Eliminating Higher-Multiplicity Intersections:

an r-fold Whitney Trick for the Topological Tverberg Conjecture

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A Thesis

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I hereby declare that this dissertation is my own work, and it does not contain other peoples work without this being so stated; and this thesis does not contain my previous work without this being stated, and that the bibliography contains all the literature that I used in writing the dissertation, and that all references refer to this bibliography. I declare that this is a true copy of my thesis, including any final revisions, as approved by my thesis committee, and that this thesis has not been submitted for a higher degree to any other University or Institution.

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Biographical Sketch

After visiting Raman Parimala at Emory during the fall 2011, Isaac completed a Master in Mathematics at EPFL, under the direction of Eva Bayer. There, he also started a PhD in 2012 in the group of Uli Wagner. He joined IST Austria in 2013 as a transferred student.

His research is at the intersection of three fields: combinatorics, algebraic topology and geometric topology.

During the course of his PhD, he developed a theory of elimination of (geometric) higher-multiplicity intersections which has applications to the topological Tverberg conjecture. For this, he was awarded the IST Austria prize for *Outstanding Scientific Achievement* in 2015.

List of Publications

Preprints and Extended Abstracts

- (with Uli Wagner) *Eliminating Higher-Multiplicity Intersections, I. A Whitney Trick for Tverberg-Type Problems*, arXiv:1508.02349.

  Extended abstract in Proc. 30th Ann. Symp. on Computational Geometry, *SoCG 2014*.

- (with Xavier Goaoc, Pavel Patak, Zuzana Patakova, Martin Tancer and Uli Wagner) *On Generalized Heawood Inequalities for Manifolds: a Van Kampen–Flores-type Nonembeddability Result*.

  Extended abstract in Proc. 31th Inter. Symp. on Computational Geometry, *SoCG 2015*.

- (with Uli Wagner) *Eliminating Higher-Multiplicity Intersections, II. Completeness of the Deleted Product Criterion in the r-Metastable Range*, arXiv:1601.00876.

  Extended abstract in Proc. 32th Inter. Symp. on Computational Geometry, *SoCG 2016*.

- (with Sergey Avvakumov, Arkadiy Skopenkov and Uli Wagner) *Eliminating Higher-Multiplicity Intersections, III. Codimension 2*, arXiv:1511.03501.

Source of this Thesis

This thesis is based on the preprint [31], jointly written with Uli Wagner. Results of Chapters 4 and 5 were presented at SoCG'14 [30].

Abstract

Motivated by *topological Tverberg-type problems* in topological combinatorics and by classical results about embeddings (maps without double points), we study the question whether a finite simplicial complex K can be mapped into $\mathbb{R}^d$ without triple, quadruple, or, more generally, *r-fold points* (image points with at least r distinct preimages), for a given multiplicity $r \geq 2$. In particular, we are interested in maps $f\colon K \to \mathbb{R}^d$ that have no *global* r-fold intersection points, i.e., no r-fold points with preimages in r *pairwise disjoint* simplices of K, and we seek necessary and sufficient conditions for the existence of such maps.

We present higher-multiplicity analogues of several classical results for embeddings, in particular of the completeness of the *Van Kampen obstruction* for embeddability of k-dimensional complexes into $\mathbb{R}^{2k}$, $k \geq 3$. Specifically, we show that under suitable restrictions on the dimensions (viz., if $\dim K = (r-1)k$ and $d = rk$ for some $k \geq 3$), a well-known *deleted product criterion* (*DPC*) is not only necessary but also sufficient for the existence of maps without global r-fold points. Our main technical tool is a higher-multiplicity version of the classical *Whitney trick*, by which pairs of isolated r-fold points of *opposite sign* can be eliminated by local modifications of the map, assuming codimension $d - \dim K \geq 3$.

An important guiding idea for our work was that sufficiency of the DPC, together with an old result of Özaydin's on the existence of equivariant maps, might yield an approach to disproving the remaining open cases of the the long-standing *topological Tverberg conjecture*, i.e., to construct maps from the N-simplex σ^N to $\mathbb{R}^d$ without r-Tverberg points when r *not a prime power* and $N = (d+1)(r-1)$. Unfortunately, our proof of the sufficiency of the DPC requires codimension $d - \dim K \geq 3$, which is not satisfied for $K = \sigma^N$.

In 2015, Frick [16] found a very elegant way to overcome this "codimension 3 obstacle" and to construct the first counterexamples to the topological Tverberg conjecture for all parameters (d, r) with $d \geq 3r + 1$ and r not a prime power, by a reduction[1] to a suitable lower-dimensional skeleton, for which the codimension 3 restriction is satisfied and maps without r-Tverberg points exist by Özaydin's result and sufficiency of the DPC.

In this thesis, we present a different construction (which does not use the constraint method) that yields counterexamples for $d \geq 3r$, r not a prime power.

[1] Using a clever trick, discovered independently by Gromov [18] and Blagojević–Frick–Ziegler [8], called the 'constraint method'.

Acknowledgments

Foremost, I would like to thank Uli Wagner for introducing me to the exciting interface between topology and combinatorics, and for our subsequent years of fruitful collaboration.

In our creative endeavors to eliminate intersection points, we had the chance to be joined later by Sergey Avvakumov and Arkadiy Skopenkov, which led us to new surprises in dimension 12.

My stay at EPFL and IST Austria was made very agreeable thanks to all these wonderful people: Cyril Becker, Marek Filakovský, Peter Franek, Radoslav Fulek, Peter Gaži, Kristóf Huszár, Marek Krčál, Zuzana Masárová, Arnaud de Mesmay, Filip Moric, Michal Rybar, Martin Tancer, and Stephan Zhechev.

Finally, I would like to thank my thesis committee Herbert Edelsbrunner and Roman Karasev for their careful reading of the present manuscript and for the many improvements they suggested.

Contents

Chapter 1

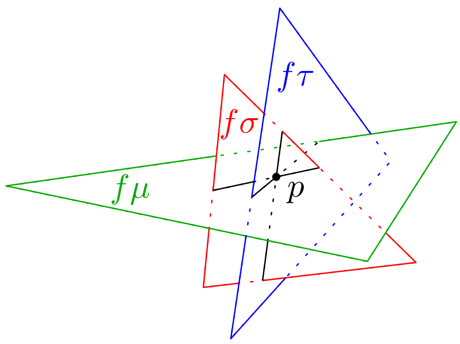
Introduction

Let $k, d, r \geq 2$ be integers, and K be a k-dimensional finite simplicial complex, with geometric realization $|K|$. The overall theme of this thesis is the following problem:

Problem. When does there exists a continuous map $f : |K| \to \mathbb{R}^d$ *without r-fold intersections points*? I.e., we are looking for a map f such that for all r-tuples $x_1, ..., x_r \in |K|$ of distinct points of the polyhedron $|K|$ we *never* have

$$f(x_1) = \cdots = f(x_r).$$

For $r = 2$, the above Problem is the classical question of *embeddability* of a simplicial complex into a Euclidean space (see [37, 43] for surveys). To illustrate $r = 3$, the image on the right shows a triple intersection point of a 2-complex mapped to $\mathbb{R}^3$. The complex K consists of three (disjoint) triangles σ, τ, μ mapped by an affine map f to $\mathbb{R}^3$ such that all three triangles intersect in the image by f on a single point $p \in \mathbb{R}^3$.



The preimages by f of p consist of one point on σ, one point on τ and one point on μ. In particular, these preimages have *disjoint support* in $|K|$. We shall stress this property of "preimages having disjoint support" by saying that that p is a *global* triple intersection point. This special kind of intersection point is going to be our main concern. More formally,

Definition. Let $f : |K| \to \mathbb{R}^d$ be a continuous map.

(a) A point $p \in \mathbb{R}^d$ is an ***r*-fold intersection point** of f if there exists $x_1, ..., x_r \in |K|$ distinct points such that

$$f(x_1) = \cdots = f(x_r) = p,$$

or, equivalently, if $f^{-1}\{p\}$ contains at least r points from $|K|$.

(b) An r-fold intersection point p is called **global** if the x_i have *disjoint support*, i.e., there exists $\sigma_1, \ldots, \sigma_r$ pairwise disjoint simplices of K such that $x_i \in \mathring{\sigma}_i$, for all i. Here,

- we denote by $\mathring{\sigma}_i$ the **interior** of σ_i, i.e., $|\sigma_i| \setminus |\partial\sigma_i|$,
- given $x \in |K|$, the unique simplex $\sigma \in K$ such that $x \in \mathring{\sigma}$ is called the **support** of x, denoted $\mathrm{supp}(x)$.

(c) A map $f : |K| \to \mathbb{R}^d$ is an ***r*-embedding** if f has no r-fold intersection point.

(d) A map $f : |K| \to \mathbb{R}^d$ is an **almost *r*-embedding** if f has no *global* r-fold intersection point.

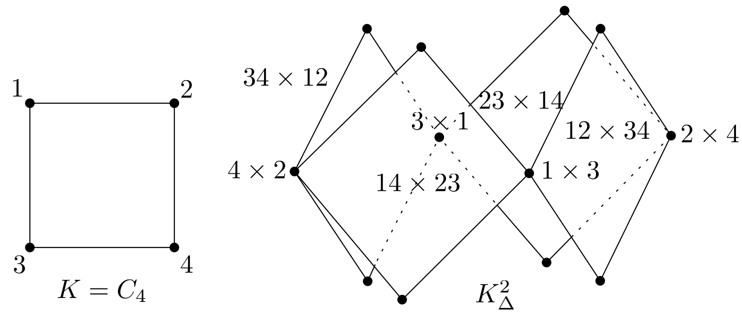


Figure 1.1: The 2-fold deleted product of the cyclic graph C_4 is made of four squares: 14×23, 12×34, 23×14, and 34×12.

Remark. For the existence of almost r-embedding, working in the topological or PL categories is equivalent: the existence of a topological embedding implies the existence of a PL-embedding. Indeed, every continuous map $g\colon K \to \mathbb{R}^d$ can be approximated arbitrarily closely by a PL-map, and if g has no *global* r-intersection points, then the same holds for any map sufficiently close to g.

Therefore, during the course of this thesis, *we shall restrict ourselves to the PL category.*

1.1 The Deleted Product Criterion

There is a well-known *necessary condition* for the existence of almost r-embeddings (i.e., maps without (global) r-fold intersection). It is formulated in terms of the (simplicial) ***r*-fold deleted product** of a complex K, which is defined as the polytopal cell complex

$$\begin{aligned} K^r_\Delta &:= \{(x_1,\dots,x_r) \in |K|^r \mid \mathrm{supp}(x_i) \cap \mathrm{supp}(x_j) = \emptyset \text{ for } 1 \le i < j \le r\} \\ &= \{\sigma_1 \times \cdots \times \sigma_r \mid \sigma_1,\dots,\sigma_r \text{ pairwise disjoint simplices of } K\} \end{aligned}$$

The deleted product K^r_Δ is a regular polyhedral cell complex (a subcomplex of the cartesian product $|K|^r$), whose cells are products $\sigma_1 \times \cdots \times \sigma_r$ of pairwise disjoint simplices of K.

Example. In Figure 1.1, we reproduce an example from de Longueville [12, p. 111]: the 2-fold deleted product of C_4, the cyclic graph on four vertices.

The space $(C_4)^2_\Delta$ is made of four squares which are attached at a few boundary points. For instance, the squares 14×23 and 34×12 both share the point 4×2 on their boundary.

An important observation about $(C_4)^2_\Delta$ is that to each point (x, y) on the square 12×34 there is a corresponding point (y, x) on the square 34×12. More generally, there is a natural involution on $(C_4)^2_\Delta$ obtained from the transposition $(x, y) \mapsto (y, x)$.

More explicit computations of 2-fold deleted products can be found in Matousek [33, p. 110]. For instance, the 2-fold deleted product of a simplex σ^d is PL-homeomorphic to the sphere S^{d-1}.

Definition 1.1.1. If X and Y are spaces on which a finite group G acts (all group actions will be from the right) then we will use the notation $F\colon X \to_G Y$ for maps that are **equivariant**, i.e., that commute with the group actions, $F(x \cdot g) = F(x) \cdot g$ for all $x \in X$ and $g \in G$).

Lemma 1.1.2 (Necessity of the Deleted Product Criterion)**.** *Let K be a finite simplicial complex, and let $d \ge 1$ and $r \ge 2$ be integers. If there exists an almost r-embedding $f : K \to \mathbb{R}^d$, then there exists an equivariant map*

$$\widetilde{f}\colon K^r_\Delta \to_{\mathfrak{S}_r} S^{d(r-1)-1},$$

where $S^{d(r-1)-1} = \big\{(y_1,\dots,y_r) \in (\mathbb{R}^d)^r \mid \sum_{i=1}^r y_i = 0, \sum_{i=1}^r \|y_i\|_2^2 = 1\big\}$, and the symmetric group $\mathfrak{S}_r$ acts on both spaces by permuting the factors.

We recall the standard proof, which uses several notions that we will need later.

Proof. Given $f\colon K \to \mathbb{R}^d$, we induce a *Gauss map*

$$f^r\colon K^r_\Delta \to (\mathbb{R}^d)^r \quad \text{by} \quad f^r(x_1,\dots,x_r) := (fx_1,\dots fx_r).$$

The map f is an almost r-embedding if and only if f^r avoids the **thin diagonal**

$$\delta_r(\mathbb{R}^d) := \{(y,\dots,y) \mid y \in \mathbb{R}^d\} \subset (\mathbb{R}^d)^r. \tag{1.1}$$

The sphere $S^{d(r-1)-1}$ is, by definition, the unit sphere in the orthogonal complement $\delta_r(\mathbb{R}^d)^\perp \cong \mathbb{R}^{d(r-1)}$ (the orthogonal complement of the vector space $\delta_r(\mathbb{R}^d)$ inside $(\mathbb{R}^d)^r$, relative to the usual scalar product).

There is an equivariant homotopy equivalence

$$\rho\colon (\mathbb{R}^d)^r \setminus \delta_r(\mathbb{R}^d) \simeq S^{d(r-1)-1},$$

which is obtained as follows: first orthogonally project $(\mathbb{R}^d)^r \setminus \delta_r(\mathbb{R}^d)$ onto $\delta_r(\mathbb{R}^d)^\perp \setminus \{0\}$, and then radially retract the latter to $S^{d(r-1)-1}$.

More concretely, the retraction is the composition

$$\rho = \mu \circ \nu \quad \text{given by}$$

$$\nu(y_1,\dots,y_r) = (\bar{y}_1,\dots,\bar{y}_r), \quad \text{where} \quad \bar{y}_j = y_j - \sum_{i=1}^r y_i, \text{ for } 1 \le j \le r, \quad \text{and}$$

$$\mu(\bar{y}_1,\dots,\bar{y}_r) = (\bar{y}_1,\dots,\bar{y}_r)/(\sum_{i=1}^r \|\bar{y}_i\|_2^2).$$

Both f^r and ρ are $\mathfrak{S}_r$-equivariant hence so is their composition

$$\widetilde{f} := \rho \circ f^r\colon K^r_\Delta \to_{\mathfrak{S}_r} S^{d(r-1)-1}. \tag{1.2}$$

□

Remarks 1.1.3. (a) The action of $\mathfrak{S}_r$ is free on K^r_Δ for all r, but not free on $S^{d(r-1)-1}$ unless $r = 2$. (A group action is **free** if only the identity element fixed any points. In symbols, $(\exists x | g * x = x) \Rightarrow (g = 1)$).

(b) Some authors prefer to work with *deleted joins* (which are again simplicial complexes) instead of deleted products as configuration spaces for Tverberg-type problems. However, it is known that deleted products provide necessary conditions that are at least as strong as those provided by deleted joins; see, e.g., [34, Sec. 3.3].

For further background on the broader *configuration space/test map* framework, see, e.g., [33, Ch. 6] or [55, 56].

In symbols, Lemma 1.1.2 becomes

$$\exists \text{ almost } r\text{-embedding } K \to \mathbb{R}^d \quad \Rightarrow \quad \exists K^r_\Delta \to_{\mathfrak{S}_r} S^{d(r-1)-1}. \tag{1.3}$$

Problem 1.1.4 (Our main concern in this thesis.)**.** When can the implication (1.3) be reversed? I.e., when does the existence of an equivariant map $K^r_\Delta \to_{\mathfrak{S}_r} S^{d(r-1)-1}$ *imply* the existence of an almost r-embedding $K \to \mathbb{R}^d$?

To reformulate this question one more time: when is the existence of the equivariant map $K^r_\Delta \to_{\mathfrak{S}_r} S^{d(r-1)-1}$ not only a necessary (Lemma 1.1.2), but also a *sufficient* condition for the existence of an almost r-embedding $K \to \mathbb{R}^d$?

In the present manuscript, we shall focus on proving the completeness of the deleted product criterion in its “critial dimension” (see Chapter 5).

Theorem 1.1.5 (Sufficiency of the Deleted Product Criterion for the critical dimension)**.** *Let $k \geq 3$, $r \geq 2$ and K be a finite $(r-1)k$-dimensional simplicial complex.*

There exists an almost r-embedding $K \to \mathbb{R}^{rk}$ if and only if there exists an $\mathfrak{S}_r$-equivariant map $K^r_\Delta \to_{\mathfrak{S}_r} S^{rk(r-1)-1}$.

Remarks 1.1.6. (a) Theorem 1.1.5 corresponds to the first "interesting case" dimension-wise. Indeed, given a generic PL-map from a $(r-1)k$-dimensional complex K to $\mathbb{R}^{rk+1}$, i.e.,

$$f\colon K \to \mathbb{R}^{rk+1} \text{ in general position}$$

then the set of r-fold intersections of f is of dimension[1] less than

$$r \cdot (r-1)k - (r-1) \cdot (rk+1) = -(r-1) < 0.$$

Hence, almost r-embedding always exists for $K \to \mathbb{R}^{rk+1}$.

(b) Despite being the first "non-trivial" case of Problem 1.1.4, Theorem 1.1.5 has striking consequences in the context of the topological Tverberg conjecture (Section 1.3).

(c) Theorem 1.1.5 is a generalized version of the classical van Kampen–Shapiro–Wu embeddability criterion:

Theorem 1.1.7 (van Kampen–Shapiro–Wu Embeddability Criterion [41, 46, 53])**.** *Let $k \geq 3$ and K be a finite k-dimensional simplicial complex.*

There exists an embedding $K \hookrightarrow \mathbb{R}^{2k}$ if and only if there exists an $\mathfrak{S}_2$-equivariant map $K^2_\Delta \to_{\mathfrak{S}_2} S^{2k-1}$.

r-embeddings of manifolds. For the case when the simplicial complex K happens to be a PL-manifold, Theorem 1.1.5 becomes boring: a PL-manifold $M^{(r-1)k}$ *always* r-embeds into $\mathbb{R}^{rk}$.

Proposition 1.1.8. *Let $r \geq 2$, $k \geq 3$, and let M a PL-manifold of dimension $(r-1)k$. Then there exists a r-embedding $M \to \mathbb{R}^{rk}$.*

This is proven in Chapter 3. It parallels the "standard" behavior for $r = 2$: a PL-manifolds M^k *always* embeds into $\mathbb{R}^{2k}$. This is the famous Whitney's *Embedding in Double Dimension Theorem* [52].

Almost r-embeddings to r-embeddings. For the general case of a simplicial complex $K^{(r-1)k}$, the problem of turning an almost r-embedding $K^{(r-1)k} \to \mathbb{R}^{rk}$ into an r-embedding (i.e., removing the remaining "local" r-fold intersections) is rather subtle, and the "obvious" candidate for an equivariant hypothesis ($K^r_\Delta \to_{\mathfrak{S}_r} S^{rk(r-1)-1}$) does not appear to be sufficient if one wants the final r-embedding to be in general position. This will be the subject of a later publication.

Two extensions. Theorem 1.1.5 has been extended in two ways (discussed in separated publications):

1. Together with S. Avvakumov and A. Skopenkov, we extend in [2] the techniques of [31] in order to replace the hypothesis "$k \geq 3$" by "$k \geq 2$" in Theorem 1.1.5. The trade-off being that we then require $r \geq 3$ (which was to be expected: the deleted product criterion famously fails for the embeddability problem of a 2-complex into $\mathbb{R}^4$ [15]).

[1] We inductively use that in general position two polyhedra P and Q intersect in $\mathbb{R}^d$ a polyhedron of dimension less $\dim P + \dim Q - d$ (see Chapter 5 in [38]).

Another (equivalent) way to compute it is: given a set $\{P_i\}$ of r polyhedra in $\mathbb{R}^d$, then the codimension of their intersection is generically greater than the sum of their codimensions, i.e.,

$$\operatorname{codim}_{\mathbb{R}^d}(P_1 \cap \cdots \cap P_r) \geq \sum_i \operatorname{codim}_{\mathbb{R}^d}(P_i).$$

2. What happens below the "critical range"? (I.e., when the set of r-folds intersections is not solely composed of isolated points, but can generically have positive dimension). For this situation, one can still extend Thm 1.1.5 over a wide range of dimensions.

Theorem 1.1.9 (Sufficiency of the Deleted Product Criterion in the r-Metastable Range, [32])**.** *Let $d, m, r \geq 2$ be integers satisfying the* ***r-metastable range*** *equation*

$$rd \geq (r+1)m + 3, \tag{1.4}$$

and let K be a finite m-dimensional simplicial complex.

There exists an almost r-embedding $K \to \mathbb{R}^d$ if and only if there exists an $\mathfrak{S}_r$-equivariant map $K^r_\Delta \to_{\mathfrak{S}_r} S^{d(r-1)-1}$.

Theorem 1.1.5 is the special case of Theorem 1.1.9 with

$$m = (r-1)k \quad \text{and} \quad d = rk.$$

The reader might recognize that Theorem 1.1.9 is a "higher-multiplicity" analogue (modulo the word "almost") of the famous Haefliger–Weber embeddability criterion:

Theorem 1.1.10 (Haefliger–Weber [20,42,50])**.** *Let $d, m \geq 2$ be integers satisfying the* ***2-metastable range*** *equation*

$$2d \geq 3m + 3,$$

and let K be a finite m-dimensional simplicial complex.

There exists an embedding $K \hookrightarrow \mathbb{R}^d$ if and only if there exists an $\mathfrak{S}_2$-equivariant map $K^2_\Delta \to_{\mathfrak{S}_2} S^{d-1}$.

1.2 A Generalized Whitney Trick

Our main tool to deal with intersections of higher multiplicity is a generalized version of the *Whitney trick*.

In this section, we briefly discuss the classical Whitney trick and state the generalization we shall use to prove Theorem 1.1.5. This part ot the introductory chapter gets more technical than the rest and some of the concepts that we are about to discuss will only be formally defined later in Chapter 2. In particular we refer the reader to that chapter for notions such as *ambient isotopy* (Ch. 2.1), *general position* (Ch. 2.2), or *intersection signs* (Ch. 2.3).

The classical Whitney trick allows one to eliminate a pair of isolated double points of *opposite sign* of a map by an ambient isotopy fixed outside a small ball, provided the codimension is at least 3.

We are mostly concerned with the PL category, whereas Whitney's original paper [52] is set in the smooth category.

Weber [49, p. 179] was the first to adapt the trick to the PL category, the textbook of Rourke and Sanderson is also an excellent source [38, Lemma 5.12].

Theorem 1.2.1 (Whitney Trick). *Suppose that M_1 and M_2 are connected, orientable PL-manifolds of dimensions m_1 and m_2 (both at least 3), and that*

$$f \colon M_1 \sqcup M_2 \to \mathbb{R}^d$$

is a PL-map in general position defined on their ***disjoint union*** *$M_1 \sqcup M_2$ with*

$$m_1 + m_2 = d,$$

and

$$x, y \in fM_1 \cap fM_2$$

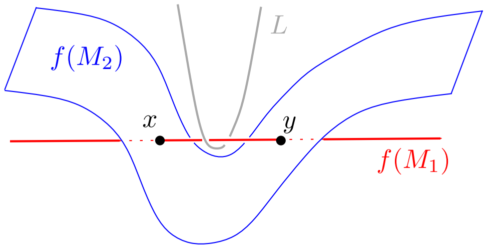


Figure 1.2: fM_1 and fM_2 intersecting in two double points x, y of opposite signs, and a potential obstacle L.

are two double points of opposite intersection signs. Then there exists an ambient PL-isotopy H_t of $\mathbb{R}^d$ such that

$$fM_1 \cap H_1(fM_2) = \big(fM_1 \cap fM_2\big) \setminus \{x, y\}.$$

Moreover, the isotopy can be chosen to be local, in the following sense: Given any closed polyhedron $L \subset \mathbb{R}^d$ of dimension $\ell \leq d-3$ and with $x, y \notin L$, there exists a PL-ball $B^d \subset \mathbb{R}^d$ disjoint from L such that H is fixed outside of $\mathring{B}^d$.

Remarks 1.2.2. (a) A consequence of the "moreover" part is that we can always assume that H_t is constant on both $f\partial M_i$. Indeed, by general position, $x, y \in f\mathring{M}_1 \cap f\mathring{M}_2$.

(b) The sign of a *single* double point depends on the choice of orientations of the M_i and that of $\mathbb{R}^d$, but having a *pair* $\{x, y\}$ of opposite signs is independent of such choices.

(c) Figure 1.2 illustrates the *Whitney Trick* in a low-dimensional situation. The idea of the trick is to "push" fM_2 upwards until the two intersections points x and y disappear, while keeping the boundary of fM_2 fixed. In low (co)dimensions, doing this might require passing over some obstacles and/or introducing new double points. If $d - m_i \geq 3$, $i = 1, 2$ (or, equivalently, if both m_i at least 3), then these problems can be avoided.

(d) The hypotheses of the Whitney trick can be weakened, e.g., one of the M_i can be allowed to have dimension $m_i = d - 2$, but then one needs to impose additional technical conditions like local flatness and simple connectivity of the complement $\mathbb{R}^d \setminus f(M_i)$; see, e.g., [38, Lemma 5.12].

(e) Theorem 1.2.1 was initially proven by Whitney [52] as a step towards his famous *Embedding in Double Dimension Theorem*: Any smooth m-manifold embeds in $\mathbb{R}^{2m}$.

The proof structure for that theorem is as follows[2]: Given a generic smooth map $f \colon M \to \mathbb{R}^{2m}$

(1) Draw a path in M joining the two preimages of a self-intersection point $p = f(x) = f(y)$.

(2) "Along that path", modify f as to introduce a new self-intersection point q of sign opposite to p.

(3) Use the Whitney Trick to remove the pair p, q.

Sometimes, the second step (2) is also referred to as the "Whitney Trick" (but we should avoid that confusion here).

The proof of Whitney's Embedding Theorem is much simpler in the PL category (See [38, Thm 5.5], or Chapter 3).

In Chapter 4, we prove the following analogue of Theorem 1.2.1 for r-fold points:

[2]See [35, p. 50] for a more precise scheme.

Theorem 1.2.3 (**Higher-Multiplicity Whitney Trick**)**.** *Let $r \geq 2$, and let $M_1, \ldots, M_r$ be connected, orientable PL-manifolds, of respective dimensions* $\dim M_i = m_i$*, such that*

$$\sum_{i=1}^{r} m_i = d(r-1) \tag{1.5}$$

and

$$d - m_i \geq 3, \qquad 1 \leq i \leq r. \tag{1.6}$$

Let

$$f : M_1 \sqcup \cdots \sqcup M_r \to \mathbb{R}^d$$

be a PL-map in general position defined on their disjoint union, and suppose that

$$x, y \in fM_1 \cap fM_2 \cap \cdots \cap fM_r$$

are two r-fold points of opposite intersection signs.

