Curvature driven interface evolution: Uniqueness properties of weak solution concepts

by

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Abstract

The present thesis is concerned with the derivation of weak-strong uniqueness principles for curvature driven interface evolution problems not satisfying a comparison principle. The specific examples being treated are two-phase Navier–Stokes flow with surface tension, modeling the evolution of two incompressible, viscous and immiscible fluids separated by a sharp interface, and multiphase mean curvature flow, which serves as an idealized model for the motion of grain boundaries in an annealing polycrystalline material. Our main results—obtained in joint works with Julian Fischer, Tim Laux and Theresa M. Simon—state that prior to the formation of geometric singularities due to topology changes, the weak solution concept of Abels (Interfaces Free Bound. 9, 2007) to two-phase Navier–Stokes flow with surface tension and the weak solution concept of Laux and Otto (Calc. Var. Partial Differential Equations 55, 2016) to multiphase mean curvature flow (for networks in $\mathbb{R}^2$ or double bubbles in $\mathbb{R}^3$) represents the unique solution to these interface evolution problems within the class of classical solutions, respectively.

To the best of the author's knowledge, for interface evolution problems not admitting a geometric comparison principle the derivation of a weak-strong uniqueness principle represented an open problem, so that the works contained in the present thesis constitute the first positive results in this direction. The key ingredient of our approach consists of the introduction of a novel concept of relative entropies for a class of curvature driven interface evolution problems, for which the associated energy contains an interfacial contribution being proportional to the surface area of the evolving (network of) interface(s). The interfacial part of the relative entropy gives sufficient control on the interface error between a weak and a classical solution, and its time evolution can be computed, at least in principle, for any energy dissipating weak solution concept. A resulting stability estimate for the relative entropy essentially entails the above mentioned weak-strong uniqueness principles.

The present thesis contains a detailed introduction to our relative entropy approach, which in particular highlights potential applications to other problems in curvature driven interface evolution not treated in this thesis.

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Für Lisa

About the Author

The author of the present thesis received a BSc in Mathematics at Freie Universität Berlin in 2015 and an MSc in Mathematics at Humboldt-Universität zu Berlin in 2017. He joined the research group of Prof. Julian Fischer at IST Austria as a PhD student in September 2017. His main field of research is the mathematical theory of partial differential equations, with particular research interests in interface evolution problems, mathematical fluid mechanics, stochastic homogenization and stochastic partial differential equations. After his time at IST Austria, he will be a Hausdorff Postdoc at the Hausdorff Center for Mathematics, University of Bonn.

List of Collaborators and Publications

Chapter 3 contains the paper "Weak-strong uniqueness for the Navier-Stokes equation for two fluids with surface tension" (without introduction), which is a joint work with Julian Fischer and published in *Archive for Rational Mechanics and Analysis* **236**, 967–1087 (2020).

Chapter 4 contains an edited and revised version of the paper "The local structure of the energy landscape in multiphase mean curvature flow: Weak-strong uniqueness and stability of evolutions" (without introduction), which is a joint work with Julian Fischer, Tim Laux and Theresa M. Simon. The first preprint version is uploaded to the arXiv (identifier 2003.05478).

Chapter 5 contains the paper "Weak-strong uniqueness for the mean curvature flow of double bubbles" (without introduction), which is a joint work with Tim Laux. The preprint can be found on the arXiv (identifier 2108.01733).

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CHAPTER 1

Introduction

1.1 Curvature driven interface evolution: Applications

Interfaces evolving under the effect of extrinsic curvature quantities such as mean curvature are ubiquitous in a wide variety of applications. We briefly discuss some of these fascinating topics including applications from image processing, biology, flame propagation in combustion processes, fluid mechanics, and finally applications from materials science.

- The first example concerns the removal of noise and the improvement of features in a given image (cf. Sethian [144, Section 16]). Typical goals are to smooth out small-scale oscillations in boundaries of distinct regions or to blend into the background color scattered points of noise. A specific challenge in this context consists of keeping sharp interfaces present in the image while trying to blur such noise or small-scale oscillations of boundaries. One strategy which overcomes this challenge is to let the image defining intensity function evolve by a speed function depending in a suitable way on the mean curvature of the contour lines of this intensity function. We refer to the books of Sethian [144], Sapiro [134], Aubert and Kornprobst [14] as well as Cao [28] for precise representations of such schemes.

- Curvature driven interface evolution also appears as a model to explain the effect of the surface geometry of supporting structures (e.g., the geometry of scaffolds in tissue engineering with applications to defect healing) on observed growth patterns for in vitro tissue formation (cf. Rumpler *et al.* [133]). For instance, Rumpler *et al.* [133] show that numerical experiments modeling tissue growth within various two-dimensional scaffold structures (i.e., triangular, square, hexagonal and circular shapes) based on the hypothesis that the tissue interface evolves proportional to its mean curvature, indeed match the results from corresponding in vitro cell growth experiments. For further results on curvature driven tissue growth and relevant applications, we refer to the works of Bidan *et al.* [21], Bidan *et al.* [22] as well as Guyot *et al.* [77].

- One way to model flame propagation within combustion processes is to view the evolving flame front as a sharp interface separating the burnt and unburnt regions (cf. Sethian [144, Section 18.1]). A simplified model taking into account, amongst other things, a dependence of the normal speed of the flame front on its mean curvature as a result of heat conduction is due to Markstein [112]. Apart from this, the modeling of combustion processes also has to incorporate the fluid dynamics within both the burnt and unburnt regions, and has to account for several effects coupling the motion of the fluids and the evolution of the flame front, which affect each other. For a discussion of such models and further references to the relevant literature, we refer to the book of Sethian [144] as well as the papers by Sethian [143] and Osher and Sethian [122].

- Further applications of curvature driven interface evolution in the context of fluid mechanics concerns, e.g., the analysis of the evolution of two incompressible, immiscible and viscous fluids (e.g., the motion of droplets of oil in surrounding water) under surface tension effects (see, e.g., Denisova [49], Sussmann, Smereka and Osher [147], Chang, Hou, Merriman and Osher [32] and Prüss and Simonett [127]), or the propagation of a "cold" flame front (see, e.g., Zhu and Sethian [150]) meaning that effects of the combustion zone on the fluid dynamics within the burnt and unburnt regions are neglected.

- A very prominent example of curvature driven interface evolution is given by the motion of grain boundaries (i.e., the interfaces between crystals with differing orientation) during grain growth in an annealing polycrystalline material like a piece of aluminum (cf. Mullins [120] or Brakke [23, Appendix A]). The driving force behind this coarsening process is the reduction of (in principle anisotropic) surface energy in form of surface tension associated with each grain boundary. The resulting evolution of the grain boundaries takes place in the direction of their center of curvature with a speed being proportional to their mean curvature (see, e.g., Beck and Sperry [18] and Rhines, Craig and DeHoff [130]).

- We conclude by mentioning the important example of liquid-solid interface evolution in solidification processes like crystal growth or the formation of dendritic patterns (cf. Sethian [144, Section 18.2]). The modeling of such phenomena requires to account for interactions between the evolving liquid-solid interface and the associated temperature field, which solves a heat equation away from the interface. Curvature appears in these models in form of boundary conditions for the temperature field along the moving liquid-solid interface as well as for the jump of the heat flux in normal direction across it (see, e.g., the works of Mullins and Sekerka [121], Langer [94], Ben-Jacobi, Goldenfield, Langer and Schon [20] or Gurtin [76]).

1.2 Curvature driven interface evolution: A small sample of mathematical models

We present in this section three important mathematical models accounting for curvature driven interface evolution: *i)* two-phase Navier–Stokes flow with surface tension, *ii)* multiphase mean curvature flow, and *iii)* the Stefan problem with isotropic Gibbs–Thomson law together with its quasi-static version, the Mullins–Sekerka equation. Our selection is mainly motivated by the fact that the main results presented and announced in this thesis (cf. Section 1.4) are precisely concerned with these models. The focus of the following discussion lies on the mathematical formulation of these problems and a brief discussion of selected parts of the existence theory for weak and strong solution concepts.

Of course, this section only provides a very small glimpse into the rich variety of mathematical models accounting for curvature driven interface evolution. For instance, we neglect at this stage considerations concerning evolution problems incorporating anisotropic resp.

crystalline surface energies (see, e.g., [29], [19], [73], [90], [31] or [30]) or the alternative representation of the evolving interface in terms of diffuse interface approximations (see, e.g., [33], [46], [9], [4], [7], [5] or [6]) instead of sharp interfaces. For an excellent overview article on the topic, we refer the reader to Garcke [72].

1.2.1 Two-phase Navier–Stokes flow with surface tension

As a first example, we consider the flow of two immiscible, incompressible and viscous fluids incorporating surface tension effects. For such a system of two fluids, the most basic model featuring a sharp interface is described as follows. Because of the immiscibility, the evolution of the interface between the fluids is governed by a transport equation along the fluid flow. The motion of each single fluid is moreover modeled by means of the incompressible Navier–Stokes equation. Finally, surface tension acts on the sharp interface by exerting a force which is proportional to the mean curvature vector of the interface.

In terms of a mathematical formulation, we consider a full space setting with time horizon $T \in (0,\infty)$. The two fluids fill two disjoint evolving open domains $\Omega^+ = (\Omega(t))_{t\in[0,T]}$ and $\Omega^- = (\mathbb{R}^d \setminus \overline{\Omega(t)})_{t\in[0,T]}$ in $\mathbb{R}^d$, $d \in \{2,3\}$, respectively. The evolving interface separating the two regions is denoted by $(I(t))_{t\in[0,T]}$. Imposing a no-slip boundary condition at the interface for the velocity fields of the single fluids, one may treat them as a single vector field $u = u(x,t)$. One may also denote the pressure as a single scalar field $p = p(x,t)$, which in general however jumps across the interface. Denoting for each $t \in [0,T]$ by $\chi = \chi(\cdot,t)$ the indicator function of the domain $\Omega(t)$, by $\mu_\pm$ the shear viscosities of the two fluids, by $\rho_\pm$ their densities, by $\sigma > 0$ a surface tension constant, by $\mathrm{H}_I = \mathrm{H}_I(\cdot,t)$ the mean curvature vector of the interface $I(t)$, as well as by $|\nabla\chi| = |\nabla\chi|(\cdot,t)$ the corresponding surface measure, one obtains the following PDE formulation in $\mathbb{R}^d \times [0,T]$ for the above two-phase fluid system

$$\partial_t\chi + (u\cdot\nabla)\chi = 0, \tag{1.1a}$$

$$\partial_t\big(\rho(\chi)u\big) + \nabla\cdot\big(\rho(\chi)u\otimes u\big) = -\nabla p + \nabla\cdot\big(\mu(\chi)\big(\nabla u + (\nabla u)^{\mathsf{T}}\big)\big) + \sigma\mathrm{H}_I|\nabla\chi|, \tag{1.1b}$$

$$\nabla\cdot u = 0, \tag{1.1c}$$

where we introduced the abbreviations $\rho(\chi) := \rho_+\chi + \rho_-(1-\chi)$ and $\mu(\chi) := \mu_+\chi + \mu_-(1-\chi)$. Writing $\mathrm{n} = \mathrm{n}(\cdot,t)$ for the unit normal along $I(t)$ pointing inside $\Omega(t)$, the right hand side of (1.1b) encodes the Young–Laplace law along the interface:

$$\big[\mathrm{n}\cdot\big(\mu(\chi)\big(\nabla u + (\nabla u)^{\mathsf{T}}\big) - p\mathrm{Id}\big)\big]_{I(t)} = \sigma\mathrm{H}_I \tag{1.2}$$

for all $t \in [0,T]$, where $[\cdot]_{I(t)}$ denotes the jump across the interface. In words, the normal component of the jump of the viscous stress tensor across the interface equals the mean curvature vector of the interface times the surface tension constant.

The energy functional for the free boundary problem (1.1a)–(1.1c) is given by

$$E[\chi,u](t) := \int \frac{1}{2}\rho\big(\chi(\cdot,t)\big)|u(\cdot,t)|^2\,\mathrm{d}x + \sigma\int_{I(t)} 1\,\mathrm{d}\mathcal{H}^{d-1}, \quad t\in[0,T], \tag{1.3}$$

and thus consists of a combination of kinetic energy due to the two fluids and interfacial energy due to surface tension. We discuss in the following a selection of results from the existence theory (either local-in-time for strong solutions or global-in-time for weak solutions) of finite energy solutions to (1.1a)–(1.1c) satisfying the associated energy dissipation inequality (or rather its time-integrated version)

$$\frac{\mathrm{d}}{\mathrm{d}t}E[\chi,u](t) + \int \frac{1}{2}\mu\big(\chi(\cdot,t)\big)\big|\nabla u(\cdot,t) + (\nabla u)^{\mathsf{T}}(\cdot,t)\big|^2\,\mathrm{d}x \le 0. \tag{1.4}$$

The existence and uniqueness of local-in-time strong solutions in Sobolev–Slobodeckij spaces is due to Denisova [49], building on earlier works by Denisova ([47] and [48]) as well as Denisova and Solonnikov ([50] and [51]) on the linearized problem in Sobolev–Slobodeckij spaces resp. Hölder spaces. As always in the context of free boundary problems, a major difficulty for the construction of solutions stems from the fact that the domains on which the PDEs are formulated are themselves part of the problem. In the context of strong solutions, the standard approach to handle this issue is to transform the problem under consideration into a setting with a fixed domain. The above mentioned works by Denisova and Solonnikov achieve this by passing to Lagrangian coordinates.

Instead of employing Lagrangian coordinates, another strategy consists of a parametrization of the evolving interface in terms of a height function over the initial interface, and to transform the free boundary problem into a setting with a fixed domain by means of this height function. The resulting evolution problem in the fixed domain is of highly nonlinear nature, and a major step in the analysis then consists of a careful study of the corresponding linearized problems in order to facilitate in the end a contraction mapping principle argument for the nonlinear problem. This idea dates back to a work of Hanzawa [78] on the existence of strong solutions for the one-phase Stefan problem, which is why the associated transformation to a fixed domain is usually referred to in the literature as the *Hanzawa transform*.

In the case that the initial interface is given by a graph over $\mathbb{R}^{d-1}\times\{0\}$ and assuming a smallness condition on the initial data, Prüss and Simonett [126] establish based on this approach a short time existence and uniqueness result for the two-phase Navier–Stokes flow with surface tension. In contrast to the works of Denisova and Solonnikov, their methods even allow to deduce a smoothing effect for positive times, namely guaranteeing real analyticity of the interface as well as real analyticity of the velocity field and the pressure away from the interface for positive times. The same result holds true if one allows gravity to act on the system, see Prüss and Simonett [127] who also provide in their work [125] an analysis on the Rayleigh–Taylor instability. For a general local-in-time existence and uniqueness result (i.e., without employing a parametrization assumption over a flat model interface $\mathbb{R}^{d-1}\times\{0\}$), we refer to the work of Köhne, Prüss and Wilke [89].

Thinking of the possibility of the pinch-off of a liquid drop into two separate drops, or the coalescence of two drops into a single drop, a mathematical formulation of the evolution problem in terms of a strong PDE solution concept necessarily breaks down from the first topology change onwards. For a global-in-time existence result with respect to general initial data, one therefore has to resort to weaker descriptions of the dynamics.

A first guess in this direction is provided by the desired energy dissipation inequality (1.4). On one side, it suggests to require $u \in L^\infty(0,T;L^2(\mathbb{R}^d;\mathbb{R}^d))$ and $\nabla u \in L^2(0,T;L^2(\mathbb{R}^d;\mathbb{R}^{d\times d}))$ for the solenoidal velocity field u (with (1.1c) interpreted in a distributional sense). On the other side, it suggests to represent the underlying evolving geometry $\Omega^+ = (\Omega(t))_{t\in[0,T]}$ in terms of a time-dependent family of sets of finite perimeter, so that the associated interfaces $I(t)$ are given by the reduced boundaries $\partial^*\Omega(t)$ for all $t \in [0,T]$. In other words, one may require $\chi \in L^\infty(0,T;BV(\mathbb{R}^d;\{0,1\}))$ for the corresponding time-dependent indicator function. How do these choices reflect in a weak formulation of (1.1a)–(1.1b)?

In a smooth setting (i.e., $I(t)$ being a smooth and closed manifold without boundary), the surface divergence theorem (cf. [128, Equation (2.31)]) implies the following distributional representation of the mean curvature vector $\mathrm{H}_I = \mathrm{H}_I(\cdot,t)$

$$\int_{I(t)} \mathrm{H}_I \cdot \varphi \,\mathrm{d}\mathcal{H}^{d-1} = -\int_{I(t)} (\mathrm{Id}{-}\mathrm{n}\otimes\mathrm{n}) : \nabla\varphi \,\mathrm{d}\mathcal{H}^{d-1} \tag{1.5}$$

for all compactly supported and smooth test vector fields $\varphi \in C^\infty_{\mathrm{cpt}}(\mathbb{R}^d;\mathbb{R}^d)$. Observing that the right hand side of (1.5) even makes sense in the setting of sets of finite perimeter, whereas the left hand side in general does not, one may try to solve (1.1b) in form of

$$\begin{aligned}
&\int \rho\big(\chi(\cdot,T')\big)u(\cdot,T')\cdot\varphi(\cdot,T')\,\mathrm{d}x - \int \rho\big(\chi(\cdot,0)\big)u(\cdot,0)\cdot\varphi(\cdot,0)\,\mathrm{d}x \\
&= \int_0^{T'}\int \rho(\chi)u\cdot\partial_t\varphi\,\mathrm{d}x\,\mathrm{d}t + \int_0^{T'}\int \rho(\chi)u\otimes u : \nabla\varphi\,\mathrm{d}x\,\mathrm{d}t \qquad (1.6)\\
&\quad - \int_0^{T'} \mu(\chi)\big(\nabla u+(\nabla u)^{\mathsf{T}}\big) : \nabla\varphi\,\mathrm{d}x\,\mathrm{d}t - \sigma\int_0^{T'}\int_{I(t)} (\mathrm{Id}-\mathrm{n}\otimes\mathrm{n}) : \nabla\varphi\,\mathrm{d}\mathcal{H}^{d-1}\,\mathrm{d}t
\end{aligned}$$

for almost every $T' \in [0,T]$ and all solenoidal $\varphi \in C^\infty_{\mathrm{cpt}}(\mathbb{R}^d\times[0,T];\mathbb{R}^d)$. A weak formulation of the transport equation (1.1a) making sense in the energy space for (χ,u) is in turn given by requiring

$$\int \chi(\cdot,T')\phi(\cdot,T')\,\mathrm{d}x - \int \chi(\cdot,0)\phi(\cdot,0)\,\mathrm{d}x = \int_0^{T'}\int \chi\big(\partial_t\phi + (u\cdot\nabla)\phi\big)\,\mathrm{d}x\,\mathrm{d}t \qquad (1.7)$$

for almost every $T' \in [0,T]$ and all $\phi \in C^\infty_{\mathrm{cpt}}(\mathbb{R}^d\times[0,T])$.

However, there is an immediate problem when one tries to construct solutions to (1.1a)–(1.1b) in the sense of the BV formulation (1.6)–(1.7) by, say, passing to the limit in an approximating sequence of solutions (χ_k,u_k), $k\in\mathbb{N}$, associated with a regularized version of the problem. Indeed, the energy dissipation inequality (1.4) only allows to infer boundedness of the sequence $(\chi_k)_{k\in\mathbb{N}}$ in the space $BV_{\mathrm{loc}}(\mathbb{R}^d)$ which in turn implies weak* convergence of a subsequence in this space (cf. [12, Section 3.1]). In particular, in such a setting we in general only know that a subsequence of $\nabla\chi_k$ converges to $\nabla\chi$ weakly* in the sense of finite Radon measures on $\mathbb{R}^d$. Without additional information (e.g., convergence of the total variations $|\nabla\chi_k|(\mathbb{R}^d)$ to the total variation of the weak* limit $|\nabla\chi|(\mathbb{R}^d)$ which then allows to exploit Reshetnyak's continuity result [12, Theorem 2.39]), this in turn is not sufficient to pass to the limit in the *nonlinear* functional (1.5) representing the mean curvature vector (cf. the discussion of Abels [2, Section 3]).

Following the work of Plotnikov [124], Abels [1] deals with this problem by passing to an even weaker notion of solutions; a framework which he refers to as varifold solutions (cf. Abels [2, Definition 3.2]). The key ingredient is the concept of an oriented varifold, which is a finite Radon measure V on the product space $\mathbb{R}^d\times\mathbb{S}^{d-1}$ (where $\mathbb{S}^{d-1}$ denotes the unit sphere). Any oriented varifold may be equivalently expressed in terms of its disintegration $V = |V|\otimes(\nu_x)_{x\in\mathbb{R}^d}$ (cf. [12, Theorem 2.28]), where $|V|$ denotes the associated local mass density (i.e., $|V|(U) := V(U\times\mathbb{S}^{d-1})$ for all Borel measurable $U\subset\mathbb{R}^d$) and $(\nu_x)_{x\in\mathbb{R}^d}$ is a family of probability measures on $\mathbb{S}^{d-1}$. Since the data associated with the sets of finite perimeter represented through the indicator functions χ_k can be lifted to an oriented varifold $V_k := |\nabla\chi_k|\otimes(\delta_{\frac{\nabla\chi_k}{|\nabla\chi_k|}(x)})_{x\in\mathbb{R}^d}$, one may pass to the weak* limit in the sense of oriented varifolds. Due to

$$-\sigma\int_{\mathbb{R}^d}\Big(\mathrm{Id}-\frac{\nabla\chi_k}{|\nabla\chi_k|}\otimes\frac{\nabla\chi_k}{|\nabla\chi_k|}\Big) : \nabla\varphi\,\mathrm{d}|\nabla\chi_k| = -\sigma\int_{\mathbb{R}^d\times\mathbb{S}^{d-1}}(\mathrm{Id}-s\otimes s) : \nabla\varphi(x)\,\mathrm{d}V_k(x,s),$$

this in turn allows to pass to the limit in the "varifold formulation" of the mean curvature functional.

These arguments motivate Abels [1] to consider the following generalization of (1.6)

$$\begin{aligned}
&\int \rho\big(\chi(\cdot,T')\big)u(\cdot,T')\cdot\varphi(\cdot,T')\,\mathrm{d}x - \int \rho\big(\chi(\cdot,0)\big)u(\cdot,0)\cdot\varphi(\cdot,0)\,\mathrm{d}x \\
&= \int_0^{T'}\int \rho(\chi)u\cdot\partial_t\varphi\,\mathrm{d}x\,\mathrm{d}t + \int_0^{T'}\int \rho(\chi)u\otimes u : \nabla\varphi\,\mathrm{d}x\,\mathrm{d}t \qquad (1.8)\\
&\quad - \int_0^{T'} \mu(\chi)\big(\nabla u+(\nabla u)^{\mathsf{T}}\big) : \nabla\varphi\,\mathrm{d}x\,\mathrm{d}t - \sigma\int_0^{T'}\int_{\mathbb{R}^d\times\mathbb{S}^{d-1}} (\mathrm{Id}-s\otimes s) : \nabla\varphi(x,t)\,\mathrm{d}V_t(x,s)\,\mathrm{d}t
\end{aligned}$$

in terms of a time-dependent family of oriented varifolds $(V_t)_{t\in[0,T]}$, which is coupled to the time-dependent indicator function χ through the natural compatibility condition

$$\int_{\mathbb{R}^d\times\mathbb{S}^{d-1}} s\cdot\psi(x)\,\mathrm{d}V_t(x,s) = \int_{I(t)} \mathrm{n}(x,t)\cdot\psi(x)\,\mathrm{d}\mathcal{H}^{d-1}(x) \tag{1.9}$$

for all $t\in[0,T]$ and all $\psi\in C^\infty_{\mathrm{cpt}}(\mathbb{R}^d;\mathbb{R}^d)$. Abels [1] then proves for general initial data the global-in-time existence of varifold solutions (χ,u,V) satisfying (1.7)–(1.9), and which dissipate the energy given by

$$E[\chi,u,V](t) := \int \frac{1}{2}\rho\big(\chi(\cdot,t)\big)|u(\cdot,t)|^2\,\mathrm{d}x + \sigma\int_{\mathbb{R}^d\times\mathbb{S}^{d-1}} 1\,\mathrm{d}V_t(x,s), \quad t\in[0,T]. \tag{1.10}$$

His analysis even provides that χ solves the transport equation (1.1a) in a renormalized sense of DiPerna and Lions [54], and that one may include a class of non-Newtonian fluids by passing to a concept of measure-valued varifold solutions (cf. Abels [1, Definition 1.2 and Theorem 1.6]).

1.2.2 Evolution by multiphase mean curvature flow

As a second example, we consider evolution by multiphase mean curvature flow. In short, this evolution problem concerns the evolution of a network of interfaces which is formed by an underlying partition of a domain into several phases. Any given point on one of these interfaces is required to move with a velocity which is proportional to the mean curvature vector at this point. The proportionality factor accounts for surface tension at the respective interface, and it may vary from interface to interface. (We will only consider the isotropic regime in the following.) A major motivation behind studying multiphase mean curvature flow is that it represents idealized grain boundary motion in a recrystallized metal which underwent a process of heat treatment (cf. Mullins [120]). The analysis of multiphase systems which evolve by the mean curvature flow rule is also a mathematically intriguing problem due to its inherent singular character: junctions form at points where more than two phases meet, parts of the boundary and even whole phases vanish during the evolution of the system, and so on. For a visual illustration, we refer to the grain growth movies on Brakke's webpage (http://facstaff.susqu.edu/brakke).

