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On eigenvector statistics in the spherical and truncated unitary ensembles*

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Abstract

We study the overlaps between right and left eigenvectors for random matrices of the spherical ensemble, as well as truncated unitary ensembles in the regime where half of the matrix at least is truncated. These two integrable models exhibit a form of duality, and the essential steps of our investigation can therefore be performed in parallel.

In every case, conditionally on all eigenvalues, diagonal overlaps are shown to be distributed as a product of independent random variables with explicit distributions. This enables us to prove that the scaled diagonal overlaps, conditionally on one eigenvalue, converge in distribution to a heavy-tail limit, namely, the inverse of a γ_2 distribution. We also provide formulae for the conditional expectation of diagonal and off-diagonal overlaps, either with respect to one eigenvalue, or with respect to the whole spectrum. These results, analogous to what is known for the complex Ginibre ensemble, can be obtained in these cases thanks to integration techniques inspired from a previous work by Forrester & Krishnapur.

Keywords: eigenvectors overlaps; non-Hermitian random matrices; truncated unitary matrices; spherical ensemble.
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Contents

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1 Introduction

1.1 Spherical and Truncated Unitary Ensembles

This work considers two ensembles of random matrices defined as follows.

(i) The spherical ensemble consists of products $G_1 G_2^{-1}$, where G_1, G_2 are i.i.d. complex Ginibre matrices. We denote the $N \times N$ complex Ginibre ensemble by $\mathrm{CGE}(N)$ and the corresponding spherical ensemble by $\mathrm{Sph}(N)$. The name *spherical* comes from a geometric description of the eigenvalues, stated as Fact 2.5 and illustrated on Figure 1.

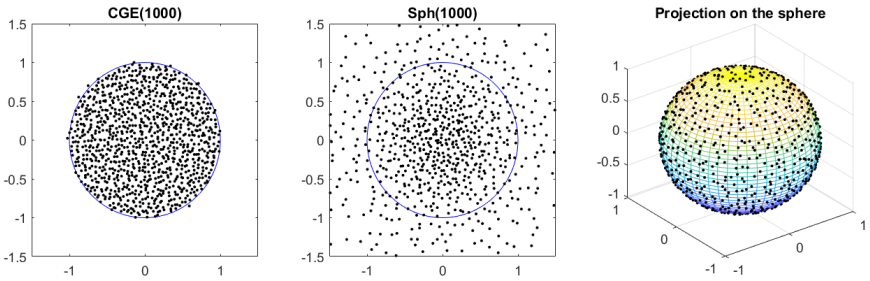

Figure 1: Scaled eigenvalues of $\mathrm{CGE}(1000)$ and $\mathrm{Sph}(1000)$; the third picture is the preimage of the latter by the stereographic projection (2.9).

(ii) The truncated unitary ensemble consists of truncations of unitary matrices distributed according to the Haar measure (CUE). It therefore depends on two parameters determining the size of the original CUE matrix and the size of the truncation, as exemplified on Figure 2. We denote by $\mathrm{TUE}(N, M)$ the ensemble of truncations of size N of matrices distributed according to $\mathrm{CUE}(N + M)$. Our results are only valid when $N \leqslant M$, that is, when the truncated matrix is **at most half as large** as the original matrix. In the relevant limits, both parameters are assumed to go to infinity; a regime of particular interest is when $\frac{M}{N} \to \kappa \geqslant 1$.

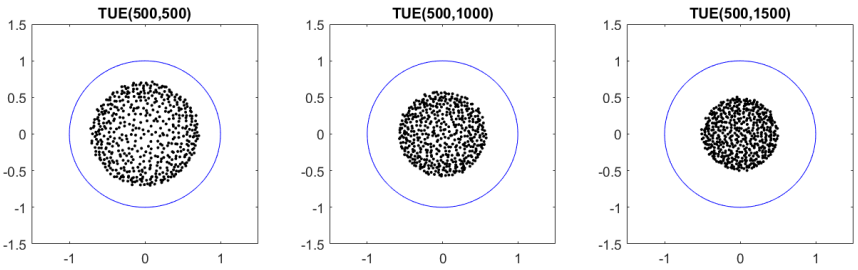

Figure 2: Eigenvalues of $\mathrm{TUE}(N, M)$ for $N = 500$ and $M = 500, 1000, 1500$.

The reason for treating these two ensembles in the same paper is the strong analogy between them, underlined and exemplified by [11], that extends to the overlap distribution. All results are presented in details for the spherical case in Section 2, while the corresponding results in the truncated unitary case are found in Section 3 – with less detail given whenever the two computations are exactly the same.

1.2 The matrix of overlaps

The matrix of overlaps associated to the bi-orthogonal family of left and right eigenvectors of a non-Hermitian random matrix has been introduced and studied by Chalker

& Mehlig in [6, 7], then more recently in a series of papers involving a variety of methods from mathematics and physics [1, 4, 5, 8, 10, 12, 14, 18, 20]. Most of these works deal with Gaussian ensembles, the complex Ginibre ensemble in particular. But more generally, the diagrammatic approach performed in [4, 18] supports very interesting conjectures much beyond the Gaussian cases, and it is one of the aims of the present work to establish some of these in the particular cases of $\mathrm{Sph}(N)$ and $\mathrm{TUE}(N, M)$.

The matrix of overlaps is defined as follows: for a given matrix $G \in \mathscr{M}_N(\mathbb{C})$ with simple spectrum $\{\lambda_1, \dots, \lambda_N\}$ (note that the random spectra are almost surely simple in the cases we consider), if $R_j = |R_j\rangle$ is the right eigenvector associated to λ_j and $L_j = \langle L_j|$ the left eigenvector associated to the same eigenvalue, that is, for every j,

$$GR_j = \lambda_j R_j \quad \& \quad L_j G = \lambda_j L_j, \tag{1.1}$$

are chosen such that they form a bi-orthogonal family, i.e. under the condition

$$\langle L_i | R_j \rangle = L_i R_j = \delta_{ij}, \tag{1.2}$$

then we define the matrix of overlaps $\mathscr{O}$ by

$$\mathscr{O}_{ij} = \langle L_i | L_j \rangle \langle R_j | R_i \rangle = (L_i L_j^*)(R_j^* R_i). \tag{1.3}$$

It is now clearly established that the matrix of overlaps quantifies the stability of the spectrum in various ways. We refer to the introduction of [5] as well as Section 2.4 of the review [3] for a more detailed presentation of overlaps and their relevance to both mathematics and physics.

1.2.1 Results.

The results we obtain in the spherical and truncated unitary cases are analogous to some of the results obtained in [5] for the complex Ginibre ensemble $\mathrm{CGE}(N)$. We recall these results, and point out which statement of the present paper corresponds to each one.

(i) **A decomposition of the distribution of diagonal overlaps.** The first notable fact is that, conditionally on the spectrum $\Lambda \in \mathbb{C}^N$, diagonal overlaps can be decomposed as a product of independent variables. In the complex Ginibre ensemble, Theorem 2.2 from [5] states that, conditionally on the event $\{\Lambda = (\lambda_1, \dots, \lambda_N)\}$, the distribution of diagonal overlaps is given by

$$\mathscr{O}_{11}^{\mathrm{CGE}(N)} \overset{d}{=} \prod_{i=2}^{N} \left(1 + \frac{|Z_i|^2}{N|\lambda_i - \lambda_1|^2}\right), \tag{1.4}$$

where $(Z_i)_{i=2}^N$ are i.i.d. standard complex Gaussian. Instead of Gaussian variables, the analogous statements in the spherical and truncated unitary ensembles involve i.i.d. variables whose distribution is specific to each case. Namely, in $\mathrm{Sph}(N)$, we have

$$\mathscr{O}_{11}^{\mathrm{Sph}(N)} \overset{d}{=} \prod_{k=2}^{N} \left(1 + \frac{(1+|\lambda_1|^2)(1+|\lambda_k|^2)}{|\lambda_1 - \lambda_k|^2} X_N^{(k)}\right), \tag{1.5}$$

where the $X_N^{(k)}$ are i.i.d. variables whose distribution is defined in (2.5); and in $\mathrm{TUE}(N, M)$,

$$\mathscr{O}_{11}^{\mathrm{TUE}(N,M)} \overset{d}{=} \prod_{k=2}^{N} \left(1 + \frac{(1-|\lambda_1|^2)(1-|\lambda_k|^2)}{|\lambda_1 - \lambda_k|^2} Y_M^{(k)}\right), \tag{1.6}$$

where the $Y_M^{(k)}$ are i.i.d. variables whose distribution is defined in (3.5). These decompositions are stated as Theorem 2.6 and 3.5 respectively.

(ii) **Conditional expectations of overlaps.** It follows from the decomposition of the law explained above that the expectation of diagonal overlaps also takes a product form. For $\mathrm{CGE}(N)$, following (1.4), this reads:

$$\mathbb{E}_{\Lambda}^{\mathrm{CGE}(N)}\left(\mathscr{O}_{11}\right) = \prod_{k=2}^{N}\left(1+\frac{1}{N|\lambda_i-\lambda_k|^2}\right), \tag{1.7}$$

which had been obtained earlier by Chalker & Mehlig [6, 7] by a direct computation. In the cases under consideration here, analogous identities derive from equations (1.5) and (1.6); they are stated in Theorem 2.6 and 3.5 respectively. We are also able to get explicit formulae for the conditional expectations with respect to only one eigenvalue, see Propositions 2.10 and 3.8. Moreover, expressions of the same kind can be obtained for off-diagonal overlaps, although no decomposition in independent variables holds in that case. In the Ginibre ensemble, the formula is

$$\mathbb{E}_{\Lambda}^{\mathrm{CGE}(N)}\left(\mathscr{O}_{12}\right) = -\frac{1}{|\lambda_1-\lambda_2|^2}\prod_{k=3}^{N}\left(1+\frac{1}{N(\lambda_1-\lambda_k)(\overline{\lambda_2-\lambda_k})}\right). \tag{1.8}$$

The analogous results for $\mathrm{Sph}(N)$ and $\mathrm{TUE}(N,M)$ are stated as Theorem 2.11 and 3.9 respectively, and also appear in the synoptic table below.

(iii) **Limit theorems for diagonal overlaps.** In the complex Ginibre ensemble, Theorem 1.1 from [5] states that conditionally on the event $\{\lambda_1 = z\}$ with $z \in \mathbb{D}$, the scaled diagonal overlap $\mathscr{O}_{11}$ converges to the inverse of a γ_2 distribution:

$$\frac{1}{N}\mathscr{O}_{11}^{\mathrm{CGE}(N)} \xrightarrow[N\to\infty]{d} (1-|z|^2)\frac{1}{\gamma_2}. \tag{1.9}$$

This heavy-tail limit appears to be universal, as illustrated on Figure 3. In particular, the exact same convergence holds at the origin for the spherical and truncated unitary ensembles, which is stated as Proposition 2.8 and 3.7 respectively. Unlike the complex Ginibre case, where $\mathscr{O}_{11}^{-1}$ follows a beta distribution when $\{\lambda_1 = 0\}$, the distribution of the overlap for fixed N does not take an especially simple form here; nevertheless, the asymptotical result can be worked out in an analogous way.

The specific structure of the spherical ensemble allows one to extend this result to the whole complex plane in this case, yielding the following Theorem.

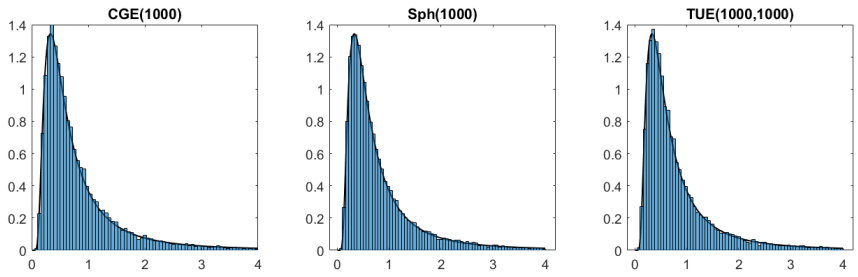

Figure 3: Histograms of scaled diagonal overlaps for $\mathrm{CGE}(N), \mathrm{Sph}(N)$ and $\mathrm{TUE}(N,N)$ respectively, with $N = 1000$ and over 30 experiments (for each experiment, the overlaps of all eigenvalues in a given domain, chosen arbitrarily inside the bulk, have been considered).

Theorem 1.1. *Conditionally on the event* $\{\lambda_1 = z_1\}$ *with* $z_1 \in \mathbb{C}$,

$$\mathbb{E}_{\{\lambda_1=z_1\}}\left(\mathscr{O}_{11}^{\mathrm{Sph}(N)}\right) = N \quad \textit{and} \quad \frac{1}{N}\mathscr{O}_{11}^{\mathrm{Sph}(N)} \xrightarrow[N\to\infty]{d} \frac{1}{\gamma_2}. \tag{1.10}$$

It is to be expected that a similar statement holds for $\mathrm{TUE}(N, M)$ in the bulk of its limit density of eigenvalues. More precisely, we expect the following to hold.

