



Global Renormalised Solutions and Equilibration of Reaction–Diffusion Systems with Nonlinear Diffusion

Klemens Fellner¹ · Julian Fischer² · Michael Kniely³ · Bao Quoc Tang¹ 

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Abstract

The global existence of renormalised solutions and convergence to equilibrium for reaction–diffusion systems with nonlinear diffusion are investigated. The system is assumed to have quasi-positive nonlinearities and to satisfy an entropy inequality. The difficulties in establishing global renormalised solutions caused by possibly degenerate diffusion are overcome by introducing a new class of weighted truncation functions. By means of the obtained global renormalised solutions, we study the large-time behaviour of complex balanced systems arising from chemical reaction network theory with nonlinear diffusion. When the reaction network does not admit boundary equilibria, the complex balanced equilibrium is shown, by using the entropy method, to exponentially attract renormalised solutions in the same compatibility class. This convergence extends even to a range of nonlinear diffusion, where global existence is an open problem, yet we are able to show that solutions to approximate systems converge exponentially to equilibrium uniformly in the regularisation parameter.

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✉ Bao Quoc Tang
quoc.tang@uni-graz.at ; baotangquoc@gmail.com

Klemens Fellner
klemens.fellner@uni-graz.at

Julian Fischer
julian.fischer@ist.ac.at

Michael Kniely
michael.kniely@wias-berlin.de

¹ Institute of Mathematics and Scientific Computing, University of Graz, Heinrichstrasse 36, 8010 Graz, Austria

² Institute of Science and Technology Austria, Am Campus 1, 3400 Klosterneuburg, Austria

³ Weierstrass Institute for Applied Analysis and Stochastics, Mohrenstrasse 39, 10117 Berlin, Germany

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1 Introduction and Main Results

In this paper, we consider the evolution of concentrations $u = (u_1, \dots, u_I), I \in \mathbb{N}$, in a bounded domain $\Omega \subset \mathbb{R}^d$ with Lipschitz boundary $\partial\Omega$ subject to nonlinear diffusion and reactions as modelled by the parabolic system

$$\begin{cases} \partial_t u_i - d_i \Delta u_i^{m_i} = f_i(u) & \text{in } Q_T, \\ \nabla u_i^{m_i} \cdot \nu = 0 & \text{on } \Gamma_T, \\ u_i(\cdot, 0) = u_{i,0} & \text{in } \Omega. \end{cases} \tag{1.1}$$

Here and below, we set $Q_T := \Omega \times (0, T)$ and $\Gamma_T := \partial\Omega \times (0, T)$ for any $T > 0$, and we employ ν to denote the unit outward normal vector to $\partial\Omega$. The diffusion constants d_i are assumed to satisfy $d_i > 0$ throughout the paper. The admissible range of the nonlinear diffusion exponents m_i depends on the specific situation and will be clearly stated at every occurrence below. For instance, for proving the existence of global renormalised solutions, we demand $m_i \in (0, 2)$, while the exponential equilibration of renormalised solutions is shown for $m_i > \frac{(d-2)_+}{d}$ with $(d-2)_+ := \max\{0, d-2\}$. We will comment on the bounds on m_i later on. Moreover, we generally assume $u_{i,0} \in L^1(\Omega), u_{i,0} \geq 0$, and impose the following assumptions on the reaction terms $f_i(u), i = 1, \dots, I$:

- (F1) (Local Lipschitz continuity) $f_i : \mathbb{R}_+^I \rightarrow \mathbb{R}$ is locally Lipschitz continuous.
- (F2) (Quasi-positivity) $f_i(u) \geq 0$ for all $u = (u_1, \dots, u_I) \in \mathbb{R}_+^I$ with $u_i = 0$.

(F3) (Entropy inequality) There exist constants $(\mu_i)_{i=1,\dots,I} \in \mathbb{R}^I$ satisfying

$$\sum_{i=1}^I f_i(u)(\log u_i + \mu_i) \leq C \sum_{i=1}^I (1 + u_i \log u_i) \tag{1.2}$$

for all $u \in \mathbb{R}_+^I$, where $C > 0$ is independent of u .

The second assumption (F2) guarantees non-negativity of solutions provided that the initial data are non-negative, which is a natural assumption since we consider here u_i as concentrations or densities. Assumption (F3), on the other hand, implies existence and control of the *entropy*

$$\sum_{i=1}^I \int_{\Omega} u_i (\log u_i + \mu_i - 1) \, dx$$

of the system, and (F3) is only slightly stronger than assuming control of the *mass*, i.e.

$$\sum_{i=1}^I f_i(u) \leq 0. \tag{1.3}$$

As it is one of the goals of this article to provide a generalisation of the results in Fischer (2015) to nonlinear diffusion models, we use the same entropy structure as in Fischer (2015), where *Boltzmann entropy–dissipating* systems are considered. Observing that (1.2) itself is just a condition on the involved reactions $f_i(u)$, it is reasonable to employ the same entropy for both the linear and the nonlinear diffusive case. Other options for an entropy were shown to be feasible if one focuses on different kinds of models such as cross-diffusion population Chen et al. (2018); Chen and Jüngel (2019) or energy–reaction–diffusion systems Fischer et al. (2022); Hopf (2022).

In general, assumptions (F1), (F2), and either (F3) or (1.3) of the nonlinearities f_i are not enough to obtain suitable a priori estimates which allow to extend local strong solutions globally, see, e.g. Pierre and Schmitt (2000). On the one hand, for the case of linear diffusion (i.e. $m_i = 1$ for all $i = 1, \dots, I$), the global existence for (1.1) under extra assumptions on nonlinearities has been extensively investigated. For instance, global bounded solutions were shown under (F3) for the case $d = 1$ with cubic nonlinearities and $d = 2$ with quadratic nonlinearities, see Goudon and Vasseur (2010); Tang (2018). Recent works Souplet (2018); Caputo et al. (2019); Fellner et al. (2020) showed under either (F3) or (1.3) that bounded solutions can be obtained for (slightly super-)quadratic nonlinearities in *all dimensions*. We refer the reader to the extensive review Pierre (2010) for related results concerning global existence of bounded or weak solutions. It is remarked that in all such results, additional assumptions on nonlinearities must be imposed. A breakthrough has been made in Fischer (2015) where global *renormalised solutions* were obtained without any extra assumptions on the nonlinearities. The notion of a renormalised solution was initially introduced for Boltzmann’s equation in DiPerna and Lions (1989), and it has been

applied subsequently to many other problems, see, e.g. Villani (1996); Dal Maso et al. (1999); Desvillettes et al. (2007). On the other hand, the case of nonlinear, possibly degenerate diffusion is much less understood. Up to our knowledge, there are only a few works (Laamri and Pierre 2017; Laamri and Perthame 2020 and Fellner et al. 2020) which showed global existence of (very) weak or bounded solutions for porous medium type of diffusion, i.e. $m_i > 1$, under (1.3) and some restricted growth conditions on the nonlinearities. However, renormalised solutions to a system as in (1.1) but imposing more regularity on the initial data and the underlying domain are provided in Lankeit and Winkler (2022). Instead of an entropy condition as in (1.2), the authors of Lankeit and Winkler (2022) assume an additional bound on cross-absorptive reaction terms. Furthermore, Fellner et al. (2020) proved the convergence to equilibrium for (1.1) modelling a single reversible chemical reaction. Recent results on the global existence of renormalised solutions also include reaction–cross-diffusion models in population dynamics Chen and Jüngel (2019) and energy–reaction–diffusion systems Fischer et al. (2022).

The present study shows the existence of global renormalised solutions to (1.1) featuring nonlinear diffusion of porous medium type or fast diffusion *without any extra conditions on the nonlinearities* (up to assuming (F1)–(F3)). In addition, we show that these solutions, in case (1.1) models general complex balanced chemical reaction networks, converge exponentially to equilibrium with explicitly computable rates. Our paper seems to be the first extensive contribution to the nonlinear diffusion-type system (1.1) establishing—for specific parameter regimes—the existence of global renormalised (or even weak and bounded) solutions as well as exponential convergence to equilibrium in various L^p spaces. A brief summary of our results is given in Table 1 at the end of this section.

The first part of this paper is concerned with the global existence of renormalised solutions to (1.1). We consider the following definition of renormalised solutions.

Definition 1.1 (Renormalised solutions) Let $m_i \in [0, 2]$ for all $1 \leq i \leq I$ and $u_0 = (u_{i,0})_i \in L^1(\Omega)^I$ be (componentwise) non-negative. A non-negative function $u = (u_1, \dots, u_I) : \Omega \times (0, \infty) \rightarrow \mathbb{R}_+^I$ is called a renormalised solution to (1.1) with initial data u_0 if the following holds:

- $u_i \in L_{\text{loc}}^\infty([0, \infty), L^1(\Omega))$ and $\nabla u_i \frac{m_i}{2} \in L_{\text{loc}}^2([0, \infty), L^2(\Omega)^I)$,
- for any smooth function $\xi : \mathbb{R}_+^I \rightarrow \mathbb{R}$ with compactly supported derivative $D\xi$, any function $\psi \in C^\infty(\bar{\Omega} \times [0, \infty))$, and a.e. $T > 0$, we have

$$\begin{aligned} & \int_{\Omega} \xi(u(\cdot, T))\psi(\cdot, T) \, dx - \int_{\Omega} \xi(u_0)\psi(\cdot, 0) \, dx - \int_{Q_T} \xi(u)\partial_t \psi \, dx \, dt \\ &= - \sum_{i,j=1}^I d_i m_i \int_{Q_T} \psi \partial_i \partial_j \xi(u) u_i^{m_i-1} \nabla u_i \cdot \nabla u_j \, dx \, dt \\ & \quad - \sum_{i=1}^I d_i m_i \int_{Q_T} \partial_i \xi(u) u_i^{m_i-1} \nabla u_i \cdot \nabla \psi \, dx \, dt \end{aligned}$$

$$+ \sum_{i=1}^I \int_{Q_T} \partial_i \xi(u) f_i(u) \psi \, dx \, dt, \tag{1.4}$$

- for a.e. $0 \leq s < T$, we have

$$\begin{aligned} \sum_{i=1}^I \int_{\Omega} u_i (\log u_i + \mu_i - 1) \, dx \Big|_s^T &\leq - \sum_{i=1}^I \frac{4d_i}{m_i^2} \int_s^T \int_{\Omega} |\nabla(u_i)^{\frac{m_i}{2}}|^2 \, dx \, dt \\ &+ \sum_{i=1}^I \int_s^T \int_{\Omega} f_i(u) (\log u_i + \mu_i) \, dx \, dt, \end{aligned} \tag{1.5}$$

- and for a.e. $t > 0$, we have

$$\sum_{i=1}^I \int_{\Omega} q_i u_i(x, t) \, dx = \sum_{i=1}^I \int_{\Omega} q_i u_{i,0}(x) \, dx, \tag{1.6}$$

provided there exist some numbers $q_i \in \mathbb{R}$ such that $\sum_{i=1}^I q_i f_i(z) = 0$ for all $z \in \mathbb{R}^I$.

Remark 1.2 • We notice that all integrals in (1.4) are well defined due to the compact support of $D\xi$ and as the integrals relating to gradients of solutions can be rewritten as

$$\begin{aligned} &\int_{Q_T} \psi \partial_i \partial_j \xi(u) u_i^{m_i-1} \nabla u_i \cdot \nabla u_j \, dx \, dt \\ &= \frac{4}{m_i m_j} \int_{Q_T} \psi \partial_i \partial_j \xi(u) u_i^{\frac{m_i}{2}} u_j^{1-\frac{m_j}{2}} \nabla u_i^{\frac{m_i}{2}} \cdot \nabla u_j^{\frac{m_j}{2}} \, dx \, dt \end{aligned}$$

and

$$\int_{Q_T} \partial_i \xi(u) u_i^{m_i-1} \nabla u_i \nabla \psi \, dx \, dt = \frac{2}{m_i} \int_{Q_T} \partial_i \xi(u) u_i^{\frac{m_i}{2}} \nabla u_i^{\frac{m_i}{2}} \nabla \psi \, dx \, dt$$

which are both finite thanks to the boundedness of $\nabla u_i^{\frac{m_i}{2}}$ in $L^2(Q_T)$ and the restriction $m_i \in [0, 2]$ for all $1 \leq i \leq I$. Note that $m_i < 0$ or $m_i > 2$ is not admissible as the integrals above may degenerate in this case even for small values of u_i .

- In the case of linear diffusion, the weak entropy law (1.5) and conservation laws (1.6) in Definition 1.1 are redundant since they can be proved using (1.4), see Fischer (2017). Unfortunately, this seems to be impossible in our present case due to the degeneracy of the diffusion. Nevertheless, if the renormalised solution turns out to be bounded, one can prove these laws directly (see the Proof of Theorem 1.7). Proving (1.5) and (1.6) using only (1.4) in the case of nonlinear diffusion remains an interesting open problem.

The first main result of this article is the following theorem. We stress that even though Definition 1.1 allows for $m_i \in [0, 2]$, the construction of a renormalised solution demands $m_i \in (0, 2)$ for all $i = 1, \dots, I$.

Theorem 1.3 (Global existence of renormalised solutions) *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with Lipschitz boundary $\partial\Omega$. Assume $m_i \in (0, 2)$ for all $i = 1, \dots, I$, and that conditions (F1)–(F3) hold. Then, for any non-negative initial data $u_0 = (u_{i,0})_i \in L^1(\Omega)^I$ subject to*

$$\sum_{i=1}^I \int_{\Omega} u_{i,0} \log u_{i,0} \, dx < \infty, \quad (1.7)$$

there exists a global renormalised solution to (1.1).

Remark 1.4 • Relation (1.6) usually corresponds to mass conservation laws in chemical reactions, see, e.g. Fellner and Tang (2018).

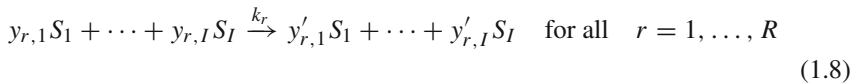
- The uniqueness of renormalised solutions is widely open. For a weaker notion called weak–strong uniqueness, i.e. the renormalised solution is unique as long as a strong solution exists, we refer the interested reader to Fischer (2017).
- The fact that (1.5) and (1.6) are satisfied by any renormalised solution plays an important role in the next part of this paper where we show that all renormalised solutions, instead of only some of them, converge to equilibrium.
- Assumption (1.7) on the initial data $u_0 \in L^1(\Omega)^I$ is used on a technical level at the end of the proof of Lemma 2.4 when showing that the candidate $u = (u_i)_i$ for a renormalised solution satisfies $u_i^{\frac{m_i}{2}} \in L^2(0, T; H^1(\Omega))$ for all $i \in \{1, \dots, I\}$. Moreover, the additional regularity (1.7) carries over to the renormalised solution (cf. (1.5)) and, in particular, guarantees that the initial entropy (cf. (1.13)) is finite.
- A similar situation is considered in Laamri and Pierre (2017) with homogeneous Dirichlet conditions. In fact, (Laamri and Pierre 2017, Theorem 2.6) is comparable to our Theorem 1.3 but instead of (1.7), the authors assume that an *a priori* L^1 estimate is available.

The strategy for constructing a global renormalised solution follows the ideas in Fischer (2015). Given a sequence of approximate solutions u^ε , one proves compactness of the family u^ε by deriving bounds on truncations $\varphi_i^E(u^\varepsilon)$, $E \in \mathbb{N}$, $i = 1, \dots, I$, and subsequently employing an Aubin–Lions lemma. The smooth functions φ_i^E defined in (2.6) essentially truncate the mappings $u \mapsto u_i$ if u becomes too large (measured in terms of E) while leaving $u \mapsto u_i$ unchanged for sufficiently small u . The main difficulty in our current situation is caused by the different diffusion exponents m_i for each species u_i , which makes the truncations φ_i^E provided in Fischer (2015) not applicable. We overcome this issue by modifying the functions φ_i^E in such a way that the truncation is not decided by $\sum_j u_j$ but by a weighted sum $\sum_j E^{-\alpha_j^i} u_j$ with suitable constants $\alpha_j^i \geq 1$ chosen depending on the diffusion exponents m_i . We then pass to the limit $\varepsilon \rightarrow 0$ in the equation for $\varphi_i^E(u^\varepsilon)$ at the cost of an additional defect measure which, nevertheless, vanishes in the subsequent limit $E \rightarrow \infty$. Finally, an

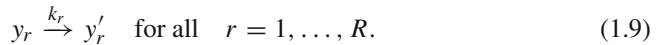
equation for $\xi(\varphi_i^E(u))$ is derived, where ξ is subject to the same assumptions as in Definition 1.1, and the limit $E \rightarrow \infty$ is performed resulting in the desired equation (1.4) for $\xi(u)$.

The restriction $m_i \in (0, 2)$ for $i = 1, \dots, I$ is, on the one hand, necessary within the construction of the truncations φ_i^E , see, e.g. (2.7), (2.9), and (2.10). On the other hand, it is crucial in showing the vanishing of the defect measure when establishing the renormalised solution. We do not know whether this restriction is purely technical or due to a deeper reason. It is remarked that the same condition is heavily used in Laamri and Pierre (2017), where global weak solutions to (1.1) are investigated. In particular, models with logarithmic diffusion $m_i = 0$ and ultra-fast diffusion processes are not included in the subsequent results. Systems of porous medium type are, nevertheless, covered to a large extent within Theorem 1.7.

In the second part of this paper, we study the large-time behaviour of reaction–diffusion systems of type (1.1) modelling chemical reaction networks subject to the complex balance condition. The main vocabulary is introduced below but we refer to Desvillettes et al. (2017) for a more detailed discussion of the involved concepts. We assume that there are I chemicals S_1, \dots, S_I reacting via the following R reactions



where $y_{r,i}, y'_{r,i} \in \{0\} \cup [1, \infty)$ are stoichiometric coefficients, and $k_r > 0$ are reaction rate constants. Utilising the notation $y_r = (y_{r,i})_{i=1,\dots,I}$ and $y'_r = (y'_{r,i})_{i=1,\dots,I}$, we can rewrite the reactions in (1.8) as



Denote by $u_i(x, t)$ the concentration of S_i at position $x \in \Omega$ and time $t \geq 0$. The reaction network (1.8) results in the following reaction–diffusion system with nonlinear diffusion for $u = (u_1, \dots, u_I)$:

$$\begin{cases} \partial_t u_i - d_i \Delta u_i^{m_i} = f_i(u) & \text{in } Q_T, \\ \nabla u_i^{m_i} \cdot \nu = 0 & \text{on } \Gamma_T, \\ u_i(x, 0) = u_{i,0}(x) & \text{in } \Omega, \end{cases} \tag{1.10}$$

in which the reaction term $f_i(u)$ is determined using the *mass action law*,

$$f_i(u) = \sum_{r=1}^R k_r u^{y_r} (y'_{r,i} - y_{r,i}) \quad \text{where} \quad u^{y_r} = \prod_{j=1}^I u_j^{y_{r,j}}. \tag{1.11}$$

It is obvious that $f_i(u)$ is locally Lipschitz continuous; hence, (F1) is satisfied. For (F2), it is easy to check that $f_i(u) \geq 0$ holds true for all $u \in \mathbb{R}_+^I$ satisfying $u_i = 0$ since we assume $k_r > 0$ and $y \in (\{0\} \cup [1, \infty))^I$ for all $y \in \{y_r, y'_r\}_{r=1,\dots,R}$. Before verifying (F3), we recall the definition of a *complex balanced equilibrium*. A constant

state $u_\infty = (u_{i,\infty})_i \in [0, \infty)^I$ is called a complex balanced equilibrium for (1.10)–(1.11) if

$$\sum_{\{r:y_r=y\}} k_r u_\infty^{y_r} = \sum_{\{s:y'_s=y\}} k_s u_\infty^{y'_s} \quad \text{for all } y \in \{y_r, y'_r\}_{r=1,\dots,R}. \quad (1.12)$$

Intuitively, this condition means that for any complex the total outflow from the complex and the total inflow into the complex are balanced at such an equilibrium. Using complex balanced equilibria, one can show that the nonlinearities in (1.11) satisfy (F3) by choosing $\mu_i = -\log u_{i,\infty}$ with a strictly positive complex balanced equilibrium u_∞ (see Desvillettes et al. 2017, Proposition 2.1). Therefore, one can apply Theorem 1.3 to obtain global renormalised solutions to (1.10) and (1.11).