Then there exist $r-1$ ambient PL-isotopies $H_t^2, \ldots, H_t^r$ of $\mathbb{R}^d$ such that

$$fM_1 \cap H_1^2(fM_2) \cap \cdots \cap H_1^r(fM_r) = \big(fM_1 \cap fM_2 \cap \cdots \cap fM_r\big) \setminus \{x, y\}$$

Moreover, these isotopies can be chosen to be local*, in the following sense: Given any closed polyhedron $L \subset \mathbb{R}^d$ of dimension $\ell \leq d-3$ and with $x, y \notin L$, there exists a PL-ball $B^d \subset \mathbb{R}^d$ disjoint from L such that H^i is fixed outside of $\mathring{B}^d$, $2 \leq i \leq r$.*

1.3 Topological Tverberg Conjecture

A central problem in combinatorial topology [33, p. 154] is the

Conjecture 1.3.1 (Topological Tverberg Conjecture)**.** *Let $r, d \geq 2$ and $N := (r-1)(d+1)$. Any* continuous *map from the N-simplex σ^N to $\mathbb{R}^d$ has one global r-fold intersection.*

I.e., there exists no almost r-embedding $\sigma^N \to \mathbb{R}^d$.

The conjecture is a generalization of classical

Theorem 1.3.2 (Tverberg theorem [45])**.** *For $N = (d+1)(r-1)$, any* affine *map from the N-simplex σ^N to $\mathbb{R}^d$ has a global r-fold intersection.*

with the word "affine" replaced by "continuous".

The conjecture is still open in low dimension $d \leq 11$, but it is known to admit counterexamples in higher dimensions $d \geq 12$ [2, 16, 31].

More precisely, the conjecture has counterexamples for $d \geq 2r$, and r not a prime power. In the present manuscript, we shall content ourselves with producing a counterexample of larger dimension (see Chapter 6).

Theorem 1.3.3 (Counterexamples to Tverberg Conjecture for $d \geq 3r$)**.** *Suppose $r \geq 6$ is not a prime power and let $N = (3r+1)(r-1)$. Then there exists an almost r-embedding $f\colon \sigma^N \to \mathbb{R}^{3r}$.*

The smallest counterexample obtained from Thm 1.3.3 is an almost 6-embedding $\sigma^{95} \to \mathbb{R}^{18}$. Currently, the smallest counterexample availabe in the litterature is an almost 6-embedding $f\colon \sigma^{65} \to \mathbb{R}^{12}$ [2]. In particular, the planar case of the Tverberg Conjecture is still wide open for non-prime power. E.g., is there a drawing of the complete graph on 16 vertices avoiding 6-fold intersections of the forms (a) 4 triangles and 2 edges and (b) 5 triangles and a vertex?

A bit of history of the Topological Tverberg Conjecture.

1966 Tverberg proves Theorem 1.3.2.

1979 Conjecture 1.3.1 is raised by Bajmoczy and Bárány [3] (who also proves it for $r = 2$), and simultaneously by Tverberg [19, Problem 84]. (In 1976, the conjecture was already raised by Bárány in a letter to Tverberg.)

1981 Bárány, Shlosman, and Szűcs [6] prove the conjecture for all primes r.

1987 In his landmark unpublished manuscript, Özaydin [36] proves the conjecture for all prime powers r.

Further proofs of the prime power case are later given by Volovikov [47], Živaljević [56], and Sarkaria [40].

2010 Gromov [18] observes that *the topological Tverberg theorem, whenever available, implies the van Kampen–Flores theorem*, by a very elegant (and in hindsight very simple proof) relying on the pigeon-hole principle (see Thm 6.1.2 in Chapter 6).

2014 Blagojević–Frick–Ziegler [8] independently rediscover Thm 6.1.2, and extend the underlying pigeonholing's idea, developing the *constraint method* that allows to obtain simple proofs of many Tverberg-type results as corollaries of the prime power case of the topological Tverberg conjecture.

In the same year, together with Uli Wagner, we announce [30] that the deleted product criterion (Lemma 1.1.2) is not only a necessary, but also a *sufficient* condition for the existence of almost r-embedding in a "critical range" of dimensions. This theory is built as an approach to reverse one implication from Özaydin's work, which would *disprove* the Topological Tverberg in the remaining open cases (i.e., r not a prime power), but at that time, we do not know how to overcome a final "codimension 3" barrier of our geometric construction.

2015 Frick [16] is the first to realize that all the pieces of the puzzle (leading to a first family of counterexamples) have been found: one can combine Özaydin's work, Gromov's trick and [30], and thus produce counterexamples to the topological Tverberg Conjecture, for all $d \geq 3r + 1$, r not a prime power (see Chapter 6.1).

Later that year, together with Uli Wagner [31], we find another way of building counterexamples, that do not require Gromov's trick, but use a notion of "prismatic maps", leading to a new family of counterexamples, for all $d \geq 3r$, r not a prime power (see Chapter 6.2).

In November of that year, together with Sergey Avvakumov and Arkadiy Skopenkov [2], we extend our construction to work for all $d \geq 2r$, r not a prime power. This extension relies on a codimension 2 version of Thm 1.1.5.

The above account is just the trail of one aspect of the Tverberg Conjecture: there are numerous close relatives and other variants of Tverberg-type problems and results, e.g., the *Colored Tverberg Problem* [4, 5, 9, 56, 57] and generalized *Van Kampen–Flores-type results* [39, 48].

Chapter 2

PL Topology & Intersection Signs

This chapter presents some background facts and definitions from piecewise-linear (PL) topology that will be useful for us latter. We suggest to the reader to simply skim through it.

For a very readable and compact introduction to the area, see the survey article [10]. For more details see, e.g., the textbooks [26, 38] or the lecture notes [54] (my personal favorite). We refer the reader to any of these sources for much of the basic terminology, such as **PL-manifolds** and **regular neighborhoods**. A **polyhedron** will always mean the underlying polyhedron of some geometric simplicial complex in some $\mathbb{R}^d$.

2.1 Isotopies, Ambient Isotopies, and Unknotting

(Ambient) Isotopies. One of the facts that make working in codimension at least 3 easier is that *isotopic* embeddings are also *ambient isotopic*, see below. This fails in codimension 2; for instance, any two PL-knots (embeddings of S^1) in S^3 are isotopic, but not necessarily ambient isotopic.

Let X be a polyhedron, and let Q be a PL-manifold. A **(PL) isotopy** of X in Q is a PL-embedding $F\colon X \times [0,1] \to Q \times [0,1]$ that is **level-preserving**, i.e., such that $F(X \times t) \subseteq Q \times t$ for all $t \in [0,1]$. An isotopy determines embeddings $F_t\colon X \hookrightarrow Q$ by $F(x,t) = (F_t(x), t)$ for $x \in X$ and $t \in [0,1]$.

An isotopy F is **fixed** on a subspace $Y \subseteq X$ if $F(y,t) = (F_0(y), t)$ for all $t \in [0,1]$ and $y \in Y$. An isotopy F is **allowable** if $F^{-1}(\partial Q \times [0,1]) = X_0 \times [0,1]$ for some closed subpolyhedron $X_0 \subseteq X$.

Two embeddings $f, g\colon X \hookrightarrow Q$ are **(allowably) isotopic (keeping Y fixed)** if there is an (allowable) isotopy (fixed on Y) F of X in Q such that $F_0 = f$ and $F_1 = g$.

An **ambient PL-isotopy** of H of Q is a level-preserving PL-homeomorphism $H\colon Q \times [0,1] \to Q \times [0,1]$ such that H_0 is the identity on Q. Two PL-embeddings $f, g\colon X \hookrightarrow Q$ are **ambient isotopic (keeping $Y \subseteq Q$ fixed)** if there is an ambient isotopy H of Q, fixed on Y, with $g = H_1 \circ f$. An ambient isotopy H of Q **extends** an isotopy F of X in Q if $F_t = H_t \circ F_0$ for all $t \in [0,1]$.

Proper Embedding. Let M and Q be PL-manifolds, possibly with boundary. A PL-embedding $f : M \to Q$ is **proper** if $f^{-1}(\partial Q) = \partial M$. An isotopy is proper if it is proper as an embedding.

From isotopy to ambient isotopy.

Theorem 2.1.1 (**Hudson** [24, Thm 1])**.** *Let M and Q be PL-manifolds, M compact, and let $F : M \times [0,1] \to Q \times [0,1]$ be a proper isotopy of M in Q, fixed on ∂M. If $\dim Q - \dim M \geq 3$, then there is an ambient isotopy of Q, fixed on ∂Q, that extends F.*

We will also need the following result concerning embeddings of compact polyhedra:[1]

Proposition 2.1.2 (**Hudson** [27, Corollary 1.3])**.** *Let X be a compact polyhedron and let Q be a PL-manifold. Let $f, g\colon X \to Q$ be allowably isotopic embeddings keeping $Y \subseteq X$ fixed, with $X_0 = f^{-1}(\partial Q) \subseteq Y$. If $\dim X \leq \dim Q - 3$, then f and g are ambient isotopic keeping $f(Y) \cup \partial Q$ fixed.*

Unknotting of balls and spheres. A **(PL) (q, m)-manifold pair** (Q, M) is a pair of PL-manifolds M and Q of dimensions m and q, respectively such that $M \subseteq Q$ properly.

A pair (B^q, B^m) of PL-balls (respectively, a pair (S^q, S^m) of PL-spheres), $m \leq q$, is **unknotted** if it is PL-homeomorphic to the **standard ball pair** $([-1,1]^q, [-1,1]^m \times 0)$ (respectively, to the **standard sphere pair** $(\partial[-1,1]^{q+1}, \partial([-1,1]^m \times 0))$.)

Theorem 2.1.3 (**Zeeman** [54, Ch. IV, Theorem 9])**.** *If $q - m \geq 3$ then every PL-ball pair (B^q, B^m) and every PL-sphere pair (S^q, S^m) are unknotted.*

We will also need the following relative version:

Corollary 2.1.4 (**Zeeman** [54, Ch. IV, Corollary 1, p. 16])**.** *If $q - m \geq 3$, then any two proper embeddings $B^m \subseteq B^q$ that agree on ∂B^m are ambient isotopic, keeping ∂B^q fixed.*

From homotopy to ambient isotopy.

Theorem 2.1.5 (**Zeeman** [54, Ch X, p 198, Thm 10.1])**.** *Let M and Q be compact manifolds of dimensions q and m, respectively, and let $f, g : M \to Q$ be two proper embeddings. Suppose that f is homotopic to g relative to ∂M. Then if $q - m \geq 3$, M is $(2m - q + 1)$-connected, and Q is $(2m - q + 2)$-connected, then f and g are ambient isotopic keeping ∂Q fixed.*

Theorem 2.1.6 (**Irwin** [54, Ch. VIII, p. 4, Thm. 23])**.** *Assume M is compact and let $f : M \to Q$ be a continuous map such that $f|_{\partial M}$ is a piecewise-linear embedding of ∂M in ∂Q. Then f is homotopic to a proper embedding keeping ∂M fixed provided*

$$q - m \geq 3, \quad M \text{ is } (2m - q)\text{-connected}, \quad Q \text{ is } (2m - q + 1)\text{-connected.}$$

Local Flatness A manifold M^m properly embedded in a manifold Q^q is **locally flat at $x \in M \subset Q$** if the pair (Q, M) can be triangulated by (L, K) with x as a vertex and such that the pair of balls

$$(\mathrm{star}(x, L), \mathrm{star}(x, K))$$

is homeomorphic to a standard pair of balls. A manifold pair (Q, M) is a **locally flat manifold pair** if it is locally flat at every point.

A direct consequence of Theorem 2.1.3 is that, provided $q - m \geq 3$, any pair (Q^q, M^q) is locally flat.

2.2 General Position and Transversality

There are many variants of general position. For the purposes of studying r-fold points and r-Tverberg points, the following definitions are convenient.

[1] In [27], the result is stated in a stronger form: The conclusion remains true under the weaker hypothesis that f and g are **allowably concordant** keeping Y fixed. (The notion of an allowable concordance F between $f = F_0$ and $g = F_1$ fixing Y is a generalization of an allowable isotopy fixing Y, where the requirement that F preserve levels is relaxed to the conditions $F(X \times t) \subseteq Q \times t$ for $t = 0, 1$ and $F(X \times t) \subseteq Q \times (0, 1)$ for $t \in (0, 1)$, see [27, Section 1].)

General position in $\mathbb{R}^d$. A collection $\mathcal{A}$ of affine subspaces of $\mathbb{R}^d$ is **in general position** if for every $r \geq 2$ and pairwise distinct $A_1, \ldots, A_r \in \mathcal{A}$,

$$\dim\left(\bigcap_{i=1}^r A_i\right) = \max\left\{-1, \left(\sum_{i=1}^r \dim(A_i)\right) - d(r-1)\right\}. \tag{2.1}$$

A set S of points in $\mathbb{R}^d$ is **in general position** if, for every $r \geq 2$ and pairwise disjoint subsets $S_1, \ldots, S_r \subseteq S$, the affine hulls $\mathrm{aff}(S_i)$, $1 \leq i \leq r$, are in general position.[2]

A collection $\mathcal{P} = \{P_1, \ldots, P_r\}$ of convex polyhedra in $\mathbb{R}^d$ is in general position if $\mathrm{aff}(F_1), \ldots \mathrm{aff}(F_r)$ are in general position for every choice of nonempty faces $F_i \subseteq P_i$, $1 \leq i \leq r$.

If K is a simplicial complex and $f\colon K \to \mathbb{R}^d$ is a simplexwise-linear map, then we say that f is in general position if the images of the vertices of K are pairwise distinct and in general position. A PL-map $f\colon K \to \mathbb{R}^d$ is in general position if there is some subdivision K' of K such that f is simplexwise-linear and in general position as a map $K' \to \mathbb{R}^d$.

If K is a finite simplicial complex and $f\colon K \to \mathbb{R}^d$ is a continuous map then, by a simple compactness and perturbation argument, for every $\varepsilon > 0$, there exists a PL-map $g\colon K \to \mathbb{R}^d$ in general position such that $\|f - g\|_\infty \leq \varepsilon$.

General position in PL-manifolds. Defining general position without reference to a particular triangulation and, more generally, for maps into PL-manifolds M other than $\mathbb{R}^d$, is more involved. We follow the presentation [54, Ch. VI], which is very suitable for dealing with r-fold points.

Let $f\colon X \to Q$ be a PL-map from a polyhedron to a PL-manifold. For $r \geq 2$, let us say that a point $x \in X$ is ***r*-singular** if it is the preimage of an r-fold image point y of f, i.e., if $|f^{-1}(f(x))| \geq r$. The **(closed) *r*-singular set** $S_r(f) \subseteq X$ is defined as the closure of the set of r-singular points of f. Each $S_r(f)$ is a subpolyhedron of X ([54, Ch. VI, Lemma 31, p. 19]). The set $S_2(f)$ is also sometimes simply called the **singular set** of f and denoted $S(f)$.

Suppose $\dim X = m$ and $\dim Q = q$. Then a PL-map $f\colon X \to Q$ is said to be **in general position** if $\dim S_r(f) \leq m - (r-1)(q-m)$ for every $r \geq 2$. If $X_0 \subseteq X$ is a subpolyhedron then f is said to be in general position for the pair (X, X_0) if f and $f|_{X_0}$ are both in general position and, if $\dim X_0 < \dim X$ then $\dim(S_r(f) \cap X_0) < m - (r-1)(q-m)$ for every r.

Theorem 2.2.1 ([54, Ch. VI, Theorem 18, p. 27])**.** *Let $f\colon X \to \mathring{Q}$ be a PL-map, $\dim X < \dim Q$, and let $X_0 \subseteq X$ be a subpolyhedron. If $f|_{X_0}$ is in general position then for every $\varepsilon > 0$ there exists a map $g\colon X \to Q$ that is in general position for the pair (X, X_0), and $f \simeq g$ are homotopic through an ε-small homotopy that keeps X_0 fixed.*

We will also need the following version of being in general position with respect to a given polyhedron:

Theorem 2.2.2 ([54, Ch. VI, Theorem 15, p. 7])**.** *Let Q be a PL-manifold of dimension m, and let $X_0 \subseteq X$ and $Y \subseteq Q$ be polyhedra. Given an embedding $f\colon X \to Q$ such that $f(X \setminus X_0) \subseteq \mathring{Q}$, for every $\varepsilon > 0$ there is an embedding $g\colon X \to Q$ such that $g|_{X \setminus X_0}$ is **in general position with respect to Y**, in the sense that*

$$\dim(g(X \setminus X_0) \cap Y) \leq \dim(X \setminus X_0) + \dim Y - \dim Q,$$

and f and g are ambient isotopic through an ε-small ambient isotopy fixing ∂Q and $f(X_0)$.

Transversality. Suppose that $M_1, \ldots, M_r$ are properly embedded PL-submanifolds of a PL-manifold Q, $\dim M_i = m_i$, $1 \leq i \leq r$, and $\dim Q = q$. We say that the M_i are **mutually transverse** (or that they **intersect transversely**) if they locally intersect like r affine subspaces in general position.

More precisely, the M_i intersect transversely at a point $y \in \mathring{Q}$ [respectively, $y \in \partial Q$] if there is a neighborhood N of y in Q and a PL-homeomorphism $h\colon \mathring{N} \cong \mathbb{R}^q$ [respectively, $h\colon N \cong \mathbb{R}^{q-1} \times \mathbb{R}_+$]

[2] Note that this is stronger than requiring that every subset of at most $d+1$ points in S is affinely independent; e.g. the vertices of a regular hexagon are not in general position in the stronger sense.

such that the images $h(M_i \cap \mathring{N})$, $1 \leq i \leq r$, are affine subspaces in general position [respectively, intersections of such subspaces with the upper halfspace $\mathbb{R}^{q-1} \times \mathbb{R}_+$]. The M_i are mutually transverse if they intersect transversely at every $y \in \bigcap_{i=1}^r M_i$. (In particular, if $\bigcap_{i=1}^r M_i \neq \emptyset$, then this implies that $\sum_i m_i \geq d(r-1)$.)

In general, transversality for PL-manifolds is much more subtle than the corresponding theory in the smooth case, see e.g., the discussion in [1].[3]

In the present paper, we will only use the following simple fact: If $M_1, \ldots, M_r$ are pairwise disjoint PL-manifolds, $\dim M_i = m_i$, $\sum_i m_i = d(r-1)$, and if $f\colon M_1 \sqcup \ldots \sqcup M_r \to \mathbb{R}^d$ is a PL-map in general position, then the images $f(\sigma_i)$ are mutually transverse at every r-fold point (necessarily an r-Tverberg point) y of f; indeed, for suitable subdivisions of the M_i on which f is simplexwise linear, there are simplices σ'_i of the subdivisions, $1 \leq i \leq r$, such that the images $f(\sigma'_i)$ are linear m_i-simplices in general position whose relative interiors intersect exactly at y. All operations that we will perform will preserve transversality of the intersections.

2.3 Oriented Intersections and Intersection Signs

In this section, we review the induced orientation on the intersection of oriented simplices in general position in $\mathbb{R}^d$ and the resulting intersection product on piecewise-linear chains (this is a particular case of Lefschetz intersection theory [29]). We first fix the notation and state the basic properties that we will need later (Lemmas 2.3.1 and 2.3.2). The definition and the proofs of the two lemmas, which boil down to elementary linear algebra, are included here for the sake of completeness but are deferred until the end of this subsection, and the reader may wish to skip them at first reading.

Let $\sigma_1, \ldots, \sigma_r$ be oriented simplices or, more generally, convex polyhedra in general position in $\mathbb{R}^d$, $\dim \sigma_i = m_i$, $1 \leq i \leq r$ (see Figure 2.1 for an illustration in the case $r = d = 3$, $m_1 = m_2 = m_3 = 2$).

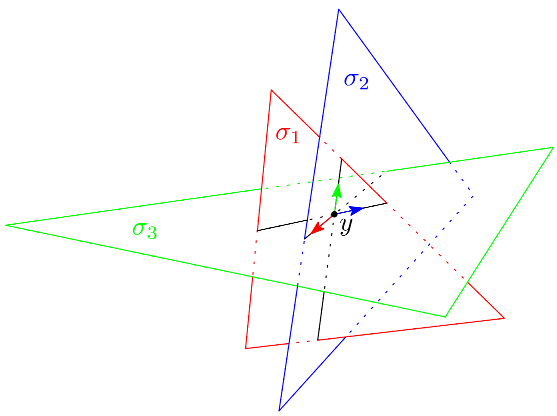


Figure 2.1: Three triangles in general position intersecting at y.

Then the intersection $\bigcap_i \sigma_i$ is either empty or a convex polyhedron of dimension $(\sum_{i=1}^r m_i) - d(r-1)$. In the latter case, given orientations of the ambient space $\mathbb{R}^d$ and of each σ_i, we can define (see Definition 2.3.3 below) an **induced orientation** on

$$\sigma_1 \cap \ldots \cap \sigma_r,$$

which depends on the order of the σ_i and on the choices of the orientations. We will also speak of the **oriented intersection** of the σ_i in $\mathbb{R}^d$, and occasionally write $(\sigma_1 \cap \ldots \cap \sigma_r)_{\mathbb{R}^d}$ to stress

[3] A particularly striking fact is the failure of relative PL-transversality: Hudson [25] showed that for every m, n, q with $m + n?q = 4k$, $m, n \geq 8k + 2$, there are transverse PL-spheres $S^m, S^n \subseteq S^q$ which can not be extended to transverse embeddings of balls $B^{m+1}, B^{n+1} \subseteq B^{q+1}$.

dependence of the orientation on that of the ambient space. If the dimensions satisfy

$$\sum_{i=1}^{r} m_i = d(r-1), \tag{2.2}$$

then the intersection is either empty, or it consists of a single point y that lies in the relative interior of each σ_i, and the induced orientation amounts to associating an **(r-fold) intersection sign** in $\{-1,+1\}$ to y, denoted by

$$\operatorname{sign}_y(\sigma_1,\ldots,\sigma_r),$$

or by $\operatorname{sign}_y^{\mathbb{R}^d}(\sigma_1,\ldots,\sigma_r)$, if we want to stress the ambient space.

The following lemma summarizes several properties that we will need in this paper.

Lemma 2.3.1. *Suppose we have chosen an orientation of $\mathbb{R}^d$, and let $\sigma_1,\ldots,\sigma_r$ be oriented simplices in general position in $\mathbb{R}^d$, $\dim\sigma_i = m_i$, $1\le i\le r$.*

(a) ***Orientation reversal**: Reversing the orientation of one σ_i (denoted by $-\sigma_i$) also reverses the orientation of the intersection,*

$$\sigma_1\cap\ldots\cap\sigma_{i-1}\cap(-\sigma_i)\cap\sigma_{i+1}\ldots\cap\sigma_r = -(\sigma_1\cap\ldots\cap\sigma_r).$$

If we reverse the orientation of $\mathbb{R}^d$ (denoted by $-\mathbb{R}^d$) then the orientation of the intersection changes by a factor of $(-1)^{r-1}$,

$$(\sigma_1\cap\ldots\cap\sigma_r)_{-\mathbb{R}^d} = (-1)^{r-1}(\sigma_1\cap\ldots\cap\sigma_r)_{\mathbb{R}^d}.$$

(b) ***Skew commutativity**: For pairwise oriented intersections,*

$$\sigma_2\cap\sigma_1 = (-1)^{(d-m_1)(d-m_2)}\sigma_1\cap\sigma_2.$$

Thus, in general, if $\pi\in\mathfrak{S}_r$ then

$$\sigma_{\pi(1)}\cap\ldots\cap\sigma_{\pi(r)} = (-1)^{\sum_{(i,j)\in\operatorname{Inv}(\pi)}(d-m_i)(d-m_j)}\sigma_1\cap\ldots\cap\sigma_r,$$

where $\operatorname{Inv}(\pi) := \{(i,j)\in[r]^2 \mid i<j, \pi(i)>\pi(j)\}$ is the set of inversions *of π.*

(c) ***Restriction:** Consider the oriented pairwise intersections $\sigma_1\cap\sigma_2,\ldots,\sigma_1\cap\sigma_r$ as oriented convex subpolytopes of (the affine hull of) σ_1. If we compute the $(r-1)$-fold oriented intersection of these within σ_1, the result is the same as the r-fold oriented intersection of $\sigma_1,\ldots,\sigma_r$ inside $\mathbb{R}^d$,*

$$(\sigma_1\cap\ldots\cap\sigma_r)_{\mathbb{R}^d} = \big((\sigma_1\cap\sigma_2)_{\mathbb{R}^d}\cap\ldots\cap(\sigma_1\cap\sigma_r)_{\mathbb{R}^d}\big)_{\sigma_1}.$$

(d) *Suppose the dimensions satisfy (2.2), i.e., that $\sigma_1\cap\ldots\cap\sigma_r$ consists of a single point y. Then the product $P := \sigma_1\times\ldots\times\sigma_r$ is a convex polytope of dimension $d(r-1)$ that intersects the thin diagonal $\delta_r(\mathbb{R}^d)$ transversely at the point $(y,\ldots,y)\in(\mathbb{R}^d)^r$. Moreover, the orientations of the σ_i determine an orientation of P, and the orientation of $\mathbb{R}^d$ determines orientations of both $(\mathbb{R}^d)^r$ and of $\delta_r(\mathbb{R}^d)$ (see Equation (2.6) below), and with respect to these orientations,*[4]

$$\operatorname{sign}_y^{\mathbb{R}^d}(\sigma_1,\ldots,\sigma_r) = \varepsilon_{d,m_1,\ldots,m_r}\cdot\operatorname{sign}_{(y,\ldots,y)}^{(\mathbb{R}^d)^r}(\sigma_1\times\ldots\times\sigma_r,\delta_r(\mathbb{R}^d)), \tag{2.3}$$

where $\varepsilon_{d,m_1,\ldots,m_r}\in\{-1,+1\}$ is a sign that depends only on the dimensions. In the special case that $d=rk$ and all $m_i=(r-1)k$, $r\ge2$ and $k\ge1$, we abbreviate the notation for the sign to $\varepsilon_{r,k}$, and it is given by

$$\varepsilon_{r,k} = \begin{cases} -1 & \text{if } k \text{ is odd and } r \text{ is } 2 \bmod 4,\\ 1 & \text{otherwise.}\end{cases} \tag{2.4}$$

[4] For $r=2$, this is well-known, and can be found in [41, §3].

Intersections of chains. We will also need to consider oriented intersections and intersection signs for more general geometric objects, in particular for PL-submanifolds of $\mathbb{R}^d$ and for images of such manifolds under PL-maps in general position.

A convenient framework is the following. An m-dimensional **PL-chain** in $\mathbb{R}^d$ is a formal linear combination $c = \sum_j a_j\sigma_j$, where the a_j are integers (only finitely many nonzero) and each σ_j is an m-dimensional convex polyhedron, modulo the relation that $(-a)\sigma = a(-\sigma)$ for integers a and convex polyhedra σ.

Suppose now that $c_1, \ldots, c_r$ are PL-chains in $\mathbb{R}^d$, $\dim c_i = m_i$ and $c_i = \sum_{i,j} a_{ij}\sigma_{ij}$, $1 \le i \le r$, and that the chains are in general position, i.e., $\sigma_{1j_1}, \ldots, \sigma_{rj_r}$ are in general position for any choice of σ_{ij_i} in c_i. Then, by multilinearity, we can define the **oriented intersection** of the chains as the chain

$$c_1 \cap \ldots \cap c_r := \sum_{j_1,\ldots,j_r} \left(\prod_{i=1}^{r} a_{ij_i}\right) \sigma_{1j_1} \cap \ldots \cap \sigma_{rj_r},$$

with the understanding that $\sigma_{1j_1} \cap \ldots \cap \sigma_{rj_r} = 0$ if the intersection is empty.

