Bing's house and is therefore non-collapsible/ cannot burn.*
- *The burning of* $\mathrm{ax}(H_t(\mathbb{S}^2))$ *is not continuous in t with respect to the Hausdorff distance.*
- *The topology of the burned axis changes from a point to Bing's house with two rooms at a single $t_0 \in [0,1]$.*
- *The isotopy H_t can be chosen to be generic in the sense of singularity theory as developed by Arnol'd and Thom [2], see in particular [23].*

Proof. Bing constructed his house as a deformation retract from a solid cube, see Figure 6. The isotopy of the sphere we consider is the boundary of this deformation. However instead of reducing to a two dimensional object we skip the last step so that every point in the deformation the set remains a topological (solid) ball and its boundary a sphere. The end point of this deformation is a thickened version Bing's house. We will only consider the medial axis in the interior of the sphere and not the exterior. The medial axis of a thickened version of Bing's house is Bing's house itself. The deformation is depicted in Figure 7. The essential topological change only happens near the end of the deformation when the room wraps around the corridor, see Figure 8. When the bisector between the corridor and the wall disappears and is replaced by the bisector between the two parts of the room that are wrapping around the corridor, the medial axis becomes non-collapsible. This transition can be made generic in terms of the transitions of the singularities [23]. ◀

We illustrate this deformation in our video; see also Figures 6, 7, and 8. These animations were made using the λ-medial axis (see `https://github.com/cdfillmore/lambda_medial_axis`) and the open source software Blender [13]. Here λ is chosen very small to ensure that the λ-medial axis is a good approximation of the medial axis.

▶ **Corollary 2.** *Collapsing or pruning the medial axis of a domain such that it becomes one-dimensional, as proposed in e.g. [7], is not always possible, even if the boundary of a domain is a smooth sphere.*

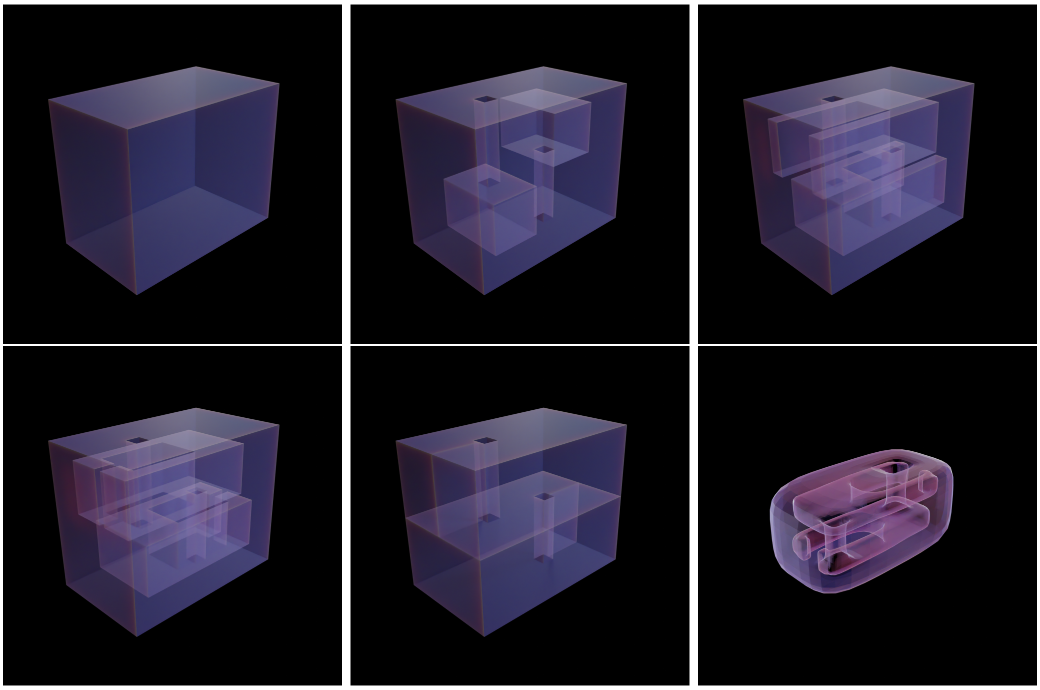

Figure 6 The deformation retract of a solid cube (topological ball) to Bing's house. In the final frame we show the smoothed version of a thickened Bing's house used in the computation.