In general, there might exist infinitely many solutions to (1.12). To uniquely identify a *positive* equilibrium, we need a set of conservation laws. More precisely, we consider the Wegscheider matrix (or stoichiometric coefficient matrix)

$$W = [(y'_r - y_r)_{r=1,\dots,R}]^T \in \mathbb{R}^{R \times I},$$

and define $m = \dim(\ker(W))$. If $m > 0$, then we can choose a matrix $\mathbb{Q} \in \mathbb{R}^{m \times I}$, where the rows form a basis of $\ker(W)$. Note that this implies $\mathbb{Q}(y'_r - y_r) = 0$ for all $r = 1, \dots, R$. Therefore, by recalling $f(u) = (f_i(u))_{i=1,\dots,I} = \sum_{r=1}^R k_r u^{y_r} (y'_r - y_r)$ we obtain

$$\mathbb{Q} f(u) = \sum_{r=1}^R k_r u^{y_r} \mathbb{Q}(y'_r - y_r) = 0 \quad \text{for any } u \in \mathbb{R}^I.$$

By using the homogeneous Neumann boundary condition, we can then (formally) compute that

$$\frac{d}{dt} \int_{\Omega} \mathbb{Q} u(t) \, dx = \int_{\Omega} \mathbb{Q} f(u) \, dx = 0$$

and, consequently, for a.e. $t > 0$,

$$\mathbb{Q} \int_{\Omega} u(t) \, dx = \mathbb{Q} \int_{\Omega} u_0 \, dx.$$

In other words, systems (1.10) and (1.11) possess m linearly independent mass conservation laws. By the vocabulary used in the literature on chemical reaction network theory, for any fixed non-negative mass vector $M \in \mathbb{R}^m$, the set $\{u \in \mathbb{R}_+^I : \mathbb{Q} u = M\}$ is called the *compatibility class* corresponding to M . In case that for each strictly positive vector $M \in \mathbb{R}^m$, there exists a strictly positive complex balanced equilibrium, one says that the chemical reaction network is complex balanced. This terminology is well defined since it was proved in Horn (1972/73) that if one equilibrium is complex balanced, then all equilibria are complex balanced. It was further shown that for each

non-negative initial mass $M \in \mathbb{R}^m$ with $M \neq 0$, there exists a unique strictly positive complex balanced equilibrium $u_\infty = (u_{i,\infty})_i \in \mathbb{R}_+^I$ in the compatibility class corresponding to M (cf. Horn and Jackson 1972; Feinberg 1979).

We stress that additionally there might exist many so-called *boundary equilibria*, which are complex balanced equilibria lying on $\partial\mathbb{R}_+^I$. It is remarked that there exists a large class of complex balanced chemical reaction networks, called *concordant networks*, which do not have boundary equilibria, see Shinar and Feinberg (2013). For the sake of brevity, from now on we call the unique strictly positive complex balanced equilibrium simply the *complex balanced equilibrium*. Note that all the previous considerations—although established for ODE models for chemical reaction networks—also apply to our PDE setting thanks to the homogeneous Neumann boundary conditions, which allow for spatially homogeneous equilibria.

Theorem 1.5 (Exponential equilibration of renormalised solutions) *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with Lipschitz boundary $\partial\Omega$. Assume that the reaction network (1.9) is complex balanced, suppose that (F1)–(F3) hold, and let any non-negative initial data $u_0 = (u_{i,0})_i \in L^1(\Omega)^I$ with $\sum_{i=1}^I \int_\Omega u_{i,0} \log u_{i,0} \, dx < \infty$ be given.*

- *If $m_i \in (0, 2)$ for all $1 \leq i \leq I$, then there exists a global renormalised solution to (1.10)–(1.11).*
- *If $m_i > \frac{(d-2)_+}{d}$ for all $1 \leq i \leq I$ and assuming that there exist no boundary equilibria, all renormalised solutions to (1.10)–(1.11) converge exponentially to the positive equilibrium $u_\infty \in \mathbb{R}_+^I$ in the same compatibility class as u_0 with a rate which can be explicitly computed up to a finite dimensional inequality, i.e.*

$$\sum_{i=1}^I \|u_i(t) - u_{i,\infty}\|_{L^1(\Omega)} \leq C e^{-\lambda t} \quad \text{for all } t \geq 0,$$

for positive constants $C > 0, \lambda > 0$ depending on the initial entropy, the domain Ω , the diffusion coefficients d_i and exponents m_i , and the vectors of stoichiometric coefficients $\{y_r, y_r'\}_{r=1,\dots,R}$.

We emphasise that the convergence to equilibrium in Theorem 1.5 is proved for *all renormalised solutions*, not only the ones obtained via an approximation procedure. This is possible because the proof of the convergence uses only the weak entropy–entropy dissipation law (1.5) and the conservation laws (1.6) which are satisfied by all renormalised solutions. It is noted that, when a system possesses boundary equilibria, the convergence to the positive equilibrium (more precisely, the instability of boundary equilibria) is very subtle and strongly connected to the famous *Global Attractor Conjecture*, see, for instance, Desvillettes et al. (2017); Fellner and Tang (2018); Craciun (2015).

Theorem 1.5, up to our knowledge, is the first result of trend to equilibrium for general complex balanced reaction networks with nonlinear diffusion. A special case of a single reversible reaction was considered recently in Fellner et al. (2020) in which the authors utilised a so-called *indirect diffusion effect* (see, e.g. Einav et al. 2020), which seems difficult to be generalised to general systems. The proof of Theorem 1.5

uses the relative entropy

$$\mathcal{E}[u|u_\infty] = \sum_{i=1}^I \int_{\Omega} \left(u_i \log \frac{u_i}{u_{i,\infty}} - u_i + u_{i,\infty} \right) dx \quad (1.13)$$

and its corresponding entropy dissipation

$$\mathcal{D}[u] = \sum_{i=1}^I d_i m_i \int_{\Omega} u_i^{m_i-2} |\nabla u_i|^2 dx + \sum_{r=1}^R k_r u_\infty^{y_r} \int_{\Omega} \Psi \left(\frac{u^{y_r}}{u_\infty^{y_r}}; \frac{u^{y_r'}}{u_\infty^{y_r'}} \right) dx \quad (1.14)$$

where $\Psi(x; y) = x \log(x/y) - x + y$. It holds, at least formally, that

$$-\frac{d}{dt} \mathcal{E}[u|u_\infty] = \mathcal{D}[u]$$

along any trajectory of (1.10) and (1.11). The degeneracy of the nonlinear diffusion makes the classical logarithmic Sobolev inequality not applicable. Our main idea here is to utilise some generalised logarithmic Sobolev inequalities (see Lemmas 3.3 and 3.4), which are suited for nonlinear diffusion, and the established results in Fellner and Tang (2018) for the case of linear diffusion, to firstly derive an entropy–entropy dissipation inequality of the form

$$\mathcal{D}[u] \geq C(\mathcal{E}[u|u_\infty])^\alpha \quad (1.15)$$

where $\alpha = \max_{i=1,\dots,I} \{1, m_i\}$. Note that this functional inequality is proved for all non-negative functions $u : \Omega \rightarrow \mathbb{R}_+^I$ satisfying the conservation laws $\mathbb{Q} \int_{\Omega} u dx = |\Omega| \mathbb{Q} u_\infty$, and therefore is suitable for renormalised solutions, which have very low regularity. If $\alpha = 1$ in (1.15), one immediately gets exponential convergence of the relative entropy to zero and, consequently, exponential convergence of the solutions to equilibrium in L^1 thanks to a Csiszár–Kullback–Pinsker-type inequality. If $\alpha > 1$, we first obtain an algebraic decay of the relative entropy to zero. Thanks to this, we can explicitly compute a finite time $T_0 > 0$ from which onwards (in time) the averages of concentrations are strictly bounded below by a positive constant. This helps to compensate the degeneracy of the diffusion and, therefore, to show that solutions with such lower bounds satisfy the linear entropy–entropy dissipation inequality, i.e.

$$\mathcal{D}[u(t)] \geq C \mathcal{E}[u(t)|u_\infty] \quad \text{for all } t \geq T_0,$$

which recovers exponential convergence to equilibrium. \square

The use of renormalised solutions allows to deal with a large class of nonlinearities, but on the other hand it restricts Theorem 1.5 to the case $\frac{(d-2)_+}{d} < m_i < 2$. When the nonlinearities are of polynomial type, it was shown in Laamri and Pierre (2017); Fellner et al. (2020), under the assumption of the mass dissipation (1.3) instead of the entropy inequality (1.2), that one can get global weak or even bounded solutions if the

porous medium exponents m_i are large enough. In the following theorem, we show an analogue result under the entropy inequality condition (1.2). Moreover, the solution is shown to converge exponentially to the positive complex balanced equilibrium.

Definition 1.6 (Weak solutions) A vector of non-negative functions $u = (u_1, \dots, u_I) : \Omega \times (0, \infty) \rightarrow \mathbb{R}_+^I$ is called a (global) weak solution to (1.1) if

$$u_i \in L^{m_i+1}(Q_T), \quad \nabla u_i^{\frac{m_i}{2}} \in L^2(Q_T), \quad f_i(u) \in L^1(Q_T),$$

$$\partial_t u_i \in L^1(Q_T) + L^{\frac{2(m_i+1)}{2m_i+1}}(0, T; (W^{1,2(m_i+1)}(\Omega))'),$$

and

$$\int_{Q_T} \partial_t u_i \psi \, dx \, dt + \frac{d_i}{2} \int_{Q_T} u_i^{\frac{m_i}{2}} \nabla u_i^{\frac{m_i}{2}} \cdot \nabla \psi \, dx \, dt = \int_{Q_T} f_i(u) \psi \, dx \, dt \quad (1.16)$$

for all $T > 0, i = 1, \dots, I$, and test functions $\psi \in L^\infty(Q_T) \cap L^{2(m_i+1)}(0, T; W^{1,2(m_i+1)}(\Omega))$.

Theorem 1.7 (Existence and convergence of weak and bounded solutions) *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with Lipschitz boundary $\partial\Omega$. Assume that the assumptions (F1)–(F3) hold and that the nonlinearities are bounded by polynomials, i.e. there exist $C > 0$ and $\rho_j \geq 1$ for all $j = 1, \dots, I$ such that*

$$|f_i(u)| \leq C \left(1 + \sum_{j=1}^I |u_j|^{\rho_j} \right) \quad \text{for all } u \in \mathbb{R}^I, \, i = 1, \dots, I.$$

- If

$$m_i \geq \rho_i - 1 \quad \text{for all } i = 1, \dots, I, \quad (1.17)$$

then system (1.1) has a global non-negative weak solution for each non-negative initial data $u_0 = (u_{i,0})_{i=1,\dots,I}$ such that $u_{i,0} \log u_{i,0} \in L^1(\Omega) \cap H^{-1}(\Omega)$ for all $i = 1, \dots, I$.

- Under an even stronger condition with $\rho := \max_i \rho_i$, namely

$$\min_{i=1,\dots,I} m_i > \max\{\rho - 1, 1\} \quad \text{for } d \leq 2,$$

$$\min_{i=1,\dots,I} m_i > \max\left\{\rho - \frac{4}{d+2}, 1\right\} \quad \text{for } d \geq 3, \quad (1.18)$$

the weak solution to (1.1) subject to bounded non-negative initial data $u_0 \in L^\infty(\Omega)^I$ is locally bounded in time, i.e.

$$\|u_i\|_{L^\infty(\Omega \times (0,T))} \leq C_T \quad \text{for all } i = 1, \dots, I,$$

where C_T is a constant growing at most polynomially in $T > 0$.

Assume now that the reaction network (1.9) is complex balanced and define

$$\rho = \rho_i := \max_{r=1, \dots, R} \{|y_r|, |y'_r|\} \quad \text{for all } i = 1, \dots, I, \quad (1.19)$$

where $|y_r| = \sum_{j=1}^I y_{r,j}$. Assume moreover that (1.9) has no boundary equilibria.

- If (1.17) holds and $0 \leq u_0$ such that $u_{i,0} \log u_{i,0} \in L^1(\Omega) \cap H^{-1}(\Omega)$ for all $i = 1, \dots, I$, all bounded weak solutions (locally in time) to (1.10) and (1.11) converge exponentially to the complex balanced equilibrium in $L^1(\Omega)$, i.e.

$$\sum_{i=1}^I \|u_i^\varepsilon(t) - u_{i,\infty}\|_{L^1(\Omega)} \leq C e^{-\lambda t} \quad \text{for all } t \geq 0,$$

for some $C > 0, \lambda > 0$ depending on the initial entropy, the domain Ω , the diffusion coefficients d_i and exponents m_i , and the vectors of stoichiometric coefficients $\{y_r, y'_r\}_{r=1, \dots, R}$.

- If (1.18) holds and $0 \leq u_0 \in L^\infty(\Omega)^I$, weak solutions to (1.10) and (1.11) are bounded locally in time and converge exponentially to equilibrium in any L^p norm with $p \in [1, \infty)$, i.e.

$$\sum_{i=1}^I \|u_i(t) - u_{i,\infty}\|_{L^p(\Omega)} \leq C_p e^{-\lambda_p t} \quad \text{for all } t \geq 0, \quad (1.20)$$

with positive constants $C_p, \lambda_p > 0$ depending on p , the initial entropy, the domain Ω , the diffusion coefficients d_i and exponents m_i , and the vectors of stoichiometric coefficients $\{y_r, y'_r\}_{r=1, \dots, R}$.

Remark 1.8 In addition to Remark 1.4, we point out that another reason for assuming $u_{i,0} \log u_{i,0} \in L^1(\Omega) \cap H^{-1}(\Omega)$ for all $i \in \{1, \dots, I\}$ (rather than demanding just $u_0 \in (L^1(\Omega) \cap H^{-1}(\Omega))^I$ as in Laamri and Perthame 2020) is the supposed entropy inequality (1.2). More precisely, we need this hypothesis on the initial data in the proof of Lemma 3.6, which is itself a crucial ingredient of the proof of Theorem 1.7.

As one can see from Theorems 1.5 and 1.7, the global existence and trend to equilibrium for (1.10) and (1.11) are well established for either $\frac{(d-2)_+}{d} < m_i < 2$ for all $i = 1, \dots, I$, or m_i large enough in the sense of (1.17) or (1.18) for all $i = 1, \dots, I$. There exists, therefore, a gap in which the global existence of any kind of solution remains open. Remarkably, the proof of the convergence to equilibrium in this paper does not rely on the restriction $m_i < 2$, and it therefore is applicable to any global solution to (1.10) and (1.11) as long as it satisfies the entropy law (1.5) and the mass conservation laws (1.6). We, thus, arrive at the following result on the convergence to equilibrium for approximating systems of (1.10) and (1.11) uniformly in the approximation parameter.

Theorem 1.9 (Uniform convergence of approximate solutions) *Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with Lipschitz boundary $\partial\Omega$. Assume that (1.10) and (1.11) subject to (F1)–(F3), non-negative initial data $u_0 = (u_{i,0})_i \in L^1(\Omega)^I$ with $\sum_{i=1}^I \int_{\Omega} u_{i,0} \log u_{i,0} dx < \infty$, and $m_i > \frac{(d-2)_+}{d}$ for all $i = 1, \dots, I$ admits a complex balanced equilibrium but no boundary equilibria. For any $\varepsilon > 0$, let $u^\varepsilon = (u_i^\varepsilon)_{i=1, \dots, I}$ be the solution to the approximate system*

$$\begin{cases} \partial_t u_i^\varepsilon - d_i \Delta (u_i^\varepsilon)^{m_i} = \frac{f_i(u^\varepsilon)}{1 + \varepsilon |f(u^\varepsilon)|} & \text{in } Q_T, \\ \nabla (u_i^\varepsilon)^{m_i} \cdot \nu = 0 & \text{on } \Gamma_T, \\ u_i^\varepsilon(x, 0) = u_{i,0}^\varepsilon(x) & \text{in } \Omega, \end{cases} \tag{1.21}$$

where $u_{i,0}^\varepsilon \in L^\infty(\Omega)$ satisfies

$$\max_i \lim_{\varepsilon \rightarrow 0} \|u_{i,0}^\varepsilon - u_{i,0}\|_{L^1(\Omega)} = 0 \quad \text{and} \quad \max_i \sup_{\varepsilon > 0} \int_{\Omega} u_{i,0}^\varepsilon \log u_{i,0}^\varepsilon dx < \infty. \tag{1.22}$$

Moreover, the approximated initial data u_0^ε are chosen such that, for all $\varepsilon > 0$,

$$\mathbb{Q} \int_{\Omega} u_0^\varepsilon dx = \mathbb{Q} \int_{\Omega} u_0 dx. \tag{1.23}$$

(This implies that system (1.21) admits a unique positive complex balanced equilibrium $u_\infty \in \mathbb{R}_+^I$, which is independent of $\varepsilon > 0$.)

Then, u^ε converges exponentially to u_∞ with a rate which is uniform in ε , i.e.

$$\sum_{i=1}^I \|u_i^\varepsilon(t) - u_{i,\infty}\|_{L^1(\Omega)} \leq C e^{-\lambda t} \quad \text{for all } t \geq 0,$$

where $C > 0, \lambda > 0$ are constants depending on the initial entropy, the domain Ω , the diffusion coefficients d_i and exponents m_i , and the vectors of stoichiometric coefficients $\{y_r, y'_r\}_{r=1, \dots, R}$ but independent of $\varepsilon > 0$.

To prove Theorem 1.9, we make use of the same relative entropy as in (1.13),

$$\mathcal{E}[u^\varepsilon | u_\infty] = \sum_{i=1}^I \int_{\Omega} \left(u_i^\varepsilon \log \frac{u_i^\varepsilon}{u_{i,\infty}} - u_i^\varepsilon + u_{i,\infty} \right) dx,$$

Table 1 Summarisation of global existence and convergence to equilibrium for mass action reaction–diffusion systems (1.10) and (1.11) with nonlinear diffusion in various parameter regimes

Diffusion exponents	Global existence	Convergence to equilibrium
$0 < m \leq \frac{(d-2)_+}{d}$	Renormalised solution	unknown
$\frac{(d-2)_+}{d} < m < 2$	Renormalised solution	Exponential in L^1
$2 \leq m < \rho - 1$	unknown	Exponential in L^1 for approximate solutions <i>uniformly in the approximation parameter</i>
$\rho - 1 \leq m \leq \rho - \frac{4}{d+2}$	Weak solution	Exponential in L^1
$\rho - \frac{4}{d+2} < m$	Bounded solution	Exponential in any L^p , $p < \infty$

where u^ε is the solution to (1.21). The corresponding non-negative entropy dissipation can be calculated as

$$\mathcal{D}[u^\varepsilon] = \sum_{i=1}^I d_i \int_{\Omega} m_i (u_i^\varepsilon)^{m_i-2} |\nabla u_i|^2 \, dx + \sum_{r=1}^R k_r u_\infty^{y_r} \int_{\Omega} \frac{1}{1 + \varepsilon |f(u^\varepsilon)|} \Psi \left(\frac{(u^\varepsilon)^{y_r}}{u_\infty^{y_r}}; \frac{(u^\varepsilon)^{y'_r}}{u_\infty^{y'_r}} \right) \, dx.$$

Since there is no uniform-in- ε L^∞ bound for u^ε available, the factor $\frac{1}{1 + \varepsilon |f(u^\varepsilon)|}$ is not bounded below uniformly in $\varepsilon > 0$. Therefore, it seems to be impossible to use the entropy dissipation (for limit solutions) in (1.14) as a lower bound for $\mathcal{D}[u^\varepsilon]$. We overcome this issue by using the ideas in Fellner and Tang (2018). Roughly speaking, we deal with the second sum in $\mathcal{D}[u^\varepsilon]$ by estimating it below by a term involving only spatial averages of u^ε (rather than u^ε pointwise as above), and then exploiting the fact that these averages are bounded uniformly in $\varepsilon > 0$.

We summarise the global existence and convergence to equilibrium for the mass action system (1.10) and (1.11) in Table 1 (assuming the existence of a unique strictly positive complex balanced equilibrium and the absence of boundary equilibria). Note that in our framework the global existence of any kind of solution remains open as soon as there exist $i, j \in \{1, \dots, I\}$ such that the corresponding diffusion exponents satisfy $m_i \geq 2$ and $m_j < \rho - 1$, where ρ is defined in (1.19). In order to keep the presentation simple, we assume within Table 1 that all diffusion exponents $m_i = m$ coincide.

The rest of this paper is organised as follows. In Sect. 2, we show the global existence of renormalised solutions for the general system (1.1). The proofs of Theorems 1.5, 1.7, and 1.9 on the convergence to equilibrium of chemical reaction networks are presented in Sect. 3. Some technical proofs of auxiliary results are postponed to Appendix.