As indicated above, we are mostly interested in the case where $c_i = f(\sigma_i)$ is the image[5] of an m_i-simplex or, more generally, of an m_i-dimensional PL-manifold σ_i under a a PL-map f in general position (this includes the case that σ_i is a submanifold of $\mathbb{R}^d$, we take f to be the inclusion map).

Note that the dimension of $c_1 \cap \ldots \cap c_r$ equals $\ell := \sum_i m_i - d(r-1)$. In particular, if the dimensions satisfy (2.2), then $\ell = 0$, and the intersection chain is a formal linear combination $\sum_y a_y y$ of points. In this case, we define the **algebraic intersection number** of the chains as the sum

$$c_1 \bullet \ldots \bullet c_r := \sum_y a_y \in \mathbb{Z},$$

where the sum ranges over all r-fold intersection points y in $c_1 \cap \ldots \cap c_r$.

In particular, if all (nonzero) coefficients in the chains c_i are ± 1 (for instance, this happens if each $c = f(\sigma_i)$ is the image of an oriented m_i-dimensional PL-manifold, $m_i < d$) then for each point y in the intersection, its coefficient a_y is ± 1 as well, and we call a_y the **(r-fold) intersection sign** of the chains at y, denoted

$$\operatorname{sign}_y(c_1, \ldots, c_r) \in \{-1, +1\}.$$

Thus, in this case, $c_1 \bullet \ldots \bullet c_r = \sum_y \operatorname{sign}_y(c_1, \ldots, c_r)$.

Even more generally, the intersection product could be defined inside an ambient oriented PL-manifold M (possibly with boundary) instead of $\mathbb{R}^d$; however, we will only need this in the special case that $M = \sigma_1$ is itself a simplex in $\mathbb{R}^d$ (as in Lemma 2.3.1 (c)), in which case we understand the intersection in σ to mean the intersection in the oriented affine subspace spanned by σ_1.

By multilinearity, the properties in Lemma 2.3.1 carry over to chains in a straightforward way.[6]

We will also need the following well-known fact about intersection numbers and boundaries:

Lemma 2.3.2. *Suppose c_1 and c_2 are PL-chains in general position in $\mathbb{R}^d$, $dim(c_i) = m_i$, $i = 1, 2$, and that $m_1 + m_2 = d + 1$. Then $\partial c_1 \bullet c_2 = (-1)^{m_1} c_1 \bullet \partial c_2$.*

We now proceed to review the definition of oriented intersections and prove the two lemmas.

Orientations. Specifying an orientation of an m-dimensional convex polyhedron σ in $\mathbb{R}^d$, $m > 0$, amounts to choosing an ordered basis[7] $B = [b_1|\ldots|b_m] \in \mathbb{R}^{d\times m}$ of the m-dimensional linear subspace $L(\sigma)$ parallel to σ. Given two such bases B and B', there is a unique invertible matrix

[5]More precisely we mean the image chain, i.e., we slightly abuse notation here and use $f(\sigma_i)$ to denote the formal linear combination $\sum_\tau f(\tau)$, where τ ranges over all the m_i-simplices in a subdivision of σ_i on which f is simplexwise-linear, and each τ carries the orientation inherited from that of σ_i; a more precise but more cumbersome notation for this image chain would be $f_\#(\sigma_i)$.

[6]In Part (d) the product of the chains is $c_1 \times \ldots \times c_r := \sum_{j_1,\ldots,j_r} \left(\prod_{i=1}^r a_{ij_i}\right) \sigma_{1j_1} \times \ldots \times \sigma_{rj_r}$.

[7]Here, we think of an ordered basis B as a $(d \times m)$-matrix, whose columns are the basis vectors.

$R \in \mathbb{R}^{m\times m}$ with $B' = BR$, and we say that B' and B define the same or the opposite orientation of σ, denoted by $B' \sim B$ or $B' \sim \mathrm{op}(B)$, respectively, depending on whether $\det(R)$ is positive or negative. Equivalently, we can view orientations in terms of exterior algebra. Given a basis B, consider the decomposable nonzero vector $\beta = b_1 \wedge \ldots \wedge b_m \in \bigwedge^m \mathbb{R}^d$. For two bases B and B', the corresponding exterior products satisfy $\beta' = \det(R)\cdot\beta$, and we will write $\beta' \sim \beta$ or $\beta' \sim -\beta$ depending on whether β' and β differ by a positive or negative factor.

If $m = 0$, i.e., if σ is a point, then an orientation is given by a sign in $\{-1, +1\}$ assigned to that point, and $\beta \in \bigwedge^0 \mathbb{R}^d \cong \mathbb{R}$ is just a nonzero scalar.

Note also that if $\tau \subseteq \sigma$ is a convex subpolyhedron of dimension ℓ, then for any orientation $\alpha \in \bigwedge^\ell \mathbb{R}^d$ of τ, we can choose[8] $\gamma \in \bigwedge^{m-\ell} \mathbb{R}^d$ such that $\alpha \wedge \gamma$ is an orientation of σ.

Moreover, the orientation of the boundary $\partial\sigma$ is given as follows: Let τ be a facet of σ, let $v = q - p \in \mathbb{R}^d$ be a vector connecting a point p in the relative interior of σ to a point $q \in \tau$ (we can think of v as pointing "outwards" from σ at τ), and let $\alpha \in \bigwedge^{m-1} \mathbb{R}^d$ be any orientation of τ. Then the orientation of τ in $\partial\sigma$ is given by $\pm\alpha$ depending on whether $v \wedge \alpha$ determines the chosen orientation of σ or its opposite.

Definition 2.3.3. Let $r \geq 2$, and let $\sigma_1, \ldots, \sigma_r$ be convex polyhedra in general position in $\mathbb{R}^d$, $m_i := \dim L_i$, $1 \leq i \leq r$. Suppose we have also chosen an orientation of $\mathbb{R}^d$.

If $\sigma_1 \cap \ldots \cap \sigma_r = \emptyset$, we consider the oriented intersection to be formally zero.

Else, $\sigma_1 \cap \ldots \cap \sigma_r$ is a convex polyhedron of dimension $\ell := (\sum_{i=1}^r m_i) - d(r-1) \geq 0$, by general position, and we proceed as follows:

(i) In the case $r = 2$ of pairwise intersections, choose an arbitrary orientation $\alpha \in \bigwedge^\ell \mathbb{R}^d$ of $\sigma_1 \cap \sigma_2$, and choose $\beta_i \in \bigwedge^{m_i-\ell} \mathbb{R}^d$ such that $\alpha \wedge \beta_i$ determines the chosen orientation of σ_i, $i = 1, 2$. Then the **induced orientation** on $\sigma_1 \cap \sigma_2$ is given by α or $-\alpha$, respectively, depending on whether $\alpha \wedge \beta_1 \wedge \beta_2 \in \bigwedge^d \mathbb{R}^d$ determines the chosen orientation of $\mathbb{R}^d$ or the opposite one.[9] The convex polyhedron $\sigma_1 \cap \sigma_2$ with this induced orientation is called the **oriented intersection** of σ_1 and σ_2.

(ii) In general, the **oriented intersection** of $\sigma_1, \ldots, \sigma_r$ is defined inductively by

$$\sigma_1 \cap \ldots \cap \sigma_r := (\sigma_1 \cap \ldots \cap \sigma_{r-1}) \cap \sigma_r. \tag{2.5}$$

(By Lemma 2.3.5 below, we can ignore the parentheses and take the intersections in any order.)

Remark 2.3.4. One can unravel the inductive definition (2.5) as follows: Choose an orientation $\alpha \in \bigwedge^\ell \mathbb{R}^d$ for $\sigma_1 \cap \ldots \cap \sigma_r$, and extend it by $\gamma_i \in \bigwedge^{d-m_i} \mathbb{R}^d$ to *some* orientation $\alpha \wedge \gamma_i$ of $\bigcap_{j\neq i} \sigma_j$, $1 \leq i \leq r$ (not necessarily the induced orientation). By general position, this determines signs $\varepsilon, \varepsilon_1, \ldots, \varepsilon_r \in \{-1, +1\}$ such that $\varepsilon\alpha \wedge \gamma_r \wedge \ldots \wedge \gamma_1 \in \bigwedge^d \mathbb{R}^d$ yields the chosen orientation of $\mathbb{R}^d$, and $\varepsilon_i\alpha \wedge \gamma_r \wedge \ldots \wedge \widehat{\gamma_i} \wedge \ldots \wedge \gamma_1 \in \bigwedge^{m_i} \mathbb{R}^d$ yields the chosen orientation of σ_i, where the notation "$\widehat{\gamma_i}$" means that the factor γ_i is omitted. Then the induced orientation of $\sigma_1 \cap \ldots \cap \sigma_r$ is given by $\varepsilon^{r-1} (\prod_{i=1}^r \varepsilon_i)\, \alpha$.

Proof. For $r = 2$, this follows immediately from Definition 2.3.3 (i). For $r \geq 3$, let $\alpha' = \alpha \wedge \gamma_r$. Then, by assumption, $\varepsilon_i\alpha' \wedge \gamma_{r-1} \wedge \ldots \wedge \widehat{\gamma_i} \wedge \ldots \wedge \gamma_1$ yields the chosen orientation of σ_i, $1 \leq i < r$, and $\varepsilon\alpha' \wedge \gamma_{r-1} \wedge \ldots \wedge \gamma_1$ yields that of $\mathbb{R}^d$. Thus, by induction, $\sigma_1 \cap \ldots \cap \sigma_{r-1}$ is oriented by $\varepsilon'\alpha' = \varepsilon'\alpha \wedge \gamma_r$, where $\varepsilon' = \varepsilon^{r-2} \left(\prod_{i=1}^{r-1} \varepsilon_i\right)$. Moreover, σ_r is oriented by $\varepsilon_r\alpha \wedge \gamma_{r-1} \wedge \ldots \wedge \gamma_r$, so $(\sigma_1 \cap \ldots \cap \sigma_{r-1}) \cap \sigma_r$ is oriented by $\varepsilon\varepsilon'\varepsilon_r\alpha = \varepsilon^{r-1} (\prod_{i=1}^r \varepsilon_i)\, \alpha$. □

[8] Write $\alpha = a_1 \wedge \ldots \wedge a_\ell$ for some basis $A = [a_1|\ldots|a_\ell] \in \mathbb{R}^{d\times\ell}$ of $L(\tau)$, choose $C = [c_1|\ldots|c_{m-\ell}]$ such that $[A|C]$ is a basis of $L(\sigma)$, and set $\gamma = c_1 \wedge \ldots \wedge c_{m-\ell}$.

[9] It is routine to check that this does not depend on the choice of α or of the β_i. Indeed, if we chose a different orientation $\alpha' \sim \varepsilon\alpha$ for $\sigma_1 \cap \sigma_2$, $\varepsilon \in \{-1, +1\}$ then for any choice of corresponding "complementary" β_i', we have $\beta_i' \sim \varepsilon\beta_i$ and hence $\alpha' \wedge \beta_1'\beta_2' \sim \varepsilon\alpha\beta_1\beta_2$.

Lemma 2.3.5 (Associativity). *If $\sigma_1, \sigma_2, \sigma_3$ are oriented simplices in general position in $\mathbb{R}^d$ then we can take oriented pairwise intersections in any order and get the same induced orientation,*

$$(\sigma_1 \cap \sigma_2) \cap \sigma_3 = \sigma_1 \cap (\sigma_2 \cap \sigma_3).$$

Proof of Lemma 2.3.5 and of Lemma 2.3.1 (a)–(c). We may assume that $\sigma_1 \cap \ldots \cap \sigma_r \neq \emptyset$, else all properties are trivially satisfied. Moreover, Lemma 2.3.1 (a) and (b) follow directly from the definition.

We proceed to prove Lemma 2.3.5 and Lemma 2.3.1 (c) at the same time. We use the notation from Remark 2.3.4 (applied with $r = 3$). By Lemma 2.3.1 (a), both equations we want to establish are invariant under reversing the orientations of some σ_i or of $\mathbb{R}^d$, so we may assume that the signs ε and ε_i, $1 \leq i \leq 3$, are all equal to $+1$. That is, we may assume that $\mathbb{R}^d$ is oriented by $\alpha \wedge \gamma_3 \wedge \gamma_2 \wedge \gamma_1$, and that $\alpha \wedge \gamma_3 \wedge \gamma_2$, $\alpha \wedge \gamma_3 \wedge \gamma_1$, and $\alpha \wedge \gamma_2 \wedge \gamma_1$ determine the chosen orientations of σ_1, σ_2, and σ_3, respectively.

It follows directly from the definition that the induced orientation of $\sigma_1 \cap \sigma_2$ is given by $\alpha \wedge \gamma_3$, and that of $(\sigma_1 \cap \sigma_2) \cap \sigma_3$ is given by α.

Moreover, $\alpha \wedge \gamma_3 \wedge \gamma_1 \sim \alpha \wedge \gamma_1 \wedge (-1)^{(d-m_1)(d-m_3)} \gamma_3$, $\alpha \wedge \gamma_2 \wedge \gamma_1 \sim \alpha \wedge \gamma_1 \wedge (-1)^{(d-m_1)(d-m_2)} \gamma_2$, and $\alpha \wedge \gamma_3 \wedge \gamma_2 \wedge \gamma_1 \sim \alpha \wedge \gamma_1 \wedge (-1)^{(d-m_1)(d-m_3)} \gamma_3 \wedge (-1)^{(d-m_1)(d-m_2)} \gamma_2$. Thus, again applying the definition, the orientation of $\sigma_2 \cap \sigma_3$ is given by $\alpha \wedge \gamma_1$, and hence that of $\sigma_1 \cap (\sigma_2 \cap \sigma_3)$ by A, which proves Lemma 2.3.5.

Similarly, the orientation of $\sigma_1 \cap \sigma_3$ is given by $\alpha \wedge \gamma_2$ since $\alpha \wedge \gamma_3 \wedge \gamma_2 \sim \alpha \wedge \gamma_2 \wedge (-1)^{(d-m_2)(d-m_3)} \gamma_3$ and $\alpha \wedge \gamma_3 \wedge \gamma_2 \wedge \gamma_1 \sim \alpha \wedge \gamma_2 \wedge (-1)^{(d-m_3)(d-m_2)} \gamma_3 \wedge \gamma_1$. Therefore, the orientation of

$$\big((\sigma_1 \cap \sigma_2)_{\mathbb{R}^d} \cap (\sigma_1 \cap \sigma_3)_{\mathbb{R}^d}\big)_{\sigma_1}$$

is given by A as well, which proves Lemma 2.3.1 (c). □

Proof of Lemma 2.3.1 (d). Suppose the orientation of σ_i is given by $B_i \in \mathbb{R}^{d \times m_i}$, $1 \leq i \leq r$, and that of $\mathbb{R}^d$ by $B \in \mathbb{R}^{d \times d}$. Then the orientations of $P := \sigma_1 \times \ldots \times \sigma_r$, of the thin diagonal $\delta_r(\mathbb{R}^d)$, and of $(\mathbb{R}^d)^r$, respectively, are given by matrices $M_P \in \mathbb{R}^{dr \times d(r-1)}$, $M_\delta \in \mathbb{R}^{dr \times d}$, and $M \in \mathbb{R}^{dr \times dr}$, where

$$M_P = \begin{bmatrix} B_1 & 0 & \cdots & 0 \\ 0 & B_2 & \cdots & 0 \\ \vdots & & \ddots & \\ 0 & 0 & \cdots & B_r \end{bmatrix}, \quad M_\delta = \begin{bmatrix} B \\ B \\ \vdots \\ B \end{bmatrix}, \quad \text{and} \quad M = \begin{bmatrix} B & 0 & \cdots & 0 \\ 0 & B & \cdots & 0 \\ \vdots & & \ddots & \\ 0 & 0 & \cdots & B \end{bmatrix}. \tag{2.6}$$

The pairwise intersection sign $\text{sign}_{(y,\ldots,y)}(P, \delta_r(\mathbb{R}^d))$ equals ± 1 depending on whether the determinants of $[M_P|M_\delta]$ and of M have the same or the opposite sign, i.e.,

$$\text{sign}\det[M_P|M_\delta] = \text{sign}_{(y,\ldots,y)}(P, \delta_r(\mathbb{R}^d)) \cdot \text{sign}\det M.$$

Note that reversing the orientation of one σ_i reverses the orientation of P, and reversing the orientation of $\mathbb{R}^d$ reverses the orientation of $\delta_r(\mathbb{R}^d)$ and changes the orientation of $(\mathbb{R}^d)^r$ by a factor of $(-1)^r$. Therefore, by Lemma 2.3.1 (a), Equation (2.3) is invariant under such orientation reversals. Thus, we can proceed similarly to Remark 2.3.4, choose bases $C_i \in \mathbb{R}^{d \times (d-m_i)}$ of $L(\bigcap_{j \neq i} \sigma_j)$, $1 \leq i \leq r$, and we may assume that $B = [C_r|\ldots|C_1]$ and $B_i = [C_r|\ldots|\widehat{C}_i|\ldots|C_1]$. Hence,

$$\text{sign}_y(\sigma_1, \ldots, \sigma_r) = +1.$$

Moreover,

$$[M_P|M_\delta] = \begin{bmatrix} [C_r|\ldots|C_2] & 0 & \cdots & 0 & [C_r|\ldots|C_1] \\ 0 & [C_r|\ldots|C_3|C_1] & \cdots & 0 & [C_r|\ldots|C_1] \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & [C_{r-1}|\ldots|C_1] & [C_r|\ldots|C_1] \end{bmatrix}$$

By subtracting columns from one another (which does not change the orientation class), we can bring $[M_P|M_\delta]$ into the form

$$\begin{bmatrix} [C_r|\dots|C_2] & 0 & \cdots & 0 & [\,0\,|\dots|\,0\,|C_1] \\ 0 & [C_r|\dots|C_3|C_1] & \cdots & 0 & [\,0\,|\dots|C_2|\,0\,] \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots & [C_{r-1}|\dots|C_1] & [C_r|\,0\,|\dots|\,0\,] \end{bmatrix},$$

and this matrix can be transformed into

$$\begin{bmatrix} [C_r|\dots|C_1] & 0 & \cdots & 0 \\ 0 & [C_r|\dots|C_1] & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & [C_r|\dots|C_1] \end{bmatrix} = M;$$

by a sequence of $t_{d,m_1,\dots,m_r} := \sum_{i=1}^{r}(r-i)d(d-m_i) \;+ \sum_{1\le i<j\le r}(d-m_i)(d-m_j)$ column transpositions, which proves (2.3) if we set

$$\varepsilon_{d,m_1,\dots,m_r} := (-1)^{t_{d,m_1,\dots,m_r}}. \tag{2.7}$$

In the special case that $m_i = m = (r-1)k$ and $d = rk$, $k \ge 1$, the number of transpositions equals

$$t_{r,k} := d(d-m)\binom{r}{2} + (d-m)^2\binom{r}{2} = \frac{(r-1)r(r+1)k^2}{2},$$

and it is easy to verify that setting $\varepsilon_{r,k} := (-1)^{t_{r,k}}$ yields (2.4). □

Proof of Lemma 2.3.2. By multilinearity, it suffices to prove the formula for simplices σ_1, σ_2 in general position in $\mathbb{R}^d$, $\dim(\sigma_i) = m_i$, $m_1 + m_2 = d+1$, $\sigma_1 \cap \sigma_2 \neq \emptyset$. By general position, $\sigma_1 \cap \sigma_2$ is a line segment with endpoints $p \in \partial\sigma_1 \cap \sigma_2$ and $q \in \sigma_1 \cap \partial\sigma_2$, where p lies in the relative interior of σ_1 and of some facet τ_2 of σ, and q lies in the relative interiors of σ_2 and some facet τ_1 of σ_1, see Figure 2.2. We need to show that $\operatorname{sign}_q(\tau_1,\sigma_2) = (-1)^{m_1}\operatorname{sign}_p(\sigma_1,\tau_2)$.

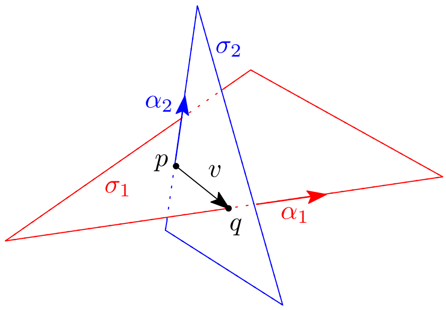


Figure 2.2: Two triangles in general position in $\mathbb{R}^3$.

Suppose the orientation of σ_i is given by $\beta_i \in \bigwedge^{m_i}\mathbb{R}^d$ and the orientation of τ_i in $\partial\sigma_i$ is given by $\alpha_i \in \bigwedge^{m_i-1}\mathbb{R}^d$, $i = 1,2$, and that $\mathbb{R}^d$ is oriented by $\beta \in \bigwedge^d \mathbb{R}^d$. Then, by definition, the intersection signs $\operatorname{sign}_q(\tau_1,\sigma_2)$ and $\operatorname{sign}_p(\sigma_1,\tau_2)$ are determined by

$$\beta \sim \operatorname{sign}_q(\tau_1,\sigma_2)\cdot\alpha_1\wedge\beta_2 \sim \operatorname{sign}_p(\sigma_1,\tau_2)\cdot\beta_1\wedge\alpha_2 \tag{2.8}$$

Let $v := q - p$. Then, by definition of the orientation of the boundary, $\beta_1 \sim v\wedge\alpha_1$ and $\beta_2 \sim (-v)\wedge\alpha_2$. It follows that $\beta_1\wedge\alpha_2 \sim v\wedge\alpha_1\wedge\alpha_2 \sim (-1)^{m_1}\alpha_1\wedge(-v)\wedge\alpha_2 \sim (-1)^{m_1}\alpha_1\wedge\beta_2$. □

Chapter 3

r-Embeddings of PL-Manifolds

3.1 Introduction

In his famous 1934 paper, Whitney [52] showed that any smooth m-dimensional manifold M^m can be embedded inside $\mathbb{R}^{2m}$. This result is sometimes called "Embedding in Double Dimension Theorem".

Here, we prove an analogous result for *r-embeddings* of PL-manifolds. An **r-embedding** of a PL-manifold M^m to $\mathbb{R}^d$ is a PL-map $f : M^m \to \mathbb{R}^d$ in general position and whose **r-singular set**

$$\Sigma_r(f) = \{x \in M^m \mid |f^{-1}fx| \geq r\},$$

is empty. (Here $|\cdot|$ denotes the cardinality of a set).

If $p \in \mathbb{R}^d$ has more that r preimages by f (i.e., $|f^{-1}(p)| \geq r$), then we say that p is an **r-intersection point** of f.

If M is oriented, then any r-intersection point p of f has an associated **r-intersection sign**, denoted $\text{sign}(p)$.

Our goal here is to prove:

Theorem 3.1.1 (Whitney r-embedding Theorem)**.** *Let $k \geq 3$ and $r \geq 2$. For any $(r-1)k$-dimensional PL-manifold $M^{(r-1)k}$, there exists an r-embedding*

$$f \colon M^{(r-1)k} \to \mathbb{R}^{rk}.$$

Remark 3.1.2.

- For $r = 2$, we recover Whitney's Embedding in Double Dimension (albeit with a codimension 3 requirement). It would be interesting to know if our result holds for smaller codimensions (i.e., $k < 3$).
- In the statement of the Theorem, the target manifold $\mathbb{R}^{rk}$ can be replaced by any 1-connected rk-dimensional PL-manifold. (This will be obvious from the proof.)

An elementary proof of Theorem 3.1.1 can be constructed by following closely the "standard" Penrose–Whitehead–Zeeman proof of the Embedding in Double Dimension Theorem for the PL category [38, Thm 5.5, p. 63]. Here, we make a detour to prove an interesting Lemma:

Lemma 3.1.3. *Let $k \geq 3$ and $r \geq 2$. Let $M^{(r-1)k}$ be a $(r-1)k$-dimensional* connected *manifold, and let $f : M^{(r-1)k} \to \mathbb{R}^{rk}$ be a PL-map in general position.*

(a) ***Removing an r-intersection point**: Let $p \in \mathbb{R}^{rk}$ be an r-intersection point of f. Then there exists $g \colon M^{(r-1)k} \to \mathbb{R}^{rk}$ in general position and with*

$$\Sigma_r(g) = \Sigma_r(f) \setminus f^{-1}(p).$$

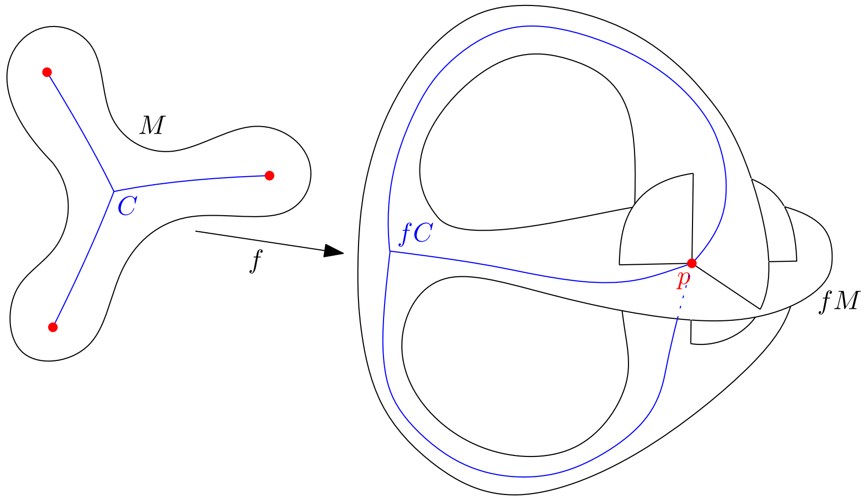


Figure 3.1: A triple intersection point p and a "cone" C over its preimages.

(b) ***Adding an*** r***-intersection point:*** *Let* $p \in \mathbb{R}^{rk} \setminus fM$*. There exists* $g : M^{(r-1)k} \to \mathbb{R}^{rk}$ *in general position with* $|g^{-1}(p)| = r$ *and with*

$$\Sigma_r(g) = \Sigma_r(f) \sqcup g^{-1}(p).$$

Furthermore, if M *is orientable, then the* r*-intersection sign associated to* p *can be prescribed*

The proof of Lemma 3.1.3 is presented in the next section.

Corollary 3.1.4. *Let* $k \geq 3$ *and* $r \geq 2$*, and let* $M^{(r-1)k}$ *be an oriented PL-manifold.*
For any $z \in \mathbb{Z}$*, there exists a PL-map in general position* $f\colon M^{(r-1)k} \to \mathbb{R}^{rk}$*, such that the number of* r*-intersection points of* f *(counted with intersection sign) is* z*.*

Proofs of Theorem 3.1.1 and Corollary 3.1.4. Immediate by Lemma 3.1.3. ☐

Corollary 3.1.4 belongs to PL topology, and contrasts with the following result about *smooth* immersions of Lashof and Smale:

Theorem 3.1.5 ([28, Corollary A])**.** *Let* M^{2k} *be a smooth closed oriented* m*-dimensional manifold, then the number of triple intersection points of an immersion* $M^{2k} \to \mathbb{R}^{3k}$ *(counted with intersection sign) is independent of the immersion.*

3.2 Proof of Lemma 3.1.3

We will need the following result, an instance of a technique called *engulfing*,

Theorem 3.2.1 (Zeeman's Engulfing, [54, Ch. VII, Thm 20])**.** *Let* M *be an* m*-dimensional* k*-connected PL-manifold with* $k \leq m-3$*. Let* X *be a compact* x*-dimensional subpolyhedron in the interior of* M*. If* $x \leq k$*, then there exists a collapsible subpolyhedron* C *in the interior of* M *with*

$$X \subseteq C \quad \text{and} \quad \dim(C) \leq x+1.$$

We now start the proof of Lemma 3.1.3.

