



Ergodic Decomposition of Dirichlet Forms via Direct Integrals and Applications

Lorenzo Dello Schiavo¹

Received: 15 March 2020 / Accepted: 8 September 2021 / Published online: 6 October 2021
© The Author(s) 2021

Abstract

We study direct integrals of quadratic and Dirichlet forms. We show that each quasi-regular Dirichlet space over a probability space admits a unique representation as a direct integral of irreducible Dirichlet spaces, quasi-regular for the same underlying topology. The same holds for each quasi-regular strongly local Dirichlet space over a metrizable Luzin σ -finite Radon measure space, and admitting carré du champ operator. In this case, the representation is only projectively unique.

Mathematics Subject Classification (2010) Primary 37A30; Secondary 31C25 · 60J25 · 60J35

1 Introduction

Let $(X, \tau, \mathcal{X}, \mu)$ be a locally compact Polish Radon measure space, and $(E, \mathcal{D}(E))$ be a regular Dirichlet form on $(X, \tau, \mathcal{X}, \mu)$. As it is well-known, $(E, \mathcal{D}(E))$ is properly associated with a right process

$$\mathbf{M} := (\Omega, \mathcal{F}, (M_t)_{t \geq 0}, (P_x)_{x \in X_\partial}, \xi)$$

with state space X , life-time ξ , and cemetery ∂ .

For a μ -measurable subset $A \subset X$, we say that

- A is **M-invariant** (e.g. [23, Dfn. IV.6.1]) if there exists $\Omega_{A^c} \in \mathcal{F}$ with $P_x \Omega_{A^c} = 0$ for every $x \in A$ and

$$\Omega_{A^c} \supset \left\{ \omega \in \Omega : A^c \cap \overline{\{M_s(\omega) : s \in [0, t]\}} \neq \emptyset \text{ for some } 0 \leq t < \xi \right\};$$

A large part of this work was written while the author was a member of the Institut für angewandte Mathematik of the University of Bonn. Research supported by the Collaborative Research Center 1060. The author gratefully acknowledges funding of his current position by the Austrian Science Fund (FWF) grant F65, and by the European Research Council (ERC, grant No. 716117, awarded to Prof. Dr. Jan Maas).

✉ Lorenzo Dello Schiavo
lorenzo.delloschiavo@ist.ac.at

¹ IST Austria, Am Campus 1, 3400 Klosterneuburg, Österreich, Austria

- A is E -invariant (also cf. Dfn. 3.1 below) if $\mathbb{1}_A f \in \mathcal{D}(E)$ for any $f \in \mathcal{D}(E)$ and

$$E(f, g) = E(\mathbb{1}_A f, \mathbb{1}_A g) + E(\mathbb{1}_{A^c} f, \mathbb{1}_{A^c} g), \quad f, g \in \mathcal{D}(E)$$

If the form $(E, \mathcal{D}(E))$ is additionally strongly local, then the process \mathbf{M} is a Markov diffusion, and the following are μ -essentially equivalent, see Rmk. 3.5 below:

- A is \mathbf{M} -invariant;
- A is E -invariant;
- A is E -quasi-clopen, i.e., simultaneously E -quasi-open and E -quasi-closed (see e.g., [15, p. 70]).

We say that a set $A \subset X$ is μ -trivial if it is μ -measurable and either $\mu A = 0$ or $\mu A^c = 0$. The process \mathbf{M} is *irreducible* if every \mathbf{M} -invariant set is μ -trivial.

When \mathbf{M} is not irreducible, it is natural — in the study of the pathwise properties of \mathbf{M} — to restrict our attention to “minimal” \mathbf{M} -invariant subsets of X . In the local case, thanks to the quasi-topological characterization of \mathbf{M} -invariance, such sets may be thought of as the “connected components” of the space X as seen by \mathbf{M} .

This description is in fact purely heuristic, since it may happen that all such “minimal” \mathbf{M} -invariant sets are μ -negligible. The question arises, whether these ideas for the study of \mathbf{M} -invariance can be made rigorous by resorting to the Dirichlet form $(E, \mathcal{D}(E))$ associated with \mathbf{M} . More precisely, we look for a decomposition $((E_\zeta, \mathcal{D}(E_\zeta)))_{\zeta \in Z}$ of $(E, \mathcal{D}(E))$ over some index set Z , and we require that

- $(E_\zeta, \mathcal{D}(E_\zeta))$ is a Dirichlet form on (X, τ) additionally irreducible (Dfn. 3.1) for every $\zeta \in Z$;
- we may reconstruct $(E, \mathcal{D}(E))$ from $((E_\zeta, \mathcal{D}(E_\zeta)))_{\zeta \in Z}$ in a unique way.

Because of the first property, such a decomposition — if any — would deserve the name of *ergodic decomposition* of $(E, \mathcal{D}(E))$.

For instance, let us consider the standard Dirichlet form E^g on a (second-countable) Riemannian manifold (M, g) , i.e. the one generated by the (negative halved) Laplace–Beltrami operator Δ_g and properly associated with the Brownian motion on M . In this case, we expect that Z is a discrete space, indexing the connected components of M , and that

$$E^g = \bigoplus_{\zeta \in Z} E_\zeta^g,$$

where $(E_\zeta^g, \mathcal{D}(E_\zeta^g))$ is but the standard form of the connected component of index ζ . This simple example suggests that, in the general case of our interest, we should expect that $(E, \mathcal{D}(E))$ is recovered from the decomposition $((E_\zeta, \mathcal{D}(E_\zeta)))_{\zeta \in Z}$ as a “direct integral”,

$$E = \int_Z^\oplus E_\zeta.$$

Our purpose is morefold:

- to introduce a notion of *direct integral of Dirichlet forms*, and to compare it with the existing notions of *superposition of Dirichlet forms* [5, §V.3.1] (also cf. [15, §3.1(2°), p. 113] and [29]), and of *direct integral of quadratic forms* [3];
- to discuss an Ergodic Decomposition Theorem for quasi-regular Dirichlet forms, a counterpart for Dirichlet forms to the Ergodic Decomposition Theorems for group actions, e.g. [6, 9, 16];

- to provide rigorous justification to the assumption — quite standard in the literature about (quasi-)regular Dirichlet forms —, that one may consider irreducible forms with no loss of generality;
- to establish tools for the generalization to arbitrary (quasi-regular) Dirichlet spaces of results currently available only in the irreducible case, e.g. the study [22] of invariance under order-isomorphism, cf. [11].

For strongly local Dirichlet forms, our ergodic decomposition result takes the following form.

Theorem *Let $(X, \tau, \mathcal{X}, \mu)$ be a metrizable Luzin Radon measure space, and consider a quasi-regular strongly local Dirichlet form $(E, \mathcal{D}(E))$ on $L^2(\mu)$ admitting carré du champ operator. Then, there exist*

- (i) *a σ -finite measure space (Z, \mathcal{Z}, ν) ;*
- (ii) *a family of measures $(\mu_\zeta)_{\zeta \in Z}$ so that $(X, \tau, \mathcal{X}, \mu_\zeta)$ is a (metrizable Luzin) Radon measure space for ν -a.e. $\zeta \in Z$, the map $\zeta \mapsto \mu_\zeta A$ is ν -measurable for every $A \in \mathcal{X}$ and*

$$\mu A = \int_Z \mu_\zeta A \, d\nu(\zeta) , \quad A \in \mathcal{X} ,$$

- and for $\nu^{\otimes 2}$ -a.e. (ζ, ζ') , with $\zeta \neq \zeta'$, the measures μ_ζ and $\mu_{\zeta'}$ are mutually singular;
- (iii) *a family of quasi-regular strongly local irreducible Dirichlet forms $(E_\zeta, \mathcal{D}(E_\zeta))$ on $L^2(\mu_\zeta)$;*

so that

$$L^2(\mu) = \int_Z^\oplus L^2(\mu_\zeta) \, d\nu(\zeta) \quad \text{and} \quad E = \int_Z^\oplus E_\zeta \, d\nu(\zeta) .$$

Additionally, the disintegration is (ν -essentially) projectively unique, and unique if μ is totally finite.

Plan of the work Firstly, we shall discuss the notion of direct integral of (non-negative definite) quadratic forms on abstract Hilbert spaces, Section 2.3, and specialize it to direct integrals of Dirichlet forms, Section 2.5, via disintegration of measures. In Section 3.2 we show existence and uniqueness of the ergodic decomposition for regular and quasi-regular (not necessarily local) Dirichlet forms on probability spaces. The results are subsequently extended to strongly local quasi-regular Dirichlet forms on σ -finite spaces and admitting carré du champ operator, Section 3.3. Examples are discussed in Section 3.4; an application is discussed in Section 3.5.

Bibliographical note Our reference of choice for direct integrals of Hilbert spaces is the monograph [12] by J. Dixmier. For some results we shall however need the more general concept of direct integrals of Banach spaces, after [9, 17]. For the sake of simplicity, all such results are confined to the Appendix.

2 Direct Integrals

Every Hilbert space is assumed to be *separable* and a real vector space.

2.1 Quadratic Forms

Let $(H, \| \cdot \|)$ be a Hilbert space with scalar product $\langle \cdot | \cdot \rangle$.

By a *quadratic form* (Q, D) on H we shall always mean a symmetric positive semi-definite — if not otherwise stated, densely defined — bilinear form. To (Q, D) we associate the non-relabelled quadratic functional $Q: H \rightarrow \mathbb{R} \cup \{+\infty\}$ defined by

$$Q(u) := \begin{cases} Q(u, u) & \text{if } u \in D \\ +\infty & \text{otherwise} \end{cases}, \quad u \in H .$$

Additionally, for every $\alpha > 0$, we set

$$\begin{aligned} Q_\alpha(u, v) &:= Q(u, v) + \alpha \langle u | v \rangle, & u, v \in D, \\ Q_\alpha(u) &:= Q(u) + \alpha \|u\|^2, & u \in H. \end{aligned}$$

For $\alpha > 0$, we let $\mathcal{D}(Q)_\alpha$ be the completion of D , endowed with the Hilbert norm $Q_\alpha^{1/2}$. The following result is well-known.

Lemma 2.1 *Let (Q, D) be a quadratic form on H . The following are equivalent:*

- (i) (Q, D) is closable, say, with closure $(Q, \mathcal{D}(Q))$;
- (ii) the identical inclusion $\iota: D \rightarrow H$ extends to a continuous injection $\iota_\alpha: \mathcal{D}(Q)_\alpha \rightarrow H$ satisfying $\|\iota_\alpha\|_{\text{op}} \leq \alpha^{-1}$;
- (iii) Q is lower semi-continuous w.r.t. the strong topology of H ;
- (iv) Q is lower semi-continuous w.r.t. the weak topology of H .

Proof (i) \iff (ii) is [18, Lem. VIII.3.14a, p. 461]. (i) \iff (ii) is noted in [23, Rmk. I.3.2. (ii)]. For (ii) \iff (iv) note that every convex subset of a Hilbert space is weakly closed if and only if it is strongly closed and apply this fact to the sublevel sets of $Q: H \rightarrow \mathbb{R} \cup \{+\infty\}$. □

To every closed quadratic form $(Q, \mathcal{D}(Q))$ we associate a non-negative self-adjoint operator $-L$, with domain defined by the equality $\mathcal{D}(\sqrt{-L}) = \mathcal{D}(Q)$, such that $Q(u, v) = \langle -Lu | v \rangle$ for all $u, v \in \mathcal{D}(L)$. We denote the associated strongly continuous contraction semigroup by $T_t := e^{tL}, t > 0$, and the associated strongly continuous contraction resolvent by $G_\alpha := (\alpha - L)^{-1}, \alpha > 0$. By Hille–Yosida Theorem, e.g. [23, p. 27],

$$Q_\alpha(G_\alpha u, v) = \langle u | v \rangle_H, \quad u \in H, v \in \mathcal{D}(Q), \tag{1a}$$

$$T_t = H\text{-}\lim_{\alpha \rightarrow \infty} e^{t\alpha(G_\alpha - 1)}. \tag{1b}$$

$$Q(u, v) = \lim_{\beta \rightarrow \infty} Q^{(\beta)}(u, v) := \lim_{\beta \rightarrow \infty} \langle \beta u - \beta G_\beta u | v \rangle_H, \quad u, v \in H. \tag{1c}$$

2.2 Direct Integrals

Let $(H_\zeta)_{\zeta \in Z}$ be a family of Hilbert spaces indexed by some index set Z . If Z is at most countable, the *direct sum* of the Hilbert spaces H_ζ is defined as

$$\bigoplus_{\zeta \in Z} H_\zeta := \left\{ (h_\zeta)_{\zeta \in Z} : h_\zeta \in H_\zeta \text{ for all } \zeta \in Z \text{ and } \sum_{\zeta \in Z} \|h_\zeta\|_{H_\zeta}^2 < \infty \right\}. \tag{2}$$

The *direct integral* of a family of Hilbert spaces is a natural generalization of the concept of direct sum of Hilbert spaces to the case when the indexing set Z is more than countable. In

this case, the requirement of ℓ_2 -summability in the definition of direct sum is replaced by a requirement of L^2 -integrability (see below for the precise definitions), which implies that Z should be taken to be a measure space (Z, \mathcal{Z}, ν) . When Z is an at most countable discrete space, and ν is the counting measure, then the direct integral of the Hilbert spaces $(H_\zeta)_\zeta$ is isomorphic to their direct sum (2).

Since their introduction by J. von Neumann in [24, §3], direct integrals have become a main tool in operator theory, and in particular in the classification of von Neumann algebras.

In order to make the definition of direct integral precise, let us first introduce some measure-theoretical notions.

Definition 2.2 (Measure spaces) A measurable space (X, \mathcal{X}) is

- *separable* if \mathcal{X} contains all singletons in X , i.e. $\{x\} \in \mathcal{X}$ for each $x \in X$;
- *separated* if \mathcal{X} separates points in X , i.e. for every $x, y \in X$ there exists $A \in \mathcal{X}$ with $x \in A$ and $y \in A^c$;
- *countably separated* if there exists a countable family of sets in \mathcal{X} separating points in X ;
- *countably generated* if there exists a countable family of sets in \mathcal{X} generating \mathcal{X} as a σ -algebra;
- a *standard Borel space* if there exists a Polish topology τ on X so that \mathcal{X} coincides with the Borel σ -algebra induced by τ .

For any subset X_0 of a measurable space (X, \mathcal{X}) , the trace σ -algebra on X_0 is $\mathcal{X} \cap X_0 := \{A \cap X_0 : A \in \mathcal{X}\}$. A σ -finite measure space (X, \mathcal{X}, μ) is *standard* if there exists $X_0 \in \mathcal{X}$, μ -conegligible and so that $(X_0, \mathcal{X} \cap X_0)$ is a standard Borel space. We denote by $(X, \mathcal{X}^\mu, \hat{\mu})$ the (Carathéodory) completion of (X, \mathcal{X}, μ) . A $[-\infty, \infty]$ -valued function is called μ -measurable if it is measurable w.r.t. \mathcal{X}^μ . For measures μ_1, μ_2 we write $\mu_1 \sim \mu_2$ to indicate that μ_1 and μ_2 are equivalent, i.e. mutually absolutely continuous. A σ -ideal \mathcal{N} of a measure space (X, \mathcal{X}, μ) is any sub- σ -algebra of \mathcal{X} closed w.r.t. \cap , i.e. satisfying $A \cap N \in \mathcal{N}$ whenever $A \in \mathcal{X}$ and $N \in \mathcal{N}$. In particular, the family \mathcal{N}_μ of μ -negligible subsets of (X, \mathcal{X}) is always a σ -ideal of $(X, \mathcal{X}^\mu, \hat{\mu})$.

For functions $f, g : X \rightarrow \mathbb{R} \cup \{\pm\infty\}$ we denote by $f^+ := 0 \vee f$, resp. $f^- := -(0 \wedge f)$, the positive, resp. negative part of f , and by $f \wedge g$, resp. $f \vee g$, the pointwise minimum, resp. maximum, of f and g .

We now recall the main definitions concerning direct integrals of separable Hilbert spaces, referring to [12, §§II.1, II.2] for a systematic treatment.

Definition 2.3 (Measurable fields, [12, §II.1.3, Dfn. 1, p. 164]) Let (Z, \mathcal{Z}, ν) be a σ -finite measure space, $(H_\zeta)_{\zeta \in Z}$ be a family of separable Hilbert spaces, and F be the linear space $F := \prod_{\zeta \in Z} H_\zeta$. We say that $\zeta \mapsto H_\zeta$ is a ν -measurable field of Hilbert spaces (with underlying space S) if there exists a linear subspace S of F with

- (a) for every $u \in S$, the function $\zeta \mapsto \|u_\zeta\|_\zeta$ is ν -measurable;
- (b) if $v \in F$ is such that $\zeta \mapsto \langle u_\zeta \mid v_\zeta \rangle_\zeta$ is ν -measurable for every $u \in S$, then $v \in S$;
- (c) there exists a sequence $(u_n)_n \subset S$ such that $(u_{n,\zeta})_n$ is a total sequence¹ in H_ζ for every $\zeta \in Z$.

¹A sequence $(u_n)_n$ in a Banach space B is called *total* if the strong closure of its linear span coincides with B .

Any such S is called a *space of ν -measurable vector fields*. Any sequence in S possessing property (c) is called a *fundamental sequence*.

Proposition 2.4 ([12, §II.1.4, Prop. 4, p. 167]) *Let \mathcal{S} be a subfamily of F satisfying both Definition 2.3(a) and (c) with S in place of \mathcal{S} . Then, there exists exactly one space of ν -measurable vector fields S so that $\mathcal{S} \subset S$.*

Definition 2.5 (Direct integrals, [12, §II.1.5, Prop. 5, p. 169]) *Let $\zeta \mapsto H_\zeta$ be a ν -measurable field of Hilbert spaces with underlying space S . A ν -measurable vector field $u \in S$ is called (ν -)square-integrable if*

$$\|u\| := \left(\int_Z \|u_\zeta\|_\zeta^2 \, d\nu(\zeta) \right)^{1/2} < \infty . \tag{3}$$

Two square-integrable vector fields u, v are called (ν -)equivalent if $\|u - v\| = 0$. The space H of equivalence classes of square-integrable vector fields, endowed with the non-re-labeled quotient norm $\|\cdot\|$, is a Hilbert space [12, §II.1.5, Prop. 5(i), p. 169], called the *direct integral of $\zeta \mapsto H_\zeta$ (with underlying space S)* and denoted by

$$H = \int_Z^{\oplus} H_\zeta \, d\nu(\zeta) . \tag{4}$$

The superscript ‘ S ’ is omitted whenever S is apparent from context.

In the following, it will occasionally be necessary to distinguish an element u of H from one of its representatives modulo ν -equivalence, say \hat{u} in S . In this case, we shall write $u = [\hat{u}]_H$. When the specification of the variable ζ is necessary, given $u \in H$, resp. $\hat{u} \in S$, we shall write $\zeta \mapsto u_\zeta$ in place of u , resp. $\zeta \mapsto \hat{u}_\zeta$ in place of \hat{u} . In most cases, the distinction between u and \hat{u} is however immaterial, similarly to the distinction between the class of a function in $L^2(\nu)$ and any of its ν -representatives. Therefore in most cases we shall simply write u in place of both u and \hat{u} .

Lemma 2.6 *Let (Z, \mathcal{Z}, ν) be a σ -finite countably generated measure space. Then, the space H in Eq. 4 is separable.*

Proof It suffices to note that $L^2(\nu)$ is separable, e.g. [14, 365X(p)]. Then, the proof of [12, §II.1.6, Cor., p. 172] applies *verbatim*. □

Remark 2.7 In general, the space H in Eq. 4 depends on S , cf. [12, p. 169, after Dfn. 3]. It is nowadays standard to define *the* direct integral of $\zeta \mapsto H_\zeta$ as the one with underlying space S generated (in the sense of Proposition 2.4) by an algebra \mathcal{S} of ‘simple functions’, see e.g. [17, §6.1, p. 61] or the [Appendix](#). Here, we prefer the original definition in [12], since we shall make a (possibly) different choice of S , more natural when addressing direct integrals of Dirichlet forms.

Let H be a direct integral Hilbert space defined as in Eq. 4. We now turn to the discussion of bounded operators in $\mathcal{B}(H)$ and their representation by measurable fields of bounded operators.

Definition 2.8 (Measurable fields of bounded operators, decomposable operators) *A field of bounded operators $\zeta \mapsto B_\zeta \in \mathcal{B}(H_\zeta)$ is called ν -measurable (with underlying space S)*

if $\zeta \mapsto B_\zeta u_\zeta \in H_\zeta$ is a ν -measurable vector field with underlying space S for every ν -measurable vector field u with underlying space S .

A ν -measurable vector field of bounded operators is called ν -essentially bounded if ν -esssup $_{\zeta \in Z} \|B_\zeta\|_{\text{op}, \zeta} < \infty$. In this case, the direct integral operator $B: H \rightarrow H$ of $\zeta \mapsto B_\zeta$ given by

$$B: [\hat{u}]_H \mapsto [\zeta \mapsto B_\zeta \hat{u}_\zeta]_H \tag{5}$$

is well-defined (in the sense that it does not depend on the choice of the representative $\hat{u} \in S$ of $[\hat{u}]_H \in H$), and a bounded operator in $\mathcal{B}(H)$. Its operator norm $\|B\|_{\text{op}}$ satisfies $\|B\|_{\text{op}} = \nu$ -esssup $_{\zeta \in Z} \|B_\zeta\|_{\text{op}, \zeta}$, [12, §II.2.3, Prop. 2, p. 181]. Conversely, a bounded operator $B \in \mathcal{B}(H)$ is called *decomposable*, [12, §II.2.3, Dfn. 2, p. 182], if B is represented by a ν -essentially bounded ν -measurable field of bounded operators $\zeta \mapsto B_\zeta$ in the sense of Eq. 5, in which case we write

$$B = \int_Z^\oplus B_\zeta \, d\nu(\zeta) .$$

The next statement is readily deduced from e.g. [8, Thm. 2] or [21, Thm. 1.10]. For the reader’s convenience, a short proof is included.

Lemma 2.9 *Let H be defined as in Eq. 4, $B \in \mathcal{B}(H)$ be decomposable, and D_B be the closed disk of radius $\|B\|_{\text{op}}$ in the complex plane. Then, for every $\varphi \in \mathcal{C}(D_B)$ the continuous functional calculus $\varphi(B)$ of B is decomposable and*

$$\varphi(B) = \int_Z^\oplus \varphi(B_\zeta) \, d\nu(\zeta) . \tag{6}$$

Proof Well-posedness follows by [12, §II.2.3, Prop. 2, p. 181]. The proof is then a straightforward application of [12, §II.2.3, Prop. 3, p. 182] and [12, §II.2.3, Prop. 4(ii), p. 183] by approximation of φ with suitable polynomials, since $\sigma(B)$ is compact. \square

Remark 2.10 Arguing with Tietze Extension Theorem, it is possible to show that the direct-integral representation (6) of $\varphi(B)$ only depends on the values of φ on the spectrum $\sigma(B)$ of B .

2.3 Direct Integrals of Quadratic Forms

The main object of our study are direct integrals of quadratic forms. Before turning to the case of Dirichlet forms on concrete Hilbert spaces (L^2 -spaces), we give the main definitions in the general case of quadratic forms on abstract Hilbert spaces.

Definition 2.11 (Direct integral of quadratic forms) Let (Z, \mathcal{Z}, ν) be a σ -finite countably generated measure space. For $\zeta \in Z$ let (Q_ζ, D_ζ) be a closable (densely defined) quadratic form on a Hilbert space H_ζ . We say that $\zeta \mapsto (Q_\zeta, D_\zeta)$ is a ν -measurable field of quadratic forms on Z if

- (a) $\zeta \mapsto H_\zeta$ is a ν -measurable field of Hilbert spaces on Z with underlying space S_H ;
- (b) $\zeta \mapsto \mathcal{D}(Q_\zeta)_1$ is a ν -measurable field of Hilbert spaces on Z with underlying space S_Q ;
- (c) S_Q is a linear subspace of S_H under the identification of $\mathcal{D}(Q_\zeta)$ with a subspace of H_ζ granted by Lemma 2.1.