In mathematical terms, evolution by multiphase mean curvature flow may be phrased in a full space setting with time horizon $T\in(0,\infty)$ as follows. Let $P\geq 2$ be an integer denoting the number of phases, and consider a family $\Omega = (\Omega_1(t),\ldots,\Omega_P(t))_{t\in[0,T]}$ representing an evolving partition of $\mathbb{R}^d$ in the sense that for all $t\in[0,T]$ the family $(\Omega_1(t),\ldots,\Omega_P(t))$ consists of P pairwise disjoint open sets which partition $\mathbb{R}^d$. For each $t\in[0,T]$ and each pair of distinct phases $i,j\in\{1,\ldots,P\}$, the common boundary of the ith phase $\Omega_i(t)$ and the jth phase $\Omega_j(t)$ describes the associated interface and is denoted by $I_{i,j}(t)$. Denoting by $\mathrm{V}_{I_{i,j}} = \mathrm{V}_{I_{i,j}}(\cdot,t)$ and $\mathrm{H}_{I_{i,j}} = \mathrm{H}_{I_{i,j}}(\cdot,t)$ the normal velocity vector field and the mean curvature vector field along an interface $I_{i,j}(t)$, respectively, the geometric evolution problem then simply reads

$$\mathrm{V}_{I_{i,j}} = \mathrm{H}_{I_{i,j}} \quad \text{along each interface } I_{i,j}(t),\ t\in[0,T],\ i\neq j\in\{1,\ldots,P\}. \tag{1.11}$$

Evolution by mean curvature can be derived as the L^2-gradient flow of interfacial surface area (cf. Garcke [72, Section 2.3]). In the multiphase case (1.11), interfacial energy is given by a weighted sum of the surface areas of the evolving interfaces . The weights account for surface tension at the interfaces, and are represented by a symmetric matrix $\sigma\in\mathbb{R}^{P\times P}$ such that (at least) $\sigma_{i,i} = 0$ for all $i \in \{1, \ldots, P\}$ as well as $\sigma_{i,j} > 0$ for all $i, j \in \{1, \ldots, P\}$ with $i \neq j$. The associated energy is then defined by

$$E[\Omega](t) := \frac{1}{2} \sum_{i,j=1, i\neq j}^{P} \sigma_{i,j} \int_{I_{i,j}(t)} 1 \, \mathrm{d}\mathcal{H}^{d-1}, \quad t \in [0, T]. \tag{1.12}$$

Under the assumption that only exactly three phases may meet along codimension two junctions, and under the assumption that along each of these triple junctions the *Herring angle condition* is satisfied in form of (where $\mathrm{n}_{i,j} = \mathrm{n}_{i,j}(\cdot, t)$ denotes the unit normal along the interface $I_{i,j}(t)$ pointing from the ith to the jth phase)

$$\sigma_{i,j}\mathrm{n}_{i,j} + \sigma_{j,k}\mathrm{n}_{j,k} + \sigma_{k,i}\mathrm{n}_{k,i} = 0 \tag{1.13}$$

for the associated pairwise distinct phases $i, j, k \in \{1, \ldots, P\}$, the energy (1.12) is then formally subject to the energy dissipation inequality

$$\frac{\mathrm{d}}{\mathrm{d}t} E[\Omega](t) + \frac{1}{2} \sum_{i,j=1, i\neq j}^{P} \sigma_{i,j} \int_{I_{i,j}(t)} |\mathrm{V}_{I_{i,j}}|^2 \, \mathrm{d}\mathcal{H}^{d-1} \leq 0. \tag{1.14}$$

After these preliminaries, we next turn to local-in-time existence and uniqueness of strong solutions to multiphase mean curvature flow. We start with results concerning the evolution of planar networks (i.e., $d = 2$) with equal surface tensions (i.e., the classical 120° Herring angle condition at triple junctions). In this context, the first result establishing local-in-time existence and uniqueness of strong solutions in Hölder spaces is due to Bronsard and Reitich [25], who restrict themselves to the specific case of three regular curves $\gamma^i\colon [0, 1] \times [0, T] \to D \subset \mathbb{R}^2$, $(x, t) \mapsto \gamma^i(x, t)$, $i \in \{1, 2, 3\}$, joining at a single triple junction, and where $D \subset \mathbb{R}^2$ is an open and convex domain with smooth boundary. In their work, the requirement (1.11) is expressed in terms of an evolution equation for the curves

$$\partial_t \gamma^i = \frac{\partial_{xx}\gamma^i}{|\partial_x \gamma^i|^2}, \quad i \in \{1, 2, 3\}, \tag{1.15}$$

which corresponds to a special choice for the tangential velocity. The resulting system of equations is then amended with compatibility conditions up to second order. This in particular includes the 120° Herring angle condition, which in turn necessitates a non-trivial tangential velocity at the triple junction in order to allow for the motion of the junction. Indeed, if the velocity vector at the triple junction for each curve would only consist of the associated normal component, the velocity vector would have to vanish as a consequence.

Given an initial triod which is parametrized by three curves such that the required compatibility conditions up to second order hold, Bronsard and Reitich [25] (cf. also Mantegazza, Novaga and Tortorelli [111]) then show local-in-time existence and uniqueness of strong solutions to (1.15). A global-in-time existence result for strong solutions in a perturbative regime of initial conditions close to minimal (Steiner) configurations is due to Kinderlehrer and Liu [88]. Finally, the result of Bronsard and Reitich [25] can be extended to a local-in-time existence and uniqueness result for the full network case in the plane, see the work of Mantegazza, Novaga, Pluda and Schulze [110].

Results in the framework of strong solutions in Hölder spaces are also available for double bubbles moving by mean curvature in ambient dimension $d = 3$. In the special case that the three surfaces are represented by graphs over a fixed domain, this follows from the works of Freire ([71] and [70]). For general double bubble clusters, local-in-time existence and uniqueness is due to Depner, Garcke and Kohsaka [52] (cf. in this context also the works of Schulze and White [138] as well as Gößwein, Menzel and Pluda [75]).

There is also considerable interest in short-time existence results for the planar network flow when considering *non-regular* networks as initial data. The main motivation stems from configurations which arise from topology changes in the evolving partition (e.g., the collision of two triple junctions) at which the above mentioned existence results for strong solutions necessarily stop to hold. With this in mind, a short-time existence result for the curvature flow of non-regular initial networks can thus be interpreted as a *restarting* result for the curvature flow of a network past a singularity, which explains its importance. The first work accomplishing this task is the paper by Ilmanen, Neves and Schulze [83]. An alternative approach is provided in the very recent work of Lira, Mazzeo, Pluda and Saez [106].

Let us now turn to the existence theory for weak solutions to multiphase mean curvature flow (1.11). The seminal work in this direction is the one of Brakke [23], who provides a global-in-time existence result for general initial data in a geometric measure theory context. His notion of solutions roughly speaking consists of a localized version of the energy dissipation inequality (1.14), the so-called Brakke inequality, which is formulated in terms of an evolving integral varifold with locally bounded first variation. Instead of diving into the technical GMT details of the solution concept in the sense of Brakke, we refer the reader to the inequality (1.19) below for a BV formulation of Brakke's inequality.

Since solutions to multiphase mean curvature flow in the sense of Brakke are defined by means of a localized energy dissipation principle alone, a sudden and arbitrary loss of surface measure at any stage of the time evolution is admissible with the definition. Therefore, the recently obtained existence result of Kim and Tonegawa [87] constitutes a significant improvement since they succeeded in proving that a variant of Brakke's original scheme converges towards a non-trivial Brakke flow. Non-triviality of the evolution more precisely follows from the fact that the authors can bound the total variation measure of the evolving varifolds from below by the boundary measure of a partition of $\mathbb{R}^d$ formed by a finite family of open sets (as introduced above). Moreover, the Lebesgue measure of this partition is shown to depend continuously on the time parameter, thus preventing an arbitrary and sudden loss of measure for the total variation of the evolving varifolds.

We finally comment on the BV formulation of energy dissipating solutions (1.14) to multiphase mean curvature flow (1.11). This solution concept is modeled on a time-evolving partition $\Omega = (\Omega_1(t), \ldots, \Omega_P(t))_{t\in[0,T]}$ of $\mathbb{R}^d$, such that for all $t \in [0,T]$ each phase $\Omega_1(t), \ldots, \Omega_P(t)$ is a set of finite perimeter, and all phases except for, say, the Pth phase have finite mass. Denoting for each $i \in \{1,\ldots,P\}$ by $\chi_i \in L^\infty(0,T;BV_{\mathrm{loc}}(\mathbb{R}^d;\{0,1\}))$ the time-dependent indicator function associated with the ith phase, one then first requires the existence of normal velocity vector fields $\mathrm{V}_i \in L^2(0,T;L^2(\mathbb{R}^d,\mathrm{d}|\nabla\chi_i|;\mathbb{R}^d))$ such that it holds in a distributional sense

$$\partial_t\chi_i + (\mathrm{V}_i\cdot\nabla)\chi_i = 0 \quad \text{in } \mathbb{R}^d\times[0,T] \text{ for all } i\in\{1,\ldots,P\}. \tag{1.16}$$

A weak formulation of the evolution law (1.11) is given by imposing

$$\begin{aligned} &\sum_{i,j=1,i\neq j}^{P} \sigma_{i,j}\int_0^{T'}\int_{I_{i,j}(t)} \mathrm{V}_i\cdot\varphi\,\mathrm{d}\mathcal{H}^{d-1}\,\mathrm{d}t \\ &= -\sum_{i,j=1,i\neq j}^{P} \sigma_{i,j}\int_0^{T'}\int_{I_{i,j}(t)} (\mathrm{Id}-\mathrm{n}_{i,j}\otimes\mathrm{n}_{i,j}):\nabla\varphi\,\mathrm{d}\mathcal{H}^{d-1}\,\mathrm{d}t \end{aligned} \tag{1.17}$$

for almost every $T' \in [0,T]$ and all $\varphi \in C^\infty_{\mathrm{cpt}}(\mathbb{R}^d\times[0,T];\mathbb{R}^d)$. The energy dissipation inequality (1.14) is finally imposed by defining the interface normal velocities $\mathrm{V}_{I_{i,j}}$ through restriction of V_i to $I_{i,j}$.

Conditional global-in-time existence results of such BV solutions to multiphase mean curvature flow are established by Laux and Otto [98] as well as Laux and Simon [101] (see also

Laux and Lelmi [96] for the case of arbitrary mobilities). These works study the convergence of numerical schemes for multiphase mean curvature flow based on an additional energy convergence assumption in the spirit of the seminal work of Luckhaus and Sturzenhecker [109] on an implicit time-discretization for two-phase mean curvature flow. More precisely, the work of Laux and Otto [98] is concerned with the convergence of the thresholding scheme of Merriman, Bence and Osher ([113] and [114]), and is based on the minimizing movements reformulation of this scheme due to Esedoğlu and Otto [58]. Laux and Simon [101] in contrast establish the convergence of solutions of the vector-valued Allen–Cahn equation.

Laux and Otto [99] in addition show the convergence of the thresholding scheme (again under an additional energy convergence assumption) towards a BV formulation of multiphase mean curvature flow which is in the spirit of Brakke's varifold solution concept with a localized energy dissipation principle at its heart. More precisely, next to working with a family of time-dependent indicator functions $\chi_i \in L^\infty(0,T;BV_{\mathrm{loc}}(\mathbb{R}^d;\{0,1\}))$ as in the previous formulation, their BV formulation of Brakke flow requires the existence of a single vector field $\mathrm{H}\colon \mathbb{R}^d\times[0,T] \to \mathbb{R}^d$, which on one side has the interpretation of a mean curvature vector in form of the weak formulation

$$\begin{aligned}
&\sum_{i,j=1,i\neq j}^{P} \sigma_{i,j} \int_0^T \int_{I_{i,j}(t)} \mathrm{H}\cdot\varphi \,\mathrm{d}\mathcal{H}^{d-1}\,\mathrm{d}t \\
&= -\sum_{i,j=1,i\neq j}^{P} \sigma_{i,j} \int_0^T \int_{I_{i,j}(t)} (\mathrm{Id}{-}\mathrm{n}_{i,j}\otimes \mathrm{n}_{i,j}) : \nabla\varphi \,\mathrm{d}\mathcal{H}^{d-1}\,\mathrm{d}t
\end{aligned} \tag{1.18}$$

for all $\varphi \in C^\infty_{\mathrm{cpt}}(\mathbb{R}^d\times[0,T];\mathbb{R}^d)$, and on the other side gives rise to a localized energy dissipation inequality

$$\begin{aligned}
&\frac{1}{2}\sum_{i,j=1,i\neq j}^{P} \sigma_{i,j} \int_{I_{i,j}(T)} \zeta(\cdot,T)\,\mathrm{d}\mathcal{H}^{d-1} - \frac{1}{2}\sum_{i,j=1,i\neq j}^{P} \sigma_{i,j} \int_{I_{i,j}(0)} \zeta(\cdot,0)\,\mathrm{d}\mathcal{H}^{d-1} \\
&\leq \frac{1}{2}\sum_{i,j=1,i\neq j}^{P} \sigma_{i,j} \int_0^T \int_{I_{i,j}(t)} -\zeta|\mathrm{H}|^2 + (\mathrm{H}\cdot\nabla)\zeta + \partial_t\zeta \,\mathrm{d}\mathcal{H}^{d-1}\,\mathrm{d}t
\end{aligned} \tag{1.19}$$

for all non-negative $\zeta \in C^\infty_{\mathrm{cpt}}(\mathbb{R}^d\times[0,T];[0,\infty))$.

For reviews on the BV formulation of multiphase mean curvature flow (either in the sense of (1.16)–(1.17) or in the sense of (1.18)–(1.19)), we finally refer the reader to Laux [95] and Laux and Otto [100].

1.2.3 The two-phase Stefan problem with isotropic Gibbs–Thomson law

Consider the problem

$$\partial_t(u+\chi) = \Delta u \quad \text{in } \mathbb{R}^d\times[0,T], \tag{1.20}$$

where $\chi = \chi(\cdot,t)$ again denotes the characteristic function of an evolving open set $\Omega(t)$ in $\mathbb{R}^d$, $t\in[0,T]$. This PDE is referred to as the (two-phase) *Stefan problem*, which is a model for liquid-solid interface evolution accounting for freezing and melting processes. Interpreting the variable $u = u(\cdot,t)$ as the deviation from the melting temperature of the material, a simple modeling assumption consists of imposing $u = 0$ along the liquid-solid interface $I(t)$ for all $t\in[0,T]$. This boundary condition leads to the *classical (two-phase) Stefan problem.*

However, this model does not account, e.g., for the phenomenon of supercooling (resp. superheating), meaning that the fluid (resp. the solid) withstands temperatures below its freezing point (resp. above its melting point). To accommodate for such effects, one incorporates surface tension in form of

$$u = \sigma H_I \quad \text{along the liquid-solid interface } I(t), \quad t \in [0,T], \tag{1.21}$$

which is referred to as the isotropic Gibbs–Thomson law. In this context, $H_I = H_I(\cdot,t)$ denotes the mean curvature of the interface $I(t)$ oriented with respect to the normal pointing inside $\Omega(t)$. The energy functional for the evolution problem (1.20)–(1.21) is then given by

$$E[\chi,u](t) := \int \frac{1}{2}|u(\cdot,t)|^2 \,\mathrm{d}x + \sigma \int_{I(t)} 1 \,\mathrm{d}\mathcal{H}^{d-1}, \quad t \in [0,T], \tag{1.22}$$

and formally subject to the energy dissipation inequality

$$\frac{\mathrm{d}}{\mathrm{d}t} E[\chi,u](t) + \int |\nabla u(\cdot,t)|^2 \,\mathrm{d}x \leq 0. \tag{1.23}$$

In terms of strong solutions, one expresses the distributional formulation (1.20)–(1.21) in form of the free boundary problem

$$\partial_t u - \Delta u = 0 \qquad \text{in } \mathbb{R}^d \setminus I(t), \tag{1.24}$$

$$u = \sigma H_I \qquad \text{along } I(t), \tag{1.25}$$

$$\mathrm{V}_I = -\big[(\mathrm{n}\cdot\nabla)u\big]_{I(t)}\mathrm{n} \qquad \text{along } I(t) \tag{1.26}$$

for all $t \in [0,T]$, where $\mathrm{V}_I = \mathrm{V}_I(\cdot,t)$ again denotes the normal velocity vector of the interface $I(t)$, $\mathrm{n} = \mathrm{n}(\cdot,t)$ the unit normal along $I(t)$ pointing inside $\Omega(t)$, and $[\cdot]_{I(t)}$ the jump across the interface $I(t)$ in the direction of the normal n.

Local-in-time existence of strong solutions to (1.24)–(1.26) was first provided by Radkevich [129] in a Hölder space setting. However, uniqueness of solutions within the considered function spaces was left open. Escher, Prüss and Simonett [55] took care of this issue establishing a local-in-time existence and uniqueness result, proving in particular for positive times real analyticity of the interface as well as smoothness of u away from the interface. Strictly speaking, the results of Escher, Prüss and Simonett [55] are restricted to a model geometry at the initial time (i.e., that the initial interface is given by a graph over the flat model interface $\mathbb{R}^{d-1}\times\{0\}$) and a smallness assumption on the initial data. The extension to general geometries for the initial interface is announced in Escher, Prüss and Simonett [55]; however, the author of this thesis was not able to locate this paper in the literature. Both works of Radkevich [129] and Escher, Prüss and Simonett [55] are based on the approach by the Hanzawa transform.

Turning to global-in-time existence of weak solutions to (1.20)–(1.21), one way to proceed is to consider again a BV formulation. Consistent with the energy dissipation inequality (1.23), one tries to construct $u \in L^\infty(0,T;L^2(\mathbb{R}^d))$ with $\nabla u \in L^2(0,T;L^2(\mathbb{R}^d;\mathbb{R}^d))$ as well as $\chi \in L^\infty(0,T;BV(\mathbb{R}^d;\{0,1\}))$ satisfying

$$\int (u{+}\chi)(\cdot,T')\phi(\cdot,T') \,\mathrm{d}x - \int (u{+}\chi)(\cdot,0)\phi(\cdot,0) \,\mathrm{d}x = \int_0^{T'} \int (u{+}\chi)\partial_t\phi - \nabla u \cdot \nabla\phi \,\mathrm{d}x \,\mathrm{d}t \tag{1.27}$$

for almost every $T' \in [0,T]$ and all $\phi \in C^\infty_{\mathrm{cpt}}(\mathbb{R}^d\times[0,T])$, as well as

$$-\sigma \int_0^{T'} \int_{I(t)} (\mathrm{Id}{-}\mathrm{n}\otimes\mathrm{n}) : \nabla\varphi \,\mathrm{d}\mathcal{H}^{d-1} \,\mathrm{d}t = -\int_0^{T'} \int \chi\nabla\cdot(u\varphi) \,\mathrm{d}x \,\mathrm{d}t \tag{1.28}$$

for almost every $T' \in [0,T]$ and all $\varphi \in C^\infty_{\mathrm{cpt}}(\mathbb{R}^d \times [0,T]; \mathbb{R}^d)$. Recalling (1.5), the identity (1.28) indeed represents a weak formulation of the isotropic Gibbs–Thomson law (1.21).

In the context of this BV formulation, Luckhaus ([107] and [108]) provides a global-in-time existence result for general initial data by considering a suitable implicit time-discretization. His scheme (cf. the discussion of Röger [131, Section 4]) avoids a loss of interfacial surface area in the limit of vanishing time-step size, thus enabling to take the limit in the approximations of (1.28) without the need to pass to a varifold solution concept. This in turn is achieved by selecting in each approximation step a global minimizer to an associated time-discrete functional. The resulting limit consequently enjoys additional regularity properties which, as noted by Röger [131, Section 1], may render the resulting limit as too restrictive.

Motivated by this observation, Röger [131] proposes a varifold solution concept which generalizes the BV formulation (1.28) of the isotropic Gibbs–Thomson law (cf. [131, Definition 1.1 and Proposition 3.1] for the interpretation of the mean curvature vector, and [131, Proposition 3.3] for the consistency with the BV formulation). He then establishes a global-in-time existence result for general initial data in this varifold context based on a suitable modification of the scheme of Luckhaus [108]. The limit passage in the varifold formulation of the isotropic Gibbs–Thomson law is facilitated by a geometric measure theory result of Schätzle [135] concerning hypersurfaces whose mean curvature is represented through an ambient Sobolev function (which precisely resembles the case of (1.21)).

We conclude this section by mentioning that an important variant of the Stefan problem with isotropic Gibbs–Thomson law (1.20)–(1.21) consists of its quasi-static version

$$\partial_t \chi = \Delta u \quad \text{in } \mathbb{R}^d \times [0,T]. \tag{1.29}$$

Amended with the isotropic Gibbs–Thomson law, the resulting evolution problem is typically referred to as the Mullins–Sekerka flow. Energy for the Mullins–Sekerka flow is given by interfacial surface area alone

$$E[\chi,u](t) := \sigma \int_{I(t)} 1 \,\mathrm{d}\mathcal{H}^{d-1}, \quad t \in [0,T], \tag{1.30}$$

which is formally dissipated in form of

$$\frac{\mathrm{d}}{\mathrm{d}t} E[\chi,u](t) + \int |\nabla u(\cdot,t)|^2 \,\mathrm{d}x \leq 0. \tag{1.31}$$

The Mullins–Sekerka equation (1.29) with isotropic Gibbs–Thomson law (1.21) may in fact be derived as the H^{-1}-gradient flow of the interface energy functional (1.30), cf. the review article of Garcke [72, Section 2.5].

For local-in-time existence and uniqueness of strong solutions for the Mullins–Sekerka equation based on the Hanzawa transformation approach, we refer the reader to the works of Bazalii [17], Chen, Hong and Yi [35], and Escher and Simonett ([56] and [57]). In the context of strong solutions, one may even allow for contact point dynamics with a fixed-in-time contact angle of 90° as the work of Abels, Rauchecker and Wilke [8] shows.

A BV formulation of the Mullins–Sekerka equation (1.29) with isotropic Gibbs–Thomson law (1.21) is obtained by requiring next to (1.28) the following obvious modification of (1.27)

$$\int \chi(\cdot,T')\phi(\cdot,T') \,\mathrm{d}x - \int \chi(\cdot,0)\phi(\cdot,0) \,\mathrm{d}x = \int_0^{T'} \int \chi \partial_t \phi - \nabla u \cdot \nabla \phi \,\mathrm{d}x \,\mathrm{d}t \tag{1.32}$$

for almost every $T' \in [0,T]$ and all $\phi \in C^\infty_{\mathrm{cpt}}(\mathbb{R}^d \times [0,T])$. As Luckhaus and Sturzenhecker remark in their seminal work [109], their implicit time-discretization scheme can be used to provide a global-in-time existence result for general initial data in this setting. Convergence towards (1.28) is facilitated by an additional energy convergence assumption. Without the latter, Röger [132] establishes in the spirit of his work on the two-phase Stefan problem [131] a global-in-time existence result for general initial data in a varifold solution context.

1.3 Uniqueness of weak solution concepts: What is known?

The main aim of this section is to summarize what has been known so far concerning uniqueness properties of weak solution concepts with respect to the three models considered in Section 1.2. To the best of the author's knowledge, this in fact restricts to the well-posedness of the level set approach to (two-phase) mean curvature flow. Uniqueness of weak solutions in this context heavily relies on the availability of a comparison principle as well as partly on techniques which are specific to the problem of mean curvature flow. However,

- neither two-phase Navier–Stokes flow with surface tension (1.1a)–(1.1c),
- nor multiphase (i.e., $P \geq 3$ phases) mean curvature flow (1.11),
- nor the two-phase Stefan problem with isotropic Gibbs–Thomson law (1.20)–(1.21) resp. the two-phase Mullins–Sekerka equation (1.29) with isotropic Gibbs–Thomson law (1.21)

satisfy a comparison principle, which is one of the main reasons why for these curvature driven interface evolution models no uniqueness result for any associated weak solution concept has been established so far (to the best of the author's knowledge).

We conclude this section presenting an interesting result due to Jerrard and Smets [85], who derived a weak-strong uniqueness principle for binormal curvature flow of curves in $\mathbb{R}^3$. Even though they consider a geometric evolution equation for a codimension two object (hence, in this sense not an interface evolution problem), we decided to include their result in this section since their approach is based on a quantitative "weak-strong stability estimate" which turns out to be the analogue for curves in $\mathbb{R}^3$ of our approach to the question of weak-strong uniqueness for curvature driven interface evolution as presented in Chapter 2.

1.3.1 Well-posedness of the level set approach to mean curvature flow

The formulation of the level set approach to evolution by (two-phase) mean curvature flow is basically the following. Consider an initial compact interface $I(0)$, and assume that there exists a continuous $g\colon \mathbb{R}^d \to \mathbb{R}$ such that $I(0)$ can be represented as the zero level set of g, i.e., it holds

$$I(0) = \big\{x \in \mathbb{R}^d \colon g(x) = 0\big\}. \tag{1.33}$$

We then consider for a function $u\colon \mathbb{R}^d \times [0,\infty) \to \mathbb{R}$ the PDE

$$\partial_t u = \Big(\mathrm{Id} - \frac{\nabla u}{|\nabla u|} \otimes \frac{\nabla u}{|\nabla u|}\Big) : \nabla^2 u \qquad \text{in } \mathbb{R}^d \times (0,\infty), \tag{1.34}$$

$$u(\cdot,0) = g \qquad \text{in } \mathbb{R}^d, \tag{1.35}$$

and define for all $t \in [0,\infty)$

$$I(t) := \big\{x \in \mathbb{R}^d \colon u(x,t) = 0\big\}. \tag{1.36}$$

Neglecting for the moment technical issues due to the degeneracy of the parabolic equation (1.34) and even its lack of a meaning whenever $\nabla u = 0$, let us first motivate why (1.34) indeed encodes that the interfaces (1.36) given by the zero level sets of u evolve by their mean curvature. To this end, a straightforward computation reveals that for smooth solutions u of (1.34) satisfying $\nabla u \neq 0$, we may rewrite the PDE (1.34) in form of

$$\partial_t u = \Big(\mathrm{Id} - \frac{\nabla u}{|\nabla u|} \otimes \frac{\nabla u}{|\nabla u|}\Big) : \nabla^2 u = |\nabla u| \nabla \cdot \Big(\frac{\nabla u}{|\nabla u|}\Big). \tag{1.37}$$

Since the vector field $\frac{\nabla u(\cdot,t)}{|\nabla u(\cdot,t)|}$ restricted to $I(t)$ represents a unit normal along $I(t)$, and since $\frac{\partial_t u(\cdot,t)}{|\nabla u(\cdot,t)|}$ represents the normal speed of $I(t)$ with respect to this unit normal, we indeed obtain that (1.34) in form of (1.37) encodes evolution by mean curvature of the level sets (1.36) for smooth u with $\nabla u \neq 0$.