Conjecture 1.2. *In the limit* $\frac{M}{N} \to \kappa \geqslant 1$, *conditionally on* $\{\lambda_1 = z_1\}$ *with* $|z_1|^2 < \frac{1}{\kappa+1}$,

$$\frac{1}{N}\mathscr{O}_{11}^{\mathrm{TUE}(N,M)} \xrightarrow[N\to\infty]{d} \left(1 - (1+\kappa)|z|^2\right)\frac{1}{\gamma_2}. \tag{1.11}$$

Such a result appears to be within the reach of known methods but to require a much longer study. We provide here two partial results: convergence to the proper limit at the origin (Proposition 3.7) and a derivation of the right scaling parameter in the conditional expectation (limit (3.12) in Proposition 3.8). This second step confirms a specific case of the general expressions conjectured in [4].

(iv) **Conditional expectation of a mixed moment.** The conditional expectation of $\mathrm{Tr}\, G^*G$ with respect to Λ also exhibits a remarkable decomposition in all three ensembles. One reason for considering this particular quantity, which is the simplest 'mixed moment', is that it is obtained from the eigenvalues and the overlaps by a simple identity:

$$\mathrm{Tr}\, GG^* = \sum_{i,j=1}^{N} \lambda_i \overline{\lambda_j} \mathscr{O}_{ij}. \tag{1.12}$$

The link between mixed moments and overlaps motivated the work of Walters & Starr [20], and similarly, more general mixed moments are linked to the generalized overlaps considered in [8]. In the complex Ginibre case, the distribution of $\mathrm{Tr}\, G^*G$ is straightforward to describe: it suffices to write

$$\mathrm{Tr}\, GG^* = \mathrm{Tr}(TT^*) = \sum_{i\leqslant j} |T_{ij}|^2 = \sum_{i=1}^{N} |\lambda_i|^2 + \sum_{i<j} |T_{ij}|^2, \tag{1.13}$$

and to note the fact that the upper-diagonal entries $(T_{ij})_{i<j}$ of the Schur transform are i.i.d. Gaussian and independent of the eigenvalues. A very simple formula for the conditional expectation follows. The spherical and truncated unitary ensembles yield slightly more intricate expressions, stated as Proposition 2.12 and 3.10 respectively.

We summarize all results relative to (iii) and (iv) in the table below, Section 1.4. It follows from (1.12) that the third column is related to the first two by elementary linear relations – a fact which is not directly seen from the quenched expressions.

1.3 Method, notations and conventions

1.3.1 Overlaps and Schur form.

We present here the first steps of the general method, already used in previous works such as [5–7, 10], that is now to be applied to the spherical and truncated unitary cases.

We first note that the conditions (1.2) can be achieved by choosing R_i as the columns of P and L_i as the rows of P^{-1} for a given diagonalization $G = P\Delta P^{-1}$; the overlaps are independent of this choice. Moreover, overlaps are unchanged by an unitary change of

1.4 Synoptic table: conditional expectations

	$\mathbb{E}_\Lambda\left(\mathscr{O}_{11}\right)$	$\mathbb{E}_\Lambda\left(\mathscr{O}_{12}\right)$	$\mathbb{E}_\Lambda\left(\frac{1}{N}\operatorname{Tr} GG^*\right)$
Complex Ginibre $\mathrm{CGE}(N)$	$\displaystyle\prod_{k=2}^{N}\left(1+\frac{1}{N\lvert\lambda_1-\lambda_k\rvert^2}\right)$ Chalker & Mehlig [6, 7]	$\displaystyle-\frac{1}{N\lvert\lambda_1-\lambda_2\rvert^2}\prod_{k=3}^{N}\left(1+\frac{1}{N(\lambda_1-\lambda_k)(\overline{\lambda_2-\lambda_k})}\right)$ Chalker & Mehlig [6, 7]	$\displaystyle\frac{1}{N}\sum_{k=1}^{N}\lvert\lambda_k\rvert^2+\frac{N-1}{2N}$ From eq. (1.13)
Spherical Ensemble $\mathrm{Sph}(N)$	$\displaystyle\prod_{k=2}^{N}\left(1+\frac{(1+\lvert\lambda_1\rvert^2)(1+\lvert\lambda_k\rvert^2)}{N\lvert\lambda_1-\lambda_k\rvert^2}\right)$ Theorem 2.6	$\displaystyle-\frac{1}{N\lvert\lambda_1-\lambda_2\rvert^2}\prod_{k=3}^{N}\left(1+\frac{(1+\lambda_1\overline{\lambda_2})(1+\lvert\lambda_k\rvert^2)}{N(\lambda_1-\lambda_k)(\overline{\lambda_2-\lambda_k})}\right)$ Theorem 2.11	$\displaystyle\prod_{k=1}^{N}\left(1+\frac{1+\lvert\lambda_k\rvert^2}{N}\right)-2$ Proposition 2.12
Truncated Unitary $\mathrm{TUE}(N,M)$	$\displaystyle\prod_{k=2}^{N}\left(1+\frac{(1-\lvert\lambda_1\rvert^2)(1-\lvert\lambda_k\rvert^2)}{M\lvert\lambda_1-\lambda_k\rvert^2}\right)$ Theorem 3.5	$\displaystyle-\frac{1}{M\lvert\lambda_1-\lambda_2\rvert^2}\prod_{k=3}^{N}\left(1+\frac{(1-\lambda_1\overline{\lambda_2})(1-\lvert\lambda_k\rvert^2)}{M(\lambda_1-\lambda_k)(\overline{\lambda_2-\lambda_k})}\right)$ Theorem 3.9	$\displaystyle\prod_{i=1}^{N}\left(1+\frac{1-\lvert\lambda_i\rvert^2}{M}\right)-\left(1+\frac{N}{M}\right)$ Proposition 3.10

Table 1: Quenched expectations.

The above table contains the expression of quenched expectations (that is, conditional expectations with respect to $\Lambda=\{\lambda_1,\dots,\lambda_N\}$) of overlaps in the three relevant integrable ensembles. The first column is the consequence of an identity in distribution involving independent variables; not so for off-diagonal overlaps (second column). The third column is related to the first two by the linear relations implied by (1.12).

basis, and therefore one can study directly the overlaps of the Schur form

$$T = U^* G U = \begin{pmatrix} \lambda_1 & T_{1,2} & \dots & T_{1,N} \\ 0 & \lambda_2 & \dots & T_{2,N} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & \lambda_N \end{pmatrix}. \tag{1.14}$$

By exchangeability of the eigenvalues, we can also limit ourselves to studying the variables $\mathscr{O}_{11}$ and $\mathscr{O}_{12}$, whose definitions only involve the first two left and right eigenvectors of T, chosen such that

$$\begin{aligned} R_1 &= (1, 0, \dots, 0)^{\mathrm{t}}, \qquad & R_2 &= (a, 1, 0, \dots, 0)^{\mathrm{t}}, \\ L_1 &= (b_1, \dots, b_N), & L_2 &= (d_1, \dots, d_N). \end{aligned}$$

Biorthogonality (1.2) gives $b_1 = 1$, $d_1 = 0$, $d_2 = 1$ and $a = -b_2$. Thanks to the upper-triangular form of T, the coefficients b_i, d_i are obtained according to a straightforward recurrence. Indeed, if we consider the sequences of sub-vectors:

$$\begin{aligned} B_k &= (1, b_2, \dots, b_k) \qquad && \text{so that } L_1 = B_N, \\ D_k &= (0, 1, d_3, \dots, d_k) \qquad && \text{so that } L_2 = D_N, \\ u_k &= (T_{1,k}, \dots, T_{k-1,k})^{\mathrm{t}} \qquad && \text{(subset of the } k\text{th column of } T\text{)}. \end{aligned}$$

The recurrence formula is

$$\begin{cases} b_{n+1} &= \frac{1}{\lambda_1 - \lambda_{n+1}} B_n u_{n+1}, \quad n \geqslant 1, \\ d_{n+1} &= \frac{1}{\lambda_2 - \lambda_{n+1}} D_n u_{n+1}, \quad n \geqslant 2. \end{cases} \tag{1.15}$$

The first overlaps, according to (1.3), are then given by the expressions

$$\mathscr{O}_{11} = \sum_{i=1}^{N} |b_i|^2, \qquad \mathscr{O}_{12} = -\overline{b_2} \sum_{i=1}^{N} b_i \overline{d_i}. \tag{1.16}$$

In order to deduce from the recurrence (1.15) a decomposition in distribution (resp. a decomposition of the conditional expectation with respect to all eigenvalues) of the overlaps in different ensembles, we need the distribution of the Schur form to be known and to allow to perform such a computation explicitly. For instance, in the complex Ginibre case treated in [5], the upper-triangular entries $(T_{ij})_{i<j}$ are i.i.d. complex Gaussian variables with variance $1/N$, so that u_{k+1} is a k-dimensional Gaussian vector with independent coordinates, and independent of $u_2, \dots, u_k$. The cases at stake here are more intricate, but still integrable: indeed, it was proved in [11] that the Schur forms of both $\mathrm{Sph}(N)$ and $\mathrm{TUE}(N, M)$ also have explicit densities expressed in the form of a determinant; a structure which allows an analogous analysis in these non-Gaussian cases.

1.4.1 Notations and conventions.

Throughout the paper, N is the size of the system (i.e. the number of eigenvalues); the spectrum is $\Lambda = \Lambda_N = (\lambda_1, \dots, \lambda_N)$. For any $n \leqslant N$, we denote by T_n the $n \times n$ top-left submatrix of the Schur form T, and by u_n the first $n-1$ coordinates of the last column vector of T_n, so that

$$T_n = \begin{pmatrix} T_{n-1} & u_n \\ 0 & \lambda_n \end{pmatrix}, \qquad T = T_N.$$

$\mathbb{E}_A$ denotes the conditional expectation with respect to A (if A is a random variable or a sigma algebra), or the expectation for the conditional probability (if A is an event); the context should prevent any ambiguity to arise. In particular, $\mathbb{E}_\Lambda$ is the conditional expectation with respect to the spectrum Λ. When conditioning on Λ, we will also use the following filtration, adapted to the nested structure of the Schur transform:

$$\mathscr{F}_n = \sigma\left(u_k, 2 \leqslant k \leqslant n\right) = \sigma\left(T_{i,j}, 1 \leqslant i < j \leqslant n\right).$$

(This convention differs from the one chosen in [5]. In particular, $b_2 = \frac{T_{12}}{\lambda_1 - \lambda_2} \in \mathscr{F}_2$, and $\mathscr{F}_1$ is trivial.) With any suitable function V, the generalized Gamma and Meijer functions are defined as

$$\Gamma_V(\alpha) := \int_{\mathbb{R}_+} t^{\alpha-1} e^{-V(t)} \mathrm{d}t, \qquad G_V(k) = \Gamma_V(1) \cdots \Gamma_V(k).$$

We also define the partial sums

$$e_V^{(m)}(X) = \sum_{k=0}^{m} \frac{X^k}{\Gamma_V(k+1)},$$

and the generalized Gamma distributions $\gamma_V(\alpha)$, with density

$$\frac{1}{\Gamma_V(\alpha)} t^{\alpha-1} e^{-V(t)} \mathbb{1}_{\mathbb{R}_+} \tag{1.17}$$

with respect to the Lebesgue measure. We will use the fact, established for instance in [9, 15, 16], that a point process in $\mathbb{C}$ with joint density given by

$$\frac{1}{Z_N} \prod_{1 \leqslant i < j \leqslant N} |\lambda_i - \lambda_j|^2 e^{-\sum_{i=1}^N V(|\lambda_i|^2)}, \tag{1.18}$$

where $Z_N = G_V(N)$, is such that the following identity in distribution holds:

Proposition 1.3 (Kostlan's property)**.** *$\{|\lambda_1|^2, \ldots, |\lambda_N|^2\} \overset{d}{=} \{\gamma_V(1), \ldots, \gamma_V(N)\}$ where the latter variables are independent, and $\gamma_V(k)$ is distributed according to (1.17) with $\alpha = k$.*

What we need here is a specific form of Kostlan's property, obtained by applying Proposition 1.3 to the conditioned measure.

Proposition 1.4. *Conditionally on the event $\{\lambda_1 = 0\}$, the identity in distribution $\{|\lambda_2|^2, \ldots, |\lambda_N|^2\} \overset{d}{=} \{\gamma_V(2), \ldots, \gamma_V(N)\}$ holds, where the latter variables are independent, and $\gamma_V(k)$ is distributed according to (1.17) with $\alpha = k$.*

Other notations or conventions relative specifically to the spherical or truncated unitary case are mentioned in the corresponding section.