We denote by

$$Q = \int_Z^{S_Q \oplus} Q_\zeta \, d\nu(\zeta)$$

the *direct integral* of $\zeta \mapsto (Q_\zeta, \mathcal{D}(Q_\zeta))$, i.e. the quadratic form defined on H as in Eq. 4 given by

$$\begin{aligned} \mathcal{D}(Q) &:= \left\{ [\hat{u}]_H : \hat{u} \in S_Q, \int_Z Q_{\zeta,1}(\hat{u}_\zeta) \, d\nu(\zeta) < \infty \right\}, \\ Q(u, v) &:= \int_Z Q_\zeta(u_\zeta, v_\zeta) \, d\nu(\zeta), \quad u, v \in \mathcal{D}(Q). \end{aligned} \tag{7}$$

Remark 2.12 (Separability) It is implicit in our definition of ν -measurable field of Hilbert spaces that H_ζ is *separable* for every $\zeta \in Z$. Therefore, when considering ν -measurable fields of domains as in Definition 2.1(b), $\mathcal{D}(Q_\zeta)_1$ is taken to be $(Q_\zeta)_1^{1/2}$ -separable by *assumption*.

Proposition 2.13 *Let $(Q, \mathcal{D}(Q))$ be a direct integral of quadratic forms. Then,*

- (i) $(Q, \mathcal{D}(Q))$ is a densely defined closed quadratic form on H ;
- (ii) $\zeta \mapsto G_{\zeta,\alpha}, \zeta \mapsto T_{\zeta,t}$ are ν -measurable fields of bounded operators for every $\alpha, t > 0$;
- (iii) Q has resolvent and semigroup respectively defined by

$$\begin{aligned} G_\alpha &:= \int_Z^{S_H \oplus} G_{\zeta,\alpha} \, d\nu(\zeta), \quad \alpha > 0; \\ T_t &:= \int_Z^{S_H \oplus} T_{\zeta,t} \, d\nu(\zeta), \quad t > 0. \end{aligned} \tag{8}$$

Proof (i) Since $\zeta \mapsto H_\zeta$ is a ν -measurable family of Hilbert spaces by Definition 2.11(a), the map $\zeta \mapsto \|u_\zeta\|_\zeta$ is ν -measurable for every $u \in S_H$ by Definition 2.3(a). Analogously, the map $\zeta \mapsto Q_{\zeta,1}^{1/2}(u_\zeta)$ is ν -measurable for every $u \in S_Q$. Together with the polarization identity for $\mathcal{D}(Q)_1$, this yields the measurability of the maps

$$\zeta \mapsto Q_{\zeta,\alpha}(u_\zeta, v_\zeta), \quad u, v \in \mathcal{D}(Q), \quad \alpha > 0.$$

As a consequence, $\zeta \mapsto \mathcal{D}(Q_\zeta)_\alpha$ is a ν -measurable field of Hilbert spaces (on Z , with underlying space S_Q) for every $\alpha > 0$. Thus, it admits a direct integral of Hilbert spaces

$$D_\alpha := \int_Z^{S_Q \oplus} \mathcal{D}(Q_\zeta)_\alpha \, d\nu(\zeta), \quad \alpha > 0.$$

For $\alpha > 0$ let $(u_n^\alpha)_n$ be a fundamental sequence of ν -measurable vector fields for D_α and $(u_n)_n$ be a fundamental sequence of ν -measurable vector fields for H .

Since (Q_ζ, D_ζ) is closable on H_ζ for every $\zeta \in Z$, the extension of the canonical inclusion $\iota_{\zeta,1}: \mathcal{D}(Q_\zeta)_1 \rightarrow H_\zeta$ is injective and non-expansive² for every $\zeta \in Z$ by Lemma 2.1.

²We say that a linear map $\iota: H_1 \rightarrow H_2$ between Hilbert spaces $(H_1, \|\cdot\|_1)$ and $(H_2, \|\cdot\|_2)$ is *non-expansive* if $\|\iota h\|_2 \leq \|h\|_1$ for every $h \in H_1$.

By Definition 2.11, D_α and H are defined on the same underlying space S . Therefore, the maps

$$\zeta \mapsto \left\langle \iota_{\zeta,\alpha} u_{i,\zeta}^\alpha \mid u_{j,\zeta} \right\rangle_\zeta = \left\langle u_{i,\zeta}^\alpha \mid u_{j,\zeta} \right\rangle_\zeta, \quad i, j \in \mathbb{N}, \quad \alpha > 0,$$

are ν -measurable. Together with the uniform boundedness of $\iota_{\zeta,\alpha}$ in $\zeta \in Z$, this yields the decomposability of the operator $\iota_\alpha : D_\alpha \rightarrow H$ defined by

$$\iota_\alpha := \int_Z^{\oplus} \iota_{\zeta,\alpha} \, d\nu(\zeta), \quad \alpha > 0.$$

By [12, §II.2.3, Example, p. 182] and the injectivity of $\iota_{\zeta,\alpha}$ for every $\zeta \in Z$ and every $\alpha > 0$, the operator $\iota_\alpha : D_\alpha \rightarrow H$ is injective. In particular, the composition of ι_1 with the inclusion of $\mathcal{D}(Q)$ into H is injective, thus Q is closed.

Finally, since $(u_n^\alpha)_n$ is $Q_{\zeta,\alpha}$ -total in $\mathcal{D}(Q_\zeta)_\alpha$ for every $\zeta \in Z$ by Definition 2.3(c), it is additionally H_ζ -total for every $\zeta \in Z$ by H_ζ -density of $\mathcal{D}(Q_\zeta)$ in H_ζ . As a consequence, $(u_n^\alpha)_n$ is fundamental also for H , thus $\mathcal{D}(Q)$ is H -dense in H .

(ii) For fixed $\alpha > 0$ consider the field of linear operators $\zeta \mapsto G_{\zeta,\alpha}$. The map (cf. Eq. 1a)

$$\zeta \mapsto Q_{\zeta,\alpha}(G_{\zeta,\alpha} u_{i,\zeta}^\alpha, u_{j,\zeta}^\alpha) = \left\langle u_{i,\zeta}^\alpha \mid u_{j,\zeta}^\alpha \right\rangle_\zeta$$

is ν -measurable for every $i, j \in \mathbb{N}$ since u_n^α is a ν -measurable vector field. Since $\|G_{\alpha,\zeta}\|_{\text{op}} \leq \alpha^{-1}$ and $(u_n^\alpha)_n$ is a fundamental sequence of ν -measurable vector fields for H , then $\zeta \mapsto G_{\zeta,\alpha}$ is a ν -measurable field of bounded operators by [12, §II.2.1, Prop. 1, p. 179] and the operator G_α defined in Eq. 8 is decomposable for every $\alpha > 0$.

By Lemma 2.9 any image of G_α via its continuous functional calculus is itself decomposable.

For every $\zeta \in Z$ one has $T_{\zeta,t} = \lim_{\beta \rightarrow \infty} e^{t\beta(G_{\zeta,\beta}-1)}$ strongly in H_ζ by Eq. 1b, hence

$$\zeta \mapsto \left\langle T_{\zeta,t} u_{i,\zeta}^\alpha \mid u_{j,\zeta}^\alpha \right\rangle_\zeta = \lim_{\beta \rightarrow \infty} \left\langle e^{t\beta(G_{\zeta,\beta}-1)} u_{i,\zeta}^\alpha \mid u_{j,\zeta}^\alpha \right\rangle_\zeta$$

is a pointwise limit of ν -measurable functions, thus it is ν -measurable, for every $i, j \in \mathbb{N}$ and every $t > 0$. As a consequence, $\zeta \mapsto T_{\zeta,t}$ is a ν -measurable field of bounded operators for every $t > 0$, again by [12, §II.2.1, Prop. 1, p. 179].

(iii) It suffices to show Eq. 1 for $(Q, \mathcal{D}(Q))$, G_α and T_t defined in Eq. 8.

Now, by definition of $(Q, \mathcal{D}(Q))$ one has for every $\alpha > 0$

$$\begin{aligned} Q_\alpha(G_\alpha u, v) &= \int_Z Q_\zeta((G_\alpha u)_\zeta, v_\zeta) \, d\nu(\zeta) + \alpha \int_Z \langle (G_\alpha u)_\zeta \mid v_\zeta \rangle_\zeta \, d\nu(\zeta) \\ &= \int_Z Q_{\zeta,\alpha}((G_\alpha u)_\zeta, v_\zeta) \, d\nu(\zeta) \\ &= \int_Z Q_{\zeta,\alpha}(G_{\alpha,\zeta} u_\zeta, v_\zeta) \, d\nu(\zeta) \end{aligned}$$

By [12, §II.2.3, Cor., p. 182] and decomposability of G_α , one has $G_{\alpha,\zeta} = G_{\zeta,\alpha}$ for ν -a.e. $\zeta \in Z$, whence, by Eq. 1a applied to $(Q_\zeta, \mathcal{D}(Q_\zeta))$ and $G_{\zeta,\alpha}$,

$$Q_\alpha(G_\alpha u, v) = \int_Z \langle u_\zeta \mid v_\zeta \rangle_\zeta \, d\nu(\zeta) = \langle u \mid v \rangle,$$

which concludes the proof of Eq. 1a for G_α .

Let us show (1b) for T_t . Define the operators

$$T_t^{(\beta)} := \int_Z^{\oplus} e^{t\beta(G_{\zeta,\beta}-1)} \, d\nu(\zeta), \quad \beta, t > 0,$$

and note that $\sup_{\beta>1} \|T_t^{(\beta)}\|_{\text{op}} < \infty$ for every $t > 0$. By Eq. 1b applied to $T_{\zeta,t}$ for every $\zeta \in Z$ and every $t > 0$ we have that $T_{\zeta,t} = \lim_{\beta \rightarrow \infty} e^{t\beta(\beta G_{\zeta,\beta-1})}$ strongly in H_{ζ} . On the one hand, we may now apply [12, §II.2.3, Prop. 4(ii), p. 183] to conclude that

$$H\text{-}\lim_{\beta \rightarrow \infty} T_t^{(\beta)} = \int_Z^{\oplus} \left(H_{\zeta}\text{-}\lim_{\beta \rightarrow \infty} e^{t\beta(\beta G_{\zeta,\beta-1})} \right) d\nu(\zeta) = \int_Z^{\oplus} T_{\zeta,t} d\nu(\zeta) =: T_t . \tag{9}$$

strongly in H . On the other hand, by Lemma 2.9 we have that

$$T_t^{(\beta)} := \int_Z^{\oplus} e^{t\beta(\beta G_{\zeta,\beta-1})} d\nu(\zeta) = e^{t\beta(\beta G_{\beta-1})} , \quad \beta, t > 0 . \tag{10}$$

Taking the strong H -limit as $\beta \rightarrow \infty$ in Eq. 10 yields the assertion by comparison with Eq. 9. □

Remark 2.14 (cf. [12, §II.1.3, Rmk. 3 p. 166 and §II.1.4, Rmk. p. 168]) Each of the above statements holds with identical proof if one substitutes ‘ ν -measurable’ with ‘measurable’.

Remark 2.15 Under the assumptions of Proposition 2.13, assertion (i) of the same Proposition implies that the space of ν -measurable vector fields S_H is uniquely determined by S_Q as a consequence of Proposition 2.4. Thus, everywhere in the following when referring to a direct integral of quadratic forms we shall — with abuse of notation — write S in place of both S_H and S_Q .

The next proposition completes the picture, by providing a direct-integral representation for the generator of the form $(Q, \mathcal{D}(Q))$ in Eq. 7. Since we shall not need this result in the following, we omit an account of direct integrals of unbounded operators, referring the reader to [21, §1]. Once the necessary definitions are established, a proof is straightforward.

Proposition 2.16 *Let $(Q, \mathcal{D}(Q))$ be defined as in Eq. 7. Then, its generator $(L, \mathcal{D}(L))$ has the direct-integral representation*

$$L = \int_Z^{\oplus} L_{\zeta} d\nu(\zeta) . \tag{11}$$

Remark 2.17 (Comparison with [3]) As for quadratic forms, Eq. 11 is understood as a direct-integral representation of the Hilbert space $\mathcal{D}(L)$, endowed with the graph norm, by the measurable field of Hilbert spaces $\zeta \mapsto \mathcal{D}(L_{\zeta})$, each endowed with the relative graph norm. The set-wise identification of $\mathcal{D}(L)$ as a linear subspace of H as in Eq. 4 is already shown in [3, Prop. 1.6].

2.4 Dirichlet Forms

We recall a standard setting for the theory of Dirichlet forms, following [23].

Assumption 2.18 The quadruple $(X, \tau, \mathcal{X}, \mu)$ is so that

- (a) (X, τ) is a metrizable Luzin space with Borel σ -algebra \mathcal{X} ;
- (b) $\hat{\mu}$ is a Radon measure on $(X, \tau, \mathcal{X}^{\mu})$ with full support.

By [14, 415D(iii), 424G] any space (X, \mathcal{X}, μ) satisfying Assumption 2.18 is, in particular, σ -finite standard. The *support* of a (μ) -measurable function $f: X \rightarrow \mathbb{R}$ (possibly

defined only on a μ -conegligible set) is defined as the measure-theoretical support $\text{supp}[|f| \cdot \mu]$. Every such f has support, independent of the μ -representative of f , cf. [23, p. 148].

A closed positive semi-definite quadratic form $(Q, \mathcal{D}(Q))$ on $L^2(\mu)$ is a (symmetric) Dirichlet form if

$$f \in \mathcal{D}(Q) \implies f^+ \wedge 1 \in \mathcal{D}(Q) \text{ and } Q(f^+ \wedge 1) \leq Q(f) . \tag{12}$$

We shall denote Dirichlet forms by $(E, \mathcal{D}(E))$. A Dirichlet form $(E, \mathcal{D}(E))$ is

- local if $E(f, g) = 0$ for every $f, g \in \mathcal{D}(E)$ with $\text{supp}[f] \cap \text{supp}[g] = \emptyset$;
- strongly local if $E(f, g) = 0$ for every $f, g \in \mathcal{D}(E)$ with g constant on a neighborhood of $\text{supp}[f]$;
- regular if (X, τ) is (additionally) locally compact and there exists a core \mathcal{C} for $(E, \mathcal{D}(E))$, i.e. a subset $\mathcal{C} \subset \mathcal{D}(E) \cap \mathcal{C}_0(\tau)$ both $E_1^{1/2}$ -dense in $\mathcal{D}(E)$ and uniformly dense in $\mathcal{C}_0(\tau)$.

On spaces that are not necessarily locally compact, the interplay between a Dirichlet form $(E, \mathcal{D}(E))$ and the topology τ on X is specified by the following definitions.

For an increasing sequence $(F_k)_k$ of Borel subsets $F_k \subset X$ set

$$\mathcal{D}(E, (F_k)_k) := \{ f \in \mathcal{D}(E) : f \equiv 0 \text{ } \mu\text{-a.e. on } F_k^c \text{ for some } k \in \mathbb{N} \} .$$

The sequence $(F_k)_k$ is called an E -nest if each F_k is closed and $\mathcal{D}(E, (F_k)_k)$ is dense in $\mathcal{D}(E)_1$. A set $N \subset X$ is E -exceptional if $N \subset \bigcap_k F_k^c$ for some E -nest $(F_k)_k$. A property of points in X holds E -quasi-everywhere if it holds for every point in N^c for some E -exceptional set N . A function $f: X \rightarrow \mathbb{R}$ is E -quasi-continuous if there exists an E -nest $(F_k)_k$ so that the restriction of f to F_k is continuous for every $k \in \mathbb{N}$.

Finally, a form $(E, \mathcal{D}(E))$ is

- quasi-regular on (X, τ) if there exist: (QR₁) an E -nest $(F_k)_k$ of compact sets; (QR₂) an $E_1^{1/2}$ -dense subset of $\mathcal{D}(E)_1$ the elements of which all have an E -quasi-continuous μ -version; (QR₃) an E -exceptional set $N \subset X$ and a countable family $(f_n)_n$ of E -quasi-continuous functions $f_n \in \mathcal{D}(E)$ separating points in N^c .

We refer to [7] or [15, §A.4] for the notion of quasi-homeomorphism of Dirichlet forms.

We say that $(E, \mathcal{D}(E))$ has carré du champ operator $(\Gamma, \mathcal{D}(E))$, if $\Gamma: \mathcal{D}(E)^{\otimes 2} \rightarrow L^1(\mu)$ is a non-negative definite symmetric continuous bilinear operator so that

$$E(f, gh) + E(fh, g) - E(fg, h) = 2 \int_X h \Gamma(f, g) d\mu , \quad f, g, h \in \mathcal{D}(E) \cap L^\infty(\mu) . \tag{13}$$

Finally, let $\mathcal{D}(E)_e$ be the linear space of all functions $u \in L^0(\mu)$ so that there exists an $E_1^{1/2}$ -Cauchy sequence $(u_n)_n \subset \mathcal{D}(E)$ with $\lim_n u_n = u$ μ -a.e. We denote by $(\mathcal{D}(E)_e, E)$ the space $\mathcal{D}(E)_e$ endowed with the extension of E to $\mathcal{D}(E)_e$ called the extended Dirichlet space of $(E, \mathcal{D}(E))$. For proofs of well-posedness in this generality, see [19, p. 693].

If $(E, \mathcal{D}(E))$ has semigroup $T_\bullet: L^2(\mu) \rightarrow L^2(\mu)$, we denote as well by $T_\bullet: L^\infty(\mu) \rightarrow L^\infty(\mu)$ the extension of the semigroup to $L^\infty(\mu)$.

We say that $(E, \mathcal{D}(E))$ is

- conservative if $T_t \mathbb{1} = \mathbb{1}$ μ -a.e. for all $t \geq 0$;
- transient if $\mathcal{D}(E)_e$ is a Hilbert space with inner product E ;
- recurrent if $\mathbb{1} \in \mathcal{D}(E)_e$ and $E(\mathbb{1}) = 0$.

These definitions are equivalent to the standard ones (e.g. [15, p. 55]) by [15, Thm.s 1.6.2, 1.6.3, p. 58], a proof of which may be adapted to the case of spaces satisfying Assumption 2.18.

2.5 Direct-integral Representation of L^2 -Spaces

In order to introduce direct-integral representations of Dirichlet forms, we need to construct direct integrals of concrete Hilbert spaces in such a way to additionally preserve the Riesz structure of Lebesgue spaces implicitly used to phrase the sub-Markovianity property (12). To this end, we shall need the concept of a disintegration of measures.

Disintegrations Let (X, \mathcal{X}, μ) and (Z, \mathcal{Z}, ν) be (non-trivial) measure spaces. A map $s: (X, \mathcal{X}) \rightarrow (Z, \mathcal{Z})$ is *inverse-measure-preserving* if $s_*\mu := \mu \circ s^{-1} = \nu$. Hereafter, we fix an inverse-measure-preserving map $s: (X, \mathcal{X}) \rightarrow (Z, \mathcal{Z})$.

Definition 2.19 (Disintegrations, [14, 452E]) A *pseudo-disintegration of μ over ν* is any family of non-zero measures $(\mu_\zeta)_{\zeta \in Z}$ on (X, \mathcal{X}) so that $\zeta \mapsto \mu_\zeta A$ is ν -measurable and

$$\mu A = \int_Z \mu_\zeta A \, d\nu(\zeta) , \quad A \in \mathcal{X} .$$

A pseudo-disintegration is

- *separated* if there exists a family of pairwise disjoint sets $(A_\zeta)_{\zeta \in Z} \subset \mathcal{X}^\mu$ so that A_ζ is μ_ζ -conegligible for ν -a.e. $\zeta \in Z$, henceforth called a *separating family* for $(\mu_\zeta)_{\zeta \in Z}$.

A *disintegration of μ over ν* is a pseudo-disintegration additionally so that μ_ζ is a sub-probability measure for every $\zeta \in Z$. A disintegration is

- *ν -essentially unique* if the measures μ_ζ are uniquely determined for ν -a.e. $\zeta \in Z$;
- *consistent with s* if

$$\mu(A \cap s^{-1}(B)) = \int_B \mu_\zeta A \, d\nu(\zeta) , \quad A \in \mathcal{X} , B \in \mathcal{Z} ; \tag{14}$$

- *strongly consistent with s* if $s^{-1}(\zeta)$ is μ_ζ -conegligible for ν -a.e. $\zeta \in Z$.

If $(\mu_\zeta)_{\zeta \in Z}$ is a pseudo-disintegration of μ over ν , then

$$\int_X g \, d\mu = \int_Z \int_X g(x) \, d\mu_\zeta(x) \, d\nu(\zeta) \tag{15}$$

whenever the left-hand side makes sense, [14, 452F]. We note that a disintegration $(\mu_\zeta)_{\zeta \in Z}$ of μ over ν strongly consistent with a map s is automatically separated, with separating family $(s^{-1}(\zeta))_{\zeta \in Z}$.

Direct integrals and disintegrations Let (X, \mathcal{X}, μ) be σ -finite standard, (Z, \mathcal{Z}, ν) be σ -finite countably generated, and $(\mu_\zeta)_{\zeta \in Z}$ be a pseudo-disintegration of μ over ν . Denote by

- $\mathcal{L}^0(\mu)$ the space of μ -measurable real-valued functions (*not*: μ -classes) on X ;
- $\mathcal{L}^\infty(\mu)$ the space of uniformly bounded (*not*: μ -essentially uniformly bounded) functions in $\mathcal{L}^0(\mu)$;
- $\mathcal{L}^p(\mu)$ the space of p -integrable functions in $\mathcal{L}^0(\mu)$.

For a family $\mathcal{A} \subset \mathcal{L}^0(\mu)$, let $[\mathcal{A}]_\mu$ denote the family of the corresponding μ -classes.

Let now $F := \prod_{\zeta \in Z} L^2(\mu_\zeta)$. We denote by $\delta: \mathcal{L}^2(\mu) \rightarrow F$ the diagonal embedding of $\mathcal{L}^2(\mu)$ into F , regarded up to μ_ζ -classes, viz. $\delta: f \mapsto (\zeta \mapsto \delta(f)_\zeta)$, where

$$\delta(f)_\zeta := \begin{cases} [f]_{\mu_\zeta} & \text{if } f \in \mathcal{L}^2(\mu_\zeta) \\ \mathbf{0}_{L^2(\mu_\zeta)} & \text{otherwise} \end{cases} . \tag{16}$$

In general, it does not hold that $f \in \mathcal{L}^2(\mu_\zeta)$ for every $\zeta \in Z$, thus we need to adjust the obvious definition of $\delta(f)$ as above in such a way that $\delta(f) \in F$, that is $\delta(f)_\zeta \in \mathcal{L}^2(\mu_\zeta)$ for every $\zeta \in Z$. Note that δ is thus *not* linear. However, since $f \in \mathcal{L}^2(\mu)$, then $\delta(f)_\zeta = [f]_{\mu_\zeta}$ for ν -a.e. $\zeta \in Z$ by Eq. 15. It will be shown in Proposition 2.25 below that δ is well-defined as linear morphism mapping μ -classes to H -classes.

Further let \mathcal{A} be satisfying

$$\mathcal{A} \text{ is a linear subspace of } \mathcal{L}^2(\mu), \text{ and } [\mathcal{A}]_\mu \text{ is dense in } L^2(\mu). \tag{17}$$

Since $[\mathcal{A}]_\mu$ is dense in $L^2(\mu)$ and the latter is separable, then there exists a countable family $\mathcal{U} \subset \mathcal{A}$ so that $[\mathcal{U}]_{\mu_\zeta}$ is total in $L^2(\mu_\zeta)$ for ν -a.e. $\zeta \in Z$.