The level set approach (1.34)–(1.36) to mean curvature flow was first considered by Osher and Sethian [122], who devised numerical algorithms based on this formulation to compute the motion of interfaces propagating with normal speeds depending on their mean curvature. A rigorous and well-posed weak solution concept for the level set approach (typically referred to in the literature as *viscosity solutions*) was then developed independently in the seminal works of Chen, Giga and Goto [36] and Evans and Spruck [60]. The problem of the degeneracy of the PDE (1.34) is dealt with by treating the problem from the viewpoint of the theory of viscosity solutions to second order nonlinear PDEs (cf. the user's guide to this theory by Crandall, Ishii and Lions [42]), which, however, also needs to be extended in order to provide a weak meaning of (1.34) whenever $\nabla u = 0$.

The main result of the works by Chen, Giga and Goto [36] and Evans and Spruck [60] then consists of the construction of a *global-in-time unique* viscosity solution to the level set approach (1.34), so that moreover (1.36) is indeed well-defined by showing that these sets are independent of the choice of g for the initial condition (1.33). These results are established as consequences of two cornerstone principles underlying the theory of viscosity solutions to second order nonlinear PDEs: the availability of a comparison principle in terms of sub- and supersolutions as well as stability of the viscosity formulation with respect to limit passages. Apart from well-posedness of viscosity solutions, one can also show consistency with smooth solutions in form of a weak-strong uniqueness principle, cf. for instance Evans and Spruck [60, Theorem 6.1]. The latter relies on specific properties of the signed distance function for sets whose boundary evolves smoothly by mean curvature flow.

Despite of providing a well-posed theory of viscosity solutions for the level set approach to mean curvature flow (1.34), there is a well-known shortcoming of the solution concept of Chen, Giga and Goto [36] and Evans and Spruck [60], respectively. The problem consists of the fact that the level set (1.36) in general may develop a non-trivial interior and thus fails to describe an actual interface in form of a hypersurface. Interestingly, this phenomenon, which is typically referred to as the *fattening* of level sets in the literature, can be explained in terms of non-uniqueness of evolutions for an *intrinsic* viscosity formulation of mean curvature flow due to Soner [146]. More precisely, Soner [146] first develops a viscosity formulation which is purely based on the signed distance function associated with the underlying evolving sets, then provides a global-in-time existence result for a maximal viscosity solution in this intrinsic sense [146, Theorem 10.4], next shows that his weak formulation in general allows for non-unique evolution [146, Section 8] by providing specific examples, and finally establishes that this non-uniqueness is directly related to the fattening of level sets [146, Corollary 11.2] of viscosity solutions in the sense of Chen, Giga and Goto [36] and Evans and Spruck [60].

Yet another view on viscosity solutions of Chen, Giga and Goto [36] and Evans and Spruck [60] is provided by Ilmanen's notion of *set-theoretic subsolutions* to evolution by mean curvature flow (cf. [81] or [82, Paragraph 10]). A family of closed subsets $(I(t))_{t\in[0,\infty)}$ of $\mathbb{R}^d$ is called a set-theoretic subsolution to mean curvature flow if an *avoidance principle* with respect to smooth solutions holds true: for all $t' \in [0,\infty)$, all $s \in (0,\infty)$ and all smoothly evolving compact interfaces $(S(t))_{t\in[t',t'+s]}$ moving by mean curvature flow it is required

$$I(t') \cap S(t') = \emptyset \quad \Longrightarrow \quad I(t) \cap S(t) = \emptyset \quad \text{for all } t \in [t', t'+s]. \tag{1.38}$$

Based on a comparison principle for set-theoretic subsolutions [82, Lemma 10.2], Ilmanen then proceeds by showing that any viscosity solution in the sense of Chen, Giga and Goto [36] and Evans and Spruck [60] is in fact a maximal set-theoretic subsolution, and vice versa (cf. [82, Lemma 10.4]).

Ilmanen also provides in his work [82] an *inclusion principle* stating that the support of any codimension one Brakke flow, whose support at the initial time is contained in the zero level set of a viscosity solution, remains contained in the zero level set of this viscosity solution at all later times (cf. [82, Theorem 10.7]). The proof of this fact is based on an avoidance principle for codimension one Brakke flows and that any codimension one Brakke flow in fact is a set-theoretic subsolution to mean curvature flow (cf. [82, Lemma 10.5 and Lemma 10.6]).

We conclude our discussion of the level set approach to mean curvature flow by mentioning that the theory of Chen, Giga and Goto [36] and Evans and Spruck [60] can be generalized to the setting of mean curvature flow of surfaces with arbitrary codimension, which is due to Ambrosio and Soner [13]. Their work in particular includes the higher codimension analogue of Ilmanen's inclusion principle with respect to Brakke flows, as well as a consistency check in form of a weak-strong uniqueness principle (the latter based on specific properties of the squared distance function to the smoothly evolving surface).

1.3.2 Binormal curvature flow of curves in $\mathbb{R}^3$

Consider a finite time horizon $T \in (0,\infty)$ and a family of embedded arc-length parametrized closed curves $(\gamma(\cdot,t)\colon \mathbb{R}/L\mathbb{Z} \to \mathbb{R}^3,\, s \mapsto \gamma(s,t))_{t\in[0,T]}$ in $\mathbb{R}^3$ for some $L > 0$. For a smoothly evolving family of such curves, the binormal curvature flow is represented by

$$\partial_t\gamma = \partial_s\gamma \times \partial_{ss}\gamma, \tag{1.39}$$

with $\times$ denoting the cross product of vectors in $\mathbb{R}^3$. Existence of solutions to (1.39) in the parametrized setting can only be guaranteed for short times due to the possibility of the formation of self-intersections and/or collisions in finite time (cf. the discussion of Jerrard and Smets [85]). To account for evolutions past such singularities, a suitable weak formulation of (1.39) has to be considered.

One way to proceed is based on the following identity satisfied by smooth solutions to evolution by binormal curvature flow (1.39)

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L \varphi\big(\gamma(s,t)\big)\cdot\partial_s\gamma(s,t)\,\mathrm{d}s = -\int_0^L \big(\nabla(\nabla\times\varphi)\big)(x)\big|_{x=\gamma(s,t)} : \partial_s\gamma(s,t)\otimes\partial_s\gamma(s,t)\,\mathrm{d}s \tag{1.40}$$

for all $\varphi \in C^\infty_{\mathrm{cpt}}(\mathbb{R}^3;\mathbb{R}^3)$, cf. Jerrard and Smets [85, Lemma 1]. The importance of (1.40) stems from the observation that it can be generalized to a varifold setting. To this end, one considers oriented varifolds V (i.e., a finite Radon measure on the product space $(x,\tau) \in \mathbb{R}^3\times\mathbb{S}^2$) satisfying the compatibility condition

$$\int_{\mathbb{R}^3\times\mathbb{S}^2} \varphi(x)\cdot\tau\,\mathrm{d}V(x,\tau) = \sum_{j=1}^{\infty}\int_0^L \varphi\big(\gamma_j(s)\big)\cdot\partial_s\gamma_j(s)\,\mathrm{d}s \tag{1.41}$$

for a family $(\gamma_j\colon \mathbb{R}/L\mathbb{Z} \to \mathbb{R}^3)_{j\geq1}$ of injective Lipschitz closed curves in $\mathbb{R}^3$ such that $\sum_{j=1}^\infty \mathrm{length}(\gamma_j) < \infty$. In the language of Federer and Fleming, the previous identity asserts that the first moment of the oriented varifold with respect to the "tangential variable" $\tau \in \mathbb{S}^2$ represents a so-called finite mass integral one current without boundary in $\mathbb{R}^3$.

Jerrard and Smets (cf. [85, Definition 2]) then introduce a weak solution concept for binormal curvature flow of curves in $\mathbb{R}^3$ in terms of a time-dependent family of oriented varifolds $(V(\cdot,\cdot,t))_{t\in[0,\infty)}$ subject to the compatibility condition (1.41), for which it is required that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^3\times\mathbb{S}^2} \varphi(x)\cdot\tau\,\mathrm{d}V(x,\tau,t) = -\int_{\mathbb{R}^3\times\mathbb{S}^2} \big(\nabla(\nabla\times\varphi)\big)(x) : \tau\otimes\tau\,\mathrm{d}V(x,\tau,t) \tag{1.42}$$

holds true for all $\varphi \in C^\infty_{\mathrm{cpt}}(\mathbb{R}^3;\mathbb{R}^3)$ in analogy to (1.40), as well as that for all $t \in [0,\infty)$ the mass $|V|(\mathbb{R}^3\times\mathbb{S}^2,t)$ is bounded from above by the mass of the underlying finite mass integral one current without boundary in $\mathbb{R}^3$ at the initial time $t = 0$. Jerrard and Smets then proceed by establishing a global-in-time existence result for general initial data [85, Theorem 1] for this weak solution concept of binormal curvature flow of curves in $\mathbb{R}^3$.

Apart from a global-in-time existence result, Jerrard and Smets also prove a weak-strong uniqueness principle [85, Theorem 2] showing that as long as a smooth solution to binormal curvature flow of curves in $\mathbb{R}^3$ in the sense of (1.39) exists, any weak solution in the above varifold sense (1.42) starting from the same initial smooth curve has to coincide with this smooth solution (in the sense of the identity (1.45) below). The interesting point in connection with the topics of this thesis is that their qualitative uniqueness result is derived as a consequence of a quantitative "weak-strong stability estimate" [85, Theorem 3]. This stability estimate is moreover formulated in terms of a "distance measure" between a smooth and a varifold solution to binormal curvature flow of curves in $\mathbb{R}^3$, which in fact is an analogue for curves of our approach to weak-strong uniqueness for curvature driven interface evolution (cf. Section 2.1 for two-phase evolution problems as well as the second part of Section 2.3 for its generalization to varifold solution concepts).

More precisely, Jerrard and Smets introduce for a given smooth solution $\Gamma = (\gamma(\cdot,t))_{t\in[0,T]}$ and a given varifold solution $V = (V(\cdot,\cdot,t))_{t\in[0,\infty)}$ with associated initial finite mass integral one current without boundary T_0 the functionals

$$E_1[V|\Gamma](t) := |T_0| - \int_{\mathbb{R}^3\times\mathbb{S}^2} \xi(x,t)\cdot\tau \,\mathrm{d}V(x,\tau,t) \tag{1.43}$$

$$\geq \int_{\mathbb{R}^3\times\mathbb{S}^2} 1 - \xi(x,t)\cdot\tau \,\mathrm{d}V(x,\tau,t) =: E_2[V|\Gamma](t), \tag{1.44}$$

where $\xi\colon \mathbb{R}^3\times[0,T] \to \mathbb{R}^3$ is some vector field associated with the strong solution Γ, and where the lower bound (1.44) is a consequence of the requirement for a varifold solution that $|V|(\mathbb{R}^3\times\mathbb{S}^2,t) \leq |T_0|$ for all $t \in [0,\infty)$. For $E_1[V|\Gamma]$ resp. $E_2[V|\Gamma]$ to be a suitable error functional, it is required that for each $t \in [0,T]$ the vector field $\xi(\cdot,t)$ at least represents an extension of the tangent vector field $\partial_s\gamma(\cdot,t)$ subject to the length constraints $|\xi(\cdot,t)| \leq 1$ in $\mathbb{R}^3$ and $|\xi(x,t)| = 1$ if and only if $x = \gamma(s,t)$ for some $s \in \mathbb{R}/L\mathbb{Z}$. Indeed, the second condition ensures that $E_2[V|\Gamma] \geq 0$, whereas the combination of the first and the last condition guarantees that $E_1[V|\Gamma] = E_2[V|\Gamma] = 0$ implies that the varifold solution V is given by the natural varifold lift of the smooth solution Γ, i.e.,

$$\int_{\mathbb{R}^3\times\mathbb{S}^2} \psi(x,\tau)\,\mathrm{d}V(x,\tau,t) = \int_0^L \psi\big(\gamma(s,t),\partial_s\gamma(s,t)\big)\,\mathrm{d}s \tag{1.45}$$

for almost every $t \in [0,T]$ and all $\psi \in C^\infty_{\mathrm{cpt}}(\mathbb{R}^3\times\mathbb{S}^2)$, cf. [85, Proof of Theorem 2].

With these definitions in place, the already mentioned "weak-strong stability estimate" of Jerrard and Smets for binormal curvature flow of curves in $\mathbb{R}^3$ then takes the form (for some suitably constructed vector field ξ)

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}E_1[V|\Gamma](t)\right| \lesssim E_2[V|\Gamma](t) \leq E_1[V|\Gamma](t), \quad t \in [0,T], \tag{1.46}$$

so that an application of Gronwall's inequality and the properties of the functionals $E_1[V|\Gamma]$ resp. $E_2[V|\Gamma]$ imply the asserted qualitative weak-strong uniqueness.

1.4 Informal statement of main results

In general, global-in-time uniqueness can not be expected in the class of weak solutions for curvature driven interface evolution problems due to singularities. An example in the context of multiphase mean curvature flow is already included in the work of Brakke [23, Section C.4 and Figure 5], cf. also our work [65, Figure 3] or the work of Lira, Mazzeo, Pluda and Saez [106, Figure 3], where the same example is discussed. The example more precisely concerns the singular planar configuration at which four phases meet at a quadruple junction with equal angles of 90°, for which two possible continuations of the evolution exist (by splitting the quadruple junction into two separate triple junctions at which the correct angle condition holds). On the other side, uniqueness may not be guaranteed even prior to the first topology change and the resulting singular configuration. The prime example for this consists of Brakke's notion of varifold solutions to multiphase mean curvature flow as discussed in the second part of Section 1.2, for which a sudden and arbitrary loss of surface measure at any stage of the time evolution is admissible with the definition of his solution concept. In particular, one can enforce non-uniqueness by hand at any time by simply replacing the evolving varifold with the empty varifold.

In summary, the best one can hope for in the context of weak solution concepts for curvature driven interface evolution problems is a conditional uniqueness result. A way to formalize this consists of so-called weak-strong uniqueness principles: *prior to the onset of geometric singularities due to topology changes, weak solutions are unique in the class of strong solutions.* In other words, in the presence of a weak-strong uniqueness principle non-uniqueness of the weak solution concept under consideration may only arise at the first singular time of the unique strong solution. For general initial data, the first singular time of a strong solution is of course expected to be finite. For example, one could think of grain boundaries in an annealing metal which may collapse, or a liquid drop which may pinch-off into two separate drops.

As already remarked at the beginning of the preceding Section 1.3, for interface evolution problems not admitting a geometric comparison principle, as it is for instance the case in multiphase geometric evolution equations or two-phase fluid flow, the derivation of a weak-strong uniqueness principle or a weak-strong stability estimate represented to the best of the author's knowledge an open problem. The works presented in this thesis are precisely concerned with such problems, and thus, again to the best of the author's knowledge, are the first to provide a positive result in this direction. We summarize in the following theorem the main results of this thesis.

Theorem (Weak-strong uniqueness and stability of evolutions for two-phase Navier–Stokes flow with surface tension and multiphase mean curvature flow; joint works with Julian Fischer, Tim Laux, and Theresa M. Simon)**.** *Energy dissipating weak solutions to*

- *two-phase Navier–Stokes flow with surface tension* (1.1a)–(1.1c) *in the sense of Abels' [1] varifold solutions* (u, χ, V)*,*
- *planar multiphase mean curvature flow* (1.11) *in the sense of BV solutions* $(\chi_i, \mathrm{V}_i)_{i\in\{1,\ldots,P\}}$ *of Laux and Otto [98] resp. Laux and Simon [101],*
- *multiphase mean curvature flow* (1.11) *of double bubbles in* $\mathbb{R}^3$ *in the sense of BV solutions* $(\chi_i, \mathrm{V}_i)_{i\in\{1,2,3\}}$ *of Laux and Otto [98] resp. Laux and Simon [101],*

satisfy a weak-strong uniqueness principle: *for each of these three curvature driven interface evolution problems, as long as a strong solution exists, any energy dissipating weak solution in the above sense starting from the same initial data has to coincide with the unique strong solution.*

Moreover, these qualitative uniqueness results are derived as a consequence of an associated weak-strong stability estimate*, which is formulated in terms of a novel distance measure between a strong and a weak solution. This distance measure is capable of controlling the* *interface error between a strong and a weak solution in a sufficiently strong sense, and its interfacial contribution has the structure of a relative entropy with respect to the energy functional given by interfacial surface area.*

Precise versions of this result with references to the specific formulations of the underlying solution concepts are given in Theorem 3.1, Theorem 4.1 and Theorem 5.1, respectively. Let us mention in this context that a similar result holds true for energy dissipating weak solutions of the Mullins–Sekerka equation (1.29) with isotropic Gibbs–Thomson law (1.21) in the sense of the BV formulation (1.32) and (1.28), for which we refer to our forthcoming work [67].

The remainder of this thesis is structured as follows. We conclude this introduction with the upcoming Section 1.5 by briefly discussing the concept of relative entropies in the classical setting of a strictly convex and dissipated energy functional. Chapter 2 then provides a rather extensive account on our novel notion of relative entropies for a class of interface evolution problems, and thus serves as a unified framework behind our strategy for the derivation of the main results of this thesis. Chapter 3, Chapter 4 and Chapter 5 finally contain the proofs of our main results in the order they are mentioned in the previous theorem (cf. in this regard the List of Collaborators and Publications section in the preamble to this thesis).

1.5 The relative entropy method: Classical setting

Weak-strong uniqueness principles happen to be true in a lot of classical applications from mathematical continuum mechanics, at least if the problem under consideration satisfies an energy dissipation principle. (However, there are counterexamples to this rule-of-thumb as the work of Colombo, De Lellis, and De Rosa [41] shows.) For instance, in the case of the incompressible Navier–Stokes equations, weak-strong uniqueness for energy dissipating weak solutions was established by Leray [105] and Serrin [142]. Many more examples for the validity of a weak-strong uniqueness principle are known in the context of mathematical fluid mechanics. For a survey, we refer to the review article of Wiedemann [149].

One common feature to most of these results is that they rely on the *relative entropy method*, which in turn originates in the works of Dafermos [43] and DiPerna [53] on conservation laws. A relative entropy is a nonlinear functional measuring the "distance" between a weak (denoted for concreteness by u) and a fixed strong solution (say v). In the case where the problem under consideration is equipped with a dissipated strictly convex energy (or entropy) functional $E[\,\cdot\,]$, one may obtain such a distance measure by subtracting the first order approximation to $E[\,\cdot\,]$ around the "base point" v

$$E[u|v] := E[u] - DE[v](u-v) - E[v]. \tag{1.47}$$

Convexity of the energy $E[\,\cdot\,]$ implies non-negativity of the relative entropy $E[u|v] \geq 0$, whereas strict convexity of $E[\,\cdot\,]$ ensures on top that $E[u|v] = 0$ if and only if $u = v$. Finally, in order to control the time evolution of the quantity $E[u|v]$ one in principle relies on only two ingredients: *i)* the dissipation of energy in form of $\frac{\mathrm{d}}{\mathrm{d}t}E[u] \leq -\mathcal{D}[u]$ for a non-negative dissipation functional $\mathcal{D}[u] \geq 0$, and *ii)* the possibility of using (in general nonlinear) functionals of the more regular strong solution v as a test function in the weak formulation of the weak solution u. In classical applications of the relative entropy method (e.g., conservation laws or fluid mechanics), one then leverages on the properties of the functional $E[u|v]$ to deduce an estimate of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}E[u|v] \leq C(v)E[u|v]. \tag{1.48}$$

An application of Gronwall's lemma in turn allows to infer from (1.48) weak-strong uniqueness and stability in form of the estimate

$$E[u|v](t) \leq e^{C(v)t}E[u|v](0). \tag{1.49}$$

With the general structure (1.47)–(1.49) of the relative entropy approach in place, we provide for illustration purposes three specific examples describing how the general structure materializes in some classical problem settings:

- (Conservation laws) We consider a scalar conservation law in spatial dimension $d = 1$ with smooth and strictly convex flux function $F\colon \mathbb{R} \to \mathbb{R}$ given by

  $$\partial_t u + \partial_x\big(F(u)\big) = 0 \quad \text{in } \mathbb{R}\times(0,\infty). \tag{1.50}$$

  More precisely, we are interested in so-called entropy solutions (cf. for what follows the book of Evans [59, Subsection 11.4.3]) whose main defining condition consists of requiring in a distributional sense

  $$\partial_t\big(\eta(u)\big) \leq -\partial_x\big(q(u)\big) \quad \text{in } \mathbb{R}\times(0,\infty) \tag{1.51}$$

  for all entropy/entropy-flux pairs (η, q), meaning that the map η is smooth, strictly convex, and it holds

  $$q' = F'\eta', \tag{1.52}$$

  where f' denotes the derivative of a differentiable $f\colon \mathbb{R} \to \mathbb{R}$. For an entropy solution u and a given entropy/entropy-flux pair (η, q), we define the associated entropy functional $E_\eta[u] := \int \eta(u)\,\mathrm{d}x$.

  Following Dafermos [43] and DiPerna [53], we then introduce for each entropy/entropy-flux pair (η, q) a relative entropy

  $$\eta(u|v) := \eta(u) - \eta'(v)(u-v) - \eta(v), \quad u, v \in \mathbb{R},$$

  as well as a relative entropy-flux

  $$q(u|v) := q(u) - \eta'(v)\big(F(u)-F(v)\big) - q(v), \quad u, v \in \mathbb{R}.$$

  It is important to observe that, for each fixed $v \in \mathbb{R}$, the pair $(\eta(\cdot|v), q(\cdot|v))$ represents again an admissible entropy/entropy-flux pair of (1.50), which in view of the entropy condition (1.51) is then a key ingredient for the computation of the time evolution of the error functional

  $$E_\eta[u|v] := \int \eta(u|v)\,\mathrm{d}x. \tag{1.53}$$

  Note also that for bounded entropic solutions u and bounded strong solutions v, the relative entropy $E_\eta[u|v]$ is comparable to the L^2 distance between u and v thanks to the entropy η being smooth and strictly convex.

  As usual in the context of the relative entropy approach, some form of improved regularity is required at the level of the strong solution v. For instance, a stability estimate for $E_\eta[u|v]$ holds true once v is at least Lipschitz continuous (cf. Serre and Vasseur [140, Section 3]). However, once one allows for discontinuities in the solution v (e.g., due to shock waves) an estimate of the form (1.48) for $E_\eta[u|v]$ in general fails. We refer the reader to Serre and Vasseur [140, Section 3.1] for a concrete counterexample.

During the last decade, a refined version of the relative entropy method—the theory of weak-strong stability up to a (time-dependent) shift—has been developed in order to incorporate discontinuities in the strong solution v. The basic idea is to shift the discontinuity of v by a (time-dependent) velocity adapted to the entropic solution u. For results based on this idea in the context of conservation laws, we refer to the works of Leger [103], Leger and Vasseur [104], Serre and Vasseur ([139] and [141]), Kang and Vasseur [86], Krupa and Vasseur ([92] and [93]), and finally Krupa [91].

- (Incompressible viscous fluid flow) We consider the Navier–Stokes equations ($\Omega \subset \mathbb{R}^d$)

$$\partial_t u + (u\cdot\nabla)u = \mu\Delta u - \nabla p + f \qquad \text{in } \Omega\times[0,T], \tag{1.54}$$

$$\nabla\cdot u = 0 \qquad \text{in } \Omega\times[0,T]. \tag{1.55}$$

Weak solutions in the sense of Leray [105] and Hopf [80] are required to satisfy an energy inequality of the form (strictly speaking, the integral version of it)

$$\frac{\mathrm{d}}{\mathrm{d}t}E[u] \le -\int_\Omega \mu|\nabla u|^2\,\mathrm{d}x + \int_\Omega f\cdot u\,\mathrm{d}x \tag{1.56}$$

with respect to kinetic energy

$$E[u] := \int_\Omega \frac{1}{2}|u|^2\,\mathrm{d}x. \tag{1.57}$$

In this setting, the relative entropy ansatz (1.47) then simply boils down to

$$E[u|v] := \int_\Omega \frac{1}{2}|u-v|^2\,\mathrm{d}x = E[u] - \int_\Omega v\cdot(u-v)\,\mathrm{d}x - E[v]. \tag{1.58}$$

As already mentioned, weak-strong uniqueness based on an estimate for the time evolution of the L^2 distance is due to Leray [105] in the full space setting and due to Serrin [142] for domains.

For the incompressible Euler equations, the situation is known to be drastically different. Scheffer [136] was the first to construct nontrivial weak solutions which are compactly supported in time, see also the work of Shnirelman [145]. This phenomenon for the Euler equations was later studied by De Lellis and Székelyhidi [44] as an instance of Gromov's h-principle (see also [45]). Their insights and techniques paved the way for a series of works establishing further striking non-uniqueness results in mathematical fluid mechanics. E.g., the resolution of Onsager's conjecture by Isett [84] and Buckmaster, De Lellis, Székelyhidi, and Vicol [27], the non-uniqueness of distributional solutions with bounded kinetic energy of the 3D incompressible Navier–Stokes equations due to Buckmaster and Vicol [26], or the non-uniqueness of entropy solutions for the isentropic compressible Euler equations due to Chiodaroli [37] and Chiodaroli, De Lellis, and Kreml [38], to mention just a few of them.