2 Global Existence of Renormalised Solutions

2.1 Existence of Approximate Solutions

Let $m_i \in (0, 2)$ for all $i \in \{1, \dots, I\}$ and $\varepsilon > 0$ and consider the following approximating system for $u^\varepsilon = (u_1^\varepsilon, \dots, u_I^\varepsilon)$,

$$\partial_t u_i^\varepsilon - d_i \Delta (u_i^\varepsilon)^{m_i} = f_i^\varepsilon(u^\varepsilon) := \frac{f_i(u^\varepsilon)}{1 + \varepsilon |f(u^\varepsilon)|}, \quad \nabla (u_i^\varepsilon)^{m_i} \cdot \nu = 0, \quad u_i^\varepsilon(\cdot, 0) = u_{i,0}^\varepsilon, \tag{2.1}$$

where $u_{i,0}^\varepsilon \in L^\infty(\Omega)$ is non-negative and $u_{i,0}^\varepsilon \xrightarrow{\varepsilon \rightarrow 0} u_{i,0}$ in $L^1(\Omega)$. Moreover, we demand

$$\sup_{\varepsilon > 0} \int_{\Omega} u_{i,0}^\varepsilon \log u_{i,0}^\varepsilon \, dx < +\infty. \tag{2.2}$$

With this approximation, it is easy to check that the approximated nonlinearities f_i^ε still satisfy assumptions (F1)–(F3).

Lemma 2.1 *Provided $m_i \in (0, 2)$ for all $i \in \{1, \dots, I\}$ and $\varepsilon > 0$, there exists a global weak solution $u^\varepsilon = (u_i^\varepsilon)_i \in L^\infty_{\text{loc}}([0, \infty), L^\infty(\Omega))^I$ to the approximate system (2.1) with $\nabla (u_i^\varepsilon)^{\frac{m_i}{2}} \in L^2_{\text{loc}}([0, \infty), L^2(\Omega))$ and $\partial_t u_i^\varepsilon \in L^2_{\text{loc}}([0, \infty), (H^1(\Omega))'$). In detail,*

$$\int_{t_0}^{t_1} \left\langle \frac{d}{dt} u_i^\varepsilon, \psi \right\rangle_{(H^1(\Omega))', H^1(\Omega)} \, dt = -d_i m_i \int_{t_0}^{t_1} \int_{\Omega} (u_i^\varepsilon)^{m_i-1} \nabla u_i^\varepsilon \cdot \nabla \psi \, dx \, dt + \int_{t_0}^{t_1} \int_{\Omega} \frac{f_i(u^\varepsilon)}{1 + \varepsilon |f(u^\varepsilon)|} \psi \, dx \, dt \tag{2.3}$$

for all $\psi \in L^2_{\text{loc}}([0, \infty), H^1(\Omega))$, $i \in \{1, \dots, I\}$, and a.e. $0 \leq t_0 < t_1$. Moreover, each u_i^ε is a.e. non-negative and

$$\begin{aligned} & \sum_{i=1}^I \int_{\Omega} u_i^\varepsilon(t_1) [\log u_i^\varepsilon(t_1) + \mu_i - 1] \, dx + C \int_{t_0}^{t_1} \int_{\Omega} \sum_{i=1}^I \left| \nabla (u_i^\varepsilon)^{\frac{m_i}{2}} \right|^2 \, dx \, dt \\ & \leq e^{C(t_1-t_0)} \left(\int_{\Omega} \sum_{i=1}^I u_i^\varepsilon(t_0) [\log u_i^\varepsilon(t_0) + \mu_i - 1] \, dx + C(t_1 - t_0) \right) \end{aligned} \tag{2.4}$$

for a.e. $0 \leq t_0 < t_1$.

Proof For each $\varepsilon > 0$, we see by recalling $|f(u^\varepsilon)| = \sum_{i=1}^I |f_i(u^\varepsilon)|$ that

$$\frac{f_i(u^\varepsilon(x, t))}{1 + \varepsilon |f(u^\varepsilon(x, t))|} \leq \frac{1}{\varepsilon} \quad \text{for all } (x, t) \in Q_T.$$

The existence of bounded weak solutions to (2.1) is therefore standard. However, since we are not able to find a precise reference, a proof is given in Appendix 1. By multiplying (2.1) by $\log(u_i^\varepsilon) + \mu_i$ (or more rigorously by $\log(u_i^\varepsilon + \delta) + \mu_i$ for some $\delta > 0$, then let $\delta \rightarrow 0$), summing the resultants over $i = 1, \dots, I$, and then integrating over Ω , we obtain

$$\begin{aligned} & \frac{d}{dt} \int_{\Omega} \sum_{i=1}^I u_i^\varepsilon (\log u_i^\varepsilon + \mu_i - 1) \, dx + \sum_{i=1}^I \frac{4d_i}{m_i} \int_{\Omega} \left| \nabla (u_i^\varepsilon)^{\frac{m_i}{2}} \right|^2 \, dx \\ &= \int_{\Omega} \frac{1}{1 + \varepsilon |f(u^\varepsilon)|} \sum_{i=1}^I f_i(u^\varepsilon) (\log u_i^\varepsilon + \mu_i) \, dx \\ &\leq C \sum_{i=1}^I \int_{\Omega} u_i^\varepsilon (\log u_i^\varepsilon + \mu_i - 1) \, dx + C \end{aligned} \tag{2.5}$$

where we used (F3) and $x \leq \delta x \log x + C_\delta$ for all $x \geq 0$ and any $\delta > 0$ at the last step. Hence, by integrating (2.5) over (t, T) and using Gronwall’s inequality, we obtain the desired estimate (2.4). The uniform bound on $u_i^\varepsilon \log u_i^\varepsilon$ in $L^\infty(0, T; L^1(\Omega))$ and $|\nabla (u_i^\varepsilon)^{\frac{m_i}{2}}|$ in $L^2(0, T; L^2(\Omega))$ follow immediately from (2.4). \square

2.2 Existence of Renormalised Solutions

As it can be seen from Lemma 2.1, the *a priori* estimates of u_i^ε are not enough to extract a convergent subsequence. Following the idea from Fischer (2015), we consider another approximation of u_i^ε by defining

$$\varphi_i^E(v) = (v_i - 3E) \xi \left(\sum_{j=1}^I \frac{v_j}{E^{\alpha_j^i}} - 1 \right) + 3E \tag{2.6}$$

for $E \in \mathbb{N}$, and a smooth function $\xi : \mathbb{R} \rightarrow [0, 1]$ satisfying $\xi \equiv 1$ on $(-\infty, 0)$ and $\xi \equiv 0$ on $(1, \infty)$. The constants α_j^i are given by (recall that $m_j \in (0, 2)$ for all $j \in \{1, \dots, I\}$)

$$\alpha_i^i := 1, \quad \alpha_j^i := \frac{2}{\min(m_j, 2 - m_j)} \text{ for } j \neq i. \tag{2.7}$$

Remark 2.2 The α_j^i are chosen in (2.7) for the sake of simplicity. In fact, we can choose any α_j^i such that

$$\alpha_i^i = 1, \quad \text{and} \quad \alpha_j^i \geq \max \left\{ \frac{2 - m_i}{2 - m_j}, \frac{m_i}{m_j} \right\}.$$

See the proof of (E2) in Lemma 2.3.

Lemma 2.3 *The smooth truncations φ_i^E defined in (2.6) have the following properties:*

(E1) $\varphi_i^E \in C^2(\mathbb{R}_+^I)$.

(E2) *For any $i \in \{1, \dots, I\}$, there exists a constant $K_i > 0$ such that*

$$\max_{1 \leq j, k \leq I} \sup_{E \geq 1} \sup_{v \in \mathbb{R}_+^I} v_j^{\frac{m_j}{2}} v_k^{1-\frac{m_k}{2}} |\partial_j \partial_k \varphi_i^E(v)| \leq K_i. \tag{2.8}$$

(E3) *For every E , the set $\text{supp } D\varphi_i^E$ is bounded.*

(E4) *For all $j \in \{1, \dots, I\}$ and all $v \in \mathbb{R}_+^I$, there holds $\lim_{E \rightarrow \infty} \partial_j \varphi_i^E(v) = \delta_{ij}$.*

(E5) *There exists a constant $K > 0$ such that*

$$\max_{1 \leq j \leq I} \sup_{E \geq 1} \sup_{v \in \mathbb{R}_+^I} |\partial_j \varphi_i^E(v)| \leq K.$$

(E6) $\varphi_i^E(v) = v_i$ for any $v \in \mathbb{R}_+^I$ with $\sum_{j=1}^I v_j E^{-\alpha_j} \leq 1$.

(E7) *For every $K > 0$ and every j, k , there holds*

$$\lim_{E \rightarrow \infty} \sup_{|v| \leq K} |\partial_j \partial_k \varphi_i^E(v)| = 0.$$

(E8) $\varphi_i^E(v) = v_i$ for any $v \in \mathbb{R}_+^I$ with $\sum_{j=1}^I \varphi_j^E(v) E^{-\alpha_j} \leq 1$.

Proof Properties (E1), (E3), (E4), and (E6) are immediate. To prove (E2), we first compute

$$\begin{aligned} \partial_i \varphi_i^E(v) &= (v_i - 3E)\xi'(\dots)E^{-\alpha_i} + \xi(\dots), \\ \partial_j \varphi_i^E(v) &= (v_i - 3E)\xi'(\dots)E^{-\alpha_j} \end{aligned}$$

for $j \neq i$, where for brevity we write (\dots) instead of $(\sum_{j=1}^I E^{-\alpha_j} v_j - 1)$. From that, one further gets the second derivatives

$$\begin{aligned} \partial_i^2 \varphi_i^E(v) &= (v_i - 3E)\xi''(\dots)E^{-2\alpha_i} + 2\xi'(\dots)E^{-\alpha_i}, \\ \partial_i \partial_j \varphi_i^E(v) &= (v_i - 3E)\xi''(\dots)E^{-\alpha_i - \alpha_j} + \xi'(\dots)E^{-\alpha_j}, \\ \partial_j \partial_k \varphi_i^E(v) &= (v_i - 3E)\xi''(\dots)E^{-\alpha_j - \alpha_k} \end{aligned}$$

for $j \neq i$ and $k \neq i$.

To show (2.8), we will consider the following cases:

- When $i = j = k$, we have

$$\begin{aligned} v_i^{\frac{m_i}{2}} v_i^{1-\frac{m_i}{2}} |\partial_i^2 \varphi_i^E(v)| &= v_i |\partial_i^2 \varphi_i^E(v)| \\ &\leq |v_i| |v_i - 3E| E^{-2\alpha_i} |\xi''(\dots)| + 2v_i E^{-\alpha_i} |\xi'(\dots)|. \end{aligned}$$

Employing the identities $\xi'(\dots) = \xi''(\dots) = 0$ for $v_i \geq 2E^{\alpha_i}$ as well as the bound $|\xi'(\dots)| + |\xi''(\dots)| \leq C$, we can further estimate

$$v_i^{\frac{m_j}{2}} v_i^{1-\frac{m_i}{2}} |\partial_i^2 \varphi_i^E(v)| \leq CE^{-\alpha_i} |E^{\alpha_i} + E| + C.$$

Therefore, by choosing $\alpha_i^i = 1$, we have the desired estimate (2.8) for $i = j = k$.

- When $k = i$ and $j \neq i$, we estimate using $\alpha_i^i = 1, \xi'(\dots) = \xi''(\dots) = 0$ when $v_j \geq 2E^{\alpha_j}$ or $v_i \geq 2E$, and $|\xi'(\dots)| + |\xi''(\dots)| \leq C$,

$$\begin{aligned} & v_j^{\frac{m_j}{2}} v_i^{1-\frac{m_i}{2}} |\partial_j \partial_i \varphi_i^E(v)| \\ & \leq v_j^{\frac{m_j}{2}} v_i^{1-\frac{m_i}{2}} |\xi'(\dots)| E^{-\alpha_j} + v_j^{\frac{m_j}{2}} v_i^{1-\frac{m_i}{2}} |\xi''(\dots)| |v_i - 3E| E^{-\alpha_j-1} \\ & \leq CE^{\frac{m_j}{2}\alpha_j} E^{1-\frac{m_i}{2}} |\xi'(\dots)| E^{-\alpha_j} + CE^{\frac{m_j}{2}\alpha_j} E^{1-\frac{m_i}{2}} |\xi''(\dots)| |E + 3E| E^{-\alpha_j-1} \\ & \leq CE^{\alpha_j(\frac{m_j}{2}-1)+(1-\frac{m_i}{2})}. \end{aligned}$$

Thus, (2.8) is proved in the case $k = i \neq j$ if we choose α_j^i such that

$$\alpha_j^i \left(\frac{m_j}{2} - 1\right) + \left(1 - \frac{m_i}{2}\right) \leq 0 \quad \text{or equivalently} \quad \alpha_j^i \geq \frac{2 - m_i}{2 - m_j}. \tag{2.9}$$

- When $j = i$ and $k \neq i$, we estimate similarly to the previous cases

$$v_i^{\frac{m_j}{2}} v_k^{1-\frac{m_k}{2}} |\partial_i \partial_k \varphi_i^E(v)| \leq CE^{\frac{m_j}{2} - \alpha_k^i \frac{m_k}{2}}.$$

If we choose

$$\frac{m_i}{2} - \alpha_k^i \frac{m_k}{2} \leq 0 \quad \text{or equivalently} \quad \alpha_k^i \geq \frac{m_i}{m_k}, \tag{2.10}$$

then obviously (2.8) holds true for $j = i \neq k$.

- Finally, when $j \neq i$ and $k \neq i$, we estimate

$$\begin{aligned} v_j^{\frac{m_j}{2}} v_k^{1-\frac{m_k}{2}} |\partial_j \partial_k \varphi_i^E(v)| & \leq CE^{\alpha_j^i \frac{m_j}{2}} E^{\alpha_k^i (1-\frac{m_k}{2})} |\xi''(\dots)| E^{-\alpha_j^i - \alpha_k^i} E \\ & \leq CE^{\alpha_j^i(\frac{m_j}{2}-1) - \alpha_k^i \frac{m_k}{2} + 1}. \end{aligned}$$

It is easy to see that from (2.9) and (2.10) it follows

$$\alpha_j^i \left(\frac{m_j}{2} - 1\right) - \alpha_k^i \frac{m_k}{2} + 1 \leq 0,$$

and hence (2.8) is proved in this case.

From (2.9) and (2.10), we see that $\alpha_j^i > 1$ for all $j \neq i$; hence, (E5) and (E7) follow.

For (E8), it is enough to show that $\sum_{j=1}^I \varphi_j^E(v) E^{-\alpha_j^i} \leq 1$ yields $\sum_{j=1}^I v_j E^{-\alpha_j^i} \leq 1$. Set

$$y := \sum_{j=1}^I \frac{v_j}{E^{\alpha_j^i}} - 1 \quad \text{and} \quad z := \sum_{j=1}^I \frac{1}{E^{\alpha_j^i}}.$$

From $\varphi_j^E(v) = (v_j - 3E)\xi(y) + 3E$ and $\sum_{j=1}^I \varphi_j^E(v) E^{-\alpha_j^i} \leq 1$, we deduce

$$\xi(y)(y + 1) + 3E(1 - \xi(y))z \leq 1.$$

Due to $\xi \leq 1$ and $Ez \geq 1$, we are now led to

$$y\xi(y) \leq (1 - 3Ez)(1 - \xi(y)) \leq 0.$$

This entails $y \leq 0$ and, thus, completes the proof of the lemma. □

Lemma 2.4 Consider non-negative functions $u_0 = (u_{i,0})_i \in L^1(\Omega)^I$ which satisfy

$$\sum_{i=1}^I \int_{\Omega} u_{i,0} \log u_{i,0} \, dx < \infty.$$

Let $u^\varepsilon = (u_1^\varepsilon, \dots, u_I^\varepsilon)$ for $\varepsilon \rightarrow 0$ be the sequence of solutions to the regularised problems as stated in Lemma 2.1.

Then, there exists a subsequence u^ε converging a.e. on $\Omega \times [0, \infty)$ to a limit $u \in L^\infty_{\text{loc}}([0, \infty), L^1(\Omega))^I$ with $u_i \log u_i \in L^\infty_{\text{loc}}([0, \infty), L^1(\Omega))$ for all $i \in \{1, \dots, I\}$. Furthermore, $(u_i^\varepsilon)^{\frac{m_i}{2}} \rightharpoonup u_i^{\frac{m_i}{2}}$ weakly in $L^2(0, T; H^1(\Omega))$ for all $i \in \{1, \dots, I\}$ and $T > 0$.

Proof Due to the lack of uniform-in- ε estimates of the nonlinearities, it is difficult to show directly, for instance, by means of an Aubin–Lions lemma, that u_i^ε has a convergent subsequence. Following the ideas from Fischer (2015), we first prove that $\varphi_i^E(u^\varepsilon)$ converges (up to a subsequence) to z_i^E as $\varepsilon \rightarrow 0$, and then that z_i^E converges to u_i (up to a subsequence) as $E \rightarrow \infty$. In combination with the convergence of $\varphi_i^E(u^\varepsilon)$ to u_i^ε for $E \rightarrow \infty$, this leads to the desired result.

Due to the bound $\varphi_i^E \leq 3E$, it follows at once that $\{\varphi_i^E(u^\varepsilon)\}$ is bounded in $L^2(0, T; L^2(\Omega))$ uniformly in $\varepsilon > 0$ for each $E \in \mathbb{N}$. Next, by the chain rule we have

$$\nabla \varphi_i^E(u^\varepsilon) = \sum_{j=1}^I \partial_j \varphi_i^E(u^\varepsilon) \nabla u_j^\varepsilon = \sum_{j=1}^I \frac{2}{m_j} \partial_j \varphi_i^E(u^\varepsilon) (u_j^\varepsilon)^{1-\frac{m_j}{2}} \nabla (u_j^\varepsilon)^{\frac{m_j}{2}}.$$

Thus,

$$\begin{aligned} \int_{Q_T} |\nabla \varphi_i^E(u^\varepsilon)|^2 dx dt &\leq I \sum_{j=1}^I \frac{4}{m_j^2} \int_{Q_T} |\partial_j \varphi_i^E(u^\varepsilon)|^2 |u_j^\varepsilon|^{2-m_j} \left| \nabla(u_j^\varepsilon)^{\frac{m_j}{2}} \right|^2 dx dt \\ &\leq C(E) \sum_{j=1}^I \int_{Q_T} \left| \nabla(u_j^\varepsilon)^{\frac{m_j}{2}} \right|^2 dx dt \end{aligned}$$

thanks to the compact support of $D\varphi_i^E$, $m_j < 2$, and the $L^2(0, T; L^2(\Omega))$ bound on $\nabla(u_j^\varepsilon)^{\frac{m_j}{2}}$ from Lemma 2.1. As a consequence, we know that $\{\varphi_i^E(u^\varepsilon)\}$ is bounded (uniformly in ε) in $L^2(0, T; H^1(\Omega))$.

To apply the Aubin–Lions Lemma, we need an estimate concerning the time derivative of $\varphi_i^E(u^\varepsilon)$. Using $\partial_j \varphi_i^E(u^\varepsilon)\psi$ as a test function in (2.3) and summing over $j \in \{1, \dots, I\}$, we obtain for almost all $t_2 > t_1 \geq 0$,

$$\begin{aligned} &\int_{\Omega} \varphi_i^E(u^\varepsilon(\cdot, t_2))\psi(\cdot, t_2) dx - \int_{\Omega} \varphi_i^E(u^\varepsilon(\cdot, t_1))\psi(\cdot, t_1) dx - \int_{t_1}^{t_2} \int_{\Omega} \varphi_i^E(u^\varepsilon)\partial_t \psi dx dt \\ &= - \sum_{j,k=1}^I \frac{4d_j}{m_k} \int_{t_1}^{t_2} \int_{\Omega} \psi \left[\partial_j \partial_k \varphi_i^E(u^\varepsilon)(u_j^\varepsilon)^{\frac{m_j}{2}} (u_k^\varepsilon)^{1-\frac{m_k}{2}} \right] \nabla(u_j^\varepsilon)^{\frac{m_j}{2}} \cdot \nabla(u_k^\varepsilon)^{\frac{m_k}{2}} dx dt \\ &\quad - \sum_{j=1}^I 2d_j \int_{t_1}^{t_2} \int_{\Omega} \partial_j \varphi_i^E(u^\varepsilon)(u_j^\varepsilon)^{\frac{m_j}{2}} \nabla(u_j^\varepsilon)^{\frac{m_j}{2}} \cdot \nabla \psi dx dt \\ &\quad + \sum_{j=1}^I \int_{t_1}^{t_2} \int_{\Omega} \partial_j \varphi_i^E(u^\varepsilon) \frac{f_j(u^\varepsilon)}{1 + \varepsilon|f(u^\varepsilon)|} \psi dx dt. \end{aligned} \tag{2.11}$$

The third term on the right-hand side is clearly bounded uniformly in ε for each fixed $E \in \mathbb{N}$ since $D\varphi_i^E$ has a compact support. The first and second terms are bounded uniformly in ε thanks to the boundedness of $\nabla(u_k^\varepsilon)^{\frac{m_k}{2}}$ in $L^2(0, T; L^2(\Omega))$ for all $k = 1, \dots, I$, and properties (E2), (E3) in Lemma 2.3. It follows then that $\partial_t \varphi_i^E(u^\varepsilon)$ is bounded uniformly (w.r.t. $\varepsilon > 0$) in $L^1(0, T; (W^{1,\infty}(\Omega))')$ for each fixed $E \in \mathbb{N}$.