Thus for every \mathcal{A} as in Eq. 17 there exists a unique space of ν -measurable vector fields $S = S_{\mathcal{A}}$ containing $\delta(\mathcal{A})$, generated by $\delta(\mathcal{A})$ in the sense of Proposition 2.4. We denote by H the corresponding direct integral of Hilbert spaces

$$H := \int_Z^{\oplus} L^2(\mu_\zeta) \, d\nu(\zeta) . \tag{18}$$

Since S is unique, it is in fact independent of \mathcal{A} . Indeed, let $\mathcal{A}_0, \mathcal{A}_1$ be satisfying (17) and note that $\mathcal{A} := \mathcal{A}_0 \oplus \mathcal{A}_1$ satisfies (17) as well. Thus, $\delta(\mathcal{A}_0), \delta(\mathcal{A}_1) \subset S_{\mathcal{A}}$, and so $S_{\mathcal{A}} = S_{\mathcal{A}_0} = S_{\mathcal{A}_1}$ by uniqueness.

Remark 2.20 (cf. [17, §7.2, p. 84]) The direct integral H constructed in Eq. 18 is a *Banach lattice* (e.g. [14, 354A(b)]) for the order

$$h \geq \mathbf{0}_H \iff h_\zeta \geq \mathbf{0}_{L^2(\mu_\zeta)} \quad \text{for } \nu\text{-a.e. } \zeta \in Z .$$

In particular, for every $g, h \in H$, the fields $h^+, h^-, g \wedge h$, and $g \vee h$, respectively defined by

$$h^\pm: \zeta \mapsto h_\zeta^\pm, \quad g \wedge h: \zeta \mapsto (g_\zeta \wedge h_\zeta), \quad g \vee h: \zeta \mapsto (g_\zeta \vee h_\zeta),$$

are ν -measurable fields representing elements of H .

In the following, we shall occasionally write — here, $\mathbf{1}_H$ is merely a shorthand —

$$\mathbf{0}_H \leq h \leq \mathbf{1}_H$$

to indicate that

$$0 \leq h_\zeta \leq 1 \quad \mu_\zeta\text{-a.e. for } \nu\text{-a.e. } \zeta \in Z .$$

Remark 2.21 For arbitrary measurable spaces, the standard choice for \mathcal{A} is the algebra of μ -integrable simple functions. If $(X, \tau, \mathcal{X}, \mu)$ were a locally compact Polish Radon measure space, one might take for instance $\mathcal{A} = \mathcal{C}_c(\tau)$, the algebra of continuous compactly supported functions. In fact, for the purposes of the present section, we might as well

choose $\mathcal{A} = \mathcal{L}^2(\mu)$, as the largest possible choice, or \mathcal{A} a countable \mathbb{Q} -vector subspace of $\mathcal{L}^1(\mu) \cap \mathcal{L}^\infty(\mu)$, as a smallest possible one. When dealing with direct integrals of regular Dirichlet forms however, the natural choice for \mathcal{A} is that of a special standard core \mathcal{C} for the resulting direct-integral form.

Remark 2.22 (Comparison with [3]) We note that for every \mathcal{A} as in Eq. 17, $[\mathcal{A}]_\mu$ is a *determining class* in the sense of [3, p. 402]. Conversely, every determining class \mathcal{L}_0 is contained in a minimal linear space of functions $[\mathcal{A}]_\mu$ satisfying (17).

Remark 2.23 (Caveat) Whereas the space H does not depend on \mathcal{A} , in general it *does* depend on $S_{\mathcal{A}}$, cf. Rmk. 2.7. Furthermore, H depends on the chosen pseudo-disintegration too, and thus H need *not* be isomorphic to $L^2(\mu)$, as shown in the next example.

Example 2.24 Let $\{*\}$ denote the one-point space, set $\mu := 2\delta_*$, and note that $L^2(\mu) \cong \mathbb{R}$. On the other hand, if $Z := (\{0, 1\}, \nu)$ is the two-point space with uniform measure ν , and $\mu_\zeta := \delta_*$ for $\zeta \in Z$, then $(\mu_\zeta)_{\zeta \in Z}$ is a (pseudo-)disintegration of μ , yet $H \cong L^2(\mu_0) \oplus L^2(\mu_1) \cong \mathbb{R}^2$.

Proposition 2.25 *Let (X, \mathcal{X}, μ) be σ -finite standard, (Z, \mathcal{Z}, ν) be σ -finite countably generated, and $(\mu_\zeta)_{\zeta \in Z}$ be a pseudo-disintegration of μ over ν . Then, the morphism*

$$\begin{aligned} \iota: L^2(\mu) &\longrightarrow H := \int_Z^{S^{\oplus}} L^2(\mu_\zeta) \, d\nu(\zeta) \\ [f]_\mu &\longmapsto [\delta(f)]_H \end{aligned} \tag{19}$$

- (i) *is well-defined, linear, and an isometry of Hilbert spaces, additionally unitary if $(\mu_\zeta)_{\zeta \in Z}$ is separated;*
- (ii) *is a Riesz homomorphism (e.g. [14, 351H]). In particular,*
 - *for each $f \in \mathcal{L}^2(\mu)$, it holds that $(\iota[f]_\mu)_\zeta \geq 0_{L^2(\mu_\zeta)}$ for ν -a.e. $\zeta \in Z$ if and only if $f \geq 0$ μ -a.e.;*
 - *for each $f, g \in \mathcal{L}^2(\mu)$, it holds that $(\iota[f \wedge g]_\mu)_\zeta = (\iota[f]_\mu)_\zeta \wedge (\iota[g]_\mu)_\zeta$ for ν -a.e. $\zeta \in Z$.*

Proof As usual, we denote by $\|\cdot\|$ the norm on H , and by $\|\cdot\|_2$, resp. $\|\cdot\|_{2,\zeta}$, the norm on $L^2(\mu)$, resp. $L^2(\mu_\zeta)$. Let \mathcal{A} be satisfying (17), and define a map $\hat{\iota}: \mathcal{A} \rightarrow H$ by $\hat{\iota}: f \mapsto [\delta(f)]_H$.

By definition of $\hat{\iota}$ and δ , by definition (3) of $\|\cdot\|$, and by the property (15) of the disintegration,

$$\begin{aligned} \|\hat{\iota}f_1 - \hat{\iota}f_2\|^2 &= \int_Z \|f_1 - f_2\|_{2,\zeta}^2 \, d\nu(\zeta) \\ &= \int_Z \int_X |f_1 - f_2|^2 \, d\mu_\zeta \, d\nu(\zeta) \qquad f_1, f_2 \in \mathcal{A} . \\ &= \|f_1 - f_2\|_2^2 \end{aligned}$$

As a consequence, $\hat{\iota}: \mathcal{A} \rightarrow H$ descends to a *linear* isometry $\iota: [\mathcal{A}]_\mu \rightarrow H$, and the latter extends to the non-re-labeled (linear) isometry (19) by density of $[\mathcal{A}]_\mu$ in $L^2(\mu)$.

Assume now that $(\mu_\zeta)_{\zeta \in Z}$ is separated with separating family $(A_\zeta)_{\zeta \in Z}$, and fix $h \in (\text{im } \iota)^\perp$. Let $\hat{h} \in S$ be an H -representative of h . For each $\zeta \in Z$, let $\tilde{h}_\zeta \in \mathcal{L}^2(\mu_\zeta)$ be a representative of \hat{h}_ζ , and define a function $\tilde{h}: X \rightarrow \mathbb{R}$ by

$$\tilde{h}(x) := \begin{cases} \tilde{h}_\zeta(x) & \text{if } x \in A_\zeta, \zeta \in Z, \\ 0 & \text{otherwise} \end{cases}.$$

This definition is well-posed since the sets A_ζ 's are pairwise disjoint.

Claim: $\tilde{h} \equiv 0$ μ -a.e. With slight abuse of notation, set $\delta(\tilde{h})_\zeta := [\tilde{h}]_{\mu_\zeta}$ for $\zeta \in Z$, and $\delta(\tilde{h}) := (\zeta \mapsto \delta(\tilde{h})_\zeta)$. By construction, $\delta(\tilde{h}) = \hat{h}$, therefore $\delta(\tilde{h}) \in S$, and so

$$0 = \langle h | \iota[f]_\mu \rangle = \int_Z \int_X \tilde{h} f \, d\mu_\zeta \, d\nu(\zeta), \quad f \in \mathcal{L}^2(\mu),$$

where the right-hand side is well-defined since $\hat{h} \in S$. As a consequence,

$$f \mapsto \int_Z \int_X \tilde{h} f \, d\mu_\zeta \, d\nu(\zeta)$$

is the $\mathbf{0}$ -functional on $L^2(\mu)$. By the Riesz Representation Theorem for $L^2(\mu)$, and by arbitrariness of $[f]_\mu \in L^2(\mu)$, we thus have $\tilde{h} \equiv 0$ μ -a.e.

As a consequence of the claim, $h = [\delta(\tilde{h})]_H = \hat{i}(\tilde{h}) = \mathbf{0}_H$. By arbitrariness of $h \in (\text{im } \iota)^\perp$, we may conclude that $(\text{im } \iota)^\perp = \{\mathbf{0}_H\}$, i.e. that ι is surjective.

We show the first assertion in (ii). A proof of the second assertion is similar, and therefore it is omitted.

Argue by contradiction that there exists $f \in \mathcal{L}^2(\mu)$ with $f \geq 0$ μ -a.e., yet such that $[(\iota[f]_\mu)_\zeta]^- \neq \mathbf{0}_{L^2(\mu_\zeta)}$ for all ζ in some $B \in \mathcal{Z}$ with $\nu B > 0$. In particular, since $\zeta \mapsto [(\iota[f]_\mu)_\zeta]^-$ is a ν -measurable field by Remark 2.20, the following integral is well-defined and strictly positive

$$\int_B \left\| [(\iota[f]_\mu)_\zeta]^- \right\|_{L^2(\mu_\zeta)}^2 \, d\nu(\zeta) > 0.$$

Then, by Eq. 15,

$$\begin{aligned} \int_X f^2 \, d\mu &= \int_X (f^+)^2 \, d\mu = \int_Z \int_X (f^+)^2 \, d\mu_\zeta \, d\nu(\zeta) = \int_Z \left\| [f^+]_{\mu_\zeta} \right\|_{L^2(\mu_\zeta)}^2 \, d\nu(\zeta) \\ &< \int_Z \left\| [f^+]_{\mu_\zeta} \right\|_{L^2(\mu_\zeta)}^2 \, d\nu(\zeta) + \int_B \left\| [(\iota[f]_\mu)_\zeta]^- \right\|_{L^2(\mu_\zeta)}^2 \, d\nu(\zeta). \end{aligned}$$

Since $[f^+]_{\mu_\zeta} = [f]_{\mu_\zeta}^+$ for every $\zeta \in Z$, and by definition of ι , continuing from the previous inequality, we have that

$$\begin{aligned} \int_X f^2 \, d\mu &< \int_Z \left\| [f^+]_{\mu_\zeta} \right\|_{L^2(\mu_\zeta)}^2 \, d\nu(\zeta) + \int_B \left\| [(\iota[f]_\mu)_\zeta]^- \right\|_{L^2(\mu_\zeta)}^2 \, d\nu(\zeta) \\ &= \int_Z \left\| [f]_{\mu_\zeta}^+ \right\|_{L^2(\mu_\zeta)}^2 \, d\nu(\zeta) + \int_B \left\| [f]_{\mu_\zeta}^- \right\|_{L^2(\mu_\zeta)}^2 \, d\nu(\zeta) \\ &\leq \int_Z \left(\left\| [f]_{\mu_\zeta}^+ \right\|_{L^2(\mu_\zeta)}^2 + \left\| [f]_{\mu_\zeta}^- \right\|_{L^2(\mu_\zeta)}^2 \right) \, d\nu(\zeta) \\ &= \int_Z \left\| [f]_{\mu_\zeta} \right\|_{L^2(\mu_\zeta)}^2 \, d\nu(\zeta) \\ &= \left\| \iota[f]_\mu \right\|_H^2 \end{aligned}$$

by definition of H . The inequality contradicts the fact, shown in (i), that $\iota: L^2(\mu) \rightarrow H$ is an isometry, and therefore $\| [f]_\mu \|_{L^2(\mu)}^2 = \|\iota[f]_\mu\|_H^2$. □

2.6 Direct Integrals of Dirichlet Forms

Let (X, \mathcal{X}, μ) be σ -finite standard, (Z, \mathcal{Z}, ν) be σ -finite countably generated, and $(\mu_\zeta)_{\zeta \in Z}$ be a pseudo-disintegration of μ over ν . Further let $\zeta \mapsto (Q_\zeta, \mathcal{D}(Q_\zeta))$ be a ν -measurable field of quadratic forms, each densely defined in $L^2(\mu_\zeta)$ with separable domain, and denote by $(Q, \mathcal{D}(Q))$ their direct integral in the sense of Definition 2.11.

Definition 2.26 We say that $(Q, \mathcal{D}(Q))$ is *compatible* with the pseudo-disintegration $(\mu_\zeta)_{\zeta \in Z}$ if the space S_Q underlying $\zeta \mapsto \mathcal{D}(Q_\zeta)_1$ is of the form $S_{\mathcal{A}}$ for some \mathcal{A} as in Eq. 17 and additionally satisfying $\mathcal{A} \subset \mathcal{D}(Q)$.

Note that, if S_Q is of the form $S_{\mathcal{A}}$ for $\mathcal{A} \subset \mathcal{D}(Q)$ and satisfying (17), then S_H is of the form $S_{\mathcal{A}}$ as well by Remark 2.15.

Definition 2.27 (Diagonal restriction) Let $(Q, \mathcal{D}(Q))$ be a direct integral of quadratic forms compatible with a pseudo-disintegration $(\mu_\zeta)_{\zeta \in Z}$. The form

$$Q_{\text{res}} = Q := \int_Z^{\oplus} Q_\zeta \, d\nu(\zeta) , \quad \mathcal{D}(Q_{\text{res}}) := \mathcal{D}(Q) \cap \iota(L^2(\mu))$$

is a closed (densely defined) quadratic form on $\iota(L^2(\mu))$, called the *diagonal restriction* of $(Q, \mathcal{D}(Q))$.

Remark 2.28 (Comparison with [3]) We note that the form $(Q_{\text{res}}, \mathcal{D}(Q_{\text{res}}))$ coincides with the form $(\mathcal{E}, D(\mathcal{E}))$ defined in [3, Thm. 1.2]. As a consequence, at least in this case, the closability of \mathcal{E} in [3, Thm. 1.2] follows from our Proposition 2.13.

Our first result on direct integrals of concrete quadratic forms is as follows.

Proposition 2.29 Let (X, \mathcal{X}, μ) be σ -finite standard, (Z, \mathcal{Z}, ν) be σ -finite countably generated, and $(\mu_\zeta)_{\zeta \in Z}$ be a separated pseudo-disintegration of μ over ν . Further let $(E, \mathcal{D}(E))$ be a direct integral of closed quadratic forms $\zeta \mapsto (E_\zeta, \mathcal{D}(E_\zeta))$ compatible with $(\mu_\zeta)_{\zeta \in Z}$. Then, $(E, \mathcal{D}(E))$ is a Dirichlet form on $L^2(\mu)$ if and only if $(E_\zeta, \mathcal{D}(E_\zeta))$ is so on $L^2(\mu_\zeta)$ for ν -a.e. $\zeta \in Z$.

Remark 2.30 Since $(\mu_\zeta)_{\zeta \in Z}$ is separated, the isometry ι in Proposition 2.25 is a unitary operator, thus there exists $\iota^{-1}: H \rightarrow L^2(\mu)$, where H is as in Eq. 19. For the sake of clarity, *only* in the proof of Proposition 2.29 below, we distinguish between the quadratic form $(E, \mathcal{D}(E))$ on H and the quadratic form $(\iota^*E, \mathcal{D}(\iota^*E))$ on $L^2(\mu)$ defined by

$$\iota^*E(f, g) := E(\iota f, \iota g) , \quad f, g \in \mathcal{D}(\iota^*E) := \iota^{-1}(\mathcal{D}(E)) .$$

By definition of ι^*E and since $\iota: L^2(\mu) \rightarrow H$ is unitary, we have that $(\iota^*E)_1 = \iota^*E_1$. Therefore, $(\iota^*E, \mathcal{D}(\iota^*E))$ is closed on $L^2(\mu)$, since $(E, \mathcal{D}(E))$ is closed on H by Proposition 2.13(i). Furthermore, the Hilbert spaces $\mathcal{D}(E)_1$ and $\mathcal{D}(\iota^*E)_1$ are intertwined via the unitary isomorphism ι .

In the statement of Proposition 2.29 above and everywhere after its proof — with a slight abuse of notation — these two quadratic forms are identified. Again for the sake of clarity, the statement of the proposition equivalently reads as follows: $(\iota^*E, \mathcal{D}(\iota^*E))$ is a Dirichlet form on $L^2(\mu)$ if and only if $(E_\zeta, \mathcal{D}(E_\zeta))$ is a Dirichlet form on $L^2(\mu_\zeta)$ for ν -a.e. $\zeta \in Z$.

Proof of Proposition 2.29 By e.g. [15, Thm. 1.4.1], the closed quadratic form $(\iota^*E, \mathcal{D}(\iota^*E))$ on $L^2(\mu)$, resp. $(E_\zeta, \mathcal{D}(E_\zeta))$ on $L^2(\mu_\zeta)$, is a Dirichlet form if and only if the associated semigroup $T_\bullet^\iota: L^2(\mu) \rightarrow L^2(\mu)$, resp. $T_{\zeta,\bullet}: L^2(\mu_\zeta) \rightarrow L^2(\mu_\zeta)$, is sub-Markovian, viz.

$$0 \leq T_t^\iota u \leq 1 \quad \mu\text{-a.e.}, \quad u \in L^2(\mu) : 0 \leq u \leq 1 \quad \mu\text{-a.e.}, \quad t > 0, \quad (20a)$$

$$\text{resp. } 0 \leq T_{\zeta,t} v_\zeta \leq 1 \quad \mu_\zeta\text{-a.e.}, \quad v_\zeta \in L^2(\mu_\zeta) : 0 \leq v_\zeta \leq 1 \quad \mu_\zeta\text{-a.e.}, \quad t > 0. \quad (20b)$$

Thus, it suffices to show that T_\bullet^ι is sub-Markovian if and only if $T_{\zeta,\bullet}$ is so for ν -a.e. $\zeta \in Z$.

Since $(E, \mathcal{D}(E))$ and $(\iota^*E, \mathcal{D}(\iota^*E))$ are intertwined by the unitary isomorphism ι , it is not difficult to show that their semigroups T_\bullet and T_\bullet^ι are intertwined as well, viz.

$$T_t^\iota = \iota^* T_t := T_t \circ \iota, \quad t > 0.$$

Furthermore, since $\iota: L^2(\mu) \rightarrow H$ is a Riesz homomorphism by Proposition 2.25(ii),

$$T_\bullet^\iota \text{ is sub-Markovian} \iff \mathbf{0}_H \leq T_t h \leq \mathbf{1}_H, \quad h \in H : \mathbf{0}_H \leq h \leq \mathbf{1}_H, \quad (21)$$

where, by definition of the Banach lattice structure on H in Remark 2.20,

$$\mathbf{0}_H \leq T_t h \leq \mathbf{1}_H \iff \begin{aligned} &0 \leq (T_t h)_\zeta \leq 1 \quad \mu_\zeta\text{-a.e.}, \quad t > 0 \\ &\text{for } \nu\text{-a.e. } \zeta \in Z, \quad h \in H : \mathbf{0}_H \leq h \leq \mathbf{1}_H. \end{aligned} \quad (22)$$

By Proposition 2.13(iii) we have that $(T_t h)_\zeta = T_{\zeta,t} h_\zeta$ for ν -a.e. $\zeta \in Z$ for every $h \in H$. Thus, combining (21) and (22) we may conclude that

$$T_\bullet^\iota \text{ is sub-Markovian} \iff \begin{aligned} &0 \leq T_{\zeta,t} h_\zeta \leq 1 \quad \mu_\zeta\text{-a.e.}, \quad t > 0 \\ &\text{for } \nu\text{-a.e. } \zeta \in Z, \quad h \in H : \mathbf{0}_H \leq h \leq \mathbf{1}_H. \end{aligned} \quad (23)$$

The reverse implication in Eq. 23 together with Eq. 20b immediately show that, if $T_{\zeta,\bullet}$ is sub-Markovian for ν -a.e. $\zeta \in Z$, then T_\bullet^ι is sub-Markovian. The converse implication is not immediate, since the right-hand side of Eq. 23, to be compared with Eq. 20b, contains the additional consistency constraint that $\zeta \mapsto h_\zeta$ be a measurable field representing an element $h \in H$.

In order to show that, if T_\bullet^ι is sub-Markovian, then $T_{\zeta,\bullet}$ is sub-Markovian for ν -a.e. $\zeta \in Z$ is sub-Markovian, we argue as follows. Since $T_{\zeta,\bullet}: L^2(\mu_\zeta) \rightarrow L^2(\mu_\zeta)$ is bounded, and since the unit contraction operator $v \mapsto 0 \vee v \wedge 1$ operates continuously on $L^2(\mu_\zeta)$, it suffices to show that, for any sequence $(v_n^\zeta)_n \subset L^2(\mu_\zeta)$ total in $L^2(\mu_\zeta)$,

$$0 \leq T_{\zeta,t}(0 \vee v_n^\zeta \wedge 1) \leq 1, \quad n \in \mathbb{N}.$$

Let $(u_n)_n \subset H$ be a fundamental sequence of ν -measurable vector fields for H , and recall that $(u_{n,\zeta})_n$ is total in $L^2(\mu_\zeta)$ for every $\zeta \in Z$ by Definition 2.3(c). Applying (23) to each element u_n of this sequence proves the assertion. \square

Proposition 2.29 motivates the next definition. A simple example follows.

Definition 2.31 A quadratic form $(E, \mathcal{D}(E))$ on $L^2(\mu)$ is a *direct integral of Dirichlet forms* $\zeta \mapsto (E_\zeta, \mathcal{D}(E_\zeta))$ on $L^2(\mu_\zeta)$ if it is a direct integral of the Dirichlet forms $\zeta \mapsto$

$(E_\zeta, \mathcal{D}(E_\zeta))$ additionally compatible with the *separated* pseudo-disintegration $(\mu_\zeta)_\zeta$ in the sense of Definition 2.26.

Example 2.32 Let $X = \mathbb{R}^2$ with standard topology, Borel σ -algebra, and the 2-dimensional Lebesgue measure Leb^2 . Consider a Dirichlet form measuring energy only in the first coordinate, viz.

$$E(f) := \int_{\mathbb{R}^2} |\partial_1 f(x_1, x_2)|^2 \, d\text{Leb}^2(x_1, x_2)$$

with $f \in L^2(\mathbb{R}^2)$ and $f(\cdot, x_2) \in W^{1,2}(\mathbb{R})$ for Leb^1 -a.e. $x_2 \in \mathbb{R}$. Then, $(E, \mathcal{D}(E))$ is the direct integral $x_2 \mapsto (E_{x_2}, W^{1,2}(\mathbb{R}))$, where x_2 ranges in $Z = \mathbb{R}$ the real line, $(X, \mathcal{X}, \mu_{x_2})$ is again the standard real line for every $x_2 \in \mathbb{R}$, and

$$E_{x_2}(f) := \int_{\mathbb{R}} |df(\cdot, x_2)|^2 \, d\text{Leb}^1, \quad \text{for } \text{Leb}^1\text{-a.e. } x_2 \in \mathbb{R}.$$

2.7 Superposition of Dirichlet Forms

We recall here the definition of a *superposition of Dirichlet forms* in the sense of [5, §V.3.1]. Let (Z, \mathcal{Z}, ν) be σ -finite, $(X, \mathcal{X}, \mu_\zeta)$ be σ -finite and $(E_\zeta, \mathcal{D}(E_\zeta))$ be a Dirichlet form on $L^2(\mu_\zeta)$, for every $\zeta \in Z$. Assume that

- (SP₁) $\zeta \mapsto \mu_\zeta A$ is ν -measurable for every $A \in \mathcal{X}$;
- (SP₂) $\zeta \mapsto E_\zeta(\hat{f}) \in [0, \infty]$ is ν -measurable for every measurable $\hat{f}: X \rightarrow [-\infty, \infty]$.