Proof of Part (a). *Following [38, Thm 5.5, p. 63]:*

The first part of the construction is illustrated in Figure 3.1.

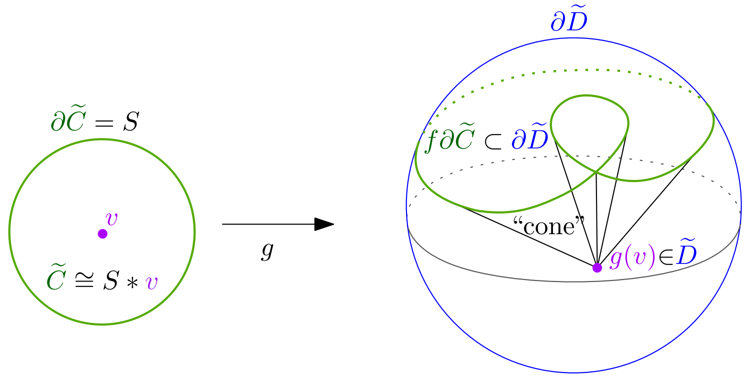


Figure 3.2: On the left, we decompose $\widetilde{C}$ as the join of a boundary sphere S and a vertex v. On the right $\widetilde{D}$ is similarly decomposed as $\partial\widetilde{D}$ and a vertex $g(v)$. We then use these two decompositions to define the map $g : S * v \to \partial\widetilde{D} * g(v)$ as a "cone".

Using Theorem 3.2.1, we find in M a collapsible 1-subpolyhedron C containing the set of isolated points $f^{-1}(p)$. By general position[1], we can furthermore assume that $C \setminus f^{-1}(p)$ is disjoint from $\Sigma_2(f)$.

Using a second time Theorem 3.2.1, we find in $\mathbb{R}^{rk}$ a collapsible 2-subpolyhedron D containing $f(C)$. By general position[2], we can assume that $f^{-1}(D) = C$.

Let us choose triangulations for M and $\mathbb{R}^{rk}$ such that C, fC and D are subcomplexes, and f is simplicial.

Let us derive these triangulations twice (to ensure that C, fC and D are full subcomplexes).

Using these new triangulations of M and $\mathbb{R}^{rk}$, we construct *simplicial neighborhoods* (see definition [38, Ch. 3, p. 32])

$$\widetilde{C} \text{ of } C \text{ in } M, \quad \text{and} \quad \widetilde{D} \text{ of } D \text{ in } \mathbb{R}^{rk}.$$

By construction, $\widetilde{C}$ and $\widetilde{D}$ are *regular neighborhoods* of C in M and D in $\mathbb{R}^{rk}$ (see [38, Ch. 3, p. 33]). Furthermore, since both C and D are collapsible, $\widetilde{C}$ and $\widetilde{D}$ are balls.

By construction,

$$f\partial\widetilde{C} \subset \partial\widetilde{D} \quad \text{and} \quad f\widetilde{C} \subset \widetilde{D}.$$

We define $g : M \to \mathbb{R}^{rk}$ as follows:

- On $M \setminus \widetilde{C}$: We simply define $g = f$.
- On $\widetilde{C}$ (see Figure 3.2): We decompose the $(r-1)k$-ball $\widetilde{C}$ as the joint of a $((r-1)k-1)$-sphere S and a point v:

  $$\widetilde{C} = S * v.$$

  Then, we define that g maps $v \in \widetilde{C}$ to a generic point $g(v)$ in the interior of $\widetilde{D}$, and we extend the map g on $\widetilde{C} \setminus \{v\}$ by the *cone construction*[3], using that g is already defined on $\partial\widetilde{C} = S$.

By construction, $g|_{\widetilde{C}}$ has *no* r-intersection points. This concludes the proof of Part (a).

[1] More precisely, by applying Theorem 2.2.2 with $X_0 = f^{-1}p$, $X = C$ and $Y = \Sigma_2(f)$. Sometimes, this is called "shifting C in general position relative to $f^{-1}p$".

[2] Again, we use Theorem 2.2.2 with $X_0 = f(C)$, $X = D$, $Y = f(M)$.

[3] More precisely, we represent $\widetilde{D}$ as the join $\partial\widetilde{D} * g(v)$. The image of a point $\lambda c + (1-\lambda)v \in \widetilde{C} * v$ (with $\lambda \in [0,1]$) is then

$$g(\lambda c + (1-\lambda)v) := \lambda g(c) + (1-\lambda)g(v) \in \partial\widetilde{D} * g(v),$$

see [38, Ex. 1.6.(3), p. 5]

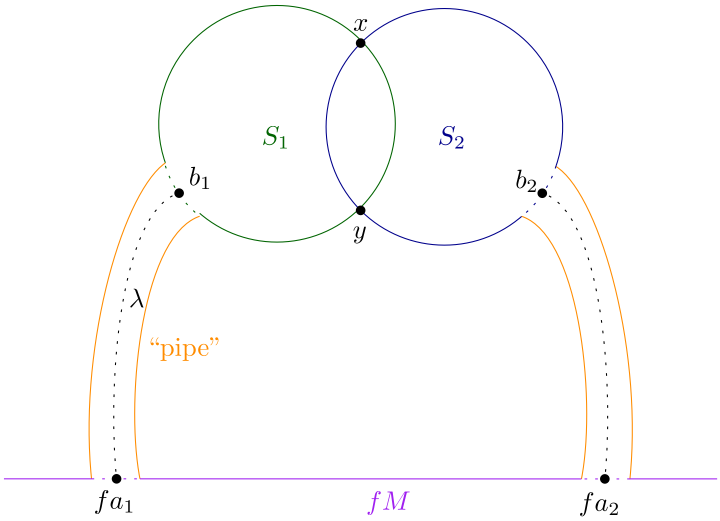


Figure 3.3: Construction for $r = 2$: Two k-spheres S_1 and S_2 are intersecting in two points x, y in $\mathbb{R}^{2k}$. By piping each sphere to fM, we add two new intersection points (of opposite signs) to fM. Sometimes this construction is referred to as an "anti-Whitney trick".

Proof of Part (b).

The trick is to add two new points of opposite signs (i.e., to perform an "anti-Whitney trick"), and then to use Part (a) to remove *one* of the newly added points. We illustrate the construction (for $r = 2$) in Figure 3.3. The first step is the following claim (which is obvious):

Claim 3.2.2. *There exists in $\mathbb{R}^{rk} \setminus fM$ a collection of r PL-spheres of dimension $(r-1)k$ in general position*

$$S_1, \dots, S_r$$

such that there intersection

$$S_1 \cap \cdots \cap S_r$$

consists of two points x, y of opposite r-intersection signs (once an orientation for each S_i is chosen).

We can furthermore assume that $y = p$.

Using this collection of spheres, we define g as follows:

- We select r generic points $a_1, \dots, a_r \in M$.
- We select r generic points $b_1 \in S_1, \dots, b_r \in S_r$.
- We define g as the modified version of f obtained by *piping*[4] $fa_i \in fM$ to $b_i \in S_i$, for all $r = 1, \dots, r$.

Note that during the piping step, we can preserve any chosen orientations on M and on the S_i (see [38, Piping, p. 67-68])

The resulting map $g : M \to \mathbb{R}^{rk}$ has two more r-intersection points x and y.

We then use the already proven Part (a) of Lemma 3.1.3 to remove the r-intersection point x of g. The resulting map has the required properties. This concludes the proof of Part (b). □

[4] More precisely, we select a generic path λ from fa_i to b_i in $\mathbb{R}^{rk}$. We then remove a small neighborhood A of $fa_i \subset fM$ and B of $b_i \subset S_i$, and run a "pipe" $S^{(r-1)k-1} \times \lambda$ from ∂A to ∂B. (I.e., we geometrically construct a connected sum).

Chapter 4

An r-fold Whitney Trick

4.1 Introduction

Our goal in this Chapter is to prove Theorem 1.2.3 from the Introduction, which we restate here for convenience:

Theorem 4.1.1 (**Higher-Multiplicity Whitney Trick**). *Let $r \geq 2$, and let $M_1, \dots, M_r$ be connected, orientable PL-manifolds, of respective dimensions* $\dim M_i = m_i$*, such that*

$$\sum_{i=1}^{r} m_i = d(r-1) \quad \textit{and} \quad d - m_i \geq 3, \quad \textit{for } 1 \leq i \leq r. \tag{4.1}$$

Let

$$f : M_1 \sqcup \cdots \sqcup M_r \to \mathbb{R}^d$$

be a PL-map in general position defined on their disjoint union, and suppose that

$$x, y \in fM_1 \cap \cdots \cap fM_r$$

are two r-fold points of opposite intersection signs.

Then there exist $r-1$ ambient PL-isotopies $H^2, \dots, H^r$ of $\mathbb{R}^d$ such that

$$fM_1 \cap H_1^2(fM_2) \cap \cdots \cap H_1^r(fM_r) = \big(fM_1 \cap fM_2 \cap \cdots \cap fM_r\big) \setminus \{x, y\}$$

Moreover, these isotopies can be chosen to be local*, in the following sense: Given any closed polyhedron $L \subset \mathbb{R}^d$ of dimension $\ell \leq d-3$ and with $x, y \notin L$, there exists a PL-ball $B^d \subset \mathbb{R}^d$ disjoint from L such that H^i is fixed outside of $\mathring{B}^d$, $2 \leq i \leq r$.*

Remark 4.1.2. The condition $\sum_{i=1}^{r} m_i = d(r-1)$ can be written as $\sum_{i=1}^{r}(d - m_i) = d$, i.e., the sum of the codimensions is equal to the dimension of the ambient space $\mathbb{R}^d$ (this, in turn, implies that the r-fold intersections are isolated points). Using $d - m_i \geq 3$, we also get $d \geq 3r$.

The proof is by induction on r. The base case $r = 2$ is the PL version of the Whitney Trick (see Weber [49]).

Thus, inductively, we may assume that $r \geq 3$ and that the theorem holds for $r-1$. We proceed in three steps, each of which is explained in detail in the corresponding section.

4.2 We show how we can restrict ourselves to a standard local situation, in which m_i-dimensional balls σ_i properly contained in a d-ball B^d, $1 \leq i \leq r$, intersect in precisely two r-intersection points x and y of opposite signs.

4.3 If we restrict ourselves to the sub-ball $\sigma_1 \subseteq B^d$, then x and y, seen as $(r-1)$-intersection points between $\sigma_1 \cap \sigma_2, \ldots, \sigma_1 \cap \sigma_r$ inside the m_i-ball σ_1, still have opposite signs. Moreover, we show that we can modify each $\sigma_1 \cap \sigma_i$, $2 \leq r \leq r$, by an ambient isotopy of B^d (which corresponds to performing a pair of complementary ambient surgeries on σ_i) so that the pairwise intersections $\sigma_1 \cap \sigma_i$ become connected.

4.4 Inductively, we remove the $(r-1)$-intersection points between $\sigma_1 \cap \sigma_2, \ldots, \sigma_1 \cap \sigma_r \subseteq \sigma_1$ by ambient isotopies of σ_1 and then extend these to ambient isotopies of B^d, using that σ_1 is unknotted in B^d, so that $B^d \cong \sigma_1 * S^{d-n_1-1}$ (this "unknottedness" always occurs in codimension ≥ 3).

4.2 Reduction to a Standard Local Situation

The first step of the proof of Theorem 4.1.1 is to reduce the problem to the following local situation:

Definition 4.2.1. We say that $B \subset \mathbb{R}^d$ and $\sigma_1, \ldots, \sigma_r \subset B$ form a **standard local situation** around two r-fold points x, y if the following properties are satisfied:

1. $B \subset \mathbb{R}^d$ is a d-dimensional PL-ball, with x, y in the interior $\mathring{B}$.
2. For $1 \leq i \leq r$, σ_i is an m_i-dimensional PL-ball properly embedded (see Section 2.1) into B, with
$$\sum_{i=1}^{r} m_i = d(r-1). \tag{1.5}$$
3. $\sigma_1, \ldots, \sigma_r$ are mutually transverse (see Section 2.1), $\sigma_1 \cap \ldots \cap \sigma_r = \{x, y\}$, and for each index set $J \subseteq \{1, \ldots, r\}$ with $|J| \geq 2$, $\bigcap_{j \in J} \sigma_j$ is the disjoint union of two PL-balls $B_{J,x} \ni x$ and $B_{J,y} \ni y$ (each properly embedded in B and of dimension $d - \sum_{j \in J}(d - m_j)$, by transversality).

Lemma 4.2.2 (Reduction to a standard local situation). *Let $M_1, \ldots, M_r$ be connected PL-manifolds (possibly with boundary) of respective dimensions $\dim M_i = m_i$, $1 \leq i \leq r$, such that $\sum_{i=1}^{r} m_i = d(r-1)$ and*
$$d - m_i \geq 3, \qquad 1 \leq i \leq r. \tag{1.6}$$
Suppose that $f \colon M_1 \sqcup \ldots \sqcup M_r \to \mathbb{R}^d$ is a PL-map in general position defined on the disjoint union of the M_i, and let
$$x, y \in f(M_1) \cap \ldots \cap f(M_r)$$
be two r-fold points of f.

Then there exists a d-dimensional PL-ball $B \subset \mathbb{R}^d$ such that B and $\sigma_i := f(M_i) \cap B$, $1 \leq i \leq r$, form a standard local situation around x and y.

Moreover if $L \subseteq \mathbb{R}^d$ is any compact polyhedron of dimension at most $d-3$ and disjoint from x and y then we can choose B to be disjoint from L.

Furthermore, if B' is a d-dimensional PL ball such that $x, y \in \mathring{B}'$ and x and y lie in the same connected component of $f(M_i) \cap \mathring{B}'$, $1 \leq i \leq r$, then we can choose B to be contained in $\mathring{B}'$.

Proof. For each i, let us use the abbreviation S_{M_i} for the closed singular set of $f|_{M_i}$ (see Section 2.1), so that $f(S_{M_i})$ is the closure of the set of double points of $f|_{M_i}$. Since f is in general position, the images $f(M_i)$ intersect transversely at x and at y, each pairwise intersection $f(M_i) \cap f(M_j)$ has dimension $m_i + m_j - d$, and $f(S_{M_i})$ has dimension at most $2m_i - d$ and is at positive distance from x and y.

For each i, we choose a PL-path $\lambda_i \subseteq f(M_i)$ connecting x and y. By choosing λ_i to be in general position within $f(M_i)$, we can guarantee that λ_i intersects the other $f(M_j)$, $j \neq i$, only in x and y, and that λ_i is disjoint from $f(S_{M_i})$, see Figure 4.1; here, we use that, by (1.6), both $f(M_i) \cap f(M_j)$ and $f(S_{M_i})$ have codimension at least 3 within $f(M_i)$ (in fact, codimension 2 would be enough).

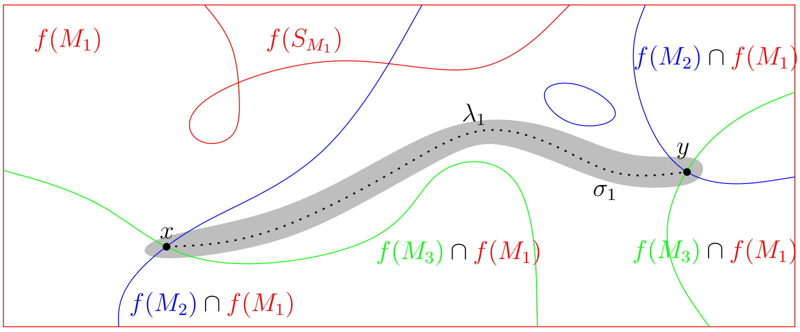


Figure 4.1: On $f(M_1)$, the path λ_1 joins x and y. Any sufficiently small regular neighborhood σ_1 of λ_1 in $f(M_1)$ is an m_1-dimensional PL-ball.

The union $\lambda_1 \cup \lambda_2$ is an embedded circle in $\mathbb{R}^d$, and, again using general position,[1] we can fill it with an embedded 2-dimensional PL-disk D_{12} that intersects $f(M_1)$ and $f(M_2)$ precisely in λ_1 and λ_2, respectively, that intersects all other $f(M_i)$, $i \neq 1, 2$ precisely in $\{x, y\}$, and that is disjoint from all $f(S_i)$ (see Figure 4.2); here, we require codimension at least 3.

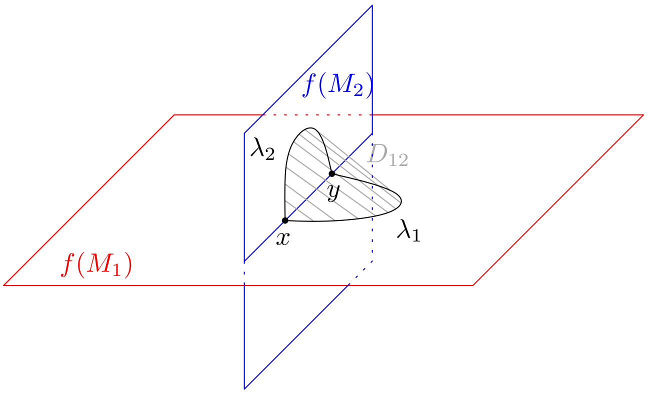


Figure 4.2: The disk D_{12} fills the circle $\lambda_1 \cup \lambda_2$.

Repeating the same construction on each successive circle $\lambda_i \cup \lambda_{i+1}$, $1 \leq i \leq r-1$, we get the sequence of filling disks

$$D_{12}, D_{23}, \ldots, D_{(r-1)r}.$$

By (1.6) (see remark 4.1.2), we have $d \geq 3r \geq 6$, so by general position[2], these filling disks are internally disjoint and their union is a disk D with boundary $\lambda_1 \cup \lambda_r$.

We pick a regular neighborhood B of D; this neighborhood is a d-dimensional PL-ball. If we pick this neighborhood sufficiently small then B intersects each image $f(M_i)$ in an m_i-dimensional PL-ball σ_i that is a regular neighborhood of λ_i, and we get Property 3 of the standard local situation since the images $f(M_i)$ intersect transversely at x and at y.

Furthermore, if L and B' are as in the statement of the lemma, then we can choose the paths λ_i and the disks $D_{i(i+1)}$ to be contained in $\mathring{B}'$ and to avoid L, and hence the same holds for any sufficiently small regular neighborhood B of D. □

[1] Indeed, we can take D_{12} to be the cone over $\lambda_1 \cup \lambda_2$ with an apex in general position.

[2] We repeatedly use Theorem 2.2.2 to make the disks disjoint while keeping their boundaries fixed. The general position argument works because we are dealing with objects of dimension 2 (=discs) inside a space of dimension $d \geq 6$.

Remark 4.2.3. If we apply the preceding lemma to a finite collection of pairwise disjoint pairs $\{x, y\}$ of r-fold points, then by general position, we can choose the resulting disks D, and hence the corresponding regular neighborhoods B to be pairwise disjoint.

Using Lemma 4.2.2, Theorem 4.1.1 reduces to the following:

Proposition 4.2.4. *Suppose that $B \subset \mathbb{R}^d$ and $\sigma_1, \dots, \sigma_r \subset B$ form a standard local situation around two r-fold points $x, y \in \mathring{B}$, and that the codimension condition* (1.6) *is satisfied.*

Suppose furthermore that x and y have opposite intersection sign, i.e., for some (and then every) choice of orientations of $\mathbb{R}^d$ and of the σ_i,

$$\operatorname{sign}_x(\sigma_1, \dots, \sigma_r) = -\operatorname{sign}_y(\sigma_1, \dots, \sigma_r).$$

Then there exist $r-1$ PL-ambient isotopies

$$H^2, \dots, H^r \colon B \times [0,1] \to B \times [0,1],$$

each fixing ∂B pointwise, such that

$$\sigma_1 \cap H_1^2(\sigma_2) \cap \dots \cap H_1^r(\sigma_r) = \emptyset.$$

Proof of Theorem 4.1.1 using Proposition 4.2.4. Using Lemma 4.2.2, we show that if Proposition 4.2.4 holds for a given multiplicity $r \geq 2$, then so does Theorem 4.1.1.

Suppose the hypotheses of Theorem 4.1.1 are satisfied. Apply Lemma 4.2.2 to get a PL d-ball B disjoint from L and such that B and $\sigma_i := B \cap f(M_i)$ form a standard local situation around the pair x, y of r-fold points in question. By assumption, these points have opposite signs (here, we use that intersection signs are determined locally, so that it does not matter whether we restrict $f(M_i)$ to its intersection with B). Let $H_t^2, \dots, H_t^r \colon B \to B$ be the isotopies guaranteed by Proposition 4.2.4. Since they are fixed pointwise on ∂B, we can extend each H_t^i to an isotopy of $\mathbb{R}^d$ by letting it fix every point outside of B; slightly abusing notation, we denote the resulting isotopies by the same symbol. Then the intersection $f(M_1) \cap H_1^2(f(M_2)) \cap \ldots \cap H_1^r(f(M_r))$ does not contain any points from B (in particular, it does not contain x or y), and it coincides with $f(M_1) \cap \ldots \cap f(M_r)$ outside of $\mathring{B}$. □

4.3 Restriction to σ_1, Piping and Unpiping

To prove Proposition 4.2.4, the idea is to restrict ourselves to σ_1, and to consider x and y as $(r-1)$-fold intersection points of the pairwise intersections $\sigma_1 \cap \sigma_2, \dots, \sigma_1 \cap \sigma_r$ inside the m_1-dimensional ball σ_1). The plan is to solve the situation inductively inside σ_1, and then to extend the solution, i.e., the resulting isotopies of σ_1 fixing $\partial\sigma_1$, to isotopies of B, using that σ_1 is unknotted in B.

Each $\sigma_1 \cap \sigma_i$ is a PL-manifold with boundary properly embedded in σ_1, of codimension

$$m_1 - \dim \sigma_1 \cap \sigma_i = d - m_i \geq 3, \qquad 2 \leq i \leq r.$$

We now fix orientations of $\sigma_1, \dots, \sigma_r$ and of B and consider the induced orientations on $\sigma_1 \cap \sigma_i$, $2 \leq r$. By Lemma 2.3.1,

$$\operatorname{sign}_x^{\sigma_1}(\sigma_1 \cap \sigma_2, \dots, \sigma_1 \cap \sigma_r) = \operatorname{sign}_x^B(\sigma_1, \dots, \sigma_r),$$

and likewise for y. Thus, with respect to the induced orientations, x and y have opposite intersection signs as $(r-1)$-fold intersection points of $\sigma_1 \cap \sigma_2, \dots, \sigma_1 \cap \sigma_r$ in σ_1.

However, there is a caveat that prevents us from directly proceeding by induction: The pairwise intersections are *not connected*; indeed, by the hypotheses of Proposition 4.2.4, each $\sigma_1 \cap \sigma_i$ is the disjoint union of two PL-balls $B_{i,x} \ni x$ and $B_{i,y} \ni y$ of dimension $m_1 + m_i - d$, $2 \leq i \leq r$, see Figure 4.3.

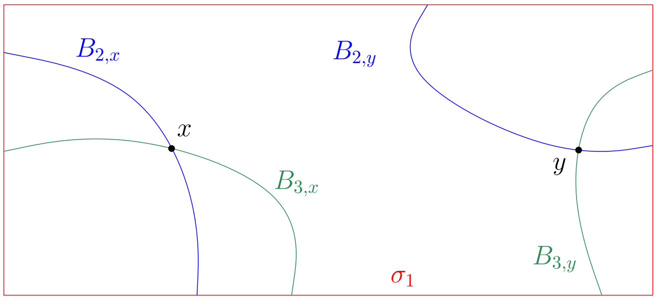


Figure 4.3: The pairwise intersections $\sigma_1 \cap \sigma_i$, $2 \le i \le r$ are not connected.

Thus, the fact that x and y have opposite signs is no longer independent of the choice of orientations; indeed, if we revert the orientation on one of the components of $\sigma_1 \cap \sigma_2$, say, then the signs become the same. More importantly, in this situation there are simply no ambient isotopies $H_t^3, \dots, H_t^r \colon \sigma_1 \to \sigma$ fixing $\partial\sigma_1$ that eliminate the intersection points. For example, in the case $r = 3$ depicted in Figure 4.3, the ball $B_{2,x}$ and the boundary $\partial B_{3,x}$ are *linked* in σ_1, i.e., for any homeomorphism fixing $\partial\sigma_1$, we have $B_{2,x} \cap h(B_{3,x}) \neq \emptyset$.

To remedy this shortcoming, we apply two operations, *piping* and *unpiping*, to be described presently, to the simplices $\sigma_2, \dots, \sigma_r$ to force connectivity of the intersections $\sigma_1 \cap \sigma_i$, $2 \le i \le r$. These operations correspond to a pair of *complementary surgeries* (see below) performed on each σ_i, $2 \le i \le r$. First, we perform a 1-surgery on σ_i to produce a manifold σ_i^*, and then we perform a complementary 2-surgery on σ^* to obtain a manifold σ_i^{**} that is again an m_i-dimensional ball. Moreover, these surgeries are performed in an ambient way inside B^d, keeping the boundaries of the σ_i and of B^d fixed and not affecting the intersection points x and y, such that $\sigma_1 \cap \sigma_i^* = \sigma_1 \cap \sigma_i^{**}$ is connected. We now describe this in more detail.

Surgeries and Handles. Let M be an m-dimensional PL-manifold (possibly with boundary). Suppose that we have a PL-embedding of $\alpha \colon S^{p-1} \hookrightarrow \mathring{M}$, and that α can be extended to an embedding $\psi \colon S^{p-1} \times B^{m-p+1} \hookrightarrow \mathring{M}$, where we identify S^{p-1} with $S^{p-1} \times 0 \subset S^{p-1} \times B^{m-p+1}$. Then we can use the fact that $\partial(S^{p-1} \times B^{m-p+1}) = \partial(B^p \times S^{m-p}) = S^{p-1} \times S^{m-p}$, remove the interior of the image $\psi(S^{p-1} \times B^{m-p+1})$ from M, and patch the resulting "hole" by attaching $B^p \times S^{m-p}$ via the attaching map $\psi|_{S^{p-1}\times S^{m-p}} \colon S^{p-1} \times S^{m-p} \to \mathring{M}$, i.e., form the new manifold

$$M' := M \setminus \operatorname{int} \psi(S^{p-1} \times B^{m-p+1}) \cup_{\psi|_{S^{p-1}\times S^{m-p}}} B^p \times S^{m-p}.$$

We refer to this operation as **attaching a hollow p-handle** $B^p \times S^{m-p}$ to M or performing a **p-surgery** on M along α. (Note that this does not affect the boundary ∂M.)

If $M \subset \partial W$ is PL-embedded on the boundary of an $(m+1)$-dimensional PL-manifold W, then the operation just described corresponds to attaching a **solid p-handle** $B^p \times B^{m-p+1}$ to W to obtain a new $(m+1)$-manifold W', as described in [38, Chapter 6, p.74] (where the embedded sphere $\alpha(S^{p-1})$ is called the *a-sphere* of the solid p-handle). The p-surgery describes how M and ∂W change when attaching the p-handle to W. We remark that our use of the adjectives *hollow* and *solid* is slightly nonstandard (in [38, Chapter 6], solid handles are simply called handles).

Suppose now that after obtaining M' from M by a p-surgery along α as described above, we perform a $(p+1)$-surgery on M' along an embedding $\beta \colon S^p \hookrightarrow \mathring{M}'$ to obtain another manifold M''. We say that these two surgeries are **complementary** if the embedded spheres $\beta(S^p)$ and $\{0\} \times S^{m-p}$ in M' are in general position and have algebraic intersection number ± 1 (with respect to some arbitrarily chosen orientations); we call the sphere $\{0\} \times S^{m-p}$ the **cocore sphere** of the p-surgery. (This corresponds to complementarity of the solid p-handle attached to W and the solid $(p+1)$-handle attached to W', as described in [38, Chapter 6, pp. 76–80], where the cocore sphere $\{0\} \times S^{m-p}$ is called the *b-sphere*; it is the boundary of the cocore ball $\{0\} \times B^{m-p+1}$ of the solid p-handle attached to W.)