1.4.2 Two general facts

We conclude this introduction with two results that are used similarly in both cases. The first one is a general identity that holds for any determinantal point process with joint density given by (1.18) with a radially invariant potential such that the generalized Gamma function is well defined.

Proposition 1.5 (Product statistics)**.** *For any function $g \in L^2(\mu)$,*

$$\mathbb{E}_{\lambda_1}\left(\prod_{i=2}^{N} g(\lambda_i)\right) = \frac{1}{Z_N^{(1)}} \det\left(\int z^{i-1} \overline{z}^{j-1} |\lambda_1 - z|^2 g(z) e^{-V(|z|^2)} \mathrm{d}m(z)\right)_{i,j=1}^{N-1},$$

where

$$Z_N^{(1)} = G_V(N) e_V^{(N-1)}(|\lambda_1|^2).$$

Proof. The determinant comes from Andréief identity [2] applied to the conditioned measure. It remains to compute the normalization constant by setting $g = 1$. We find

$$Z_N^{(1)} = \det\left(f_{i,j}\right)_{i,j=1}^{N-1} =: D_{N-1}$$

where the determinants D_N are Hermitian, tridiagonal and nested (i.e. D_k is the principal minor of D_{k+1}), with

$$\begin{aligned} f_{k,k} &= |\lambda_1|^2\Gamma_V(k) + \Gamma_V(k+1) \\ f_{k+1,k} &= \overline{f_{k,k+1}} = -\lambda_1\Gamma_V(k+1). \end{aligned}$$

This gives the initial value $Z_2^{(1)} = D_1 = |\lambda_1|^2\Gamma_V(1) + \Gamma_V(2) = G_V(2)e_V^{(1)}(|\lambda_1|^2)$ and the induction

$$D_k = f_{k,k}D_{k-1} - |f_{k,k-1}|^2 D_{k-2} = \left(|\lambda_1|^2\Gamma_V(k) + \Gamma_V(k+1)\right) D_{k-1} - |\lambda_1|^2\Gamma_V(k)^2 D_{k-2},$$

whose solution is the formula provided for $Z_N^{(1)}$. Indeed, by induction, if D_{k-2}, D_{k-1} are given by the expected formula, then

$$\begin{aligned} D_k &= |\lambda_1|^2\Gamma_V(k)\left(D_{k-1} - \Gamma_V(k)D_{k-2}\right) + \Gamma_V(k+1)D_{k-1} \\ &= G_V(k+1)\left(e_V^{(k-1)} + \frac{|\lambda_1|^{2k}}{\Gamma_V(k+1)}\right) \\ &= G_V(k+1)e_V^{(k)}, \end{aligned}$$

which yields the claim, for $k = N-1$. □

This three-terms induction structure will appear again in Propositions 2.10 and 3.8 – that is, whenever we use Proposition 1.5 to perform an explicit computation. We will systematically call the determinant at stake D_k, and use the shorthands

$$\tilde{D}_k := \frac{1}{G_V(k+1)}D_k, \qquad \gamma_k := \frac{\Gamma_V(k)}{\Gamma_V(k+1)}, \tag{1.19}$$

with respect to the appropriate V. One more general result we shall need as a technical input is an elementary exercise on sequences and series.

Lemma 1.6. *Let* $\left(u_{k,n}^{(m)}\right)_{1\leqslant k\leqslant n}^{m\in\mathbb{N}}$ *be a countable family of double-indexed real positive sequences such that*

$$\forall m, k \geqslant 1, \qquad u_{k,n}^{(m)} \xrightarrow[n\to\infty]{} 0.$$

Then there exists a sequence $(k_n)_{n\geqslant 1}$ *such that* $1 \leqslant k_n \leqslant n$, $k_n \xrightarrow[n\to\infty]{} \infty$, *and for any* $m \in \mathbb{N}$,

$$\sum_{i=1}^{k_n} u_{i,n}^{(m)} \xrightarrow[n\to\infty]{} 0.$$

Proof of Lemma 1.6. We first prove the statement for one double-indexed sequence $(u_{k,n})_{1\leqslant k\leqslant n}$. We define, for $1 \leqslant k \leqslant n$, the partial sums $S_{k,n} = \sum_{i=1}^{k} u_{i,n}$, and the following sequence, iteratively:

$$n_1 := 1, \qquad n_{j+1} := \min\left\{ l \;\middle|\; \forall n \geqslant l,\ S_{j+1,n} \leqslant \frac{1}{2}S_{j,n_j} \right\}.$$

By assumption on $u_{k,n}$, the sequence $(n_j)_{j\geqslant 1}$ is well defined, increasing, and goes to infinity. Moreover, by construction we see that S_{j,n_j} converges to zero. It is straightforward to check that the sequence

$$k_n := \max\{j \in [\![1, n]\!] \;|\; n_j \leqslant n\}$$

is such that $1 \leqslant k_n \leqslant n$, $k_n \xrightarrow[n\to\infty]{} \infty$, and

$$\forall n \in [\![n_j, n_{j+1}-1]\!], \qquad S_{k_n,n} = S_{j,n} \leqslant \frac{1}{2} S_{j-1,n_{j-1}},$$

so that $S_{k_n,n}$ converges to 0; thus, the Lemma is established for one sequence. We extend this to a countable family of double-indexed sequences $u_{k,n}^{(m)}$ by defining $v_{k,n} := \sum_{m=1}^{k} u_{k,n}^{(m)}$, which converges to 0 for every fixed k; by the above argument, there exists a sequence k_n such that

$$\sum_{j=1}^{k_n} v_{j,n} \xrightarrow[n\to\infty]{} 0, \quad \text{so that} \quad \forall m \in \mathbb{N},\ \sum_{j=1}^{k_n} u_{j,n}^{(m)} \xrightarrow[n\to\infty]{} 0.$$

Indeed, every term being positive, as $k_n \to \infty$, the latter sum can be bounded by the first one as soon as $k_n \geqslant m$. This concludes the proof of Lemma 1.6. □

In fact, the argument we will make relies on the multiplicative version of Lemma 1.6, namely: if a countable family of double-indexed sequences $p_{k,n}^{(m)}$ is such that $p_{k,n}^{(m)} \to 1$ for every fixed k and m, then there exists a sequence $(k_n)_{n\geqslant 1}$, going to infinity, such that for every m

$$\prod_{j=1}^{k_n} p_{j,n}^{(m)} \to 1.$$

Note that this existential statement does not give any estimate on the growth rate of (k_n).

2 Spherical ensemble

This section contains the proof of all claims related to the spherical ensemble $\mathrm{Sph}(N)$. These proofs rely on a few estimates that are found in Subsection 2.3.

2.1 Schur form and eigenvalues

We first present a few general results in order to illustrate the method; the tools and definitions that follow are specific to the spherical case. We recall that the Schur transfom T of a matrix from $\mathrm{Sph}(N)$ is distributed with density proportional to

$$\prod_{1\leqslant i<j\leqslant N} |\lambda_i - \lambda_j|^2 \frac{1}{\det(I_N + TT^*)^{2N}} \tag{2.1}$$

with respect to the Lebesgue measure on all complex matrix elements, diagonal ($\mathrm{d}\Lambda = \mathrm{d}\lambda_1 \cdots \mathrm{d}\lambda_N$) and upper-triangular ($\mathrm{d}u_2 \cdots \mathrm{d}u_n$). We introduce the Hermitian, definite-positive matrices

$$H_n := I_n + T_n T_n^*, \qquad S_{n-1} := (1+|\lambda_n|^2)^{1/2} H_{n-1}^{1/2}. \tag{2.2}$$

The following lemma is the essential tool used in [11].

Lemma 2.1. *The determinant of $H_n = I_n + T_n T_n^*$ can be recursively decomposed as*

$$\det(H_n) = (1+|\lambda_n|^2) \det(H_{n-1}) \left(1 + \frac{1}{1+|\lambda_n|^2} u_n^* H_{n-1}^{-1} u_n\right). \tag{2.3}$$

Proof. We first write

$$\det(H_n) = \det \begin{pmatrix} I_{n-1} + T_{n-1}T_{n-1}^* + u_n u_n^* & \overline{\lambda_n} u_n \\ \lambda_n u_n^* & 1+|\lambda_n|^2 \end{pmatrix}.$$

Elementary operations on columns brings this matrix to an upper-triangular form, so that

$$\begin{aligned}\det(H_n) &= (1+|\lambda_n|^2)\det\left(I_{n-1}+T_{n-1}T_{n-1}^*+\frac{1}{1+|\lambda_n|^2}u_nu_n^*\right)\\ &= (1+|\lambda_n|^2)\det(H_{n-1})\det\left(I_{n-1}+\frac{1}{1+|\lambda_n|^2}u_nu_n^*H_{n-1}^{-1}\right).\end{aligned}$$

The claim follows by Sylvester's identity, $\det(I+AB)=\det(I+BA)$, for matrices A,B of such sizes that the second determinant is reduced to a scalar. □

For any $p>n$, we denote by $\mathscr{V}_p^{(n)}$ a random vector with density

$$\frac{1}{C_{n,p}}\frac{1}{(1+v^*v)^p} \tag{2.4}$$

with respect to the Lebesgue measure on $\mathbb{C}^n$; the value of $C_{n,p}$ is given by (2.22). For any $m\geqslant 0$, we denote by X_m a real random variable with density

$$\frac{m+1}{(1+x)^{m+2}}\mathbb{1}_{\mathbb{R}_+} \tag{2.5}$$

with respect to the Lebesgue measure. In particular $\mathbb{E}X_m=\frac{1}{m}$, and if v_i is a coordinate of $\mathscr{V}_p^{(n)}$, it follows from Lemma 2.16 that

$$|v_i|^2 \overset{d}{=} X_{p-n-1}.$$

Note that the i.i.d. variables that appear in Theorem 2.6 follow the distribution of X_m with $m=N$.

Lemma 2.2. *Identity holds between the following expressions, for $p\geqslant n$ and f,g integrable functions of the matrix elements:*

$$\begin{aligned}&\int\frac{f(\Lambda_n,u_2,\dots,u_{n-1})g(u_n)}{\det(H_n)^p}\mathrm{d}T_n\\ &\qquad= C_{n-1,p}\int\frac{f(\Lambda_n,u_2,\dots,u_{n-1})\mathbb{E}\left(g(S_{n-1}\mathscr{V}_p^{(n-1)})\right)}{(1+|\lambda_n|^2)^{p-n+1}\det(H_{n-1})^{p-1}}\mathrm{d}T_{n-1}\mathrm{d}\lambda_n,\end{aligned}$$

where $H_n, S_{n-1}, \mathscr{V}_p^{(n)}$ are defined in (2.2) and (2.4).

Proof. Lemma 2.1 and the change of variable $u_n=S_{n-1}v_n$ bring the left hand side to the form

$$\int\frac{f(\Lambda_n,u_2,\dots,u_{n-1})g(S_{n-1}v_n)}{(1+|\lambda_n|^2)^{p-n+1}\det(H_{n-1})^{p-1}(1+v_n^*v_n)^p}\mathrm{d}T_{n-1}\mathrm{d}v_n\mathrm{d}\lambda_n.$$

Recall that u_n, and therefore v_n, are column vectors of size $n-1$. The claim follows by definition of the random vector $\mathscr{V}_p^{(n-1)}$. □

A first relevant fact that can be deduced from the above Lemma is the distribution of every top-left submatrix of the Schur form T.

Proposition 2.3. *Conditionally on Λ and for $2\leqslant n\leqslant N$, the submatrix T_n of the Schur transform is distributed with density proportional to*

$$\frac{1}{\det(I_n+T_nT_n^*)^{N+n}} \tag{2.6}$$

with respect to the Lebesgue measure on upper-triangular matrix elements ($\mathrm{d}u_2\cdots\mathrm{d}u_n$).

Proof. The claim is known for $n = N$. We deduce it for all n by a backward recurrence; indeed, as long as $n - 1 \geqslant 2$, the claim for $n - 1$ follows from the claim for n by Lemma 2.2 with $g = 1$ and generic f. □

We can also derive the joint eigenvalue density of the spherical ensemble from the density of its Schur form, as was done in [11].

Theorem 2.4. *The joint density of eigenvalues for the spherical ensemble is proportional to*

$$\prod_{i<j} |\lambda_i - \lambda_j|^2 \frac{1}{\prod_{i=1}^N (1 + |\lambda_i|^2)^{N+1}} \tag{2.7}$$

with respect to the Lebesgue measure on $\mathbb{C}^N$.