- (Compressible viscous fluid flow) As a third and last example, we consider the following compressible Navier–Stokes system ($\Omega \subset \mathbb{R}^d$)

$$\partial_t\rho + \nabla\cdot(\rho u) = 0 \qquad \text{in } \Omega\times[0,T], \tag{1.59}$$

$$\partial_t(\rho u) + \nabla\cdot(\rho u\otimes u) = \nabla\cdot\mathbb{S}(\nabla u) - \nabla p(\rho) + \rho f \qquad \text{in } \Omega\times[0,T], \tag{1.60}$$

with the viscous stress tensor defined by $\mathbb{S}(\nabla u) := \mu\big(\nabla u + (\nabla u)^\mathsf{T} - \frac{2}{3}(\nabla\cdot u)\mathrm{Id}\big) + \eta(\nabla\cdot u)\mathrm{Id}$, and the pressure $p = p(\rho) \in C[0,\infty)\cap C^2(0,\infty)$ being subject to the conditions ($\gamma > \frac{3}{2}$)

$$p(0) = 0, \quad p' > 0 \text{ in } (0,\infty), \quad \frac{p'(\rho)}{\rho^{\gamma-1}} \to a > 0 \text{ as } \rho\to\infty,$$

$$H(\rho) := \rho\int_0^\rho \frac{p(z)}{z^2}\,\mathrm{d}z < \infty \text{ for all } \rho > 0.$$

In this context, Feireisl, Jin, and Novotný [62] (see also the work of Germain [74]) provide a weak-strong uniqueness principle for weak solutions of (1.59)–(1.60) with finite energy

$$E[\rho,u] := \int_\Omega \frac{1}{2}\rho|u|^2 + H(\rho)\,\mathrm{d}x.$$

Defining the auxiliary quantity $H(\rho|r) := H(\rho) - H'(r)(\rho{-}r) - H(r)$, the associated relative entropy for a given strong solution (r,v) is given by

$$E[\rho,u|r,v] := \int_\Omega \frac{1}{2}\rho|u{-}v|^2 + H(\rho|r)\,\mathrm{d}x.$$

We conclude by mentioning that their analysis can be extended to the case of the full Navier–Stokes–Fourier system, see the work of Feireisl and Novotný [63].

The main contribution of the works contained in this thesis consists of a suitable adaptation of the classical relative entropy ansatz (1.47) to a certain class of curvature driven interface evolution equations. The next chapter is devoted to a detailed exposition of our relative entropy approach for such problems, adopting in the process a viewpoint which is as general as possible.

CHAPTER 2

The relative entropy approach for a class of interface evolution problems

To the best of the author's knowledge, no analogue of the relative entropy method has been used or developed for interface evolution problems in general. In our recent works [64] and [65], we introduced a notion of a relative entropy for two-phase and multiphase evolution problems, respectively, for which the total energy functional is dissipated and contains an interfacial energy contribution being proportional to the surface area of the evolving interface. (In the multiphase case, the constant of proportionality is allowed to vary for interfaces corresponding to different pairs of phases.) In these works, our novel concept of a relative entropy for interface evolution problems serves the crucial purpose of overcoming the lack of a geometric comparison principle.

It is the aim of this chapter to give a rather precise and extensive account on the main ideas and principles underlying our approach to the uniqueness problem for such curvature driven interface evolution problems. In particular, we want to emphasize those parts of our arguments which do not specifically rely on the precise formulation of the free boundary problem at hand (i.e., an equation for the normal velocity vector). The ideas presented in this chapter therefore constitute a unified framework connecting all the results on uniqueness properties of weak solution concepts mentioned in Section 1.4 above.

For the purposes of most of this chapter, we put ourselves in the most simple situation and consider only energy functionals given by the surface area of the evolving interface (up to proportionality constants, which in the multiphase case may vary for interfaces corresponding to different pairs of phases). In particular, we will neglect further contributions to the energy functional (e.g., kinetic energy in two-phase fluid flow with sharp interface). We will also exclude for the sake of the discussion the possibility that the evolving interface may intersect the boundary of a given domain, and instead consider a full-space setting (under a finite mass assumption for all except of one of the phases). The reader may keep in mind the case of evolution by mean curvature as a prime example.

In light of these simplifying restrictions, we conclude this chapter by a discussion of the robustness of our relative entropy approach to curvature driven interface evolution problems. More precisely, we remark how the ideas and principles of the following two sections extend and/or apply to other settings, including two-phase fluid flow driven by surface tension (in particular, the extension to varifold solution concepts), interface evolution in bounded domains with the possibility of boundary contact of the interface with a fixed-in-time contact angle, and finally the derivation of convergence rates for diffuse interface models to a sharp interface limit. Parts of this discussion include an outlook on possible future projects.

2.1 The relative entropy method: The case of two phases

For the purposes of this section, we fix the following setup. Let $T \in (0, \infty)$ be a finite time horizon, let $\Omega = (\Omega(t))_{t\in[0,T]}$ be a family of sets of finite perimeter in $\mathbb{R}^d$ (modeling the phase of a weak solution) with finite mass, and let the reduced boundary $I(t) := \partial^*\Omega(t)$ of $\Omega(t)$ be the associated interface for all $t \in [0, T]$. We are interested in interface evolution problems with an interfacial energy contribution proportional to the surface area of the interface

$$E[\Omega](t) := \sigma \int_{I(t)} 1 \,\mathrm{d}\mathcal{H}^{d-1}, \quad t \in [0, T], \tag{2.1}$$

with the proportionality factor given by surface tension $\sigma > 0$. Moreover, consider a "smoothly evolving" family of open and bounded sets $\bar\Omega = (\bar\Omega(t))_{t\in[0,T]}$ (modeling the phase of a strong solution) with "smoothly evolving" interfaces $\bar I(t) := \partial\bar\Omega(t)$ for all $t \in [0, T]$. (One way to encode a smooth evolution would be to assume that $\bar\Omega(0) \subset \mathbb{R}^d$ is an open and bounded set with finitely many connected components and smooth boundary $\partial\bar\Omega(0)$, and that there exists a smooth space-time diffeomorphism $\Psi\colon \mathbb{R}^d\times[0, T] \to \mathbb{R}^d\times[0, T]$ such that $\bar\Omega(t)\times\{t\} = \Psi(\bar\Omega(0), t)$ for all $t \in [0, T]$.)

We aim to introduce a quantity $E[\Omega|\bar\Omega]$ which is based on the energy functional (2.1), mimics the structural properties of a classical relative entropy as in (1.47), and gives sufficient control on the interface error between the two evolving interfaces $(I(t))_{t\in[0,T]}$ and $(\bar I(t))_{t\in[0,T]}$. To this end, we make use of duality to rewrite the energy (2.1) in form of

$$E[\Omega](t) = \sup_{\xi\in C^1_{\mathrm{cpt}}(\mathbb{R}^d;\mathbb{R}^d),\, \|\xi\|_{L^\infty}\leq 1} \int_{\Omega(t)} \nabla\cdot\sigma\xi \,\mathrm{d}x.$$

This in turn motivates the following ansatz

$$E[\Omega|\bar\Omega](t) := \sigma \int_{I(t)} 1 - \mathrm{n}(\cdot, t)\cdot\xi(\cdot, t) \,\mathrm{d}\mathcal{H}^{d-1}, \quad t \in [0, T], \tag{2.2}$$

where $\mathrm{n}(\cdot, t)$ denotes the (measure-theoretic) unit normal along the reduced boundary $I(t)$ pointing inside the phase $\Omega(t)$, and ξ is a smooth space-time vector field such that along the smooth interface $\bar I(t)$ it coincides with the inward pointing unit normal vector field

$$\xi(\cdot, t) = \bar{\mathrm{n}}(\cdot, t) \quad \text{on } \bar I(t). \tag{2.3}$$

Away from the interface, the length of ξ is moreover required to decrease quadratically in the distance to the interface

$$|\xi(\cdot, t)| \leq 1 - c\min\{\mathrm{dist}^2(\cdot, \bar I(t)), 1\} \quad \text{in } \mathbb{R}^d \tag{2.4}$$

for some $c \in (0, 1]$. One should recall at this point that a similar construction to (2.2) was already employed by Jerrard and Smets [85] for a codimension two problem, namely the derivation of a weak-strong uniqueness principle for the evolution of curves in $\mathbb{R}^3$ by binormal curvature flow.

The requirements on the vector field ξ are coercivity conditions in the sense that they ensure non-negativity $E[\Omega|\bar{\Omega}] \geq 0$, and that the validity of $E[\Omega|\bar{\Omega}](t) = 0$ for $t \in [0, T]$ implies $I(t) \subset \bar{I}(t)$. Moreover, property (2.4) immediately entails that

$$\int_{I(t)} \min\{\operatorname{dist}^2(\cdot, \bar{I}(t)), 1\} \,\mathrm{d}\mathcal{H}^{d-1} \leq c^{-1}\sigma^{-1} E[\Omega|\bar{\Omega}](t). \tag{2.5}$$

Note that $E[\Omega|\bar{\Omega}](t)$ also yields tilt-excess type control of the error in the interface normals since trivially

$$\int_{I(t)} |\mathrm{n}(\cdot, t) - \xi(\cdot, t)|^2 \,\mathrm{d}\mathcal{H}^{d-1} \leq 2\sigma^{-1} E[\Omega|\bar{\Omega}](t). \tag{2.6}$$

Finally, the ansatz (2.2) indeed resembles the structural form (1.47) of a classical relative entropy as one may compute by an integration by parts and the properties of the vector field ξ

$$\begin{aligned} E[\Omega|\bar{\Omega}](t) &= E[\Omega](t) + \int_{\Omega(t)} (\nabla \cdot \sigma\xi)(\cdot, t) \,\mathrm{d}x \\ &= E[\Omega](t) - \int (\chi - \bar{\chi})(\cdot, t)\big(\nabla \cdot (-\sigma\xi)\big)(\cdot, t) \,\mathrm{d}x - E[\bar{\Omega}](t), \end{aligned} \tag{2.7}$$

where $\chi(\cdot, t)$ and $\bar{\chi}(\cdot, t)$ denote the characteristic functions of the phases $\Omega(t)$ and $\bar{\Omega}(t)$, respectively.

In particular, in order to control the time evolution of the interface error functional $E[\Omega|\bar{\Omega}]$, one in principle only relies on inserting $\nabla \cdot \sigma\xi$ as a test function into the evolution equation for the time-evolving phase $\Omega(t)$ of the weak solution, and an energy dissipation principle to control the contribution of the term $\frac{\mathrm{d}}{\mathrm{d}t}E[\Omega]$. With respect to the former, we of course need in addition an appropriate control on the time evolution of the vector field ξ. To this end, it turns out to be beneficial (in the two-phase case mostly for clarity of exposition and efficient organization of terms in the time evolution of the relative entropy) to introduce a second vector field B, which shall represent a smooth space-time vector field whose normal component along $\bar{I}$ equals the normal velocity $\mathrm{V}_{\bar{I}}$ of the smoothly evolving interface $\bar{I}$:

$$((B \cdot \bar{\mathrm{n}})\, \bar{\mathrm{n}})(\cdot, t) = \mathrm{V}_{\bar{I}}(\cdot, t).$$

Since the vector field ξ extends the unit normal of the interface $\bar{I}$, which itself gets *transported* and *rotated* as a consequence of the motion of the interface, one may guess that the differential operator $\partial_t\xi + (B \cdot \nabla)\xi + (\nabla B)^{\mathsf{T}}\xi$ captures the evolution of the vector field ξ (up to admissible error terms in the distance to the interface $\bar{I}$). We refer to the next subsection for an explanation of how to capitalize on the structure of this differential operator.

Of course, the arguments needed to eventually arrive at an estimate of the form (1.48) are at some point specific to the geometric evolution equation under consideration. However, a large part of the involved computations are in principle generic, and in order to underline this fact, we will present in the next subsection the problem independent part of the computation of the time evolution of the error functional $E[\Omega|\bar{\Omega}]$. In this context, it is an interesting observation that these computations will naturally lead to (the BV formulation of) the mean curvature functional associated with the interface $I(t)$ tested against $B(\cdot, t)$, i.e., formally

$$-\int_{I(t)} (\mathrm{Id} - \mathrm{n} \otimes \mathrm{n})(\cdot, t) : \nabla B(\cdot, t) \,\mathrm{d}\mathcal{H}^{d-1} = \int_{I(t)} \mathrm{H}_I(\cdot, t) \cdot B(\cdot, t) \,\mathrm{d}\mathcal{H}^{d-1} \tag{2.8}$$

with $\mathrm{H}_I(\cdot, t)$ denoting the mean curvature vector of the interface $I(t)$, and where B is the above mentioned velocity vector field. This may serve as one explanation why the relative entropy approach based on the error functional (2.2) is particularly well suited to *curvature driven* interface evolution problems. In terms of the examples from Section 1.2, the left hand side of (2.8) interpreted as the functional

$$C^1_{\mathrm{cpt}}(\mathbb{R}^d;\mathbb{R}^d) \ni \varphi \mapsto -\int_{I(t)} (\mathrm{Id}-\mathrm{n}\otimes\mathrm{n})(\cdot,t) : \nabla\varphi \,\mathrm{d}\mathcal{H}^{d-1} \tag{2.9}$$

is naturally an integral constituent of the corresponding weak (i.e., BV) formulation:

- In the case of evolution by mean curvature, the functional (2.9) is directly linked to the normal velocity of the evolving interface I. For this example, the normal component along $\bar{I}$ of the velocity vector field B is given by the mean curvature vector of the smoothly evolving interface $\bar{I}$.
- In the case of the Mullins–Sekerka equation, the functional (2.9) appears in the weak formulation of the Gibbs–Thomson law, and the normal component along $\bar{I}$ of B may be chosen as the jump across $\bar{I}$ of the Neumann data for the temperature field of the smoothly evolving solution.
- In the case of two-phase Navier–Stokes flow with sharp interface, the functional (2.9) is part of the weak formulation of the Young–Laplace law, and thus represents a coupling term between the evolution equation for the domain occupied by one of the fluids and the evolution equation for the fluid velocity. In this setting, one fixes the normal component of B along $\bar{I}$ as the normal component of the fluid velocity of the smoothly evolving solution.

2.1.1 The time evolution of the two-phase relative entropy

In the above setting, denote by $\chi(\cdot,t)$ the indicator function of the phase $\Omega(t)$ for all $t\in[0,T]$. We further write $\mathrm{V}_I(\cdot,t)$ for the normal velocity vector field of the associated evolving interface $I(t)$, i.e., it holds in a distributional sense

$$\partial_t\chi + (\mathrm{V}_I\cdot\nabla)\chi = 0. \tag{2.10}$$

Based on the representation (2.7) of the relative entropy functional (2.2), one may then compute (we omit for notational convenience the dependence on the time variable)

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}E[\Omega|\bar\Omega] &= \frac{\mathrm{d}}{\mathrm{d}t}E[\Omega] + \frac{\mathrm{d}}{\mathrm{d}t}\int \chi(\nabla\cdot\sigma\xi)\,\mathrm{d}x \\
&= \frac{\mathrm{d}}{\mathrm{d}t}E[\Omega] - \sigma\int_I (\nabla\cdot\xi)(\mathrm{V}_I\cdot\mathrm{n})\,\mathrm{d}\mathcal{H}^{d-1} - \sigma\int_I \mathrm{n}\cdot\partial_t\xi\,\mathrm{d}\mathcal{H}^{d-1}.
\end{aligned}$$

We next add zero to the last right hand side term of the previous display in order to generate the proposed PDE for the time evolution of the vector field ξ, which yields after adding another zero in a second step

$$\begin{aligned}
-\sigma\int_I \mathrm{n}\cdot\partial_t\xi\,\mathrm{d}\mathcal{H}^{d-1} &= -\sigma\int_I \mathrm{n}\cdot\big(\partial_t\xi+(B\cdot\nabla)\xi+(\nabla B)^\mathsf{T}\xi\big)\,\mathrm{d}\mathcal{H}^{d-1} \\
&\quad + \sigma\int_I \mathrm{n}\cdot(B\cdot\nabla)\xi\,\mathrm{d}\mathcal{H}^{d-1} + \sigma\int_I \xi\cdot(\mathrm{n}\cdot\nabla)B\,\mathrm{d}\mathcal{H}^{d-1} \\
&= -\sigma\int_I (\mathrm{n}-\xi)\cdot\big(\partial_t\xi+(B\cdot\nabla)\xi+(\nabla B)^\mathsf{T}\xi\big)\,\mathrm{d}\mathcal{H}^{d-1} \\
&\quad - \sigma\int_I \xi\cdot\big(\partial_t\xi+(B\cdot\nabla)\xi\big)\,\mathrm{d}\mathcal{H}^{d-1} \\
&\quad + \sigma\int_I \xi\cdot\big((\mathrm{n}-\xi)\cdot\nabla\big)B\,\mathrm{d}\mathcal{H}^{d-1} + \sigma\int_I \mathrm{n}\cdot(B\cdot\nabla)\xi\,\mathrm{d}\mathcal{H}^{d-1}.
\end{aligned}$$

By an application of the product rule and by adding zero, one may rewrite the last right hand side term of the previous display in a form which generates the BV formulation of the mean curvature functional (2.9)

$$\begin{aligned}\sigma\int_I \mathrm{n}\cdot(B\cdot\nabla)\xi\,\mathrm{d}\mathcal{H}^{d-1} &= \sigma\int_I \mathrm{n}\cdot\big(\nabla\cdot(\xi\otimes B)\big)\,\mathrm{d}\mathcal{H}^{d-1} - \sigma\int_I(\mathrm{n}\cdot\xi-1)(\nabla\cdot B)\,\mathrm{d}\mathcal{H}^{d-1}\\ &\quad -\sigma\int_I(\mathrm{Id}-\mathrm{n}\otimes\mathrm{n}):\nabla B\,\mathrm{d}\mathcal{H}^{d-1} - \sigma\int_I \mathrm{n}\cdot(\mathrm{n}\cdot\nabla)B\,\mathrm{d}\mathcal{H}^{d-1}.\end{aligned}$$

Moreover, due to an integration by parts in the first right hand side term of the previous display, the symmetry relation $\nabla\cdot(\nabla\cdot(\xi\otimes B)) = \nabla\cdot(\nabla\cdot(B\otimes\xi))$, as well as the product rule, we may also compute

$$\begin{aligned}\sigma\int_I \mathrm{n}\cdot\big(\nabla\cdot(\xi\otimes B)\big)\,\mathrm{d}\mathcal{H}^{d-1} &= -\sigma\int \chi\nabla\cdot(\nabla\cdot(\xi\otimes B))\,\mathrm{d}x\\ &= -\sigma\int \chi\nabla\cdot(\nabla\cdot(B\otimes\xi))\,\mathrm{d}x\\ &= \sigma\int_I \mathrm{n}\cdot\big(\nabla\cdot(B\otimes\xi)\big)\,\mathrm{d}\mathcal{H}^{d-1}\\ &= \sigma\int_I(\nabla\cdot\xi)(\mathrm{n}\cdot B)\,\mathrm{d}\mathcal{H}^{d-1} + \sigma\int_I \mathrm{n}\cdot(\xi\cdot\nabla)B\,\mathrm{d}\mathcal{H}^{d-1}.\end{aligned}$$

The combination of the previous four displays thus yields the following preliminary identity for the time evolution of the relative entropy functional

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}E[\Omega|\bar\Omega] &= \frac{\mathrm{d}}{\mathrm{d}t}E[\Omega] - \sigma\int_I(\nabla\cdot\xi)\big((\mathrm{V}_I-B)\cdot\mathrm{n}\big)\,\mathrm{d}\mathcal{H}^{d-1}\\ &\quad -\sigma\int_I(\mathrm{Id}-\mathrm{n}\otimes\mathrm{n}):\nabla B\,\mathrm{d}\mathcal{H}^{d-1}\\ &\quad -\sigma\int_I(\mathrm{n}-\xi)\cdot\big(\partial_t\xi+(B\cdot\nabla)\xi+(\nabla B)^\mathsf{T}\xi\big)\,\mathrm{d}\mathcal{H}^{d-1}\\ &\quad -\sigma\int_I \xi\cdot\big(\partial_t\xi+(B\cdot\nabla)\xi\big)\,\mathrm{d}\mathcal{H}^{d-1}\\ &\quad -\sigma\int_I(\mathrm{n}\cdot\xi-1)(\nabla\cdot B)\,\mathrm{d}\mathcal{H}^{d-1}\\ &\quad -\sigma\int_I(\mathrm{n}-\xi)\cdot\big((\mathrm{n}-\xi)\cdot\nabla\big)B\,\mathrm{d}\mathcal{H}^{d-1}.\end{aligned} \tag{2.11}$$

The first three right hand side terms of (2.11) are precisely those requiring a further processing on an individual basis for a given specific two-phase free boundary problem. This possibly involves further restrictions on the pair of vector fields (ξ, B) in addition to the already stated properties. For instance, the reader may consult the first part of Section 4.2 for the argument in the context of evolution by mean curvature. There, the additional condition $(B\cdot\xi+\nabla\cdot\xi)(\cdot,t) = O(\min\{\mathrm{dist}(\cdot,\bar I(t)),1\})$ shows up, which is natural recalling that ξ represents an extension of the unit normal of $\bar I$.

The last four right hand side terms of (2.11), however, can already be dealt with in the general setting of this subsection as follows. Under the assumption of a uniform bound on the gradient of the velocity vector field B it is a trivial consequence of the definition (2.2) and the coercivity property (2.6) that the last two right hand side terms of (2.11) are controlled by the two-phase relative entropy $E[\Omega|\bar\Omega]$. Appealing in addition to the coercivity property (2.5), and provided that the following error bounds hold true for the time evolution of the vector field ξ as well as its length

$$\big(\partial_t\xi+(B\cdot\nabla)\xi+(\nabla B)^{\mathsf{T}}\xi\big)(\cdot,t) = O\big(\min\big\{\mathrm{dist}(\cdot,\bar{I}(t)),1\big\}\big) \qquad \text{in } \mathbb{R}^d, \tag{2.12}$$

$$\big(\partial_t|\xi|^2+(B\cdot\nabla)|\xi|^2\big)(\cdot,t) = O\big(\min\big\{\mathrm{dist}^2(\cdot,\bar{I}(t)),1\big\}\big) \qquad \text{in } \mathbb{R}^d, \tag{2.13}$$

also the fourth and fifth right hand side term of (2.11) are controlled by the two-phase relative entropy $E[\Omega|\bar{\Omega}]$.

It thus remains to argue how to establish the estimates (2.12) and (2.13), which in turn requires to provide an explicit construction of the pair of vector fields (ξ, B) given a smoothly evolving phase $(\bar{\Omega}(t))_{t\in[0,T]}$ with smoothly evolving interface $(\bar{I}(t))_{t\in[0,T]}$. In the two-phase setting, this is rather straightforward and essentially a consequence of the assumed regularity of the interface. More precisely, we may appeal first to the tubular neighborhood theorem in order to fix a scale $\bar{r}\in(0,1)$ such that within the space-time tubular neighborhood $\bigcup_{t\in[0,T]}\{x\in\mathbb{R}^d\colon \mathrm{dist}(x,\bar{I}(t))<\bar{r}\}\times\{t\}$ the associated signed distance function $s_{\bar{I}}(\cdot,t)$ to $\bar{I}(t)$ (with its orientation fixed by requiring $\nabla s_{\bar{I}}(\cdot,t)=\bar{\mathrm{n}}(\cdot,t)$ along $\bar{I}(t)$) and the projection $P_{\bar{I}}(\cdot,t)$ onto the nearest point on $\bar{I}(t)$ are smooth space-time functions. Denoting further by η a smooth quadratic cut-off satisfying $\min\{r^2,1\}\leq 1-\eta(r)\leq C\min\{r^2,1\}$ for all $r\in\mathbb{R}$ and some constant $C>1$ as well as $\operatorname{supp}\eta\subset[-1,1]$, and by $\mathrm{V}_{\bar{I}}(\cdot,t)$ the normal velocity vector field of the smoothly evolving interface $\bar{I}(t)$ for all $t\in[0,T]$, we then simply define for all $(x,t)\in\mathbb{R}^d\times[0,T]$

$$\xi(x,t) := \eta\big(\bar{r}^{-1}s_{\bar{I}}(x,t)\big)\nabla s_{\bar{I}}(x,t), \tag{2.14}$$

$$B(x,t) := \eta\big(\bar{r}^{-1}s_{\bar{I}}(x,t)\big)\mathrm{V}_{\bar{I}}\big(P_{\bar{I}}(x,t),t\big). \tag{2.15}$$

Note that (2.3) and (2.4) are immediate consequences of the definition (2.14). The estimate (2.12) follows from the properties of the cut-off η, the definitions (2.14) resp. (2.15), the chain rule, and differentiating with respect to the spatial variable the identity

$$\partial_t s_{\bar{I}}(x,t) + \big(\mathrm{V}_{\bar{I}}\big(P_{\bar{I}}(x,t),t\big)\cdot\nabla\big)s_{\bar{I}}(x,t) = 0, \quad (x,t)\colon \mathrm{dist}(x,\bar{I}(t))<\bar{r}, \tag{2.16}$$

which in turn is a well-known property for smooth evolutions. The estimate (2.13) finally follows from $|\xi(x,t)|^2=\eta^2\big(\bar{r}^{-1}s_{\bar{I}}(x,t)\big)$, the properties of η, the definition (2.15), the chain rule, and the evolution equation for the signed distance from the previous display. Observe carefully that this argument even works when including a smooth "tangential component" in the definition (2.15) of the velocity B, which may prove helpful in applications.

2.1.2 Control in the limit of vanishing interface measure: The bulk error

Once one succeeded in providing a stability estimate in form of (1.48) with respect to the relative entropy (2.2), one may deduce from it a "weak-strong inclusion principle" for the underlying interface evolution problem:

$$\begin{aligned} &I(0)\subset\bar{I}(0) \text{ up to } \mathcal{H}^{d-1} \text{ null sets} \\ &\qquad\Longrightarrow\quad I(t)\subset\bar{I}(t) \text{ up to } \mathcal{H}^{d-1} \text{ null sets for a.e. } t\in[0,T]. \end{aligned} \tag{2.17}$$

In words, the property of the interface of the weak solution being contained in the interface of the strong solution is stable with respect to the flow.

However, it is clear that any argument which is solely based on the error functional (2.2) can not provide a full weak-strong uniqueness principle in form of

$$\begin{aligned} &\Omega(0)=\bar{\Omega}(0) \text{ up to a Lebesgue null set} \\ &\qquad\Longrightarrow\quad \Omega(t)=\bar{\Omega}(t) \text{ up to a Lebesgue null set for a.e. } t\in[0,T]. \end{aligned} \tag{2.18}$$

For example, it holds $E[\Omega|\bar\Omega](t) = 0$ if $\Omega(t) = \emptyset$ whereas (2.18) is obviously violated for non-trivial evolution of the strong solution.