Let us now consider

- a measure $\lambda \ll \nu$ on (Z, \mathcal{Z}) so that $\mu := \int_Z \mu_\zeta \, d\lambda(\zeta)$ is σ -finite on (X, \mathcal{X}) (therefore: (X, \mathcal{X}, μ) is standard);
- the subspace \mathbb{D} of all functions $f \in L^2(\mu)$ so that

$$\mathcal{E}(f) := \int_Z E_\zeta(f) \, d\nu(\zeta) < \infty, \tag{24}$$

and let us further assume that

- (SP₃) \mathbb{D} is dense in $L^2(\mu)$.

Then, it is claimed in [5, p. 214] that Eq. 24 is well-defined and depends only on the μ -class of f , and it is shown in [5, Prop. V.3.1.1] that

Definition 2.33 $(\mathcal{E}, \mathbb{D})$ is a Dirichlet form on $L^2(\mu)$, called the *superposition of* $\zeta \mapsto E_\zeta$.

Note that we may always choose $\lambda = \nu$ provided that the integral measure μ defined above is σ -finite. If this is not the case, we may recast the definition by letting $\nu := \lambda$. In this way, we may always assume with no loss of generality that μ is given, and that (μ_ζ) is a pseudo-disintegration of μ over ν .

Remark 2.34 In fact, [5] requires all functions in (SP₁)-(SP₂) to be \mathcal{Z} -measurable, rather than only ν -measurable. Here, we relax this condition to ‘ ν -measurability’ in order to simplify the proof of the reverse implication in the next Proposition 2.35. Our definition of ‘superposition’ is equivalent to the one in [5].

Proposition 2.35 *Let (X, \mathcal{X}, μ) be σ -finite standard, (Z, \mathcal{Z}, ν) be σ -finite countably generated, $(\mu_\zeta)_{\zeta \in Z}$ be a separated pseudo-disintegration of μ over ν , and $(E_\zeta, \mathcal{D}(E_\zeta))$ be a Dirichlet form on $L^2(\mu_\zeta)$ for every $\zeta \in Z$.*

Then, the following are equivalent:

- (i) *there exists the superposition $(\mathcal{E}, \mathbb{D})$ of $\zeta \mapsto E_\zeta$ and the space \mathbb{D} in Eq. 24 is $\mathcal{E}_1^{1/2}$ -separable;*
- (ii) *there exists a direct integral of Dirichlet forms $(E, \mathcal{D}(E))$ of the forms $\zeta \mapsto E_\zeta$.*

Furthermore, if either one holds, then $(E, \mathcal{D}(E))$ and $(\mathcal{E}, \mathbb{D})$ are isomorphic Dirichlet spaces.

Proof We only show that (i) implies (ii). A proof of the reverse implication is similar, and it is therefore omitted. For simplicity, set throughout the proof $H_\zeta := L^2(\mu_\zeta)$, with norm $\|\cdot\|_\zeta$, for every $\zeta \in Z$.

Assume (i). It follows from (SP₁) that $\zeta \mapsto \|f\|_\zeta$ is ν -measurable for every $f \in \mathbb{D}$, and thus from (SP₂) that $\zeta \mapsto E_{\zeta,1}(f, g)$ is ν -measurable for every $f, g \in \mathbb{D}$ by polarization. By $\mathcal{E}_1^{1/2}$ -separability of \mathbb{D} , there exists a countable \mathbb{Q} -linear space $\mathcal{U} \subset \mathcal{L}^2(\mu)$ so that $[\mathcal{U}]_\mu$ is $\mathcal{E}_1^{1/2}$ -dense in \mathbb{D} , and dense in $L^2(\mu)$ by (SP₃). Since $(\mu_\zeta)_{\zeta \in Z}$ is separated by assumption, it follows by Proposition 2.25 that $[\mathcal{U}]_{\mu_\zeta}$ is dense in $L^2(\mu_\zeta)$ for ν -a.e. $\zeta \in Z$.

As a consequence, the quadratic form $(E_\zeta, [\mathcal{U}]_{\mu_\zeta})$ is densely defined on H_ζ . Since $(E_\zeta, \mathcal{D}(E_\zeta))$ is closed, the closure $(E_\zeta^{\text{res}}, \mathcal{D}(E_\zeta^{\text{res}}))$ of $(E_\zeta, [\mathcal{U}]_{\mu_\zeta})$ is well-defined and a Dirichlet form on $H_\zeta = L^2(\mu_\zeta)$.

Again since $(\mu_\zeta)_{\zeta \in Z}$ is separated, we may then construct a form $(E, \mathcal{D}(E))$ as the direct integral of Dirichlet forms (Dfn. 2.31) of the forms $\zeta \mapsto (E_\zeta^{\text{res}}, \mathcal{D}(E_\zeta^{\text{res}}))$ with underlying space of measurable vector fields $S = S_{\mathcal{U}}$ generated by $\delta(\mathcal{U})$ in the sense of Proposition 2.4. By construction, the pre-Hilbert spaces $([\mathcal{U}]_\mu, \mathcal{E}_1^{1/2})$ and $([\delta(\mathcal{U})]_{\mathcal{D}(E)_1}, E_1^{1/2})$ are linearly and latticially isometrically isomorphic. The isomorphism extends to a unitary lattice isomorphism between $(\mathbb{D}, \mathcal{E}_1^{1/2})$ and $\mathcal{D}(E)_1$. The last assertion follows provided we show the following claim.

Claim: $\mathcal{D}(E_\zeta) = \mathcal{D}(E_\zeta^{\text{res}})$ for ν -a.e. $\zeta \in Z$. Argue by contradiction that there exists $B \in \mathcal{Z}^\nu$, with $\nu B > 0$ and so that $\mathcal{D}(E_\zeta^{\text{res}}) \subsetneq \mathcal{D}(E_\zeta)$ for every $\zeta \in B$. We may assume with no loss of generality that $B \in \mathcal{Z}$. Furthermore, since (Z, \mathcal{Z}, ν) is σ -finite countably generated, we may and shall assume that $\nu B < \infty$. Denote by $\mathcal{D}(E_\zeta^{\text{res}})_1^{\perp \zeta}$ the $(E_\zeta)_1^{1/2}$ -orthogonal complement of $\mathcal{D}(E_\zeta^{\text{res}})_1$ in $\mathcal{D}(E_\zeta)_1$. By the axiom of choice, there exists $\hat{h}_\zeta := (\zeta \mapsto \hat{h}_\zeta)$ with $\hat{h}_\zeta \in \mathcal{D}(E_\zeta^{\text{res}})_1^{\perp \zeta}$ and $\|\hat{h}_\zeta\|_{\mathcal{D}(E_\zeta)_1} = 1$ for all $\zeta \in B$ and $\hat{h}_\zeta := \mathbf{0}_{\mathcal{D}(E_\zeta)}$ for $\zeta \in B^c$. By closability of $(E_\zeta, \mathcal{D}(E_\zeta))$, the domain $\mathcal{D}(E_\zeta)_1$ embeds identically via $\iota_{\zeta,1}$ into H_ζ , thus \hat{h}_ζ may be regarded as an element $\iota_{\zeta,1}\hat{h}_\zeta$ of H_ζ for every $\zeta \in Z$. Since $\|\iota_{\zeta,1}\|_{\text{op}} \leq 1$ for every $\zeta \in Z$, then

$$0 < \|\iota_{\zeta,1}\hat{h}_\zeta\|_\zeta \leq 1, \quad \zeta \in B. \tag{25}$$

Since $(E, \mathcal{D}(E))$ is in particular a direct integral of quadratic forms, $\zeta \mapsto \iota_{\zeta,1}$ is a ν -measurable field of bounded operators, and thus $\zeta \mapsto \iota_{\zeta,1}\hat{h}_\zeta$ is a ν -measurable field of vectors.

Now, let H be defined as in Eq. 18. Setting $\bar{h} := (\zeta \mapsto \iota_{\zeta,1} \hat{h}_{\zeta})$, it follows by Eq. 25 that

$$\|\bar{h}\|^2 := \int_Z \|\iota_{\zeta,1} \hat{h}_{\zeta}\|_{\zeta}^2 \, d\nu(\zeta) = \int_B \|\iota_{\zeta,1} \hat{h}_{\zeta}\|_{\zeta}^2 \, d\nu(\zeta) \in (0, \nu B) .$$

In particular, for the equivalence class $h := [\bar{h}]_H$, we have that $h \neq \mathbf{0}_H$. By Proposition 2.25, there exists $\tilde{h} \in \mathcal{L}^2(\mu)$ representing $h \in H$, and thus satisfying $\mathbf{0}_{\mathbb{D}} \neq [\tilde{h}]_{\mu} \in \mathbb{D}$. On the other hand though,

$$\begin{aligned} \mathcal{E}_1([\tilde{h}]_{\mu}, [\tilde{u}]_{\mu}) &:= \int_Z E_{\zeta,1}([\tilde{h}]_{\mu_{\zeta}}, [\tilde{u}]_{\mu_{\zeta}}) \, d\nu(\zeta) && \tilde{u} \in \mathcal{U} , \\ &= \int_Z E_{\zeta,1}(\hat{h}_{\zeta}, \delta(\tilde{u})_{\zeta}) \, d\nu(\zeta) = 0 , \end{aligned}$$

by definition of \hat{h} . By $\mathcal{E}_1^{1/2}$ -density of $[\mathcal{U}]_{\mu}$ in \mathbb{D} , the latter implies that $h = \mathbf{0}_{\mathbb{D}}$, the desired contradiction. □

Remark 2.36 If the disintegration in Proposition 2.35 is not separated, then $(\mathcal{E}, \mathbb{D})$ is still isomorphic, as a quadratic form, to the diagonal restriction $(Q_{\text{res}}, \mathcal{D}(Q_{\text{res}}))$ (Dfn. 2.27) of the direct integral of quadratic forms $(E, \mathcal{D}(E))$.

3 Ergodic Decomposition

Everywhere in this section, let $(X, \tau, \mathcal{X}, \mu)$ be satisfying Assumption 2.18. We are interested in the notion of invariant sets for a Dirichlet form.

Definition 3.1 (Invariant sets, irreducibility, cf. [15, p. 53]) Let $(E, \mathcal{D}(E))$ be a Dirichlet form on $L^2(\mu)$. We say that $A \subset X$ is *E-invariant* if it is μ -measurable and any of the following equivalent³ conditions holds.

- (a) $T_t(\mathbb{1}_A f) = \mathbb{1}_A T_t f$ μ -a.e. for any $f \in L^2(\mu)$ and $t > 0$;
- (b) $T_t(\mathbb{1}_A f) = 0$ μ -a.e. on A^c for any $f \in L^2(\mu)$ and $t > 0$;
- (c) $G_{\alpha}(\mathbb{1}_A f) = 0$ μ -a.e. on A^c for any $f \in L^2(\mu)$ and $\alpha > 0$;
- (d) $\mathbb{1}_A f \in \mathcal{D}(E)$ for any $f \in \mathcal{D}(E)$ and

$$E(f, g) = E(\mathbb{1}_A f, \mathbb{1}_A g) + E(\mathbb{1}_{A^c} f, \mathbb{1}_{A^c} g) , \quad f, g \in \mathcal{D}(E) \tag{26}$$

- (e) $\mathbb{1}_A f \in \mathcal{D}(E)_e$ for any $f \in \mathcal{D}(E)_e$ and Eq. 26 holds for any $f, g \in \mathcal{D}(E)_e$.

The form $(E, \mathcal{D}(E))$ is *irreducible* if, whenever A is E -invariant, then either $\mu A = 0$ or $\mu A^c = 0$.

As shown by Example 2.32, the form $(E, \mathcal{D}(E))$ constructed in Proposition 2.29 is hardly ever irreducible, even if $(E_{\zeta}, \mathcal{D}(E_{\zeta}))$ is so for every $\zeta \in Z$.

³See [15, Lem. 1.6.1, p. 53], the proof of which adapts *verbatim* to our more general setting.

3.1 The Algebra of Invariant Sets

Invariants sets of symmetric Markov processes on locally compact Polish spaces are studied in detail by H. Ôkura in [25]. In particular, he notes the following. For $A \in \mathcal{X}$ set

$$[A]_E := \left\{ \tilde{A} \in \mathcal{X}^\mu : \mathbb{1}_{\tilde{A}} \text{ is an } E\text{-quasi-continuous version of } \mathbb{1}_A \right\} .$$

For arbitrary $A \in \mathcal{X}$ it can happen that $[A]_E = \emptyset$ or that $A \notin [A]_E$. If however $(E, \mathcal{D}(E))$ is regular, then $[A]_E$ is non-empty for every E -invariant set A . Suppose now $A_0, A_1 \in \mathcal{X}$ and $[A_0]_E \neq \emptyset$. Then, one has the following dichotomy, [25, Rmk. 1.1(ii)],

- $[A_0]_E = [A_1]_E$ if (and only if) $\mu(A_0 \Delta A_1) = 0$;
- $[A_0]_E \cap [A_1]_E = \emptyset$ if (and only if) $\mu(A_0 \Delta A_1) > 0$.

As a consequence, when describing an E -invariant set A of a regular Dirichlet form $(E, \mathcal{D}(E))$, we may use interchangeably the E -class $[A]_E$ — i.e. the finest object representing A , as far as E is concerned — and the μ -class $[A]_\mu$ representing A in the measure algebra of (X, \mathcal{X}, μ) . This motivates to allow A in our definition of invariant set to be μ -measurable, rather than only measurable.

We turn now to the study of invariant sets via direct integrals. We aim to show that, under suitable assumptions on μ , a Dirichlet form $(E, \mathcal{D}(E))$ on $L^2(\mu)$ may be decomposed as a direct integral $\zeta \mapsto (E_\zeta, \mathcal{D}(E_\zeta))$ with $(E_\zeta, \mathcal{D}(E_\zeta))$ irreducible for every $\zeta \in Z$.

To this end, we need to construct a *measure space* (Z, \mathcal{Z}, ν) “indexing” E -invariant sets. Let us start with a heuristic argument, showing how this cannot be done naïvely, at least in the general case when (X, \mathcal{X}, μ) is merely σ -finite.

Let \mathcal{X}_0 be the family of μ -measurable E -invariant subsets of X , and note that \mathcal{X}_0 is a σ -sub-algebra of \mathcal{X}^μ , e.g. [15, Lem. 1.6.1, p. 53]. Let μ_0 be the restriction of $\hat{\mu}$ to (X, \mathcal{X}_0) .

The space $(X, \mathcal{X}_0, \mu_0)$ — our candidate for (Z, \mathcal{Z}, ν) — is generally *not* σ -finite, nor even semi-finite. For instance, in the extreme case when $(E, \mathcal{D}(E))$ is irreducible and $\mu X = \infty$, then \mathcal{X}_0 is the minimal σ -algebra on X , the latter is an atom, and thus μ_0 is purely infinite.

Since (X, \mathcal{X}, μ) is σ -finite, every disjoint family of μ -measurable non-negligible subsets is at most countable [14, 215B(iii)], thus $(X, \mathcal{X}_0, \mu_0)$ has up to countably many disjoint atoms.

However, even in the case when $(X, \mathcal{X}_0, \mu_0)$ has *no* atoms, μ_0 might again be purely infinite. This is the case of Example 2.32, where $\mathcal{X}_0 = \{\emptyset, \mathbb{R}\} \otimes \mathcal{B}(\mathbb{R})^{\text{Leb}^1}$ is the product σ -algebra of the minimal σ -algebra on the first coordinate with the Lebesgue σ -algebra on the second coordinate, and where μ_0 coincides with the $\hat{\mu}$ -measure of horizontal stripes.

This latter example shows that, again even when $(X, \mathcal{X}_0, \mu_0)$ has no atoms, the complete locally determined version [14, 213D] of $(X, \mathcal{X}_0, \mu_0)$ is trivial. Thus, in this generality, there is no natural way to make $(X, \mathcal{X}_0, \mu_0)$ into a more amenable measure space while retaining information on E -invariant sets.

The situation improves as soon as (X, \mathcal{X}, μ) is a probability space, in which case so is $(X, \mathcal{X}_0, \mu_0)$. The reasons for this fact are better phrased in the language of von Neumann algebras.

Remark 3.2 (Associated von Neumann algebras) Denote by \mathcal{M} the space $L^\infty(\mu)$ regarded as the (commutative, unital) von Neumann algebra of multiplication operators in $B(L^2(\mu))$. Then, $L^\infty(\mu_0)$ is a (commutative) von Neumann sub-algebra of \mathcal{M} , denoted by \mathcal{M}_0 . Two key observations are as follows:

- since $(X, \mathcal{X}_0, \mu_0)$ is now (semi-)finite, \mathcal{M}_0 is unital as well;
- by Definition 3.1(d), the algebra \mathcal{M}_0 acts by multiplication also on $\mathcal{D}(E)$, and the action $\mathcal{M}_0 \curvearrowright L^2(\mu)$ is compatible with the action $\mathcal{M}_0 \curvearrowright \mathcal{D}(E)$ by restriction.

The next definition, borrowed from [4], encodes a notion of “smallness” of the σ -algebra \mathcal{X} w.r.t. μ .

Definition 3.3 [4, Dfn. A.1] Let $\mathcal{X}^* \subset \mathcal{X}$ be a countably generated σ -subalgebra. We say that:

- \mathcal{X} is μ -essentially countably generated by \mathcal{X}^* if for each $A \in \mathcal{X}$ there is $A^* \in \mathcal{X}^*$ with $\mu(A \Delta A^*) = 0$;
- \mathcal{X} is μ -essentially countably generated if it is so by some \mathcal{X}^* as above.

By our Assumption 2.18, \mathcal{X} is countably generated, thus \mathcal{X}_0 is μ_0 -essentially countably generated by $\mathcal{X}^* := \mathcal{X} \cap \mathcal{X}_0$. We denote by μ_0^* the restriction of μ_0 to \mathcal{X}^* . As noted in [4, p. 418], atoms of \mathcal{X}^* are, in general, larger (in cardinality, *not* in measure) than atoms of \mathcal{X} . It is therefore natural to pass to a suitable quotient space. Following [4, Dfn. A.5], we define an equivalence relation \sim on X by

$$x_1 \sim x_2 \quad \text{if and only if} \quad x_1 \in A \iff x_2 \in A \quad \text{for every } A \in \mathcal{X}^* . \tag{27}$$

Further let $p: X \rightarrow Z := X / \sim$ be the quotient map, $\mathcal{Z} := \{B \subset Z : p^{-1}(B) \in \mathcal{X}^*\}$ be the quotient σ -algebra induced by p , and $\nu := p_{\#}\mu_0^*$ be the quotient measure. Similarly to [4, p. 416], it follows by definition of \sim that every $A \in \mathcal{X}^*$ is p -saturated. In particular:

$$\emptyset \neq A \subset p^{-1}(p(x)) \implies A = p^{-1}(p(x)) , \quad A \in \mathcal{X}^* . \tag{28}$$

As a consequence \mathcal{X}^* and \mathcal{Z} are isomorphic and thus both are countably generated, since \mathcal{X}^* is by assumption. Furthermore, (Z, \mathcal{Z}) is separable by construction, and thus it is countably separated.

3.2 Ergodic Decomposition of Forms: Probability Measure Case

We are now ready to state our main result, a decomposition theorem for Dirichlet forms over their invariant sets.

Theorem 3.4 (Ergodic decomposition: regular case) *Let $(X, \tau, \mathcal{X}, \mu)$ be a locally compact Polish probability space, and $(E, \mathcal{D}(E))$ be a τ -regular Dirichlet form on $L^2(\mu)$. Then, there exist*

- (i) a probability space (Z, \mathcal{Z}, ν) and a measurable map $s: (X, \mathcal{X}) \rightarrow (Z, \mathcal{Z})$;
- (ii) a ν -essentially unique disintegration $(\mu_\zeta)_{\zeta \in Z}$ of $\hat{\mu}$ w.r.t. ν , strongly consistent with s , and so that, when $s^{-1}(\zeta)$ is endowed with the subspace topology and the trace σ -algebra inherited from $(X, \tau, \mathcal{X}^\mu)$, then $(s^{-1}(\zeta), \mu_\zeta)$ is a Radon probability space for ν -a.e. $\zeta \in Z$;
- (iii) a ν -measurable field $\zeta \mapsto (E_\zeta, \mathcal{D}(E_\zeta))$ of τ -regular irreducible Dirichlet forms $(E_\zeta, \mathcal{D}(E_\zeta))$ on $L^2(\mu_\zeta)$;

so that

$$L^2(\mu) = \int_Z^\oplus L^2(\mu_\zeta) \, d\nu(\zeta) \quad \text{and} \quad E = \int_Z^\oplus E_\zeta \, d\nu(\zeta) . \tag{29}$$

Proof (i) Let (Z, \mathcal{Z}, ν) be the quotient space of $(X, \mathcal{X}^*, \mu_0^*)$ defined in Section 3.1, and recall that (Z, \mathcal{Z}) is countably separated. Note that $\text{id}_X: (X, \mathcal{X}^\mu, \hat{\mu}) \rightarrow (X, \mathcal{X}_0^{\mu_0}, \hat{\mu}_0)$ is inverse-measure-preserving [14, 235H(b)], thus so is

$$s := p \circ \text{id}_X: (X, \mathcal{X}^\mu, \hat{\mu}) \rightarrow (Z, \mathcal{Z}, \nu) . \tag{30}$$

Since $(X, \tau, \mathcal{X}^\mu, \hat{\mu})$ is Radon (in the sense of [14, 411H(b)]) and (Z, \mathcal{Z}, ν) is a probability space (in particular: strictly localizable [14, 322C]), there exists a disintegration $(\mu_\zeta)_{\zeta \in Z}$ of $\hat{\mu}$ over ν consistent with s , and so that $(X, \tau, \mathcal{X}, \mu_\zeta)$ is a Radon probability space [14, 452O, 452G(a)]. Since (Z, \mathcal{Z}) is countably separated, $(\mu_\zeta)_{\zeta \in Z}$ is in fact strongly consistent with s [14, 452G(c)]. By definition of strong consistency, we may restrict μ_ζ to $s^{-1}(\zeta)$; the Radon property is preserved by this restriction [14, 416R(b)]. Since s factors through p , one has $s^{-1}(\zeta) \in \mathcal{X}^* \subset \mathcal{X}$ for every $\zeta \in Z$. In particular, $s^{-1}(\zeta)$ is a Borel subset of the metrizable Luzin space (X, τ) , and thus a metrizable Luzin space itself by [27, §II.1, Thm. 2, p. 95]. It follows that $\text{supp}[\mu_\zeta]$, endowed with the subspace topology inherited from (X, τ) and the induced Borel σ -algebra, satisfies Assumption 2.18. The disintegration is ν -essentially unique, similarly to [4, Thm. A.7, Step 1, p. 420].

This shows (i)–(ii). The proof of (iii) is divided into several steps.