Therefore, by applying an Aubin–Lions lemma to the sequence $\{\varphi_i^E(u^\varepsilon)\}_{\varepsilon>0}$, for fixed $E \in \mathbb{N}$, there exists a subsequence (not relabeled) of $\varphi_i^E(u^\varepsilon)$ converging strongly in $L^2(0, T; L^2(\Omega))$ and, thus, almost everywhere as $\varepsilon \rightarrow 0$. Using a diagonal sequence argument, one can extract a further subsequence such that $\varphi_i^E(u^\varepsilon)$ converges a.e. in Q_T to a measurable function z_i^E for all $E \in \mathbb{N}$ and $i = 1, \dots, I$.

We next prove that z_i^E converges a.e. to some measurable function u_i as $E \rightarrow \infty$. First, since $\sum_{j=1}^I u_j^\varepsilon \log u_j^\varepsilon$ is uniformly bounded w.r.t. to $\varepsilon > 0$ in $L^\infty(0, T; L^1(\Omega))$, it follows that $\varphi_i^E(u^\varepsilon) \log \varphi_i^E(u^\varepsilon)$ is uniformly bounded w.r.t. to $\varepsilon > 0$ and $E \in \mathbb{N}$ in $L^\infty(0, T; L^1(\Omega))$. This is trivial in case $E \leq 1$, $\varphi_i^E(u^\varepsilon) \leq 1$, or $\sum_{j=1}^I u_j^\varepsilon E^{-\alpha_j} \leq 1$. Otherwise, there exists some j such that $u_j^\varepsilon E^{-\alpha_j} > 1/I$ holds true. And as $\alpha_j > 1$, we derive

$$\varphi_i^E(u^\varepsilon) \leq 3E \leq 3EIu_j^\varepsilon E^{-\alpha_j} \leq 3Iu_j^\varepsilon.$$

The bound $\varphi_i^E(u^\varepsilon) > 1$ ensures $\varphi_i^E(u^\varepsilon) \log \varphi_i^E(u^\varepsilon) \leq 3Iu_j^\varepsilon \log(3Iu_j^\varepsilon)$ and the claim follows. Thus, $z_i^E \log z_i^E$ is uniformly bounded in $L^\infty(0, T; L^1(\Omega))$ w.r.t. $E \in \mathbb{N}$ by Fatou’s Lemma. Secondly, $z_j^E(x, t) = z_j^{\tilde{E}}(x, t)$ for all $\tilde{E} > E$, if $\sum_{j=1}^I z_j^E(x, t) E^{-\alpha_j} < 1$. Indeed,

$$\sum_{j=1}^I z_j^E(x, t) E^{-\alpha_j} = \lim_{\varepsilon \rightarrow 0} \sum_{j=1}^I \varphi_j^E(u^\varepsilon(x, t)) E^{-\alpha_j},$$

guarantees that $\sum_{j=1}^I \varphi_j^E(u^\varepsilon(x, t)) E^{-\alpha_j} < 1$ holds true for sufficiently small $\varepsilon > 0$. Thanks to property (E8) in Lemma 2.3, it follows that $\varphi_i^E(u^\varepsilon(x, t)) = u_i^\varepsilon(x, t) = \varphi_i^{\tilde{E}}(u^\varepsilon(x, t))$ for small enough ε and all $\tilde{E} > E$. Therefore, due to $z_i^E(x, t) = \lim_{\varepsilon \rightarrow 0} \varphi_i^E(u^\varepsilon(x, t))$, we obtain $z_i^E(x, t) = z_i^{\tilde{E}}(x, t)$ for all $\tilde{E} > E$ as desired. As a result, if $\sum_{j=1}^I z_j^E(x, t) E^{-\alpha_j} < 1$ for some (x, t) , then $\lim_{E \rightarrow \infty} z_i^E(x, t)$ exists and is finite for all $i = 1, \dots, I$. Using the fact that $\sum_{j=1}^I z_j^E \log z_j^E$ is bounded uniformly w.r.t. $E \in \mathbb{N}$ in $L^\infty(0, T; L^1(\Omega))$, we have

$$\lim_{E \rightarrow \infty} \mathcal{L}^{n+1} \left(\left\{ (x, t) \in Q_T : \sum_{j=1}^I z_j^E(x, t) E^{-\alpha_j} \geq 1 \right\} \right) = 0$$

where \mathcal{L}^{n+1} is the Lebesgue measure in \mathbb{R}^{n+1} . Hence, the limit $u_i(x, t) = \lim_{E \rightarrow \infty} z_i^E(x, t)$ exists for a.e. $(x, t) \in Q_T$. Moreover, $u_i \log u_i \in L^\infty(0, T; L^1(\Omega))$ due to Fatou’s lemma and $z_i^E \log z_i^E \in L^\infty(0, T; L^1(\Omega))$. Since $\sum_{j=1}^I u_j^\varepsilon$ is uniformly bounded in $L^1(Q_T)$, we find

$$\begin{aligned} 0 &= \lim_{E \rightarrow \infty} \mathcal{L}^{n+1} \left(\left\{ (x, t) \in Q_T : \sum_{j=1}^I u_j^\varepsilon(x, t) E^{-\alpha_j} \geq 1 \right\} \right) \\ &\geq \lim_{E \rightarrow \infty} \mathcal{L}^{n+1} \left(\left\{ (x, t) \in Q_T : u_i^\varepsilon(x, t) \neq \varphi_i^E(u^\varepsilon)(x, t) \right\} \right) \end{aligned}$$

and this limit is uniform in $\varepsilon > 0$. Now, we estimate for $\delta > 0$

$$\begin{aligned} &\mathcal{L}^{d+1} \left(\left\{ (x, t) \in Q_T : |u_i^\varepsilon(x, t) - u_i(x, t)| > \delta \right\} \right) \\ &\leq \mathcal{L}^{d+1} \left(\left\{ (x, t) \in Q_T : u_i^\varepsilon(x, t) \neq \varphi_i^E(u^\varepsilon)(x, t) \right\} \right) \\ &\quad + \mathcal{L}^{d+1} \left(\left\{ (x, t) \in Q_T : |\varphi_i^E(u^\varepsilon)(x, t) - z_i^E(x, t)| > \frac{\delta}{2} \right\} \right) \end{aligned}$$

$$+ \mathcal{L}^{d+1} \left(\left\{ (x, t) \in Q_T : |z_i^E(x, t) - u_i(x, t)| > \frac{\delta}{2} \right\} \right).$$

As we have proved that $\varphi_i^E(u^\varepsilon) \rightarrow z_i^E$ a.e. in Q_T for $\varepsilon \rightarrow 0$ and fixed $E \in \mathbb{N}$ and that $z_i^E \rightarrow u_i$ a.e. in Q_T as $E \rightarrow \infty$, we infer the convergence in measure of u_i^ε to u_i for $\varepsilon \rightarrow 0$, and convergence a.e. of another subsequence.

The uniform bound on $u_i^\varepsilon \log u_i^\varepsilon$ in $L^\infty(0, T; L^1(\Omega))$ and the convergence $u_i^\varepsilon \rightarrow u_i$ a.e. ensure that $u_i^\varepsilon \rightarrow u_i$ strongly in $L^p(0, T; L^1(\Omega))$ for all $p \geq 1$. One can easily prove this by truncating u_i^ε at a sufficiently large threshold. By the strong convergence of u_i^ε to u_i in $L^1(0, T; L^1(\Omega))$, we are able to prove the distributional convergence of $(u_i^\varepsilon)^{\frac{m_i}{2}}$ to $(u_i)^{\frac{m_i}{2}}$ by noting that $m_i < 2$. The uniform $L^2(0, T; H^1(\Omega))$ bound on $(u_i^\varepsilon)^{\frac{m_i}{2}}$ then leads to a subsequence $(u_i^\varepsilon)^{\frac{m_i}{2}}$ which converges to $(u_i)^{\frac{m_i}{2}}$ weakly in $L^2(0, T; H^1(\Omega))$. Thanks to the weak lower semicontinuity of the $L^2(0, T; L^2(\Omega))$ norm, we also deduce

$$\int_0^T \int_\Omega \left| \nabla u_i^{\frac{m_i}{2}} \right|^2 dx dt \leq \liminf_{\varepsilon \rightarrow 0} \int_0^T \int_\Omega \left| \nabla (u_i^\varepsilon)^{\frac{m_i}{2}} \right|^2 dx dt$$

for all $i \in \{1, \dots, I\}$, where the right-hand side is bounded via the uniform $L^2(0, T; H^1(\Omega))$ bound on $(u_i^\varepsilon)^{\frac{m_i}{2}}$. This completes the proof of the lemma. \square

At this point, we expect the function $u = (u_1, \dots, u_I)$ in Lemma 2.4 to be a renormalised solution to (1.1). We will first derive an equation admitting $\varphi_i^E(u)$ as a solution. This equation is already “almost” matching the formulation for a renormalised solution (1.4) except for a “defect measure”.

Lemma 2.5 *Let $u = (u_1, \dots, u_I)$ be the functions constructed in Lemma 2.4. Then, for any $\psi \in C^\infty([0, T], C_0^\infty(\overline{\Omega}))$, the truncated functions $\varphi_i^E(u)$, $i \in \{1, \dots, I\}$, satisfy*

$$\begin{aligned} & \int_\Omega \varphi_i^E(u(\cdot, T))\psi(\cdot, T) dx - \int_\Omega \varphi_i^E(u(\cdot, 0))\psi(\cdot, 0) dx - \int_{Q_T} \varphi_i^E(u)\partial_t \psi dx dt \\ &= - \int_{Q_T} \psi d\mu_i^E(x, t) - \sum_{j=1}^I d_j m_j \int_{Q_T} \partial_j \varphi_i^E(u) u_j^{m_j-1} \nabla u_j \cdot \nabla \psi dx dt \\ &+ \sum_{j=1}^I \int_{Q_T} \partial_j \varphi_i^E(u) f_j(u) \psi dx dt \end{aligned} \tag{2.12}$$

where μ_i^E is a signed Radon measure satisfying

$$\lim_{E \rightarrow \infty} |\mu_i^E|(\overline{\Omega} \times [0, T]) = 0 \tag{2.13}$$

for all $T > 0$ and $i \in \{1, \dots, I\}$.

Proof Choosing $t_1 = 0, t_2 = T$, and using the convergence $u^\varepsilon \rightarrow u$ a.e. in Q_T , the weak convergence $\nabla(u_i^\varepsilon)^{\frac{m_j}{2}} \rightharpoonup \nabla u_i^{\frac{m_j}{2}}$ in $L^2(0, T; L^2(\Omega))$, and the fact that $D\varphi_i^E(u^\varepsilon)$ vanishes when u^ε is too large, we straightforwardly obtain the convergence of the left-hand side and the last two terms on the right-hand side of (2.11). It remains to establish

$$\begin{aligned}
 & -4 \sum_{j,k=1}^I \frac{d_j}{m_k} \int_{Q_T} \psi \left[\partial_j \partial_k \varphi_i^E(u^\varepsilon) (u_j^\varepsilon)^{\frac{m_j}{2}} (u_k^\varepsilon)^{1-\frac{m_k}{2}} \right] \nabla(u_j^\varepsilon)^{\frac{m_j}{2}} \cdot \nabla(u_k^\varepsilon)^{\frac{m_k}{2}} \, dx \, dt \\
 & \quad \rightarrow - \int_{Q_T} \psi \, d\mu_i^E(x, t)
 \end{aligned}$$

with a signed Radon measure μ_i^E satisfying (2.13). By denoting

$$\mu_{i,\varepsilon}^E := 4 \sum_{j,k=1}^I \frac{d_j}{m_k} \left[\partial_j \partial_k \varphi_i^E(u^\varepsilon) (u_j^\varepsilon)^{\frac{m_j}{2}} (u_k^\varepsilon)^{1-\frac{m_k}{2}} \right] \nabla(u_j^\varepsilon)^{\frac{m_j}{2}} \cdot \nabla(u_k^\varepsilon)^{\frac{m_k}{2}} \, dx \, dt,$$

we can use property (E2) of the truncation function φ_i^E and the ε -uniform bound of $\nabla(u_i^\varepsilon)^{\frac{m_j}{2}}$ in $L^2(0, T; L^2(\Omega))$, to obtain

$$|\mu_{i,\varepsilon}^E|(\overline{Q_T}) \leq C$$

for all $\varepsilon > 0$. Therefore, by passing to a subsequence, we know that $\mu_{i,\varepsilon}^E$ weak-* converges on $\overline{Q_T}$ to a signed Radon measure μ_i^E as $\varepsilon \rightarrow 0$. It remains to prove (2.13). Due to Young's inequality, we have

$$\begin{aligned}
 & |\mu_{i,\varepsilon}^E|(\overline{Q_T}) \\
 & \leq C \sum_{j,k=1}^I \sum_{K=1}^\infty \int_{Q_T} |\partial_j \partial_k \varphi_i^E(u^\varepsilon)| (u_j^\varepsilon)^{\frac{m_j}{2}} (u_k^\varepsilon)^{1-\frac{m_k}{2}} \chi_{\{K-1 \leq |u^\varepsilon| < K\}} \left| \nabla(u_j^\varepsilon)^{\frac{m_j}{2}} \right|^2 \, dx \, dt \\
 & \leq C \sum_{l=1}^I \sum_{K=1}^\infty v_{l,K}^\varepsilon(\overline{Q_T}) \sup_{\substack{1 \leq j,k \leq I \\ K-1 \leq |v| < K}} |\partial_j \partial_k \varphi_i^E(v)| v_j^{\frac{m_j}{2}} v_k^{1-\frac{m_k}{2}}
 \end{aligned}$$

where

$$v_{l,K}^\varepsilon := \chi_{\{K-1 \leq |u^\varepsilon| < K\}} \left| \nabla(u_l^\varepsilon)^{\frac{m_l}{2}} \right|^2 \, dx \, dt.$$

We stress that $v_{l,K}^\varepsilon$ is uniformly bounded w.r.t. l, K , and ε . Consequently, we may pass to a subsequence $v_{l,K}^\varepsilon$ which converges weak-* on $\overline{Q_T}$ to a Radon measure $v_{l,K}$

as $\varepsilon \rightarrow 0$. Together with the weak- $*$ convergence of $\mu_{i,\varepsilon}^E$ to μ_i^E on $\overline{Q_T}$, we derive

$$\begin{aligned} |\mu_i^E|(\overline{Q_T}) &\leq \liminf_{\varepsilon \rightarrow 0} |\mu_{i,\varepsilon}^E|(\overline{Q_T}) \\ &\leq C \sum_{l=1}^I \sum_{K=1}^{\infty} v_{l,K}(\overline{Q_T}) \sup_{\substack{1 \leq j, k \leq I \\ K-1 \leq |v| < K}} |\partial_j \partial_k \varphi_i^E(v)| v_j^{\frac{m_j}{2}} v_k^{1-\frac{m_k}{2}}. \end{aligned}$$

Moreover,

$$\sum_{K=1}^{\infty} v_{l,K}^{\varepsilon}(\overline{Q_T}) = \int_{Q_T} \left| \nabla (u_j^{\varepsilon})^{\frac{m_j}{2}} \right|^2 dx dt,$$

which is bounded uniformly in ε . Fatou’s Lemma applied to the counting measure on \mathbb{N} now entails

$$\sum_{K=1}^{\infty} v_{l,K}(\overline{Q_T}) \leq \liminf_{\varepsilon \rightarrow 0} \sum_{K=1}^{\infty} v_{l,K}^{\varepsilon}(\overline{Q_T}) < +\infty.$$

Therefore, employing the dominated convergence theorem, we can finally estimate

$$\begin{aligned} \lim_{E \rightarrow \infty} |\mu_i^E|(\overline{Q_T}) &\leq C \sum_{l=1}^I \sum_{K=1}^{\infty} v_{l,K}(\overline{Q_T}) \lim_{E \rightarrow \infty} \sup_{\substack{1 \leq j, k \leq I \\ K-1 \leq |v| < K}} |\partial_j \partial_k \varphi_i^E(v)| v_j^{\frac{m_j}{2}} v_k^{1-\frac{m_k}{2}} \\ &= C \sum_{l=1}^I \sum_{K=1}^{\infty} v_{l,K}(\overline{Q_T}) \cdot 0 = 0 \end{aligned}$$

thanks to properties (E2) and (E7) of the truncation function φ_i^E . □

To prove Theorem 1.3, we use the following technical lemma whose proof can be found in Fischer (2015).

Lemma 2.6 (Fischer 2015, Lemma 4)(A weak chain rule for the time derivative)
 Let Ω be a bounded domain with Lipschitz boundary. Assume that $T > 0$, $v \in L^2(0, T; H^1(\Omega)^I)$, and $v_0 \in L^1(\Omega)^I$. Let v_i be a Radon measure on $\overline{Q_T}$, $w_i \in L^1(Q_T)$, and $z_i \in L^2(0, T; L^2(\Omega)^I)$ for $1 \leq i \leq I$. Assume moreover that for any $\psi \in C^\infty(\overline{Q_T})$ with compact support we have

$$\begin{aligned} &\int_{\Omega} v_i(T) \psi(T) dx - \int_{Q_T} v_i \frac{d}{dt} \psi dx dt - \int_{\Omega} v_0 \psi(0) dx \\ &= \int_{\overline{Q_T}} \psi dv_i + \int_{Q_T} w_i \psi dx dt + \int_{Q_T} z_i \cdot \nabla \psi dx dt. \end{aligned}$$

Let $\xi : \mathbb{R}^I \rightarrow \mathbb{R}$ be a smooth function with compactly supported first derivatives. Then, we have for all $\psi \in C^\infty(\overline{Q_T})$ with compact support

$$\begin{aligned} & \left| \int_{\Omega} \xi(v(T))\psi(T) \, dx \right. \\ & \quad - \int_{Q_T} \xi(v) \frac{d}{dt} \psi \, dx \, dt - \int_{\Omega} \xi(v_0)\psi(0) \, dx - \sum_{i=1}^I \int_{Q_T} \psi \partial_i \xi(v) w_i \, dx \, dt \\ & \quad \left. - \sum_{i=1}^I \int_{Q_T} \partial_i \xi(v) z_i \cdot \nabla \psi \, dx \, dt - \sum_{i,k=1}^I \int_{Q_T} \psi \partial_i \partial_k \xi(v) z_i \cdot \nabla v_k \, dx \, dt \right| \\ & \leq C(\Omega) \|\psi\|_{\infty} (\sup_u |D\xi(u)|) \sum_{i=1}^I |v_i|(\overline{Q_T}). \end{aligned}$$

We are now ready to prove the existence of global renormalised solutions.