This observation motivates to introduce a second error functional which directly controls the L^1 error between the two solutions Ω and $\bar\Omega$, and thus takes care of the lack of coercivity of the relative entropy (2.2) in the limit of vanishing interface measure for the weak solution. To this end, denoting again by $\chi(\cdot,t)$ (resp. $\bar\chi(\cdot,t)$) the indicator function of the phase $\Omega(t)$ of the weak solution (resp. the phase $\bar\Omega(t)$ of the strong solution) one defines for all $t \in [0,T]$

$$E_{\mathrm{bulk}}[\Omega|\bar\Omega](t) := \int (\chi-\bar\chi)(\cdot,t)\,\vartheta(\cdot,t)\,\mathrm{d}x, \tag{2.19}$$

where $\vartheta\colon \mathbb{R}^d\times[0,T] \to [-1,1]$ is a smooth weight subject to (at least) the following requirements:

$$\min\{\mathrm{dist}(\cdot,\bar I(t)),1\} \le |\vartheta(\cdot,t)| \le C\min\{\mathrm{dist}(\cdot,\bar I(t)),1\}, \tag{2.20}$$

$$\vartheta(\cdot,t) < 0 \text{ in } \bar\Omega(t), \quad \vartheta(\cdot,t) > 0 \text{ in } \mathbb{R}^d\setminus\overline{\bar\Omega(t)} \tag{2.21}$$

for some $C \ge 1$ and all $t \in [0,T]$, where $\overline{\bar\Omega(t)}$ denotes the closure of $\bar\Omega(t)$. Note that the sign conditions of (2.21) are precisely what is needed to ensure non-negativity of the error functional (2.19)

$$E_{\mathrm{bulk}}[\Omega|\bar\Omega](t) = \int |(\chi-\bar\chi)(\cdot,t)|\,|\vartheta(\cdot,t)|\,\mathrm{d}x = \int_{\Omega(t)\Delta\bar\Omega(t)} |\vartheta(\cdot,t)|\,\mathrm{d}x \ge 0. \tag{2.22}$$

The second equality of the previous display together with (2.20) moreover show that the error functional (2.19) is a slight modification of the well-known distance functional employed in the famous works of Almgren, Taylor and Wang [10] and Luckhaus and Sturzenhecker [109], respectively. In particular, for any $t \in [0,T]$

$$E_{\mathrm{bulk}}[\Omega|\bar\Omega](t) = 0 \quad\Longrightarrow\quad \Omega(t) = \bar\Omega(t) \text{ up to a Lebesgue null set.} \tag{2.23}$$

In order to deduce a weak-strong uniqueness principle of the form (2.18), the goal therefore is to establish a stability estimate à la Gronwall in terms of $E_{\mathrm{bulk}}[\Omega|\bar\Omega]$. In the spirit of the previous subsection, we briefly discuss the part of the argument which is independent of the specific geometric evolution equation under consideration. Appealing to the evolution equation (2.10) of the phase of the weak solution, and noting that (2.21) implies $\vartheta(\cdot,t) = 0$ along $\bar I(t)$ for all $t \in [0,T]$, it follows (we again omit the dependence on the time variable)

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}E_{\mathrm{bulk}}[\Omega|\bar\Omega] &= -\int_I (\mathrm{V}_I\cdot\mathrm{n})\vartheta\,\mathrm{d}\mathcal{H}^{d-1} + \int (\chi-\bar\chi)\partial_t\vartheta\,\mathrm{d}x \\ &= -\int_I (\mathrm{V}_I\cdot\mathrm{n})\vartheta\,\mathrm{d}\mathcal{H}^{d-1} - \int (\chi-\bar\chi)(B\cdot\nabla)\vartheta\,\mathrm{d}x \\ &\quad + \int (\chi-\bar\chi)\big(\partial_t\vartheta+(B\cdot\nabla)\vartheta\big)\,\mathrm{d}x,\end{aligned}$$

where B denotes the velocity vector field from the computation of the time evolution of the two-phase relative entropy (2.2). By an application of the product rule, an integration by parts, and again using that $\vartheta(\cdot,t) = 0$ along $\bar I(t)$ for all $t \in [0,T]$, we further compute

$$\begin{aligned}-\int (\chi-\bar\chi)(B\cdot\nabla)\vartheta\,\mathrm{d}x &= \int (\chi-\bar\chi)(\nabla\cdot B)\vartheta\,\mathrm{d}x - \int (\chi-\bar\chi)\nabla\cdot(\vartheta B)\,\mathrm{d}x \\ &= \int (\chi-\bar\chi)(\nabla\cdot B)\vartheta\,\mathrm{d}x + \int_I (B\cdot\mathrm{n})\vartheta\,\mathrm{d}\mathcal{H}^{d-1}.\end{aligned}$$

Hence, the previous two displays imply

$$\frac{\mathrm{d}}{\mathrm{d}t}E_{\mathrm{bulk}}[\Omega|\bar{\Omega}] = -\int_I \big((\mathrm{V}_I - B)\cdot \mathrm{n}\big)\vartheta \,\mathrm{d}\mathcal{H}^{d-1} + \int (\chi - \bar{\chi})\big(\partial_t\vartheta + (B\cdot\nabla)\vartheta\big)\,\mathrm{d}x + \int (\chi-\bar{\chi})(\nabla\cdot B)\vartheta\,\mathrm{d}x. \tag{2.24}$$

Under the assumptions that

$$\big(\partial_t\vartheta + (B\cdot\nabla)\vartheta\big)(\cdot,t) = O\big(\min\{\mathrm{dist}(\cdot,\bar{I}(t)),1\}\big) \quad \text{in } \mathbb{R}^d \tag{2.25}$$

and that the gradient of the velocity vector field B is uniformly bounded, the last two right hand side terms of (2.24) are immediately controlled by the error functional (2.19) due to its definition and (2.20). Realizing (2.25) in the two-phase setting is in turn straightforward by means of the following procedure. Fixing a smooth truncation of the identity $\bar{\vartheta}\colon \mathbb{R}\to[-1,1]$ satisfying $\min\{|r|,1\} \leq |\bar{\vartheta}(r)| \leq C\min\{|r|,1\}$ for some $C\geq 1$ and all $r\in[-1,1]$, $|\bar{\vartheta}(r)| = 1$ for all $r\in\mathbb{R}\setminus[-1,1]$, as well as $\bar{\vartheta}(r)<0$ for $r>0$ resp. $\bar{\vartheta}(r)>0$ for $r<0$, we define for all $(x,t)\in\mathbb{R}^d\times[0,T]$

$$\vartheta(x,t) := \bar{\vartheta}\big(\bar{r}^{-1}s_{\bar{I}}(x,t)\big), \tag{2.26}$$

where the tubular neighborhood scale $\bar{r}\in(0,1]$ and the signed distance function $s_{\bar{I}}$ are chosen as in the previous subsection. Note that the conditions (2.20)–(2.21) are obviously satisfied due to the definition (2.26). Validity of the approximate evolution equation (2.25) is in turn a consequence of the properties of the smooth truncation of the identity $\bar{\vartheta}$, the chain rule, the evolution equation (2.16) of the signed distance function, and the choice (2.15) of the velocity vector field B from the previous subsection.

Post-processing the first right hand side term of (2.24) again has to be performed on a case-by-case basis for each specific free boundary problem at hand. This may require additional restrictions on the weight ϑ. It is furthermore expected that one relies on an already closed stability estimate for the interfacial relative entropy (2.2) in order to close the Gronwall argument for the bulk error functional (2.19). (This is indeed the case for all the results of this thesis.)

2.2 The relative entropy method: The case of multiple phases

Let $P\geq 3$, and consider for the case of multiple phases a time-evolving partition $\Omega = (\Omega_1(t),\ldots,\Omega_P(t))_{t\in[0,T]}$ of $\mathbb{R}^d$ such that for all $t\in[0,T]$ each phase $\Omega_1(t),\ldots,\Omega_P(t)$ is a set of finite perimeter, and all phases except for, say, the Pth phase have finite mass. For distinct phases $i,j\in\{1,\ldots,P\}$ and all $t\in[0,T]$, we denote by $I_{i,j}(t) := \partial^*\Omega_i(t)\cap\partial^*\Omega_j(t)$ the interface between the ith and the jth phase. We also write $\mathrm{n}_{i,j}(\cdot,t)$ for the (measure-theoretic) unit normal along $I_{i,j}(t)$ pointing from the ith to the jth phase, i.e., the restriction to $I_{i,j}(t)$ of the (measure-theoretic) unit normal $\mathrm{n}_{\partial^*\Omega_j(t)}$ along the reduced boundary $\partial^*\Omega_j(t)$ pointing inside $\Omega_j(t)$. We are then interested in multiphase interface evolution problems with an interfacial energy contribution given by

$$E[\Omega](t) := \frac{1}{2}\sum_{i,j=1,i\neq j}^{P}\sigma_{i,j}\int_{I_{i,j}(t)} 1\,\mathrm{d}\mathcal{H}^{d-1}, \quad t\in[0,T], \tag{2.27}$$

with proportionality factors given by a (symmetric) matrix of surface tensions $\sigma\in\mathbb{R}^{d\times d}_{>0}$. We also consider a smoothly evolving partition $\bar{\Omega} = (\bar{\Omega}_1(t),\ldots,\bar{\Omega}_P(t))_{t\in[0,T]}$ of $\mathbb{R}^d$, with all the phases except for the Pth phase having finite mass, and the interfaces $\bar{I}_{i,j}(t) := \partial\bar{\Omega}_i(t)\cap\partial\bar{\Omega}_j(t)$ being smooth for all $t \in [0,T]$ and all distinct $i,j \in \{1,\ldots,P\}$.

The multiphase analogue for the interface error functional (2.2) is simply given by the ansatz

$$E[\Omega|\bar{\Omega}](t) := \frac{1}{2}\sum_{i,j=1,i\neq j}^{P} \sigma_{i,j}\int_{I_{i,j}(t)} 1 - \mathrm{n}_{i,j}(\cdot,t)\cdot\xi_{i,j}(\cdot,t)\,\mathrm{d}\mathcal{H}^{d-1}, \quad t\in[0,T]. \tag{2.28}$$

Basic coercivity of this error functional follows from requiring the smooth vector fields $\xi_{i,j}$ to coincide along the interface $\bar{I}_{i,j}(t)$ with the unit normal $\bar{\mathrm{n}}_{i,j}(\cdot,t)$ pointing from $\bar{\Omega}_i(t)$ to $\bar{\Omega}_j(t)$ for all $t\in[0,T]$, and to satisfy away from the interface $\bar{I}_{i,j}(t)$ the length constraint $|\xi_{i,j}(\cdot,t)| \leq 1 - c\min\{\mathrm{dist}^2(\cdot,\bar{I}_{i,j}(t)),1\}$ for all $t\in[0,T]$ and some $c\in(0,1]$. However, without further conditions on the family of vector fields $(\xi_{i,j})_{i\neq j}$ we can not simply proceed by an integration by parts as in (2.7) to rewrite the ansatz (2.28) into a form structurally resembling the one of classical relative entropies (1.47).

The additional necessary ingredient in the multiphase case is given by the following algebraic relation: we assume that there exists a family of smooth vector fields $(\xi_i)_{i\in\{1,\ldots,P\}}$ such that

$$\sigma_{i,j}\xi_{i,j} = \xi_i - \xi_j. \tag{2.29}$$

Provided the structural requirement (2.29) is satisfied, one may then use the skew-symmetry relation $\mathrm{n}_{i,j} = -\mathrm{n}_{j,i}$ to rewrite the multiphase relative entropy (2.28) as follows

$$\begin{aligned} E[\Omega|\bar{\Omega}](t) &= E[\Omega](t) - \frac{1}{2}\sum_{i,j=1,i\neq j}^{P}\int_{I_{i,j}(t)} \mathrm{n}_{i,j}(\cdot,t)\cdot(\xi_i{-}\xi_j)(\cdot,t)\,\mathrm{d}\mathcal{H}^{d-1} \\ &= E[\Omega](t) + \sum_{i=1}^{P}\int_{\partial^*\Omega_i(t)} \mathrm{n}_{\partial^*\Omega_i(t)}\cdot\xi_i(\cdot,t)\,\mathrm{d}\mathcal{H}^{d-1}. \end{aligned}$$

Instead of surface integrals over individual interfaces (with weights depending on the associated pairs of phases), the second term on the right hand side of the previous display now involves surface integrals over the phase boundaries. Hence, performing first an integration by parts and adding zero in a second step (exploiting also the fact that $(\Omega_1(t),\ldots,\Omega_P(t))$ resp. $(\bar{\Omega}_1(t),\ldots,\bar{\Omega}_P(t))$ are partitions of $\mathbb{R}^d$ for all $t\in[0,T]$) yields

$$E[\Omega|\bar{\Omega}](t) = E[\Omega](t) - \sum_{i=1}^{P}\int_{\Omega_i(t)} \nabla\cdot\xi_i(\cdot,t)\,\mathrm{d}x \tag{2.30}$$

$$= E[\Omega](t) - \sum_{i,j=1,i\neq j}^{P}\int_{\Omega_i(t)\cap\bar{\Omega}_j(t)} \nabla\cdot(\xi_i{-}\xi_j)(\cdot,t)\,\mathrm{d}x - E[\bar{\Omega}](t), \tag{2.31}$$

which is the multiphase generalization of (2.7). Again, the merit of having the representation (2.30) is that, on top of requiring an energy dissipation principle, we only rely on testing the evolution equation of the ith phase $\Omega_i(t)$ of the weak solution with the test function $\nabla\cdot\xi_i$ in order to compute the time derivative of the error functional $E[\Omega|\bar{\Omega}]$.

2.2.1 The time evolution of the multiphase relative entropy

We briefly argue how to produce the multiphase analogue of (2.11). To this end, let us denote by $\chi_i(\cdot,t)$ the indicator function of the ith phase $\Omega_i(t)$ for all $t\in[0,T]$. We further write $\mathrm{V}_i(\cdot,t)$ for the normal velocity vector field of the associated evolving phase boundary $\partial^*\Omega_i(\cdot,t)$, i.e., it holds in a distributional sense

$$\partial_t\chi_i + (\mathrm{V}_i\cdot\nabla)\chi_i = 0. \tag{2.32}$$

Starting point for the computation of the time evolution of the multiphase relative entropy (2.28) is the representation (2.30), which together with the evolution equations (2.32) implies (we again omit the dependence on the time variable)

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}E[\Omega|\bar\Omega] &= \frac{\mathrm{d}}{\mathrm{d}t}E[\Omega] - \sum_{i=1}^P \frac{\mathrm{d}}{\mathrm{d}t}\int \chi_i(\nabla\cdot\xi_i)\,\mathrm{d}x \\ &= \frac{\mathrm{d}}{\mathrm{d}t}E[\Omega] + \sum_{i=1}^P \int_{\partial^*\Omega_i}(\nabla\cdot\xi_i)(\mathrm{V}_i\cdot\mathrm{n}_{\partial^*\Omega_i})\,\mathrm{d}\mathcal{H}^{d-1} + \sum_{i=1}^P\int_{\partial^*\Omega_i}\mathrm{n}_{\partial^*\Omega_i}\cdot\partial_t\xi_i\,\mathrm{d}\mathcal{H}^{d-1}.\end{aligned}$$

By the same argument which allowed to proceed from (2.28) to (2.30) based on the condition (2.29) and the skew-symmetry relation $\mathrm{n}_{i,j} = -\mathrm{n}_{j,i}$, it holds

$$\sum_{i=1}^P\int_{\partial^*\Omega_i}\mathrm{n}_{\partial^*\Omega_i}\cdot\partial_t\xi_i\,\mathrm{d}\mathcal{H}^{d-1} = -\frac{1}{2}\sum_{i,j=1,i\neq j}^P\sigma_{i,j}\int_{I_{i,j}}\mathrm{n}_{i,j}\cdot\partial_t\xi_{i,j}\,\mathrm{d}\mathcal{H}^{d-1}.$$

Defining along $I_{i,j}$ the interface velocity $\mathrm{V}_{I_{i,j}} := (\mathrm{V}_i\cdot\mathrm{n}_{i,j})\mathrm{n}_{i,j}$ for all distinct phases $i,j\in\{1,\ldots,P\}$, we also obtain for the same reasons

$$\sum_{i=1}^P\int_{\partial^*\Omega_i}(\nabla\cdot\xi_i)(\mathrm{V}_i\cdot\mathrm{n}_{\partial^*\Omega_i})\,\mathrm{d}\mathcal{H}^{d-1} = -\frac{1}{2}\sum_{i,j=1,i\neq j}^P\sigma_{i,j}\int_{I_{i,j}}(\nabla\cdot\xi_{i,j})(\mathrm{V}_{I_{i,j}}\cdot\mathrm{n}_{i,j})\,\mathrm{d}\mathcal{H}^{d-1},$$

so that the combination of the previous three displays entails

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}E[\Omega|\bar\Omega] = \frac{\mathrm{d}}{\mathrm{d}t}E[\Omega] &- \frac{1}{2}\sum_{i,j=1,i\neq j}^P\sigma_{i,j}\int_{I_{i,j}}\mathrm{n}_{i,j}\cdot\partial_t\xi_{i,j}\,\mathrm{d}\mathcal{H}^{d-1} \\ &- \frac{1}{2}\sum_{i,j=1,i\neq j}^P\sigma_{i,j}\int_{I_{i,j}}(\nabla\cdot\xi_{i,j})(\mathrm{V}_{I_{i,j}}\cdot\mathrm{n}_{i,j})\,\mathrm{d}\mathcal{H}^{d-1}.\end{aligned}$$

The next step consists of introducing a *single* velocity vector field B, which shall represent a smooth space-time vector field whose normal component along $\bar I_{i,j}$ equals the normal velocity of the smoothly evolving interface $\bar I_{i,j}$ for all distinct phases $i,j\in\{1,\ldots,P\}$. Noting then that (2.29) together with the skew-symmetry relation $\mathrm{n}_{i,j} = -\mathrm{n}_{j,i}$ again enables to switch back and forth between surface integrals over individual interfaces and volume integrals over individual phases, this time in form of

$$\begin{aligned}\frac{1}{2}\sum_{i,j=1,i\neq j}^P\sigma_{i,j}\int_{I_{i,j}}\mathrm{n}_{i,j}\cdot\big(\nabla\cdot(\xi_{i,j}\otimes B)\big)\,\mathrm{d}\mathcal{H}^{d-1} &= -\sum_{i=1}^P\int\chi_i\nabla\cdot\big(\nabla\cdot(\xi_i\otimes B)\big)\,\mathrm{d}x \\ &= -\sum_{i=1}^P\int\chi_i\nabla\cdot\big(\nabla\cdot(B\otimes\xi_i)\big)\,\mathrm{d}x \\ &= \frac{1}{2}\sum_{i,j=1,i\neq j}^P\sigma_{i,j}\int_{I_{i,j}}\mathrm{n}_{i,j}\cdot\big(\nabla\cdot(B\otimes\xi_{i,j})\big)\,\mathrm{d}\mathcal{H}^{d-1},\end{aligned}$$

one may otherwise simply follow the exact same arguments leading to (2.11) (with obvious notational modifications) in order to produce the identity

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}E[\Omega|\bar{\Omega}] = \frac{\mathrm{d}}{\mathrm{d}t}E[\Omega] &- \frac{1}{2}\sum_{i,j=1,i\neq j}^{P}\sigma_{i,j}\int_{I_{i,j}}(\nabla\cdot\xi_{i,j})\big((\mathrm{V}_{I_{i,j}}-B)\cdot\mathrm{n}_{i,j}\big)\,\mathrm{d}\mathcal{H}^{d-1} \\ &- \frac{1}{2}\sum_{i,j=1,i\neq j}^{P}\sigma_{i,j}\int_{I_{i,j}}(\mathrm{Id}-\mathrm{n}_{i,j}\otimes\mathrm{n}_{i,j}):\nabla B\,\mathrm{d}\mathcal{H}^{d-1} \\ &- \frac{1}{2}\sum_{i,j=1,i\neq j}^{P}\sigma_{i,j}\int_{I_{i,j}}(\mathrm{n}_{i,j}-\xi_{i,j})\cdot\big(\partial_t\xi_{i,j}+(B\cdot\nabla)\xi_{i,j}+(\nabla B)^{\mathsf{T}}\xi_{i,j}\big)\,\mathrm{d}\mathcal{H}^{d-1} \\ &- \frac{1}{2}\sum_{i,j=1,i\neq j}^{P}\sigma_{i,j}\int_{I_{i,j}}\xi_{i,j}\cdot\big(\partial_t\xi_{i,j}+(B\cdot\nabla)\xi_{i,j}\big)\,\mathrm{d}\mathcal{H}^{d-1} \\ &- \frac{1}{2}\sum_{i,j=1,i\neq j}^{P}\sigma_{i,j}\int_{I_{i,j}}(\mathrm{n}_{i,j}\cdot\xi_{i,j}-1)(\nabla\cdot B)\,\mathrm{d}\mathcal{H}^{d-1} \\ &- \frac{1}{2}\sum_{i,j=1,i\neq j}^{P}\sigma_{i,j}\int_{I_{i,j}}(\mathrm{n}_{i,j}-\xi_{i,j})\cdot\big((\mathrm{n}_{i,j}-\xi_{i,j})\cdot\nabla\big)B\,\mathrm{d}\mathcal{H}^{d-1}.\end{aligned} \tag{2.33}$$

As in the two-phase setting, the further processing of the first three right hand side terms is subject to the specific evolution problem under consideration, and may in particular put further conditions on the vector fields $((\xi_{i,j})_{i\neq j}, B)$. We refer to the second part of Section 4.2 for the example of evolution by multiphase mean curvature flow.

For the remaining four right hand side terms, provided that the gradient of B is uniformly bounded and it holds for all distinct $i,j\in\{1,\ldots,P\}$

$$|\xi_{i,j}(\cdot,t)| \leq 1 - c\min\{\mathrm{dist}^2(\cdot,\bar{I}_{i,j}(t)),1\} \qquad \text{in } \mathbb{R}^d, \tag{2.34}$$

$$\big(\partial_t\xi_{i,j}+(B\cdot\nabla)\xi_{i,j}+(\nabla B)^{\mathsf{T}}\xi_{i,j}\big)(\cdot,t) = O\big(\min\big\{\mathrm{dist}(\cdot,\bar{I}_{i,j}(t)),1\big\}\big) \qquad \text{in } \mathbb{R}^d, \tag{2.35}$$

$$\big(\partial_t|\xi_{i,j}|^2+(B\cdot\nabla)|\xi_{i,j}|^2\big)(\cdot,t) = O\big(\min\big\{\mathrm{dist}^2(\cdot,\bar{I}_{i,j}(t)),1\big\}\big) \qquad \text{in } \mathbb{R}^d, \tag{2.36}$$

one immediately observes that these four terms are directly controlled by the multiphase relative entropy $E[\Omega|\bar{\Omega}]$ from (2.28). However, in contrast to the two-phase setting, the actual construction of a family of vector fields $((\xi_{i,j})_{i\neq j}, B)$ satisfying at least (2.29) as well as (2.34)–(2.36) is a substantially more difficult task. This is due to the—even on the level of strong solutions—inherent singular structure of the underlying network of interfaces (e.g., triple and/or higher-order junctions will be present in general), and thus requires additional ideas. For a realization of such a construction in the context of multiphase mean curvature flow of networks in $\mathbb{R}^2$, or mean curvature flow of a double bubble in $\mathbb{R}^3$, we refer to Sections 4.4–4.6 and Sections 5.2–5.4, respectively.

2.2.2 Relation to the method of paired calibrations

The crucial algebraic requirement (2.29) provides an interesting connection to a well-known notion from minimal surface theory (see, e.g., Harvey and Lawson [79] or Morgan [118]) resp. the theory of the partition problem (see, e.g., Lawlor and Morgan [102] or Brakke [24]): the concept of *calibrations* resp. *paired calibrations*.

In the context of the partition problem and employing the language of Lawlor and Morgan [102], a family of (time-independent) vector fields $(\xi_i)_{i\in\{1,\ldots,P\}}$ is called a paired calibration for a (time-independent) partition $\bar{\Omega} = (\bar{\Omega}_1,\ldots,\bar{\Omega}_P)$ of a bounded domain $D\subset\mathbb{R}^d$

if—next to the algebraic requirement (2.29), the extension property $\xi_{i,j} = \bar{\mathrm{n}}_{i,j}$ along $\bar{I}_{i,j}$, and the global length constraint $|\xi_{i,j}| \leq 1$—it satisfies the divergence constraint $\nabla \cdot (\xi_i - \xi_j) = 0$ throughout D for all distinct $i, j \in \{1, \ldots, P\}$. The interest in this concept stems from the classical fact that the existence of a paired calibration implies global minimality of the interfacial energy (2.27) for the underlying partition amongst all partitions of D with the same boundary data, see again Lawlor and Morgan [102].

Indeed, thanks to the additional constraint on the divergence of $\xi_i - \xi_j$, the argument leading to the identity (2.31) shows that for all other partitions $\Omega = (\Omega_1, \ldots, \Omega_P)$ of the domain $D \subset \mathbb{R}^d$ with the same boundary data along ∂D as the calibrated partition $\bar{\Omega}$, it holds

$$
\begin{aligned}
E[\Omega] &= E[\bar{\Omega}] + E[\Omega|\bar{\Omega}] + \sum_{i,j=1, i\neq j}^{P} \int_{\Omega_i \cap \bar{\Omega}_j} \nabla \cdot (\xi_i - \xi_j) \,\mathrm{d}x \\
&= E[\bar{\Omega}] + E[\Omega|\bar{\Omega}],
\end{aligned}
\tag{2.37}
$$

so that the claim follows from recalling that $E[\Omega|\bar{\Omega}] \geq 0$.

The above reasoning towards global minimality does not rely on any of the properties of the relative entropy functional $E[\Omega|\bar{\Omega}]$ except for its non-negativity. We develop in a forthcoming work [66] a local analogue of the concept of paired calibrations, and leverage on the local version of this concept to show that flat partitions of a bounded domain in the plane (i.e., interfaces are straight line segments joining at triple junctions with the correct angle condition) are local Dirichlet minimizer for the interface energy functional with respect to the L^1 topology (for given boundary data). The idea behind this result stems from the observation that for local minimality it may suffice to enforce the divergence constraint $\nabla \cdot (\xi_i - \xi_j) = 0$ only in a small neighborhood around the interface $\bar{I}_{i,j}$. For the remaining contributions from the bulk terms $\int_{\Omega_i \cap \bar{\Omega}_j} \nabla \cdot (\xi_i - \xi_j) \,\mathrm{d}x$ appearing on the right hand side of (2.37), we then argue that, at least for sufficiently small perturbations of the phases in L^1, they can be absorbed by the relative entropy functional exploiting its coercivity properties.