Proof. Let h be a bounded and continuous function of the spectrum Λ_n. We use Lemma 2.1 with $p = 2N$, $g = 1$ and

$$f_0(\Lambda_n, u_2, \dots, u_n) := \prod_{i<j} |\lambda_i - \lambda_j|^2 h(\Lambda_n),$$

which yields

$$\mathbb{E}\left(h(\Lambda_n)\right) = C_{N-1,2N} \int \frac{f_n(\Lambda_n, u_2, \dots, u_{n-1})}{(1 + |\lambda_n|^2)^{N+1} \det(H_{n-1})^{2N-1}} \mathrm{d}T_{n-1} \mathrm{d}\lambda_n.$$

We then use Lemma 2.1 again with

$$f_{n-1}(\Lambda_{n-1}, u_2, \dots, u_{n-1}) := \int \frac{f_n(\Lambda_n, u_2, \dots, u_{n-1})}{(1 + |\lambda_n|^2)^{p-n+1}} \mathrm{d}\lambda_n,$$

and so on; this recurrence leads to the expression

$$\mathbb{E}\left(h(\Lambda_N)\right) = C \int \prod_{i<j} |\lambda_i - \lambda_j|^2 \frac{h(\Lambda_N)}{\prod_{i=1}^N (1 + |\lambda_i|^2)^{N+1}} \mathrm{d}\Lambda_N,$$

which is equivalent to the claim. □

Theorem 2.4 can be rephrased by saying that the eigenvalues of $\mathrm{Sph}(N)$ are distributed according to (1.18) with potential $V(t) = (N + 1)\ln(1 + t)$. A straightforward computation shows that

$$\gamma_V(\alpha) \stackrel{d}{=} \frac{1}{\beta_{N+1-\alpha,\alpha}} - 1. \tag{2.8}$$

Origin of the name spherical.

The stereographic projection from $\mathbb{S}^2$ to $\mathbb{C}$ is defined by

$$\rho(\mathbf{w}) = \tan\left(\frac{\phi}{2}\right) e^{i\theta}, \qquad \text{where} \quad \mathbf{w} = \begin{pmatrix} \sin\phi\cos\theta \\ \sin\phi\sin\theta \\ \cos\phi \end{pmatrix} \in \mathbb{S}^2, \tag{2.9}$$

and its inverse map from $\mathbb{C}$ to $\mathbb{S}^2$ is given by:

$$p(\lambda) = \frac{1}{1 + |\lambda|^2} \begin{pmatrix} 2\mathrm{Re}\lambda \\ 2\mathrm{Im}\lambda \\ |\lambda|^2 - 1 \end{pmatrix} \in \mathbb{S}^2, \qquad \text{where } \lambda \in \mathbb{C}. \tag{2.10}$$

The reason for the name *spherical* is that the following identity in distribution holds, as was first established in Section 3 of [17].

Fact 2.5. *Let* $(\mathbf{w}_1, \dots, \mathbf{w}_N) = (p(\lambda_1), \dots, p(\lambda_N))$ *be the images of the eigenvalues of* $G_1 G_2^{-1}$ *by the map (2.10). This point process on* $\mathbb{S}^2$ *has joint density proportional to*

$$\prod_{i<j} \|\mathbf{w}_i - \mathbf{w}_j\|_{\mathbb{R}^3}^2. \tag{2.11}$$

In other terms, the eigenvalues of $\mathrm{Sph}(N)$ can be described as the stereographic projection of a one-component plasma on $\mathbb{S}^2$, with respect to a uniform potential[1].

Proof. This is obtained by a change of variable applied to the density (2.7), noting that

$$\|p(\lambda) - p(\mu)\|_{\mathbb{R}^3}^2 = \frac{4|\lambda - \mu|^2}{(1+|\lambda|^2)(1+|\mu|^2)} \tag{2.12}$$

and that the Jacobian of p at λ is $\frac{4}{(1+|\lambda|^2)^2}$. □

2.2 Distribution and conditional expectation of overlaps

We now give the proof of the claims concerning diagonal and off-diagonal overlaps in the spherical ensemble. Some results hold conditionally on the whole spectrum Λ_N, whereas others only imply a condition on one eigenvalue.

Theorem 2.6. *Conditionally on* $\{\Lambda = (\lambda_1, \dots, \lambda_N)\}$, *the diagonal overlaps of* $\mathrm{Sph}(N)$ *are distributed as*

$$\mathscr{O}_{11} \overset{d}{=} \prod_{k=2}^{N} \left(1 + \frac{(1+|\lambda_1|^2)(1+|\lambda_k|^2)}{|\lambda_1 - \lambda_k|^2} X_N^{(k)}\right), \tag{2.13}$$

where the $X_N^{(k)}$ *are i.i.d. distributed according to (2.5) with* $m = N$. *In particular, the quenched expectation is given by*

$$\mathbb{E}_\Lambda\left(\mathscr{O}_{11}\right) = \prod_{k=2}^{N} \left(1 + \frac{(1+|\lambda_1|^2)(1+|\lambda_k|^2)}{N|\lambda_1 - \lambda_k|^2}\right). \tag{2.14}$$

Proof. For $1 \leqslant d \leqslant N$, we define the partial sums

$$\mathscr{O}_{11}^{(d)} := \sum_{i=1}^{d} |b_i|^2.$$

It follows from the general facts presented in Section 1.2 that $\mathscr{O}_{11}^{(1)} = 1$, and for any $d \geqslant 1$,

$$\mathscr{O}_{11}^{(d+1)} = \mathscr{O}_{11}^{(d)} + |b_{d+1}|^2 = \mathscr{O}_{11}^{(d)} \left(1 + \frac{1}{|\lambda_1 - \lambda_{d+1}|^2} \frac{|B_d u_{d+1}|^2}{\|B_d\|^2}\right).$$

In order to characterize the distribution of this factor, we use our preliminary results in the following order:

- Proposition 2.3 gives the distribution of T_{d+1}, so that $p = N + d + 1$ in the following steps.
- Lemma 2.2 with $n = d+1$, generic f and $g(u_{d+1}) := h(|B_d u_{d+1}|^2)$ with generic h gives that
$$|B_d u_{d+1}|^2 \overset{d}{=} |B_d S_d \mathscr{V}_{N+d+1}^{(d)}|^2$$
and is independent of $\mathscr{F}_d$.

[1] Note that the appropriate convention for the stereographic projection here is such that the unit circle is mapped to the equator of $\mathbb{S}^2$. In particular, the average proportion of eigenvalues of $\mathrm{Sph}(N)$ falling in the unit disk is $\frac{1}{2}$.

- Lemma 2.16 with $a = b = B_d^*$ and $S = S_d$ yields

$$|B_d u_{d+1}|^2 \stackrel{d}{=} \|S_d B_d^*\|^2 X_N = (1 + |\lambda_{d+1}|^2)(B_d(I_d + T_d T_d^*)B_d^*)X_N, \tag{2.15}$$

  where X_N is distributed according to (2.5) with parameter $m = N$, and independent of $\mathscr{F}_d$.

We notice that, as T is triangular and T_d, B_d are obtained from T and L_1,

$$B_d T_d = \lambda_1 B_d, \tag{2.16}$$

which implies that $B_d(I_d + T_d T_d^*)B_d^* = (1 + |\lambda_1|^2)\|B_d\|^2$. It follows that

$$\mathscr{O}_{11}^{(d+1)} \stackrel{d}{=} \mathscr{O}_{11}^{(d)} \left(1 + \frac{(1 + |\lambda_1|^2)(1 + |\lambda_{d+1}|^2)}{|\lambda_1 - \lambda_{d+1}|^2} X_N\right),$$

where X_N is independent of $\mathscr{F}_d$; we denote this variable by $X_N^{(d+1)}$ in order to avoid confusion between the different variables X_N. This implies the claim, as $\mathscr{O}_{11} = \mathscr{O}_{11}^{(N)}$. □

Diagonal overlap are (deterministically) larger than one, and typically of order N. The following proposition states that in the spherical ensemble the expectation of the diagonal overlap for an eigenvalue conditioned to be at the origin is exactly N, as is also the case in the complex Ginibre and truncated unitary ensembles.

Proposition 2.7. *Conditionally on $\{\lambda_1 = 0\}$, the expectation of the diagonal overlap $\mathscr{O}_{11}$ in the spherical ensemble $\mathrm{Sph}(N)$ is*

$$\mathbb{E}_{\{\lambda_1=0\}} \mathscr{O}_{11} = N.$$

Proof. We know from Proposition 1.3 that the squared radii are distributed like independent variables with distributions $\gamma_{V,k}$ with $V(x) = (N+1)\log(1+x)$ and $2 \leqslant k \leqslant N$. We have

$$\mathbb{E}_{\{\lambda_1=0\}} \mathscr{O}_{11} = \prod_{k=2}^{N} \mathbb{E}\left(1 + \frac{1}{N} + \frac{1}{N\gamma_{V,k}}\right),$$

and according to Lemma 2.13,

$$\mathbb{E}\left(\frac{1}{\gamma_{V,k}}\right) = \frac{\beta(N+2-k, k-1)}{\beta(N+1-k, k)} = \frac{N+1-k}{k-1},$$

so that the expectation is given by the telescopic product

$$\mathbb{E}_{\{\lambda_1=0\}} \mathscr{O}_{11} = \prod_{k=2}^{N} \frac{k}{k-1} = N$$

as was claimed. □

Proposition 2.8. *Conditionally on $\{\lambda_1 = 0\}$, the following convergence in distribution takes place:*

$$\frac{1}{N} \mathscr{O}_{11} \xrightarrow[N\to\infty]{d} \frac{1}{\gamma_2}.$$

The proof relies on the multiplicative version of the elementary Lemma 1.6 presented in the introduction.

Proof of Proposition 2.8. We first recall how convergence to γ_2^{-1} arises for the complex Ginibre ensemble; part of the argument then relies on comparison with this case, treated in [5]. The reason why this situation is more tractable is that the distribution of the diagonal overlap yields an exact expression: using a few classical identities of the beta and gamma distributions, we see that

$$N\mathscr{O}_{11}^{-1} \overset{d}{=} N\prod_{k=2}^{N}\left(1+\frac{\gamma_1^{(k)}}{\gamma_k}\right)^{-1} \overset{d}{=} N\prod_{k=2}^{N}\beta_{k,1} \overset{d}{=} N\beta_{2,N-1} \xrightarrow[N\to\infty]{d} \gamma_2.$$

Now, for any sequence of integers $(k_n)_{n\geqslant 1}$ such that

$$1 \leqslant k_n \leqslant n, \quad k_n \xrightarrow[n\to\infty]{} \infty, \tag{2.17}$$

the same product can be decomposed as

$$k_N\prod_{k=2}^{k_N}\left(1+\frac{\gamma_1^{(k)}}{\gamma_k}\right)^{-1} \times \frac{N}{k_N}\prod_{k=k_N+1}^{N}\left(1+\frac{\gamma_1^{(k)}}{\gamma_k}\right)^{-1} \overset{d}{=} k_N\beta_{2,k_N-1} \times \frac{N}{k_N}\beta_{k_N+1,N-1}.$$

It is straightforward to check that

$$k_N\beta_{2,k_N-1} \xrightarrow[N\to\infty]{d} \gamma_2, \qquad \frac{N}{k_N}\beta_{k_N+1,N} \xrightarrow[N\to\infty]{d} 1.$$

In other words, the limit distribution γ_2 essentially depends on the first k_N factors, provided k_N goes to infinity. Similarly in the spherical case, using Theorem 2.6 and Proposition 1.4, we write:

$$\begin{aligned} N\mathscr{O}_{11}^{-1} \overset{d}{=} k_N\prod_{k=2}^{k_N}\left(1+\frac{1+\gamma_V(k)}{\gamma_V(k)}X_N^{(k)}\right)^{-1} &\times \frac{N}{k_N}\prod_{k=k_N+1}^{N}\left(1+\frac{1+\gamma_V(k)}{\gamma_V(k)}X_N^{(k)}\right)^{-1} \\ &=: F(2,k_N)\times F(k_N+1,N). \end{aligned}$$

We will prove that the first factor $F(2,k_N)$ converges to γ_2 for a suitable sequence k_N that allows comparison with the complex Ginibre case, whereas the second factor $F(k_N+1,N)$ converges to 1. By the identity (2.8), the independent variables involved are distributed as follows:

$$F_{N,k} := 1+\frac{1+\gamma_V(k)}{\gamma_V(k)}X_N^{(k)} \overset{d}{=} 1+\frac{X_N}{\beta_{k,N+1-k}},$$

where X_N is defined by (2.5).