1. Measurable fields Let \mathcal{C} be a special standard core [15, p. 6] for $(E, \mathcal{D}(E))$, and $N \subset Z$ be a ν -negligible set so that $(X, \tau, \mathcal{X}, \mu_\zeta)$ is Radon for every $\zeta \in N^c$. Then, $\mathcal{C}|_{\text{supp}[\mu_\zeta]}$ is dense in $L^2(\mu_\zeta)$ for every $\zeta \in N^c$. In particular, since $L^2(\mu)$ is separable, there exists a fundamental sequence $(u_n)_n \subset \mathcal{C}$, i.e. total in $L^2(\mu_\zeta)$ for every $\zeta \in N^c$, and additionally total in $L^2(\mu)$. Since $(E, \mathcal{D}(E))$ is regular, $\mathcal{D}(E)_1$ is separable by [23, Prop. IV.3.3(i)], and therefore we can and will assume, with no loss of generality that $(u_n)_n$ is additionally $E_1^{1/2}$ -total in $\mathcal{D}(E)_1$.

Moreover, \mathcal{C} is an algebra, thus $\zeta \mapsto \langle f | g \rangle_\zeta = \mu_\zeta(fg)$ is ν -measurable by definition of disintegration for every $f, g \in \mathcal{C}$. As a consequence, by Proposition 2.4 there exists a *unique* ν -measurable field of Hilbert spaces $\zeta \mapsto L^2(\mu_\zeta)$ making ν -measurable all functions of the form $\zeta \mapsto \mu_\zeta f$ with $f \in \mathcal{C}$. We denote by $S_{\mathcal{C}}$ the underlying space of ν -measurable vector fields. Everywhere in the following, we identify $[f]_{\mu_\zeta}$ with a fixed continuous representative $f \in \mathcal{C}$, thus writing f in place of $\delta(f)$.

2. L^2 -isomorphism Since $(\mu_\zeta)_{\zeta \in Z}$ is strongly consistent with s , it is separated. Therefore, the first isomorphism in Eq. 29 follows now by Eq. 19 with underlying space $S_{\mathcal{C}}$. In the following, set $H := L^2(\mu)$ and $H_\zeta := L^2(\mu_\zeta)$.

3. Semigroups Let T_t be the semigroup associated to $(E, \mathcal{D}(E))$ and consider the natural complexification $T_t^{\mathbb{C}}$ of T_t defined on $H_{\mathbb{C}} := H \otimes_{\mathbb{R}} \mathbb{C}$. For $g \in L^\infty(\nu)$ denote by

$$M_g := \int_Z^{S_{\mathcal{C}} \times \oplus} g(\zeta) \mathbf{1}_{H_\zeta} \, d\nu(\zeta)$$

the associated *diagonalizable operator* in $B(H)$, [12, §II.2.4 Dfn. 3, p. 185]. For $B \in \mathcal{Z}$ set $M_B := M_{\mathbf{1}_B}$.

Claim: the commutator $[T_t^{\mathbb{C}}, M_g^{\mathbb{C}}]$ vanishes for $g \in L^\infty_{\mathbb{C}}(\nu)$. By [12, §II.2.3, Prop. 4(ii), p. 183] and the norm-density of simple functions in $L^\infty(\nu)$, it suffices to show that $[T_t, M_B] = 0$ for every $B \in \mathcal{Z}$. To this end, recall the discussion [12, p. 165] on ν -measurable structures induced by ν -measurable subsets of Z . Since $B \in \mathcal{Z}$, then $A := p^{-1}(B) \in \mathcal{X}_0^{\mu_0}$, and $A \in \mathcal{X}^\mu$ as well [14, 235H(c)]. Note further that, since H

is reconstructed as a direct integral with underlying space S_C , for every $h \in \mathcal{C}$ the representative h_ζ of h in H_ζ may be chosen so that $h_\zeta = h$ for every $\zeta \in Z$. Thus, for all $f, g \in \mathcal{C}$,

$$\begin{aligned} \langle M_B f \mid g \rangle_H &= \int_Z \mathbb{1}_B(\zeta) \langle \mathbb{1}_{H_\zeta} f_\zeta \mid g_\zeta \rangle_\zeta \, d\nu(\zeta) \\ &= \int_B \int_X f g \, d\mu_\zeta \, d\nu(\zeta) = \int_A f g \, d\mu \\ &= \langle \mathbb{1}_A f \mid g \rangle_H . \end{aligned}$$

By density of \mathcal{C} in H and since M_B is bounded, it follows that $M_B = \mathbb{1}_A$ as elements of $B(H)$. Thus, $[T_t, M_B] = [T_t, \mathbb{1}_A] = 0$ for every $t > 0$ by Definition 3.1(a), since A is E -invariant.

By the characterization of decomposable operators via diagonalizable operators [12, §II.2.5, Thm. 1, p. 187], T_t is decomposable, and represented by a ν -measurable field of contraction operators $\zeta \mapsto T_{\zeta,t}$.

Finally, in light of [12, §II.2.3, Prop. 4, p. 183], it is a straightforward verification that $T_{\zeta,t}$, $t > 0$, is a strongly continuous symmetric contraction semigroup on H_ζ for ν -a.e. $\zeta \in Z$, since so is T_t . Analogously to the proof of Proposition 2.29, $T_{\zeta,t}$ is sub-Markovian for ν -a.e. $\zeta \in Z$, since so is T_t .

4. Forms: construction Denote by $(E_\zeta, \mathcal{D}(E_\zeta))$ the Dirichlet form on $L^2(\mu_\zeta)$ associated to the sub-Markovian semigroup $T_{\zeta,t}$ for ν -a.e. $\zeta \in Z$. Let further $G_{\zeta,\alpha}$, $\alpha > 0$, be the associated strongly continuous contraction resolvent.

We claim that $\mathcal{C} \subset \mathcal{D}(E_\zeta)$ for ν -a.e. $\zeta \in Z$.

Firstly, note that $\zeta \mapsto E_\zeta(f, g)$ is ν -measurable, since it is the ν -a.e.-limit of the measurable functions $\zeta \mapsto E_\zeta^{(\beta)}(f, g) := \langle f - \beta G_{\zeta,\beta} f \mid g \rangle_\zeta$ as $\beta \rightarrow \infty$ by Eq. 1c. By [23, p. 27],

$$E_\zeta(f, g) = \lim_{\beta \rightarrow \infty} \left\langle f - \beta \int_0^\infty e^{-\beta t} T_{\zeta,t} f \, dt \mid g \right\rangle_\zeta , \quad f, g \in \mathcal{C} .$$

Now,

$$\begin{aligned} \int_Z E_\zeta(f) \, d\nu(\zeta) &= \int_Z \lim_{\beta \rightarrow \infty} \left\langle f - \beta \int_0^\infty e^{-\beta t} T_{\zeta,t} f \, dt \mid f \right\rangle_\zeta \, d\nu(\zeta) \\ &\leq \liminf_{\beta \rightarrow \infty} \int_Z \left\langle f - \beta \int_0^\infty e^{-\beta t} T_{\zeta,t} f \, dt \mid f \right\rangle_\zeta \, d\nu(\zeta) \end{aligned}$$

by Fatou’s Lemma. It is readily checked that, since $\|T_{\zeta,t}\|_{\text{op}} \leq 1$, we may exchange the order of both integration and H_ζ -scalar products by Fubini’s Theorem. Thus,

$$\begin{aligned} \int_Z E_\zeta(f) \, d\nu(\zeta) &\leq \liminf_{\beta \rightarrow \infty} \int_Z \left\langle f - \beta \int_0^\infty e^{-\beta t} T_{\zeta,t} f \, dt \mid f \right\rangle_\zeta \, d\nu(\zeta) \\ &= \int_Z \|f\|_\zeta^2 \, d\nu(\zeta) - \limsup_{\beta \rightarrow \infty} \int_0^\infty \int_Z \beta e^{-\beta t} \langle T_{\zeta,t} f \mid f \rangle_\zeta \, d\nu(\zeta) \, dt . \end{aligned}$$

By the representation of T_t via $\zeta \mapsto T_{\zeta,t}$ established in Step 3,

$$\int_Z E_\zeta(f) \, d\nu(\zeta) \leq \int_Z \int_X f^2 \, d\mu_\zeta \, d\nu(\zeta) - \limsup_{\beta \rightarrow \infty} \int_0^\infty \beta e^{-\beta t} \langle T_t f \mid f \rangle_H \, dt .$$

Finally, by Eqs. 14, [23, p. 27] and 1c,

$$\begin{aligned} \int_Z E_\zeta(f) \, d\nu(\zeta) &\leq \liminf_{\beta \rightarrow \infty} \beta \int_0^\infty \langle f - e^{-\beta t} T_t f \mid f \rangle_H \, dt \\ &= \liminf_{\beta \rightarrow \infty} \beta \left\langle f - \int_0^\infty e^{-\beta t} T_t f \, dt \mid f \right\rangle_H = E(f) < \infty . \end{aligned}$$

This shows that $E_\zeta(f) < \infty$ for every $f \in \mathcal{C}$ for ν -a.e. $\zeta \in Z$, thus $\mathcal{C} \subset \mathcal{D}(E_\zeta)$ ν -a.e.

Claim: \mathcal{C} is a core for $(E_\zeta, \mathcal{D}(E_\zeta))$ for ν -a.e. $\zeta \in Z$. It suffices to show that the inclusion $\mathcal{C} \subset \mathcal{D}(E_\zeta)$ is $(E_\zeta)_1^{1/2}$ -dense for ν -a.e. $\zeta \in Z$. Argue by contradiction that there exists a ν -measurable non-negligible set B so that the inclusion $\mathcal{C} \subset \mathcal{D}(E_\zeta)$ is not $(E_\zeta)_1^{1/2}$ -dense for every $\zeta \in B$, and let $\mathcal{C}^{\perp_\zeta}$ be the $(E_\zeta)_1^{1/2}$ -orthogonal complement of \mathcal{C} in $\mathcal{D}(E_\zeta)$. By the axiom of choice we may construct $h \in \prod_{\zeta \in Z} H_\zeta$ so that $h_\zeta \in \mathcal{C}^{\perp_\zeta} \setminus \{0\}$ for $\zeta \in B$ and $h_\zeta = 0$ for $\zeta \in B^c$. Further let $(u_n)_n \subset \mathcal{C}$ be as in Step 1. Then, $\zeta \mapsto \langle u_n \mid h_\zeta \rangle_\zeta = 0$ is ν -measurable for every n . As a consequence, $\zeta \mapsto h_\zeta$ is ν -measurable (i.e., it belongs to $S_{\mathcal{C}}$) by [12, §II.1.4, Prop. 2, p. 166]. By the first isomorphism in Eq. 29, established in Step 1, $\zeta \mapsto h_\zeta$ represents an element h in H . Since $(u_n)_n$ is total in H , there exists n so that

$$0 \neq \langle u_n \mid h \rangle_H = \int_Z \langle u_n \mid h_\zeta \rangle_\zeta \, d\nu(\zeta) = 0 ,$$

a contradiction. Since functions in \mathcal{C} are continuous, the form $(E_\zeta, \mathcal{D}(E_\zeta))$ is regular for ν -a.e. $\zeta \in Z$. In particular, $\mathcal{D}(E_\zeta)_1$ is separable for ν -a.e. $\zeta \in Z$ by [23, Prop. IV.3.3].

We note that, by the above claim and [23, Prop. IV.3.3(i)], $\mathcal{D}(E_\zeta)_1$ is separable for every $\zeta \in Z$, and so the observation in Remark 2.12 is satisfied.

5. Forms: direct integral By Step 1, resp. Step 4, $\zeta \mapsto L^2(\mu_\zeta)$, resp. $\zeta \mapsto \mathcal{D}(E_\zeta)_1$, is a ν -measurable field of Hilbert spaces with underlying space $S_{\mathcal{C}}$. In particular, $\zeta \mapsto (E_\zeta, \mathcal{D}(E_\zeta))$ satisfies Definition 2.11, and we may consider the direct integral of quadratic forms

$$\tilde{E}(f) := \int_Z^{\oplus} E_\zeta(f) \, d\nu(\zeta) \tag{31}$$

defined by Eq. 7.

We claim that $(\tilde{E}, \mathcal{D}(\tilde{E})) = (E, \mathcal{D}(E))$. This is a consequence of Proposition 2.13(iii), since (8) was shown in Step 3 for T_t . Definition 2.26 holds with $\mathcal{A} = \mathcal{C}$ by construction.

6. Forms: irreducibility Let A_ζ be E_ζ -invariant, with $\hat{\mu}_\zeta A_\zeta > 0$.

With no loss of generality, we may and will assume that $A_\zeta \in \mathcal{X}$. Up to removing a ν -negligible set of ζ 's, we have that $A_\zeta \subset s^{-1}(\zeta)$, by strong consistency of the disintegration. Thus, by Eq. 14,

$$\mu A_\zeta = \mu(A_\zeta \cap s^{-1}(\zeta)) = \int_{\{\zeta\}} \mu_\zeta A_\zeta \, d\nu(\zeta) = \mu_\zeta A_\zeta \cdot \nu\{\zeta\} . \tag{32}$$

Claim: $A_\zeta \in \mathcal{X}_0$. Assume first that $\nu\{\zeta\} = 0$. Then, A_ζ is contained in the μ -negligible invariant set $s^{-1}(\zeta)$, hence, it is E -invariant, i.e. $A_\zeta \in \mathcal{X}_0$. Assume now $\nu\{\zeta\} > 0$. By Eq. 32, $\mu A_\zeta > 0$, thus $A_\zeta \neq \emptyset$ and $\mathbb{1}_{A_\zeta} \neq \mathbf{0}_H$. By [12, §II.2.3, Prop. 3, p. 182] and the direct-integral representation of T_t established in Step 3,

$$\mathbb{1}_{A_\zeta} T_t = \int_Z^{\oplus} \mathbb{1}_{A_\zeta} T_{\zeta',t} \, d\nu(\zeta') .$$

By strong consistency, A_ζ is $\mu_{\zeta'}$ -negligible for every $\zeta' \neq \zeta$, thus in fact

$$\mathbb{1}_{A_\zeta} T_t = \nu\{\zeta\} \mathbb{1}_{A_\zeta} T_{\zeta,t} ,$$

whence, by E_ζ -invariance of A_ζ ,

$$\mathbb{1}_{A_\zeta} T_t = \nu\{\zeta\} \mathbb{1}_{A_\zeta} T_{\zeta,t} = \nu\{\zeta\} T_{\zeta,t} \mathbb{1}_{A_\zeta} = T_t \mathbb{1}_{A_\zeta} ,$$

and so A_ζ is E -invariant, and thus $A_\zeta \in \mathcal{X}_0$.

Now, since $A_\zeta \in \mathcal{X}$ by assumption, then $A_\zeta \in \mathcal{X}^* := \mathcal{X} \cap \mathcal{X}_0$. Together with $A_\zeta \subset s^{-1}(\zeta)$, this implies that either $A_\zeta = \emptyset$, or $A_\zeta = s^{-1}(\zeta)$ by Eq. 28. Thus, it must be $A_\zeta = s^{-1}(\zeta)$, since $\hat{\mu}_\zeta A_\zeta > 0$ by assumption. Since $s^{-1}(\zeta)$ is $\hat{\mu}_\zeta$ -conegligible, this shows that $(E_\zeta, \mathcal{D}(E_\zeta))$ is irreducible. \square

In the statement of Theorem 3.4, we write that each $(E_\zeta, \mathcal{D}(E_\zeta))$ is a regular Dirichlet form on $L^2(\mu_\zeta)$ with underlying space $(X, \tau, \mathcal{X}, \mu_\zeta)$ to emphasize that the topology of the space is the given one. As it is well-known however, in studying the potential-theoretic and probabilistic properties of a Dirichlet form $(E, \mathcal{D}(E))$ on $L^2(\mu)$, one should always assume that μ has full support, which is usually not the case for μ_ζ on (X, τ) . In the present case, the restriction of μ_ζ to $s^{-1}(\zeta)$ is however harmless, since $s^{-1}(\zeta)$ is E -invariant, and therefore $s^{-1}(\zeta)^c$ is also E_ζ -exceptional.

Remark 3.5 As anticipated in Section 1, if $(E, \mathcal{D}(E))$ is regular and strongly local, then every invariant set admits an E -quasi-clopen μ -modification [15, Cor. 4.6.3, p. 194]. This suggests that, at least in the local case, one may treat E -invariant sets as “connected components” of X . Our intuition can be made rigorous by noting that E -invariant subsets of X are in bijective correspondence to compact open subsets of the spectrum $\text{spec}(\mathcal{M}_0)$ of the von Neumann algebra \mathcal{M}_0 (cf. Rmk. 3.2), endowed with its natural weak* topology. In particular, $\text{spec}(\mathcal{M}_0)$ coincides with the Stone space of the measure algebra of $(X, \mathcal{X}_0, \mu_0)$, and is thus a totally disconnected Hausdorff space. Its singletons correspond to the “minimal connected components” sought after in Section 1. At this point, we should emphasize that (Z, \mathcal{Z}) and $\text{spec}(\mathcal{M}_0)$ are different measure spaces, the points of which index “minimal invariant sets” in X . However, points in Z index — via s — sets in \mathcal{X}^* , whereas points in $\text{spec}(\mathcal{M}_0)$ index sets in \mathcal{X}_0 . In this sense at least, Z is minimal with the property of indexing such “minimal invariant sets”, while $\text{spec}(\mathcal{M}_0)$ is maximal. For this reason, one might be tempted to use $\text{spec}(\mathcal{M}_0)$ in place of (Z, \mathcal{Z}) in Theorem 3.4. The issue is that $\text{spec}(\mathcal{M}_0)$ is nearly always too large for the disintegration to be *strongly* consistent with the indexing map.

In the next result we show that the regularity of the Dirichlet form $(E, \mathcal{D}(E))$ in Theorem 3.4 may be relaxed to quasi-regularity. As usual, a proof of this result relies on the so-called transfer method.

Let $(X, \tau, \mathcal{X}, \mu)$ and $(X^\sharp, \tau^\sharp, \mathcal{X}^\sharp, \mu^\sharp)$ be measure spaces satisfying Assumption 2.18. We note *en passant* that a Hilbert isomorphism of Dirichlet spaces $\iota: \mathcal{D}(E)_1 \rightarrow \mathcal{D}(E^\sharp)_1$, additionally preserving the L^∞ -norm on $\mathcal{D}(E) \cap L^\infty(\mu)$, is automatically a lattice isomorphism, e.g. [15, Lem. A.4.1, p. 422].

Theorem 3.6 (Ergodic decomposition: quasi-regular case) *The conclusions of Theorem 3.4 remain valid if $(X, \tau, \mathcal{X}, \mu)$ is a topological probability space satisfying Assumption 2.18 and “regular” is replaced by “quasi-regular”.*

Proof By the general result [7, Thm. 3.7], there exist a locally compact Polish, Radon probability space $(X^\sharp, \tau^\sharp, \mathcal{X}^\sharp, \mu^\sharp)$ and a quasi-homeomorphism

$$j : (X, \tau, \mathcal{X}, \mu) \longrightarrow (X^\sharp, \tau^\sharp, \mathcal{X}^\sharp, \mu^\sharp)$$

so that $(E, \mathcal{D}(E))$ is quasi-homeomorphic, via j , to a regular Dirichlet form $(E^\sharp, \mathcal{D}(E^\sharp))$ on $(X^\sharp, \tau^\sharp, \mathcal{X}^\sharp, \mu^\sharp)$.

Applying Theorem 3.4 to $(E^\sharp, \mathcal{D}(E^\sharp))$ gives a disintegration $(\mu_\zeta^\sharp)_\zeta$ of μ w.r.t. ν and a direct-integral representation

$$E^\sharp = \int_Z^\oplus E_\zeta^\sharp \, d\nu(\zeta) ,$$

where $(E_\zeta^\sharp, \mathcal{D}(E_\zeta^\sharp))$ is a regular Dirichlet form on $L^2(\mu_\zeta^\sharp)$ for ν -a.e. $\zeta \in Z$.

1. Forms In the following, whenever $(F_k)_k$ is a nest, let us set $F := \bigcup_k F_k$. With no loss of generality by [15, Lem. 2.1.3, p. 69], we may and will always assume that every nest is *increasing*, and regular w.r.t. a measure apparent from context.

Let $(F_k)_k$, resp. $(F_k^\sharp)_k$, be an E -, resp. E^\sharp -, nest, additionally so that $j : F \rightarrow F^\sharp$ restricts to a homeomorphism $j : F_k \rightarrow F_k^\sharp$ for every k . Since $(F_k)_k$ is increasing, $j : F \rightarrow F^\sharp$ is bijective. Let N_1 be ν -negligible so that $(E_\zeta^\sharp, \mathcal{D}(E_\zeta^\sharp))$ is regular by Theorem 3.4.

Let $X_\partial := X \cup \{\partial\}$, where ∂ is taken to be an isolated point in X_∂ . Since j may be not surjective, in the following we extend j^{-1} on $X^\sharp \setminus j(F)$ by setting $j^{-1}(x^\sharp) = \partial$. Note that this extension is \mathcal{X}^\sharp -to- \mathcal{X} -measurable (having care to extend \mathcal{X} on X_∂ in the obvious way).

Since $j_\# \mu = \mu^\sharp$ the set $N_2 := \{\zeta \in Z : \mu_\zeta^\sharp(j(F)) < 1\}$ is ν -measurable, since j is measurable on F , and thus it is ν -negligible. In particular, $j_\#^{-1} \mu^\sharp \{\partial\} = 0$, and $j_\#^{-1} \mu_\zeta^\sharp \{\partial\} = 0$ for every $\zeta \in N_2^c$.

Set now $N := N_1 \cup N_2$. For $\zeta \in N^c$ set $\mu_\zeta := j_\#^{-1} \mu_\zeta^\sharp$ and denote by $(E_\zeta, \mathcal{D}(E_\zeta))$ the image form of $(E_\zeta^\sharp, \mathcal{D}(E_\zeta^\sharp))$ via j^{-1} on $L^2(\mu_\zeta)$, cf. [7, Eqn. (3.2)]. For $\zeta \in N$ let $(E_\zeta, \mathcal{D}(E_\zeta))$ be the 0-form on $L^2(\mu)$.

For $f^\sharp : X^\sharp \rightarrow \mathbb{R}$ denote further by $j^* f := f^\sharp \circ j : X \rightarrow \mathbb{R}$ the pullback of f^\sharp via j , and recall [7, Eqn. (3.3)]:

$$G_\alpha(j^* f^\sharp) = j^* G_\alpha^\sharp f^\sharp , \quad f^\sharp \in L^2(\mu^\sharp) . \tag{33}$$

2. Nests For $\zeta \in N^c$ let $(F_{\zeta,k}^\sharp)_k$ be a μ_ζ^\sharp -regular E_ζ^\sharp -nest witnessing the (quasi-)regularity of the form, i.e. verifying [7, Dfn. 2.8]. With no loss of generality, up to intersecting $F_{\zeta,k}^\sharp$ with F_h^\sharp if necessary, we may assume that for every k there exists $h := h_k$ so that $F_{\zeta,k}^\sharp \subset F_h^\sharp$. In particular, $j^{-1} : F_{\zeta,k}^\sharp \rightarrow F_{\zeta,k} := j(F_{\zeta,k}^\sharp)$ is a homeomorphism onto its image. Let $X_\zeta^\sharp := \text{supp}[\mu_\zeta^\sharp]$ and note that $F_{\zeta,k}^\sharp \subset X_\zeta^\sharp$ since $(F_{\zeta,k}^\sharp)_k$ is μ_ζ^\sharp -regular. Denote by j_ζ^{-1} the restriction of j^{-1} to X_ζ^\sharp .

Claim: j_ζ^{-1} is a quasi-homeomorphism for ν -a.e. $\zeta \in Z$. It suffices to show that $(F_{\zeta,k})_k$ is an E_ζ -nest for ν -a.e. $\zeta \in Z$, which holds by construction.