2.2.3 The bulk error functional in the multiphase regime

In analogy to the two-phase setting, a stability estimate of the form (1.48) for the multiphase relative entropy (2.28) implies a "weak-strong inclusion principle"

$$
\begin{aligned}
&I_{i,j}(0) \subset \bar{I}_{i,j}(0) \text{ up to } \mathcal{H}^{d-1} \text{ null sets for all } i \neq j \in \{1, \ldots, P\} \\
&\implies \quad I_{i,j}(t) \subset \bar{I}_{i,j}(t) \text{ up to } \mathcal{H}^{d-1} \text{ null sets for a.e. } t \in [0,T] \text{ and all } i \neq j \in \{1, \ldots, P\},
\end{aligned}
$$

but in general does not yet imply a weak-strong uniqueness principle

$$
\begin{aligned}
&\Omega_i(0) = \bar{\Omega}_i(0) \text{ up to Lebesgue null sets for all } i \in \{1, \ldots, P\} \\
&\implies \quad \Omega_i(t) = \bar{\Omega}_i(t) \text{ up to Lebesgue null sets for a.e. } t \in [0,T] \text{ and all } i \in \{1, \ldots, P\}.
\end{aligned}
$$

For the latter, one again relies on a stability estimate with respect to a bulk error functional, which in the multiphase regime may be defined by means of

$$
E_{\mathrm{bulk}}[\Omega|\bar{\Omega}](t) := \sum_{i=1}^{P} \int (\chi_i - \bar{\chi}_i)(\cdot, t)\, \vartheta_i(\cdot, t) \,\mathrm{d}x. \tag{2.38}
$$

Here, $\vartheta_i \colon \mathbb{R}^d \times [0,T] \to [-1,1]$ represents for each phase $i \in \{1, \ldots, P\}$ a smooth and integrable weight subject to (at least) the following conditions:

$$
\vartheta_i(\cdot, t) < 0 \text{ in } \bar{\Omega}_i(t), \quad \vartheta_i(\cdot, t) > 0 \text{ in } \mathbb{R}^d \setminus \overline{\bar{\Omega}_i(t)}, \tag{2.39}
$$

$$
\min\{\mathrm{dist}(\cdot, \partial\bar{\Omega}_i(t)), 1\} \leq |\vartheta_i(\cdot, t)| \leq C \min\{\mathrm{dist}(\cdot, \partial\bar{\Omega}_i(t)), 1\}, \tag{2.40}
$$

$$
\big(\partial_t \vartheta_i + (B \cdot \nabla)\vartheta_i\big)(\cdot, t) = O\big(\min\{\mathrm{dist}(\cdot, \bar{\Omega}_i(t)), 1\}\big) \tag{2.41}
$$

for some $C \geq 1$, all $t \in [0,T]$ and all $i \in \{1,\ldots,P\}$, and where B denotes the velocity vector field from the computation of the time evolution of the multiphase relative entropy (2.28). Constructing such a family of weights is slightly more involved than the corresponding argument from the two-phase setting because the phase boundaries $\partial\bar{\Omega}_i$ will in general contain lower-dimensional boundaries in form of, e.g., corners. We refer to Section 4.7 or Section 5.5 for an explicit construction in the context of networks of interfaces in $\mathbb{R}^2$ or double bubbles in $\mathbb{R}^3$, respectively. (The flow rule being evolution by mean curvature is in fact not essential for the construction of the weights.)

We conclude the discussion of the multiphase regime stating a preliminary representation for the time evolution of the multiphase bulk error functional (2.38). In analogy to the argument in the two-phase setting leading to the identity (2.24), one obtains the formula

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}E_{\mathrm{bulk}}[\Omega|\bar{\Omega}] = &\sum_{i,j=1,i\neq j}^{P}\int_{I_{i,j}}\vartheta_i\big((\mathrm{V}_{I_{i,j}}-B)\cdot\mathrm{n}_{i,j}\big)\,\mathrm{d}\mathcal{H}^{d-1} \\ &+\sum_{i=1}^{P}\int(\chi_i-\bar{\chi}_i)\big(\partial_t\vartheta_i+(B\cdot\nabla)\vartheta_i\big)\,\mathrm{d}x \\ &+\sum_{i=1}^{P}\int(\chi_i-\bar{\chi}_i)\vartheta_i(\nabla\cdot B)\,\mathrm{d}x.\end{aligned} \tag{2.42}$$

The last two right hand side terms are directly controlled by $E_{\mathrm{bulk}}[\Omega|\bar{\Omega}]$ due to (2.40)–(2.41), whereas further computations for the first right hand side term depend on the specific problem at hand.

2.3 Robustness of the relative entropy approach

The relative entropy approach to curvature driven interface evolution problems outlined in the previous two sections turns out to be sufficiently robust to apply it to more general settings than the ones considered before. We discuss in this section applications and/or extensions of the previously developed ideas to *i)* two-phase Navier–Stokes flow with sharp interface, *ii)* varifold solution concepts, *iii)* interface evolution equations incorporating boundary contact energies allowing for contact point dynamics with fixed-in-time contact angles, and finally, leaving the realm of sharp interface models, *iv)* the rigorous derivation of convergence rates for diffuse interface approximations.

2.3.1 Application to two-phase Navier–Stokes flow with sharp interface

We start by recalling that the energy for a weak solution (Ω, u) of the free boundary problem for the flow of two viscous, incompressible and immiscible fluids with surface tension in $\mathbb{R}^d$, $d \in \{2,3\}$, is given by

$$E[\Omega,u] := E_{\mathrm{kin}}[\Omega,u] + E[\Omega], \tag{2.43}$$

where $E[\Omega]$ represents the interfacial energy contribution due to surface tension defined by (2.1), and where the kinetic energy contribution $E_{\mathrm{kin}}[\Omega,u]$ is given by

$$E_{\mathrm{kin}}[\Omega,u] := \int\frac{\rho(\chi)}{2}|u|^2\,\mathrm{d}x. \tag{2.44}$$

Here, for each $t \in [0,T]$ we again denoted by $\chi(\cdot,t)$ the characteristic function of the phase $\Omega(t)$, and given the two densities $\rho_\pm$ of the two fluids, we defined $\rho(\chi) := \rho_+\chi+\rho_-(1-\chi)$.

Assuming for simplicity that the two shear viscosities of the two fluids are the same (for a discussion of the highly non-trivial problem of different shear viscosities, we refer the reader to the fourth subsection of Section 3.2), and denoting by $(\bar\Omega, v)$ a strong solution for the two-phase Navier–Stokes system with surface tension, the ansatz for the relative entropy functional splits into two contributions (mimicking the structure (2.43) of the total energy functional)

$$E[\Omega,u|\bar\Omega,v] := \big(E_{\mathrm{kin}}[\Omega,u]-DE_{\mathrm{kin}}[\Omega,v]\big((\Omega,u)-(\Omega,v)\big)-E_{\mathrm{kin}}[\Omega,v]\big) + E[\Omega|\bar\Omega]. \tag{2.45}$$

The second contribution of (2.45) is precisely the two-phase relative entropy from (2.2), whereas the contribution based on the kinetic energy functional (2.44) fits into the classical framework (1.47) with

$$\begin{aligned}&E_{\mathrm{kin}}[\Omega,u]-DE_{\mathrm{kin}}[\Omega,v]\big((\Omega,u)-(\Omega,v)\big)-E_{\mathrm{kin}}[\Omega,v]\\ &= \int \frac{\rho(\chi)}{2}|u-v|^2\,\mathrm{d}x = E_{\mathrm{kin}}[\Omega,u] - \int \rho(\chi)u\cdot v\,\mathrm{d}x + \int \rho(\chi)\frac{1}{2}|v|^2\,\mathrm{d}x.\end{aligned}$$

Making use of the identity from the previous display as well as the identity (2.11) in combination with the subsequent discussion, we consequently obtain the following preliminary representation for the time evolution of the total relative entropy (2.45)

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}E[\Omega,u|\bar\Omega,v] &= \frac{\mathrm{d}}{\mathrm{d}t}E[\Omega,u] - \frac{\mathrm{d}}{\mathrm{d}t}\int \rho(\chi)u\cdot v\,\mathrm{d}x + \frac{\mathrm{d}}{\mathrm{d}t}\int \rho(\chi)\frac{1}{2}|v|^2\,\mathrm{d}x \\ &\quad - \sigma\int_I (\nabla\cdot\xi)\big((\mathrm{V}_I - B)\cdot\mathrm{n}\big)\,\mathrm{d}\mathcal{H}^{d-1} - \sigma\int_I (\mathrm{Id}-\mathrm{n}\otimes\mathrm{n}):\nabla B\,\mathrm{d}\mathcal{H}^{d-1} \\ &\quad + O\big(E[\Omega|\bar\Omega]\big).\end{aligned} \tag{2.46}$$

In the context of the free boundary problem for the flow of two viscous, incompressible and immiscible fluids with surface tension, one then proceeds as follows. The structure of the second and third right hand side term of (2.46) suggests to use the vector field v as a test function in the evolution equation for $\rho(\chi)u$ and the scalar field $\frac{1}{2}|v|^2$ in the evolution equation for $\rho(\chi)$, respectively. After plugging in the energy dissipation inequality at the level of the weak solution, one then tries to combine the resulting terms with the remaining fourth and fifth right hand side term of (2.46). For the rigorous implementation of this argument, we refer the reader to Section 3.6.

2.3.2 Application to a class of varifold solution concepts

As we already explained in some detail in Section 1.2, without imposing additional assumptions (e.g., an energy convergence assumption) one may only guarantee, if at all, the existence of so-called *varifold solutions* to a given interface evolution problem. For a prominent instance of such a varifold solution concept relating to the results of this thesis, we refer to Abels' notion of generalized solutions for two-phase Navier–Stokes flow with sharp interface [1]. We aim to make the point here that for a certain class of varifold solution concepts (i.e., for which a natural compatibility condition holds true, cf. (2.47) below), the two-phase relative entropy approach (2.2) extends naturally. To this end, recall that an oriented varifold is a finite Radon measure $V \in \mathcal{M}(\mathbb{R}^d\times\mathbb{S}^{d-1})$, and we write $|V| \in \mathcal{M}(\mathbb{R}^d)$ for its local mass density: $|V|(U) := V(U\times\mathbb{S}^{d-1})$ for all Borel measurable $U \subset \mathbb{R}^d$.

For what follows, we assume that we are given a time-dependent family of oriented varifolds $(V(\cdot,\cdot,t))_{t\in[0,T]}$, which is coupled to the evolving phase $(\Omega(t))_{t\in[0,T]}$ of the weak solution by means of the compatibility condition

$$\int_{\mathbb{R}^d\times\mathbb{S}^{d-1}} s\cdot\psi(x)\,\mathrm{d}V(x,s,t) = \int_{I(t)} \mathrm{n}(x,t)\cdot\psi(x)\,\mathrm{d}\mathcal{H}^{d-1}(x) \tag{2.47}$$

for all $t \in [0,T]$ and all $\psi \in C^\infty_{\mathrm{cpt}}(\mathbb{R}^d;\mathbb{R}^d)$. The energy functional is then given by

$$E[\Omega,V](t) := \sigma\int_{\mathbb{R}^d\times\mathbb{S}^{d-1}} 1\,\mathrm{d}V(x,s,t) = \sigma\int_{\mathbb{R}^d} 1\,\mathrm{d}|V|(x,t). \tag{2.48}$$

It is a direct consequence of the compatibility condition (2.47) that for all $t \in [0,T]$ the Radon–Nikodým derivative $\theta(\cdot,t) := \frac{\mathrm{d}\mathcal{H}^{d-1}\llcorner I(t)}{\mathrm{d}|V|(\cdot,t)}$ exists and satisfies $|V|(\cdot,t)$ a.e. $\theta(\cdot,t)\in[0,1]$. In particular, we may define a non-negative error functional by means of

$$\begin{aligned} E[\Omega,V|\bar\Omega](t) &:= \sigma\int_{\mathbb{R}^d\times\mathbb{S}^{d-1}} 1 - s\cdot\xi(\cdot,t)\,\mathrm{d}V(x,s,t) \\ &= E[\Omega,V](t) - \sigma\int_{I(t)} \mathrm{n}(\cdot,t)\cdot\xi(\cdot,t)\,\mathrm{d}\mathcal{H}^{d-1} \\ &= E[\Omega|\bar\Omega](t) + \sigma\int_{\mathbb{R}^d} 1-\theta(\cdot,t)\,\mathrm{d}|V|(x,t), \end{aligned} \tag{2.49}$$

where $E[\Omega|\bar\Omega]$ denotes the two-phase relative entropy defined by (2.2).

The definition itself of $E[\Omega,V|\bar\Omega]$ ensures tilt-excess type control on the level of the "varifold normal" in form of

$$\int_{\mathbb{R}^d\times\mathbb{S}^{d-1}} |s{-}\xi|^2\,\mathrm{d}V(x,s,t) \leq 2\sigma^{-1}E[\Omega,V|\bar\Omega](t), \tag{2.50}$$

whereas the third line of the previous display guarantees that $E[\Omega,V|\bar\Omega]$ inherits the coercivity properties of the two-phase relative entropy (2.2) and that it controls the multiplicity error (or in other words, the difference between being a BV solution or a varifold solution). The second line of the previous display in turn allows for a computation of the time evolution of the error functional $E[\Omega,V|\bar\Omega]$.

In terms of a precise representation of the time evolution, we claim that it holds (omitting again the dependence on the time variable)

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}E[\Omega,V|\bar\Omega] &= \frac{\mathrm{d}}{\mathrm{d}t}E[\Omega,V] - \sigma\int_I(\nabla\cdot\xi)\big((\mathrm{V}_I{-}B)\cdot\mathrm{n}\big)\,\mathrm{d}\mathcal{H}^{d-1} \\ &\quad - \sigma\int_{\mathbb{R}^d\times\mathbb{S}^{d-1}}(\mathrm{Id}{-}s\otimes s):\nabla B\,\mathrm{d}V(x,s) \\ &\quad - \sigma\int_I(\mathrm{n}{-}\xi)\cdot\big(\partial_t\xi{+}(B\cdot\nabla)\xi{+}(\nabla B)^\mathsf{T}\xi\big)\,\mathrm{d}\mathcal{H}^{d-1} \\ &\quad - \sigma\int_I\xi\cdot\big(\partial_t\xi{+}(B\cdot\nabla)\xi\big)\,\mathrm{d}\mathcal{H}^{d-1} \\ &\quad - \sigma\int_I(\mathrm{n}\cdot\xi-1)(\nabla\cdot B)\,\mathrm{d}\mathcal{H}^{d-1} \\ &\quad - \sigma\int_{\mathbb{R}^d\times\mathbb{S}^{d-1}}(s{-}\xi)\cdot\big((s{-}\xi)\cdot\nabla\big)B\,\mathrm{d}V(x,s) \\ &\quad - \sigma\int_{\mathbb{R}^d}(\theta{-}1)(\nabla\cdot B)\,\mathrm{d}|V|(x) \\ &\quad - \sigma\int_{\mathbb{R}^d}(\theta{-}1)\big(\xi\cdot(\xi\cdot\nabla)B\big)\,\mathrm{d}|V|(x). \end{aligned} \tag{2.51}$$

Note that the last three right hand side terms do not involve any new difficulties as these are directly controlled by the error functional $E[\Omega,V|\bar\Omega]$ (provided the gradient of B is uniformly bounded). In view of the second line of (2.49) and the identity (2.11), the asserted representation (2.51) is a consequence of

$$
\begin{aligned}
&-\sigma\int_I(\mathrm{Id}-\mathrm{n}\otimes\mathrm{n}):\nabla B\,\mathrm{d}\mathcal{H}^{d-1}-\sigma\int_I(\mathrm{n}-\xi)\cdot\big((\mathrm{n}-\xi)\cdot\nabla\big)B\,\mathrm{d}\mathcal{H}^{d-1}\\
&=-\sigma\int_{\mathbb{R}^d\times\mathbb{S}^{d-1}}(\mathrm{Id}-s\otimes s):\nabla B\,\mathrm{d}V(x,s)-\sigma\int_{\mathbb{R}^d\times\mathbb{S}^{d-1}}(s-\xi)\cdot\big((s-\xi)\cdot\nabla\big)B\,\mathrm{d}V(x,s)\\
&\quad-\sigma\int_{\mathbb{R}^d}(\theta-1)(\nabla\cdot B)\,\mathrm{d}|V|(x)-\sigma\int_{\mathbb{R}^d}(\theta-1)\big(\xi\cdot(\xi\cdot\nabla)B\big)\,\mathrm{d}|V|(x).
\end{aligned}
$$

The identity of the previous display, however, follows from straightforward computations making use of the compatibility condition (2.47) and the definition of the Radon–Nikodým derivative θ.

2.3.3 Application to interface evolution with boundary contact

We next turn to two-phase interface evolution within a bounded domain $D\subset\mathbb{R}^d$ with smooth boundary ∂D. We aim to outline a potential strategy to incorporate into our general framework a class of boundary contact energies allowing for contact point dynamics with fixed-in-time contact angles $\alpha\in(0,\pi)$. The rigorous implementation of the following arguments in the context of evolution by mean curvature will be the subject of future work (in ambient spatial dimension $d\in\{2,3\}$). In the special case of $\alpha=\frac{\pi}{2}$ in the setting of planar two-phase Navier–Stokes flow with sharp interface, we refer to the forthcoming work [68] which will be a part of the PhD thesis of Alice Marveggio.

At the level of the weak solution, we consider a time-dependent family $\Omega=(\Omega(t))_{t\in[0,T]}$ of subsets $\Omega(t)\subset D$, $t\in[0,T]$, which are of finite perimeter in D. For each $t\in[0,T]$, we denote by $I(t):=\partial^*\Omega(t)$ the reduced boundary of $\Omega(t)$ in $\mathbb{R}^d$. Surface tension along the interface $I(t)\cap D$ is again accounted for by $\sigma>0$, whereas we denote by γ_+ and γ_- the analogs for the "interfaces" $I(t)\cap\partial D$ and $\partial^*(D\setminus\Omega(t))\cap\partial D$, respectively. We assume for these parameters that Young's relation $\frac{\gamma_+-\gamma_-}{\sigma}\in(-1,1)$ holds true, and that the fixed-in-time contact angle $\alpha\in(0,\pi)$, formally formed by the intersection of the tangent spaces to ∂D and $I(t)\cap D$ at a contact point through the region $D\setminus\Omega(t)$, is determined by Young's equation

$$\sigma\cos\alpha=\gamma_+-\gamma_-. \tag{2.52}$$

We then consider the energy functional defined as the sum of an interfacial energy contribution in the bulk and a boundary contact energy in form of

$$E[\Omega](t):=\sigma\int_{I(t)\cap D}1\,\mathrm{d}\mathcal{H}^{d-1}+\sigma\int_{I(t)\cap\partial D}\cos\alpha\,\mathrm{d}\mathcal{H}^{d-1}. \tag{2.53}$$

For the definition of a relative entropy in this context, we first consider a time dependent family $\bar\Omega=(\bar\Omega(t))_{t\in[0,T]}$ of open subsets $\bar\Omega(t)\subset D$, $t\in[0,T]$, with finitely many connected components. Defining for each $t\in[0,T]$ the set $\bar I(t):=\partial\bar\Omega(t)$, we assume that the closure $\overline{\bar I(t)\cap D}$ of $\bar I(t)\cap D$ and the closure $\overline{\bar I(t)\cap\partial D}$ of $\bar I(t)\cap\partial D$ are smooth manifolds with common smooth boundary $\partial(\bar I(t)\cap D)\subset\partial D$ along which the contact angle is given by α. One shall think of the data $(\bar\Omega(t))_{t\in[0,T]}$ and $(\bar I(t))_{t\in[0,T]}$ as a strong solution of the underlying interface evolution problem. We next assume that we already constructed a pair of continuous vector fields (ξ,B) on the closure of the domain D satisfying at least

$$
\begin{aligned}
\xi(\cdot,t)&=\bar{\mathrm{n}}(\cdot,t) &&\text{along } \overline{\bar I(t)\cap D}, &(2.54)\\
|\xi(\cdot,t)|&\leq 1-c\min\{\mathrm{dist}^2(\cdot,\overline{\bar I(t)\cap D}),1\} &&\text{in } D, &(2.55)\\
\big(\partial_t\xi+(B\cdot\nabla)\xi+(\nabla B)^{\mathsf{T}}\xi\big)(\cdot,t)&=O\big(\min\big\{\mathrm{dist}(\cdot,\overline{\bar I(t)\cap D}),1\big\}\big) &&\text{in } D, &(2.56)\\
\big(\partial_t|\xi|^2+(B\cdot\nabla)|\xi|^2\big)(\cdot,t)&=O\big(\min\big\{\mathrm{dist}^2(\cdot,\overline{\bar I(t)\cap D}),1\big\}\big) &&\text{in } D, &(2.57)
\end{aligned}
$$

for all $t \in [0,T]$ and some $c \in (0,1]$. We finally define for all $t \in [0,T]$

$$E[\Omega|\bar{\Omega}](t) := \sigma \int_{I(t)\cap D} 1 - \mathrm{n}(\cdot,t) \cdot \xi(\cdot,t) \,\mathrm{d}\mathcal{H}^{d-1}, \tag{2.58}$$

where $\mathrm{n}(\cdot,t)$ denotes the measure theoretic unit normal along the interface $I(t)\cap D$ pointing inside the phase $\Omega(t)$.

Before we derive the analogue of (2.11) in the present context of a fixed-in-time contact angle, we first motivate the natural boundary condition for the pair of vector fields (ξ, B) along the domain boundary ∂D. The boundary condition for the extension $\xi(\cdot,t)$ of the unit normal $\bar{\mathrm{n}}(\cdot,t)$ along the interface $\bar{I}(t)\cap D$ is chosen in such a way to ensure that the definition (2.58) of the error functional $E[\Omega|\bar{\Omega}](t)$ again mimics the properties of classical relative entropies. More precisely, we impose for all $t \in [0,T]$ the condition

$$\xi(\cdot,t)\cdot \mathrm{n}_{\partial D}(\cdot) = \cos\alpha \quad \text{along } \partial D, \tag{2.59}$$

where $\mathrm{n}_{\partial D}$ denotes the inward pointing unit normal along the domain boundary ∂D. Based on the boundary condition (2.59), we may add zero in a first step and then integrate by parts in a second step to rewrite the definition (2.58) in form of

$$E[\Omega|\bar{\Omega}](t) = E[\Omega] - \sigma \int_{\partial^*\Omega(t)} \mathrm{n}_{\partial^*\Omega(t)} \cdot \xi(\cdot,t)\,\mathrm{d}\mathcal{H}^{d-1} = E[\Omega] + \int_{\Omega(t)} (\nabla\cdot\sigma\xi)(\cdot,t)\,\mathrm{d}x. \tag{2.60}$$

This structure is again precisely what is needed in order to evaluate the time evolution of $E[\Omega|\bar{\Omega}]$ based only on an energy dissipation inequality and testing the evolution equation of the weak solution against the test function $\nabla\cdot\sigma\xi$.

For the boundary condition of the velocity vector field B, we impose for all $t\in[0,T]$

$$B(\cdot,t)\cdot \mathrm{n}_{\partial D}(\cdot) = 0 \quad \text{along } \partial D. \tag{2.61}$$

This condition is indeed natural since the evolution of contact points is restricted to the domain boundary ∂D, so that the associated velocity has to be tangential to ∂D. Note that in the case of $\alpha \neq \frac{\pi}{2}$, this inevitably necessitates a non-trivial tangential component of $B(\cdot,t)$ at contact points (tangential meaning with respect to the interface $\bar{I}(t)\cap D$). Recall from the remark below (2.16) that the construction of B in principle allows for such flexibility.