Convergence of $F(2,k_N)$ to γ_2, for a suitable sequence (k_N). For fixed k, each term $F_{N,k}$ converges to its analog in the complex Ginibre case. Indeed,

$$NX_N \xrightarrow[N\to\infty]{d} \gamma_1, \quad \text{and} \quad N\beta_{k,N-k+1} \xrightarrow[N\to\infty]{d} \gamma_k,$$

so that

$$F_{N,k} \overset{d}{=} 1+\frac{NX_N}{N\beta_{k,N-k+1}} \xrightarrow[N\to\infty]{d} 1+\frac{\gamma_1}{\gamma_k}.$$

The function $x\mapsto x^{-m}$ being smooth and bounded on $(1,\infty)$ for any integer m, we have that

$$\mathbb{E}F_{N,k}^{-m} \xrightarrow[N\to\infty]{} \mathbb{E}\left(1+\frac{\gamma_1}{\gamma_k}\right)^{-m}$$

and so, by the multiplicative version of Lemma 1.6 applied to the appropriate fraction of moments, there exists a sequence k_n verifying (2.17), such that for every m,

$$\prod_{k=2}^{k_N} \frac{\mathbb{E}F_{N,k}^{-m}}{\mathbb{E}\left(1+\frac{\gamma_1}{\gamma_k}\right)^{-m}} \to 1,$$

which implies, by comparison with the product arising in the complex Ginibre case,

$$\mathbb{E}\left(F(2,k_N)^m\right) = k_N^m \mathbb{E}\prod_{k=2}^{k_N} F_{N,k}^{-m} \sim k_N^m \mathbb{E}\prod_{k=2}^{k_N}\left(1+\frac{\gamma_1}{\gamma_k}\right)^{-m} \to (m+1)! = \mathbb{E}\gamma_2^m,$$

so that we have

$$F(2,k_N) \xrightarrow[N\to\infty]{d} \gamma_2.$$

Convergence of $F(k_N+1,N)$ to the constant 1. Let k_n be the sequence of integers used in the first part of the argument; in particular, it satisfies (2.17). We check that this is enough to ensure the convergence of $F(k_N+1,N)$ to 1. A straightforward computation, similar to the one performed in Proposition 2.7, yields

$$\mathbb{E}F_{N,k} = \frac{k}{k-1}, \qquad \mathbb{E}F_{N,k}^2 = \frac{k}{k-2},$$

so that, thanks to telescopic products, we obtain the following expressions

$$\mathbb{E}\left(\prod_{k=k_N+1}^{N} F_{N,k}\right) = \frac{N}{k_N}, \qquad \mathbb{E}\left(\prod_{k=k_N+1}^{N} F_{N,k}^2\right) = \frac{N(N-1)}{k_N(k_N-1)}.$$

As k_N verifies condition (2.17),

$$\mathbb{E}\left(F(k_N+1,N)^{-1}\right) = 1, \qquad \mathrm{Var}\left(F(k_N+1,N)^{-1}\right) = \frac{N-k_N}{N(k_N-1)} \to 0,$$

which proves that $F(k_N+1,N)^{-1} \xrightarrow[N\to\infty]{L^2} 1$, and in particular $F(k_N+1,N) \xrightarrow[N\to\infty]{d} 1$, concluding the second half of the proof. The claim of the Theorem follows by Slutsky's theorem. □

The following proposition relies on the spherical structure of $\mathrm{Sph}(N)$ and has no analog in Section 3.

Fact 2.9. *The distribution of $\mathscr{O}_{11}$ conditionally on the event $\{\lambda_1 = z \in \mathbb{C}\}$ does not depend on z.*

Proof. Recall that the Jacobian of p at $\lambda \in \mathbb{C}$ is $\frac{4}{(1+|\lambda|^2)^2}$ and that, for any $\lambda, \mu \in \mathbb{C}$, identity (2.12) holds. For any continuous and bounded function F of $N-1$ variables, evaluated in

$$l_k := \frac{4|\lambda_1-\lambda_k|^2}{(1+|\lambda_1|^2)(1+|\lambda_k|^2)} \quad k=2,\dots,N$$

we have for any $z \in \mathbb{C}$, by a straightforward change of variables,

$$\mathbb{E}_{\{\lambda_1=z\}}\left(F(l_2,\dots,l_N)\right) = \mathbb{E}_{\{w_1=p(z)\}}\left(F\left(\|w_1-w_2\|^2,\dots,\|w_1-w_N\|^2\right)\right), \tag{2.18}$$

where $(w_1,\dots,w_N)$ is a point process on the sphere with density proportional to (2.11). As the expectation on the right hand side does not depend on z (by invariance under orthogonal transformations), neither does the one on the left hand side. The claim

follows by noting that for any continuous and bounded function G, by the tower property of conditional expectation,

$$\mathbb{E}_{\{\lambda_1=z\}}G(\mathscr{O}_{11}) = \mathbb{E}_{\{\lambda_1=z\}}F(l_2,\dots,l_N),$$

where $F(l_2,\dots,l_N) := \mathbb{E}_\Lambda G(\mathscr{O}_{11})$ is indeed a function of the variables $l_2,\dots,l_N$. □

Clearly, Propositions 2.7, 2.8 and Fact 2.9 provide together a full proof of Theorem 1.1. The explicit formula of the conditional expectation can also be obtained by a direct computation that has an analog in Section 3 – we give a proof below in order to illustrate this analogy.

Proposition 2.10. *The conditional expectation of diagonal overlaps with respect to λ_1 in the spherical ensemble is*

$$\mathbb{E}_{\lambda_1}(\mathscr{O}_{11}) = N.$$

Proof. By the tower property of conditional expectation,

$$\mathbb{E}_{\lambda_1}(\mathscr{O}_{11}) = \mathbb{E}_{\lambda_1}\mathbb{E}_\Lambda(\mathscr{O}_{11}) = \mathbb{E}_{\lambda_1}\left(\prod_{k=2}^{N} 1 + \frac{(1+|\lambda_1|^2)(1+|\lambda_k|^2)}{N|\lambda_1-\lambda_k|^2}\right).$$

Using Proposition 1.5 together with the specific potential for the spherical ensemble, we find

$$\mathbb{E}_{\lambda_1}\left(\prod_{k=2}^{N} 1 + \frac{(1+|\lambda_1|^2)(1+|\lambda_k|^2)}{N|\lambda_1-\lambda_k|^2}\right) = \frac{1}{Z_N^{(1)}}\det\left(f_{i,j}\right)_{i,j=1}^{N-1} =: \frac{1}{Z_N^{(1)}}D_{N-1},$$

where the determinants D_k are Hermitian, tridiagonal and 'nested' (i.e. D_k is the principal minor of D_{k+1}, as in the proof of Proposition 1.5), with coefficients

$$\begin{aligned} f_{k,k} &= |\lambda_1|^2\Gamma_V(k) + \Gamma_V(k+1) + \frac{1}{k}(1+|\lambda_1|^2)\Gamma_V(k+1) \\ f_{k+1,k} &= \overline{f_{k,k+1}} = -\lambda_1\Gamma_V(k+1). \end{aligned}$$

This gives the initial values $D_0 = 1$, $D_1 = |\lambda_1|^2\Gamma_V(1) + \Gamma_V(2) + (1+|\lambda_1|^2)\Gamma_V(2)$ and the induction

$$\begin{aligned} D_k &= f_{k,k}D_{k-1} - |f_{k,k-1}|^2 D_{k-2} \\ &= \left(|\lambda_1|^2\Gamma_V(k) + \Gamma_V(k+1) + \frac{1}{k}(1+|\lambda_1|^2)\Gamma_V(k+1)\right)D_{k-1} - |\lambda_1|^2\Gamma_V(k)^2 D_{k-2}. \end{aligned}$$

We use the shorthands (1.19) and pose $X = |\lambda_1|^2$. Note that in the spherical case, with potential $V_N(t) = (N+1)\ln(1+t)$,

$$\Gamma_{V_N}(k) = \frac{(k-1)!(N-k)!}{N!} = \frac{1}{N}\binom{N-1}{k-1}^{-1}, \qquad \gamma_k := \frac{\Gamma_{V_N}(k)}{\Gamma_{V_N}(k+1)} = \frac{N}{k} - 1,$$

and $e_{V_N}^{(N-1)}(X) = N(1+X)^{N-1}$ coherently with Lemma 2.13. This gives $\tilde{D}_0 = N$,

$$\tilde{D}_1 = \frac{X}{\Gamma_V(2)} + \frac{1}{\Gamma_V(1)} + \frac{(1+X)}{\Gamma_V(1)} = N^2X + 2N,$$

and the three-terms induction

$$\tilde{D}_k = \left(X\gamma_k + 1 + \frac{1}{k}(1+X)\gamma_k\right)\tilde{D}_{k-1} - X\gamma_k\tilde{D}_{k-2},$$

which defines a sequence of polynomials in X; the next one is

$$\tilde{D}_2 = N\left(3 + 2NX + \frac{N(N-1)}{2}X^2\right).$$

These first terms suggest the general formula below, which can be checked by a direct computation:

$$\tilde{D}_k = N\sum_{l=0}^{k}(k-l+1)\binom{N}{l}X^l. \tag{2.19}$$

It follows from the above formula and the one for $Z_N^{(1)}$ in Proposition 1.5 that the expectation is given by

$$\frac{D_{N-1}}{Z_N^{(1)}} = \frac{\tilde{D}_{N-1}}{e_{V_N}^{(N-1)}} = \frac{N^2(1+X)^{N-1}}{N(1+X)^{N-1}} = N,$$

which is the claim. □

Theorem 2.11. *The quenched expectation of off-diagonal overlaps in the spherical ensemble is given by the formula*

$$\mathbb{E}_\Lambda\left(\mathscr{O}_{12}\right) = -\frac{1}{N|\lambda_1-\lambda_2|^2}\prod_{k=3}^{N}\left(1+\frac{(1+\lambda_1\overline{\lambda_2})(1+|\lambda_k|^2)}{N(\lambda_1-\lambda_k)(\overline{\lambda_2-\lambda_k})}\right). \tag{2.20}$$

Proof. Similarly to the diagonal case, we define the partial sums

$$\mathscr{O}_{12}^{(d)} := -\overline{b_2}\sum_{i=2}^{d} b_i\overline{d_i}.$$

It follows from the facts presented in Section 1.2 that

$$\mathscr{O}_{12}^{(2)} = -|b_2|^2 = \frac{-|u_2|^2}{|\lambda_1-\lambda_2|^2}.$$

One can check, following the proof of Theorem 2.6, that $|u_2|^2 \stackrel{d}{=} X_N$, so that

$$\mathbb{E}|u_2|^2 = \frac{1}{N} \quad \text{and} \quad \mathbb{E}_\Lambda\mathscr{O}_{12}^{(2)} = \frac{-1}{N|\lambda_1-\lambda_2|^2},$$

which initiates the recurrence. We now compute the conditional expectation of $b_{n+1}\overline{d_{n+1}}$ by integrating out the vector u_{n+1}. We use Proposition 2.3 and (2.26) from Lemma 2.15 with $a = B_n^*$, $b = D_n^*$ and $S = S_n$ such that $S_n^2 = (1+|\lambda_n|^2)(I_{n-1}+T_{n-1}T_{n-1}^*)$. It follows that

$$\begin{aligned}\mathbb{E}_{\Lambda,\mathscr{F}_{n-1}}b_{n+1}\overline{d_{n+1}} &= \frac{1}{N(\lambda_1-\lambda_{n+1})(\overline{\lambda_2-\lambda_{n+1}})}B_nS^2D_n^* \\ &= \frac{(1+|\lambda_{n+1}|^2)}{N(\lambda_1-\lambda_{n+1})(\overline{\lambda_2-\lambda_{n+1}})}(B_nD_n^* + B_nTT^*D_n^*).\end{aligned}$$

We notice that, as T is triangular and B_n, D_n are subvectors of L_1 and L_2,

$$B_nT_n = \lambda_1B_n, \quad D_nT_n = \lambda_2D_n,$$

which gives

$$-\overline{b}_2\mathbb{E}_{\Lambda,\mathscr{F}_n}b_{n+1}\overline{d_{n+1}} = \frac{(1+|\lambda_{n+1}|^2)(1+\lambda_1\overline{\lambda_2})}{N(\lambda_1-\lambda_{n+1})(\overline{\lambda_2-\lambda_{n+1}})}\mathscr{O}_{12}^{(n)}.$$

The factorization follows. □

Proposition 2.12. *The conditional expectation of* $\frac{1}{N}\operatorname{Tr} G^*G$ *with* G *distributed according to* $\mathrm{Sph}(N)$ *is given by the formula:*

$$\mathbb{E}_\Lambda\left(\frac{1}{N}\operatorname{Tr} G^*G\right) = \prod_{i=1}^{N}\left(1+\frac{1+|\lambda_i|^2}{N}\right) - 2.$$

Proof. It is clear that $\operatorname{Tr} G^*G = \operatorname{Tr} T_N^*T_N$, and that for any $n \leqslant N$,

$$\operatorname{Tr} T_nT_n^* = |\lambda_n|^2 + \|u_n\|^2 + \operatorname{Tr} T_{n-1}T_{n-1}^*,$$

so that defining

$$v_{N,n} = v_{N,n}(\lambda_1,\dots,\lambda_n) := \mathbb{E}_{N,\Lambda}\operatorname{Tr} T_nT_n^*,$$

yields a recursion with $v_{N,1} = |\lambda_1|^2$ and, using Proposition 2.3 and (2.27) from Lemma 2.15,

$$v_{N,n+1} \quad = \quad v_{N,n}\left(1+\frac{1+|\lambda_{n+1}|^2}{N}\right) + |\lambda_{n+1}|^2 + \frac{n}{N}\left(1+|\lambda_{n+1}|^2\right). \tag{2.21}$$