Finally, set $j_\zeta := j|_{j^{-1}(X_\zeta^\sharp)}$ and note that, again by [7, Eqn. (3.3)],

$$G_{\zeta,\alpha}(j_\zeta^* f^\sharp) = j_\zeta^* G_{\zeta,\alpha}^\sharp f^\sharp , \quad f^\sharp \in L^2(\mu_\zeta^\sharp) , \quad \zeta \in N^c . \tag{34}$$

3. Direct integral representation By Eq. 8 for the resolvent applied to $(E^\sharp, \mathcal{D}(E^\sharp))$,

$$G_\alpha^\sharp = \int_Z^\oplus G_{\zeta,\alpha}^\sharp \, d\nu(\zeta) . \tag{35}$$

By *Step 1* and [7, Lem. 3.3(ii)], $j_\zeta^* : L^2(\mu^\sharp) \rightarrow L^2(\mu)$ is an isomorphism for ν -a.e. $\zeta \in Z$, with inverse $(j_\zeta^{-1})^*$, and

$$j^* = \int_Z^\oplus j_\zeta^* \, d\nu(\zeta) . \tag{36}$$

Now, by a subsequent application of Eqs. 33, 35, 36 and [12, §II.2.3, Prop. 3, p. 182], and 34,

$$\begin{aligned} G_\alpha \circ j^* &= j^* \circ G_\alpha^\sharp = j^* \circ \int_Z^\oplus G_{\zeta,\alpha}^\sharp \, d\nu(\zeta) = \int_Z^\oplus j_\zeta^* \circ G_{\zeta,\alpha}^\sharp \, d\nu(\zeta) \\ &= \int_Z^\oplus G_{\zeta,\alpha} \circ j_\zeta^* \, d\nu(\zeta) . \end{aligned}$$

Thus, by a further application of [12, §II.2.3, Prop. 3, p. 182],

$$G_\alpha \circ j^* = \int_Z^\oplus G_{\zeta,\alpha} \, d\nu(\zeta) \circ j^* .$$

Cancelling j^* by its inverse $(j^{-1})^*$, this yields the direct-integral representation of G_α via $\zeta \mapsto G_{\alpha,\zeta}$. By Eq. 8 for the resolvent, this shows

$$E = \int_Z^\oplus E_\zeta \, d\nu(\zeta) .$$

4. Quasi-regularity and irreducibility By *Step 2* and [7, Thm. 3.7], the form $(E_\zeta, \mathcal{D}(E_\zeta))$ is quasi-regular for ν -a.e. $\zeta \in Z$. Again by *Step 2*, it is also irreducible, since it is isomorphic to the irreducible form $(E_\zeta^\sharp, \mathcal{D}(E_\zeta^\sharp))$. □

3.3 Ergodic Decomposition of Forms: σ -Finite Measure Case

Under some additional assumptions, we may now extend the results in Theorem 3.4 to the case when μ is only σ -finite. The main idea — borrowed from [6] — is to reduce the σ -finite case to the probability case.

General strategy Let $(E, \mathcal{D}(E))$ be a strongly local regular Dirichlet form on a locally compact Polish, Radon measure space $(X, \tau, \mathcal{X}^\mu, \hat{\mu})$ with full support. In particular, (X, \mathcal{X}, μ) is σ -finite, [14, 415D(iii)]. Assume further that $(E, \mathcal{D}(E))$ has carré du champ $(\Gamma, \mathcal{D}(E))$.

Let $\varphi \in \mathcal{D}(E)$, with $\varphi > 0$ μ -a.e. and $\|\varphi\|_{L^2(\mu)} = 1$, and set $\mu^\varphi := \varphi^2 \cdot \mu$. Here, we understand φ as a fixed E -quasi-continuous representative of its μ -class. Note that $(X, \mathcal{X}, \mu^\varphi)$ is a probability space and that μ^φ is equivalent to μ . Therefore, μ -classes and μ^φ -classes coincide. On $L^2(\mu^\varphi)$ we define a bilinear form

$$\mathcal{D}(E^\varphi) := \left\{ \int (\Gamma(f) + f^2) \, d\mu^\varphi < \infty \right\} , \quad E^\varphi(f, g) := \int \Gamma(f, g) \, d\mu^\varphi . \tag{37}$$

Suppose now that

- (a) the form $(E^\varphi, \mathcal{D}(E^\varphi))$ is a (closed) regular Dirichlet form on $L^2(\mu^\varphi)$.

Then, we may apply Theorem 3.4 to obtain

- a probability space $(Z_\varphi, \mathcal{Z}_\varphi, \nu_\varphi)$ and a measurable map $s_\varphi: X \rightarrow Z$;
- a ν_φ -essentially unique disintegration $(\mu_\zeta^{(\varphi)})_{\zeta \in Z}$ of μ^φ over ν_φ , strongly consistent with s_φ ;
- a family of regular strongly local Dirichlet forms $(E_\zeta^{(\varphi)}, \mathcal{D}(E_\zeta^{(\varphi)}))$ on $L^2(\mu_\zeta^{(\varphi)})$;

satisfying the direct-integral representation

$$E^\varphi = \int_{Z_\varphi}^\oplus E_\zeta^{(\varphi)} \, d\nu_\varphi(\zeta) . \tag{38}$$

For $\zeta \in Z$ let now $\mu_\zeta^{[\varphi]} := \varphi^{-2} \cdot \mu_\zeta^{(\varphi)}$ be a measure on (X, \mathcal{X}) and suppose further that

- (b) the form $(E_\zeta^{(\varphi)}, \mathcal{D}(E_\zeta^{(\varphi)}))$ has carré du champ operator $(\Gamma_\zeta^{(\varphi)}, \mathcal{D}(\Gamma_\zeta^{(\varphi)}))$ for ν_φ -a.e. $\zeta \in Z_\varphi$;
- (c) the form

$$\begin{aligned} \mathcal{D}(E_\zeta^{[\varphi]}) &:= \left\{ \int (\Gamma_\zeta^{(\varphi)}(f) + f^2) \, d\mu_\zeta^{[\varphi]} < \infty \right\} , \\ E_\zeta^{[\varphi]}(f, g) &:= \int \Gamma_\zeta^{(\varphi)}(f, g) \, d\mu_\zeta^{[\varphi]} , \quad f, g \in \mathcal{D}(E_\zeta^{[\varphi]}) , \end{aligned} \tag{39}$$

is a (closed) regular Dirichlet form on $L^2(\mu_\zeta^{[\varphi]})$ for ν_φ -a.e. $\zeta \in Z$.

Then, finally, we may expect to have a direct-integral representation

$$E = \int_{Z_\varphi}^\oplus E_\zeta^{[\varphi]} \, d\nu_\varphi(\zeta) . \tag{40}$$

As it turns out, the properties of the Girsanov-type transformation (37) are quite delicate.

Before discussing the technical details, let us note here that, provided we have shown the direct-integral representation in Eq. 40, it should not be expected that the latter is (essentially) unique, but rather merely essentially *projectively* unique — as it is the case for other ergodic theorems, e.g. [6, Thm. 2]. In the present setting, projective uniqueness is understood in the following sense.

Definition 3.7 We say that the direct integral representation (40) is *essentially projectively unique* if, for every φ, ψ as above:

- (a) the measurable space $(Z, \mathcal{Z}) := (Z_\varphi, \mathcal{Z}_\varphi) = (Z_\psi, \mathcal{Z}_\psi)$ is uniquely determined;
- (b) the measures ν_φ, ν_ψ are equivalent (i.e., mutually absolutely continuous);
- (c) the forms $E_\zeta^{[\varphi]}, E_\zeta^{[\psi]}$ are multiple of each other for ν_φ - (hence ν_ψ -)a.e. $\zeta \in Z$.

As it is clear, the definition only depends on the σ -ideal of ν_φ -negligible sets in \mathcal{Z} . By condition (b), this ideal does not, in fact, depend on ν_φ , hence the omission of the measure in the designation. The lack of uniqueness is shown as follows. Since $\mu_\zeta^{[\varphi]}$ is merely a σ -finite (as opposed to: probability) measure, the family $(\mu_\zeta^{[\varphi]})_{\zeta \in Z}$ is merely a pseudo-disintegration (as opposed to: a disintegration). Thus, for every measurable $g: Z \rightarrow (0, \infty)$,

$$\mu A = \int_Z \mu_\zeta^{[\varphi]} A \, d\nu_\varphi(\zeta) = \int_Z g(\zeta) \cdot \mu_\zeta^{[\varphi]} A \, d(g^{-1} \cdot \nu_\varphi)(\zeta) , \quad A \in \mathcal{X} .$$

Since g is defined on Z , the pullback function $f := (s_\varphi)^*g$ is \mathcal{X}_0 -measurable, i.e. all its level sets are E -invariant; by strong locality of $(E, \mathcal{D}(E))$, f is E -quasi-continuous, and therefore an element of the extended domain F_e of $(E, \mathcal{D}(E))$. As soon as $f \in L^2(\mu)$, then we have the direct-integral representation

$$E = \int_{Z_\varphi}^{\oplus} g(\zeta) E_\zeta^{[\varphi]} d(g^{-1} \cdot \nu_\varphi)(\zeta) , \quad g(\zeta) E_\zeta^{[\varphi]}(u) = \int_X \Gamma_\zeta^{(\varphi)}(u) d(f \cdot \mu_\zeta^{[\varphi]}) .$$

Proofs' summary The Girsanov-type transformations (37) are thoroughly studied by A. Eberle in [13], where (a) is proved. We shall therefore start by showing (b) above, Lemma 3.8 below. Informally, in the setting of Theorem 3.4, if $(E, \mathcal{D}(E))$ has carré du champ Γ , then

$$\Gamma = \int_Z^{\oplus} \Gamma_\zeta d\nu(\zeta) , \tag{41}$$

where Γ_ζ is the carré du champ of $(E_\zeta, \mathcal{D}(E_\zeta))$.

Since the range of Γ is a Banach (not Hilbert) space, we shall need the concept of direct integrals of Banach spaces. In particular, we shall need an analogue of Proposition 2.25 for L^1 -spaces, an account of which is given in the Appendix, together with a proof of the next lemma.

Lemma 3.8 *Under the assumptions of Theorem 3.4 suppose further that $(E, \mathcal{D}(E))$ admits carré du champ operator $(\Gamma, \mathcal{D}(E))$. Then, $(E_\zeta, \mathcal{D}(E_\zeta))$ admits carré du champ operator $(\Gamma_\zeta, \mathcal{D}(E_\zeta))$ for ν -a.e. $\zeta \in Z$.*

Lemma 3.9 *Under the assumptions of Theorem 3.4 suppose further that $(E, \mathcal{D}(E))$ is strongly local. Then, $(E_\zeta, \mathcal{D}(E_\zeta))$ is strongly local for ν -a.e. $\zeta \in Z$.*

Proof Note: In this proof we shall make use of results in [5]. We recall that a regular form is ‘strongly local’ in the sense of [15, p. 6] if and only if it is ‘local’ in the sense of [5, Dfn. I.V.1.2, p. 28]. This is noted e.g. in [28, §2, p. 78], after [26, Prop. 1.4]. In this respect, we always adhere to the terminology of [15, 23]. Let $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ be a regular Dirichlet form. By [5, Cor. I.5.1.4 and Rmk. I.5.1.5, p. 31], $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is strongly local if and only if $|u| \in \mathcal{D}(\mathcal{E})$ and $\mathcal{E}(|u|) = \mathcal{E}(u)$ for every $u \in \mathcal{D}(\mathcal{E})$. Further note that $|u|$ is a normal contraction of $u \in \mathcal{D}(\mathcal{E})$ for every $u \in \mathcal{D}(\mathcal{E})$. As a consequence, $|u| \in \mathcal{D}(\mathcal{E})$ and $\mathcal{E}(|u|) \leq \mathcal{E}(u)$ for every $u \in \mathcal{D}(\mathcal{E})$, see e.g. [15, Thm. 1.4.1]. In particular, a Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ is not strongly local if and only if there exists $u \in \mathcal{D}(\mathcal{E})$ with $\mathcal{E}(u) > \mathcal{E}(|u|)$.

Since $\mu_\zeta X \leq 1$ for every $\zeta \in Z$, it is not difficult to show, arguing by contradiction, that

$$\mathbb{1}_X \in \mathcal{D}(E_\zeta) , \quad E_\zeta(\mathbb{1}_X) = 0 \quad \text{for } \nu\text{-a.e. } \zeta \in Z . \tag{42}$$

Let $(u_n)_n \subset \mathcal{C}$ be the fundamental sequence constructed in Step 1 in the proof of Theorem 3.4(iii). The Dirichlet form $(E_\zeta, \mathcal{D}(E_\zeta))$ on $L^2(\mu_\zeta)$ is strongly local if and only if $E_\zeta(u_n) = E_\zeta(|u_n|)$ for every $n \geq 1$. The same holds for $(E, \mathcal{D}(E))$.

Now, argue by contradiction that there exists a ν -measurable non-negligible set $B \subset Z$ so that the form $(E_\zeta, \mathcal{D}(E_\zeta))$ is not strongly local for each $\zeta \in B$. Let $B_n := \{\zeta \in Z : E_\zeta(u_n) > E_\zeta(|u_n|)\}$ and note that $B_n \subset B$ is ν -measurable for every $n \geq 1$ since $(u_n)_n \subset \mathcal{C}$. Since $B = \cup_n B_n$ and $\nu B > 0$, there exists some fixed n_* so that $\nu B_{n_*} > 0$. Without loss of generality, up to relabeling, we may choose $n_* = 1$.

Analogously to the proof of the *Claim* in *Step 3* of Theorem 3.4(iii), set $A := p^{-1}(B_1)$ and note that it is E -invariant. Thus, finally, $\mathbb{1}_A u_1 \in \mathcal{D}(E)$ and

$$E(u_1 \mathbb{1}_A) = \int_{B_1} E_\zeta(u_1) \, d\nu(\zeta) > \int_{B_1} E_\zeta(|u_1|) \, d\nu(\zeta) = E(|u_1| \mathbb{1}_A) = E(|u_1 \mathbb{1}_A|),$$

which contradicts the strong locality of $(E, \mathcal{D}(E))$. □

Remark 3.10 The converse implications to Lemmas 3.8, and 3.9 are true in a more general setting, viz.

- (a) if $(E, \mathcal{D}(E))$ is a direct integral of Dirichlet forms $\zeta \mapsto (E_\zeta, \mathcal{D}(E_\zeta))$ each with carré du champ operator $(\Gamma_\zeta, \mathcal{D}(\Gamma_\zeta))$, then $(E, \mathcal{D}(E))$ has carré du champ given by Eq. 41, see [5, Ex. V.3.2, p. 216];
- (b) if $(E, \mathcal{D}(E))$ is a direct integral of strongly local Dirichlet forms $\zeta \mapsto (E_\zeta, \mathcal{D}(E_\zeta))$, then $(E, \mathcal{D}(E))$ is strongly local, see [5, Ex. V.3.1, p. 216].

We are now ready to prove the main result of this section.

Theorem 3.11 (Ergodic decomposition: σ -finite case) *Let $(X, \tau, \mathcal{X}, \mu)$ be a locally compact Polish Radon measure space, and $(E, \mathcal{D}(E))$ be a regular strongly local Dirichlet form on $L^2(\mu)$ with carré du champ operator $(\Gamma, \mathcal{D}(\Gamma))$. Then, there exist*

- (i) a probability space (Z, \mathcal{Z}, ν) and a measurable map $s: (X, \mathcal{X}) \rightarrow (Z, \mathcal{Z})$;
- (ii) an essentially projectively unique family of measures $(\mu_\zeta)_{\zeta \in Z}$ so that, when $s^{-1}(\zeta)$ is endowed with the subspace topology and the trace σ -algebra inherited from $(X, \tau, \mathcal{X}^\mu)$, then $(s^{-1}(\zeta), \hat{\mu}_\zeta)$ is a Radon measure space for ν -a.e. $\zeta \in Z$;
- (iii) a ν -measurable field $\zeta \mapsto (E_\zeta, \mathcal{D}(E_\zeta))$ of regular irreducible Dirichlet forms $(E_\zeta, \mathcal{D}(E_\zeta))$ on $L^2(\mu_\zeta)$;

so that

$$L^2(\mu) = \int_Z^\oplus L^2(\mu_\zeta) \, d\nu(\zeta) \quad \text{and} \quad E = \int_Z^\oplus E_\zeta \, d\nu(\zeta).$$

Proof Let $\varphi \in \mathcal{D}(E)$ with $0 < \varphi < 1$ μ -a.e. and $\|\varphi\|_{L^2} = 1$. Since $(E, \mathcal{D}(E))$ is regular strongly local on $L^2(\mu)$ and admits carré du champ $(\Gamma, \mathcal{D}(E))$, then the Girsanov-type transform $(E^\varphi, \mathcal{D}(E^\varphi))$ defined in Eq. 37 is a quasi-regular strongly local Dirichlet form on $L^2(\mu^\varphi)$ by [13, Thm.s 1.1 and 1.4(ii)], and admits carré du champ $(\Gamma, \mathcal{D}(E^\varphi))$ by construction. We note that we are applying the results in [13] in the context of [13, Example 1), p. 501]. In particular, Assumption (D3) in [13, p. 501] holds by definition.

1. Constructions Let now \mathcal{C} be a core for $(E, \mathcal{D}(E))$. Since $\varphi \leq 1$ μ -a.e., then $E_1^\varphi \leq E_1$, and the form $(E^\varphi, \mathcal{D}(E^\varphi))$ is in fact regular, with same core \mathcal{C} . Since μ^φ is a fully supported probability measure by construction, we may apply Theorem 3.4 to obtain the direct integral representation (38). For ν_φ -a.e. $\zeta \in Z_\varphi$, the form $(E_\zeta^{(\varphi)}, \mathcal{D}(E_\zeta^{(\varphi)}))$ is regular with core \mathcal{C} and irreducible by Theorem 3.4, strongly local by Lemma 3.9, and admitting carré du champ $(\Gamma_\zeta^{(\varphi)}, \mathcal{D}(E_\zeta^{(\varphi)}))$ by Lemma 3.8.

Claim: $\varphi^{-2} \vee n \in \mathcal{D}(E_\zeta^{(\varphi)})$ for ν_φ -a.e. $\zeta \in Z_\varphi$, for every $n \geq 1$. Since $E_1^\varphi \leq E_1$, one has $\varphi \in \mathcal{D}(E^\varphi)$, hence $\varphi \in \mathcal{D}(E_\zeta^{(\varphi)})$ for ν_φ -a.e. $\zeta \in Z_\varphi$ as a consequence of Eq. 7. The claim then follows by strong locality of $(E_\zeta, \mathcal{D}(E_\zeta))$ for ν_φ -a.e. $\zeta \in Z_\varphi$.

Let now $(E_\zeta^{[\varphi],n}, \mathcal{D}(E_\zeta^{[\varphi],n}))$ be defined analogously to Eq. 39 with $\mu_\zeta^{[\varphi],n} := (\varphi^{-2} \vee n) \cdot \mu_\zeta^{(\varphi)}$ in place of $\mu_\zeta^{[\varphi]}$.

By applying once more [13, Thm.s 1.1 and 1.4(ii)], the Girsanov-type transform $(E_\zeta^{[\varphi],n}, \mathcal{D}(E_\zeta^{[\varphi],n}))$ defined in Eq. 39 is a regular Dirichlet form on $L^2(\mu_\zeta^{[\varphi],n})$ with core \mathcal{C} , strongly local, irreducible, and admitting carré du champ operator $(\Gamma_\zeta^{(\varphi)}, \mathcal{D}(E_\zeta^{[\varphi],n}))$ for ν_φ -a.e. $\zeta \in Z_\varphi$, for every $n \geq 1$.

Claim: the quadratic form $(E_\zeta^{[\varphi]}, \mathcal{D}(E_\zeta^{[\varphi]}))$ defined in Eq. 39 is a regular Dirichlet form on $L^2(\mu_\zeta^{[\varphi]})$, strongly local, irreducible, and admitting carré du champ operator $(\Gamma_\zeta^{(\varphi)}, \mathcal{D}(E_\zeta^{[\varphi]}))$ for ν_φ -a.e. $\zeta \in Z_\varphi$. Firstly, note that $E_\zeta^{[\varphi]} = \sup_n E_\zeta^{[\varphi],n}$ is well-defined on $\mathcal{D}(E_\zeta^{[\varphi]}) = \bigcap_{n \geq 1} \mathcal{D}(E_\zeta^{[\varphi],n})$, thus $(E_\zeta^{[\varphi]}, \mathcal{D}(E_\zeta^{[\varphi]}))$ is a closable quadratic form by [23, Prop. I.3.7(ii)]. The Markov property, the strong locality and the existence and computation of the carré du champ operator are straightforward. Note that $\mathcal{C} \subset \mathcal{D}(E_\zeta^{[\varphi]})$, so that the latter is dense in $L^2(\mu_\zeta^{[\varphi]})$ for ν_φ -a.e. $\zeta \in Z_\varphi$. By Dominated Convergence and Eq. 15

$$\begin{aligned} E(u) &= \int_X \Gamma(u) \, d\mu = \lim_n \int_X \Gamma(u) \cdot (\varphi^{-2} \vee n) \, d\mu^\varphi \\ &= \lim_n \int_{Z_\varphi} \int_X \Gamma_\zeta^{(\varphi)}(u) \, d\mu_\zeta^{[\varphi],n} \, d\nu_\varphi(\zeta) \\ &= \int_{Z_\varphi} \int_X \Gamma_\zeta^{(\varphi)}(u) \, d\mu_\zeta^{[\varphi]} \, d\nu_\varphi(\zeta) \end{aligned}$$

which establishes the direct integral representation (40), with underlying space $S_{\mathcal{C}}$. The regularity of the forms $(E_\zeta^{[\varphi]}, \mathcal{D}(E_\zeta^{[\varphi]}))$, all with core \mathcal{C} , follows from the regularity of $(E, \mathcal{D}(E))$, exactly as in the Claim in Step 4 in the proof of Theorem 3.4.

2. Projective uniqueness By Eq. 38, E -invariant sets are also E^φ -invariant. The reverse implication follows since μ and μ^φ are equivalent. As a consequence, the σ -algebras \mathcal{X}_0 and \mathcal{X}^* defined w.r.t. μ^φ as in Section 3.1 are independent of φ and thus so are the space $(Z_\varphi, \mathcal{Z}_\varphi)$, henceforth denoted simply by (Z, \mathcal{Z}) , and the map s_φ , henceforth denoted simply by s . Let now $\varphi, \psi \in \mathcal{D}(E)$, $\varphi, \psi > 0$ μ -a.e., and replicate every construction for ψ as well. Note that $\mu^\varphi := \varphi^2 \cdot \mu$ and $\mu^\psi := \psi^2 \cdot \mu$ are equivalent.

Claim 1: $\nu_\varphi \sim \nu_\psi$. It suffices to recall that $\nu_\varphi = s_\# \mu^\varphi$ and analogously for ψ w.r.t. the same map s , hence the conclusion, since $\mu^\varphi \sim \mu^\psi$.

It follows that the σ -ideal $\mathcal{N} := \mathcal{N}_\varphi$ of ν_φ -negligible sets in \mathcal{Z} does not in fact depend on φ . In the following, we write therefore “ \mathcal{N} -negligible” in place of “ ν_φ -negligible” and “ \mathcal{N} -a.e.” in place of “ ν_φ -a.e.”.