Proof of Theorem 1.3 By applying Lemma 2.6 to (2.12), we obtain an approximate relation for the weak time derivative of $\xi(\varphi^E(u))$, which leads to the desired equation for $\xi(u)$ when passing to the limit $E \rightarrow \infty$. In detail, we set $v_i := \varphi_i^E(u)$, $(v_0)_i := \varphi_i^E(u_0)$, $v_i := -\mu_i^E$,

$$z_i := - \sum_{j=1}^I d_j m_j \partial_j \varphi_i^E(u) u_j^{m_j-1} \nabla u_j, \quad w_i := \sum_{j=1}^I \partial_j \varphi_i^E(u) f_j(u),$$

and obtain for any smooth function ξ with compactly supported first derivatives

$$\begin{aligned} & \left| \int_{\Omega} \xi(\varphi^E(u))\psi(T) \, dx - \int_{Q_T} \xi(\varphi^E(u)) \frac{d}{dt} \psi \, dx \, dt - \int_{\Omega} \xi(\varphi^E(u_0))\psi(0) \, dx \right. \\ & \quad - \sum_{i=1}^I \sum_{j=1}^I \int_{Q_T} \psi \partial_i \xi(\varphi^E(u)) \partial_j \varphi_i^E(u) f_j(u) \, dx \, dt \\ & \quad + \sum_{i=1}^I \sum_{j=1}^I \int_{Q_T} d_j m_j \partial_i \xi(\varphi^E(u)) \partial_j \varphi_i^E(u) u_j^{m_j-1} \nabla u_j \cdot \nabla \psi \, dx \, dt \tag{2.14} \\ & \quad \left. + \sum_{i,k=1}^I \sum_{j,\ell=1}^I \int_{Q_T} d_j m_j \psi \partial_i \partial_k \xi(\varphi^E(u)) \partial_j \varphi_i^E(u) \partial_\ell \varphi_k^E(u) u_j^{m_j-1} \nabla u_j \cdot \nabla u_\ell \, dx \, dt \right| \\ & \leq C(\Omega) \|\psi\|_{\infty} (\sup_u |D\xi(u)|) \sum_{i=1}^I |\mu_i^E|(\overline{Q_T}). \end{aligned}$$

We now want to pass to the limit $E \rightarrow \infty$ in (2.14) to obtain (1.4). Note first that due to (2.13), we obtain in the limit an equality instead of just an estimate. Since ξ has

compactly supported first derivatives, $\partial_j \varphi_i^E(u)$ is bounded (see (E5)), and $\varphi_i^E(u) \rightarrow u_i$ a.e. as $E \rightarrow \infty$, we directly obtain the following limits for the first line in (2.14):

$$\begin{aligned} \int_{\Omega} \xi(\varphi^E(u(T)))\psi(T) \, dx &\longrightarrow \int_{\Omega} \xi(u(T))\psi(T) \, dx, \\ \int_{Q_T} \xi(\varphi^E(u)) \frac{d}{dt} \psi \, dx \, dt &\longrightarrow \int_{Q_T} \xi(u) \frac{d}{dt} \psi \, dx \, dt, \\ \int_{\Omega} \xi(\varphi^E(u_0))\psi(0) \, dx &\longrightarrow \int_{\Omega} \xi(u_0)\psi(0) \, dx. \end{aligned}$$

By employing $m_j u_j^{m_j-1} \nabla u_j = 2u_j^{\frac{m_j}{2}} \nabla u_j^{\frac{m_j}{2}}$ and $u_j^{\frac{m_j}{2}} \in L^2(0, T; H^1(\Omega))$, we also derive

$$\begin{aligned} &\sum_{i=1}^I \sum_{j=1}^I \int_{Q_T} d_j m_j \partial_i \xi(\varphi^E(u)) \partial_j \varphi_i^E(u) u_j^{m_j-1} \nabla u_j \cdot \nabla \psi \, dx \, dt \\ &\longrightarrow \sum_{i=1}^I \int_{Q_T} d_i m_i \partial_i \xi(u) u_i^{m_i-1} \nabla u_i \cdot \nabla \psi \, dx \, dt. \end{aligned}$$

It remains to ensure the convergence of the third and fourth line of (2.14). To this end, we recall $u_j^{\frac{m_j}{2}} \in L^2(0, T; H^1(\Omega))$, and we utilise the following observation: there exists a constant $E_0 > 0$ such that for all $E > E_0$ the inequality $\sum_{i=1}^I u_i \geq E_0$ implies $\partial_i \xi(\varphi^E(u)) = \partial_i \xi(u) = 0$ and $\partial_i \partial_k \xi(\varphi^E(u)) = \partial_i \partial_k \xi(u) = 0$ for all $i, k \in \{1, \dots, I\}$. The convergence of the third and fourth line above is now a consequence of this auxiliary result, as the derivatives of ξ are zero provided $\max_i u_i$ is larger than E_0 :

$$\begin{aligned} &\sum_{i=1}^I \sum_{j=1}^I \int_{Q_T} \psi \partial_i \xi(\varphi^E(u)) \partial_j \varphi_i^E(u) f_j(u) \, dx \, dt \longrightarrow \sum_{i=1}^I \int_{Q_T} \psi \partial_i \xi(u) f_i(u) \, dx \, dt, \\ &\sum_{i,j,k,\ell=1}^I \int_{Q_T} d_j m_j \psi \partial_i \partial_k \xi(\varphi^E(u)) \partial_j \varphi_i^E(u) \partial_\ell \varphi_k^E(u) u_j^{m_j-1} \nabla u_j \cdot \nabla u_\ell \, dx \, dt \\ &\longrightarrow \sum_{i,k=1}^I \int_{Q_T} d_i m_i \psi \partial_i \partial_k \xi(u) u_i^{m_i-1} \nabla u_i \cdot \nabla u_k \, dx \, dt. \end{aligned}$$

The previous observation readily follows by choosing $E_0 > 0$ such that $\text{supp}(D\xi) \subset B_{E_0/\sqrt{I}}(0)$. Given $E > E_0$ and $\sum_{i=1}^I u_i \geq E_0$, one gets $\sum_{i=1}^I \varphi_i^E(u) \geq E_0$ from the definition of φ_i^E in (2.6). In particular, $u, \varphi^E(u) \notin B_{E_0/\sqrt{I}}(0)$, which proves the claim. \square

Finally, we show that the constructed renormalised solution satisfies the weak entropy law (1.5) and, when it is applicable, the conservation laws (1.6).

Proof of (1.5) and (1.6) Since $u_i^\varepsilon \rightarrow u_i$ a.e. in Q_T and f_i are continuous, we have

$$\begin{aligned} & \frac{1}{1 + \varepsilon|f(u^\varepsilon)|} \left[\sum_{i=1}^I f_i(u^\varepsilon)(\log u_i^\varepsilon + \mu_i) - C \sum_{i=1}^I (1 + u_i^\varepsilon \log u_i^\varepsilon) \right] \\ & \xrightarrow{\text{a.e. in } Q_T} \sum_{i=1}^I f_i(u)(\log u_i + \mu_i) - C \sum_{i=1}^I (1 + u_i \log u_i) \leq 0. \end{aligned} \tag{2.15}$$

We rewrite (2.5) as

$$\begin{aligned} & \frac{d}{dt} \int_{\Omega} \sum_{i=1}^I u_i^\varepsilon (\log u_i^\varepsilon + \mu_i - 1) \, dx + \sum_{i=1}^I \frac{4d_i}{m_i} \int_{\Omega} \left| \nabla (u_i^\varepsilon)^{\frac{m_i}{2}} \right|^2 \, dx \\ & = \int_{\Omega} \frac{1}{1 + \varepsilon|f(u^\varepsilon)|} \left(\sum_{i=1}^I f_i(u^\varepsilon)(\log u_i^\varepsilon + \mu_i) - C \sum_{i=1}^I (1 + u_i^\varepsilon \log u_i^\varepsilon) \right) \, dx \\ & \quad + \int_{\Omega} \frac{1}{1 + \varepsilon|f(u^\varepsilon)|} C \sum_{i=1}^I (1 + u_i^\varepsilon \log u_i^\varepsilon) \, dx. \end{aligned}$$

We now integrate this relation on (s, T) and use (2.15) and Fatou’s lemma for the first integral on the right-hand side, and the uniform-in- ε in Lemma 3.6 for the second integral to obtain

$$\begin{aligned} & \sum_{i=1}^I \int_{\Omega} u_i (\log u_i + \mu_i - 1) \, dx \Big|_s^T + \sum_{i=1}^I \frac{4d_i}{m_i^2} \int_s^T \int_{\Omega} \left| \nabla (u_i)^{\frac{m_i}{2}} \right|^2 \, dx \, dt \\ & \leq \sum_{i=1}^I \int_s^T \int_{\Omega} f_i(u)(\log u_i + \mu_i) \, dx \, dt, \end{aligned}$$

which is (1.5). To show (1.6), we take $t_0 = 0$ and $\psi = q_i$ in (2.3) and sum over i to get

$$\sum_{i=1}^I \int_{\Omega} q_i u_i^\varepsilon(x, t) \, dx = \sum_{i=1}^I \int_{\Omega} q_i u_{i,0}^\varepsilon(x) \, dx.$$

Letting $\varepsilon \rightarrow 0$ proves (1.6). □

3 Exponential Convergence to Equilibrium

3.1 Entropy–Entropy Dissipation Inequality

Without loss of generality, we assume that the domain Ω has unit volume, i.e. $|\Omega| = 1$. We employ the following notation:

$$\bar{u}_i := \int_{\Omega} u_i \, dx \quad \text{and} \quad \bar{u} := (\bar{u}_i)_{i=1, \dots, I}.$$

To show the convergence to equilibrium for (1.10), we exploit the so-called entropy method. Assuming the complex balanced condition, system (1.10) possesses the relative entropy functional

$$\mathcal{E}[u|u_{\infty}] = \sum_{i=1}^I \int_{\Omega} \left(u_i \log \frac{u_i}{u_{i,\infty}} - u_i + u_{i,\infty} \right) dx \tag{3.1}$$

and the corresponding entropy dissipation function

$$\mathcal{D}[u] = \sum_{i=1}^I d_i m_i \int_{\Omega} u_i^{m_i-2} |\nabla u_i|^2 \, dx + \sum_{r=1}^R k_r u_{\infty}^{y_r} \int_{\Omega} \Psi \left(\frac{u^{y_r}}{u_{\infty}^{y_r}}; \frac{u^{y'_r}}{u_{\infty}^{y'_r}} \right) dx \tag{3.2}$$

with $\Psi(x; y) = x \log(x/y) - x + y$ (see (Desvillettes et al. 2017, Proposition 2.1) for a derivation of $\mathcal{D}[u]$). Formally, we have

$$\mathcal{D}[u] = -\frac{d}{dt} \mathcal{E}[u|u_{\infty}]$$

along the trajectory of (1.10). The following lemma shows that the L^1 norm can be bounded by the relative entropy.

Lemma 3.1 *Assume that $\mathcal{E}[u|u_{\infty}] < +\infty$. Then,*

$$\|u_i\|_{L^1(\Omega)} = \bar{u}_i \leq \tilde{K} := 2 \left(\mathcal{E}(u|u_{\infty}) + \sum_{i=1}^I u_{i,\infty} \right)$$

for $i = 1, \dots, I$.

Proof The elementary inequalities $x \log(x/y) - x + y \geq (\sqrt{x} - \sqrt{y})^2 \geq \frac{1}{2}x - y$ ensure that

$$\mathcal{E}[u|u_{\infty}] = \sum_{i=1}^I \int_{\Omega} \left(u_i \log \frac{u_i}{u_{i,\infty}} - u_i + u_{i,\infty} \right) dx \geq \frac{1}{2} \sum_{i=1}^I \int_{\Omega} u_i \, dx - \sum_{i=1}^I u_{i,\infty}.$$

Taking the non-negativity of the solution u into account allows to conclude. □

Lemma 3.2 (Fellner and Tang 2017, Lemma 2.3) (Csiszár–Kullback–Pinsker-type inequality) Fix a non-negative mass $M \in \mathbb{R}_+^m$. For all measurable functions $u : \Omega \rightarrow \mathbb{R}_+^I$ satisfying the mass conservation $\mathbb{Q}\bar{u} = M$, the following inequality holds,

$$\mathcal{E}[u|u_\infty] \geq C_{CKP} \sum_{i=1}^I \|u_i - u_{i,\infty}\|_{L^1(\Omega)}^2,$$

in which $C_{CKP} > 0$ is a constant depending only on Ω and u_∞ .

The following two lemmas provide generalisations of the classical logarithmic Sobolev inequality, which are suited for nonlinear diffusion.

Lemma 3.3 Dolbeault et al. (2008)(L^m -logarithmic Sobolev inequality) For any $m \geq 1$, there exists a constant $C(\Omega, m) > 0$ such that we have

$$\int_{\Omega} \left| \nabla u^{\frac{m}{2}} \right|^2 dx \geq C(\Omega, m) \left(\int_{\Omega} u \log \frac{u}{\bar{u}} dx \right)^m.$$

Lemma 3.4 Mielke and Mitzenzweig (2018) (Generalised logarithmic Sobolev inequality) For any $m > \frac{(d-2)_+}{d}$, there exists a constant $C(\Omega, m) > 0$ such that

$$\int_{\Omega} u^{m-2} |\nabla u|^2 dx \geq C(\Omega, m) \bar{u}^{m-1} \int_{\Omega} u \log \frac{u}{\bar{u}} dx.$$

The cornerstone of the entropy method is the entropy–entropy dissipation inequality which is proved in the following lemma.

Lemma 3.5 Let $K > 0$ and $m_i > \frac{(d-2)_+}{d}$ for all $1 \leq i \leq I$. There exist constants $C > 0$ and $\alpha \geq 1$ depending on K such that for any measurable function $u : \Omega \rightarrow \mathbb{R}_+^I$ subject to $\mathbb{Q}\bar{u} = \mathbb{Q}u_\infty$ and $\mathcal{E}[u|u_\infty] \leq K$, we have

$$\mathcal{D}[u] \geq C(\mathcal{E}[u|u_\infty])^\alpha.$$

Proof First, similar to Lemma 3.1, we have the following L^1 bounds:

$$\bar{u}_i = \|u_i\|_{L^1(\Omega)} \leq M_1 := 2 \left(K + \sum_{i=1}^I u_{i,\infty} \right).$$

For all i with $\frac{(d-2)_+}{d} < m_i < 1$, we apply Lemma 3.4 to have

$$\begin{aligned} \int_{\Omega} u_i^{m_i-2} |\nabla u_i|^2 dx &\geq C(\Omega, m_i) \bar{u}_i^{m_i-1} \int_{\Omega} u_i \log \frac{u_i}{\bar{u}_i} dx \\ &\geq C(\Omega, m_i) M_1^{m_i-1} \int_{\Omega} u_i \log \frac{u_i}{\bar{u}_i} dx \end{aligned} \tag{3.3}$$

since $m_i < 1$ and $\bar{u}_i \leq M_1$. Therefore, by applying Lemma 3.3, we can estimate the entropy dissipation from below as follows:

$$\mathcal{D}[u] \geq K_1 \left[\sum_{i=1}^I \left(\int_{\Omega} u_i \log \frac{u_i}{\bar{u}_i} dx \right)^{\max\{1, m_i\}} + \sum_{r=1}^R k_r u_{\infty}^{y_r} \int_{\Omega} \Psi \left(\frac{u^{y_r}}{u_{\infty}^{y_r}}; \frac{u^{y_r'}}{u_{\infty}^{y_r'}} \right) dx \right].$$

For simplicity, we denote $\alpha = \max_{i=1, \dots, I} \{1, m_i\}$,

$$D_i = \int_{\Omega} u_i \log \frac{u_i}{\bar{u}_i} dx \quad \text{for } i = 1, \dots, I, \quad \text{and} \quad F = \sum_{r=1}^R k_r u_{\infty}^{y_r} \int_{\Omega} \Psi \left(\frac{u^{y_r}}{u_{\infty}^{y_r}}; \frac{u^{y_r'}}{u_{\infty}^{y_r'}} \right) dx.$$

If $D_i \geq 1$ for some $i \in \{1, \dots, I\}$ or $F \geq 1$, then we have

$$\mathcal{D}[u] \geq K_1 \geq \frac{K_1}{K^{\alpha}} (\mathcal{E}[u|u_{\infty}])^{\alpha} \tag{3.4}$$

using $\mathcal{E}[u|u_{\infty}] \leq K$. It remains to consider the case $D_i \leq 1$ for all $i = 1, \dots, I$ and $F \leq 1$. Here, we find

$$\mathcal{D}[u] \geq K_1 \left[\sum_{i=1}^I D_i^{\max\{1, m_i\}} + F \right] \geq K_1 \left[\sum_{i=1}^I D_i^{\alpha} + F^{\alpha} \right] \geq K_2 \left[\sum_{i=1}^I D_i + F \right]^{\alpha}. \tag{3.5}$$

By applying now the entropy–entropy production estimate for the case of linear diffusion (cf. Fellner and Tang 2018, Theorem 1.1), we have

$$\sum_{i=1}^I D_i + F \geq K_3 \mathcal{E}[u|u_{\infty}] \tag{3.6}$$

and, thus,

$$\mathcal{D}[u] \geq K_4 (\mathcal{E}[u|u_{\infty}])^{\alpha}.$$

From (3.4) and (3.6), we obtain the desired estimate. □

3.2 Convergence of Renormalised Solutions

Proof of Theorem 1.5 The global existence of renormalised solutions for (1.10) follows from Theorem 1.3. Indeed, assumptions (F1) and (F2) can be easily checked using (1.11), and (F3) follows from the identity (cf. Desvillettes et al. 2017, Proposition 2.1)

$$\sum_{i=1}^I f_i(u) (-\log u_{i, \infty} + \log u_i) = - \sum_{r=1}^R k_r u_{\infty}^{y_r} \Psi \left(\frac{u^{y_r}}{u_{\infty}^{y_r}}; \frac{u^{y_r'}}{u_{\infty}^{y_r'}} \right) \leq 0$$

recalling $\Psi(x; y) = x \log(x/y) - x + y$.

Thanks to (1.6) and (1.5), we have the conservation laws

$$\mathbb{Q}\overline{u(t)} = \mathbb{Q}\overline{u_0} \quad \text{for all } t \geq 0,$$

and

$$\mathcal{E}[u(t)|u_\infty] + \int_s^t \mathcal{D}[u(\tau)]d\tau \leq \mathcal{E}[u(s)|u_\infty] \quad \text{for all } 0 \leq s \leq t.$$

We now apply Lemma 3.5 to get

$$\mathcal{E}[u(t)|u_\infty] + C \int_s^t (\mathcal{E}[u(\tau)|u_\infty])^\alpha d\tau \leq \mathcal{E}[u(s)|u_\infty] \quad \text{for all } 0 \leq s \leq t. \tag{3.7}$$

If $\alpha = 1$, we immediately get the exponential convergence

$$\mathcal{E}[u(t)|u_\infty] \leq e^{-\lambda t} \mathcal{E}[u_0|u_\infty] \quad \text{for all } t \geq 0$$

for some constant $\lambda > 0$. For $\alpha > 1$, we first obtain the algebraic decay

$$\mathcal{E}[u(t)|u_\infty] \leq \frac{1}{(\mathcal{E}[u_0|u_\infty]^{1-\alpha} + C(\alpha - 1)t)^{1/(\alpha-1)}}. \tag{3.8}$$

Indeed, we define

$$\psi(t) = \mathcal{E}[u(t)|u_\infty] \quad \text{and} \quad \varphi(s) = \int_s^t (\mathcal{E}[u(\tau)|u_\infty])^\alpha d\tau.$$

It follows from (3.7) that

$$\psi(t) + C\varphi(s) \leq \psi(s).$$

Differentiating $\varphi(s)$ with respect to s gives

$$\frac{d}{ds}\varphi(s) = -(\psi(s))^\alpha \leq -(\psi(t) + C\varphi(s))^\alpha$$

and consequently,

$$\frac{d}{ds} [\psi(t) + C\varphi(s)] + C [\psi(t) + C\varphi(s)]^\alpha \leq 0.$$

Applying a nonlinear Gronwall inequality yields

$$\begin{aligned} \psi(t) + C\varphi(s) &\leq \frac{1}{[(\psi(t) + C\varphi(0))^{1-\alpha} + C(\alpha - 1)s]^{1/(\alpha-1)}} \\ &\leq \frac{1}{[\psi(0)^{1-\alpha} + C(\alpha - 1)s]^{1/(\alpha-1)}} \end{aligned}$$

thanks to $\psi(t) + C\varphi(0) \leq \psi(0)$ and $\alpha > 1$. Setting $s = t$ and using $\varphi(t) = 0$ leads us to the desired estimate (3.8). Owing to the Csiszár–Kullback–Pinsker inequality, it follows

$$\sum_{i=1}^I \|u_i(t) - u_{i,\infty}\|_{L^1(\Omega)}^2 \leq \frac{C_{CKP}^{-1}}{(\mathcal{E}[u_0|u_\infty]^{1-\alpha} + C(\alpha - 1)t)^{1/(\alpha-1)}}.$$

Therefore, there exists an explicit $T_0 > 0$ such that

$$\|u_i(t) - u_{i,\infty}\|_{L^1(\Omega)} \leq \frac{1}{2}u_{i,\infty} \quad \text{for all } t \geq T_0$$

and, thus,

$$\bar{u}_i(t) = \|u_i(t)\|_{L^1(\Omega)} \geq \frac{1}{2}u_{i,\infty} \quad \text{for all } t \geq T_0.$$