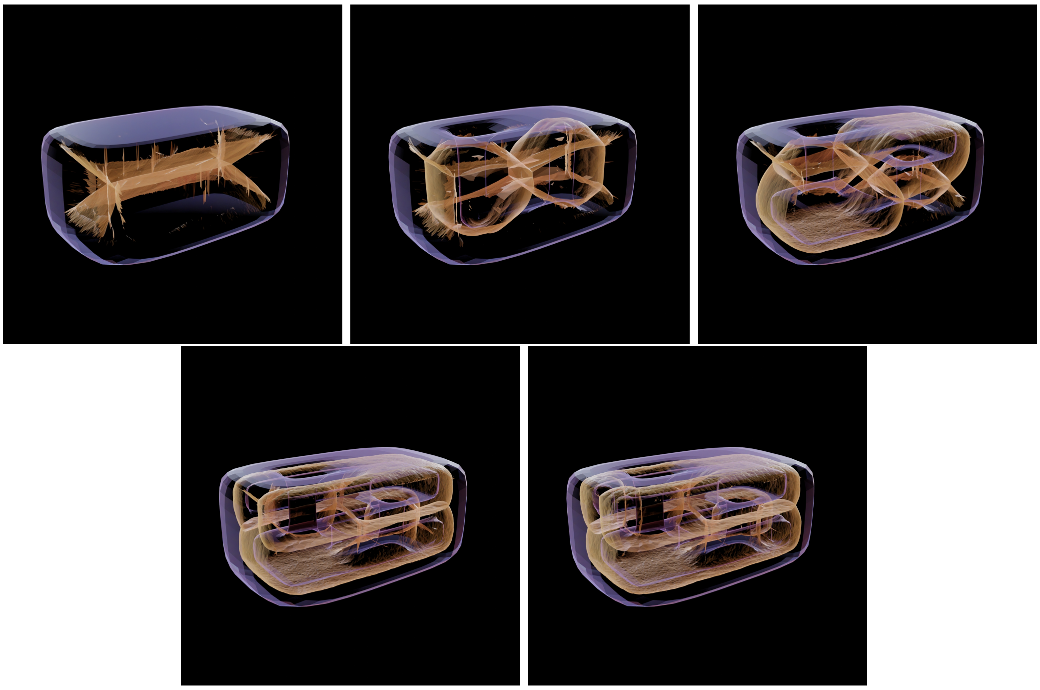

Figure 7 The evolution of the medial axis (yellow) in the interior as the solid cube is deformed into a thickened version of Bing's house (blue/purple).

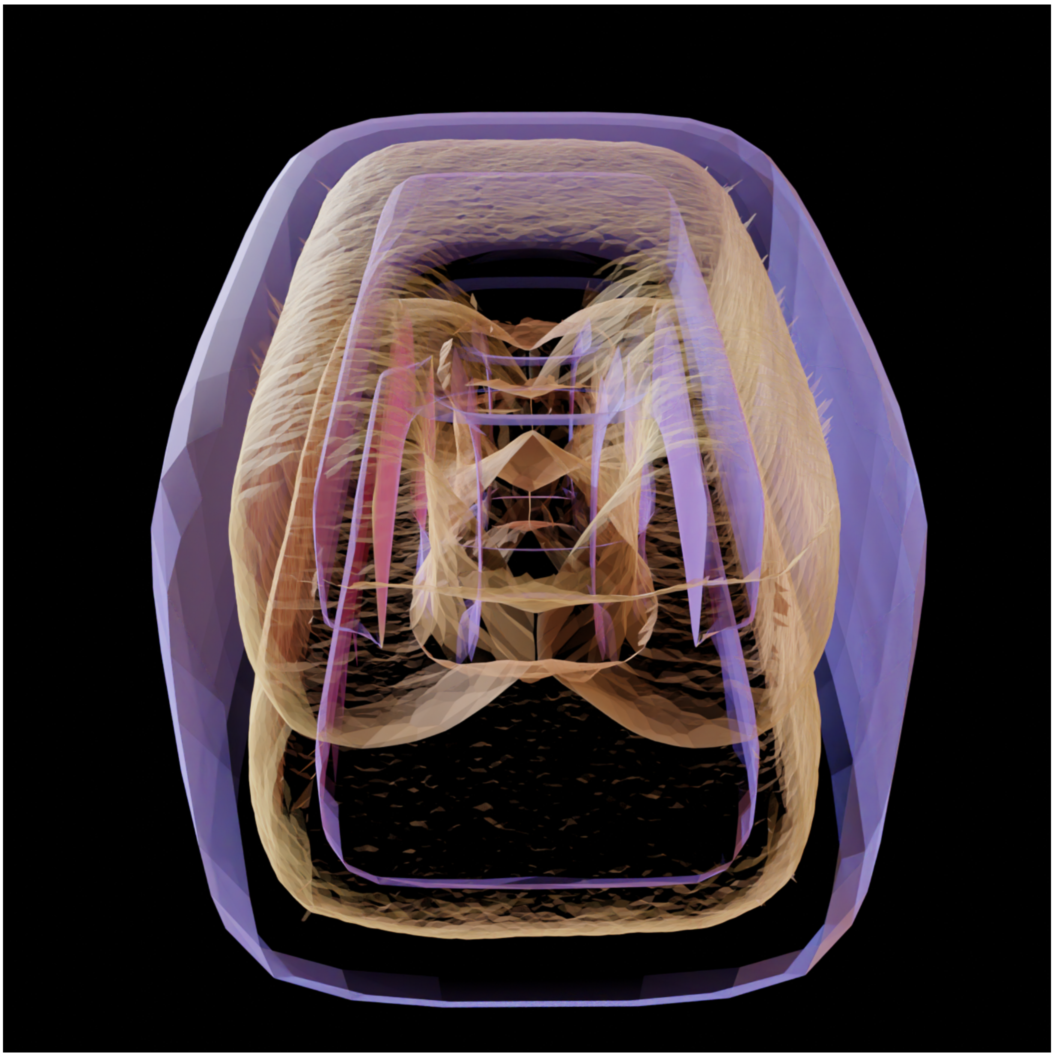

Figure 8 The critical transition of the medial axis. There are points on the medial axis equidistant to the two parts of the room that wrap around the corridor, the corridor itself and the exterior wall, which can be avoided by a small perturbation. This transition occurs between frames 4 and 5 of Figure 6.

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