Claim 2: $\mu_\zeta^{(\varphi)} \sim \mu_\zeta^{(\psi)}$ for \mathcal{N} -a.e. $\zeta \in Z$. Argue by contradiction that there exist $B \in \mathcal{Z} \setminus \mathcal{N}$ and a family $(A_\zeta)_{\zeta \in B}$ with $A_\zeta \in \mathcal{X}$ and, without loss of generality, $\mu_\zeta^{(\varphi)} A_\zeta > \mu_\zeta^{(\psi)} A_\zeta = 0$ for all $\zeta \in B$. Set further $\tilde{A} := \bigcup_{\zeta \in B} A_\zeta$ and let $A \in \mathcal{X}$ be its measurable envelope [14, 132D]. Then, by Eq. 14 and strong consistency of $(\mu_\zeta^{(\varphi)})_{\zeta \in Z}$ with s ,

$$\begin{aligned} \mu^\varphi A &= \mu^\varphi (A \cap s^{-1}(B)) = \int_B \mu_\zeta^{(\varphi)} A \, d\nu_\varphi(\zeta) = \int_B \mu_\zeta^{(\varphi)} (A \cap s^{-1}(\zeta)) \, d\nu_\varphi(\zeta) \\ &\geq \int_B \mu_\zeta^{(\varphi)} A_\zeta \, d\nu_\varphi(\zeta) > 0. \end{aligned} \tag{43}$$

Let now μ^{ψ^*} be the outer measure of μ^ψ , and analogously for $\mu_\zeta^{(\psi)^*}$. Note that, by Assumption 2.18, the Carathéodory measure induced by μ^{ψ^*} coincides with the completion measure $\widehat{\mu}^\psi = \widehat{\mu}^\psi$. Furthermore, by strong consistency of $(\mu_\zeta^{(\psi)})_{\zeta \in Z}$ with s , one has $\mu_\zeta^{(\psi)^*} \tilde{A} = \mu_\zeta^{(\psi)^*} (\tilde{A} \cap s^{-1}(\zeta)) = \mu_\zeta^{(\psi)^*} A_\zeta = 0$ by assumption. In particular, the function $\zeta \mapsto \mu_\zeta^{(\psi)^*} \tilde{A} \equiv 0$ is measurable. Thus, by [14, 452X(i)],

$$0 = \int_B \mu_\zeta^{(\psi)^*} \tilde{A} = \mu^{\psi^*} \tilde{A} = \mu^\psi A ,$$

which contradicts (43) since $\mu^\varphi \sim \mu^\psi$ by the previous claim.

By the last claim, $\mu_\zeta^{[\varphi]}$ -classes and $\mu_\zeta^{[\psi]}$ -classes coincide. Therefore, the carré du champ operator $\Gamma_\zeta^{[\varphi]} = \Gamma_\zeta^{(\varphi)}$ is independent of φ , and henceforth denoted by Γ_ζ . Thus we have

$$E(u) = \int_Z \int_X \Gamma_\zeta(u) d\mu_\zeta^{[\varphi]} dv_\varphi(\zeta) = \int_Z \int_X \Gamma_\zeta(u) d\mu_\zeta^{[\psi]} dv_\psi(\zeta) , \quad u \in \mathcal{C} ,$$

and, finally, it suffices to show the following.

Claim 3: $\mu_\zeta^{[\varphi]} = \frac{dv_\psi}{dv_\varphi} \cdot \mu_\zeta^{[\psi]}$. By construction, $(\mu_\zeta^{[\varphi]})_{\zeta \in Z}$, resp. $(\mu_\zeta^{[\psi]})_{\zeta \in Z}$, is a pseudo-disintegration of μ over v_φ , resp. v_ψ . For fixed $f \in L^1(\mu)^+$ and every $t > 0$ set $A_t := \{f/\varphi^2 = t\}$. By consistency of the disintegration of μ^φ over v_φ ,

$$\int_0^\infty \mu^\varphi(A_t \cap s^{-1}(B)) dt = \int_0^\infty \int_B \mu_\zeta^{(\varphi)} A_t dv_\varphi(\zeta) dt , \quad B \in \mathcal{Z} .$$

whence, by the level-set representation of the Lebesgue integral and Tonelli’s Theorem

$$\int_{s^{-1}(B)} f d\mu = \int_B \int_X f d\mu_\zeta^{[\varphi]} dv_\varphi(\zeta) . \tag{44}$$

Note that the left-hand side does not depend on φ . Therefore, equating (44) with the same representation for ψ and using *Claim 1* yields

$$\int_B \int_X f d\mu_\zeta^{[\varphi]} dv_\varphi(\zeta) = \int_B \int_X f d\mu_\zeta^{[\psi]} dv_\psi(\zeta) = \int_B \frac{dv_\psi}{dv_\varphi}(\zeta) \int_X f d\mu_\zeta^{[\psi]} dv_\varphi(\zeta) ,$$

and the conclusion follows, since f and B were arbitrary. □

3.4 Some Examples

Here, we specialize the results in the previous sections to some particular cases.

In order to discuss the next example, we shall need the definition of 1-capacity cap_E of a regular Dirichlet form $(E, \mathcal{D}(E))$ on a locally compact Polish space (X, τ) , for which we refer the reader to [15, §2.1]. We recall that cap_E is a Choquet capacity on X in the sense of e.g. [15, §A.1], see e.g. [15, Thm. 2.1.1]. Finally, we say that $A \subset X$ is E -capacitable if

$$\text{cap}_E(A) = \sup_{K \in \mathcal{K}_\tau : K \subset A} \text{cap}_E(K) ,$$

where \mathcal{K}_τ denotes the family of all τ -compact subsets of X .

Example 3.12 (Ergodic decomposition of forms on product spaces) Let $X = Y \times Z$ be a product of locally compact Polish spaces endowed with a probability (hence Radon) measure μ , and $(\mu_\zeta)_{\zeta \in Z}$ be a disintegration of μ over $v := \text{pr}_Y^Z \mu$ strongly consistent with the standard projection $\text{pr}^Z : X \rightarrow Z$. This includes the setting of Example 2.32.

Indeed, let $(E_\zeta, \mathcal{D}(E_\zeta))$ be regular irreducible Dirichlet forms on $L^2(\mu_\zeta)$, all with common core $\mathcal{C} \subset \mathcal{C}_0(Y)$, and assume that $\zeta \mapsto (E_\zeta, \mathcal{D}(E_\zeta))$ is a ν -measurable field of quadratic forms in the sense of Definition 2.11 with underlying ν -measurable field $S = S_{\mathcal{C}}$. Then, it is readily verified that

- (i) the direct integral $(E, \mathcal{D}(E))$ of quadratic forms $\zeta \mapsto (E_\zeta, \mathcal{D}(E_\zeta))$ is a direct integral of Dirichlet forms;
- (ii) $(E, \mathcal{D}(E))$ is a regular Dirichlet form on $L^2(\mu)$ with core $\mathcal{C} \otimes \mathcal{C}_0(Z)$ and semigroup

$$(T_t u)(y, \zeta) = (T_{\zeta,t} \otimes \text{id}_{L^2(\nu)} u)(y, \zeta) = (T_{\zeta,t} u(\cdot, \zeta))(y) ;$$

- (iii) if $A \subset Z$ is ν -measurable and $U \subset Y$ is E_ζ -capacitable for every $\zeta \in Z$, then $U \times A \subset X$ satisfies

$$\text{cap}_E(U \times A) \leq \int_A \text{cap}_{E_\zeta}(U) \, d\nu(\zeta) .$$

As a further example, we state here the ergodic decomposition theorem for mixed Poisson measures on the configuration space over a *connected* Riemannian manifold. We refer the reader to [2] for the main definitions.

Example 3.13 (Mixed Poisson measures, [2]) Let (M, g) be a Riemannian manifold with infinite volume, and $\sigma = \rho \cdot \text{vol}_g$ be a non-negative Borel measure on M with density $\rho > 0$ vol_g -a.e., and satisfying $\rho^{1/2} \in W_{\text{loc}}^{1,2}(M)$. Let further Γ_M be the configuration space over M , endowed with the vague topology and the induced Borel σ -algebra, and denote by π_σ the Poisson measure on Γ_M with intensity measure σ . Let now λ be a Borel probability measure on $\mathbb{R}_+ := (0, \infty)$ with finite second moment. The *mixed Poisson measure with intensity measure σ and Lévy measure λ* is the measure

$$\mu_{\lambda,\sigma} := \int_{\mathbb{R}_+} \pi_{s\sigma} \, d\lambda(s) .$$

In [2], Albeverio, Kondratiev, and Röckner construct a canonical Dirichlet form $(\mathcal{E}_{\mu_{\lambda,\sigma}}, \mathcal{D}(\mathcal{E}_{\mu_{\lambda,\sigma}}))$ on $L^2(\mu_{\lambda,\sigma})$ and show that

- $(\mathcal{E}_{\mu_{\lambda,\sigma}}, \mathcal{D}(\mathcal{E}_{\mu_{\lambda,\sigma}}))$ is quasi-regular strongly local, [2, Thm. 6.1];
- $(\mathcal{E}_{\mu_{\lambda,\sigma}}, \mathcal{D}(\mathcal{E}_{\mu_{\lambda,\sigma}}))$ is irreducible if and only if $\lambda = \delta_s$, i.e. $\mu_{\lambda,\sigma} = \pi_{s\sigma}$, for some $s \geq 0$, [2, Thm. 4.3];
- $\pi_{s\sigma} \perp \pi_{r\sigma}$ for all $r, s \geq 0, r \neq s$.

Applying Theorem 3.6 to the form $(\mathcal{E}_{\mu_{\lambda,\sigma}}, \mathcal{D}(\mathcal{E}_{\mu_{\lambda,\sigma}}))$ yields the direct-integral representation

$$\mathcal{E}_{\mu_{\lambda,\sigma}} = \int_{\mathbb{R}_+}^{\oplus} \mathcal{E}_{\pi_{s\sigma}} \, d\lambda(s) ,$$

where $(Z, \mathcal{Z}, \nu) = (\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+), \lambda)$, and the disintegration of $\mu_{\lambda,\sigma}$ constructed in the theorem coincides with $(\pi_{s\sigma})_{s \in \mathbb{R}_+}$.

Remark 3.14 Other examples are given by [1, Thm. 3.7] and [3], both concerned with strongly local Dirichlet forms on locally convex topological vector spaces, and by [10], concerned with a particular quasi-regular Dirichlet form on the space of probability measures over a closed Riemannian manifold.

3.5 Some Applications

We collect here some applications of the direct-integral decomposition discussed in the previous sections.

Transience/recurrence Let $(X, \tau, \mathcal{X}, \mu)$ be satisfying Assumption 2.18.

For an invariant set $A \in \mathcal{X}_0$, we denote by μ_A the restriction of μ to A , and by $(E_A, \mathcal{D}(E_A))$ the Dirichlet form

$$\mathcal{D}(E_A) := \{ \mathbb{1}_A f : f \in \mathcal{D}(E) \} , \quad E_A(f, g) := E(\mathbb{1}_A f, \mathbb{1}_A g) ,$$

well-defined on $L^2(\mu_A)$ as a consequence of Definition 3.1(d). The next result is standard. In the generality of Assumption 2.18, a proof is readily deduced from the corresponding result for μ -tight Borel right processes, shown with different techniques by K. Kuwae in [20, Thm. 1.3], in the far more general setting of quasi-regular semi-Dirichlet forms.

Corollary 3.15 (Ergodic decomposition: transience/recurrence) *Under the assumptions of Theorem 3.6, there exist E -invariant subsets X_c, X_d , and a properly E -exceptional subset N of X , so that*

- (i) $X = X_c \sqcup X_d \sqcup N$;
- (ii) *the restriction $(E_d, \mathcal{D}(E_d))$ of $(E, \mathcal{D}(E))$ to X_d is transient;*
- (iii) *the restriction $(E_c, \mathcal{D}(E_c))$ of $(E, \mathcal{D}(E))$ to X_c is recurrent.*

As an application, we have the following proposition. Similarly to Remark 3.10, some implications hold for superpositions of arbitrary Dirichlet forms.

Proposition 3.16 *Let $(X, \tau, \mathcal{X}, \mu)$ be a topological measure space satisfying Assumption 2.18, (Z, \mathcal{Z}, ν) be σ -finite countably generated, $(\mu_\zeta)_{\zeta \in Z}$ be a separated pseudo-disintegration of μ over ν , and $(E, \mathcal{D}(E))$ be a direct integral of quasi-regular Dirichlet forms $\zeta \mapsto (E_\zeta, \mathcal{D}(E_\zeta))$ on $L^2(\mu_\zeta)$. Then,*

- (i) *$(E, \mathcal{D}(E))$ is conservative if and only if $(E_\zeta, \mathcal{D}(E_\zeta))$ is conservative for ν -a.e. $\zeta \in Z$;*
- (ii) *$(E, \mathcal{D}(E))$ is transient if and only if $(E_\zeta, \mathcal{D}(E_\zeta))$ is transient for ν -a.e. $\zeta \in Z$. Furthermore, one has the direct-integral representation of Hilbert spaces*

$$\mathcal{D}(E)_e = \int_Z^\oplus \mathcal{D}(E_\zeta)_e \, d\nu(\zeta) ; \tag{45}$$

- (iii) *if $(E, \mathcal{D}(E))$ is recurrent, then $(E_\zeta, \mathcal{D}(E_\zeta))$ is recurrent for ν -a.e. $\zeta \in Z$. In the situation of Theorem 3.4 or Theorem 3.11, the converse implication holds as well.*

Proof Analogously to the proof of Theorem 3.6 we may restrict to the regular case by the transfer method. Thus we can and will assume with no loss of generality that $(X, \tau, \mathcal{X}^\mu, \hat{\mu})$ is a locally compact Polish Radon measure space with full support, and $(E, \mathcal{D}(E))$ is a direct integral of Dirichlet forms $\zeta \mapsto (E_\zeta, \mathcal{D}(E_\zeta))$ with underlying space of ν -measurable vector fields \mathcal{S}_C generated by a core \mathcal{C} for $(E, \mathcal{D}(E))$. By this assumption, the form $(E, \mathcal{D}(E))$ and all forms $(E_\zeta, \mathcal{D}(E_\zeta))$ are regular, with common core \mathcal{C} . Analogously to the proof of Theorem 3.4, if $u \in \mathcal{C}$, then we may choose u as a representative for u_ζ , thus writing u in place of u_ζ for every $\zeta \in Z$ and every $u \in \mathcal{C}$. Without loss of generality,

possibly up to enlargement of \mathcal{C} , we may assume that \mathcal{C} is special standard (e.g., [15, p. 6]). In particular, \mathcal{C} is a lattice.

(i) Let $(u_n)_n \subset \mathcal{C}_c(\tau)$ with $0 \leq u_n \leq 1$ and so that $\lim_n u_n \equiv \mathbb{1}$ locally uniformly on (X, τ) , and note that $\lim_n u_n = \mathbb{1}$ both μ - and μ_ζ -a.e. for every $\zeta \in Z$. By the direct-integral representation (8) of T_\bullet and by Eq. 19,

$$\int_X f T_t u_n \, d\mu = \int_Z \int_X f_\zeta T_{\zeta,t} u_n \, d\mu_\zeta \, dv(\zeta) , \quad \begin{aligned} f &\in L^1(\mu) \cap L^2(\mu) , \\ f &= (\zeta \mapsto f_\zeta) , \end{aligned} \quad t > 0 .$$

Letting n to infinity, it follows by several applications of the Dominated Convergence Theorem that

$$\int_X f T_t \mathbb{1} \, d\mu = \int_Z \int_X f_\zeta T_{\zeta,t} \mathbb{1} \, d\mu_\zeta \, dv(\zeta) , \quad \begin{aligned} f &\in L^1(\mu) \cap L^2(\mu) , \\ f &= (\zeta \mapsto f_\zeta) , \end{aligned} \quad t > 0 . \tag{46}$$

Now, assume that T_\bullet is *not* conservative and argue by contradiction that $T_{\zeta,\bullet}$ is conservative for ν -a.e. $\zeta \in Z$. Then, choosing $f \in L^1(\mu) \cap L^2(\mu)$, with $f > 0$ μ -a.e., in Eq. 46,

$$\begin{aligned} \int_X f \, d\mu &> \int_X f T_t \mathbb{1} \, d\mu = \int_Z \int_X f_\zeta T_{\zeta,t} \mathbb{1} \, d\mu_\zeta \, dv(\zeta) = \int_Z f_\zeta \mathbb{1} \, d\mu_\zeta \, dv(\zeta) \\ &= \int_X f \, d\mu , \end{aligned}$$

a contradiction. The reverse implication follows from Eq. 46 in a similar way.

(ii) Assume $(E_\zeta, \mathcal{D}(E_\zeta))$ is transient for every $\zeta \in N^c$ for some ν -negligible $N \subset Z$. That is, $\mathcal{D}(E_\zeta)_e$ is a Hilbert space with inner product E_ζ for every $\zeta \in N^c$. By (the proof of) [15, Lem. 1.5.5, p. 42], the space $\mathcal{D}(E_\zeta)$ is $E_\zeta^{1/2}$ -dense in $\mathcal{D}(E_\zeta)_e$ for every $\zeta \in N^c$. Thus, the space of ν -measurable vector fields S_C is underlying to each of the direct integrals

$$\begin{aligned} \mathcal{D}(E)_1 &= \int_Z^{S_C^\oplus} \mathcal{D}(E_\zeta)_1 \, dv(\zeta) , \\ F_e &:= \int_Z^{S_C^\oplus} \mathcal{D}(E_\zeta)_e \, dv(\zeta) , \\ L^2(\mu) &= \int_Z^{S_C^\oplus} L^2(\mu_\zeta) \, dv(\zeta) . \end{aligned}$$

In particular, there exists a sequence $(u_n)_n \subset \mathcal{C}$ simultaneously $\mathcal{D}(E)_1$ -, F_e - and $L^2(\mu)$ -fundamental in the sense of Definition 2.3. Denote by $\iota_{\zeta,e}$ the identity of $L^2(\mu_\zeta)$, regarded as the continuous embedding $\iota_{\zeta,e}: \mathcal{D}(E_\zeta)_1 \rightarrow \mathcal{D}(E_\zeta)_e$ and note that

$$\zeta \mapsto \langle \iota_{\zeta,e} u_n \mid u_m \rangle_{\mathcal{D}(E_\zeta)_e} = E_\zeta(u_n, u_m) = \langle u_n \mid u_m \rangle_{\mathcal{D}(E_\zeta)_1} - \langle u_n \mid u_m \rangle_{L^2(\mu_\zeta)}$$

is ν -measurable for every n, m . By [12, §II.1.4, Prop. 2, p. 166] this implies that $\zeta \mapsto \iota_{\zeta,e}$ is a ν -measurable field of bounded operators. Writing

$$\iota_e: \mathcal{D}(E)_1 \rightarrow F_e , \quad u \mapsto \int_Z^\oplus \iota_{\zeta,e} u_\zeta \, dv(\zeta) ,$$

and arguing as in the proof of Proposition 2.13, the map ι_e is injective, and thus it is a continuous embedding of $\mathcal{D}(E)_1$ into the Hilbert space (F_e, E) . As a consequence, $K := \text{cl}_{F_e}(\iota_e \mathcal{D}(E)_1)$ is a Hilbert space with scalar product E . By definition of $\mathcal{D}(E)_e$, the identity map ι_e is a continuous embedding $(\mathcal{D}(E)_e, E) \subset (K, E) \subset$

(F_e, E) . In particular, $(\mathcal{D}(E)_e, E)$ is a Hilbert space, and the form $(E, \mathcal{D}(E))$ is transient by [15, Thm. 1.6.2, p. 58].

Claim: $\mathcal{D}(E)_e = K = F_e$ and Eq. 45 holds. By (the proof of) [15, Lem. 1.5.5, p. 42], the space $\mathcal{D}(E)$ is $E^{1/2}$ -dense in $\mathcal{D}(E)_e$, thus the same holds for \mathcal{C} . It suffices to show that \mathcal{C} is $E^{1/2}$ -dense in F_e as well. We denote by \mathcal{C}^\perp the E -orthogonal complement of \mathcal{C} in F_e , resp. by $\mathcal{C}^{\perp_\zeta}$ the E_ζ -orthogonal complement of \mathcal{C} in $\mathcal{D}(E_\zeta)$. By assumption, $\mathcal{C}^{\perp_\zeta} = \{0\}$ for every $\zeta \in N^c$. Finally, by the direct-integral representation of F_e ,

$$\mathcal{C}^\perp = \int_Z^\oplus \mathcal{C}^{\perp_\zeta} \, d\nu(\zeta) = \{0\} \, ,$$

similarly to the proof of the *Claim* in *Step 4* of Theorem 3.4.

We say that $u, v \in \mathcal{C}$ are E -equivalent if $E(u - v) = 0$, and we write $u \sim v$. Let the analogous definition for $u \sim_\zeta v$ be given. By the direct-integral representation (24) of $(E, \mathcal{D}(E))$, it is readily seen that

$$u \sim v \iff u \sim_\zeta v \text{ for } \nu\text{-a.e. } \zeta \in Z \, . \tag{47}$$

Assume now that $(E, \mathcal{D}(E))$ is transient. That is, $\mathcal{D}(E)_e$ is a Hilbert space with inner product E . It suffices to show Eq. 45. Since $E^{1/2}$ is non-degenerate on $\mathcal{D}(E)_e$, it is non-degenerate on \mathcal{C} , thus $u \sim v$ if and only if $u = v$ μ -a.e. By Eq. 47, $E_\zeta^{1/2}$ is non-degenerate on \mathcal{C} for ν -a.e. $\zeta \in Z$, thus $(\mathcal{C}, E_\zeta^{1/2})$ is a pre-Hilbert space for ν -a.e. $\zeta \in Z$. For each $\zeta \in Z$ denote by K_ζ the abstract completion of $(\mathcal{C}, E_\zeta^{1/2})$, endowed with the non-reabeled extension of $E_\zeta^{1/2}$. It is a straightforward verification that there holds the direct-integral representation

$$\mathcal{D}(E)_e = \int_Z^{\text{Sc}^\oplus} K_\zeta \, d\nu(\zeta) \, . \tag{48}$$

By definition of $\mathcal{D}(E_\zeta)_e$, the completion embedding $\iota_\zeta : \mathcal{C} \rightarrow K_\zeta$ extends to a setwise injection $\bar{\iota}_\zeta : \mathcal{D}(E_\zeta)_e \rightarrow K_\zeta$. Indeed, let $u_\zeta \in \mathcal{D}(E_\zeta)_e$ and $(u_n)_n \subset \mathcal{C}$ be its approximating sequence. Since $(u_n)_n$ is, by definition, $E_\zeta^{1/2}$ -Cauchy, it converges to some $h_\zeta \in K_\zeta$ by completeness of K_ζ . Set $\bar{\iota}_\zeta(u_\zeta) := h_\zeta$, and note that the definition is well-posed since $E_\zeta^{1/2}$ is a norm in K_ζ . Thus, $\mathcal{D}(E_\zeta)_e$, identified with a subset of K_ζ via $\bar{\iota}_\zeta$, is a pre-Hilbert space with scalar product E_ζ , and in fact it holds that $\mathcal{D}(E_\zeta)_e = K_\zeta$ by $E_\zeta^{1/2}$ -density of \mathcal{C} in K_ζ . We note that the equality $\mathcal{D}(E_\zeta)_e = K_\zeta$ is not a mere isomorphism of Hilbert spaces, but rather an extension of the completion embedding ι_ζ , thus preserving the lattice property of \mathcal{C} regarded as a subspace of both $\mathcal{D}(E_\zeta)_e$ and K_ζ . Together with Eq. 48, this shows Eq. 45.

(iii) Assume $(E, \mathcal{D}(E))$ is recurrent. By [15, Thm. 1.6.3, p. 58] there exists a sequence $(u_n)_n \subset \mathcal{D}(E)$, so that $\lim_n u_n(x) = 1$ for μ -a.e. $x \in X$, and $\lim_n E(u_n) = 0$. By the Markov property for $(E, \mathcal{D}(E))$ we may assume that $u_n \in [0, 1]$. By regularity of $(E, \mathcal{D}(E))$, we may assume that $(u_n)_n \subset \mathcal{C}^+ \subset \mathcal{C}_0(\tau)$. By e.g. [19, Prop. 3.1(iii), p. 690], $E(u \vee v) \leq E(u) + E(v)$, thus, up to passing to a suitable non-reabeled subsequence, we may assume that $(u_n)_n$ is monotone non-decreasing. Then, $\lim_n u_n \equiv \mathbb{1}$ τ -locally uniformly on $\text{supp}[\mu] = X$ by Dini's Theorem, and therefore $\lim_n u_n(x) = 1$ for μ_ζ -a.e. $x \in X$ for every $\zeta \in Z$.