The main fact we will need is the following:

Lemma 4.3.1. *If M'' is obtained from M by performing a p-surgery followed by a complementary $(p+1)$-surgery, then M'' and M are PL-homeomorphic.*

This is essentially the cancellation lemma for handle theory [38, Lemma 6.4], which states that if W'' is obtained from W by attaching a p-handle and then a complementary $(p+1)$-handle, then there is a PL-homeomorphism $W \cong W''$ that is the identity outside of a neighborhood of the two handles (so that it restricts to a PL-homeomorphism $M \cong M''$).

Piping [38, pp. 67–68]. Let M_1 and M_2 be two disjoint m-dimensional submanifolds of B^d, with $d-m \geq 3$. The *piping* technique consists of forming a new submanifold M_3 homeomomorphic to the connected sum $M_1 \# M_2$ as follows [38, p. 46]: Pick two points $p_i \in M_i$, $i = 1, 2$, and choose a path λ in B^d that connects p_1 and p_2; by general position, we can assume that λ is disjoint from the M_i except at its endpoints and that λ avoids any given obstacle (closed polyhedron) of codimension at least 2. Remove the interiors of two small m-dimensional balls B_1 and B_2 around $p_1 \in M_1$ and $p_2 \in M_2$ and patch the resulting holes by a an embedded cylinder $Z \cong S^{m-1} \times [-1, +1]$ along λ, the **piping tube**, see Figure 4.4. Thus, Z intersects $M_1 \cup M_2$ precisely in $\partial T = \partial B_1 \cup \partial B^1$, and $M_3 = (M_1 \cup M_2) \setminus (\mathring{B}_1 \cup \mathring{B}_2) \cup Z$. The sphere $S^{m-1} \times \{0\} \subset Z$ is the **cocore sphere** of the piping. If both M_1 and M_2 are oriented, then the piping can be performed in such a way that M_3 is oriented compatibly with both given orientations.

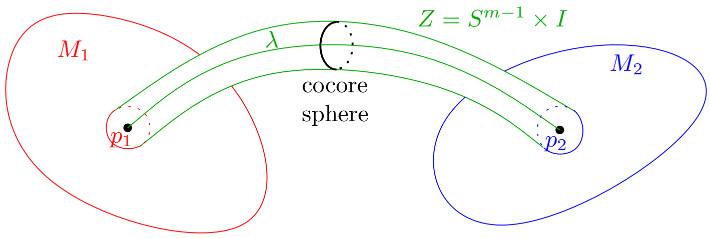


Figure 4.4: Piping of two submanifolds.

Somewhat more formally, the piping tube can be described as follows:

Proposition 4.3.2 ([38, Proposition 5.10])**.** *Let λ be as above. Let (N, N_1, N_2) be a regular neighborhood of λ in (B^d, M_1, M_2). Then there is a PL-homeomorphism*

$$h\colon (N, N_1, N_2) \cong ([-1, +1]^{d-1} \times [-2, 2], [-1, 1]^m \times 0^{d-1-m} \times \{-1\}, [-1, 1]^m \times 0^{d-1-m} \times \{1\}),$$

and h can be chosen to preserve any given orientations (for this, $d - m \geq 2$ would suffice). The piping tube can be taken to be $Z = \partial[-1, 1]^m \times 0^{d-1-m} \times [-1, 1]$.

If M_1 and M_2 are submanifolds of an m-manifold M, then piping corresponds to performing a 1-surgery on M, in an ambient way inside B^d, with the hollow 1-handle embedded as the piping tube. If M is oriented, we use that the piping tube can be given an orientation compatible with that of M at both ends, so that the resulting manifold M' is again orientable.

Moreover, the piping tube is *unique* up to ambient isotopy of B^d fixed on $M \cup \partial B^d$, in the following sense [38, Exercise, p. 68]: Consider two PL-paths λ and λ' in general position with endpoints p_1 and p_2 (and otherwise disjoint from M). By general position, using $d - m \geq 3$, there is an isotopy F between $\lambda \cup M \subset B^d$ and $\lambda' \cup M \subset B^d$, fixed on M and such that $F^{-1}(\partial Q \times [0, 1]) = \partial M \times [0, 1]$ (so F is allowable, see Section 2.1). By Proposition 2.1.2, there is an ambient isotopy H of B^d, fixed on $M \cup \partial Q$, such that $H_1(\lambda) = \lambda'$. Thus, by the uniqueness of regular neighborhoods up to ambient isotopy, any piping tube along λ is ambient isotopic to any piping tube along λ'.

Piping simultaneously in σ_1 and in B^d. We now apply this to each σ_i, $2 \le i \le r$ to make the pairwise intersections

$$\sigma_1 \cap \sigma_i \cong B_{i,x} \sqcup B_{i,y}.$$

connected: For each i, $2 \le i \le r$, we pick two points $b_{i,x} \in B_{i,x}$ and $b_{i,y} \in B_{i,y}$ and not contained in any other σ_j, $j \notin \{1, i\}$. We connect $b_{i,x}$ and $b_{i,y}$ by a path λ_i in σ_1; by general position, we may assume that λ_i avoids $\sigma_1 \cap \sigma_j$, $j \notin \{1, i\}$. We now perform an ambient 1-surgery on σ_1, i.e., we run a piping tube from σ_i to itself along λ_i, in an orientation-compatible way, as described above. We denote the resulting piped m_i-manifold by σ_i^*, see Figure 4.5.

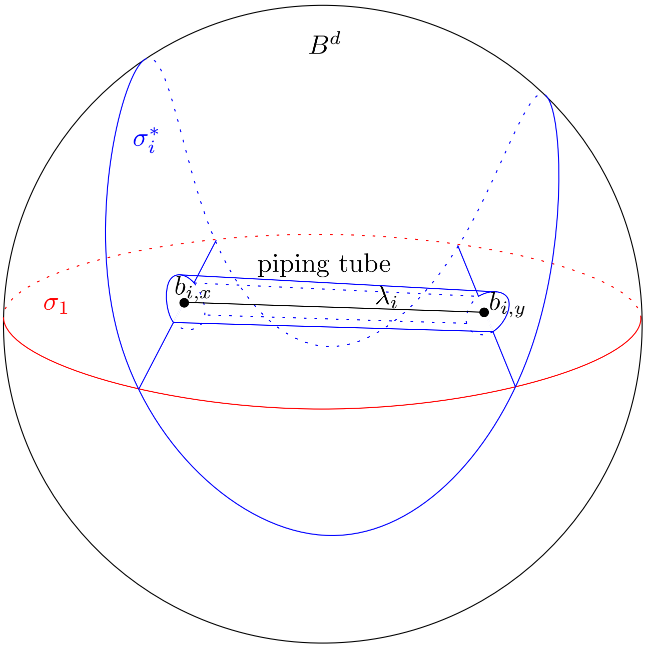


Figure 4.5: σ_i is piped along $\lambda_i \subset \sigma_1$, forming σ_i^*.

Moreover, σ_1 is unknotted in B^d, i.e., up to a homeomorphism of B^d, σ_1 is embedded as a coordinate m_1-ball. Therefore, we can take the piping tube to be transverse to σ_1. Then σ_i^* is still transverse to σ_1, and the intersection $\sigma_1 \cap \sigma_i^*$ is a piping of the two components $B_{i,x}$ and $B_{i,y}$ of $\sigma_1 \cap \sigma_i$, see Figure 4.6). Since orientations are preserved by the pipings, x and y have opposite signs as $(r-1)$-fold intersections points of the connected oriented manifolds $\sigma_1 \cap \sigma_2^*, \ldots, \sigma_1 \cap \sigma_r^*$ inside σ_1.

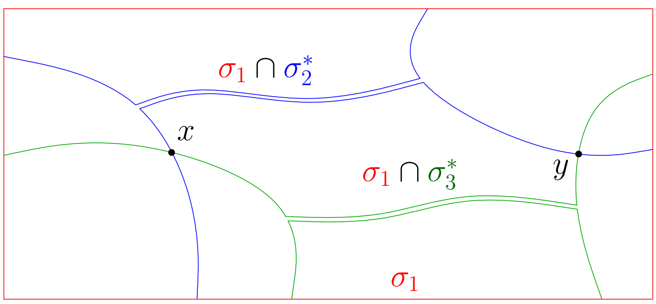


Figure 4.6: The "piped" surfaces σ_2^* and σ_3^* intersected with σ_1.

Unpiping in B^d. As explained above, piping σ_i corresponds to performing a 1-surgery on σ_1, in an ambient way inside B^d. In this way, we obtained a submanifold σ_i^*, with the same boundary as σ_i, such that $\sigma_i^* \cap \sigma_1$ is connected. However, σ_i^* is not homeomorphic to an m_i-ball, so in particular, there is no isotopy of B^d that transforms σ_i into σ_i^*.

We now describe how to amend this by performing a complementary ambient 2-surgery on σ_i^*, which we call **unpiping**, such that the resulting manifold σ_i^{**} is again an m_i-ball and such that $\sigma_1 \cap \sigma_i^{**} = \sigma_1 \cap \sigma_i^*$ does not change (hence stays connected). The basic idea is shown in Figure 4.7.

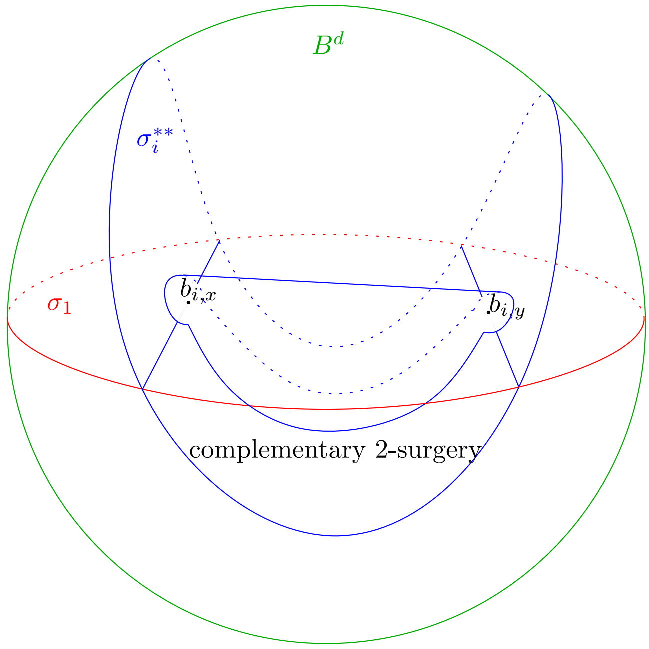


Figure 4.7: A 1-surgery can be cancelled by a complementary 2-surgery, both ambient.

Lemma 4.3.3 (Unpiping Lemma). *For each i, $2 \le i \le r$, there is an ambient isotopy $\widetilde{H}^i$ of B^d fixed on ∂B^d such that $\sigma_i^{**} := \widetilde{H}_1^i(\sigma_i)$ satisfies $\sigma_1 \cap \sigma_i^{**} = \sigma_1 \cap \sigma_i^*$ and $\sigma_i^{**} \cap \sigma_j^{**} = \sigma_i \cap \sigma_j$, $2 \le i < j \le r$.*

Proof. We need to achieve three things:

1. First, if we think of σ_i and σ_i^* as abstract (non-embedded) PL-manifolds, with σ_i^* obtained from σ_i by a 1-surgery, then in order to be able to perform a complementary 2-surgery on σ_i^* and obtain an m_i-ball σ_i^{**}, we need an embedded circle β_i in σ_i^* that intersects the cocore circle of the 1-surgery exactly once and such that that a small neighborhood of β_i in σ_i^* is PL-homeomorphic to $S^1 \times B^{m_i-1}$.

2. Moreover, in our situation, σ_i^* is an embedded submanifold of B^d and we want to perform the 2-surgery *ambiently* in B^d, i.e., we want to attach a hollow 2-handle *embedded* in B^d and internally disjoint from σ_i^* to get σ_i^{**} embedded as well.

3. Furthermore, we want to avoid introducing new intersections, so we want the embedded hollow 2-handle for σ_i^* to be disjoint from σ_1 and σ_j^*, $j \neq i$, and the handles to be disjoint from each other. In order to do this, we will show that, for each $i = 2, \ldots, r$, there is a 2-dimensional disk D_i in general position with boundary β_i such that we can choose the hollow 2-handle for σ_i^* to lie in a small regular neighborhood of D_i in B^d. Then, by general position, D_i is disjoint from σ_1, and from σ_j^* and D_j, $j \neq i$, so the same holds for any sufficiently small neighborhood of D_i, and hence for the hollow 2-handles.

We now make this more precise.

Let us first see that we can achieve the first two goals. We use the fact that σ_i is unknotted in B^d, i.e., up to a PL self-homeomorphism of B^d, σ_i is a standard coordinate m_i-ball embedded in B^d. Next, all possible pipings of σ_i are ambient isotopic keeping σ_i fixed. Thus, we may assume that σ_i^* is a "standard" piped m_i-ball in B^d, see Figure 4.8. In this "standard" situation, it is clear that we can find the desired β and that the ambient 2-surgery can be performed such that the hollow 2-handle lies in a small neighborhood of a "standard" 2-dimensional disk D_i with $\partial D_i = \beta_i$. More precisely, in this standard situation, we can find a small regular neighborhood N of D_i in

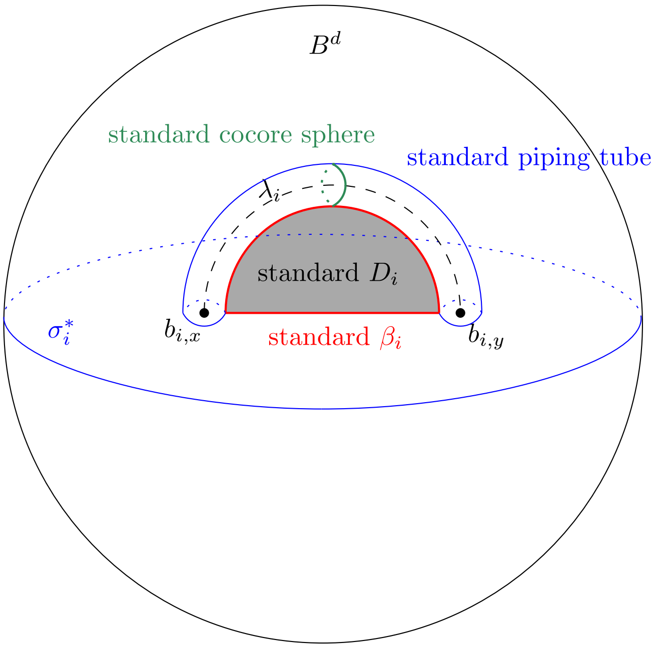


Figure 4.8: A standard piped σ_i^*.

B^d and a PL-homeomorphism

$$h\colon N \cong [-2,2]^2 \times [-1,1]^{d-2}$$

such that $h(D_i) = [-1,1]^2 \times 0^{d-2}$ and $h(N \cap \sigma_i^*) = \partial[-1,1]^2 \times [-1,1]^{m_i-1} \times 0^{d-m_i}$.

We do not control how the self-homeomorphism of B^d and the ambient isotopy that we apply to get σ_i^* into standard position affect σ_1 or the other σ_j^* and D_j, $j \neq i$, and a priori they may intersect N. However, we know that each of them is of codimension at least 3 in B^d (and hence in N) and intersects σ_i^* transversely in a submanifold of dimension at most $m_i - 3$. Thus, up to a small "parallel perturbation" of β_i in σ_i^* corresponding to a parallel translation of $h(\beta_i) = \partial[-1,1]^2 \times 0^{d-2}$ by a random vector in $0^2 \times (-\delta,\delta)^{m_i-1} \times 0^{d-1-m_i}$ for some small $\delta > 0$, we may assume that β_i is disjoint from $\sigma_1 \cap \sigma_i^*$ and from $\sigma_i^* \cap \sigma_j^*$, $1 \leq j \leq r$, $j \neq i$. Similarly, up to a small perturbation of D_i inside N and keeping β_i fixed, we may assume that the disc D_i is disjoint from

$$\sigma_1 \cup \bigcup_{j \neq i} (\sigma_j^* \cup D_j),$$

(e.g., we can think of D_i as a cone over β_i and slightly perturb the apex of the cone, if necessary). Then we can take the hollow 2-handle to be the preimage under h of

$$[-1,1]^2 \times \partial[-\varepsilon,\varepsilon]^{m_i-1} \times 0^{d-m_i},$$

which is disjoint from σ_1 as well as σ_j^* and D_j, and hence from σ_j^{**}, $j \neq i$, for $\varepsilon > 0$ sufficiently small.

Finally, σ_i and σ_i^{**} are m_i-dimensional PL-balls properly embedded in B^d, $d - m_i \geq 3$, with $\partial\sigma_i = \partial\sigma_i^{**}$, $2 \leq i \leq r$. Thus, by the relative version of Zeeman's Unknotting Theorem (Corollary 2.1.4), for each i there is an ambient isotopy $\widetilde{H}^i$ of B^d such that $\widetilde{H}^i(\sigma_i) = \sigma_i^{**}$. □

Remark 4.3.4. Instead of using the above somewhat ad-hoc elementary argument to show that we can perform the ambient 2-surgery, we could simply choose the disks D_i, $2 \leq i \leq r$, in general position and then construct the required embedded hollow 2-handles using the fact that each D_i has a normal disk bundle in B^d by [22, Corollary 4.2]. However, we prefer to avoid using PL (micro)bundles in the present paper.

4.4 Proof of the Higher-Multiplicity Whitney Trick

As shown above, it suffices to prove Proposition 4.2.4.

Proof of Proposition 4.2.4. As mentioned before, we proceed by induction on r, and the base case $r = 2$ is the PL version of the Whitney Trick (see, e.g., Weber [49]). Thus, we may assume that $r \geq 3$ and that Proposition 4.2.4 holds for multiplicity $r - 1$.

As described in Section 4.3, we pipe and then unpipe each of $\sigma_2, \ldots, \sigma_r$ to form σ_2^{**}, …, σ_r^{**}. Each σ_i^{**} is a PL-ball of dimension m_i, each pairwise intersection $\sigma_1 \cap \sigma_i^{**} = \sigma_1 \cap \sigma_i^{*}$ is a PL-cylinder $S^{m-1} \cong [0,1]$ properly embedded into σ_1 and of codimension $m_1 - \dim(\sigma_1 \cap \sigma_i^{**}) = d - m_i \geq 3$. Moreover, these cylinders intersect inside σ_1 in two $(r-1)$-fold intersection points of opposite sign,

$$\{x, y\} = (\sigma_1 \cap \sigma_2^{**}) \cap \ldots \cap (\sigma_1 \cap \sigma_r^{**}).$$

Since each $\sigma_1 \cap \sigma_i^{**}$ is connected, by Lemma 4.2.2, there is an m_i-dimensional ball $B^{m_1} \subseteq \mathring{\sigma}_1$ such that B^{m_1} and $\sigma_1 \cap \sigma_i^{**} \cap B^{m_1} = \sigma_i^{**} \cap B^{m_1}$, $2 \leq i \leq r$, form a standard local situation around x and y. By induction, there are ambient isotopies $\widehat{H}^i$ of B^{m_1}, fixed on ∂B^{m_1}, $3 \leq i \leq r$, which we can view as ambient isotopies of σ_1 fixed outside of $\mathring{B}^{m_1}$, such that

$$\sigma_1 \cap \sigma_2^{**} \cap \widehat{H}_1^3(\sigma_1 \cap \sigma_3^{**}) \cap \ldots \cap \widehat{H}_1^r(\sigma_1 \cap \sigma_r^{**}) = \emptyset.$$

Since σ_1 is unknotted in B^d, i.e., $B^d \cong \sigma_1 * S^{d-1-m_1}$, we can extend the $\widehat{H}^r$ to ambient isotopies of B^d, fixed on ∂B^d, which by some abuse of notation, we will denote by the same symbol. These ambient isotopies of B^d satisfy $\widehat{H}_t^i(\sigma_1) = \sigma_1$ for all $t \in [0,1]$ and hence

$$\sigma_1 \cap \sigma_2^{**} \cap \widehat{H}_1^3(\sigma_3^{**}) \cap \ldots \cap \widehat{H}_1^r(\sigma_1 \cap \sigma_r^{**}) = \emptyset.$$

Let $\widetilde{H}^i$ be the ambient isotopy of B^d constructed in Lemma 4.3.3, i.e., $\widetilde{H}_1(\sigma_i) = \sigma_i^{**}$, $2 \leq i \leq r$. Let H^i be the composition of $\widehat{H}^i$ and $\widetilde{H}^i$, $3 \leq i \leq r$, and set $H^2 := \widetilde{H}^2$. Then each H^i is an ambient isotopy of B^d fixed on ∂B^d, and

$$\sigma_1 \cap H_1^2(\sigma_2) \cap H_1^3(\sigma_3) \cap \ldots \cap H_1^r(\sigma_r) = \emptyset,$$

as desired. □

Chapter 5

Deleted Product Criterion in the Critical Range

5.1 Introduction

Our goal in this Chapter is to prove Theorem 1.1.5, which we restate here:

Theorem 5.1.1 (Sufficiency of the Deleted Product Criterion for the critical dimension)**.** *Let $k \geq 3$, $r \geq 2$ and K be a finite $(r-1)k$-dimensional simplicial complex.*

There exists an almost r-embedding $K \to \mathbb{R}^{rk}$ if and only if there exists a $\mathfrak{S}_r$-equivariant map $K^r_\Delta \to_{\mathfrak{S}_r} S^{rk(r-1)-1}$.

As already mentioned in Chapter 1.1, Theorem 5.1.1 is a generalization to higher multiplicity of an old result: the van Kampen–Shapiro–Wu embeddability criterion (see [15] for an exposition), which we state here in a "split-in-two" form which is convenient for our discussion (the reader might want to compare it to Theorem 1.1.7 from the Introduction):

Theorem 5.1.2 (**Van Kampen–Shapiro–Wu**)**.** *Let K be a simplicial complex with $k :=$* $\dim K \geq 3$.

(VK1) *There exists an almost-embedding $f\colon K \to \mathbb{R}^{2k}$ if and only if there exists an equivariant map $K^2_\Delta \to_{\mathfrak{S}_2} S^{2m-1}$.*

(VK2) *If there exists an almost-embedding $f\colon K \to \mathbb{R}^{2k}$ then there exists an embedding $g\colon K \hookrightarrow \mathbb{R}^{2m}$; moreover, g can be taken to be piecewise-linear.*

Remarks 5.1.3.

1. In the case of Theorem 5.1.2, the action of $\mathfrak{S}_2 = \mathbb{Z}_2$ on the sphere is by antipodality. The map $\widetilde{f}$ is induced by the almost embedding f as in (1.2). For $r = 2$ (our case here), equation (1.2) takes the simple form:

$$\widetilde{f}(x, y) := \frac{f(x) - f(y)}{\|f(x) - f(y)\|}.$$

2. Theorem 5.1.1 is only a generalization of (VK1) to higher multiplicity intersection, and it leaves open the second part of the problem: when can we turn an almost r-embedding into an r-embedding? I.e., when can (VK2) be generalized to r-fold intersection?

   This question (= how to deal with **local** r-intersection?) will be treated in a future paper.

3. Our proof of Theorem 5.1.1 is structured along the lines of the classical proof of (VK1) In particular, Theorem 5.1.1 is based on generalizations to higher multiplicity of two classical tools of Geometric Topology: the *Van Kampen finger moves* (Section 5.3) and the *Whitney trick* (Theorem 4.1.1 already presented in Chapter 4).

4. The assumption that the map F is equivariant with respect to the action of the full symmetric group $\mathfrak{S}_r$ (and not just some subgroup $H \leq \mathfrak{S}_r$) will be important when applying the r-fold Van Kampen finger moves; see Section 5.3 (Remark 5.3.4).

5. Theorem 5.1.1 required a "codimension 3" in its stated form: $\dim \mathbb{R}^{rk} - \dim K \geq 3$. This condition has recently [2] been improved to "codimension 2", i.e., the deleted product criterion is also sufficient to decide the existence of almost r-embeddings of $2(r-1)$ complex K to $\mathbb{R}^{2r}$. This extension is based on a new version of the r-fold Whitney trick adapted to work in codimension 2 and for $r \geq 3$: the *local and global disjunction lemmas.*

   This approach uses crucially the fact that $r \geq 3$. Indeed, its counterpart for $r = 2$ famously fails [15].

6. For embeddings, there is a far-reaching generalization of Theorem 5.1.2: The *Haefliger–Weber Theorem* [21,50] (see also [43] for a modern survey and extensions) guarantees that in the so-called *metastable range* $2d \geq 3m+3$, an m-dimensional complex K embeds (piecewise-linearly) into $\mathbb{R}^d$ if and only if there is an equivariant map $K^2_\Delta \to_{\mathfrak{S}_2} S^{d-1}$. In [32], we present a generalization of this to almost r-embeddings, which works in the corresponding *r-metastable range* $rd \geq (r+1)m + 3$.

7. Vanishing of the generalized Van Kampen obstruction amounts to the solvability of a certain system of inhomogeneous linear equations over the integers (see Section 5.3). As a consequence, we have the following:

   Corollary 5.1.4. *There is an algorithm which, under the assumptions of Theorem 5.1.1 , decides whether a given input $(r-1)k$-complex K admits an almost r-embedding to $\mathbb{R}^{rk}$.*

   Furthermore, if the parameters r and k are fixed, the algorithm runs in polynomial time in the size (number of simplices) of K.

Plan of this Chapter ("Strategy of Proof"). The proof of Theorem 5.1.1 is subdivided into three steps as follows (the necessary definitions will be given in the corresponding sections):

5.2 If K is an m-dimensional simplicial complex, $m = (r-1)k$ and $d = rk$, $k \geq 1$ (more generally, if $\dim K^r_\Delta = d(r-1)$, then there exists a primary equivariant obstruction $\mathfrak{o}(K^r_\Delta) \in Z^{d(r-1)}_{\mathfrak{S}_r}(K^r_\Delta; \mathcal{Z})$, the **generalized Van Kampen obstruction**, such that there exists an equivariant map $F\colon K^r_\Delta \to_{\mathfrak{S}_r} S^{d(r-1)-1}$ if and only if $\mathfrak{o}(K^r_\Delta) = 0$. Moreover, if $f\colon K \to \mathbb{R}^d$ is any PL-map in general position, then the obstruction can be represented by an **intersection number cocycle** $\mathfrak{o}(K^r_\Delta) = [\varphi_f]$, where

$$\varphi_f(\sigma_1 \times \ldots \times \sigma_r) = \pm f(\sigma_1) \bullet \ldots \bullet f(\sigma_r).$$

5.3 Starting with an arbitrary map $f\colon K \to \mathbb{R}^d$ with $\mathfrak{o}(K^r_\Delta) = [\varphi_f] = 0$, one can construct a new PL-map $g\colon K \to \mathbb{R}^d$ by using an r-fold generalization of the classical *Van Kampen finger moves.* From $\varphi_g = 0$, we conclude that, for each r-tuple of pairwise disjoint m-simplices of K, the r-fold intersection points $y \in g(\sigma_1) \cap \ldots \cap g(\sigma_r)$ appear in pairs of opposite sign.

5.4 Having obtained such a map $g\colon K \to \mathbb{R}^d$, and assuming now $k \geq 3$, we can apply the r-fold Whitney trick (Theorem 1.2.3) to remove all its global r-fold intersection points, since they appear in pairs of opposite sign. Furthermore, this process will not introduce new r-fold points. Thus, we obtain an almost r-embedding $h\colon K \to \mathbb{R}^d$.