This suggests the introduction of $w_{N,n} = v_{N,n} + N + n$, for which we see that

$$w_{N,1} = N\left(1+\frac{1+|\lambda_1|^2}{N}\right) \quad \text{and} \quad w_{N,n+1} = w_{N,n}\left(1+\frac{1+|\lambda_{n+1}|^2}{N}\right),$$

so that for every $n \leqslant N$,

$$\frac{1}{N}v_{N,n} = \prod_{i=1}^{N}\left(1+\frac{1+|\lambda_i|^2}{N}\right) - \left(1+\frac{n}{N}\right),$$

which is equivalent to the statement, when $n = N$. □

2.3 Constants and integrals

Lemma 2.13. *The normalization constant for generalized gamma variables* $\gamma_V(k)$ *with potential* $V(x) = (N+1)\log(1+x)$ *and* $1 \leqslant k \leqslant N$ *is*

$$\int_{\mathbb{R}_+}\frac{x^{k-1}}{(1+x)^{N+1}}\mathrm{d}x = \beta(N+1-k,k),$$

and $\gamma_V(k) \overset{d}{=} \frac{1}{\beta_{N+1-k,k}} - 1$. *Moreover, the associated function* $e_V^{(N-1)}$ *is given by*

$$e_V^{(N-1)} = N(1+X)^{N-1}.$$

Proof. Let us compute, for any suitable function f,

$$\begin{aligned}\int_{\mathbb{R}_+}\frac{x^{k-1}}{(1+x)^{N+1}}f(x)\mathrm{d}x &= \int_0^1 x^{N-k}(1-x)^{k-1}f\left(\frac{1}{x}-1\right)\mathrm{d}x \\ &= \beta(N+1-k,k)\mathbb{E}f\left(\frac{1}{\beta_{N+1-k,k}}-1\right),\end{aligned}$$

which implies the first claims; the next ones follow naturally. □

Lemma 2.14. *For any* $p > n$,

$$C_{n,p} := \int_{z\in\mathbb{C}^n}\frac{1}{\left(1+\sum_{i=1}^n|z_i|^2\right)^p}\mathrm{d}m(z_1)\dots\mathrm{d}m(z_n) = \pi^n\frac{(p-n-1)!}{(p-1)!}, \tag{2.22}$$

and for $p > n+1$,

$$C_{n,p}^{(1)} := \int_{z\in\mathbb{C}^n}\frac{|z_1|^2}{\left(1+\sum_{i=1}^n|z_i|^2\right)^p}\mathrm{d}m(z_1)\dots\mathrm{d}m(z_n) = \frac{1}{p-(n+1)}C_{n,p}. \tag{2.23}$$

Proof. We first compute $C_{n,p}$ by induction on n. For $n = 1, p > 1$,

$$C_{1,p} = \int_{z\in\mathbb{C}} \frac{1}{(1+|z|^2)^p} \mathrm{d}m(z) = \pi \int_{r=1}^{\infty} \frac{1}{r^p} \mathrm{d}r = \frac{\pi}{p-1}$$

and one can note that for any $\alpha > 0$,

$$\int_{z\in\mathbb{C}} \frac{1}{(1+\alpha^{-1}|z|^2)^p} \mathrm{d}m(z) = \frac{\pi\alpha}{p-1}.$$

For general n, using the above equalities with $\alpha_n = 1 + \sum_{i=1}^{n-1} |z_i|^2$,

$$\begin{aligned} C_{n,p} &= \int_{z\in\mathbb{C}^n} \frac{1}{\left(1+\sum_{i=1}^{n-1}|z_i|^2\right)^p} \times \frac{1}{\left(1+\alpha_n^{-1}|z_n|^2\right)^p} \mathrm{d}m(z_1)\dots\mathrm{d}m(z_n) \\ &= \frac{\pi}{p-1} \int_{z\in\mathbb{C}^{n-1}} \frac{1}{\left(1+\sum_{i=1}^{n-1}|z_i|^2\right)^{p-1}} \mathrm{d}m(z_1)\dots\mathrm{d}m(z_{n-1}) = \frac{\pi}{p-1} C_{n-1,p-1}. \end{aligned}$$

Equation (2.22) follows. A similar induction can be performed on $C^{(1)}_{n,p}$. The only difference is that the last step involves the following identity: for any $p > 2$,

$$C^{(1)}_{1,p} = \int_{z\in\mathbb{C}} \frac{|z|^2}{(1+|z|^2)^p} \mathrm{d}m(z) = \pi \int_{r=1}^{\infty} \frac{r-1}{r^p} \mathrm{d}r = \pi \left(\frac{1}{1-p} - \frac{1}{2-p}\right) = \frac{\pi}{(p-1)(p-2)},$$

which, in general, yields the extra factor $\frac{1}{p-(n+1)}$ in (2.23). □

Note that when we begin the recursion from [11] with $n = N-1, p = 2N$, the extra factor is $\frac{1}{N}$ at every step.

Lemma 2.15. *For any $p > n$, $a, b \in \mathbb{C}^n$ and any Hermitian positive-definite matrix S,*

$$\int_{\mathbb{C}^n} \frac{1}{(1+u^*S^{-2}u)^p} \mathrm{d}u = C_{n,p} |\det S|^2, \tag{2.24}$$

$$\int_{\mathbb{C}^n} \frac{a^*u}{(1+u^*S^{-2}u)^p} \mathrm{d}u = 0, \tag{2.25}$$

$$\int_{\mathbb{C}^n} \frac{(a^*u)(u^*b)}{(1+u^*S^{-2}u)^p} \mathrm{d}u = C_{n,p} |\det S|^2 \frac{a^*S^2b}{p-(n+1)}, \tag{2.26}$$

$$\int_{\mathbb{C}^n} \frac{\|u\|^2}{(1+u^*S^{-2}u)^p} \mathrm{d}u = C_{n,p} |\det S|^2 \frac{\operatorname{Tr} S^2}{p-(n+1)}, \tag{2.27}$$

where the constant $C_{n,p}$ is explicitly computed in Lemma 2.14.

Proof. Integral (2.24) was computed in [11]. (2.25) is zero by symmetry. For (2.26), the change of variables $u = Sv$ yields

$$|\det S|^2 \int \mathrm{d}v \frac{(a^*Sv)(v^*Sb)}{(1+v^*v)^p}.$$

We notice that

$$(a^*Sv)(v^*Sb) = v^*(Sba^*S)v = v^*Av,$$

where $A = Sba^*S$ is a matrix of rank 1. If we express $v = \sum v_i e_i$ in a unitary basis such that the vectors $(e_2, \dots, e_n)$ form a basis of $\ker(A)$ and denote $Ae_1 = \lambda_1(A)e_1 + \sum_{i\geqslant 2} \alpha_i e_i$,

$$v^*Av = \lambda_1(A)v_1^2 + \sum_{i\geqslant 2} \alpha_i v_1 v_i.$$

Therefore, after a unitary change of basis the integral becomes, using Lemma 2.14 and the fact that cross-terms $v_1 v_i$ vanish by symmetry,

$$\int \mathrm{d}v \frac{\lambda_1(A) v_1^2}{(1+v_1^2+\cdots+v_n^2)^p} = \frac{\lambda_1(A)}{p-(n+1)} C_{n,p}.$$

The value of $\lambda_1(A)$ can be obtained by writing

$$\lambda_1(A) = \operatorname{Tr} Sba^*S = a^*S^2b,$$

from which the claim (2.26) follows. The same technique applied to (2.27) yields

$$|\det S|^2 \int \mathrm{d}v \frac{\|Sv\|^2}{(1+v^*v)^p}.$$

and a unitary change of variable to a basis that diagonalizes S, together with Lemma 2.14, gives

$$\int \mathrm{d}v \frac{\lambda_1(S^2)v_1^2+\cdots+\lambda_n(S^2)v_n^2}{(1+v_1^2+\cdots+v_n^2)^p} = (\lambda_1(S^2)+\cdots+\lambda_n(S^2)) \frac{1}{p-(n+1)} C_{n,p},$$

concluding the proof of the last claim. □

Lemma 2.16. *For any $p > n$, $a \in \mathbb{C}^n$ and any Hermitian positive-definite matrix S, if $u \in \mathbb{C}^n$ is distributed with density*

$$\frac{1}{C_{n,p}|\det S|^2} \frac{1}{(1+u^*S^{-2}u)^p}$$

with respect to the Lebesgue measure on $\mathbb{C}^n$, then the following identity in distribution holds:

$$|a^*u|^2 \stackrel{d}{=} \|Sa\|^2 X_{p-n-1}.$$

Proof. By a direct change of variable, it is clear that $u \stackrel{d}{=} S\mathscr{V}_p^n$. We note that $|a^*Sv|^2 = v^*Av$ where $A = Saa^*S$ is a Hermitian matrix of rank one. A unitary change of variable brings it to the form $\lambda_1(A)v_1^2$ with $\lambda_1(A) = \operatorname{Tr} A = a^*S^2a = \|Sa\|^2$. Successive integration of the other coordinates $v_2, \ldots, v_n$ yields the result. □

3 Truncated unitary ensemble

This section contains the proof of all claims concerning the truncated unitary ensembles $\mathrm{TUE}(N, M)$ when $N \leqslant M$. Almost every step in this study is analogous to what was done in the spherical case; we therefore refer constantly to the corresponding parts of Section 2. The condition $N \leqslant M$ is an essential requirement here, as it was in [11], for the integration techniques to apply. The cases where $N > M$ require a different approach and will be the subject of a separate paper. Note that the particular case $M = 1$ can be treated by the analytical techniques exposed in [13]. It follows in particular that, in this weakly non-unitary case, the overlaps are functions of the eigenvalues, with no extra randomness.

3.1 Schur form and eigenvalues

As in Section 2, we first present a few general results in order to illustrate the method, as well as a few tools and definitions that are specific to the truncated unitary case. We first recall that the Schur transfom T is distributed with density proportional to

$$\prod_{i<j} |\lambda_i - \lambda_j|^2 \det(I_N - TT^*)^{M-N} \mathbb{1}_{TT^*<1} \tag{3.1}$$

with respect to the Lebesgue measure on all complex matrix elements, diagonal ($\mathrm{d}\Lambda = \mathrm{d}\lambda_1 \cdots \mathrm{d}\lambda_N$) and upper-triangular ($\mathrm{d}u_2 \cdots \mathrm{d}u_n$).

Provided $TT^* < 1$ (which implies the same condition on every submatrix T_n), we introduce the Hermitian, definite-positive matrices

$$H_n := I_n - T_n T_n^*, \qquad S_{n-1} := (1 - |\lambda_n|^2)^{1/2} H_{n-1}^{1/2}. \tag{3.2}$$

Note that the only differences with the matrices H_n, S_{n-1} used in the spherical case are the minus sign and the condition on the eigenvalues of TT^*.

Lemma 3.1. *The determinant of $H_n = I_n - T_n T_n^*$ can be recursively decomposed as*

$$\det(H_n) = (1 - |\lambda_n|^2) \det(H_{n-1}) \left(1 - \frac{1}{1 - |\lambda_n|^2} u_n^* H_{n-1}^{-1} u_n\right). \tag{3.3}$$

The proof is analogous to the proof of Lemma 2.1.

For any $p \geqslant 0$, we denote by $\mathscr{W}_p^{(n)}$ a random vector with density

$$\frac{1}{C_{n,p}} (1 - v^* v)^p \mathbb{1}_{v^* v < 1} \tag{3.4}$$

with respect to the Lebesgue measure on $\mathbb{C}^n$; the value of $C_{n,p}$ is given by (3.15). For any $m \geqslant 2$, we denote by Y_m a real random variable with density

$$(m-1)(1-y)^{m-2} \mathbb{1}_{(0,1)} \tag{3.5}$$

with respect to the Lebesgue measure, i.e. it follows a $\beta_{1,m-1}$ distribution; in particular $\mathbb{E}Y_m = \frac{1}{m}$. If w_i is a coordinate of $\mathscr{W}_p^{(n)}$, it follows from Lemma 3.14 that

$$|w_i|^2 \overset{d}{=} Y_{p+n+1}.$$

Note that the i.i.d. variables that appear in Theorem 3.5 follow the above distribution with $m = M$.