We proceed with the computation of the time evolution of $E[\Omega|\bar{\Omega}]$, pursuing the goal of arriving at a preliminary representation analogous to (2.11). Denoting for every $t\in[0,T]$ by $\chi(\cdot,t)$ the indicator function of the phase $\Omega(t)$, and denoting by $\mathrm{V}_{I\cap D}(\cdot,t)$ an associated velocity vector field for the interface $I(t)\cap D$ so that it holds (in a distributional sense)

$$\partial_t\chi = -(\mathrm{V}_{I\cap D}\cdot\mathrm{n})\,\mathrm{d}\mathcal{H}^{d-1} \llcorner \big(I\cap D\big), \tag{2.62}$$

we again first compute by means of (2.60) and (2.62), omitting the dependence on the time variable,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}E[\Omega|\bar{\Omega}] &= \frac{\mathrm{d}}{\mathrm{d}t}E[\Omega] - \sigma\int_{I\cap D}(\nabla\cdot\xi)(\mathrm{V}_{I\cap D}\cdot\mathrm{n})\,\mathrm{d}\mathcal{H}^{d-1} \\ &\quad - \sigma\int_{I\cap D}\mathrm{n}\cdot\partial_t\xi\,\mathrm{d}\mathcal{H}^{d-1} - \sigma\int_{I\cap\partial D}\mathrm{n}_{\partial D}\cdot\partial_t\xi\,\mathrm{d}\mathcal{H}^{d-1}.\end{aligned}$$

The last two right hand side terms of the previous display may be equivalently expressed—based on by now routine arguments—as follows

$$
\begin{aligned}
&-\sigma\int_{I\cap D}\mathrm{n}\cdot\partial_t\xi\,\mathrm{d}\mathcal{H}^{d-1}-\sigma\int_{I\cap\partial D}\mathrm{n}_{\partial D}\cdot\partial_t\xi\,\mathrm{d}\mathcal{H}^{d-1}\\
&=-\sigma\int_{I\cap D}(\mathrm{n}-\xi)\cdot\big(\partial_t\xi+(B\cdot\nabla)\xi+(\nabla B)^{\mathsf{T}}\xi\big)\,\mathrm{d}\mathcal{H}^{d-1}-\sigma\int_{I\cap D}\xi\cdot\big(\partial_t\xi+(B\cdot\nabla)\xi\big)\,\mathrm{d}\mathcal{H}^{d-1}\\
&\quad-\sigma\int_{I\cap\partial D}\mathrm{n}_{\partial D}\cdot\big(\partial_t\xi+(B\cdot\nabla)\xi\big)\,\mathrm{d}\mathcal{H}^{d-1}+\sigma\int_{I\cap D}\xi\cdot\big((\mathrm{n}-\xi)\cdot\nabla\big)B\,\mathrm{d}\mathcal{H}^{d-1}\\
&\quad+\sigma\int_{I\cap D}\mathrm{n}\cdot(B\cdot\nabla)\xi\,\mathrm{d}\mathcal{H}^{d-1}+\sigma\int_{I\cap\partial D}\mathrm{n}_{\partial D}\cdot(B\cdot\nabla)\xi\,\mathrm{d}\mathcal{H}^{d-1}.
\end{aligned}
$$

Recalling the fact that the tangential gradient of $\mathrm{n}_{\partial D}$ satisfies $(\mathrm{n}_{\partial D}\cdot\nabla^{\tan})\mathrm{n}_{\partial D}=0$ and $(\nabla^{\tan}\mathrm{n}_{\partial D})^{\mathsf{T}}\mathrm{n}_{\partial D}=0$ along ∂D, it follows from an application of the product rule, the boundary condition (2.61), the time-independence of $\mathrm{n}_{\partial D}$, and finally the boundary condition (2.59)

$$
-\sigma\int_{I\cap\partial D}\mathrm{n}_{\partial D}\cdot\big(\partial_t\xi+(B\cdot\nabla)\xi\big)\,\mathrm{d}\mathcal{H}^{d-1}=\sigma\int_{I\cap\partial D}\big((\mathrm{Id}-\mathrm{n}_{\partial D}\otimes\mathrm{n}_{\partial D})\xi\big)\cdot(B\cdot\nabla)\mathrm{n}_{\partial D}\,\mathrm{d}\mathcal{H}^{d-1}.
$$

Appealing to the product rule and adding zero twice moreover entails

$$
\begin{aligned}
&\sigma\int_{I\cap D}\mathrm{n}\cdot(B\cdot\nabla)\xi\,\mathrm{d}\mathcal{H}^{d-1}+\sigma\int_{I\cap\partial D}\mathrm{n}_{\partial D}\cdot(B\cdot\nabla)\xi\,\mathrm{d}\mathcal{H}^{d-1}\\
&=\sigma\int_{I\cap D}\mathrm{n}\cdot\big(\nabla\cdot(\xi\otimes B)\big)\,\mathrm{d}\mathcal{H}^{d-1}+\sigma\int_{I\cap\partial D}\mathrm{n}_{\partial D}\cdot\big(\nabla\cdot(\xi\otimes B)\big)\,\mathrm{d}\mathcal{H}^{d-1}\\
&\quad-\sigma\int_{I\cap D}(\mathrm{Id}-\mathrm{n}\otimes\mathrm{n}):\nabla B\,\mathrm{d}\mathcal{H}^{d-1}-\sigma\int_{I\cap\partial D}(\mathrm{n}_{\partial D}\cdot\xi)(\nabla\cdot B)\,\mathrm{d}\mathcal{H}^{d-1}\\
&\quad-\sigma\int_{I\cap D}(\mathrm{n}\cdot\xi-1)(\nabla\cdot B)\,\mathrm{d}\mathcal{H}^{d-1}-\sigma\int_{I\cap D}\mathrm{n}\cdot(\mathrm{n}\cdot\nabla)B\,\mathrm{d}\mathcal{H}^{d-1}.
\end{aligned}
$$

Based on the by now routine procedure, using in addition only the boundary condition (2.61), we further rewrite the first two right hand side terms of the previous display in form of

$$
\begin{aligned}
&\sigma\int_{I\cap D}\mathrm{n}\cdot\big(\nabla\cdot(\xi\otimes B)\big)\,\mathrm{d}\mathcal{H}^{d-1}+\sigma\int_{I\cap\partial D}\mathrm{n}_{\partial D}\cdot\big(\nabla\cdot(\xi\otimes B)\big)\,\mathrm{d}\mathcal{H}^{d-1}\\
&=\sigma\int_{D}\chi\nabla\cdot\big(\nabla\cdot(B\otimes\xi)\big)\,\mathrm{d}x\\
&=\sigma\int_{I\cap D}(\nabla\cdot\xi)(B\cdot\mathrm{n})\,\mathrm{d}\mathcal{H}^{d-1}+\sigma\int_{I\cap D}\mathrm{n}\cdot(\xi\cdot\nabla)B\,\mathrm{d}\mathcal{H}^{d-1}\\
&\quad+\sigma\int_{I\cap\partial D}\mathrm{n}_{\partial D}\cdot(\xi\cdot\nabla)B\,\mathrm{d}\mathcal{H}^{d-1}.
\end{aligned}
$$

Splitting the vector field ξ along ∂D into tangential and normal components, applying the product rule, exploiting another time the boundary condition (2.61), as well as making use of the symmetry of $\nabla^{\tan}\mathrm{n}_{\partial D}$, we infer

$$
\begin{aligned}
\sigma\int_{I\cap\partial D}\mathrm{n}_{\partial D}\cdot(\xi\cdot\nabla)B\,\mathrm{d}\mathcal{H}^{d-1}&=\sigma\int_{I\cap\partial D}(\mathrm{n}_{\partial D}\cdot\xi)\,\mathrm{n}_{\partial D}\cdot(\mathrm{n}_{\partial D}\cdot\nabla)B\,\mathrm{d}\mathcal{H}^{d-1}\\
&\quad-\sigma\int_{I\cap\partial D}B\cdot\big((\mathrm{Id}-\mathrm{n}_{\partial D}\otimes\mathrm{n}_{\partial D})\xi\cdot\nabla\big)\mathrm{n}_{\partial D}\,\mathrm{d}\mathcal{H}^{d-1}\\
&=\sigma\int_{I\cap\partial D}(\mathrm{n}_{\partial D}\cdot\xi)\,\mathrm{n}_{\partial D}\cdot(\mathrm{n}_{\partial D}\cdot\nabla)B\,\mathrm{d}\mathcal{H}^{d-1}\\
&\quad-\sigma\int_{I\cap\partial D}\big((\mathrm{Id}-\mathrm{n}_{\partial D}\otimes\mathrm{n}_{\partial D})\xi\big)\cdot(B\cdot\nabla)\mathrm{n}_{\partial D}\,\mathrm{d}\mathcal{H}^{d-1}.
\end{aligned}
$$

Finally, it holds due to the boundary conditions (2.59) and (2.61)

$$-\sigma\int_{I\cap\partial D}(\mathrm{n}_{\partial D}\cdot\xi)(\nabla\cdot B)\,\mathrm{d}\mathcal{H}^{d-1} = -\sigma\int_{I\cap\partial D}\cos\alpha\,(\nabla^{\tan}\cdot B)\,\mathrm{d}\mathcal{H}^{d-1} - \sigma\int_{I\cap\partial D}(\mathrm{n}_{\partial D}\cdot\xi)\,\mathrm{n}_{\partial D}\cdot(\mathrm{n}_{\partial D}\cdot\nabla)B\,\mathrm{d}\mathcal{H}^{d-1}.$$

The combination of the previous seven displays in total implies the following clean preliminary identity for the time evolution of the relative entropy (2.58)

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}E[\Omega|\bar\Omega] &= \frac{\mathrm{d}}{\mathrm{d}t}E[\Omega] - \sigma\int_{I\cap D}(\nabla\cdot\xi)\big((\mathrm{V}_{I\cap D}{-}B)\cdot\mathrm{n}\big)\,\mathrm{d}\mathcal{H}^{d-1} \\ &\quad - \sigma\int_{I\cap D}(\mathrm{Id}{-}\mathrm{n}\otimes\mathrm{n}):\nabla B\,\mathrm{d}\mathcal{H}^{d-1} - \sigma\int_{I\cap\partial D}\cos\alpha\,(\nabla^{\tan}\cdot B)\,\mathrm{d}\mathcal{H}^{d-1} \\ &\quad - \sigma\int_{I\cap D}(\mathrm{n}{-}\xi)\cdot\big(\partial_t\xi{+}(B\cdot\nabla)\xi{+}(\nabla B)^{\mathsf{T}}\xi\big)\,\mathrm{d}\mathcal{H}^{d-1} \\ &\quad - \sigma\int_{I\cap D}\xi\cdot\big(\partial_t\xi{+}(B\cdot\nabla)\xi\big)\,\mathrm{d}\mathcal{H}^{d-1} \\ &\quad - \sigma\int_{I\cap D}(\mathrm{n}\cdot\xi-1)(\nabla\cdot B)\,\mathrm{d}\mathcal{H}^{d-1} \\ &\quad - \sigma\int_{I\cap D}(\mathrm{n}{-}\xi)\cdot\big((\mathrm{n}{-}\xi)\cdot\nabla\big)B\,\mathrm{d}\mathcal{H}^{d-1}.\end{aligned} \tag{2.63}$$

The last four right hand side terms are again already controlled by $E[\Omega|\bar\Omega]$ thanks to the requirements (2.55)–(2.57) and the definition (2.58).

We conclude our discussion of a potential relative entropy approach to interface evolution problems incorporating an energy contribution of the form (2.53) by some remarks. The first concerns the claim that the functional

$$\begin{cases} C^1_{\mathrm{cpt}}(\mathbb{R}^d;\mathbb{R}^d)\ni\varphi \\ \varphi\cdot\mathrm{n}_{\partial D}=0 \text{ along } \partial D\end{cases} \mapsto -\int_{I\cap D}(\mathrm{Id}{-}\mathrm{n}\otimes\mathrm{n}):\nabla\varphi\,\mathrm{d}\mathcal{H}^{d-1} - \int_{I\cap\partial D}\cos\alpha\,(\nabla^{\tan}\cdot\varphi)\,\mathrm{d}\mathcal{H}^{d-1}$$

appearing on the right hand side of (2.63) represents a weak formulation of the mean curvature functional in the BV setting when allowing for boundary contact of the interface with contact angle $\alpha\in(0,\pi)$. To this end, we show in the smooth setting that

$$\int_{\bar I\cap D}\mathrm{H}_{\bar I\cap D}\cdot\varphi\,\mathrm{d}\mathcal{H}^{d-1} = -\int_{\bar I\cap D}(\mathrm{Id}{-}\bar{\mathrm{n}}\otimes\bar{\mathrm{n}}):\nabla\varphi\,\mathrm{d}\mathcal{H}^{d-1} - \int_{\bar I\cap\partial D}\cos\alpha\,(\nabla^{\tan}\cdot\varphi)\,\mathrm{d}\mathcal{H}^{d-1} \tag{2.64}$$

for all $\varphi\in C^1_{\mathrm{cpt}}(\mathbb{R}^d;\mathbb{R}^d)$ such that $\varphi\cdot\mathrm{n}_{\partial D}=0$ along ∂D, where $\mathrm{H}_{\bar I\cap D}$ denotes the mean curvature vector of the interface $\bar I\cap D$. For simplicity, we assume that the interface $\bar I\cap D$ is connected. Along the smooth contact manifold $\partial(\bar I\cap D)\subset\partial D$, we choose two unit normal vector fields $\tau_{\bar I\cap D}$ and $\tau_{\partial D}$, which in addition are tangential to $\bar I\cap D$ and ∂D, respectively, and finally satisfy $\tau_{\bar I\cap D}\cdot\tau_{\partial D}=\cos\alpha$ along $\partial(\bar I\cap D)$. There is a unique choice of these unit length vector fields by requiring $\tau_{\bar I\cap D}$ to point inside D (i.e., in the direction of the interface $\bar I\cap D$), and in that case $\tau_{\partial D}$ then points away from $\bar I\cap\partial D$. By means of the surface divergence theorem for smooth manifolds with boundary, we then obtain

$$\int_{\bar I\cap D}\mathrm{H}_{\bar I\cap D}\cdot\varphi\,\mathrm{d}\mathcal{H}^{d-1} = -\int_{\bar I\cap D}(\mathrm{Id}{-}\bar{\mathrm{n}}\otimes\bar{\mathrm{n}}):\nabla\varphi\,\mathrm{d}\mathcal{H}^{d-1} - \int_{\partial(\bar I\cap D)}\tau_{\bar I\cap D}\cdot\varphi\,\mathrm{d}\mathcal{H}^{d-1}.$$

Since φ is tangential to ∂D, we obtain by the properties of $\tau_{\bar I\cap D}$ and $\tau_{\partial D}$ that $\tau_{\bar I\cap D}\cdot\varphi = (\tau_{\bar I\cap D}\cdot\tau_{\partial D})(\tau_{\partial D}\cdot\varphi)=\cos\alpha(\tau_{\partial D}\cdot\varphi)$. Hence, by another application of the surface divergence theorem for smooth manifolds with boundary (recalling that $\tau_{\partial D}$ points away from $\bar{I} \cap \partial D$), using in the process that φ is tangential to ∂D, we deduce

$$-\int_{\partial(\bar{I}\cap D)} \tau_{\bar{I}\cap D} \cdot \varphi \, \mathrm{d}\mathcal{H}^{d-1} = -\int_{\partial(\bar{I}\cap D)} \cos\alpha (\tau_{\partial D} \cdot \varphi) \, \mathrm{d}\mathcal{H}^{d-1} = -\int_{\bar{I}\cap\partial D} \cos\alpha \, (\nabla^{\tan} \cdot \varphi) \, \mathrm{d}\mathcal{H}^{d-1}$$

so that the claim (2.64) follows.

It is clear from the boundary conditions (2.59) and (2.61) that, for a given specific interface evolution problem, the actual construction of a pair of vector fields (ξ, B) satisfying at least (2.54)–(2.57) as well as (2.59) and (2.61) requires a careful argument in a tubular neighborhood of the contact manifold $\partial(\bar{I} \cap D) \subset \partial D$. (A further processing of the first four right hand side terms of (2.63) possibly puts additional constraints on the pair of vector fields (ξ, B).) We will carry out this task in the context of evolution by mean curvature in a future work.

However, for a satisfying weak-strong uniqueness result one should also at least be able to say something about the existence of weak solutions in the BV setting. Assuming evolution by mean curvature, the BV formulation consists roughly speaking of the evolution equation (2.62) and an additional condition which directly links the velocity $\mathrm{V}_{I\cap D}$ with the weak formulation of the mean curvature functional, cf. the right hand side of (2.64). A potential strategy for an existence proof would be to study the Allen–Cahn equation on the domain D together with an appropriate non-linear Robin boundary condition along ∂D. The latter shall be formulated in a way which formally ensures that in the sharp interface limit one indeed obtains two-phase mean curvature flow with a fixed-in-time contact angle $\alpha \in (0, \pi)$. Imposing an energy convergence assumption in the spirit of the classical work by Luckhaus and Sturzenhecker [109] (cf. also the closely related work by Laux and Simon [101] for the vector-valued Allen–Cahn approximation of a BV formulation of multiphase mean curvature flow with periodic boundary data), it is tempting to ask whether one can provide a rigorous convergence proof towards the above sketched BV formulation. Investigations in this direction will be part of future work as well.

2.3.4 Phase field models: Convergence rates to sharp interface limits

Phase field models represent an alternative approach to describe the evolution of interfaces past topology changes and geometric singularities. In contrast to sharp interface models, where the evolution of a phase and its interface is, e.g., modeled by means of a characteristic function χ and the corresponding sharp phase boundary $\partial\{\chi{=}1\}$, the phase field approach is based on a smooth order function taking values in the continuum $[-1, 1]$. For most parts, the order function is required to take values close to 1 or -1, representing the bulk of the phase and its complement, respectively. The interface in turn is characterized as the region where the order function (rapidly) transitions from -1 to 1.

Such behavior may be enforced by introducing the Ginzburg–Landau energy functional

$$E_\varepsilon[\varphi_\varepsilon] := \int_{\mathbb{R}^d} \frac{\varepsilon}{2}|\nabla\varphi_\varepsilon|^2 + \frac{1}{\varepsilon}W(\varphi_\varepsilon) \, \mathrm{d}x, \tag{2.65}$$

with, say, a double well potential $W(r) := C(1-r^2)^2$ and $C > 0$ being a normalization constant. For order functions φ_ε with small energy $E_\varepsilon[\varphi_\varepsilon]$, the heuristic is that the energy contribution coming from the potential forces φ_ε to be close to ± 1 throughout most of $\mathbb{R}^d$, whereas the Dirichlet energy contribution forces the interfacial region to have finite (nonzero) extent. Moreover, it turns out that the typical width of the interfacial region scales linearly in the parameter ε.

Recall that important examples of sharp interface evolution equations arise (formally) as the gradient flow of the sharp interface energy functional (2.1): two-phase mean curvature flow being the gradient flow with respect to the L^2 scalar product, whereas the Mullins–Sekerka equation may be identified as the gradient flow with respect to the H^{-1} scalar product. It is thus natural to consider the corresponding gradient flows for the Ginzburg–Landau energy (2.65). The by a factor of $\frac{1}{\varepsilon}$ accelerated L^2 scalar product yields the Allen–Cahn equation

$$\partial_t \varphi_\varepsilon = \Delta \varphi_\varepsilon - \frac{1}{\varepsilon^2} W'(\varphi_\varepsilon), \tag{2.66}$$

whereas the H^{-1} scalar product gives rise to the Cahn–Hilliard equation

$$\partial_t \varphi_\varepsilon = \Delta u_\varepsilon, \quad u_\varepsilon = -\varepsilon \Delta \varphi_\varepsilon + \frac{1}{\varepsilon} W'(\varphi_\varepsilon). \tag{2.67}$$

The relation of the Ginzburg–Landau energy (2.65) with the sharp interface energy (2.1) is classical: it was shown by Modica and Mortola [117] and Modica [115] that the energy functional (2.65) converges as $\varepsilon \to 0$ in the precise sense of Γ-convergence to the energy functional (2.1) with surface tension

$$\sigma_W = \int_{-1}^{1} \sqrt{2W(r)}\, \mathrm{d}r. \tag{2.68}$$

The multiphase analogue of this statement is due to Baldo [16]. This in turn clearly motivates to study the convergence of the solutions to the underlying gradient flow equations. For instance, it was shown by Chen [33] and De Mottoni and Schatzman [46] that solutions to the Allen–Cahn equation (2.66) converge to strong solutions of two-phase mean curvature flow in arbitrary ambient dimension $d \geq 2$, assuming for the latter the existence of a (local-in-time) smooth solution starting from well-prepared initial data (essentially meaning that a diffuse interface of width $\sim \varepsilon$ has already developed). Contact point dynamics with a fixed-in-time $90°$ contact angle can be handled in the planar case $d = 2$ as was recently shown by Abels and Moser [7]. An extension to arbitrary ambient dimension $d \geq 2$ and fixed-in-time contact angle in a perturbative regime close to $90°$ can be found in the PhD thesis of Moser [119]. Convergence of solutions to the Cahn–Hilliard equation (2.67) to solutions of the Mullins–Sekerka problem was established by Alikakos, Bates and Chen [9] for arbitrary ambient dimension $d \geq 2$ but excluding contact points. Generalization to more complex phase field models are possible as well. For example, the case of a Stokes/Allen–Cahn system is treated by Abels and Liu [4], and the case of a Stokes/Cahn–Hilliard system in the very recent works of Abels and Marquardt [5] and [6] (all placed in the planar regime $d = 2$ without allowing boundary contact for the sharp interface in the limit).

Essentially all of the previously mentioned rigorous convergence results are facilitated by the principles of a well-established method due to De Mottoni and Schatzman [46] and Chen [34]: the combination of rigorous asymptotic expansions with a linear stability analysis for the Allen–Cahn or the Cahn–Hilliard operator, respectively. A completely different approach, however, was recently proposed by Fischer, Laux and Simon [69] in the simplest setting of the Allen–Cahn equation (2.66). For their derivation of optimal-order convergence rates towards strong solutions of two-phase mean curvature flow, they rely on a phase field analogue of the two-phase relative entropy method for sharp interface evolution problems as described above. The aim of the following discussion is to summarize their approach and to highlight parallels in the argument by adopting the general viewpoint from this chapter. To this end, we will derive the corresponding "phase field analogue" of (2.11) (cf. [69, Lemma 5]).

In terms of the underlying strong solution, we again fix a finite time horizon $T > 0$ and consider a "smoothly evolving" family of open and bounded sets $\bar{\Omega} = (\bar{\Omega}(t))_{t\in[0,T]}$ with "smoothly evolving" interfaces $\bar{I}(t) := \partial\bar{\Omega}(t)$ for all $t \in [0, T]$. Moreover, we assume that there exists a pair of vector fields (ξ, B) which is at least subject to the conditions (2.3)–(2.4) and (2.12)–(2.13). In order to define a phase field analogue of the two-phase relative entropy (2.2), we follow [69] and define for all $(x,t) \in \mathbb{R}^d \times [0,T]$

$$\psi_\varepsilon(x,t) := \int_{-1}^{\varphi_\varepsilon(x,t)} \sqrt{2W(r)}\,\mathrm{d}r, \tag{2.69}$$

which serves (up to a multiplicative factor of σ_W) as a proxy for the characteristic function of the phase $\bar{\Omega}(t)$ in each fixed time slice $t \in [0,T]$. We also introduce a normal n_ε by

$$\mathrm{n}_\varepsilon := \begin{cases} \frac{\nabla\varphi_\varepsilon}{|\nabla\varphi_\varepsilon|} & \text{if } \nabla\varphi_\varepsilon \neq 0, \\ \mathrm{s} & \text{else,} \end{cases} \tag{2.70}$$

with $\mathrm{s} \in \mathbb{S}^{d-1}$ a fixed but otherwise arbitrary unit vector. Note that because of the previous two definitions we always have

$$\mathrm{n}_\varepsilon|\nabla\varphi_\varepsilon| = \nabla\varphi_\varepsilon \quad \text{and} \quad \mathrm{n}_\varepsilon|\nabla\psi_\varepsilon| = \nabla\psi_\varepsilon. \tag{2.71}$$

Based on the Ginzburg–Landau energy (2.65), one may then define an error functional by means of

$$\begin{aligned} E_\varepsilon[\varphi_\varepsilon|\bar{\Omega}] &:= E_\varepsilon[\varphi_\varepsilon] - \int_{\mathbb{R}^d} \xi \cdot \nabla\psi_\varepsilon\,\mathrm{d}x \\ &= \int_{\mathbb{R}^d} \frac{1}{2}\Big(\sqrt{\varepsilon}|\nabla\varphi_\varepsilon| - \frac{1}{\sqrt{\varepsilon}}\sqrt{2W(\varphi_\varepsilon)}\Big)^2\,\mathrm{d}x + \int_{\mathbb{R}^d} (1 - \xi\cdot\mathrm{n}_\varepsilon)|\nabla\psi_\varepsilon|\,\mathrm{d}x. \end{aligned} \tag{2.72}$$

Note that the Modica–Mortola trick played the decisive role in order to proceed from the definition of $E_\varepsilon[\varphi_\varepsilon|\bar{\Omega}]$ to the alternative representation from the second line of the previous display. The latter in combination with (2.4) in turn implies the main coercivity properties of the error functional $E_\varepsilon[\varphi_\varepsilon|\bar{\Omega}]$, cf. [69, Lemma 4].

For a suitable representation of the time evolution of $E_\varepsilon[\varphi_\varepsilon|\bar{\Omega}]$, we first compute based on the definitions (2.72) and (2.69), the chain rule, as well as an integration by parts

$$\frac{\mathrm{d}}{\mathrm{d}t}E_\varepsilon[\varphi_\varepsilon|\bar{\Omega}] = \frac{\mathrm{d}}{\mathrm{d}t}E_\varepsilon[\varphi_\varepsilon] + \int (\nabla\cdot\xi)\sqrt{2W(\varphi_\varepsilon)}\partial_t\varphi_\varepsilon\,\mathrm{d}x - \int \nabla\psi_\varepsilon\cdot\partial_t\xi\,\mathrm{d}x.$$

We next rewrite the term involving the time derivative of the vector field ξ by appealing to the second identity of (2.71) and adding zero several times

$$\begin{aligned} -\int \nabla\psi_\varepsilon\cdot\partial_t\xi\,\mathrm{d}x &= -\int \mathrm{n}_\varepsilon\cdot\partial_t\xi\,|\nabla\psi_\varepsilon|\,\mathrm{d}x \\ &= -\int (\mathrm{n}_\varepsilon{-}\xi)\cdot\big(\partial_t\xi{+}(B\cdot\nabla)\xi{+}(\nabla B)^{\mathsf{T}}\xi\big)\,|\nabla\psi_\varepsilon|\,\mathrm{d}x \\ &\quad - \int \xi\cdot\big(\partial_t\xi{+}(B\cdot\nabla)\xi\big)\,|\nabla\psi_\varepsilon|\,\mathrm{d}x \\ &\quad + \int \xi\cdot\big((\mathrm{n}_\varepsilon{-}\xi)\cdot\nabla\big)B\,|\nabla\psi_\varepsilon|\,\mathrm{d}x \\ &\quad + \int \mathrm{n}_\varepsilon\cdot(B\cdot\nabla)\xi\,|\nabla\psi_\varepsilon|\,\mathrm{d}x. \end{aligned}$$

We further compute based on the second identity of (2.71) as well as adding zero several times

$$
\begin{aligned}
\int \mathrm{n}_\varepsilon \cdot (B\cdot\nabla)\xi\,|\nabla\psi_\varepsilon|\,\mathrm{d}x = \int \nabla\psi_\varepsilon \cdot \big(\nabla\cdot(\xi\otimes B)\big)\,\mathrm{d}x
&- \int (\mathrm{n}_\varepsilon\cdot\xi - 1)(\nabla\cdot B)\,|\nabla\psi_\varepsilon|\,\mathrm{d}x \\
&- \int (\mathrm{Id} - \mathrm{n}_\varepsilon\otimes\mathrm{n}_\varepsilon) : \nabla B\,|\nabla\psi_\varepsilon|\,\mathrm{d}x \\
&- \int \mathrm{n}_\varepsilon\otimes\mathrm{n}_\varepsilon : \nabla B\,|\nabla\psi_\varepsilon|\,\mathrm{d}x,
\end{aligned}
$$

and moreover by two integration by parts, the product rule, and again the the second identity of (2.71)

$$
\begin{aligned}
\int \nabla\psi_\varepsilon\cdot\big(\nabla\cdot(\xi\otimes B)\big)\,\mathrm{d}x &= -\int \psi_\varepsilon\nabla\cdot\big(\nabla\cdot(\xi\otimes B)\big)\,\mathrm{d}x \\
&= -\int \psi_\varepsilon\nabla\cdot\big(\nabla\cdot(B\otimes\xi)\big)\,\mathrm{d}x \\
&= \int \nabla\psi_\varepsilon\cdot\big(\nabla\cdot(B\otimes\xi)\big)\,\mathrm{d}x \\
&= \int \mathrm{n}_\varepsilon\cdot\big(\xi\cdot\nabla\big)B\,|\nabla\psi_\varepsilon|\,\mathrm{d}x + \int (\nabla\cdot\xi)(\mathrm{n}_\varepsilon\cdot B)\,|\nabla\psi_\varepsilon|\,\mathrm{d}x.
\end{aligned}
$$

Together with the first identity of (2.71) and $|\nabla\psi_\varepsilon| = \sqrt{2W(\varphi_\varepsilon)}|\nabla\varphi_\varepsilon|$, the previous four displays in combination imply

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}E_\varepsilon[\varphi_\varepsilon|\bar\Omega] = \frac{\mathrm{d}}{\mathrm{d}t}E_\varepsilon[\varphi_\varepsilon] &+ \int (\nabla\cdot\xi)\sqrt{2W(\varphi_\varepsilon)}\big(\partial_t\varphi_\varepsilon + (B\cdot\nabla)\varphi_\varepsilon\big)\,\mathrm{d}x \\
&- \int (\mathrm{Id} - \mathrm{n}_\varepsilon\otimes\mathrm{n}_\varepsilon) : \nabla B\,|\nabla\psi_\varepsilon|\,\mathrm{d}x \\
&- \int (\mathrm{n}_\varepsilon - \xi)\cdot\big(\partial_t\xi + (B\cdot\nabla)\xi + (\nabla B)^{\mathsf{T}}\xi\big)\,|\nabla\psi_\varepsilon|\,\mathrm{d}x \\
&- \int \xi\cdot\big(\partial_t\xi + (B\cdot\nabla)\xi\big)\,|\nabla\psi_\varepsilon|\,\mathrm{d}x \\
&- \int (\mathrm{n}_\varepsilon - \xi)\cdot\big((\mathrm{n}_\varepsilon - \xi)\cdot\nabla\big)B\,|\nabla\psi_\varepsilon|\,\mathrm{d}x \\
&- \int (\mathrm{n}_\varepsilon\cdot\xi - 1)(\nabla\cdot B)\,|\nabla\psi_\varepsilon|\,\mathrm{d}x,
\end{aligned} \tag{2.73}
$$

which is exactly the already mentioned phase field analogue of (2.11).