By the direct integral representation (24), it is readily seen arguing by contradiction that $\lim_n E_\zeta(u_n) = 0$ for ν -a.e. $\zeta \in Z$. As a consequence, $(E_\zeta, \mathcal{D}(E_\zeta))$ is recurrent for ν -a.e. $\zeta \in Z$, again by [15, Thm. 1.6.3, p. 58].

Suppose now that $(E, \mathcal{D}(E))$ is given as the direct integral of Dirichlet forms in Theorem 3.4, and assume that $(E_\zeta, \mathcal{D}(E_\zeta))$ is recurrent for ν -a.e. $\zeta \in Z$. We show that $(E, \mathcal{D}(E))$ is recurrent. A proof in the setting of Theorem 3.11 is nearly identical, and therefore it is omitted.

Recall the notation in Section 3.1 and argue by contradiction that $(E, \mathcal{D}(E))$ is not recurrent. By Corollary 3.15, there exists an E -invariant subset X_d , with $\mu X_d > 0$, so that $(E_d, \mathcal{D}(E_d))$ is transient. Since \mathcal{X}_0 is μ -essentially countably generated by \mathcal{X}^* , we may and shall assume without loss of generality that $X_d \in \mathcal{X}^*$, so that $B := s(X_d) \in \mathcal{Z}$. Since $\mu X_d > 0$, we have $\nu B > 0$. It is not difficult to show that the direct-integral decomposition of $L^2(\mu)$ splits as a direct sum of Hilbert spaces

$$L^2(\mu) \cong L^2(\mu_{X_d}) \oplus L^2(\mu_{X_d^c}) \cong \int_B^\oplus L^2(\mu_\zeta) \, d\nu(\zeta) \oplus \int_{B^c}^\oplus L^2(\mu_\zeta) \, d\nu(\zeta) .$$

Since X_d is E -invariant, a corresponding direct-integral decomposition of $\mathcal{D}(E)$ is induced by Corollary 3.15

$$\mathcal{D}(E)_1 \cong \mathcal{D}(E_d)_1 \oplus \mathcal{D}(E_c)_1 \cong \int_B^\oplus \mathcal{D}(E_\zeta)_1 \, d\nu(\zeta) \oplus \int_{B^c}^\oplus \mathcal{D}(E_\zeta)_1 \, d\nu(\zeta) .$$

Applying the reverse implication in (ii), it follows from the transience of $(E_d, \mathcal{D}(E_d))$ that $\mathcal{D}(E_\zeta, \mathcal{D}(E_\zeta))$ is transient for ν -a.e. $\zeta \in B$. Since $\nu B > 0$, this contradicts the assumption. □

Ergodic decomposition of measures Let $(X, \tau, \mathcal{X}, \mu)$ be a locally compact Polish probability space. Since (X, \mathcal{X}) is a standard Borel space, the space \mathfrak{M} of all σ -finite measures on (X, \mathcal{X}) is a standard Borel space as well when endowed with the σ -algebra generated by the family of sets

$$\{\eta \in \mathfrak{M} : a_1 < \eta A < a_2\} , \quad a_1, a_2 \in \mathbb{R}_+ , \quad A \in \mathcal{X} .$$

Let now $(E, \mathcal{D}(E))$ be a regular Dirichlet form on $L^2(\mu)$, and

$$\mathbf{M} := (\Omega, \mathcal{F}, (M_t)_{t \geq 0}, (P_x)_{x \in X_\partial}, \xi)$$

be the properly associated right process. We set

$$p_t(x, A) := P_x \{ \omega \in \Omega : M_t(\omega) \in A \} , \quad x \in X_\partial , \quad t \geq 0 , \quad A \in \mathcal{X}_\partial .$$

The semigroup T_\bullet of $(E, \mathcal{D}(E))$ is thus well-defined on bounded Borel measurable functions, by letting

$$T_t : \mathcal{X}_b(\mathbb{R}) \longrightarrow \mathcal{X}_b(\mathbb{R})$$

$$f \longmapsto \int_X f(y) p_t(\cdot, dy) , \quad t \geq 0 .$$

Definition 3.17 We say that a σ -finite measure η on (X, \mathcal{X}) is T_\bullet -invariant if

$$\int_X T_t f \, d\eta = \int_X f \, d\eta , \quad f \in \mathcal{X}_b(\mathbb{R}) , \quad t \geq 0 .$$

An invariant measure η is T_\bullet -ergodic if every E -invariant set is either η -negligible or η -conegligible. We denote by $\mathfrak{M}_{\text{inv}}$, resp. $\mathfrak{M}_{\text{erg}}$, the set of all σ -finite T_\bullet -invariant, resp. T_\bullet -ergodic, measures.

The formulation of the following result is adapted from [6, Thm. 1]. In light of Corollary 3.15, we may restrict to the case of recurrent Dirichlet forms.

Corollary 3.18 *Let $(X, \tau, \mathcal{X}, \mu)$ be a probability space satisfying Assumption 2.18, and let $(E, \mathcal{D}(E))$ be a recurrent quasi-regular Dirichlet form on $L^2(\mu)$. Then, there exists a properly E -coexceptional subset X_{inv} of X , and a surjective map $\pi : X_{\text{inv}} \rightarrow \mathfrak{M}_{\text{erg}}$ so that*

- (i) *for every $\lambda \in \mathfrak{M}_{\text{erg}}$ the set $\pi^{-1}(\lambda)$ is λ -conegligible;*
- (ii) *for every $\eta \in \mathfrak{M}_{\text{inv}}$,*

$$\eta = \int_{\mathfrak{M}_{\text{erg}}} \lambda \, d\bar{\eta}(\lambda), \quad \bar{\eta} := \pi_{\#}\eta ;$$

- (iii) *the map $\pi_{\#} : \mathfrak{M}_{\text{inv}} \rightarrow \mathfrak{M}(\mathfrak{M}_{\text{erg}})$ is a Borel isomorphism;*
- (iv) *for any $\eta_1, \eta_2 \in \mathfrak{M}_{\text{inv}}$ one has $\eta_1 \ll \eta_2$ if and only if $\pi_{\#}\eta_1 \ll \pi_{\#}\eta_2$, and $\eta_1 \perp \eta_2$ if and only if $\pi_{\#}\eta_1 \perp \pi_{\#}\eta_2$.*

Proof As a consequence of Theorem 3.6, we may restrict to the case when (X, τ) is a locally compact Polish space. This reduces measurability statements to the case of standard Borel spaces.

By Theorem 3.4(iii), there exists a ν -negligible set $N \in \mathcal{Z}^{\nu}$ so that, for every $\zeta \in N^c$, (a) $\mu_{\zeta}s^{-1}(\zeta) = 1$, in particular, μ_{ζ} is a probability measure (as opposed to: sub-probability); (b) $(E_{\zeta}, \mathcal{D}(E_{\zeta}))$ is a regular irreducible recurrent Dirichlet form on $L^2(\mu_{\zeta})$ over the space $\text{supp}[\mu_{\zeta}]$. Set $X_{\text{inv}} := s^{-1}(N^c)$ and note that X_{inv}^c is properly E -exceptional. Further define $\pi : x \mapsto \mu_{s(x)}$. For notational simplicity, we relabel Z as $Z \setminus N$, so that (a),(b) hold for every $\zeta \in Z$, and $X_{\text{inv}} = s^{-1}(Z)$. Assertions (ii)-(iii) are standard, e.g. [30, Thm. 6.6]. As a consequence of (ii), assertion (i) is precisely the strong consistency of $(\mu_{\zeta})_{\zeta \in Z}$ with s . The ‘only if’ part of assertion (iv) is straightforward. The ‘if’ part is a consequence of the representation in (ii), together with (i). □

Appendix

The theory of direct integrals of Banach spaces is inherently more sophisticated than the corresponding theory for Hilbert spaces. We discuss here an irreducible minimum after [17, Ch.s 5-7] and especially [9, §3]. For simplicity, we restrict ourselves to the case of σ -finite (not necessarily complete) indexing spaces (Z, \mathcal{Z}, ν) .

A decomposition $(Z_{\alpha})_{\alpha \in A}$ of (Z, \mathcal{Z}, ν) is a family of subsets $Z_{\alpha} \subset Z$ so that

$$\begin{aligned} \mathcal{Z} &= \{B \subset Z : B \cap Z_{\alpha} \in \mathcal{Z} \text{ for all } \alpha \in A\} && \text{and} \\ \nu B &= \sum_{\alpha \in A} \nu(B \cap Z_{\alpha}), && B \in \mathcal{Z}. \end{aligned}$$

Definition 1.1 (Measurable fields, cf. [17, §6.1, p. 61f.] and [9, §3.1]) Let (Z, \mathcal{Z}, ν) be a σ -finite measure space, and V be a real linear space. A ν -measurable family of semi-norms on V is a family $(q\|\cdot\|_{\zeta})_{\zeta \in Z}$ so that

- $\|\cdot\|_{\zeta}$ is a semi-norm on V for every $\zeta \in Z$;
- the map $\zeta \mapsto \|v\|_{\zeta}$ is ν -measurable for every $v \in V$.

Letting Y_{ζ} denote the Banach completion of $V / \ker \|\cdot\|_{\zeta}$, we say that a vector field $u \in \prod_{\zeta \in Z} Y_{\zeta}$ is ν -measurable if, for each $B \in \mathcal{Z}$ with $\nu B < \infty$, there exists a sequence $(u_n)_n$ of simple V -valued vector fields on B so that $\lim \|u_{\zeta} - u_{n,\zeta}\|_{\zeta} = 0$ ν -a.e. on B .

A family $(Y_\zeta)_{\zeta \in Z}$ of Banach spaces Y_ζ is a ν -measurable field of Banach spaces if there exist

- a decomposition $(Z_\alpha)_{\alpha \in A}$ of (Z, \mathcal{Z}, ν) consisting of sets of finite ν -measure;
- a family of real linear spaces $(Y^\alpha)_{\alpha \in A}$;
- for each $\alpha \in A$, a ν -measurable family of norms $\|\cdot\|_\zeta$ on Y^α ,

so that, for each $\alpha \in A$ and each $\zeta \in Z_\alpha$, the space Y_ζ is the completion of $(Y^\alpha, \|\cdot\|_\zeta)$.

Extending the above definition of ν -measurability, we say that $u \in \prod_{\zeta \in Z} Y_\zeta$ is ν -measurable if (and only if) the restriction of u to each Z_α is ν -measurable.

Let $p \in [1, \infty]$. A ν -measurable vector field u is called $L^p(\nu)$ -integrable if $\|u\|_p := \left\| (\zeta \mapsto \|u_\zeta\|_\zeta) \right\|_{L^p(\nu)}$ is finite. Two $L^p(\nu)$ -integrable vector fields u, v are ν -equivalent if $\|u - v\|_p = 0$.

The space Y_p of equivalence classes of $L^p(\nu)$ -integrable vector fields modulo ν -equivalence, endowed with the non-reabeled quotient norm $\|\cdot\|_p$, is a Banach space [9, Prop. 3.2], called the L^p -direct integral of $\zeta \mapsto Y_\zeta$ and denoted by

$$Y_p = \left(\int_Z^\oplus Y_\zeta \, d\nu(\zeta) \right)_p. \tag{49}$$

The following is a generalization of Proposition 2.25 to direct integrals of L^p -spaces. Recall (16).

Proposition 1.2 *Let (X, \mathcal{X}, μ) be σ -finite standard, (Z, \mathcal{Z}, ν) be σ -finite countably generated, and $(\mu_\zeta)_{\zeta \in Z}$ be a separated pseudo-disintegration of μ over ν . Further let \mathcal{A} be the lattice algebra of all real-valued μ -integrable simple functions on (X, \mathcal{X}) . Then, for every $p \in [1, \infty)$, the map*

$$\begin{aligned} \iota: [\mathcal{A}]_\mu &\longrightarrow Y_p := \left(\int_Z^\oplus L^p(\mu_\zeta) \, d\nu(\zeta) \right)_p \\ [s]_\mu &\longmapsto [\delta(s)]_{Y_p} \end{aligned} \tag{50}$$

extends to an isomorphism of Banach lattices $\iota_p: L^p(\mu) \rightarrow Y_p$.

A proof of the above Proposition 1.2 is quite similar to that of Proposition 2.25, and therefore it is omitted. Alternatively, a proof may be adapted from [9, §4.2], having care that:

- the algebra \mathcal{A} corresponds to the vector lattice V in [9, p. 694];
- the order on Y_p is defined analogously to Remark 2.20, cf. [9, p. 694];
- the map ι corresponds to the map defined in [9, Eqn. (4.6)];
- the surjectivity of ι_p follows as in [9, p. 696] since it only depends on the disintegration being separated. In the terminology and notation of [9], this is accounted by the fact that the decomposition β satisfies [9, Thm. 4.2(2)].

As an obvious corollary to Proposition 1.2, we obtain that the direct integral of Hilbert spaces H in Eq. 18 with underlying space of measurable vector fields generated by \mathcal{A} is identical to Y_2 as in Eq. 50. The specification of the underlying space of ν -measurable vector fields is necessary in light of Remark 2.23.

Proof of Lemma 3.8 Retain the notation established in Section 3.1 and in the proof of Theorem 3.4. Firstly, note that $L^1(\mu)$ is, trivially, an $L^\infty(\mu_0)$ -module, and $\mathcal{D}(E)$ is an $L^\infty(\mu_0)$ -module too, by Definition 3.1(d). As in Section 3.1, let p be the quotient map of Eq. 27. For $u \in L^\infty(\nu)$ denote by $p^*u \in L^\infty(\mu_0)$ the pullback of u via p . Setting $u \cdot f \mapsto p^*u \cdot f$ defines an action of $L^\infty(\nu)$ on $L^2(\mu)$ and $\mathcal{D}(E)$. Thus, since the spaces $(X, \mathcal{X}_0, \mu_0)$ and (Z, \mathcal{Z}, ν) have the same measure algebra by construction of Z , here and in the following we may replace $L^\infty(\mu_0)$ -modularity with $L^\infty(\nu)$ -modularity.

Let now $A \in \mathcal{X}_0$. Since A is E -invariant, then $\mathbb{1}_A f \in \mathcal{D}(E)$ and

$$E(\mathbb{1}_A f, g) = E(f, \mathbb{1}_A g) = E(\mathbb{1}_A f, \mathbb{1}_A g) \quad , \quad f, g \in \mathcal{D}(E) \tag{51}$$

by Definition 3.1. Replacing f with $\mathbb{1}_A f$ in Eq. 13, and applying (51) and again (13) yields

$$\mathbb{1}_A \Gamma(f, g) = \Gamma(\mathbb{1}_A f, g) \quad , \quad f, g \in \mathcal{D}(E) \cap L^\infty(\mu) \quad , \tag{52}$$

which is readily extended to $f, g \in \mathcal{D}(E)$ by approximation. Then, Eq. 52 shows that $f \mapsto \Gamma(f, g): \mathcal{D}(E)_1 \rightarrow L^1(\mu)$ is, for every fixed $g \in \mathcal{D}(E)$, a bounded $L^\infty(\nu)$ -modular operator in the sense of [17, §5.2]. By Step 1 in the proof of Theorem 3.4(iii), $\mathcal{D}(E)_1$ is a countably generated direct integral of Banach spaces, thus we may apply [17, Thm. 9.1] to obtain, for every fixed $g \in \mathcal{D}(E)$, a direct integral decomposition

$$\Gamma(\cdot, g) = \int_Z^\oplus \Gamma_{\zeta, g} \, d\nu(\zeta) : \int_Z^\oplus \mathcal{D}(E_\zeta)_1 \, d\nu(\zeta) \longrightarrow \left(\int_Z^\oplus L^1(\mu_\zeta) \, d\nu(\zeta) \right)_1 \cong L^1(\mu)$$

for some family of bounded operators $\Gamma_{\zeta, g}: \mathcal{D}(E_\zeta)_1 \rightarrow L^1(\mu_\zeta)$. Let \mathcal{C} be a core for $(E, \mathcal{D}(E))$ underlying the construction of the direct integral representation of $(E, \mathcal{D}(E))$ as in Step 1 in the proof of Theorem 3.4. It follows by symmetry of Γ that $\Gamma_{\zeta, g}(f) = \Gamma_{\zeta, f}(g)$ for every $f, g \in \mathcal{C}$ and ν -a.e. $\zeta \in Z$. In particular, the assignment $g \mapsto \Gamma_{\zeta, g}$ is linear on $\mathcal{C} \subset \mathcal{D}(E_\zeta)$ for ν -a.e. $\zeta \in Z$. A symmetric bilinear map is then induced on $\mathcal{C}^{\oplus 2}$ by setting $\Gamma_\zeta: (f, g) \mapsto \Gamma_{\zeta, g}(f)$.

Thus, finally, it suffices to show Eq. 13 for Γ_ζ and $(E_\zeta, \mathcal{D}(E_\zeta))$ for ν -a.e. $\zeta \in Z$ with $f, g, h \in \mathcal{C}$, which is readily shown arguing by contradiction, analogously to the proof of the claim in Step 4 of Theorem 3.4. □

Acknowledgements The author is grateful to Professors Sergio Albeverio and Andreas Eberle, and to Dr. Kohei Suzuki, for fruitful conversations on the subject of the present work, and for respectively pointing out the references [1, 13], and [3, 20]. Finally, he is especially grateful to an anonymous Reviewer for their very careful reading and their suggestions which improved the readability of the paper.

Funding Open access funding provided by Institute of Science and Technology (IST Austria).

Declarations

Conflict of interest The author declares that he has no conflict of interest.

Data availability The manuscript has no associated data.

Open Access This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article’s Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article’s Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit <http://creativecommons.org/licenses/by/4.0/>.

References

1. Alberverio, S.A., Kondrat'ev, Yu.G., Röckner, M.: Ergodicity for the Stochastic Dynamics of Quasi-invariant Measures with Applications to Gibbs States. *J. Funct. Anal.* **149**, 415–469 (1997). <https://doi.org/10.1006/jfan.1997.3099>
2. Alberverio, S.A., Kondrat'ev, Yu.G., Röckner, M.: Analysis and geometry on configuration spaces. *J. Funct. Anal.* **154**(2), 444–500 (1998). <https://doi.org/10.1006/jfan.1997.3183>
3. Alberverio, S.A., Röckner, M.: Classical dirichlet forms on topological vector spaces — closability and a Cameron–Martin formula. *J. Funct. Anal.* **88**, 395–436 (1990). [https://doi.org/10.1016/0022-1236\(90\)90113-Y](https://doi.org/10.1016/0022-1236(90)90113-Y)
4. Bianchini, S., Caravenna, L.: On the extremality, uniqueness, and optimality of transference plans. *Bull. Inst. Math. Acad. Sin. (N.S.)* **4**(4), 353–454 (2009)
5. Bouleau, N., Hirsch, F.: Dirichlet forms and analysis on Wiener space. De Gruyter (1991)
6. Bufetov, A.I.: Ergodic decomposition for measures quasi-invariant under a Borel action of an inductively compact group. *Sb. Math.* **205**(2), 192–219 (2014). <https://doi.org/10.1070/sm2014v205n02abeh004371>
7. Chen, Z.-Q., Ma, Z.-M., Röckner, M.: Quasi-homeomorphisms of Dirichlet forms. *Nagoya Math. J.* **136**, 1–15 (1994)
8. Chow, T.R., Gilfeather, F.: Functions of direct integrals of operators. *Proc. Amer. Math. Soc.* **29**(2), 325–330 (1971)
9. de Jeu, M., Rozendaal, J.: Disintegration of positive isometric group representations on L^p -spaces. *Positivity* **21**, 673–710 (2017)
10. Dello Schiavo, L.: The Dirichlet–Ferguson Diffusion on the Space of Probability Measures over a Closed Riemannian Manifold. arXiv:1811.11598 (2018)
11. Dello Schiavo, L., Wirth, M.: Ergodic Decompositions of Dirichlet Forms under Order Isomorphisms. arXiv:2109.00615 (2021)
12. Dixmier, J.: Von neumann algebras, North-Holland (1981)
13. Eberle, A.: Girsanov-type transformations of local Dirichlet forms: an analytic approach. *Osaka J. Math.* **33**(2), 497–531 (1996)
14. Fremlin, D.H.: Measure Theory – Volume I – IV, V Part I & II. Torres Fremlin (ed.) (2000–2008)
15. Fukushima, M., Oshima, Y., Takeda, M.: Dirichlet forms and symmetric Markov processes, De Gruyter Studies in Mathematics, vol. 19. Extended edn. De Gruyter (2011)
16. Greschonig, G., Schmidt, K.: Ergodic decomposition of quasi-invariant probability measures. *Colloq. Math.* **84**, 495–514 (2000)
17. Haydon, R., Levy, M., Raynaud, Y.: Randomly normed spaces. *Travaux en cours Hermann* (1991)
18. Kato, T.: Perturbation Theory for Linear Operators, Reprint of the 1980 Edition. Corrected Printing of the 2nd edn., Classics in Mathematics. Springer (1995)
19. Kuwae, K.: Functional calculus for dirichlet forms. *Osaka J. Math.* **35**, 683–715 (1998)
20. Kuwae, K.: Invariant sets and ergodic decomposition of local semi-Dirichlet forms. *Forum Math.* **23**, 1259–1279 (2011). <https://doi.org/10.1515/FORM.2011.046>
21. Lance, C.: Direct integrals of left Hilbert algebras. *Math. Ann.* **216**, 11–28 (1975)
22. Lenz, D., Schmidt, M., Wirth, M.: Geometric Properties of Dirichlet Forms under Order Isomorphism. arXiv:1801.08326 (2018)
23. Ma, Z.M., Röckner, M.: Introduction to the theory of (Non-Symmetric) dirichlet forms. Graduate Studies in Mathematics Springer (1992)
24. von Neumann, J.: On rings of operators. Reduction theory. *Ann. Math.* **50**(2), 401 (1949). <https://doi.org/10.2307/1969463>
25. Ôkura, H.: On the invariant sets and the irreducibility of symmetric markov processes. In: Shiryaev, A.N., Korolyuk, V.S., Watanabe, S., Fukushima, M. (eds.) Probability Theory and Mathematical Statistics – Proceedings of the 6th USSR–Japan Symposium, pp. 238–247. World Scientific (1992). <https://doi.org/10.1142/1780>
26. Schmuland, B.: On the local property for positivity preserving coercive forms. In: Ma, Z.-M., Röckner, M., Yan, J.A. (eds.) Dirichlet Forms and Stochastic Processes: Proceedings of the International Conference held in Beijing, pp. 345–354. De Gruyter Proceedings in Mathematics, China (1995). <https://doi.org/10.1515/9783110880052.345>

27. Schwartz, L.: Radon measures on arbitrary topological spaces and cylindrical measures. Oxford University Press (1973)
28. ter Elst, A.F.M., Robinson, D.W., Sikora, A., Zhu, Y.: Dirichlet Forms and Degenerate Elliptic Operators. In: Koelink, E., Van Neerven, J., De Pagter, B., Sweers, G., Luger, A., Woracek, H. (eds.) Partial Differential Equations and Functional Analysis: The Philippe Clément Festschrift, pp. 73–95. Birkhäuser (2006)
29. Tomisaki, M.: Superposition of diffusion processes. *J. Math. Soc. Jpn.* **32**(4), 671–696 (1980)
30. Varadhan, S.R.S.: Probability Theory, Courant Lecture Notes, vol. 7. Amer. Math Soc (2001)

Publisher's Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.