5.2 Equivariant Obstruction Theory and Intersection Number Cocycles

Here, we briefly review some basic elements of equivariant obstruction theory. For short and very accessible introductions, see [7] or [56, Sec. 4.1]; for a comprehensive and detailed treatment of the theory, the standard source is tom Dieck's monograph [13, Sec. II.3].

For the present section, fix parameters $r \geq 2$ and $d \geq 1$, and set $n := d(r-1)$. Let $Y := (\mathbb{R}^d)^r \setminus \delta_r(\mathbb{R}^d) \simeq_{\mathfrak{S}_r} S^{n-1}$ be the complement of the thin diagonal in $(\mathbb{R}^d)^r$, with the natural action of the symmetric group $\mathfrak{S}_r$ by permuting the factors.

We will need the fact that Y is **$(n-2)$-connected** (i.e., every map $S^{\ell-1} \to Y$ is nullhomotopic, $\ell < n$) and that, by the classical theorem of Hopf, the set $[S^{n-1}, Y]$ of homotopy classes of maps $f\colon S^{n-1} \to Y$ can be identified with the integers via the mapping degree,

$$[S^{n-1}, Y] \cong \mathbb{Z}, \qquad [f] \mapsto \deg(f). \tag{5.1}$$

More precisely, the definition of the degree involves the choice of an orientation of S^{n-1} and of a generator ζ of $H_{n-1}(Y;\mathbb{Z}) \cong \mathbb{Z}$, and in what follows we will always specify these choices.[1]

The action of $\mathfrak{S}_r$ on Y induces a natural action on $[S^{n-1}, Y]$ and hence, via the bijection (5.1), on the integers $\mathbb{Z}$ (it can be checked that the action of a permutation π is given by multiplication by $(\mathrm{sign}\pi)^d$); we will use the notation $\mathcal{Z}$ to denote the integers with this $\mathfrak{S}_r$-action.

Let X be an n-dimensional CW complex on which $\mathfrak{S}_r$ acts freely by cellular maps. The two cases that we will be interested in the present paper are $X = K^r_\Delta$, and $X = {\mathfrak{S}_r}^{*(n+1)}$.

An ℓ-dimensional cellular cochain $\varphi \in C^\ell(X;\mathcal{Z})$ is **equivariant** if it commutes with the group action, i.e., $\varphi(\sigma \cdot \pi) = \varphi(\sigma) \cdot \pi$ for every oriented ℓ-cell σ of X and $\pi \in \mathfrak{S}_r$. The equivariant cochains form a subgroup $C^\ell_{\mathfrak{S}_r}(X;\mathcal{Z})$ of the usual (nonequivariant) cochains. Moreover, the coboundary operator sends equivariant cochains to equivariant cochains, so we get subgroups $B^\ell_{\mathfrak{S}_r}(X;\mathcal{Z})$ of **equivariant coboundaries** (coboundaries of equivariant $(\ell-1)$-cochains) and $Z^\ell_{\mathfrak{S}_r}(X;\mathcal{Z})$ of **equivariant cocycles** (ℓ-cocycles that are equivariant), and the **equivariant cohomology groups** are defined by

$$H^\ell_{\mathfrak{S}_r}(X;\mathcal{Z}) = Z^\ell_{\mathfrak{S}_r}(X;\mathcal{Z}) / B^\ell_{\mathfrak{S}_r}(X;\mathcal{Z}).$$

The basic idea of (equivariant) obstruction theory is that we want to construct an (equivariant) map $F\colon X \to Y$ inductively over skeleta of X of increasing dimension, and likewise for (equivariant) homotopies between such maps (which are maps $X \times [0,1] \to Y$). If σ is an ℓ-cell of X and if we inductively assume that F is already defined on $\mathrm{skel}_{\ell-1}(X)$, hence in particular on the boundary $\partial\sigma \cong S^{\ell-1}$, then we can extend F over σ if and only if $F|_{\partial\sigma}$ is nullhomotopic.[2] If this is the case, then any choice of such an extension to σ yields a unique equivariant extension to all cells $\pi \cdot \sigma$ in the orbit of σ (since the action of $\mathfrak{S}_r$ on X is free).

Using the connectivity of Y, it is not hard to show [13, Prop. II.3.15] that there exists an equivariant map $G\colon \mathrm{skel}_{n-1}(X) \to_{\mathfrak{S}_r} Y$, and that the restrictions of any two such maps to $\mathrm{skel}_{n-2}(X)$ are equivariantly homotopic.

In the next extension step to the n-skeleton of X (which is the last since $\dim X = n$), however, we might get stuck, namely if there is an n-cell σ such that $\deg(G|_{\partial\sigma}\colon \partial\sigma \to Y) \neq 0$. If this is the case, we might try to modify the chosen G on $\mathrm{skel}_{n-1}(X)$ so as to make $G|_\sigma$ nullhomotopic. Whether it is possible to achieve this for all n-cells σ simultaneously is governed by a single n-dimensional equivariant cohomology class; see [13, Section II.3, pp. 119–120] for a proof:

Theorem 5.2.1. *Suppose that X is an n-dimensional CW complex with a free cellular action of $\mathfrak{S}_r$. Then there exists an equivariant cohomology class $\mathfrak{o}(X) \in H^n_{\mathfrak{S}_r}(X;\mathcal{Z})$, called the* ***primary equivariant obstruction****, such that the following properties are satisfied:*

(1) *There exists an equivariant map $F\colon X \to_{\mathfrak{S}_r} Y = (\mathbb{R}^d)^r \setminus \delta_r(\mathbb{R}^d)$ if and only if $\mathfrak{o}(X) = 0$.*

(2) *Let $G\colon \mathrm{skel}_{n-1}(X) \to_{\mathfrak{S}_r} Y$ be an arbitrary equivariant map, and let ζ_0 be a fixed generator of $H_{n-1}(Y;\mathbb{Z}) \cong \mathcal{Z}$. For every oriented n-cell σ of X, set*

$$\varphi_G(\sigma) := \deg(G|_{\partial\sigma}\colon \partial\sigma \to Y) \in \mathcal{Z},$$

[1] Choosing an orientation of S^{n-1} is equivalent to choosing a generator ι of $H_{n-1}(S^{n-1};\mathbb{Z}) \cong \mathbb{Z}$, and given ι and ζ, the degree $\deg(f)$ is, by definition, the unique integer such that $f_*(\iota) = \deg(f)\zeta$, where f_* is the induced map in homology.

[2] Here, we tacitly use that X is a **regular** CW complex, i.e., that all attaching maps are homeomorphisms, so that a closed ℓ-cell σ of X is a closed ℓ-disk embedded in X; for more general CW complexes, the condition would be that $F \circ \alpha_\sigma|_{S^{\ell-1}}$ needs to be nullhomotopic, where $\alpha_\sigma\colon S^{\ell-1} \to X$ is the attaching map of the cell σ.

where the mapping degree is computed with respect to ζ_0 and the orientation of $\partial\sigma \cong S^{n-1}$ is induced by that of σ. This defines an equivariant ***obstruction cocycle***

$$\varphi_G \in Z^n_{\mathfrak{S}_r}(X; \mathcal{Z})$$

which represents the primary obstruction, i.e., $\mathfrak{o}(X) = [\varphi_G]$.

In the special case that $X = K^r_\Delta$ for a finite simplicial complex K, we call $\mathfrak{o}(K^r_\Delta)$ the ***r*-fold Van Kampen obstruction**

Lemma 5.2.2. (a) *Suppose the equivariant map $G\colon \mathrm{skel}_{n-1}(X) \to_{\mathfrak{S}_r} (\mathbb{R}^d)^r \setminus \delta_r(\mathbb{R}^d)$ in Theorem* 5.2.1 (2) *is the restriction of an equivariant PL-map in general position*[3] *(denoted by the same symbol, by abuse of notation)*

$$G\colon X \to_{\mathfrak{S}_r} (\mathbb{R}^d)^r.$$

Then the value of the obstruction cocycle φ_G on each oriented n-cell σ of X is given by the (pairwise) intersection number[4]

$$\varphi_G(\sigma) := G(\sigma) \bullet \delta_r(\mathbb{R}^d). \tag{5.2}$$

(b) *Furthermore, suppose that $X = K^r_\Delta$ for a simplicial complex K and that $f\colon K \to \mathbb{R}^d$ is a PL-map in general position. In this case, we can take*

$$G = f^r\colon K^r_\Delta \to_{\mathfrak{S}_r} (\mathbb{R}^d)^r$$

as in the proof of Lemma 1.1.2, and represent $\mathfrak{o}(K^r_\Delta) = [\varphi_f]$ by the following ***intersection number cocycle*** *(denoted by φ_f instead of φ_{f^r} for simplicity) given by*

$$\begin{aligned} \varphi_f(\sigma_1 \times \ldots \times \sigma_r) &= \big(f(\sigma_1) \times \ldots \times f(\sigma_r)\big) \bullet \delta_r(\mathbb{R}^d) \\ &= \varepsilon_{d,m_1,\ldots,m_r} f(\sigma_1) \bullet \ldots \bullet f(\sigma_r) \\ &= \varepsilon_{d,m_1,\ldots,m_r} \sum_{y \in f(\sigma_1)\cap\ldots\cap f(\sigma_r)} \mathrm{sign}_y(f(\sigma_1), \ldots, f(\sigma_r)) \end{aligned} \tag{5.3}$$

where $\varepsilon_{d,m_1,\ldots,m_r}$ is the sign introduced in Lemma 2.3.1 (d), and $m_i = \dim \sigma_i$, $1 \le i \le r$.

Proof. A generator ζ_0 of $H_n(Y; \mathbb{Z})$ can be represented geometrically as the homology class $\zeta_0 = [\partial\tau_0]$ of the boundary of an oriented linear n-simplex τ_0 in $(\mathbb{R}^d)^r$ that intersects $\delta_r(\mathbb{R}^d)$ in its relative interior. For concreteness, we choose τ_0 so that it intersects $\delta_r(\mathbb{R}^d)$ positively. Then[5]

$$\deg(G|_{\partial\sigma}\colon \partial\sigma \to Y) = G(\sigma) \bullet \delta_r(\mathbb{R}^d),$$

which shows (5.2). Furthermore, (5.3) follows immediately, by Lemma 2.3.1 (d). □

[3]Note that for every $G\colon \mathrm{skel}_{n-1}(X) \to_{\mathfrak{S}_r} (\mathbb{R}^d)^r \setminus \delta_r(\mathbb{R}^d)$ there is such an extension, since $(\mathbb{R}^d)^r$ is contractible; conversely, for every PL-map $G\colon X \to_{\mathfrak{S}_r} (\mathbb{R}^d)^r$ in general position, its restriction to $\mathrm{skel}_{n-1}(X)$ avoids the thin diagonal.

[4]Calculated with respect to the orientations of $(\mathbb{R}^d)^r$ and of $\delta_r(\mathbb{R}^d)$ induced by that of $\mathbb{R}^d$ as described in Section 2.3.

[5]To see this, note that the boundaries of any two oriented linear n-simplices that intersect the diagonal positively correspond to the same generator of $H_{n-1}(Y, \mathbb{Z})$, and if we reverse the orientation of such a simplex τ, so that its intersection sign with $\delta_r(\mathbb{R}^d)$ becomes negative, then we also reverse the sign of $[\partial\tau]$ as a generator of the homology group. Furthermore, if τ is disjoint from $\delta_r(\mathbb{R}^d)$ then $[\partial\tau] = 0$ in the homology group. The first part of the lemma now follows by choosing a sufficiently fine triangulation of the cell σ on which G is simplexwise linear: Then $G_*([\partial\sigma]) = \sum_\tau G_*([\partial\tau])$, where τ ranges over all n-simplices in the triangulation, and $[\partial\tau]$ equals $+[\partial\tau_0]$, $-[\partial\tau_0]$, or zero depending on whether $h(\tau)$ intersects $\delta_r(\mathbb{R}^d)$ positively, negatively, or not at all.

5.3 r-Fold Van Kampen Finger Moves

By Lemma 5.2.2, vanishing of the r-fold Van Kampen obstruction means that for every PL-map $f\colon K \to \mathbb{R}^d$ in general position, the corresponding intersection number cocycle φ_f satisfies $\mathfrak{o}(K^r_\Delta) = [\varphi_f] = 0$ as a cohomology class, i.e., $\varphi_f \in B^{d(r-1)}_{\mathfrak{S}_r}(K^r_\Delta; \mathcal{Z})$ is an equivariant coboundary. The goal of this section is to show that in this situation, we can find a map g such that $\varphi_g = 0$ as a cocycle (see Lemma 5.3.3 below). To this end, we consider the following system of generators of the equivariant coboundaries.

Elementary coboundaries. For any dimension ℓ, we get a basis of the ℓ-dimensional equivariant cochains $C^\ell_{\mathfrak{S}_r}(K^r_\Delta; \mathcal{Z})$ as follows: Choose an ℓ-dimensional oriented cell $\eta_1 \times \cdots \times \eta_r$ of K^r_Δ (i.e., the product of pairwise disjoint simplices of K with $\sum_{i=1}^r \dim \eta_i = \ell$). We define the cochain $1_{(\eta_1\times\cdots\times\eta_r)\cdot\mathfrak{S}_r}$ to take value 1 on $\eta_1 \times \cdots \times \eta_r$ and then extend equivariantly over the $\mathfrak{S}_r$-orbit of the cell, i.e., $1_{(\eta_1\times\cdots\times\eta_r)\cdot\mathfrak{S}_r}$ takes value $(\mathrm{sign}\pi)^d$ on ℓ-cells of the form $(\eta_1 \times \cdots \times \eta_r)\cdot\pi = \pm\eta_{\pi(1)} \times \ldots \times \eta_{\pi(r)}$, $\pi \in \mathfrak{S}_r$ (where the sign depends how the action of π affects the orientation), and the cochain evaluates to zero on all other cells.

Thus, the equivariant coboundaries $B^{\ell+1}_{\mathfrak{S}_r}(K^r_\Delta; \mathcal{Z})$ are generated by **elementary equivariant coboundaries** of the form $\delta 1_{(\eta_1\times\cdots\times\eta_r)\cdot\mathfrak{S}_r}$, where $\eta_1 \times \cdots \times \eta_r$ is an oriented ℓ-cells of K^r_Δ. In particular, we have the following:

Lemma 5.3.1. *If $f\colon K \to \mathbb{R}^d$ is a PL-map in general position then $\mathfrak{o}(K^r_\Delta) = [\varphi_f] = 0$ if and only if φ_f can be written as a finite sum of elementary coboundaries,*

$$\varphi_f = \sum \pm\delta 1_{(\eta_1\times\cdots\times\eta_r)\cdot\mathfrak{S}_r}, \tag{5.4}$$

where the sum is over a finite multiset of $(d(r-1)-1)-$dimensional-cells of K^r_Δ.

Suppose now that $\dim K = m = (r-1)k$, and $d = rk$. If $\eta_1 \times \cdots \times \eta_r$ is a cell of K^r_Δ of dimension $d(r-1)-1 = rm-1$ then (up to a permutation $\pi \in \mathfrak{S}_r$ of the η_i), we may assume that

$$\eta_1 \times \cdots \times \eta_r = \mu_1 \times \sigma_2 \times \ldots \times \sigma_r, \tag{5.5}$$

where μ_1 is an $(m-1)$-simplex of K and σ_i is an m-simplex of K, $2 \le i \le r$. Consequently,

$$\delta 1_{(\mu_1\times\sigma_2\times\cdots\times\sigma_r)\cdot\mathfrak{S}_r} = \sum_{\sigma_1} 1_{(\sigma_1\times\sigma_2\times\cdots\times\sigma_r)\cdot\mathfrak{S}_r}, \tag{5.6}$$

where the sum is over all the m-simplices σ_1 of K that contain μ_1 in their boundary and that are disjoint from σ_i, $2 \le r \le r$ (where the orientation of σ_1 is chosen such that μ_1 appears positively oriented in $\partial\sigma_1$).

On the one hand, this immediately yields a proof that the condition $\mathfrak{o}(K^r_\Delta) = 0$ is efficiently testable (see the end of this section). More importantly, by the following lemma, addition of single elementary coboundary to φ_f can be emulated geometrically by a simple modification of the map f (the case $r = 2$ corresponds to the classical Van Kampen finger moves).

Lemma 5.3.2 (r-**Fold Finger Move**). *If $f\colon K \to \mathbb{R}^d$ is a PL-map in general position and if $\delta 1_{(\mu_1\times\sigma_2\times\cdots\times\sigma_r)\cdot\mathfrak{S}_r}$ is an elementary equivariant mr-dimensional coboundary then for any choice of a sign $\varepsilon \in \{-1,+1\}$, there exists a PL-map $g\colon K \to \mathbb{R}^d$ such that $\varphi_g = \varphi_f + \varepsilon\cdot\delta 1_{(\mu_1\times\sigma_2\times\cdots\times\sigma_r)\cdot\mathfrak{S}_r}$.*

Corollary 5.3.3. *Suppose K is a finite simplicial complex,* $\dim K = m \le d-1$, $\dim K^r_\Delta = d(r-1)$ *and*

$$\mathfrak{o}(K^r_\Delta) = 0.$$

Then there exists a PL-map $g\colon K \to \mathbb{R}^d$ such that

$$\varphi_g = 0$$

as a cocycle, i.e., $g(\sigma_1)\bullet \ldots \bullet g(\sigma_r) = 0$ for every $d(r-1)$-cell $\sigma_1 \times \cdots \times \sigma_r$ of K^r_Δ.

Remark 5.3.4. Lemma 5.3.2 and Corollary 5.3.3 are where we need equivariance with respect to the full symmetric group $\mathfrak{S}_r$ and not just with respect to some subgroup $H \leq \mathfrak{S}_r$. If H is some proper subgroup then we get a larger set of H-equivariant coboundaries $\delta 1_{(\mu_1 \times \sigma_2 \times \cdots \times \cdots \sigma_r)\cdot H}$ (hence the condition that φ_f is a sum of H-equivariant coboundaries is more easily fulfilled), but we do not have a analogous geometric modification of a given map f that would allow us to emulate the addition of $\delta 1_{(\mu_1 \times \sigma_2 \times \cdots \times \cdots \sigma_r)\cdot H}$ to φ_f.

Proof of Corollary 5.3.3. Let $f\colon K \to \mathbb{R}^d$ be an arbitrary PL-map in general position. Then $\mathfrak{o}(K^r_\Delta) = [\varphi_f] = 0$, so, by (5.4), we get the desired map g by a finite number of applications of Lemma 5.3.2. □

Proof of Lemma 5.3.2. Fix $f\colon K \to \mathbb{R}^d$ and an oriented $(mr-1)$-cell $\mu_1 \times \sigma_2 \times \cdots \times \sigma_r$ of K^r_Δ.

By (5.6) and (5.3), we need to construct $g\colon K \to \mathbb{R}^d$ that satisfies two conditions: First,

$$g(\sigma_1)\bullet g(\sigma_2)\bullet \ \ldots \bullet g(\sigma_r) = f(\sigma_1)\bullet f(\sigma_2)\bullet \ \ldots \bullet f(\sigma_r) + \varepsilon$$

whenever σ_1 is an m-simplex of K that is disjoint from σ_i, $2 \leq i \leq r$, and that contains μ_1 in its boundary with positive orientation. Second, $g(\tau_1)\bullet \ldots \bullet g(\tau_r) = f(\tau_1)\bullet \ldots \bullet f(\tau_r)$ for every mr-cell of K^r_Δ that is not incident to any $(mr-1)$-cell of the form $\pi \cdot (\mu_1 \times \sigma_2 \times \cdots \times \sigma_r)$, $\pi \in \mathfrak{S}_r$.

Consider a point x in the relative interior of $f(\mu_1)$. Since f is PL, in a sufficiently small neighborhood of x, f looks like a simplexwise linear map, see Figure 5.1.

Choose $(r-1)$ PL-spheres $S_2, \ldots, S_r$ in $\mathbb{R}^d$ in general position, each of dimension m, such that

$$S_2 \cap \cdots \cap S_r = S^{d-m},$$

is a PL-sphere of dimension $(d-m)$ that bounds a flat PL-ball B^{d-m+1} (a convex polytope, say) such that $f(\mu_1) \cap B^{d-m+1} = \{x\}$ (i.e., S^{d-m} is locally linked with $f(\mu_1)$).

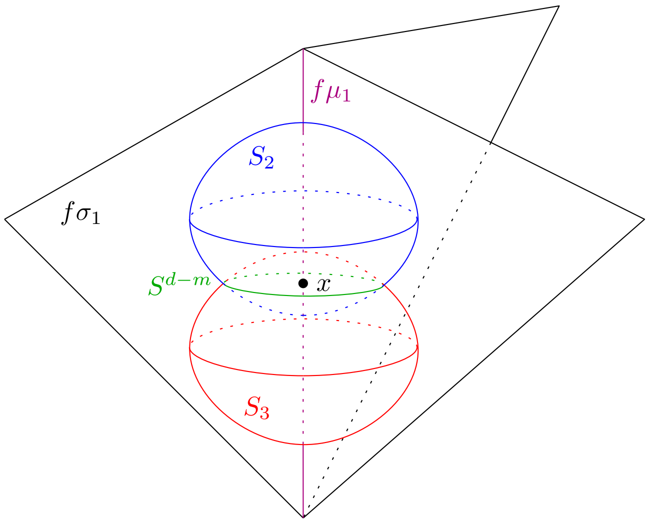


Figure 5.1: For $r = 3$, S_2 and S_3 intersects in a sphere S^{d-m}.

By choosing the spheres S_i sufficiently small, we can make sure that S^{d-m} is disjoint from the image $f(\tau)$ of any simplex τ of K that does not contain μ_1, and that S^{d-m} intersects the image $f(\sigma_1)$ of each m-simplex σ_1 containing μ_1 in a single point.

Choose the orientation of B^{d-m+1} such that $f(\mu_1)\bullet B^{d-m+1} = (-1)^m \varepsilon$, and let $S^{d-m} = \partial B^{d-m+1}$ have the induced orientation. Then, by Lemma 2.3.2, we have

$$f(\sigma_1)\bullet S^{d-m} = (-1)^m \partial f(\sigma_1)\bullet B^{d-m+1} = \varepsilon,$$

if σ_1 contains μ_1 on its boundary with positive orientation, and $f(\tau) \bullet S^{d-m} = 0$ if τ does not contain μ_1.

By Lemma 2.3.1 (a), we can choose orientations for the spheres $S_2, \ldots, S_r$ such that the induced orientation on their intersection S^{d-m} agrees with the orientation of S^{d-m} as the boundary of B^{d-m+1}. Thus, we have

$$f(\sigma_1) \bullet S_2 \bullet \ldots \bullet S_r = \varepsilon.$$

and $f(\tau) \bullet S_2 \bullet \ldots \bullet S_r = 0$ whenever τ does not contain μ_1.

To conclude, we connect each m-sphere S_i, $2 \leq i \leq r$ to $f(\sigma_i)$ by a pipe that avoids $f(K)$ except at its boundary and that preserves orientations at both ends (see Section 4.3). Piping with S^m does not change the topology, so we can view the piped $f(\sigma_i)$ as the image $g(\sigma_i)$ of σ_i under a PL-map. We get the desired map $g \colon K \to \mathbb{R}^d$ by setting $g = f$ outside of the interiors $\mathring{\sigma}_i$, $2 \leq i \leq r$. □

Proof of Corollary 5.1.4. Let R be the number of $\mathfrak{S}_r$-orbits $(\sigma_1 \times \ldots \times \sigma_r) \cdot \mathfrak{S}_r$ of $d(r-1)$-cells of K^r_Δ, and let S be the number of $\mathfrak{S}_r$-orbits $(\mu_1 \times \ldots \times \sigma_r) \cdot \mathfrak{S}_r$ of cells of K^r_Δ of dimension $d(r-1)-1$. Then we can identify $C^{d(r-1)}_{\mathfrak{S}_r}(K^r_\Delta; \mathcal{Z})$ with the free abelian group $\mathbb{Z}^R$, and we can identify the equivariant coboundaries $B^{\ell+1}_{\mathfrak{S}_r}(K^r_\Delta; \mathcal{Z})$ with the subgroup $B \leq Z^R$ generated by (vectors corresponding to) the elementary coboundaries $1_{(\mu_1 \times \ldots \times \sigma_r) \cdot \mathfrak{S}_r}$. Let $A \in \{-1, 0, 1\}^{R \times S}$ be the matrix whose columns are these generators of B.

Choose an arbitrary simplexwise linear map $f \colon K \to \mathbb{R}^d$ in general position. Then the intersection number cocycle φ_f takes only values in $\{-1, 0, +1\}$, so we can view φ_f as a vector $v \in \{-1, 0, 1\}^R$. Then $[\varphi_f] = 0$, or equivalently $\varphi_f \in B^{d(r-1)}_{\mathfrak{S}_r}(K^r_\Delta; \mathcal{Z})$ if and only if the inhomogeneous system of integer linear equations

$$Ax = v$$

has a solution $x \in \mathbb{Z}^S$. For fixed r, this system has size polynomial in the size (number of simplices) of K, and solvability of $Ax = v$ can be tested by bringing the matrix A into Smith normal form. For this, several polynomial-time algorithms are available in the literature, both deterministic (see e.g., [44]) and randomized ones (see, e.g., [14, 17]). □

5.4 Proof of Sufficiency of the Deleted Product Criterion

Proof of Theorem 5.1.1. Suppose that there is a $\mathfrak{S}_r$-equivariant map $K^r_\Delta \to_{\mathfrak{S}_r} S^{mr-1}$, or equivalently, that $\mathfrak{o}(K^r_\Delta) = 0$. By Corollary 5.3.3, there exists a PL-map $f \colon K \to \mathbb{R}^d$ in general position such that $\varphi_f = 0$, or equivalently

$$0 = f(\sigma_1) \bullet \ldots \bullet f(\sigma_r)$$

whenever $\sigma_1, \ldots, \sigma_r$ are pairwise-disjoint m-simplices of K. Thus, the global r-fold points $y \in f(\sigma_1) \cap \ldots \cap f(\sigma_r)$ occur in pairs of opposite signs (we match the pairs up arbitrarily). By the generalized Whitney Trick (Theorem 1.2.3), we can remove these pairs of r-intersection points, one pair at a time, by local ambient isotopies. Since we can choose the isotopies for each pair to have support in a PL-ball that avoids any given obstacle L of codimension at least 3, we do not introduce any new r-intersection points in the process. □

Chapter 6

Counterexamples to the Topological Tverberg Conjecture

In this chapter, we present two ways of building counterexamples to the Topological Tverberg Conjecture:

Conjecture (Topological Tverberg Conjecture)**.** *Let $d, r \geq 2$, and $N = (d+1)(r-1)$. Any map $\Delta^N \to \mathbb{R}^d$ has a global r-fold intersection.*

The conjecture is still open for $2 \leq d \leq 11$. For any $d \geq 12$, the conjecture admits counterexamples. More precisely, we can build counterexamples for $d \geq 2r$ and r not a prime power. In this section we are going to construct two families of counterexamples: the first for $d \geq 3r+1$ and the second for $d \geq 3r$. For the current best $d \geq 2r$, the reader is referred to [2].

The first family that we are going to describe (Section 6.1) is due to Frick. It combines Özaydin 1987 result, a trick of Gromov from 2010, and a theorem from this thesis (Theorem 5.1.2).

In Section 6.2 we present our own take at building counterexamples to the Tverberg Conjecture. Our counterexamples are based on a notion of "prismatic maps" which are maps whose global r-fold intersection only occurs on simplices of prescribed (co)dimension, hence our generalized Whitney trick can be applied to them.