Using this property, we estimate $\mathcal{D}[u(t)]$ for $t \geq T_0$ as follows. If $m_i < 1$, then it is similar to (3.3) that

$$\begin{aligned} \int_{\Omega} \bar{u}_i(t)^{m_i-2} |\nabla u_i(t)|^2 dx &\geq C(\Omega, m_i) \bar{u}_i(t)^{m_i-1} \int_{\Omega} u_i(t) \log \frac{u_i(t)}{\bar{u}_i(t)} dx \\ &\geq C(\Omega, m_i) \tilde{K}^{m_i-1} \int_{\Omega} u_i(t) \log \frac{u_i(t)}{\bar{u}_i(t)} dx, \end{aligned} \tag{3.9}$$

and for $m_i \geq 1$,

$$\begin{aligned} \int_{\Omega} \bar{u}_i(t)^{m_i-2} |\nabla u_i(t)|^2 dx &\geq C(\Omega, m_i) \bar{u}_i(t)^{m_i-1} \int_{\Omega} u_i(t) \log \frac{u_i(t)}{\bar{u}_i(t)} dx \\ &\geq C(\Omega, m_i) \left(\frac{u_{i,\infty}}{2}\right)^{m_i-1} \int_{\Omega} u_i(t) \log \frac{u_i(t)}{\bar{u}_i(t)} dx \end{aligned} \tag{3.10}$$

since $\bar{u}_i(t) \geq \frac{1}{2}u_{i,\infty}$ for all $t \geq T_0$. Therefore, for all $t \geq T_0$, we can estimate $\mathcal{D}[u(t)]$ as

$$\mathcal{D}[u(t)] \geq K_5 \left[\sum_{i=1}^I \int_{\Omega} u_i(t) \log \frac{u_i(t)}{\bar{u}_i(t)} dx + \sum_{r=1}^R k_r u_\infty^{y_r} \int_{\Omega} \Psi \left(\frac{u(t)^{y_r}}{u_\infty^{y_r}}; \frac{u(t)^{y_r'}}{u_\infty^{y_r'}} \right) dx \right].$$

Using again the entropy–entropy production estimate in the case of linear diffusion (cf. Fellner and Tang 2018, Theorem 1.1), we have

$$\mathcal{D}[u(t)] \geq K_6 \mathcal{E}[u(t)|u_\infty] \quad \text{for all } t \geq T_0,$$

which leads to the exponential convergence

$$\mathcal{E}[u(t)|u_\infty] \leq e^{-K_6(t-T_0)} \mathcal{E}[u(T_0)|u_\infty] \quad \text{for all } t \geq T_0.$$

Since T_0 can be explicitly computed, we in fact get the exponential convergence for all $t \geq 0$, i.e.

$$\mathcal{E}[u(t)|u_\infty] \leq C e^{-\lambda t} \mathcal{E}[u_0|u_\infty] \quad \text{for all } t \geq 0,$$

for some constants $C > 0$ and $\lambda > 0$. Thanks to the Csiszár–Kullback–Pinsker inequality, we finally obtain the desired exponential convergence to equilibrium. \square

3.3 Convergence of Weak and Bounded Solutions

We consider the approximate u^ε to the regularised system (2.1). From Lemma 2.4, we know that, up to a subsequence,

$$u_i^\varepsilon \xrightarrow{\varepsilon \rightarrow 0} u_i \quad \text{a.e. in } Q_T. \tag{3.11}$$

The next lemma establishes uniform-in- ε *a priori* bounds.

Lemma 3.6 *Under assumptions (F1)–(F3), $u \geq 0$ with $u_{i,0} \log u_{i,0} \in L^1(\Omega) \cap H^{-1}(\Omega)$, and $m_i > 0$ for all $i = 1, \dots, I$, there exists a constant $C_T > 0$ which depends on $T > 0$ but not on $\varepsilon > 0$, such that*

$$\int_{Q_T} (\log u_i^\varepsilon)^2 (u_i^\varepsilon)^{m_i+1} \, dx \, dt \leq C_T \quad \text{for all } i = 1, \dots, I. \tag{3.12}$$

Proof The proof uses the idea from Laamri and Perthame (2020). For completeness, we present it in Appendix 2. \square

Proof of Theorem 1.7 By testing (2.1) with $\psi \in L^\infty(Q_T) \cap L^{2(m_i+1)}(0, T; W^{1,2(m_i+1)}(\Omega))$, we have

$$\int_{Q_T} \partial_t u_i^\varepsilon \psi \, dx \, dt + \frac{d_i}{2} \int_{Q_T} (u_i^\varepsilon)^{\frac{m_i}{2}} \nabla (u_i^\varepsilon)^{\frac{m_i}{2}} \cdot \nabla \psi \, dx \, dt = \int_{Q_T} f_i^\varepsilon(u^\varepsilon) \psi \, dx \, dt \tag{3.13}$$

From (3.11) and (3.12), we can use Vitali’s lemma to deduce

$$(u_i^\varepsilon)^{\frac{m_i}{2}} \rightarrow u_i^{\frac{m_i}{2}} \quad \text{strongly in } L^{\frac{2(m_i+1)}{m_i}}(Q_T).$$

This combined with Lemma 2.1 gives

$$\nabla(u_i^\varepsilon)^{\frac{m_i}{2}} \rightharpoonup \nabla u_i^{\frac{m_i}{2}} \text{ weakly in } L^2(Q_T).$$

Therefore,

$$\int_{Q_T} (u_i^\varepsilon)^{\frac{m_i}{2}} \nabla(u_i^\varepsilon)^{\frac{m_i}{2}} \cdot \nabla \psi \, dx \, dt \rightarrow \int_{Q_T} u_i^{\frac{m_i}{2}} \nabla u_i^{\frac{m_i}{2}} \cdot \nabla \psi \, dx \, dt. \tag{3.14}$$

From the definition of f_i^ε in (2.1) as well as (3.11), we have

$$f_i^\varepsilon(u^\varepsilon) \rightarrow f_i(u) \text{ a.e. in } Q_T. \tag{3.15}$$

We show that the set $\{f_i^\varepsilon(u^\varepsilon)\}_{\varepsilon>0}$ is uniformly integrable. Indeed, let $K \subset Q_T$ be a measurable and compact set with its measure being denoted by $|K|$. From (1.19) and (1.17), we have

$$\begin{aligned} \int_K |f_i^\varepsilon(u^\varepsilon)| \, dx \, dt &\leq \int_K |f_i(u^\varepsilon)| \, dx \, dt \leq C \int_K \left(1 + \sum_{j=1}^I |u_j^\varepsilon|^{\rho_j} \right) \, dx \, dt \\ &\leq C \left[|K| + \int_K \sum_{j=1}^I |u_j^\varepsilon|^{m_j+1} \, dx \, dt \right]. \end{aligned} \tag{3.16}$$

It is easy to see that for any $\delta > 0$, there exists some $C_\delta > 0$ such that for all $x \geq 0$,

$$x^{m_i+1} \leq \delta(\log x)^2 x^{m_i+1} + C_\delta.$$

It follows from (3.16) and Lemma 3.6 that

$$\begin{aligned} \int_K |f_i^\varepsilon(u^\varepsilon)| \, dx \, dt &\leq C \left[|K| + \delta \int_K \sum_{i=1}^I (\log u_i^\varepsilon)^2 (u_i^\varepsilon)^{m_i+1} + IC_\delta |K| \right] \\ &\leq C_\delta |K| + C\delta. \end{aligned}$$

Therefore, for any $\tau > 0$, we first choose $\delta > 0$ fixed such that $C\delta < \tau/2$ and then $K \subset Q_T$ arbitrarily according to $C_\delta |K| < \tau/2$ to ultimately obtain

$$\int_K |f_i^\varepsilon(u^\varepsilon)| \, dx \, dt \leq \tau,$$

which is exactly the uniform integrability of $\{f_i^\varepsilon(u^\varepsilon)\}_{\varepsilon>0}$. From this and (3.15), we can apply Vitali’s lemma again to get $f_i^\varepsilon(u^\varepsilon) \rightarrow f_i(u)$ strongly in $L^1(Q_T)$ and, therefore,

$$\int_{Q_T} f_i^\varepsilon(u^\varepsilon) \psi \, dx \, dt \rightarrow \int_{Q_T} f_i(u) \psi \, dx \, dt. \tag{3.17}$$

Finally, using the boundedness of $\{u_i^\varepsilon\}_{\varepsilon>0}$ in $L^{m_i+1}(Q_T)$, of $\{\nabla(u_i^\varepsilon)^{\frac{m_i}{2}}\}_{\varepsilon>0}$ in $L^2(Q_T)$, and of $\{f_i^\varepsilon(u^\varepsilon)\}_{\varepsilon>0}$ in $L^1(Q_T)$, it follows that

$$\{\partial_t u_i^\varepsilon\}_{\varepsilon>0} \text{ is bounded in } L^1(Q_T) + L^{\frac{2(m_i+1)}{2m_i+1}}(0, T; (W^{1,2(m_i+1)}(\Omega))')$$

and, consequently,

$$\int_{Q_T} \partial_t u_i^\varepsilon \psi \, dx \, dt \rightarrow \int_{Q_T} \partial_t u_i \psi \, dx \, dt. \tag{3.18}$$

From (3.14), (3.17), and (3.18), we obtain that $u = (u_i)_{i=1,\dots,I}$ is a global weak solution to (1.1). It is remarked that a weak solution is not automatically a renormalised solution; therefore, the weak entropy–entropy dissipation relation (1.5) and the conservation laws (1.6) are not immediate consequences. On the one hand, we can use Definition 1.6 to show directly that (1.6) also holds for weak solutions. Indeed, by choosing $\psi = q_i$ as the test function for the equation of u_i and summing the resultants using $\sum_{i=1}^I q_i f_i(u) = 0$, we obtain the conservation laws (1.6).

On the other hand, for verifying the entropy law (1.5), we need to restrict ourselves to bounded weak solutions in order to make the subsequent arguments rigorous. We start with a sequence of functions $\{u_i^\gamma\}_{\gamma>0} \subset L^{2(m_i+1)}(0, T; W^{1,2(m_i+1)})$ such that $u_i^\gamma \rightarrow u_i$ a.e. in Q_T and $\nabla(u_i^\gamma)^{m_i/2} \rightharpoonup \nabla u_i^{m_i/2}$ in $L^2(Q_T)$ as $\gamma \rightarrow 0$. We may further assume that the functions $\{u_i^\gamma\}_{\gamma>0}$ are uniformly bounded, and satisfying $u_i^\gamma \geq u_i$ a.e. in Q_T . For $\delta > 0$, we test (1.16) with $\log(u_i^\gamma + \delta) + \mu_i$ and sum over $i = 1, \dots, I$ to get

$$\begin{aligned} \sum_{i=1}^I \int_{Q_T} \partial_t u_i (\log(u_i^\gamma + \delta) + \mu_i) \, dx \, dt + \sum_{i=1}^I \frac{d_i}{2} \int_{Q_T} u_i^{\frac{m_i}{2}} \nabla u_i^{\frac{m_i}{2}} \cdot \frac{\nabla(u_i^\gamma)}{u_i^\gamma + \delta} \, dx \, dt \\ = \sum_{i=1}^I \int_{Q_T} f_i(u) (\log(u_i^\gamma + \delta) + \mu_i) \, dx \, dt. \end{aligned}$$

Thanks to $u_i^{m_i/2} \nabla u_i^{m_i/2} \cdot \frac{\nabla(u_i^\gamma)}{u_i^\gamma + \delta} = \frac{2}{m_i} \frac{u_i^{m_i/2} (u_i^\gamma)^{1-m_i/2}}{u_i^\gamma + \delta} \nabla u_i^{m_i/2} \cdot \nabla(u_i^\gamma)^{m_i/2}$ and $\frac{u_i^{m_i/2} (u_i^\gamma)^{1-m_i/2}}{u_i^\gamma + \delta} \leq 1$, we conclude the convergence of the corresponding integral above taking the convergence properties of u_i^γ and ∇u_i^γ into account. As the dominated convergence theorem applies to the integral on the right-hand side, we can continuously extend the linear functional $\partial_t u_i$ to $\log(u_i + \delta)$, which is a priori not contained in the

domain of definition of $\partial_t u_i$. Therefore,

$$\begin{aligned} \sum_{i=1}^I \int_{Q_T} \partial_t u_i (\log(u_i + \delta) + \mu_i) \, dx \, dt + \sum_{i=1}^I \frac{d_i}{2} \int_{Q_T} u_i^{\frac{m_i}{2}} \nabla u_i^{\frac{m_i}{2}} \cdot \frac{\nabla u_i}{u_i + \delta} \, dx \, dt \\ = \sum_{i=1}^I \int_{Q_T} f_i(u) (\log(u_i + \delta) + \mu_i) \, dx \, dt. \end{aligned} \tag{3.19}$$

The convergence of the first two sums is now straightforward thanks to the regularity of weak solutions. To pass to the limit for the last sum, we rewrite

$$\begin{aligned} \sum_{i=1}^I \int_{Q_T} f_i(u) (\log(u_i + \delta) + \mu_i) \, dx \, dt \\ = \int_{Q_T} \left(\sum_{i=1}^I f_i(u + \delta) (\log(u_i + \delta) + \mu_i) - C \sum_{i=1}^I (1 + (u_i + \delta) \log(u_i + \delta)) \right) \, dx \, dt \\ + C \sum_{i=1}^I \int_{Q_T} (1 + (u_i + \delta) \log(u_i + \delta)) \, dx \, dt \\ - \sum_{i=1}^I \int_{Q_T} [f_i(u + \delta) - f_i(u)] (\log(u_i + \delta) + \mu_i) \, dx \, dt =: (I) + (II) - (III) \end{aligned} \tag{3.20}$$

where $u + \delta = (u_i + \delta)_{i=1, \dots, I}$ and $C > 0$ is the constant from (1.2). The convergence of (II) is straightforward:

$$\lim_{\delta \rightarrow 0} (II) = C \sum_{i=1}^I \int_{Q_T} (1 + u_i \log u_i) \, dx \, dt.$$

For (I), we use (F3) and Fatou’s lemma to obtain

$$\begin{aligned} \limsup_{\delta \rightarrow 0} \int_{Q_T} \left(\sum_{i=1}^I f_i(u + \delta) (\log(u_i + \delta) + \mu_i) - C \sum_{i=1}^I (1 + (u_i + \delta) \log(u_i + \delta)) \right) \, dx \, dt \\ \leq \int_{Q_T} \left(\sum_{i=1}^I f_i(u) (\log u_i + \mu_i) - C \sum_{i=1}^I (1 + u_i \log u_i) \right) \, dx \, dt. \end{aligned}$$

Finally, from (1.11) it follows that $|Df_i(z)| \leq C(1 + \sum_{j=1}^I |z_j|^{\rho-1})$ and, hence,

$$\limsup_{\delta \rightarrow 0} |(III)| \leq \limsup_{\delta \rightarrow 0} C \sum_{i=1}^I \int_{Q_T} \left(1 + \sum_{j=1}^I |u_j|^{\rho-1} \right) (\delta |\log(u_i + \delta)| + \delta |\mu_i|) \, dx \, dt = 0$$

thanks to (3.12) and the dominated convergence theorem. Inserting the estimates of (I), (II), and (III) into (3.20), then letting $\delta \rightarrow 0$ in (3.19), we conclude that any

bounded weak solution to (1.1) satisfies the weak entropy–entropy dissipation law (1.5).

Concerning bounded solutions, we first obtain from Lemma 3.6 that

$$\|u_i\|_{L^{m_i+1}(Q_T)} \leq C_T \quad \text{for all } i = 1, \dots, I.$$

Under assumptions (1.18) on the porous medium exponent, we can apply (Fellner et al. 2020, Lemma 2.2) to get

$$\|u_i\|_{L^\infty(Q_T)} \leq C_T \quad \text{for all } i = 1, \dots, I.$$

Note that the constant C_T grows at most polynomially in T . This implies the exponential convergence to equilibrium in $L^p(\Omega)$ for any $1 \leq p < \infty$ in (1.20). Indeed, from the exponential convergence in $L^1(\Omega)$, we can use the interpolation inequality to have

$$\begin{aligned} \|u_i(T) - u_{i,\infty}\|_{L^p(\Omega)} &\leq \|u_i(T) - u_{i,\infty}\|_{L^\infty(\Omega)}^{1-1/p} \|u_i(T) - u_{i,\infty}\|_{L^1(\Omega)}^{1/p} \\ &\leq C_T^{1-1/p} (Ce^{-\lambda T})^{1/p} \leq C_p e^{-\lambda_p T} \end{aligned}$$

for some $0 < \lambda_p < \lambda$, since C_T grows at most polynomially in T . □

3.4 Uniform Convergence of Approximate Solutions

We remark that all the involved constants in the following results *do not depend on ε* .

Lemma 3.7 *There exists a constant $L_0 > 0$ such that*

$$\sup_{t \geq 0} \max_{i=1, \dots, I} \|u_i^\varepsilon(t)\|_{L^1(\Omega)} \leq L_0. \tag{3.21}$$

Proof It follows from

$$\frac{d}{dt} \mathcal{E}[u^\varepsilon | u_\infty] = -\mathcal{D}[u^\varepsilon] \leq 0$$

that

$$\mathcal{E}[u^\varepsilon(t) | u_\infty] \leq \mathcal{E}[u_0^\varepsilon | u_\infty]. \tag{3.22}$$

Thanks to (1.22), $\sup_{\varepsilon > 0} \max_{i=1, \dots, I} \mathcal{E}[u_0^\varepsilon | u_\infty] < +\infty$. The L^1 bound (3.21) follows with the same arguments as in Lemma 3.1. □

Lemma 3.8 *There exists a constant $L_2 > 0$ such that*

$$\begin{aligned} & \sum_{i=1}^I \|\sqrt{u_i^\varepsilon} - \sqrt{u_i^\varepsilon}\|_{L^2(\Omega)}^2 + \sum_{r=1}^R \int_{\Omega} \frac{1}{1 + \varepsilon|f(u^\varepsilon)|} \left[\sqrt{\frac{u^\varepsilon}{u_\infty}}^{y_r} - \sqrt{\frac{u^\varepsilon}{u_\infty}}^{y'_r} \right]^2 dx \\ & \geq L_2 \sum_{r=1}^R \left[\frac{\overline{\sqrt{u^\varepsilon}}^{y_r}}{\sqrt{u_\infty}^{y_r}} - \frac{\overline{\sqrt{u^\varepsilon}}^{y'_r}}{\sqrt{u_\infty}^{y'_r}} \right]^2. \end{aligned}$$

Proof We introduce the shorthand notation

$$v_i^\varepsilon = \sqrt{u_i^\varepsilon}, \quad v^\varepsilon = (v_i^\varepsilon)_{i=1, \dots, I}, \quad v_{i, \infty} = \sqrt{u_{i, \infty}}, \quad v_\infty = (v_{i, \infty})_{i=1, \dots, I},$$

and $g(v^\varepsilon) = f((v^\varepsilon)^2) = f(u^\varepsilon)$. Moreover, for each $i \in \{1, \dots, I\}$, we set

$$v_i^\varepsilon(x) = \overline{v_i^\varepsilon} + \delta_i(x), \quad x \in \Omega.$$

The desired inequality in Lemma 3.8 becomes

$$\begin{aligned} & \sum_{i=1}^I \|\delta_i\|_{L^2(\Omega)}^2 + \sum_{r=1}^R \int_{\Omega} \frac{1}{1 + \varepsilon|g(v^\varepsilon)|} \left[\frac{(v^\varepsilon)^{y_r}}{v_\infty^{y_r}} - \frac{(v^\varepsilon)^{y'_r}}{v_\infty^{y'_r}} \right]^2 dx \\ & \geq L_2 \sum_{r=1}^R \left[\frac{\overline{v^\varepsilon}^{y_r}}{v_\infty^{y_r}} - \frac{\overline{v^\varepsilon}^{y'_r}}{v_\infty^{y'_r}} \right]^2. \end{aligned} \tag{3.23}$$

Using Lemma 3.7, we observe by Hölder’s inequality, noting that $|\Omega| = 1$, that

$$\overline{v_i^\varepsilon} = \sqrt{\overline{u_i^\varepsilon}} \leq \sqrt{u_i^\varepsilon} \leq \sqrt{L_0}$$

and

$$\|\delta_i\|_{L^2(\Omega)} = \sqrt{(v_i^\varepsilon)^2 - \overline{v_i^\varepsilon}^2} \leq \sqrt{u_i^\varepsilon} \leq \sqrt{L_0}.$$