Lemma 3.2. *Identity holds between the following expressions, for $p \geqslant n$ and f, g integrable functions of the matrix elements:*

$$\int f(T_{n-1}, \lambda_n) g(u_n) \det(H_n)^p \mathbb{1}_{T_n T_n^* < 1} \mathrm{d}T_n = C_{n-1,p} \times$$
$$\int f(T_{n-1}, \lambda_n) \mathbb{E}\left(g(S_{n-1} \mathscr{W}_p^{(n-1)})\right) (1 - |\lambda_n|^2)^{p+n-1} \det(H_{n-1})^{p+1} \mathbb{1}_{T_{n-1} T_{n-1}^* < 1} \mathrm{d}T_{n-1} \mathrm{d}\lambda_n,$$

where $H_n, S_{n-1}, \mathscr{W}_p^{(n)}$ are defined in (3.2) and (3.4).

We deduce from the above Lemma the distribution of every top-left submatrix of the Schur form, analogously to Proposition 2.3.

Proposition 3.3. *Conditionally on Λ and for $2 \leqslant n \leqslant N$, the submatrix T_n of the Schur transform is distributed with density proportional to*

$$\det(I_n - T_n T_n^*)^{M-n} \mathbb{1}_{T_n T_n^* \leqslant 1} \tag{3.6}$$

with respect to the Lebesgue measure on upper-triangular matrix elements ($\mathrm{d}u_2 \cdots \mathrm{d}u_n$).

We also derive the joint eigenvalue density of the truncated unitary ensemble from the density of its Schur form, as was done in [11]. The result itself was first proven in [19].

Theorem 3.4 (Życzkowski & Sommers)**.** *The joint density of eigenvalues for the truncated unitary ensemble when $M \geqslant N$ is proportional to*

$$\frac{1}{Z_{M,N}} \prod_{1 \leqslant i < j \leqslant N} |\lambda_i - z_j|^2 \prod_{i=1}^{N} (1 - |\lambda_i|^2)^{M-1} \mathbb{1}_{\mathbb{D}}(\lambda_i) \tag{3.7}$$

with respect to the Lebesgue measure on $\mathbb{C}^N$.

The proof is analogous to the one of Theorem 2.4.

Theorem 3.4 can be rephrased by saying that the eigenvalues of $\mathrm{TUE}(N, M)$ are distributed according to (1.18) with potential $V(t) = V_M(t) = -(M-1)\ln(1-t)\mathbb{1}_{(0,1)}$. A straightforward computation shows that in that case

$$\Gamma_V(\alpha) = \beta(\alpha, M), \qquad \gamma_V(\alpha) \stackrel{d}{=} \beta_{\alpha,M}. \tag{3.8}$$

Thus, Kostlan's theorem in that case asserts that the set of squared radii is distributed as a set of independent β variables. Namely,

$$\{|\lambda_1|^2, \dots, |\lambda_k|^2\} \stackrel{d}{=} \{\beta_{1,M}, \dots, \beta_{k,M}\}.$$

3.2 Distribution and conditional expectation of overlaps

Theorem 3.5. *Conditionally on $\{\Lambda = (\lambda_1, \dots, \lambda_N)\}$, diagonal overlaps in the truncated unitary ensemble $\mathrm{TUE}(N, M)$ are distributed as*

$$\mathscr{O}_{11} \stackrel{d}{=} \prod_{k=2}^{N} \left(1 + \frac{(1-|\lambda_1|^2)(1-|\lambda_k|^2)}{|\lambda_1 - \lambda_k|^2} Y_M^{(k)}\right), \tag{3.9}$$

where the $Y_M^{(k)}$ are i.i.d. distributed according to (3.5) with $m = M$. In particular, the quenched expectation is given by the formula

$$\mathbb{E}_\Lambda(\mathscr{O}_{11}) = \prod_{k=2}^{N} \left(1 + \frac{(1-|\lambda_1|^2)(1-|\lambda_k|^2)}{M|\lambda_1 - \lambda_k|^2}\right). \tag{3.10}$$

Proof. It is similar to the one of Theorem 2.6; we sketch it again to see where the differences lie. We first write

$$\mathscr{O}_{11}^{(d+1)} = \mathscr{O}_{11}^{(d)} + |b_{d+1}|^2 = \mathscr{O}_{11}^{(d)} \left(1 + \frac{1}{|\lambda_1 - \lambda_{d+1}|^2} \frac{|B_d u_{d+1}|^2}{\|B_d\|^2}\right).$$

In order to characterize the distribution of this factor, we use Proposition 3.3, then Lemma 3.2 and Lemma 3.14 with $a = b = \overline{B_d}$ and $S = S_{d+1}$ such that $S_{d+1}^2 = (1 - |\lambda_{d+1}|^2)(I_d - T_d T_d^*)$. This yields

$$|B_d u_{d+1}|^2 \stackrel{d}{=} (1 - |\lambda_{d+1}|^2) \|(I_d - T_d T_d^*)\overline{B_d}\|^2 Y_N, \tag{3.11}$$

where Y_N is distributed according to (3.5) with $m = M$, and independent of $\mathscr{F}_d$; we denote this variable by $Y_N^{(d+1)}$ to avoid confusion. The last steps of the proof follow accordingly. □

Proposition 3.6. *Conditionally on $\{\lambda_1 = 0\}$, the expectation of the diagonal overlap $\mathscr{O}_{11}$ in the truncated unitary ensemble $\mathrm{TUE}(N, M)$ is*

$$\mathbb{E}_{\{\lambda_1 = 0\}} \mathscr{O}_{11} = N.$$

Proof. We know from Proposition 1.4 that the squared radii, conditionally on the event $\{\lambda_1 = 0\}$, are distributed like independent variables with distributions $\gamma_{V,k}$ with $V(x) = -(M-1)\log(1-x)\mathbb{1}_{(0,1)}$ and $2 \leqslant k \leqslant N$. We already noticed that $\gamma_{V,k} \overset{d}{=} \beta_{k,M}$. A straightforward computation follows:

$$\mathbb{E}_{\{\lambda_1=0\}}\mathscr{O}_{11} = \mathbb{E}\prod_{k=2}^{N}\left(1+\frac{1-|\lambda_k|^2}{M|\lambda_k|^2}\right) = \prod_{k=2}^{N}\mathbb{E}\left(1-\frac{1}{M}+\frac{1}{M\beta_{k,M}}\right).$$

For any $k \geqslant 2$,

$$\mathbb{E}\left(\frac{1}{\beta_{k,M}}\right) = \frac{\beta(k-1,M)}{\beta(k,M)} = \frac{M+k-1}{k-1},$$

so that the expectation is given by the telescopic product

$$\mathbb{E}_{\{\lambda_1=0\}}\mathscr{O}_{11} = \prod_{k=2}^{N}\frac{k}{k-1} = N$$

as was claimed. □

Proposition 3.7. *Conditionally on $\{\lambda_1 = 0\}$, the following convergence in distribution takes place:*

$$\frac{1}{N}\mathscr{O}_{11} \xrightarrow[N\to\infty]{d} \frac{1}{\gamma_2}.$$

Note that $N \to \infty$ implies $M \to \infty$, as we study the truncated unitary ensemble in the regime where $N \leqslant M$. The rate at which N, M go to infinity does not have any impact on the following proof (although it is expected to play a role when conditioning on a generic z in the bulk).

Proof. The technique is similar to the proof of Proposition 2.8 and also relies on the the multiplicative version of Lemma 1.6. We decompose the distribution obtained by Theorem 3.5 in two factors

$$N\mathscr{O}_{11}^{-1} \overset{d}{=} k_N\prod_{k=2}^{k_N}\left(1+\frac{1-\gamma_V(k)}{\gamma_V(k)}Y_M^{(k)}\right)^{-1} \times \frac{N}{k_N}\prod_{k=k_N+1}^{N}\left(1+\frac{1-\gamma_V(k)}{\gamma_V(k)}Y_M^{(k)}\right)^{-1}$$
$$=: G(2,k_N)\times G(k_N+1,N).$$

As $\gamma_V(k) \overset{d}{=} \beta_{k,M}$, we have

$$G_{M,k} := 1+\frac{1-\gamma_V(k)}{\gamma_V(k)}Y_M^{(k)} \overset{d}{=} 1+\left(\frac{1}{\beta_{k,M}}-1\right)Y_M,$$

where Y_M is defined by (3.5). The proof then proceeds in two separate parts.

Convergence of $G(2,k_N)$ to γ_2 for a suitable sequence k_N. It is straightforward to check that for every k, the term $G_{M,k}$ converges to the factor playing an analogous role in the complex Ginibre case. Indeed,

$$MY_M \xrightarrow[N\to\infty]{d} \gamma_1, \quad \text{and} \quad M\beta_{k,M} \xrightarrow[N\to\infty]{d} \gamma_k,$$

so that

$$G_{M,k} \overset{d}{=} 1+\left(\frac{1}{M\beta_{k,M}}-\frac{1}{M}\right)MY_M \xrightarrow[N\to\infty]{d} 1+\frac{\gamma_1}{\gamma_k}.$$

The argument then proceeds exactly as in Proposition 2.8: by Lemma 1.6, there exists a sequence k_N that verifies (2.17) and such that we can derive the convergence

$$G(2,k_N) \xrightarrow[N\to\infty]{d} \gamma_2$$

by comparison with the complex Ginibre case.

Convergence of $G(k_N+1,N)$ to 1. It follows from the computation performed in the proof of Proposition 3.6 that

$$\mathbb{E}G_{M,k} = \frac{k}{k-1},$$

which is the same as the expectation of $F_{N,k}$ (and does not depend on M nor N). We compute the second moment, using the values

$$\mathbb{E}Y_M^2 = \frac{2}{M(M+1)}, \qquad \mathbb{E}\left(\frac{1}{\beta_{k,M}}-1\right)^2 = \frac{M(M+1)}{(k-1)(k-2)},$$

and find, as for $F_{N,k}$,

$$\mathbb{E}G_{M,k}^2 = \frac{k}{k-2},$$

so that we obtain the exact same expressions as in the spherical case. The end of the argument is strictly similar to what has been written in the proof of Proposition 2.8. □

The analog of the spherical structure of $\mathrm{Sph}(N)$ for $\mathrm{TUE}(N,M)$ is the stereographic projection on the pseudosphere (see [11]). However, the symmetries of the pseudosphere do not allow to establish an exact equivalent to Fact 2.9. The way in which the distribution of the diagonal overlap $\mathscr{O}_{11}$ depends on the eigenvalue λ_1 in the truncated unitary ensembles is a more delicate question that would require analytical tools beyond the scope of the present work. What can be obtained with techniques analogous to the spherical case is an explicit formula for the conditional expectation with respect to one eigenvalue, which checks the general form predicted in [4].

Proposition 3.8. *The conditional expectation of diagonal overlaps with respect to λ_1 in the truncated unitary ensemble $\mathrm{TUE}(N,M)$ is*

$$\mathbb{E}_{\lambda_1}(\mathscr{O}_{11}) = \frac{NF_{M-1,N}(|\lambda_1|^2) - |\lambda_1|^2\partial F_{M-1,N}(|\lambda_1|^2)}{F_{M,N}(|\lambda_1|^2)},$$

where for any integer parameters a,b,

$$F_{a,b}(X) := \sum_{l=0}^{b-1}\binom{a+l}{l}X^l.$$

In particular, in the regime where the limit $M/N\to\kappa\geqslant 1$ holds, for any z_1 s.t. $|z_1|^2<\frac{1}{\kappa+1}$,

$$\frac{1}{N}\mathbb{E}_{\{\lambda_1=z_1\}}(\mathscr{O}_{11}) \xrightarrow[M\to\infty]{} 1-(1+\kappa)|z_1|^2. \tag{3.12}$$

Note that the limit (3.12) is a special case of eq. (25) in [4]. In that same regime, the eigenvalue limit distribution is on a disk of radius $\frac{1}{\sqrt{\kappa+1}}$, such that for a typical eigenvalue, $|\lambda_1|^2$ has mean density

$$\frac{\kappa}{(1-x)^2}\mathbb{1}_{[0,\frac{1}{\kappa+1}]}.$$

This should be compared to the complex Ginibre case, where the empirical distribution of eigenvalues converge to the circular law on the unit disk, and the corresponding conditional expectation of the diagonal overlap $\mathscr{O}_{11}$ is approximately $N(1-|\lambda_1|^2)\mathbb{1}_{|\lambda_1|<1}$.