This second "prismatic" family of counterexamples is more natural with respect to the theory we have developped in the previous chapters: they hit the dimensional limit of our theory, and there is no way of improving them without improving the whole theory (and that's what we did in [2] to obtain counterexamples for $d \geq 2r$).

In search of counterexamples to the Topological Tverberg Conjecture, one shall keep in mind

Theorem (Longueville [11, Prop 2.5])**.** *If the topological Tverberg hold for r and $d+1$, then it also holds for r and d.*

In particular, if one can construct a counterexamples to the Conjecture for some d, then the Conjecture also fails for all larger values of d. That is why, when building counterexamples, we mostly care only about the minimal counterexamples that a given technique can produce: counterexamples in higher dimensions are obtained "for free" from Longueville's result.

This also means that the plane $\mathbb{R}^2$ is the "hardest" place to find counterexamples to the Conjecture, whereas in higher dimensions finding them presumably becomes easier. Indeed, all current known counterexamples are products of high-dimension topology.

6.1 Counterexamples in Dimension $\geq 3r+1$

We present the three pieces of the puzzle: Özaydin, Gromov, and our results. Then we describe how to assemble them together to produce a family of counterexamples.

***First Piece*: Özaydin Result** Here is a special case of Özaydin's result [36, Thm. 4.2].

Theorem 6.1.1. *Let r be a non-prime power, $d \geq 2$, and X be a cell complex with free $\mathfrak{S}_r$-action. If $\dim(X) \leq d(r-1)$, then there exists a $\mathfrak{S}_r$-equivariant map $X \to S^{d(r-1)-1}$.*

***Second Piece*: Gromov Trick.** In [18, p. 445], Gromov observes that *the topological Tverberg theorem, when available, implies the van Kampen–Flores theorem.* In particular, if we specialize Gromov's argument for $q = r$, $k = 3(r-1)$, $n = 3r$, and $N = (3r+2)(r-1)$:

Theorem 6.1.2. *If there exists an almost r-embedding $f : \Delta^{(3r+2)(r-1)}_{\leq 3(r-1)} \to \mathbb{R}^{3r}$, then there also exists an almost r-embedding $\Delta^{(3r+2)(r-1)} \to \mathbb{R}^{3r+1}$*

Proof. Extend f to a map $F\colon \Delta^{(3r+2)(r-1)} \to \mathbb{R}^{3r+1} \supset \mathbb{R}^{3r}$ such that

$$F^{-1}(\mathbb{R}^{3r}) = \Delta^{(3r+2)(r-1)}_{\leq 3(r-1)}. \tag{6.1}$$

By contradiction, assume that F has a global r-fold intersection. That is, there exists r pairwise disjoint faces $\sigma_1, \ldots, \sigma_r$ of $\Delta^{(3r+2)(r-1)}$ whose interior intersects in the image by F.

By the pigeon hole principle, at least one of the σ_i is of dimension at most $3(r-1)$. Say, $\dim(\sigma_1) \leq 3(r-1)$. Hence, $F(\sigma_1) \subset \mathbb{R}^{3r}$, and so every $F(\mathring{\sigma}_i)$ contains at least one point of $\mathbb{R}^{3r}$. By (6.1), we must have $\dim(\sigma_i) \leq 3(r-1)$. That is, f has a global r-fold point. □

Theorem 6.1.2 and its proof were independently discovered by Blagojević–Frick–Ziegler, see [8] where this trick is used to obtain simple proofs of numerous Tverberg-type results.

***Third Piece*: Generalized van Kampen.** Here we simply repeat the statement of Theorem 5.1.2:

Theorem 6.1.3. *Let $r, d \geq 3$, and let K by a $(r-1)k$-simplicial complex. There exists an almost r-embedding $K \to \mathbb{R}^{rk}$ if and only if there exists a $\mathfrak{S}_r$-equivariant map $K^r_\Delta \to_{\mathfrak{S}_r} S^{r(r-1)k-1}$.*

Solving the Puzzle. We describe here how to assemble all three pieces and obtain the failure of the Conjecture for $d \geq 3r+1$:

Theorem 6.1.4 (Frick [16])**.** *For any $r \geq 6$ not a prime power, there exists an almost r-embedding $\Delta^{(3r+2)(r-1)} \to \mathbb{R}^{3r+1}$.*

Proof. The dimension of the r-fold deleted product of $\Delta^{(3r+2)(r-1)}_{\leq 3(r-1)}$ is $3r(r-1)$, therefore, by Özaydin Theorem 6.1.1, there exists a $\mathfrak{S}_r$-equivariant map from this deleted product to the sphere $S^{3r(r-1)-1}$.

This, together with Theorem 6.1.3, implies the existence of an almost r-embedding $f\colon \Delta^{(3r+2)(r-1)}_{\leq 3(r-1)} \to \mathbb{R}^{3r}$.

Given the existence of f, we get by Theorem 6.1.2 an almost r-embedding $\Delta^{(3r+2)(r-1)} \to \mathbb{R}^{3r+1}$. □

6.2 Counterexamples in Dimension $\geq 3r$

Our goal in this section is to prove

Theorem 6.2.1. *Suppose $r \geq 6$ is not a prime power and let $N = (3r+1)(r-1)$. Then there exists a map $f\colon \sigma^N \to \mathbb{R}^{3r}$ without global r-fold points.*

Remarks 6.2.2. (a) The smallest counterexample obtained in this way is an almost 6-embedding $f\colon \sigma^{95} \to \mathbb{R}^{18}$.

(b) The idea of the proof of Theorem 6.2.1 is to consider a restricted family of maps, called *prismatic maps*, whose special structure guarantees that in order to study the global r-fold intersection points of a prismatic map, it suffices to consider the restriction of the map to a certain "colorful" m-dimensional subcomplex C of σ^N, where $m = 3(r-1)$.

Since the codimension $3r - m = 3$ is large enough, the r-fold Whitney trick is applicable to maps $C \to \mathbb{R}^{3r}$.

(c) The main technical part of the proof consists in showing that there are variants of the r-fold finger moves and of the r-fold Van Kampen obstruction for the restricted, prismatic setting.

(d) As a byproduct of the proof of Theorem 6.2.1, we obtain the following result:

Definition 6.2.3 (Tverberg Partitions and Type). Let $r \geq 2$, $d \geq 1$, $N = (d+1)(r-1)$, and let $f\colon \sigma^N \to \mathbb{R}^d$ be a PL-map in general position. Suppose $y \in f(\tau_1) \cap \cdots \cap f(\tau_r)$ is a global r-fold point of f and $\dim \tau_i = m_i$, $1 \leq i \leq r$. The vertex sets of the simplices τ_i form a partition of the vertex set of σ^N, hence $\sum_{i=1}^{r} m_i = d(r-1)$ and (by general position) $m_i \leq d$ for $1 \leq i \leq r$. Somewhat abusing terminology, we say that $\tau_1, \ldots, \tau_r$ form a **Tverberg partition** for f, and we call the multiset of dimensions $\{m_1, \ldots, m_r\}$ the **type** of this Tverberg partition (and of the gobal r-fold point y).

Corollary 6.2.4. *Suppose $r \geq 2$, $k \geq 1$, and $N = (rk+1)(r-1)$. Then there exists an affine map $f\colon \sigma^N \to \mathbb{R}^{rk}$ having all global r-fold points of the same type $\{m, \ldots, m\}$ ($m = (r-1)k$ is repeated r times).*

It is also well-known that for every r and d, there are affine maps[1] all of whose global r-fold points are of type $\{1\} \cup \{d\}^{r-1}$. This raises the question whether we can generally construct (affine) maps all of whose global r-fold points are of a specified type:

Question 6.2.5. Let $r \geq 2$ and $d \geq 1$. Suppose we are given integers $m_1, \ldots, m_r \in \{0, 1, \ldots, d\}$ such that $\sum_{i=1}^{r} m_i = d(r-1)$. Does there exist an affine map $f\colon \sigma^N \to \mathbb{R}^d$ such that all global r-fold points of f are of the same type $\{m_1, m_2, \ldots, m_r\}$?

This question was recently answered positively (with an explicit construction) by Moshe White [51].

6.2.1 Prismatic Maps

Fix parameters $r \geq 2$ and $k \geq 1$ and set

$$N = (rk+1)(r-1), \qquad \text{and} \qquad m = (r-1)k. \tag{6.2}$$

We note that $N + 1 = r(m+1)$, and we fix a partition of the vertices of σ^N into $m+1$ subsets

$$C_j = \{v_{1,j}, \ldots, v_{r,j}\}, \qquad 0 \leq j \leq m, \tag{6.3}$$

consisting of r vertices each; we choose and fix labeling of the vertices in each C_j as indicated. We think of the vertex subsets $C_0, \ldots, C_m$ as **color classes**, and we call a simplex τ of σ^N **colorful** if it contains at most one vertex from each color class C_j, $0 \leq j \leq m$. The colorful simplices form a subcomplex

$$C = C_0 * \cdots * C_m \subset \sigma^N. \tag{6.4}$$

Let us fix a labeling $u_0, \ldots, u_m$ of the vertices of σ^m. This yields a **projection map**

$$p\colon \sigma^N \to \sigma^m \tag{6.5}$$

[1] Specifically, such an affine map is given by the point configuration in $\mathbb{R}^d$ (the images of the vertices) consisting of $(d+1)$ small clusters of $(r-1)$ points centered at the vertices of a $(d+1)$-simplex, plus one point at the barycenter of the simplex.

by setting $p(v_{i,j}) = u_j$ for $1 \le i \le r$ and $0 \le j \le m$ and extending linearly. We note that the colorful simplices are precisely those simplices τ of σ^N such that $p|_\tau$ is injective.

We consider a particular kind of maps whose image is contained in the "prism" $\sigma^m \times \sigma^k \subset \mathbb{R}^d$, and which we call *prismatic*; to motivate the general definition, we first consider the special case of affine maps; see Figure 6.1 for an illustration in the case $k = 1, r = 3$.

Example 6.2.6. For the vertices $v_{i,j}$ of σ^N, we choose *generic* image points[2]

$$f(v_{i,j}) \in \{u_j\} \times \mathring{\sigma}^k, \qquad 1 \le i \le r, \quad 0 \le j \le m, \tag{6.6}$$

and then extend linearly on each face of σ^N to obtain an affine map (called **affine prismatic map**)

$$f\colon \sigma^N \to \sigma^m \times \mathring{\sigma}^k \subset \mathbb{R}^{rk}.$$

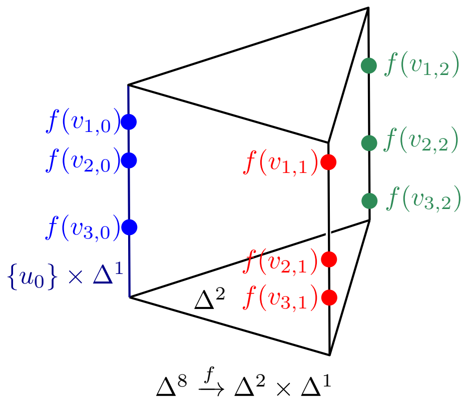


Figure 6.1: For $k = 1$ and $r = 3$ (hence $m = 2$), an affine prismatic map $f\colon \sigma^8 \to \sigma^2 \times \mathring{\sigma}^1 \subset \mathbb{R}^3$ (with images of vertices in C_0, C_1, and C_2 colored blue, red, and green, respectively). The map is extended linearly on each face of σ^8.

The following lemma summarizes the basic properties of affine prismatic maps that we will use to define prismatic maps in general:

Lemma 6.2.7. *Let $f\colon \sigma^N \to \sigma^m \times \sigma^k \subset \mathbb{R}^{rk}$ be an affine prismatic map as defined in Example 6.2.6.*

(a) *There exists a map $h\colon \sigma^N \to \sigma^k$ such that*

(REG) $$f(x) = (p(x), h(x))$$

for $x \in \sigma^N$. We view $h(x)$ as the "height" *of $f(x)$ in the prism $\sigma^m \times \sigma^k$ with* "base" *σ^m and* "vertical component" *σ^k.*

(b) *As an immediate consequence of* (a)*, f has the following properties:*

(PR1) *For every simplex τ of σ^N,*

$$f(\mathring{\tau}) \subseteq p(\mathring{\tau}) \times \mathring{\sigma}^k,$$

where p is the projection map (6.5)*, and $\mathring{\tau}$ denotes the relative interior of τ.*

(PR2) *If τ is colorful (i.e., if $p|_\tau$ is injective) then $f|_\tau$ is also injective.*

[2]The notion of genericity used here is a bit different from the notion of general position as discussed in Section 2.2 and will be discussed in more detail in the proof of Lemma 6.2.7 (c) below.

(c) *Furthermore, apart from non-generic behavior forced by the property* (PR1),[3] *the restriction of the map f to colorful simplices is in general position, in the following sense:*

(PR3) *Let ω be a q-dimensional face of σ^m, $0 \le q \le m$. Then the restriction*

$$f|_{p^{-1}(\mathring{\omega})\cap C} \colon p^{-1}(\mathring{\omega}) \cap C \to \mathring{\omega} \times \mathring{\sigma}^k \cong \mathbb{R}^{q+k}$$

is in general position. In particular, if $\tau_1, \ldots, \tau_s$, $1 \le s \le r$, are pairwise disjoint colorful simplices in $C \subset \sigma^N$ with $p(\tau_i) = \omega$ then

$$\dim \big(f(\mathring{\tau}_1) \cap \cdots \cap f(\mathring{\tau}_s)\big) = \max\{-1, \underbrace{sq - (s-1)(q+k)}_{=q-(s-1)k}\}. \tag{6.7}$$

Proof. Part (a) (and therefore also (b)) follows immediately from the definition of an affine prismatic map. The proof of (c) is by induction on the dimension q. For $q = 0$, the requirement is simply that we choose the image points $f(v_{i,j})$ to be pairwise distinct. More generally, given q-simplices τ_i, $1 \le i \le s$ as in (c), we observe that for each i and each vertex u_j of ω, the affine subspaces $A_i := \operatorname{aff}(f(\tau_i)$ and $\{u_j\} \times \mathbb{R}^k$ of $\operatorname{aff}(\omega) \times \mathbb{R}^k \subset \mathbb{R}^m \times \mathbb{R}^k = \mathbb{R}^d$ intersect transversely, at an angle bounded away from zero. Moreover, it is clear that we could achieve general position if we could perturb each image $f(v_{i,j})$ inside a small $(q+k)$-dimensional open set $U_{i,j}$ in $\operatorname{aff}(\omega) \times \mathbb{R}^k$ containing $f(v_{i,j})$. Since we want to keep the map f prismatic, we are only allowed to perturb each $f(v_{i,j})$ inside a small k-dimensional open set $O_{i,j}$ in $\{u_j\} \times \mathbb{R}^k$. However, in order to analyze the intersections of the $f(\tau_i)$, we can imagine that we first perform this perturbation within $O_{i,j}$ and then further perturb each $f(v_{i,j})$ inside a small q-dimensional open set $Q_{i,j}$ inside A_i. Together these two perturbations would amount to perturbing $f(v_{i,j})$ in a $(q+k)$-dimensional open set, as desired. However, since the second perturbation does not affect A_i, the first one alone is sufficient to bring the subspaces A_i into general position. □

Definition 6.2.8 (**Prismatic Maps**)**.** Let K denote either σ^N or the colorful subcomplex C.

A PL-map $f \colon K \to \sigma^m \times \mathring{\sigma}^k$ is **prismatic** if it satisfies Conditions (PR1) (for all simplices τ in K), (PR2), and (PR3).

A prismatic map is called **regular** if, in addition, it is of the special form (REG).

Thus, a non-regular prismatic map does not need to respect the projection onto the base σ^m (see Figure 6.2 for an example), and this additional flexibility will be convenient for some techncial arguments in what follows.

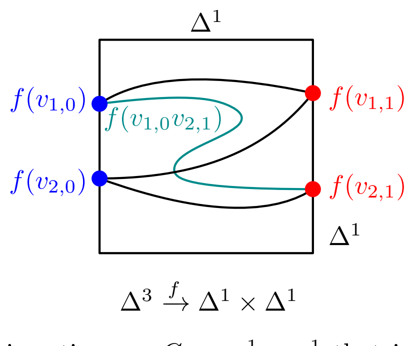


Figure 6.2: For $k = 1$, $r = 2$, a prismatic map $C \to \sigma^1 \times \sigma^1$ that is non-regular; regularity is violated for the image of the edge $v_{1,0}v_{2,1}$.

The following lemmas capture two key properties of prismatic maps.

[3]For instance, the affine map in Figure 6.1 is not, strictly speaking, in general position as a map into $\mathbb{R}^3$, since the three vertices in each color class have collinear images.

Lemma 6.2.9. *Let $f\colon \sigma^N \to \sigma^m \times \mathring{\sigma}^k \subset \mathbb{R}^{rk}$ be a prismatic map. If $y \in f(\tau_1) \cap \dots \cap f(\tau_r)$ is a global r-fold point of f then each simplex τ_i is colorful and of dimension m.*

Proof. Let ω be the unique face of σ^m such that $y \in \mathring{\omega} \times \mathring{\sigma}^k$, and let $q = \dim \omega$. Without loss of generality (up to relabeling), we may assume that the vertex set of ω is $\{u_0, \dots, u_q\}$.

By (PR1), all simplices $\tau_1, \dots, \tau_r$ must be contained in $p^{-1}(\omega)$, so their vertices are contained in $C_0 \cup \dots \cup C_q$, which is a set of size $(q+1)r$. Moreover, every simplex τ_i must contain at least one vertex from each of C_j, $0 \le j \le q$, otherwise (by (PR1) again), the image $f(\tau_i)$ and hence y would be contained in $\partial\omega \times \mathring{\sigma}^k$, contradicting the choice of ω. By straightforward counting, it follows that every τ_i contains exactly one vertex from each C_j, $0 \le j \le q$, i.e., every τ_i is colorful.

Therefore, by Condition (PR3), we have $q = m$, since for $q < m$, (6.7) and induction on q would imply that $f(\tau_1) \cap \dots \cap f(\tau_r) = \emptyset$. □

Lemma 6.2.10. *Every prismatic map $g\colon C \to \sigma^m \times \mathring{\sigma}^k$ can be extended to a prismatic map $f\colon \sigma^N \to \sigma^m \times \mathring{\sigma}^k$.*

Proof. We can construct the extension by induction on the dimension of the faces τ of $\sigma^N \setminus C$: Suppose that f is already defined on $\partial\tau$. Let $\omega = p(\tau)$. We can extend f to $\mathring{\tau}$ by coning, using that $\omega \times \sigma^k$ is convex. More precisely, fix a point $b \in \mathring{\tau}$, choose an arbitrary image $f(b) \in \mathring{\omega} \times \mathring{\sigma}^k$ and extend f linearly. □

Using these two lemmas, the proof of Theorem 6.2.1 reduces to the following:

Proposition 6.2.11. *Suppose $r \ge 6$ is not a prime power and $k \ge 3$. Then there exists a prismatic map $g\colon C \to \sigma^m \times \mathring{\sigma}^k$ without global r-fold points.*

Proof of Theorem 6.2.1 using Proposition 6.2.11. Let $r \ge 6$ is not a prime power, $k = 3$, and let g be the prismatic map whose existence is guaranteed by the proposition.

By Lemma 6.2.10, we can extend g to a prismatic map $f\colon \sigma^N \to \sigma^m \times \mathring{\sigma}^3$, and by Lemma 6.2.9, the map f has no global r-fold points since $g = f|_C$ does not have any, which proves the theorem. □

Proof of Corollary 6.2.4. The corollary follows directly from Lemma 6.2.9 and the affine prismatic maps constructed in Example 6.2.6. □

6.2.2 A Deleted Product Criterion For Prismatic Maps

Thus, it remains to prove Proposition 6.2.11. For this purpose, we will need analogues, for the restricted class of prismatic maps, of the Deleted Product Criterion, of the r-fold Van Kampen obstruction, and of r-fold finger moves. We begin by defining a suitable configuration space.

The prismatic configuration space $X \cong_{\mathfrak{S}_r} (\mathfrak{S}_r)^{*(m+1)}$. By Lemma 6.2.9, the preimages of global r-fold points of a prismatic map are supported on r pairwise disjoint colorful m-simplices $\tau_1, \dots, \tau_r$ in $C \subset \sigma^N$. Using the fixed labeling $C_j = \{v_{1,j}, \dots, v_{r,j}\}$ of the r vertices in each color class, we can encode such an r-tuple of simplices using an $(m+1)$-tuple of permutations $\pi_j \in \mathfrak{S}_r$. Slightly more generally, we have the following:

Observation 6.2.12. Suppose that $J = \{j_0, \dots, j_q\}$ is a $(q+1)$-element subset of $\{0, \dots, m\}$, $0 \le q \le m$, and that

$$(\tau_1, \dots, \tau_r)$$

is an r-tuple of pairwise disjoint q-simplices in $C_{j_0} * \dots * C_{j_q}$. Such an r-tuple of simplices corresponds bijectively to a $(q+1)$-tuple

$$\boldsymbol{\pi} = (\pi_{j_0}, \dots, \pi_{j_q}) \tag{6.8}$$

of permutations $\pi_j \in \mathfrak{S}_r$ given by

$$\tau_i \cap C_j = v_{\pi_j(i),j} \tag{6.9}$$

for $1 \le i \le r$ and $j \in J$.

Observation 6.2.13. Consider the the $(m+1)$-fold join

$$(\mathfrak{S}_r)^{*(m+1)}$$

(where we view the symmetric group $\mathfrak{S}_r$ as a zero-dimensional complex). Every point in $(\mathfrak{S}_r)^{*(m+1)}$ can be written as a formal convex combination

$$\lambda_0\pi_0 + \cdots + \lambda_m\pi_m, \tag{6.10}$$

with $\pi_j \in \mathfrak{S}_r$ and $\lambda_j \in [0,1]$, $\sum_{j=1}^m \lambda_j = 1$.

For $0 \le q \le m$, a q-dimensional face of $(\mathfrak{S}_r)^{*(m+1)}$ is uniquely described by a pair

$$(J, \boldsymbol{\pi}) \tag{6.11}$$

where $J = \{j_0, \ldots, j_q\} \subseteq \{0, \ldots, m\}$ and $\boldsymbol{\pi} = (\pi_{j_0}, \ldots, \pi_{j_q})$ as in (6.8); the corresponding face consists of all formal convex combinations of the form $\sum_{j\in J} \lambda_j\pi_j$, $0 \le \lambda_j \le 1$.

The group action. For every $m \ge 0$, let $E^m_{\mathfrak{S}_r}$ denote an m-dimensional, $(m-1)$-connected free $\mathfrak{S}_r$-cell complex. Such complexes exist for all $n \ge 0$: one can take the $(m+1)$-fold join $E^m_{\mathfrak{S}_r} = (\mathfrak{S}_r)^{*(m+1)}$, where $\mathfrak{S}_r$ is considered as a 0-dimensional complex and acts on itself by right multiplication. They have the universal property that every free $\mathfrak{S}_r$-cell complex X of dimension $\dim X \le n$ maps equivariantly into $E^n_{\mathfrak{S}_r}$ (see [33, Sec. 6.2]), and fullfill the following crucial property:

Theorem 6.2.14 (Özaydin [36]). *Let $d \ge 1$ and $r \ge 2$. There exists an equivariant map $E^{d(r-1)}_{\mathfrak{S}_r} \to_{\mathfrak{S}_r} S^{d(r-1)-1}$ if and only if r is not a prime power.*

The join $(\mathfrak{S}_r)^{*(m+1)}$ is an $E^m_{\mathfrak{S}_r}$-space, i.e., it is an m-dimensional and $(m-1)$-connected space on which the group $\mathfrak{S}_r$ acts freely, by multiplication on the right,

$$(\lambda_0\pi_0 + \cdots + \lambda_m\pi_m) \cdot \pi = \lambda_0(\pi_0\pi) + \cdots + \lambda_m(\pi_m\pi), \tag{6.12}$$

for $\pi, \pi_0, \ldots, \pi_m \in \mathfrak{S}_r$ and $\lambda_0, \ldots, \lambda_m \in [0,1]$.

There is an alternative way of looking at this space: Consider the space

$$X := \{\boldsymbol{x} = (x_1, \ldots, x_r) \in C^r_\Delta \mid p(x_1) = \cdots = p(x_r)\},$$

on which $\mathfrak{S}_r$ acts by permuting the factors.[4] The space X is a simplicial complex, whose faces can be described as follows: For $0 \le q \le m$, a q-dimensional simplex of X is of the form

$$\tau = \tau_1 \times_p \ldots \times_p \tau_r := \{\boldsymbol{x} = (x_1, \ldots, x_r) \in \tau_1 \times \cdots \times \tau_r \mid p(x_1) = \cdots = p(x_r)\}, \tag{6.13}$$

where $(\tau_1, \ldots, \tau_r)$ is an r-tuple of pairwise disjoint q-simplices of C, each of which projects via p onto the same q-dimensional face ω of the base space σ^m.

Orientations. In what follows, unless indicated otherwise, we consider the simplices τ_i and $\tau = \tau_1 \times_p \ldots \times_p \tau_r$ to be *oriented compatibly*, via the projection p (which restricts to an isomorphism on each of these simplices) with a given orientation of the corresponding face ω of the base σ^m; such an orientation can be described in terms of an ordering of the set J indexing the vertices of ω and the corresponding color classes C_j, $j \in J$.

Lemma 6.2.15. *There is a canonical equivariant simplicial homeomorphism*

$$\Phi\colon (\mathfrak{S}_r)^{*(m+1)} \cong_{\mathfrak{S}_r} X$$

which sends $\lambda_0\pi_0 + \cdots + \lambda_m\pi_m$ to $\boldsymbol{x} = (x_1, \ldots, x_r)$ given by $x_i = \sum_{j=0}^m \lambda_j v_{\pi_j(i),j}$.

[4] The definition of X is closely related to the standard *pullback* or *fiber product* of r copies of C over the common base space σ_m, except for the additional condition that we only take r-tuples of points supported in pairwise disjoint simplices; one might call X the *deleted r-fold fiber product* of C.

Proof. For $\boldsymbol{x} = (x_1, \dots, x_r) \in X$, consider the face ω of σ^m that supports the projections $p(x_i)$, and let $\{u_j \mid j \in J\}$ be the vertex set of ω. We can write $p(x_1) = \cdots = p(x_r) = \sum_{j \in J} \lambda_j u_j$, where $\lambda_j \in (0, 1)$ for $j \in J$ and $\sum_j \lambda_j$. Then each x_i is supported on a $(|J| - 1)$-dimensional colorful simplex τ_i with $\tau_i \cap C_j = 1$ for $j \in J$; since the x_i have disjoint supports, there are permutations $\pi_j \in \mathfrak{S}_r$, $j \in J$, defined by Equation (6.9), such that $x_i = \sum_{j \in J} \lambda_j v_{\pi_j(i),j}$. This defines $\Phi^{-1}(\boldsymbol{x}) = (J, \boldsymbol{\pi})$, where $\boldsymbol{\pi} = (\pi_j \mid j \in J)$.