Therefore, there exists a constant $K_7 > 0$ such that

$$\sum_{r=1}^R \left[\frac{\overline{\sqrt{u^\varepsilon}}^{y_r}}{\sqrt{u_\infty}^{y_r}} - \frac{\overline{\sqrt{u^\varepsilon}}^{y'_r}}{\sqrt{u_\infty}^{y'_r}} \right]^2 \leq K_7. \tag{3.24}$$

We decompose Ω into $\Omega = \Omega_1 \cup \Omega_2$ where

$$\Omega_1 = \{x \in \Omega : |\delta_i(x)| \leq 1 \quad \forall i = 1, \dots, I\} \quad \text{and} \quad \Omega_2 = \Omega \setminus \Omega_1.$$

Note that on Ω_1 we have

$$0 \leq v_i^\varepsilon(x) = \bar{v}_i^\varepsilon + \delta_i \leq \sqrt{L_0} + 1 \quad \text{for all } i = 1, \dots, I.$$

Therefore, there exists a constant $K_8 > 0$ such that

$$\frac{1}{1 + \varepsilon|g(v^\varepsilon(x))|} \geq K_8 \quad \text{for all } x \in \Omega_1. \tag{3.25}$$

By using Taylor’s expansion, we have for all $x \in \Omega_1$,

$$\begin{aligned} \left[\frac{(v^\varepsilon(x))^{y_r}}{v_\infty^{y_r}} - \frac{(v^\varepsilon(x))^{y'_r}}{v_\infty^{y'_r}} \right]^2 &= \left[\frac{1}{v_\infty^{y_r}} \prod_{i=1}^I [\bar{v}_i^\varepsilon + \delta_i(x)]^{y_{r,i}} - \frac{1}{v_\infty^{y'_r}} \prod_{i=1}^I [\bar{v}_i^\varepsilon + \delta_i(x)]^{y'_{r,i}} \right]^2 \\ &= \left[\frac{1}{v_\infty^{y_r}} \prod_{i=1}^I (\bar{v}_i^{y_{r,i}} + R_1 \delta_i(x)) - \frac{1}{v_\infty^{y'_r}} \prod_{i=1}^I (\bar{v}_i^{y'_{r,i}} + R_2 \delta_i(x)) \right]^2 \\ &\geq \frac{1}{2} \left[\frac{\bar{v}^{\varepsilon,y_r}}{v_\infty^{y_r}} - \frac{\bar{v}^{\varepsilon,y'_r}}{v_\infty^{y'_r}} \right]^2 - K_9 \sum_{i=1}^I |\delta_i(x)|^2 \end{aligned} \tag{3.26}$$

for some $K_9 > 0$. From (3.25) and (3.26), we can estimate the second sum on the left-hand side of (3.23) as

$$\begin{aligned} &\sum_{r=1}^R \int_\Omega \frac{1}{1 + \varepsilon|g(v^\varepsilon)|} \left[\frac{(v^\varepsilon)^{y_r}}{v_\infty^{y_r}} - \frac{(v^\varepsilon)^{y'_r}}{v_\infty^{y'_r}} \right]^2 dx \\ &\geq \sum_{r=1}^R \int_{\Omega_1} \frac{1}{1 + \varepsilon|g(v^\varepsilon)|} \left[\frac{(v^\varepsilon)^{y_r}}{v_\infty^{y_r}} - \frac{(v^\varepsilon)^{y'_r}}{v_\infty^{y'_r}} \right]^2 dx \\ &\geq K_8 \sum_{r=1}^R \int_{\Omega_1} \left(\frac{1}{2} \left[\frac{\bar{v}^{\varepsilon,y_r}}{v_\infty^{y_r}} - \frac{\bar{v}^{\varepsilon,y'_r}}{v_\infty^{y'_r}} \right]^2 - K_9 \sum_{i=1}^I |\delta_i(x)|^2 \right) dx \\ &\geq \frac{K_8}{2} |\Omega_1| \sum_{r=1}^R \left[\frac{\bar{v}^{\varepsilon,y_r}}{v_\infty^{y_r}} - \frac{\bar{v}^{\varepsilon,y'_r}}{v_\infty^{y'_r}} \right]^2 - K_8 K_9 R \sum_{i=1}^I \|\delta_i\|_{L^2(\Omega)}^2. \end{aligned} \tag{3.27}$$

On the other hand, the first sum on the left-hand side of (3.23) is estimated as

$$\begin{aligned} \sum_{i=1}^I \|\delta_i\|_{L^2(\Omega)}^2 &\geq \frac{1}{2} \sum_{i=1}^I \|\delta_i\|_{L^2(\Omega)}^2 + \frac{1}{2} \int_{\Omega_2} \sum_{i=1}^I |\delta_i(x)|^2 dx \\ &\geq \frac{1}{2} \sum_{i=1}^I \|\delta_i\|_{L^2(\Omega)}^2 + \frac{1}{2} |\Omega_2| \\ &\geq \frac{1}{2} \sum_{i=1}^I \|\delta_i\|_{L^2(\Omega)}^2 + \frac{1}{2K_7} |\Omega_2| \sum_{r=1}^R \left[\frac{\bar{v}^{\varepsilon,y_r}}{v_\infty^{y_r}} - \frac{\bar{v}^{\varepsilon,y'_r}}{v_\infty^{y'_r}} \right]^2. \end{aligned} \tag{3.28}$$

By combining (3.27) and (3.28), we have for any $\theta \in (0, 1]$,

$$\begin{aligned} \text{LHS of (3.23)} &\geq \left(\theta \frac{K_8}{2} |\Omega_1| + \frac{1}{2K_7} |\Omega_2| \right) \sum_{r=1}^R \left[\frac{\overline{v^{\varepsilon y_r}}}{\overline{v_\infty^{y_r}}} - \frac{\overline{v^{\varepsilon y_r'}}}{\overline{v_\infty^{y_r'}}} \right]^2 \\ &\quad + \left(\frac{1}{2} - \theta K_8 K_9 R \right) \sum_{i=1}^I \|\delta_i\|_{L^2(\Omega)}^2. \end{aligned} \tag{3.29}$$

Thus, by choosing $\theta = \min \left\{ 1; \frac{1}{2K_8 K_9 R} \right\}$, we finally obtain the desired inequality (3.23) with $L_2 = \min \left\{ \frac{\theta K_8}{2}; \frac{1}{2K_7} \right\}$. □

Lemma 3.9 *There exists a constant $L_3 > 0$ such that*

$$\sum_{i=1}^I \|\sqrt{u_i^\varepsilon} - \sqrt{\overline{u_i^\varepsilon}}\|_{L^2(\Omega)}^2 + \sum_{r=1}^R \left[\frac{\sqrt{\overline{u^\varepsilon} y_r}}{\sqrt{\overline{u_\infty} y_r}} - \frac{\sqrt{\overline{u^\varepsilon} y_r'}}{\sqrt{\overline{u_\infty} y_r'}} \right]^2 \geq L_3 \sum_{r=1}^R \left[\sqrt{\frac{\overline{u^\varepsilon}}{u_\infty}} - \sqrt{\frac{\overline{u^\varepsilon} y_r'}}{\overline{u_\infty}} \right]^2.$$

Proof The proof of this lemma follows by the same arguments as in (Fellner and Tang 2018, Lemma 2.7), so we omit it here. □

We are now ready to prove Theorem 1.9.

Proof of Theorem 1.9 We proceed similarly to the proof of Lemma 3.5. If $\frac{(d-2)_+}{d} < m_i < 1$, we have, thanks to Lemma 3.4 and bound (3.21),

$$\int_{\Omega} (u_i^\varepsilon)^{m_i-2} |\nabla u_i^\varepsilon|^2 dx \geq C \overline{u_i^\varepsilon}^{m_i-1} \int_{\Omega} u_i^\varepsilon \log \frac{u_i^\varepsilon}{\overline{u_i^\varepsilon}} dx \geq C L_0^{m_i-1} \int_{\Omega} u_i^\varepsilon \log \frac{u_i^\varepsilon}{\overline{u_i^\varepsilon}} dx$$

with a constant $C = C(\Omega, m_i) > 0$. If $m_i \geq 1$, we apply Lemma 3.3 to get

$$\int_{\Omega} (u_i^\varepsilon)^{m_i-2} |\nabla u_i^\varepsilon|^2 dx \geq C(\Omega, m_i) \left(\int_{\Omega} u_i^\varepsilon \log \frac{u_i^\varepsilon}{\overline{u_i^\varepsilon}} dx \right)^{m_i}.$$

Therefore, by using $\Psi(x, y) = x \log(x/y) - x + y \geq (\sqrt{x} - \sqrt{y})^2$ and noting that

$$\int_{\Omega} u_i^\varepsilon \log \frac{u_i^\varepsilon}{\overline{u_i^\varepsilon}} dx = \int_{\Omega} \left(u_i^\varepsilon \log \frac{u_i^\varepsilon}{\overline{u_i^\varepsilon}} - u_i^\varepsilon + \overline{u_i^\varepsilon} \right) dx \geq \|\sqrt{u_i^\varepsilon} - \sqrt{\overline{u_i^\varepsilon}}\|_{L^2(\Omega)}^2,$$

there exists a constant $K_{10} > 0$ such that

$$\begin{aligned} \mathcal{D}[u^\varepsilon] \geq & K_{10} \left[\sum_{i=1}^I \left(\|\sqrt{u_i^\varepsilon} - \sqrt{u_i^\varepsilon}\|_{L^2(\Omega)}^2 \right)^{\max\{1, m_i\}} \right. \\ & \left. + \sum_{r=1}^R k_r u_\infty^{y_r} \int_\Omega \frac{1}{1 + \varepsilon |f(u^\varepsilon)|} \left(\sqrt{\frac{u^\varepsilon}{u_\infty}} - \sqrt{\frac{u^\varepsilon}{u_\infty}} \right)^2 dx \right]. \end{aligned} \tag{3.30}$$

We define

$$\widehat{D}_i := \|\sqrt{u_i^\varepsilon} - \sqrt{u_i^\varepsilon}\|_{L^2(\Omega)}^2, \quad \widehat{F} := \sum_{r=1}^R k_r u_\infty^{y_r} \int_\Omega \frac{1}{1 + \varepsilon |f(u^\varepsilon)|} \left[\sqrt{\frac{u^\varepsilon}{u_\infty}} - \sqrt{\frac{u^\varepsilon}{u_\infty}} \right]^2 dx,$$

and we consider the following two cases.

Case 1. If $\widehat{F} \geq 1$ or if there exists some $i \in \{1, \dots, I\}$ such that $\widehat{D}_i \geq 1$, we estimate from (3.30) that

$$\mathcal{D}[u^\varepsilon] \geq K_{10} \geq K_{10} \frac{1}{\mathcal{E}[u_0|u_\infty]} \mathcal{E}[u^\varepsilon|u_\infty]. \tag{3.31}$$

Case 2. If $\widehat{F} \leq 1$ and $\widehat{D}_i \leq 1$ for all $i = 1, \dots, I$, then with $\alpha = \max_{i=1, \dots, I} \{1, m_i\}$, we have from (3.30) that

$$\mathcal{D}[u^\varepsilon] \geq K_{10} \left[\sum_{i=1}^I \widehat{D}_i^\alpha + \widehat{F}^\alpha \right] \geq K_{11} \left[\sum_{i=1}^I \widehat{D}_i + \widehat{F} \right]^\alpha \tag{3.32}$$

for some $K_{11} = K_{11}(K_{10}, \alpha)$. Thanks to Lemmas 3.8 and 3.9, we have

$$\begin{aligned} & \sum_{i=1}^I \widehat{D}_i + \widehat{F} \\ & \geq \min_{r=1, \dots, R} \{1, k_r u_\infty^{y_r}\} \left[\frac{1}{2} \sum_{i=1}^I \|\sqrt{u_i^\varepsilon} - \sqrt{u_i^\varepsilon}\|_{L^2(\Omega)}^2 + \frac{L_2}{2} \sum_{r=1}^R \left(\sqrt{\frac{u^\varepsilon}{u_\infty}} - \sqrt{\frac{u^\varepsilon}{u_\infty}} \right)^2 \right] \\ & \geq K_{12} \sum_{r=1}^R \left[\sqrt{\frac{u^\varepsilon}{u_\infty}} - \sqrt{\frac{u^\varepsilon}{u_\infty}} \right]^2 \end{aligned} \tag{3.33}$$

where

$$K_{12} = \frac{1}{2} \min_{r=1, \dots, R} \{L_2, 1, k_r u_\infty^{y_r}\} L_3.$$

From here, we can use (Fellner and Tang 2018, Lemma 2.5) to estimate

$$\mathcal{E}[u^\varepsilon | u_\infty] \leq K_{13} \sum_{i=1}^I \left(\sqrt{\frac{u_i^\varepsilon}{u_{i,\infty}}} - 1 \right)^2,$$

and (Fellner and Tang 2018, Eq. (11)) to get

$$\sum_{r=1}^R \left[\sqrt{\frac{u^\varepsilon}{u_\infty}}^{y_r} - \sqrt{\frac{u^\varepsilon}{u_\infty}}^{y'_r} \right]^2 \geq K_{14} \sum_{i=1}^I \left(\sqrt{\frac{u_i^\varepsilon}{u_{i,\infty}}} - 1 \right)^2 \geq \frac{K_{14}}{K_{13}} \mathcal{E}[u^\varepsilon | u_\infty]. \tag{3.34}$$

It follows from (3.32), (3.33), and (3.34) that

$$\mathcal{D}[u^\varepsilon] \geq K_{11} \left(\frac{K_{12}K_{14}}{K_{13}} \right)^\alpha \mathcal{E}[u^\varepsilon | u_\infty]^\alpha. \tag{3.35}$$

Owing to (3.31) and (3.35), there exists some $L_4 > 0$ such that

$$\mathcal{D}[u^\varepsilon] \geq L_4 \mathcal{E}[u^\varepsilon | u_\infty]^{\max\{1, \alpha\}}.$$

The rest of this proof follows exactly from that of Theorem 1.5, which helps us to eventually obtain the exponential convergence to equilibrium

$$\sum_{i=1}^I \|u_i^\varepsilon(t) - u_{i,\infty}\|_{L^1(\Omega)}^2 \leq C e^{-\lambda t}$$

where $C, \lambda > 0$ are independent of ε . □

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Appendix A: Proof of the Global Existence of Approximate Solutions

In this appendix, we provide a proof for the existence of a global solution to the approximate system (2.1). While a similar proof, with Dirichlet instead of Neumann boundary conditions, can be found in (Laamri and Pierre 2017, Proof of Lemma 2.3), we mainly follow a standard Galerkin approach as presented in many textbooks.

For $0 < \delta \leq 1$ and $1 \leq i \leq I$, we consider the regularised system

$$\partial_t u_i^{\varepsilon, \delta} - d_i \nabla \cdot \left[\left(\frac{m_i |u_i^{\varepsilon, \delta}|^{m_i-1}}{1 + \delta m_i |u_i^{\varepsilon, \delta}|^{m_i-1}} + \delta \right) \nabla u_i^{\varepsilon, \delta} \right] = f_i^\varepsilon(u^{\varepsilon, \delta}), \quad v \cdot \nabla u_i^{\varepsilon, \delta} = 0, \tag{A.1}$$

with initial data $u_i^{\varepsilon, \delta}(x, 0) = u_i^{\varepsilon, 0}(x) + \delta$, where we extend f_i^ε to a locally Lipschitz continuous function on \mathbb{R}^I by setting $f_i^\varepsilon(v) := f_i^\varepsilon(\max\{v_1, 0\}, \dots, \max\{v_I, 0\})$. For the sake of readability, we subsequently write $w_i := u_i^{\varepsilon, \delta}$ and $H^\delta(w_i) := \frac{m_i |w_i|^{m_i-1}}{1 + \delta m_i |w_i|^{m_i-1}} + \delta$.

We employ the ansatz $w_i^k(x, t) = \sum_{j=1}^k \xi_i^{j,k}(t) e_j(x)$ with $k \in \mathbb{N}$, scalar coefficient functions $\xi_i^{j,k}(t)$, and the Schauder basis $e_j \in H^1(\Omega)$ of $L^2(\Omega)$ given by the orthonormal eigenfunctions of the Laplace operator satisfying $v \cdot \nabla e_j = 0$ on $\partial\Omega$. Along with the initial condition $\xi_i^{j,k}(0) = \int_\Omega (w_0)_i e_j \, dx$, the functions $\xi_i^{j,k}$, $1 \leq j \leq k$, are solutions to the set of equations

$$\int_\Omega \partial_t w_i^k e_j \, dx = -d_i \int_\Omega H^\delta(w_i^k) \nabla w_i^k \cdot \nabla e_j \, dx + \int_\Omega f_i^\varepsilon(w_i^k) e_j \, dx \tag{A.2}$$

for $1 \leq j \leq k$, which can be recast into an ODE system for the time-dependent coefficient vector $\hat{\xi}_i^k(t) := (\xi_i^{1,k}(t), \dots, \xi_i^{k,k}(t))$. As the right-hand side continuously depends on $\hat{\xi}_i^k(t)$ via the bounded and continuous functions H^δ and f_i^ε , we know that a solution $\hat{\xi}_i^k(t) \in C^1([0, \tau], \mathbb{R}^I)$ exists for some $\tau > 0$. The energy estimate below, which is essentially obtained by multiplying (A.2) with $\xi_i^{j,k}(t)$ and summing over j , will indeed show that the solution exists for all times $t \geq 0$. Integrating (A.2) over $[0, t] \subset [0, \tau)$, we have

$$\|w_i^k(t)\|_{L^2(\Omega)}^2 + \delta d_i \int_0^t \|\nabla w_i^k(s)\|_{L^2(\Omega)}^2 \, ds \leq C \left(\|w_i^k(0)\|_{L^2(\Omega)}^2 + \int_0^t \int_\Omega \varepsilon^{-1} |w_i^k| \, dx \, ds \right),$$

where $C > 0$ is independent of k by the uniform bounds $H^\delta(w_i^k) \geq \delta$ and $f_i^\varepsilon(w_i^k) \leq \varepsilon^{-1}$. In order to control the $H^1(\Omega)$ norm of w_i^k , we first observe that e_1 is constant in space. Hence, (A.2) leads for all $t \in (0, \tau)$ to

$$\left| \int_\Omega w_i^k(x, t) \, dx \right| = \left| \int_\Omega w_i^k(x, 0) \, dx + \int_0^t \int_\Omega f_i^\varepsilon(w_i^k) \, dx \, ds \right| \leq \|w_i^k(0)\|_{L^1(\Omega)} + \frac{|\Omega|t}{\varepsilon}.$$

Poincaré’s inequality now yields

$$\|w_i^k(t)\|_{H^1(\Omega)}^2 \leq C \left(\|\nabla w_i^k(t)\|_{L^2(\Omega)}^2 + \|w_i^k(0)\|_{L^2(\Omega)}^2 + \frac{|\Omega|^2 \tau^2}{\varepsilon^2} \right).$$

As a result, we arrive at

$$\|w_i^k(t)\|_{L^2(\Omega)}^2 + \int_0^t \|w_i^k(s)\|_{H^1(\Omega)}^2 ds \leq C \left(1 + \|w_i(0)\|_{L^2(\Omega)}^2 \right), \tag{A.3}$$

where $C > 0$ is independent of k , and which shows that the solution is bounded on $[0, \tau)$ and, thus, existing for all $t \geq 0$. Note that we applied the elementary estimate $\|w_i^k(0)\|_{L^2(\Omega)} \leq \|w_i(0)\|_{L^2(\Omega)}$ in the last step.

Let $P_k : L^2(\Omega) \rightarrow L^2(\Omega)$ be the orthogonal projection in $L^2(\Omega)$ onto $\text{span}\{e_1, \dots, e_k\}$, and let $\phi \in L^2(0, T; H^1(\Omega))$ for some arbitrary but fixed $T > 0$. We recall the uniform bounds $H^\delta(w_i^k) \leq \delta^{-1} + \delta$ and $f_i^\varepsilon(w_i^k) \leq \varepsilon^{-1}$ leading to

$$\begin{aligned} \int_{Q_T} \partial_t w_i^k \phi \, dx \, dt &= \int_{Q_T} \partial_t w_i^k P_k \phi \, dx \, dt \\ &= -d_i \int_{Q_T} H^\delta(w_i^k) \nabla w_i^k \cdot \nabla (P_k \phi) \, dx \, dt + \int_{Q_T} f_i^\varepsilon(w_i^k) P_k \phi \, dx \, dt \\ &\leq C \int_0^T \|\nabla w_i^k(t)\|_{L^2(\Omega)} \|\nabla (P_k \phi)(t)\|_{L^2(\Omega)} \, dt + C \int_0^T \|(P_k \phi)(t)\|_{L^2(\Omega)} \, dt \\ &\leq C(1 + \|w_i^k\|_{L^2(0, T; H^1(\Omega))}) \|\phi\|_{L^2(0, T; H^1(\Omega))} \end{aligned}$$

with a constant $C > 0$ independent of k . We stress that the last estimate also employs the bound $\|(P_k \phi)(t)\|_{H^1(\Omega)} \leq \|\phi(t)\|_{H^1(\Omega)}$ for a.e. $t \in [0, T]$, which provides a uniform-in- k bound for $\partial_t w_i^k \in L^2(0, T; (H^1(\Omega))')$ together with (A.3).