Proof. Proposition 1.5, together with the specific potential for the truncated unitary ensemble, yields

$$\mathbb{E}_{\lambda_1}\left(\prod_{i=2}^{N}1+\frac{(1-|\lambda_1|^2)(1-|\lambda_k|^2)}{M|\lambda_1-\lambda_k|^2}\right) = \frac{1}{Z_N^{(1)}}\det\left(f_{i,j}\right)_{i,j=1}^{N-1} =: \frac{1}{G_V(N)e_V^{(N-1)}(|\lambda_1|^2)}D_{N-1},$$

where the determinants D_k are Hermitian, tridiagonal and nested, with matrix coefficients

$$f_{k,k} = |\lambda_1|^2\Gamma_V(k) + \Gamma_V(k+1) + \frac{1}{M}(1-|\lambda_1|^2)\left(\Gamma_V(k) - \Gamma_V(k+1)\right)$$
$$f_{k+1,k} = \overline{f_{k,k+1}} = -\lambda_1\Gamma_V(k+1).$$

This gives the initial values $D_0 = 1$,

$$D_1 = |\lambda_1|^2\Gamma_V(1) + \Gamma_V(2) + \frac{1}{M}(1-|\lambda_1|^2)(\Gamma_V(1) - \Gamma_V(2)),$$

and the three-terms induction $D_k = f_{k,k}D_{k-1} - |f_{k,k-1}|^2 D_{k-2}$.

As in the previous cases, we use the shorthands (1.19) and pose $X = |\lambda_1|^2$. Note that in the truncated-unitary case, with potential $V(t) = -(M-1)\ln(1-t)\mathbb{1}_{(0,1)}$,

$$\Gamma_V(k) = \beta(k,M) = \frac{1}{M}\binom{M+k-1}{k-1}^{-1}, \qquad \gamma_k := \frac{\Gamma_V(k)}{\Gamma_V(k+1)} = \frac{M}{k} + 1,$$

and $e_V^{(N-1)}(X) = MF_{M,N}(X)$, coherently with Lemma 3.11. This gives $\tilde{D}_0 = M$,

$$\tilde{D}_1 = \frac{X}{\Gamma_V(2)} + \frac{1}{\Gamma_V(1)} + \frac{(1-X)}{M}\left(\frac{1}{\Gamma_V(2)} - \frac{1}{\Gamma_V(1)}\right) = M(2+MX),$$

and the three-terms induction

$$\tilde{D}_k = \left(X\gamma_k + 1 + \frac{1}{k}(1-X)\right)\tilde{D}_{k-1} - X\gamma_k\tilde{D}_{k-2},$$

which defines a sequence of polynomials in X; the next one is

$$\tilde{D}_2 = M\left(3 + 2MX + \frac{M(M+1)}{2}X^2\right).$$

These first terms suggest the general solution below, which can be checked by a direct computation:

$$\tilde{D}_k = M\sum_{l=0}^{k}(k-l+1)\binom{M-1+l}{l}X^l,$$

so that in particular

$$\tilde{D}_{N-1}(X) = M\left(NF_{M-1,N}(X) - X\partial_X F_{M-1,N}(X)\right),$$

and the statement follows. In the regime $M/N \to \kappa$ and for fixed $x < \frac{1}{1+\kappa}$, a straightforward computation gives the asymptotics

$$F_{M,N}(x) \sim \frac{1}{(1-x)^{M+1}}, \quad F_{M-1,N}(x) \sim \frac{1}{(1-x)^M}, \quad \partial_x F_{M-1,N}(x) \sim \frac{M}{(1-x)^{M+1}},$$

so that the limit (3.12) follows in the corresponding regime. □

Theorem 3.9. *The quenched expectation of off-diagonal overlaps in* $\mathrm{TUE}(N,M)$ *with* $N \leqslant M$ *is given by the formula*

$$\mathbb{E}_\Lambda\left(\mathscr{O}_{12}\right) = -\frac{1}{M|\lambda_1-\lambda_2|^2}\prod_{k=3}^{N}\left(1 + \frac{(1-\lambda_1\overline{\lambda}_2)(1-|\lambda_k|^2)}{M(\lambda_1-\lambda_k)(\overline{\lambda_2-\lambda_k})}\right). \tag{3.13}$$

Proof. As for the proof of theorem 2.11, we consider the partial sums $\mathscr{O}_{12}^{(d)}$ and proceed by induction. It follows from the proof of Theorem 2.6, that $|u_2|^2 \stackrel{d}{=} Y_M$, so that

$$\mathbb{E}|u_2|^2 = \frac{1}{M} \quad \text{and} \quad \mathbb{E}_\Lambda \mathscr{O}_{12}^{(2)} = \frac{-1}{M|\lambda_1 - \lambda_2|^2}.$$

We then compute the conditional expectation of $b_{n+1}\overline{d_{n+1}}$ by integrating out the vector u_{n+1}, using Proposition 3.3 and (3.19) from Lemma 3.13 with $a = B_n^*$, $b = D_n^*$ and $S = S_n$. It follows that

$$\mathbb{E}_{\Lambda,\mathscr{F}_{n-1}} b_{n+1}\overline{d_{n+1}} = \frac{(1-|\lambda_{n+1}|^2)}{M(\lambda_1 - \lambda_{n+1})(\overline{\lambda_2 - \lambda_{n+1}})}(B_n D_n^* - B_n T_n T_n^* D_n^*).$$

As noted in the proof of Theorem 2.11, we have

$$B_n T_n = \lambda_1 B_n, \quad D_n T_n = \lambda_2 D_n,$$

and conclude that

$$-\bar{b}_2 \mathbb{E}_{\Lambda,\mathscr{F}_n} b_{n+1}\overline{d_{n+1}} = \frac{(1-|\lambda_{n+1}|^2)(1-\lambda_1\overline{\lambda_2})}{M(\lambda_1 - \lambda_{n+1})(\overline{\lambda_2 - \lambda_{n+1}})}\mathscr{O}_{12}^{(n)}.$$

The factorization follows. □

Proposition 3.10. *The quenched expectation of* $\operatorname{Tr} G^*G$ *with* G *distributed according to* $\mathrm{TUE}(N,M)$ *is given by the formula:*

$$\mathbb{E}_\Lambda\left(\frac{1}{N}\operatorname{Tr} G^*G\right) = \prod_{i=1}^{N}\left(1 + \frac{1-|\lambda_i|^2}{M}\right) - \left(1 + \frac{N}{M}\right).$$

Proof. As in the proof of Proposition 2.12, we define $v_{N,n} := \mathbb{E}_{N,\Lambda} \operatorname{Tr} T_n T_n^*$ and note that for any $n \leqslant N$,

$$\operatorname{Tr} T_n T_n^* = |\lambda_n|^2 + \|u_n\|^2 + \operatorname{Tr} T_{n-1} T_{n-1}^*.$$

Using (3.20) from Lemma 3.13 yields a induction with $v_{N,1} = |\lambda_1|^2$ and

$$v_{N,n+1} \quad = \quad v_{N,n}\left(1 + \frac{1-|\lambda_{n+1}|^2}{M}\right) + |\lambda_{n+1}|^2 + \frac{n}{M}\left(1 - |\lambda_{n+1}|^2\right). \tag{3.14}$$

This is an analogous recursion formula to the one obtained in Proposition 2.12 and it can be solved the same way, replacing $|\lambda_i|^2$ by $-|\lambda_i|^2$ and N by M in the denominators; this leads to the expression

$$\frac{1}{N} v_{N,n} = \prod_{i=1}^{n}\left(1 + \frac{1-|\lambda_i|^2}{M}\right) - \left(1 + \frac{n}{M}\right),$$

which is equivalent to the statement, when $n = N$. □

3.3 Constants and integrals

Lemma 3.11. *The normalization constant for generalized gamma variables* $\gamma_V(k)$ *with potential* $V(x) = -(M-1)\log(1-x)\mathbb{1}_{(0,1)}$ *and* $k \geqslant 1$ *is*

$$\int_0^1 x^{k-1}(1-x)^{M-1}\mathrm{d}x = \beta(k,M),$$

and $\gamma_V(k) \stackrel{d}{=} \beta_{k,M}$. *Moreover, the associated function* $e_V^{(N-1)}$ *is given by*

$$e_V^{(N-1)}(X) = M\sum_{k=0}^{N-1}\binom{M+k}{k}X^k =: MF_{M,N}(X),$$

where the $F_{M,N}(X)$ *corresponds to the partial sums of* $\frac{1}{(1-X)^{M+1}}$ *as formal series.*

Proof. The first computations are straightforward. The correspondence with partial sums of $\frac{1}{(1-X)^{M+1}}$ can be established either by induction, or combinatorially by noticing that this is an identity related to 'weak compositions' (unordered partitions of integers with possible zeros). □

Lemma 3.12. *For any $p \geqslant 0$, with $\mathcal{B}_n := \{(\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n \mid \sum |\lambda_i|^2 \leqslant 1\}$,*

$$D_{n,p} := \int_{z \in \mathcal{B}_n} \left(1 - \sum_{i=1}^n |\lambda_i|^2\right)^p \mathrm{d}m(\lambda_1) \dots \mathrm{d}m(z_n) = \pi^n \frac{p!}{(p+n)!}, \tag{3.15}$$

and

$$D^{(1)}_{n,p} := \int_{z \in \mathcal{B}_n} |\lambda_1|^2 \left(1 - \sum_{i=1}^n |\lambda_i|^2\right)^p \mathrm{d}m(\lambda_1) \dots \mathrm{d}m(z_n) = \frac{1}{p+n+1} D_{n,p}. \tag{3.16}$$

Proof. We first compute $D_{n,p}$ by induction on n. For $n = 1, p \geqslant 0$,

$$D_{1,p} = \int_{z \in \mathbb{D}} \left(1 - |z|^2\right)^p \mathrm{d}m(z) = \pi \int_{r=0}^1 r^p \mathrm{d}r = \frac{\pi}{p+1}.$$

Note that for any $\alpha > 0$,

$$\int_{|z|^2 < \alpha} \left(1 - \alpha^{-1}|z|^2\right)^p \mathrm{d}m(z) = \frac{\pi\alpha}{p+1}.$$

For general n, using the above equalities with $\alpha_n = 1 - \sum_{i=1}^{n-1} |\lambda_i|^2$,

$$\begin{aligned} D_{n,p} &= \int_{z \in \mathcal{B}_n} \left(1 - \sum_{i=1}^{n-1} |\lambda_i|^2\right)^p \times \left(1 - \alpha_n^{-1}|\lambda_n|^2\right)^p \mathrm{d}m(\lambda_1) \dots \mathrm{d}m(\lambda_n) \\ &= \frac{\pi}{p+1} \int_{z \in \mathcal{B}_{n-1}} \left(1 - \sum_{i=1}^{n-1} |\lambda_i|^2\right)^{p+1} \mathrm{d}m(\lambda_1) \dots \mathrm{d}m(\lambda_{n-1}) = \frac{\pi}{p+1} D_{n-1,p+1}. \end{aligned}$$

Equation (3.15) follows. A similar induction can be performed on $D^{(1)}_{n,p}$. The only difference is that the last step involves the following identity: for any $p \geqslant 0$,

$$\begin{aligned} D^{(1)}_{1,p} &= \int_{z \in \mathbb{D}} |z|^2 (1 - |z|^2)^p \mathrm{d}m(z) = \pi \int_{r=0}^1 (r-1) r^p \mathrm{d}r \\ &= \pi \left(\frac{1}{p+1} - \frac{1}{p+2}\right) = \frac{\pi}{(p+1)(p+2)}, \end{aligned}$$

which in general yields the extra factor $\frac{1}{p+n+1}$ in (3.16). □

Note that when we begin the recursion from [11] with $n = N-1, p = M-N$, the extra factor is $\frac{1}{M}$ at every step.

Lemma 3.13. *For any $p > n$, $a, b \in \mathbb{C}^N$ and any Hermitian positive-definite matrix S,*

$$\int_{S\mathcal{B}_n} (1 - u^* S^{-2} u)^p \mathrm{d}u = D_{n,p} |\det S|^2, \tag{3.17}$$

$$\int_{S\mathcal{B}_n} (a^* u)(1 - u^* S^{-2} u)^p \mathrm{d}u = 0, \tag{3.18}$$

$$\int_{S\mathcal{B}_n} (a^* u)(u^* b)(1 - u^* S^{-2} u)^p \mathrm{d}u = D_{n,p} |\det S|^2 \frac{a^* S^2 b}{n+p+1}, \tag{3.19}$$

$$\int_{S\mathcal{B}_n} \|u\|^2 (1 - u^* S^{-2} u)^p \mathrm{d}u = D_{n,p} |\det S|^2 \frac{\operatorname{Tr} S^2}{n+p+1}, \tag{3.20}$$

where the constant $D_{n,p}$ is explicitly computed in Lemma 3.12.

Lemma 3.14. *For any $p > n$, $a \in \mathbb{C}^n$ and any Hermitian positive-definite matrix S, if $u \in \mathbb{C}^n$ is distributed with density*

$$\frac{1}{C_{n,p}|\det S|^2}(1 - u^* S^{-2} u)^p$$

with respect to the Lebesgue measure on $\mathbb{C}^n$, then the following identity in distribution holds:

$$|a^* u|^2 \stackrel{d}{=} \|Sa\|^2 Y_{p+n+1}.$$

The proofs of Lemmata 3.13 and 3.14 are exactly analogous to the proofs of their spherical counterpart, Lemmata 2.15 and 2.16.

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