It is straightforward to verify that Φ^{-1} is continuous (the λ_j are the barycentric coordinates of each x_i), and Φ is equivariant since $x_{\pi(i)} = \sum_{j=0}^{m} \lambda_j v_{\pi_j(\pi(i)),j}$. □

Using this configuration space, we obtain, as an analogue of Lemma 1.1.2, the following necessary condition for the existence of *regular* prismatic maps without global r-fold points:

Lemma 6.2.16. *Suppose $f\colon C \to \sigma^m \times \mathring{\sigma}^k \subset \mathbb{R}^{rk}$ is a regular prismatic map and $h\colon C \to \mathring{\sigma}^k \subset \mathbb{R}^k$ is the corresponding height function, i.e., $f(x) = (p(x), h(x))$ for $x \in C$. Consider the induced map*

$$\widetilde{h}\colon X \to (\mathbb{R}^k)^r, \qquad \widetilde{h}(x_1, \dots, x_r) = \big(h(x_1), \dots, h(x_r)\big).$$

(a) *Suppose that $y \in f(\tau_1) \cap \cdots \cap f(\tau_r) \subset \sigma^m \times \mathring{\sigma}^k$ is a global r-fold point of f, and that z is the projection of y onto σ^k (i.e., $y = (w, z)$ for some $w \in \sigma^m$). Then the r-fold intersection point y corresponds to the pairwise intersection point $(z, \dots, z)$ of $\widetilde{h}(\tau)$ with the thin diagonal $\delta_r(\mathbb{R}^k)$, where $\tau = \tau_1 \times_p \ldots \times_p \tau_r$ is the m-simplex of X corresponding to the τ_i.*

(b) *Moreover, up to a universal sign $\varepsilon_{r,k}^{\mathrm{PRIS}}$ depending only on r and k, the intersection signs at these points agree, i.e.,*

$$\operatorname{sign}_{(z,\dots,z)}\big(\widetilde{h}(\tau), \delta_r(\mathbb{R}^k)\big) = \varepsilon_{r,k}^{\mathrm{PRIS}} \cdot \operatorname{sign}_y(f(\tau_1), \dots, f(\tau_r)). \tag{6.14}$$

(c) *In particular, if f has no global r-fold point, then there is an equivariant map*

$$\widetilde{h}\colon X \to_{\mathfrak{S}_r} (\mathbb{R}^k)^r \setminus \delta_r(\mathbb{R}^k) \simeq_{\mathfrak{S}_r} S^{m-1}. \tag{6.15}$$

Proof. This is analogous to the proof of Lemma 1.1.2. It is clear that the map $\widetilde{h}$ is equivariant. Since h is a prismatic map, any global r-fold point of f occurs as an r-intersection point of pairwise disjoint m-simplices $\tau_1, \dots, \tau_r$. Moreover, since $f = (p, h)$ is regular, we have $y = f(x_1) = \cdots = f(x_r)$ for $x_i \in \tau_i$, $1 \le i \le r$, if and only if $p(x_1) = \cdots = p(x_r)$ and $z = h(x_1) = \cdots = h(x_r)$, or equivalently $(x_1, \dots, x_r) \in \tau = \tau_1 \times_p \cdots \times_p \tau_r \subset X$ and $(z, \dots, z) \in \widetilde{h}(\tau) \cap \delta_r(\mathbb{R}^k)$. This proves (a) and hence (c), since, as before, we have an equivariant homotopy equivalence $\rho\colon (\mathbb{R}^k)^r \setminus \delta_r(\mathbb{R}^k) \simeq_{\mathfrak{S}_r} S^{(r-1)k-1} = S^{m-1}$.

It remains to prove (b). Since intersection signs are completed locally, it suffices to consider the case that the height function h and hence $f = (p, h)$ are simplexwise affine maps, and that the intersection $f(\tau_1) \cap \cdots \cap f(\tau_r)$ consists of a single point $y = (w, z)$. We may assume that the base σ^m has the standard orientation given by the identity matrix I_m, and that the orientation of each affine simplex $f(\tau_i)$ is given by $\left[\begin{smallmatrix} A_i \\ I_m \end{smallmatrix}\right]$, where $A_i \in \mathbb{R}^{k \times m}$ is the matrix describing the linear part of the affine function $h|_{\tau_i}$. Thus, the orientation of $\tilde{h}(\tau)$ is given by the matrix $[A_1 | \dots | A_r]^\top \in \mathbb{R}^{rk \times m}$, and the pairwise intersection sign of $\tilde{h}(\tau)$ and $\delta_r(\mathbb{R}^k)$ equals the determinant of the matrix

$$B := \begin{bmatrix} A_1 & I_k \\ A_2 & I_k \\ & \vdots \\ A_r & I_k \end{bmatrix} \in \mathbb{R}^{rk \times rk}$$

Moreover, by Lemma 2.3.1 (d), we have the identity

$$\operatorname{sign}_y(f(\tau_1), \dots, f(\tau_r)) = \varepsilon_{r,k} \cdot \operatorname{sign}_{(y,\dots,y)}(f(\tau_1) \times \cdots \times f(\tau_r), \delta_r(\mathbb{R}^d)) \tag{6.16}$$

between the r-fold intersection sign in $\mathbb{R}^d$ and the pairwise intersection sign with the thin diagonal in $(\mathbb{R}^d)^r$, where $\varepsilon_{r,k}$ is the universal sign introduced in (2.4). Furthermore, the pairwise intersection sign on the right-hand side of (6.16) is equal to the determinant of the matrix

$$A := \begin{bmatrix} A_1 & & & & I_k & \\ I_m & & & & & I_m \\ & A_2 & & & I_k & \\ & I_m & & & & I_m \\ & & \ddots & & & \\ & & & A_r & I_k & \\ & & & I_m & & I_m \end{bmatrix} \in \mathbb{R}^{rd \times rd}$$

We can modify this matrix A, without changing its determinant, to obtain the matrices A' and A'' described below, as follows: First we get A' by successively subtracting the columns of A corresponding to each submatrix A_i from the last m columns. Next, we eliminate the copies of the A_i appearing in the left part of A' by subtracting suitable linear combinations of the rows corresponding to the remaining copies of I_m. In this way, we obtain A'', where

$$A' = \begin{bmatrix} A_1 & & & & I_k & -A_1 \\ I_m & & & & & 0 \\ & A_2 & & & I_k & -A_2 \\ & I_m & & & & 0 \\ & & \ddots & & & \\ & & & A_r & I_k & -A_r \\ & & & I_m & & 0 \end{bmatrix}, \quad \text{and} \quad A'' = \begin{bmatrix} 0 & & & & I_k & -A_1 \\ I_m & & & & & 0 \\ & 0 & & & I_k & -A_2 \\ & I_m & & & & 0 \\ & & \ddots & & & \\ & & & 0 & I_k & -A_r \\ & & & I_m & & 0 \end{bmatrix}$$

Finally, by multiplying the last $m = k(r-1)$ columns of A'' by -1 and by a total of $km\binom{r+1}{2}$ row transpositions, we can transform A'' into

$$A''' = \begin{bmatrix} I_m & & & & & \\ & I_m & & & & \\ & & \ddots & & & \\ & & & I_m & & \\ & & & & I_k & A_1 \\ & & & & I_k & A_2 \\ & & & & \vdots & \\ & & & & I_k & A_r \end{bmatrix} = \begin{bmatrix} I_{rm} & \\ & B \end{bmatrix}$$

Thus,

$$\operatorname{sign}_y(f(\tau_1), \dots, f(\tau_r)) = \varepsilon_{r,k} \det A = \varepsilon^{\mathrm{PRIS}}_{r,k} \det A''' = \varepsilon^{\mathrm{PRIS}}_{r,k} \det B = \varepsilon^{\mathrm{PRIS}}_{r,k} \operatorname{sign}_{(z,\dots,z)}\big(\widetilde{h}(\tau), \delta_r(\mathbb{R}^k)\big),$$

as we set out to show, where

$$\varepsilon^{\mathrm{PRIS}}_{r,k} := \varepsilon_{r,k} \cdot (-1)^{k^2(r-1)\binom{r+1}{2} + k(r-1)}. \tag{6.17}$$

□

Moreover, for codimension $k \geq 3$, we will prove the following partial converse of Lemma 6.2.16:

Theorem 6.2.17 (Sufficiency of the Prismatic Deleted Product Criterion). *Let $r \geq 2$, $N = (rk+1)(r-1)$ and $m = (r-1)k$.*

If $k \geq 3$ and if there exists a $\mathfrak{S}_r$-equivariant map

$$X \to_{\mathfrak{S}_r} (\mathbb{R}^k)^r \setminus \delta_r(\mathbb{R}^k) \simeq_{\mathfrak{S}_r} S^{m-1} \tag{6.18}$$

then there exists a prismatic map
$$C \to \sigma^m \times \mathring{\sigma}^k$$
without global r-fold point.

We believe that it should be possible to strengthen the conclusion of the theorem and obtain a regular prismatic map. However, the current form of the theorem serves our purposes and, together with Özaydin's Theorem 6.2.14, implies Proposition 6.2.11, and hence the existence of counterexamples to the topological Tverberg conjecture in dimension $3r$ (Theorem 6.2.1):

Proof of Proposition 6.2.11 using Theorem 6.2.17. Suppose $r \geq 6$ is not a prime power and $k \geq 3$. Then Theorem 6.2.14 implies that there exists an an equivariant map $X \to_{\mathfrak{S}_r} S^{m-1}$. Consequently, by Theorem 6.2.17, there exists a prismatic map $C \to \sigma^m \times \mathring{\sigma}^k$ without global r-fold point. □

The proof of Theorem 6.2.17 is structured along similar lines as the proof of Theorem 5.1.1.

In a first step, by Theorem 5.2.1, there is a primary obstruction $\mathfrak{o}(X) \in H^m_{\mathfrak{S}_r}(X; \mathcal{Z})$ such that there exists an equivariant map $X \to_{\mathfrak{S}_r} (\mathbb{R}^k)^r \setminus \delta_r(\mathbb{R}^k)$ if and only if $\mathfrak{o}(X) = 0$. Moreover, by Lemma 6.2.16, any regular prismatic map $f = (p, h) \colon C \to \sigma^m \times \mathring{\sigma}^k$ induces an equivariant map $\widetilde{h} \colon X \to (\mathbb{R}^k)^r$ in general position, and thus, by Lemma 5.2.2, the obstruction $\mathfrak{o}(X) = [\varphi_f]$ is represented by the *prismatic intersection number cocycle* φ_f defined on m-cells $\tau = \tau_1 \times_p \cdots \times_p \tau_r$ of X by
$$\varphi_f(\tau) = \tilde{h}(\tau) \bullet \delta_r(\mathbb{R}^k) = \varepsilon^{\mathrm{PRIS}}_{r,k} f(\tau_1) \bullet \ \ldots \bullet f(\tau_r), \tag{6.19}$$
where the last equality follows from (6.14). Note that, while the middle term of this equation makes sense only for *regular* prismatic maps, the right-hand side is defined for arbitrary prismatic maps, and we will use this as the definition of the intersection cocycle for arbitrary prismatic maps f.

The main technical lemma to prove Theorem 6.2.17 is the following:

Lemma 6.2.18 (Prismatic Finger Moves). *Suppose $r \geq 2$, $k \geq 1$, $m = (r-1)k$ and $N = (kr+1)(r-1)$. Suppose furthermore that $f \colon C \to \sigma^m \times \mathring{\sigma}^k$ is a prismatic map, that η is an oriented $(m-1)$-simplex of X, and that $\delta 1_{\eta \cdot \mathfrak{S}_r}$ is the corresponding equivariant m-dimensional coboundary (see Section 5.3).*

Then there exists a prismatic map $f' \colon C \to \sigma^m \times \mathring{\sigma}^k$ such that
$$\varphi_{f'} = \varphi f - \delta 1_{\eta \cdot \mathfrak{S}_r}.$$

Proof of Theorem 6.2.17 using Lemma 6.2.18. We start by choosing and fixing an arbitrary *regular* prismatic map $f = (p, h) \colon C \to \sigma^m \times \mathring{\sigma}^k$ (e.g., an affine prismatic map as described in Example 6.2.6). By assumption, there exists an an equivariant map $X \to_{\mathfrak{S}_r} S^{m-1}$. This is equivalent to the vanishing of the primary obstruction, $\mathfrak{o}X = [\varphi_f] = 0$, which means that the prismatic intersection number cocycle φ_f can be written as a finite sum of elementary equivariant coboundaries. By repeatedly applying Lemma 6.2.18, once for each elementary coboundary in the sum, we thus arrive at a prismatic map f' such that $\varphi_{f'} = 0$ as a cocycle, i.e.,
$$f'(\tau_1) \bullet \ \ldots \bullet f'(\tau_r) = 0$$
for every r-tuple of pairwise disjoint m-simplices of C. Thus, we can arbitrarily pair up the global r-fold points in $f'(\tau_1) \cap \cdots \cap f'(\tau_r)$ into pairs of opposite sign. To conclude, we eliminate each pair by applying the r-fold Whitney trick, without introducing new global r-fold points; this is possible since the codimension $d - \dim C = k$ is at least 3.

More precisely, suppose $x, y \in f'(\tau_1) \cap \cdots \cap f'(\tau_r)$ is a pair of global r-fold points of f' of opposite sign. By the r-fold Whitney trick there are are ambient isotopies $H^2, \ldots, H^r$ of $\mathbb{R}^d$ such that
$$f'(\tau_1) \cap H^2_1(f'(\tau_2) \cap \cdots \cap H^r_1(f'(\tau_r) = f'(\tau_1) \cap f'(\tau_2) \cap \cdots \cap f'(\tau_r) \setminus \{x, y\}.$$
Moreover, we can choose these isotopies to be fixed outside an open d-ball B that avoids all other faces of C and is contained in $\mathring{\sigma}^m \times \mathring{\sigma}^k$; in particular, each isotopy fixes the boundary of the prism

$\sigma^m \times \sigma^k$. Thus, if we define a new PL-map $f''\colon C \to \sigma^m \times \mathring{\sigma}^k$ by setting $f''(x) = H^i(f'(x))$ for $x \in \mathring{\tau}_i$, $2 \le i \le r$, and $f''(x) = f'(x)$ otherwise, then f'' is again a prismatic map and has the same global r-fold points as f', except for $\{x, y\}$. By applying this procedure a finite number of times, we arrive at a prismatic map $g\colon C \to \sigma^m \times \mathring{\sigma}^k$ that has no global r-fold points at all. □

It remains to prove Lemma 6.2.18. This is done in the following subsection.

6.2.3 r-Fold Linking Numbers and Prismatic Finger Moves

Throughout this subsection, let $r \ge 2$, $k \ge 1$, and let $m = (r-1)k$.

Suppose that $\Sigma_1, \dots, \Sigma_r$ are r PL-spheres of dimension $m-1$ contained in a PL-sphere S^{rk-1} and in general position with respect to one another. Suppose furthermore that we have chosen orientations for each of the Σ_i and for S^{rk-1}.

By Alexander duality (see, e.g., [23, Theorem 3.44]),

$$H_{k-1}(S^{rk-1} \setminus \Sigma_r) \cong H^{m-1}(\Sigma_r) \cong \mathbb{Z}.$$

In order to fix a specific isomorphism with the integers, we fix a generator ζ of $H_{k-1}(S^{rk-1} \setminus \Sigma_r)$ as follows: Choose a small k-dimensional PL-disk D in S^{rk-1} that intersects Σ_r transversely in a single point, and orient D such that this pairwise intersection point has positive sign; then ζ is represented by ∂D.

By the general position assumption, $\Sigma_1 \cap \dots \cap \Sigma_r = \emptyset$. The orientations of the Σ_i induce an orientation of the intersection $\Sigma_1 \cap \dots \cap \Sigma_{r-1}$, as described in Section 2.3. Moreover, this oriented intersection is a $(k-1)$-cycle (in fact, a closed $(k-1)$-dimensional PL-manifold) and thus defines a homology class

$$[\Sigma_1 \cap \dots \cap \Sigma_{r-1}] \in H_{k-1}(S^{rk-1} \setminus \Sigma_r) \cong \mathbb{Z}.$$

Definition 6.2.19. Via the choice of the generator ζ, we can write $[\Sigma_1 \cap \dots \cap \Sigma_{r-1}] = \ell \cdot \zeta$ for a uniquely defined integer $\ell = \ell(\Sigma_1, \dots, \Sigma_r) \in \mathbb{Z}$, which we call the ***r*-fold linking number** of $\Sigma_1, \dots, \Sigma_r$ in S^{rk-1}.

We remark that the r-fold linking number depends on the order of the Σ_i and on the choice of the orientations.

Next, suppose that $\sigma_1, \dots, \sigma_r$ are r PL-balls of dimension $m = (r-1)k$ properly embedded in a PL-ball B^{rk}. Then we can apply the previous definition to the $(m-1)$-dimensional PL-spheres $\Sigma_i = \partial\sigma_i$ in $S^{rk-1} = \partial B^{rk}$ (with the induced orientations on the boundaries).

Lemma 6.2.20. *In the setting described above, the r-fold linking number $\ell(\partial\sigma_1, \dots, \partial\sigma_r)$ of the $\partial\sigma_i$ in $S^{rk-1} = \partial B^{rk}$ is equal to the algebraic r-fold intersection number $\sigma_1 \bullet \dots \bullet \sigma_r$ of the σ_i in B^{rk}.*

Proof. The argument is similar to the one for the standard 2-fold intersection and linking numbers (see, e.g., [38, Lemma 5.15]).

First, we note that the inclusion map $\iota\colon \partial B^{rk} \setminus \partial\sigma_r \hookrightarrow B^{rk} \setminus \sigma_r$ induces an isomorphism $\iota_*\colon H_{k-1}(\partial B^{rk} \setminus \partial\sigma_r) \cong H_{k-1}(B^{rk} \setminus \sigma_r)$; in particular, $\iota_*(\zeta)$ is a generator of $H_{k-1}(B^{rk} \setminus \sigma_r)$. Thus, r-fold linking number $\ell = \ell(\partial\sigma_1, \dots, \partial\sigma_r)$ can be equivalently defined as the unique integer such that $[\partial\sigma_1 \cap \dots \cap \partial\sigma_r] = \ell \cdot \iota_*(\zeta) \in H_{k-1}(B^{rk} \setminus \sigma_r)$.

The generator $\iota_*(\zeta)$ is represented by the boundary ∂D of the k-dimensional disk $D \subset S^{rk-1}$ used above. Alternatively, we can slightly translate this disk into the interior to obtain a small k-dimensional PL-disk D' in $\mathring{B}^{rk}$ that intersects σ_r transversely in a single point and that is oriented so that this pairwise intersection has positive sign; then $\iota_*(\zeta) = [\partial D'] \in H_{k-1}(B^{rk} \setminus \sigma_r)$.

By Lemma 2.3.5, the r-fold intersection number $\sigma_1 \bullet \dots \bullet \sigma_r$ equals the 2-fold intersection number $\omega \bullet \sigma_r$, where

$$\omega := \sigma_1 \cap \dots \cap \sigma_{r-1}$$

is the oriented intersection of the first $(r-1)$ terms, which is an oriented k-dimensional PL-manifold with boundary $\partial\omega = \partial\sigma_1 \cap \dots \cap \partial\sigma_{r-1}$, properly embedded in B^{rk}.

Consider an intersection point

$$y \in \omega \cap \sigma_r = \sigma_1 \cap \dots \cap \sigma_r$$

with 2-fold intersection sign $\mathrm{sign}_y(\omega, \sigma_r) \in \{-1, +1\}$. Choose a small k-dimensional disk $D_y \subset \omega$ containing y in its interior, with the orientation induced from ω. Then $\mathrm{sign}_y(\omega, \sigma_r) = \mathrm{sign}_y(D_y, \sigma_r)$, and the sphere ∂D_y (with the induced orientation) represents the element

$$\mathrm{sign}_y(\omega, \sigma_r) \cdot \iota_*(\zeta) \in H_{k-1}(B^{rk} \setminus \sigma_r).$$

Choosing such a k-ball D_y for each $y \in \omega \cap \sigma_r$, we can consider

$$\omega \setminus \Big(\bigcup_{y \in \omega \cap \sigma_r} \mathring{D}_y \Big).$$

This is an oriented k-dimensional manifold with boundary and hence a k-dimensional chain in $B^{rk} \setminus \sigma_r$ witnessing that the two $(k-1)$-cycles

$$\partial\omega = \partial\sigma_1 \cap \dots \cap \partial\sigma_{r-1}$$

and

$$\bigcup_{y \in \sigma_1 \cap \dots \cap \sigma_r} \partial D_y.$$

are homologous in $B^{rk} \setminus \sigma_r$. Thus, they define the same homology class

$$[\partial\sigma_1 \cap \dots \cap \partial\sigma_{r-1}] = \sum_{y \in \sigma_1 \cap \dots \cap \sigma_r} \mathrm{sign}_y(\omega, \sigma_r) \cdot \iota_*(\zeta) \in H_{k-1}(B^{rk} \setminus \sigma_r).$$

Therefore, the linking number $\ell(\partial\sigma_1, \dots, \partial\sigma_r)$ is equal to the intersection number $\sigma_1 \bullet \dots \bullet \sigma_r = \sum_{y \in \sigma_1 \cap \dots \cap \sigma_r} \mathrm{sign}_y(\omega, \sigma_r)$, as we set out to show. □

Modifying the r-fold linking number As before, let $\Sigma_1, \dots, \Sigma_r$ be r PL-spheres of dimension $m-1$ in general position in a PL-sphere S^{rk-1}. We describe a down-to-earth way of changing their r-fold linking number by ± 1.

Let $\varepsilon \in \{-1, +1\}$. Choose $(r-1)$ small PL-spheres $S_1, .., S_{r-1}$ of dimension $m-1$ embedded in general position in S^{rk-1}. We arrange the spheres and orient them in such a way that their oriented intersection

$$S_1 \cap \dots \cap S_{r-1}$$

is an oriented $(k-1)$-sphere S that links precisely once with Σ_r, with the chosen sign ε, i.e.,

$$[S_1 \cap \dots \cap S_{r-1}] = \varepsilon\zeta \in H_{k-1}(S^{rk-1} \setminus \Sigma_r).$$

This embedding can be performed in a small neighbourhood of an affine piece of Σ_r in S^{rk-1}. In particular, we choose the spheres S_i so that they are disjoint from all Σ_j, $1 \le i, j \le r-1$.

Finally, for $1 \le i \le r-1$, we connect Σ_i to S_i by an orientation-preserving pipe (see Section 4.3), as in the proof of Lemma 5.3.2 to obtain a new $(m-1)$-dimensional PL-sphere $\Sigma_i' = \Sigma_i \# S_i$. By construction, this has the effect of modifying the r-fold linking number by ε, i.e.,

$$\ell(\Sigma_1', \dots, \Sigma_r') = \ell(\Sigma_1, \dots, \Sigma_r) + \varepsilon.$$

In particular, suppose that $\sigma_1, \dots, \sigma_r$ are m-dimensional PL-balls properly containd in B^{rk}, and that we modify the spheres $\Sigma_i = \partial\sigma_i$ in ∂B^{rk} as just described. Suppose furthermore that we arbitrarily choose m-dimensional PL-balls σ_i' in B^{rk} with $\partial\sigma_i' = \Sigma_i'$ (this is always possible, e.g., by coning over Σ_i' from the center of B^{rk}). Then, by Lemma 6.2.20 the r-fold intersection number of the balls in B^{rk} also changes by ε, i.e.,

$$\sigma_1' \bullet \dots \bullet \sigma_r' = \sigma_1 \bullet \dots \bullet \sigma_r + \varepsilon. \tag{6.20}$$

We are now ready to prove the last remaining lemma.

Proof of Lemma 6.2.18. Let $f\colon C \to \sigma^m \times \mathring{\sigma}^k$ be a prismatic map, and let η be an oriented $(m-1)$-simplex of X. We know that $\eta = \eta_1 \times_p \cdots \times_p \eta_r$ for r pairwise disjoint $(m-1)$-simplices of C that project onto the same $(m-1)$-simplex $\omega = p(\eta_1) = \cdots = p(\eta_r)$ of the base σ^m of the prism.

In analogy with the previously described way of changing linking numbers, we modify f to obtain a new new prismatic map $f'\colon : C \to \sigma^m \times \mathring{\sigma}^k$ as follows:

- We select $r-1$ small oriented PL-spheres $S_1, ..., S_{r-1}$ of dimension $m-1$ in general position in $\omega \times \mathring{\sigma}^k$; we choose these sphere so that their intersection $S_1 \cap \cdots \cap S_{r-1}$ is a flat $(k-1)$-dimensional PL-sphere S "linking" with $f(\eta_r)$ exactly once and with negative sign, i.e., if we fill this sphere with k-dimensional PL-ball then this ball intersects $f(\eta_r)$ exactly once, with negative intersection sign.

- For $1 \leq i \leq r-1$, we connect $f(\eta_i)$ to S_i by an orientation-preserving pipe to create a new $(m-1)$-dimensional ball in $\omega \times \mathring{\sigma}^k$ with the same boundary as $f(\eta_i)$.

- We define f' to agree with f on all faces of C of dimension less than $m-1$ and on all $(m-1)$-simplices of C except for $\eta_1, \ldots, \eta_{r-1}$. On η_i, $1 \leq i \leq r-1$, we define f' so that $f'(\eta_i)$ equals the result of piping $f(\eta_i)$ with S_i (this possible, since $f(\eta_i)$ and the result of the piping are two PL-balls in $\omega \times \sigma^k$ with the same boundary).[5]

- Finally, let τ be an m-dimensional simplex of C. If τ does not contain any one of the simplices $\eta_1, \ldots, \eta_{r-1}$ on its boundary, then we define $f'|_\tau = f|_\tau$. Otherwise, we redefine f' on τ so that $f'(\tau)$ is an m-dimensional ball properly contained in $\sigma^m \times \sigma_k$; this is always possible, for instance by coning over $f'(\partial\tau)$ from a point in general position in the interior of $\sigma^m \times \sigma^k$.

It is clear that the resulting map f' is prismatic. We claim that its prismatic intersection number cocycle satisfies

$$\varphi_{f'} = \varphi_f - \delta 1_{\eta \cdot \mathfrak{S}_r}.$$

To see this, consider an m-simplex $\tau_1 \times_p \cdots \times_p \tau_r$ of X corresponding to an r-tuple of pairwise disjoint m-simplices $\tau_1, \ldots, \tau_r$ of C. Up to the universal sign $\varepsilon^{\mathrm{PRIS}}_{r,k}$, the value of $\varphi_{f'}(\tau_1 \times_p \cdots \times_p \tau_r)$ equals the intersection number $f'(\tau_1) \bullet \ \ldots \bullet f'(\tau_r)$ in the rk-ball $\sigma^m \times \sigma^k$, or equivalently, the linking number $\ell(f'(\partial\tau_1), \ldots, f'(\partial\tau_r))$ in $\partial(\sigma^m \times \sigma^k)$.

If there is one τ_j that contains none of the η_i in its boundary, then

$$\ell(f'(\partial\tau_1), \ldots, f'(\partial\tau_r)) = \ell(f(\partial\tau_1), \ldots, f(\partial\tau_r))$$

is unchanged.

Otherwise, up to a permutation of the indices, we may assume that η_i is contained in the boundary of τ_i, $1 \leq i \leq r$. In this case, as discussed above, the piping of η_i, $1 \leq i \leq r-1$ has the effect that

$$\ell(f'(\partial\tau_1), \ldots, f'(\partial\tau_r)) = \ell(f(\partial\tau_1), \ldots, f(\partial\tau_r)) - 1,$$

i.e., $\varphi_{f'}(\tau_1 \times_p \cdots \times_p \tau_r) - \delta 1_{\eta \cdot \mathfrak{S}_r}(\tau_1 \times_p \cdots \times_p \tau_r)$. By equivariance, the same is true if η_i is contained in the boundary of $\tau_{\pi(i)}$, $1 \leq i \leq r$. This proves the claim and hence the lemma. □

This also completes the proofs of Theorems 6.2.17 and 6.2.1.

[5]For $k \geq 3$, there even exists an ambient homotopy H^i of $\omega \times \sigma^k$, fixed on the boundary, such that we can take $f'|_{\eta_i} = H^i_1 \circ f|_{\eta_i}$, but we will not need this.

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