We may choose a (not relabelled) subsequence w_i^k being weakly converging for $k \rightarrow \infty$ to some w_i in $L^2(0, T; H^1(\Omega))$ and in $H^1(0, T; (H^1(\Omega))')$. Strong convergence $w_i^k \rightarrow w_i$ in $L^2(0, T; L^2(\Omega)) \cong L^2(Q_T)$ now follows by means of the Aubin–Lions lemma. Choosing another subsequence, we obtain for all $i = 1, \dots, I$, $w_i^k \rightarrow w_i$ and, hence, $H^\delta(w_i^k) \rightarrow H^\delta(w_i)$ and $f_i^\varepsilon(w_i^k) \rightarrow f_i^\varepsilon(w_i)$ a.e. in Ω due to the continuity of H^δ and f_i^ε . The boundedness of H^δ and f_i^ε further guarantees $H^\delta(w_i^k) \rightarrow H^\delta(w_i)$ and $f_i^\varepsilon(w_i^k) \rightarrow f_i^\varepsilon(w_i)$ strongly in $L^2(Q_T)$. This allows us to pass to the limit $k \rightarrow \infty$ in an integrated version of (A.2) resulting in

$$\begin{aligned} \int_{\Omega} w_i \phi \, dx \Big|_0^T - \int_{Q_T} w_i \partial_t \phi \, dx \, dt + d_i \int_{Q_T} H^\delta(w_i) \nabla w_i \cdot \nabla \phi \, dx \, dt \\ = \int_{Q_T} f_i^\varepsilon(w_i) \phi \, dx \, dt \end{aligned}$$

for all $\phi \in C^\infty(\overline{Q_T})$. By a density argument, (A.1) holds as an identity in $L^2(0, T; (H^1(\Omega))')$.

Thanks to the quasi-positivity assumption (F2), the solution w_i is also non-negative. For $p > 1$, we have

$$\begin{aligned} & \partial_t \|w_i\|_{L^p(\Omega)}^p + d_i p(p-1) \int_{\Omega} H^\delta(w_i) w_i^{p-2} |\nabla w_i|^2 dx \\ & = p \int_{\Omega} w_i^{p-1} f_i^\varepsilon(w) dx \leq p \|f_i^\varepsilon(w)\|_{L^p(\Omega)} \left(\|w_i(t)\|_{L^p(\Omega)}^p \right)^{1-\frac{1}{p}}. \end{aligned} \tag{A.4}$$

Applying the Gronwall lemma $y' \leq \alpha(t)y^{1-1/p} \Rightarrow y(t) \leq \left(y(0)^{1/p} + \frac{1}{p} \int_0^t \alpha(s) ds \right)^p$ for $y(t) = \|w_i(t)\|_{L^p(\Omega)}^p$ entails

$$\begin{aligned} \|w_i(t)\|_{L^p(\Omega)}^p & \leq \left(\|w_i(0)\|_{L^p(\Omega)} + \int_0^t \|f_i^\varepsilon(w(s))\|_{L^p(\Omega)} ds \right)^p \\ & \leq \left(\|w_i(0)\|_{L^p(\Omega)} + \|f_i^\varepsilon(w)\|_{L^p(Q_T)} T^{\frac{p-1}{p}} \right)^p. \end{aligned}$$

Taking the root of order p on both sides and letting $p \rightarrow \infty$, this leads to

$$\sup_{t \in (0, T)} \|w_i(t)\|_{L^\infty(\Omega)} \leq \|w_i(0)\|_{L^\infty(\Omega)} + T \|f_i^\varepsilon(w)\|_{L^\infty(Q_T)},$$

which means

$$\|w_i\|_{L^\infty(Q_T)} \leq C_{\varepsilon, T} \quad \text{for all } i = 1, \dots, I, \tag{A.5}$$

where $C_{\varepsilon, T}$ is independent of $\delta > 0$. Therefore,

$$H^\delta(w_i) = \frac{m_i w_i^{m_i-1}}{1 + \delta m_i w_i^{m_i-1}} + \delta \geq \frac{m_i w_i^{m_i-1}}{1 + m_i C_{\varepsilon, T}^{m_i-1}} + \delta.$$

By integrating (A.4) on $(0, T)$, and using (A.5), we get

$$\begin{aligned} & m_i \int_{Q_T} w_i^{m_i-1} w_i^{p-2} |\nabla w_i|^2 dx dt + \delta \int_{Q_T} w_i^{p-2} |\nabla w_i|^2 dx dt \\ & \leq \frac{1}{d_i(p-1)} \left(\|w_i(0)\|_{L^p(\Omega)}^p + \int_{Q_T} f_i^\varepsilon(w) w_i^{p-1} dx dt \right) \\ & \leq \frac{1}{d_i(p-1)} \left(\|w_i(0)\|_{L^p(\Omega)}^p + |\Omega| T \|f_i^\varepsilon(w)\|_{L^\infty(Q_T)} C_{\varepsilon, T}^{p-1} \right), \end{aligned} \tag{A.6}$$

which implies, in particular,

$$\left\| \nabla \left(w_i^{\frac{m_i+p-1}{2}} \right) \right\|_{L^2(Q_T)} \leq C_{\varepsilon, T} \tag{A.7}$$

for all $p > 1$. By testing equation (A.1) with $\phi \in L^2(0, T; H^1(\Omega))$ and keeping (A.5) in mind, we see that $\{\partial_t w_i\}$ is bounded in $L^2(0, T; (H^1(\Omega))')$ uniformly in $\delta > 0$. Hence, from this, (A.5), and (A.7), we can apply a nonlinear Aubin–Lions lemma, see e.g. (Moussa 2016, Theorem 1), to ensure the existence of a subsequence (not relabelled) such that $w_i = u_i^{\varepsilon, \delta} \rightarrow u_i^\varepsilon$ strongly in $L^2(Q_T)$ as $\delta \rightarrow 0$. From here onwards, we will write $u_i^{\varepsilon, \delta}$ instead of w_i . Thanks to the L^∞ bound (A.5), this convergence in fact holds in $L^p(Q_T)$ for any $1 \leq p < \infty$. It remains to pass to the limit $\delta \rightarrow 0$ in the weak formulation of (A.1) with $\psi \in L^2(0, T; H^1(\Omega))$,

$$\begin{aligned} & \int_{t_0}^{t_1} \langle \partial_t u_i^{\varepsilon, \delta}, \psi \rangle_{(H^1(\Omega))', H^1(\Omega)} dt + d_i \int_{t_0}^{t_1} \int_{\Omega} \frac{m_i (u_i^{\varepsilon, \delta})^{m_i-1}}{1 + \delta m_i (u_i^{\varepsilon, \delta})^{m_i-1}} \nabla u_i^{\varepsilon, \delta} \cdot \nabla \psi dx dt \\ & + d_i \delta \int_{t_0}^{t_1} \int_{\Omega} \nabla u_i^{\varepsilon, \delta} \cdot \nabla \psi dx dt = \int_{t_0}^{t_1} \int_{\Omega} f_i^\varepsilon(u_i^{\varepsilon, \delta}) \psi dx dt. \end{aligned} \tag{A.8}$$

The convergence of the first term on the left-hand side and the last term on the right-hand side of (A.8) is immediate. From

$$\nabla u_i^{\varepsilon, \delta} = \frac{2}{m_i + p - 1} (u_i^{\varepsilon, \delta})^{\frac{3-m_i-p}{2}} \nabla (u_i^{\varepsilon, \delta})^{\frac{m_i+p-1}{2}},$$

we can choose $1 < p < 3 - m_i$ (since $m_i < 2$) to get

$$\|\nabla u_i^{\varepsilon, \delta}\|_{L^2(Q_T)} \leq C_{\varepsilon, T} \tag{A.9}$$

thanks to the uniform bounds (A.5) and (A.7). For the third term on the left-hand side of (A.8), we estimate

$$\left| \delta \int_{t_0}^{t_1} \int_{\Omega} \nabla u_i^{\varepsilon, \delta} \cdot \nabla \psi dx dt \right| \leq \delta \|\nabla u_i^{\varepsilon, \delta}\|_{L^2(Q_T)} \|\nabla \psi\|_{L^2(Q_T)} \leq C_{\varepsilon, T} \delta \|\nabla \psi\|_{L^2(Q_T)},$$

and, therefore, it converges to zero as $\delta \rightarrow 0$. Finally, the convergence

$$\int_{t_0}^{t_1} \int_{\Omega} \frac{m_i (u_i^{\varepsilon, \delta})^{m_i-1}}{1 + \delta m_i (u_i^{\varepsilon, \delta})^{m_i-1}} \nabla u_i^{\varepsilon, \delta} \cdot \nabla \psi dx dt \xrightarrow{\delta \rightarrow 0} \int_{t_0}^{t_1} \int_{\Omega} m_i (u_i^\varepsilon)^{m_i-1} \nabla u_i^\varepsilon \cdot \nabla \psi dx dt$$

follows from $\frac{m_i (u_i^{\varepsilon, \delta})^{m_i-1}}{1 + \delta m_i (u_i^{\varepsilon, \delta})^{m_i-1}} \xrightarrow{\delta \rightarrow 0} m_i (u_i^\varepsilon)^{m_i-1}$ a.e. in Q_T , (A.5), and (A.9).

Appendix B: Proof of a Uniform Bound on Approximate Solutions

We provide a proof of Lemma 3.6. Define the new variables

$$w_i^\varepsilon = u_i^\varepsilon \log u_i^\varepsilon + (\mu_i - 1)u_i^\varepsilon + e^{-\mu_i} \geq 0$$

and

$$z_i^\varepsilon = (\log u_i^\varepsilon + \mu_i)(u_i^\varepsilon)^{m_i} - \frac{(u_i^\varepsilon)^{m_i} - e^{-\mu_i m_i}}{m_i} \geq 0,$$

where the non-negativity follows from elementary calculus. Direct computations give

$$\partial_t w_i^\varepsilon - d_i \Delta z_i^\varepsilon = (\log u_i^\varepsilon + \mu_i) f_i^\varepsilon(u^\varepsilon) - d_i m_i (u_i^\varepsilon)^{m_i-2} |\nabla u_i^\varepsilon|^2$$

leading to

$$\partial_t \left(\sum_{i=1}^I w_i^\varepsilon \right) - \Delta \left(\sum_{i=1}^I d_i z_i^\varepsilon \right) = \sum_{i=1}^I (\log u_i^\varepsilon + \mu_i) f_i^\varepsilon(u^\varepsilon) - \sum_{i=1}^I d_i m_i (u_i^\varepsilon)^{m_i-2} |\nabla u_i^\varepsilon|^2.$$

Thanks to (1.2) and an elementary estimate, there exist constants $A, \beta > 0$ such that

$$\sum_{i=1}^I (\log u_i^\varepsilon + \mu_i) f_i^\varepsilon(u^\varepsilon) \leq C \sum_{i=1}^I (1 + u_i^\varepsilon \log u_i^\varepsilon) \leq A + \beta \sum_{i=1}^I w_i^\varepsilon.$$

Defining the non-negative function

$$\tilde{F} = A + \beta \sum_{i=1}^I w_i^\varepsilon - \sum_{i=1}^I (\log u_i^\varepsilon + \mu_i) f_i^\varepsilon(u^\varepsilon) + \sum_{i=1}^I d_i m_i (u_i^\varepsilon)^{m_i-2} |\nabla u_i^\varepsilon|^2 \geq 0,$$

we infer that

$$\partial_t \left(\sum_{i=1}^I w_i^\varepsilon \right) - \Delta \left(\sum_{i=1}^I d_i z_i^\varepsilon \right) = A + \beta \sum_{i=1}^I w_i^\varepsilon - \tilde{F},$$

while Gronwall’s inequality gives

$$\sum_{i=1}^I \int_{Q_T} w_i^\varepsilon \, dx \, dt + \int_{Q_T} \tilde{F} \, dx \, dt \leq C(T, \|w_0^\varepsilon\|_{L^1(\Omega)}). \tag{B.1}$$

By introducing $v(x, t) = e^{-\beta t} \sum_{i=1}^I w_i^\varepsilon$, we obtain

$$\partial_t v - \Delta \left(e^{-\beta t} \sum_{i=1}^I d_i z_i^\varepsilon \right) = A e^{-\beta t} - e^{-\beta t} \tilde{F}.$$

Therefore, by applying (Laamri and Perthame 2020, Lemma 1) and (B.1), we have

$$\int_{Q_T} e^{-2\beta t} \left(\sum_{i=1}^I w_i^\varepsilon \right) \left(\sum_{i=1}^I d_i z_i^\varepsilon \right) dx dt \leq \frac{1}{2} \left\| \sum_{i=1}^I w_{i,0}^\varepsilon \right\|_{H^{-1}(\Omega)}^2 + C \int_0^T \left(e^{-\beta t} \sum_{i=1}^I \int_{\Omega} d_i z_i^\varepsilon dx + \int_{\Omega} \tilde{F} dx \right) \left(A e^{-\beta t} + \sum_{i=1}^I \int_{\Omega} w_{i,0}^\varepsilon \right) dt.$$

From the definition of w_i^ε and z_i^ε , it is easy to see that for any $\delta > 0$, there exists some $C_\delta > 0$ such that

$$\sum_{i=1}^I d_i z_i^\varepsilon \leq \delta \left(\sum_{i=1}^I w_i^\varepsilon \right) \left(\sum_{i=1}^I d_i z_i^\varepsilon \right) + C_\delta$$

and, hence,

$$\int_{Q_T} \left(\sum_{i=1}^I w_i^\varepsilon \right) \left(\sum_{i=1}^I d_i z_i^\varepsilon \right) dx dt \leq C(T, \|w_0^\varepsilon\|_{L^1(\Omega) \cap H^{-1}(\Omega)}).$$

The desired estimate (3.12) is now an immediate consequence.

References

- Caputo, M.C., Goudon, T., Vasseur, A.F.: Solutions of the 4-species quadratic reaction-diffusion system are bounded and C^∞ -smooth in any space dimension. *Anal. PDE* **12**(7), 1773–1804 (2019)
- Chen, X., Daus, E.S., Jüngel, A.: Global existence analysis of cross-diffusion population systems for multiple species. *Arch. Ration. Mech. Anal.* **227**(2), 715–747 (2018)
- Chen, X., Jüngel, A.: Global renormalized solutions to reaction-cross-diffusion systems with self-diffusion. *J. Differential Equ.* **267**(10), 5901–5937 (2019)
- Craciun, G.: Toric differential inclusions and a proof of the global attractor conjecture. Preprint, 2015. [arXiv:1501.02860](https://arxiv.org/abs/1501.02860)
- Dal Maso, G., Murat, F., Orsina, L., Prignet, A.: Renormalized solutions of elliptic equations with general measure data. *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* **28**(4), 741–808 (1999)
- Desvillettes, L., Fellner, K., Pierre, M., Vovelle, J.: Global existence for quadratic systems of reaction-diffusion. *Adv. Nonlinear Stud.* **7**(3), 491–511 (2007)
- Desvillettes, L., Fellner, K., Tang, B.Q.: Trend to equilibrium for reaction-diffusion systems arising from complex balanced chemical reaction networks. *SIAM J. Math. Anal.* **49**(4), 2666–2709 (2017)
- DiPerna, R.J., Lions, P.-L.: On the Cauchy problem for Boltzmann equations: global existence and weak stability. *Ann. of Math.* **130**(2), 321–366 (1989)
- Dolbeault, J., Gentil, I., Guillin, A., Wang, F.-Y.: L^q -functional inequalities and weighted porous media equations. *Potential Anal.* **28**(1), 35–59 (2008)
- Einav, A., Morgan, J., Tang, B.Q.: Indirect diffusion effect in degenerate reaction-diffusion systems. *SIAM J. Math. Anal.* **52**(5), 4314–4361 (2020)
- Feinberg, M.: Lectures on chemical reaction networks. Notes of lectures given at the Mathematics Research Center, University of Wisconsin, pp. 49 (1979)
- Fellner, K., Latos, E., Tang, B.Q.: Global regularity and convergence to equilibrium of reaction-diffusion systems with nonlinear diffusion. *J. Evol. Equ.* **20**(3), 957–1003 (2020)

- Fellner, K., Morgan, J., Tang, B.Q.: Global classical solutions to quadratic systems with mass control in arbitrary dimensions. *Ann. Inst. H. Poincaré Anal. Non Linéaire* **37**(2), 281–307 (2020)
- Fellner, K., Tang, B.Q.: Explicit exponential convergence to equilibrium for nonlinear reaction-diffusion systems with detailed balance condition. *Nonlinear Anal.* **159**, 145–180 (2017)
- Fellner, K., Tang, B.Q.: Convergence to equilibrium of renormalised solutions to nonlinear chemical reaction-diffusion systems. *Z. Angew. Math. Phys.* **69**(3), Paper No. 54, 30 (2018)
- Fischer, J.: Global existence of renormalized solutions to entropy-dissipating reaction-diffusion systems. *Arch. Ration. Mech. Anal.* **218**(1), 553–587 (2015)
- Fischer, J.: Weak-strong uniqueness of solutions to entropy-dissipating reaction-diffusion equations. *Nonlinear Anal.* **159**, 181–207 (2017)
- Fischer, J., Hopf, K., Knily, M., Mielke, A.: Global existence analysis of energy-reaction-diffusion systems. *SIAM J. Math. Anal.* **54**(1), 220–267 (2022)
- Goudon, T., Vasseur, A.: Regularity analysis for systems of reaction-diffusion equations. *Ann. Sci. Éc. Norm. Supér. (4)* **43**(1), 117–142 (2010)
- Hopf, K.: Weak-strong uniqueness for energy-reaction-diffusion systems. *Math. Models Methods Appl. Sci.* **32**(5), 1015–1069 (2022)
- Horn, F.J.M.: Necessary and sufficient conditions for complex balancing in chemical kinetics. *Arch. Ration. Mech. Anal.* **49**, 172–186 (1972)
- Horn, F.J.M., Jackson, R.: General mass action kinetics. *Arch. Ration. Mech. Anal.* **47**, 81–116 (1972)
- Laamri, E.-H., Perthame, B.: Reaction-diffusion systems with initial data of low regularity. *J. Differential Equ.* **269**(11), 9310–9335 (2020)
- Laamri, E.H., Pierre, M.: Global existence for reaction-diffusion systems with nonlinear diffusion and control of mass. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, **34**(3), 571–591 (2017)
- Lankeit, J., Winkler, M.: Global existence in reaction-diffusion systems with mass control under relaxed assumptions merely referring to cross-absorptive effects. *J. Evol. Equ.* **22**(1), Paper No. 14, 23 (2022)
- Mielke, A., Mittnenzweig, M.: Convergence to equilibrium in energy-reaction-diffusion systems using vector-valued functional inequalities. *J. Nonlinear Sci.* **28**(2), 765–806 (2018)
- Moussa, A.: Some variants of the classical Aubin-Lions lemma. *J. Evol. Equ.* **16**(1), 65–93 (2016)
- Pierre, M.: Global existence in reaction-diffusion systems with control of mass: a survey. *Milan J. Math.* **78**(2), 417–455 (2010)
- Pierre, M., Schmitt, D.: Blowup in reaction-diffusion systems with dissipation of mass. *SIAM Rev.* **42**(1), 93–106 (2000)
- Shinar, G., Feinberg, M.: Concordant chemical reaction networks and the species-reaction graph. *Math. Biosci.* **241**(1), 1–23 (2013)
- Souplet, P.: Global existence for reaction-diffusion systems with dissipation of mass and quadratic growth. *J. Evol. Equ.* **18**(4), 1713–1720 (2018)
- Tang, B.Q.: Global classical solutions to reaction-diffusion systems in one and two dimensions. *Commun. Math. Sci.* **16**(2), 411–423 (2018)
- Villani, C.: On the Cauchy problem for Landau equation: sequential stability, global existence. *Adv. Differential Equ.* **1**(5), 793–816 (1996)