In the specific context of the Allen–Cahn equation (2.66), Fischer, Laux and Simon [69] proceed from (2.73) by a suitable post-processing of the first three right hand side terms of (2.73), cf. again [69, Lemma 5]. The derived stability estimate in terms of $E_\varepsilon[\varphi_\varepsilon|\bar\Omega]$ is then a key input for the derivation of a stability estimate of a "phase field version" of the Luckhaus–Sturzenhecker type error functional (2.22). More precisely, given a weight ϑ associated with the smoothly evolving phase $\bar\Omega$ and (at least) subject to the conditions (2.20)–(2.21) and (2.25), one may define

$$
E_{\mathrm{bulk}}[\psi_\varepsilon|\bar\Omega] := \int (\psi_\varepsilon - \sigma_W\bar\chi)\vartheta\,\mathrm{d}x. \tag{2.74}
$$

By the sign condition (2.21) and the definition (2.69), this again yields a non-negative functional provided $\psi_\varepsilon \in [0,\sigma_W]$, or equivalently $\varphi_\varepsilon \in [-1,1]$. The latter, however, can be ensured in the context of the Allen–Cahn equation (2.66) thanks to a maximum principle argument (provided the values of the initial phase field satisfy the same restriction, which we of course assume). With the definition (2.74) in place, let us briefly describe how to obtain the analogue of the representation (2.24). To this end, one first computes by means of the chain rule, the definition (2.69), and the fact that $\vartheta(\cdot,t) = 0$ along $\bar{I}(t)$ for all $t \in [0,T]$,

$$\frac{\mathrm{d}}{\mathrm{d}t}E_{\mathrm{bulk}}[\psi_\varepsilon|\bar{\Omega}] = \int \vartheta\sqrt{2W(\varphi_\varepsilon)}\partial_t\varphi_\varepsilon\,\mathrm{d}x + \int (\psi_\varepsilon - \sigma_W\bar{\chi})\partial_t\vartheta\,\mathrm{d}x.$$

Adding zero, applying the product rule, and integrating by parts (using again in the process that $\vartheta(\cdot,t) = 0$ along $\bar{I}(t)$ for all $t \in [0,T]$) furthermore yields

$$\begin{aligned}\int (\psi_\varepsilon - \sigma_W\bar{\chi})\partial_t\vartheta\,\mathrm{d}x &= \int (\psi_\varepsilon - \sigma_W\bar{\chi})\big(\partial_t\vartheta+(B\cdot\nabla)\vartheta\big)\,\mathrm{d}x \\ &\quad - \int (\psi_\varepsilon - \sigma_W\bar{\chi})\big(\nabla\cdot(B\vartheta)\big)\,\mathrm{d}x \\ &\quad + \int (\psi_\varepsilon - \sigma_W\bar{\chi})\vartheta(\nabla\cdot B)\,\mathrm{d}x \\ &= \int \vartheta\sqrt{2W(\varphi_\varepsilon)}(B\cdot\nabla)\varphi_\varepsilon\,\mathrm{d}x \\ &\quad + \int (\psi_\varepsilon - \sigma_W\bar{\chi})\big(\partial_t\vartheta+(B\cdot\nabla)\vartheta\big)\,\mathrm{d}x \\ &\quad + \int (\psi_\varepsilon - \sigma_W\bar{\chi})\vartheta(\nabla\cdot B)\,\mathrm{d}x,\end{aligned}$$

where for the precise representation of the first right hand side term of the second identity we also used the first identity of (2.71) and $|\nabla\psi_\varepsilon| = \sqrt{2W(\varphi_\varepsilon)}|\nabla\varphi_\varepsilon|$. The combination of the previous two displays finally entails the following analogue of (2.24)

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}E_{\mathrm{bulk}}[\psi_\varepsilon|\bar{\Omega}] &= \int \vartheta\sqrt{2W(\varphi_\varepsilon)}\big(\partial_t\varphi_\varepsilon+(B\cdot\nabla)\varphi_\varepsilon\big)\,\mathrm{d}x + \int (\psi_\varepsilon - \sigma_W\bar{\chi})\big(\partial_t\vartheta+(B\cdot\nabla)\vartheta\big)\,\mathrm{d}x \\ &\quad + \int (\psi_\varepsilon - \sigma_W\bar{\chi})\vartheta(\nabla\cdot B)\,\mathrm{d}x. \end{aligned} \tag{2.75}$$

In the specific context of the Allen–Cahn equation (2.66), Fischer, Laux and Simon [69] suitably post-process the first right hand side term of the previous display in order to derive by a Gronwall argument a stability estimate for the error functional $E_{\mathrm{bulk}}[\psi_\varepsilon|\bar{\Omega}]$, cf. [69, Step 2, Proof of Theorem 1]. We stress again that their argument requires to appeal to an already established stability estimate for $E_\varepsilon[\varphi_\varepsilon|\bar{\Omega}]$, as expected.

As already noted by Fischer, Laux and Simon [69], the above outlined alternative approach to convergence rates of diffuse interface approximations neither relies on the comparison principle (in an essential way) nor on asymptotic expansions techniques and a linear stability analysis of the Allen–Cahn operator. This may raise the hope that their approach can also be successfully applied to more general or different types of phase field models. We conclude this subsection by providing a short list of examples in this direction:

- We start by mentioning the recent result of Laux and Liu [97], who employ, amongst other techniques, the principles of the above outlined approach in their study of nematic-isotropic phase transitions in the context of Landau–De Gennes theory of liquid crystals.

- It is by no means trivial to extend the techniques of Fischer, Laux and Simon [69] to the setting of the vector-valued Allen–Cahn problem. This is highly relevant since the sharp interface limit is given by *multiphase* mean curvature flow. We already mentioned in this context the work of Baldo [16] proving Γ-convergence of the associated energy functionals. A formal convergence result employing formally matched asymptotic expansions is due to Bronsard and Reitich [25], whereas a rigorous (qualitative) convergence result (under an energy convergence assumption) towards the BV formulation of multiphase mean curvature flow is the content of the work of Laux and Simon [101].

To the best of the author's knowledge, there is currently no rigorous convergence result available in the literature establishing convergence rates towards a strong solution of multiphase mean curvature flow; at least in settings which allow for the occurrence of triple junctions in the sharp interface limit (otherwise, the reader may consult the PhD thesis of Moser [119]). It would be interesting to see whether some of the principles and ideas of our multiphase relative entropy approach could prove helpful in the investigation of this open problem.

- In his PhD thesis, Moser [119] established for the first time a rigorous convergence result in the $\varepsilon \to 0$ limit for the Allen–Cahn equation with non-linear Robin boundary condition towards two-phase mean curvature flow in a bounded domain, for which a fixed-in-time contact angle is prescribed at points where the sharp interface intersects the boundary of the domain. A formal result based on formally matched asymptotic expansions is due to Owen and Sternberg [123], whereas the Γ-convergence result for the underlying energy functionals is due to Modica [116].

  The results of Moser [119] hold true in ambient spatial dimension $d = 2$ on the time interval of existence of a strong solution of the sharp interface limit model (with well-prepared initial data). Moreover, his arguments are based on the classical approach due to De Mottoni and Schatzman [46], and so far are "limited" to a perturbative regime around the case of a fixed-in-time 90° contact angle. It is an intriguing question whether one can perform a suitable extension of the approach by Fischer, Laux and Simon [69] in order to derive convergence rates in a non-perturbative regime for the contact angle (possibly in a first step under rather restrictive assumptions on the structure of the boundary contact energy at the level of the diffuse interface approximation, cf. the setting of Moser [119, Section 1.3]). An investigation of this problem will be the subject of future work.

- Let us next consider the example of Navier–Stokes/Allen–Cahn systems in $\mathbb{R}^d$, where $d \in \{2,3\}$, in their simplest form given by

$$
\begin{aligned}
\partial_t v_\varepsilon + (v_\varepsilon \cdot \nabla) v_\varepsilon &= \Delta v_\varepsilon - \nabla p_\varepsilon - \nabla \cdot \big(\mathrm{n}_\varepsilon \otimes \mathrm{n}_\varepsilon\, \varepsilon |\nabla \varphi_\varepsilon|^2\big) && \text{in } \mathbb{R}^d \times (0,T),\\
\nabla \cdot v_\varepsilon &= 0 && \text{in } \mathbb{R}^d \times (0,T),\\
\partial_t \varphi_\varepsilon + (v_\varepsilon \cdot \nabla)\varphi_\varepsilon &= m_0 \varepsilon^\theta \Big(\Delta \varphi_\varepsilon - \frac{1}{\varepsilon^2} W'(\varphi_\varepsilon)\Big) && \text{in } \mathbb{R}^d \times (0,T),
\end{aligned}
$$

  with mobility constant $m_0 > 0$ and exponent $\theta \in \{0,1\}$. Formally matched asymptotic expansions suggest convergence of the above diffuse interface models to a two-phase Navier–Stokes problem with sharp interface. More precisely, and focusing only on the equation for the normal velocity vector $\mathrm{V}_{\bar I}$ of the sharp interface $\bar I$ in the limit, one formally obtains in the case of $\theta = 0$

$$
\mathrm{V}_{\bar I} = (\bar{\mathrm{n}} \cdot v)\bar{\mathrm{n}} + m_0 \mathrm{H}_{\bar I} \quad \text{on } \bar I, \tag{2.76}
$$

  whereas in the case of $\theta = 1$ one obtains pure transport along the fluid flow

$$
\mathrm{V}_{\bar I} = (\bar{\mathrm{n}} \cdot v)\bar{\mathrm{n}} \quad \text{on } \bar I. \tag{2.77}
$$

In the former case of (2.76), there is the already mentioned work of Abels and Liu [4] proving a rigorous convergence result (including convergence rates in strong norms) in a simplified setting where the full Navier–Stokes system is replaced by the (quasi-stationary) Stokes system. Even in this setting, their analysis requires substantial efforts and is again based on methods in the spirit of the work of De Mottoni and Schatzman [46]. At the time of this writing, a rigorous convergence result (in either case of $\theta = 0$ or $\theta = 1$) addressing the full Navier–Stokes system remains an open issue.

- We finally mention that any rigorous convergence result based on a relative entropy technique à la Fischer, Laux and Simon [69] establishing convergence rates for diffuse interface models incorporating the Cahn–Hilliard equation (2.67) would be of interest.

CHAPTER 3

Weak-strong uniqueness for two-phase Navier–Stokes flow

Abstract. We consider the evolution of two fluids separated by a sharp interface in the presence of surface tension – like, for example, the evolution of oil bubbles in water. Our main result is a weak-strong uniqueness principle for the corresponding free boundary problem for the incompressible Navier-Stokes equation: As long as a strong solution exists, any varifold solution must coincide with it. In particular, in the absence of physical singularities the concept of varifold solutions – whose global in time existence has been shown by Abels [1] for general initial data – does not introduce a mechanism for non-uniqueness. The key ingredient of our approach is the construction of a relative entropy functional capable of controlling the interface error. If the viscosities of the two fluids do not coincide, even for classical (strong) solutions the gradient of the velocity field becomes discontinuous at the interface, introducing the need for a careful additional adaption of the relative entropy.

3.1 Main results & definitions

The main result of the present work is the derivation of a weak-strong uniqueness principle for varifold solutions to the free boundary problem for the Navier–Stokes equation for two immiscible incompressible fluids with surface tension: As long as a strong solution to the free boundary problem (1.1a)-(1.1c) exists, any varifold solution must coincide with it. In particular, the concept of varifold solutions developed by Abels [1] (see Definition 3.2 below for a precise definition) does not introduce an additional mechanism for non-uniqueness, at least as long as a classical solution exists. At the same time, the concept of varifold solutions of Abels allows for the construction of globally existing solutions [1], while any concept of strong solutions is limited to the absence of geometric singularities and therefore – at least in three spatial dimensions $d = 3$ – to short-time existence results.

Furthermore, we prove a quantitative stability result (3.1) for varifold solutions with respect to changes in the data: As long as a classical solution exists, any varifold solution with slightly perturbed initial data remains close to it.

Theorem 3.1 (Weak-strong uniqueness principle). *Let $d \in \{2,3\}$. Let (χ_u, u, V) be a varifold solution to the free boundary problem for the incompressible Navier–Stokes equation for two fluids (1.1a)-(1.1c) in the sense of Definition 3.2 on some time interval $[0, T_{vari})$. Let (χ_v, v) be a strong solution to (1.1a)-(1.1c) in the sense of Definition 3.6 on some time interval $[0, T_{strong})$ with $T_{strong} \leq T_{vari}$. Let the relative entropy $E[\chi_u, u, V|\chi_v, v](t)$ be defined as in Proposition 3.10.*

Then there exist constants $C, c > 0$ such that the stability estimate

$$E\big[\chi_u, u, V\big|\chi_v, v\big](T) \leq C(E[\chi_u, u, V|\chi_v, v](0))^{e^{-CT}} \tag{3.1}$$

holds for almost every $T \in [0, T_{strong})$, provided that the initial relative entropy satisfies $E[\chi_u, u, V|\chi_v, v](0) \leq c$. The constants $c > 0$ and $C > 0$ depend only on the data and the strong solution.

In particular, if the initial data of the varifold solution and the strong solution coincide, the varifold solution must be equal to the strong solution in the sense that

$$\chi_u(\cdot, t) = \chi_v(\cdot, t) \qquad \text{and} \qquad u(\cdot, t) = v(\cdot, t)$$

hold almost everywhere for almost every $t \in [0, T_{strong})$, and the varifold is given for almost every $t \in [0, T_{strong})$ by

$$\mathrm{d}V_t = \delta_{\frac{\nabla \chi_v}{|\nabla \chi_v|}} \mathrm{d}|\nabla \chi_v|.$$

We emphasize that our main result in Theorem 3.1 remains valid if we allow for a density-dependent bulk force like gravity, i.e., if we add a term of the form $\rho(\chi)\mathrm{g}$ on the right-hand side of (1.1b). Details are provided in Remark 3.35.

The following notion of varifold solutions for the free boundary problem associated with the flow of two immiscible incompressible viscous fluids with surface tension has been introduced by Abels [1]. For Newtonian fluids, the global-in-time existence of such varifold solutions has been proven for quite general initial data in two and three spatial dimensions in [1]. For the notion of an oriented varifold, see the section on notation just prior to Section 3.2.

Definition 3.2 (Varifold solution for the two-phase Navier–Stokes equation). *Let a surface tension constant $\sigma > 0$, the densities and shear viscosities of the two fluids $\rho^{\pm}$, $\mu^{\pm} > 0$, a finite time $T_{vari} > 0$, a solenoidal initial velocity profile $v_0 \in L^2(\mathbb{R}^d; \mathbb{R}^d)$, and an indicator function of the volume occupied initially by the first fluid $\chi_0 \in \mathrm{BV}(\mathbb{R}^d)$ be given.*

A triple (χ, v, V) consisting of a velocity field v, an indicator function χ of the volume occupied by the first fluid, and an oriented varifold V with

$$\begin{aligned}
&v \in L^2([0, T_{vari}]; H^1(\mathbb{R}^d; \mathbb{R}^d)) \cap L^\infty([0, T_{vari}]; L^2(\mathbb{R}^d; \mathbb{R}^d)),\\
&\chi \in L^\infty([0, T_{vari}]; \mathrm{BV}(\mathbb{R}^d; \{0, 1\})),\\
&V \in L^\infty_w([0, T_{vari}]; \mathcal{M}(\mathbb{R}^d \times \mathbb{S}^{d-1})),
\end{aligned}$$

is called a varifold solution *to the free boundary problem for the Navier-Stokes equation for two fluids with initial data (χ_0, v_0) if the following conditions are satisfied:*

i) The velocity field v has vanishing divergence $\nabla\cdot v = 0$ and the equation for the momentum balance

$$\begin{aligned}\int_{\mathbb{R}^d}&\rho(\chi(\cdot,T))v(\cdot,T)\cdot\eta(\cdot,T)\,\mathrm{d}x - \int_{\mathbb{R}^d}\rho(\chi_0)v_0\cdot\eta(\cdot,0)\,\mathrm{d}x\\ &= \int_0^T\int_{\mathbb{R}^d}\rho(\chi)v\cdot\partial_t\eta\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_{\mathbb{R}^d}\rho(\chi)v\otimes v:\nabla\eta\,\mathrm{d}x\,\mathrm{d}t\\ &\quad-\int_0^T\int_{\mathbb{R}^d}\mu(\chi)(\nabla v+\nabla v^T):\nabla\eta\,\mathrm{d}x\,\mathrm{d}t\\ &\quad-\sigma\int_0^T\int_{\mathbb{R}^d\times\mathbb{S}^{d-1}}(\mathrm{Id}-s\otimes s):\nabla\eta\,\mathrm{d}V_t(x,s)\,\mathrm{d}t\end{aligned}\tag{3.2a}$$

is satisfied for almost every $T\in[0,T_{vari})$ and every smooth vector field $\eta\in C^\infty_{cpt}(\mathbb{R}^d\times[0,T_{vari});\mathbb{R}^d)$ with $\nabla\cdot\eta=0$. For the sake of brevity, we have used the abbreviations $\rho(\chi):=\rho^+\chi+\rho^-(1-\chi)$ and $\mu(\chi):=\mu^+\chi+\mu^-(1-\chi)$.

ii) The indicator function χ of the volume occupied by the first fluid satisfies the weak formulation of the transport equation

$$\int_{\mathbb{R}^d}\chi(\cdot,T)\varphi(\cdot,T)\,\mathrm{d}x-\int_{\mathbb{R}^d}\chi_0\varphi(\cdot,0)\,\mathrm{d}x=\int_0^T\int_{\mathbb{R}^d}\chi\left(\partial_t\varphi+(v\cdot\nabla)\varphi\right)\mathrm{d}x\,\mathrm{d}t\tag{3.2b}$$

for almost every $T\in[0,T_{vari})$ and all $\varphi\in C^\infty_{cpt}(\mathbb{R}^d\times[0,T_{vari}))$.

iii) The energy dissipation inequality

$$\begin{aligned}\int_{\mathbb{R}^d}&\frac{1}{2}\rho(\chi(\cdot,T))|v(\cdot,T)|^2\,\mathrm{d}x+\sigma|V_T|(\mathbb{R}^d\times\mathbb{S}^{d-1})\\ &+\int_0^T\int_{\mathbb{R}^d}\frac{\mu(\chi)}{2}\left|\nabla v+\nabla v^T\right|^2\mathrm{d}x\,\mathrm{d}t\\ &\leq\int_{\mathbb{R}^d}\frac{1}{2}\rho(\chi_0(\cdot))|v_0(\cdot)|^2\,\mathrm{d}x+\sigma|\nabla\chi_0(\cdot)|(\mathbb{R}^d)\end{aligned}\tag{3.2c}$$

is satisfied for almost every $T\in[0,T_{vari})$, and the energy

$$E[\chi,v,V](t):=\int_{\mathbb{R}^d}\frac{1}{2}\rho(\chi(\cdot,t))|v(\cdot,t)|^2\,\mathrm{d}x+\sigma|V_t|(\mathbb{R}^d\times\mathbb{S}^{d-1})\tag{3.2d}$$

is a nonincreasing function of time.

iv) The phase boundary $\partial\{\chi(\cdot,t)=0\}$ and the varifold V satisfy the compatibility condition

$$\int_{\mathbb{R}^d\times\mathbb{S}^{d-1}}\psi(x)s\,\mathrm{d}V_t(x,s)=\int_{\mathbb{R}^d}\psi(x)\,\mathrm{d}\nabla\chi(x)\tag{3.2e}$$

for almost every $T\in[0,T_{vari})$ and every smooth function $\psi\in C^\infty_{cpt}(\mathbb{R}^d)$.

Let us continue with a few comments on the relation between the varifold V_t and the interface described by the indicator function $\chi(\cdot,t)$.

Remark 3.3. *Let $V_t\in\mathcal{M}(\mathbb{R}^d\times\mathbb{S}^{d-1})$ denote the non-negative measure representing (at time t) the varifold associated to a varifold solution (χ,v,V) to the free boundary problem for the incompressible Navier-Stokes equation for two fluids. The compatibility condition* (3.2e) *entails that* $|\nabla\chi_u(t)|$ *is absolutely continuous with respect to* $|V_t|_{\mathbb{S}^{d-1}}$*. Hence, we may define the Radon–Nikodym derivative*

$$\theta_t := \frac{\mathrm{d}|\nabla\chi_u(t)|}{\mathrm{d}|V_t|_{\mathbb{S}^{d-1}}}, \tag{3.3}$$

which is a $|V_t|_{\mathbb{S}^{d-1}}$*-measurable function with* $|\theta_t(x)| \leq 1$ *for* $|V_t|_{\mathbb{S}^{d-1}}$*-almost every* $x \in \mathbb{R}^d$*. In particular, we have*

$$\int_{\mathbb{R}^d} f(x)\,\mathrm{d}|\nabla\chi(\cdot,t)|(x) = \int_{\mathbb{R}^d} \theta_t(x) f(x)\,\mathrm{d}|V_t|_{\mathbb{S}^{d-1}}(x) \tag{3.4}$$

for every $f \in L^1(\mathbb{R}^d, |\nabla\chi(\cdot,t)|)$ *and almost every* $t \in [0, T_{vari})$.

The compatibility condition between the varifold V_t and the interface described by the indicator function $\chi(\cdot,t)$ has the following consequence.

Remark 3.4. *Consider a varifold solution* (χ, v, V) *to the free boundary problem for the incompressible Navier-Stokes equation for two fluids. Let* E_t *be the measurable set* $\{x \in \mathbb{R}^d\colon \chi(x,t) = 1\}$*. Note that for almost every* $t \in [0, T_{vari})$ *this set is then a Caccioppoli set in* $\mathbb{R}^d$*. Let* $\mathrm{n}(\cdot,t) = \frac{\nabla\chi}{|\nabla\chi|}$ *denote the measure theoretic unit normal vector field on the reduced boundary* $\partial^* E_t$*. By means of the compatibility condition (3.2e) and the definition (3.3) we obtain*

$$\frac{\mathrm{d}\int_{\mathbb{S}^{d-1}} s\,\mathrm{d}V_t(\cdot,s)}{\mathrm{d}|V_t|_{\mathbb{S}^{d-1}}(\cdot)} = \begin{cases} \theta_t(x)\mathrm{n}(x,t) & \text{for } x \in \partial^* E_t, \\ 0 & \text{else,} \end{cases} \tag{3.5}$$

for almost every $t \in [0, T_{vari})$ *and* $|V_t|_{\mathbb{S}^{d-1}}$*-almost every* $x \in \mathbb{R}^d$.

In order to define a notion of strong solutions to the free boundary problem for the flow of two immiscible fluids, let us first define a notion of smoothly evolving domains.

Definition 3.5 (Smoothly evolving domains and surfaces)**.** *Let* Ω_0^+ *be a bounded domain of class* C^3 *and consider a family* $(\Omega_t^+)_{t\in[0,T_{strong})}$ *of open sets in* $\mathbb{R}^d$*. Let* $I(t) = \partial\Omega_t^+$ *and* $\Omega_t^- = \mathbb{R}^d \setminus (\Omega_t^+ \cup I(t))$ *for every* $t \in [0, T_{strong}]$*.*

We say that Ω_t^+*,* Ω_t^- *are* smoothly evolving domains *and that* $I(t)$ *are* smoothly evolving surfaces *if we have* $\Omega_t^+ = \Psi^t(\Omega_0^+)$*,* $\Omega_t^- = \Psi^t(\Omega_0^-)$*, and* $I(t) = \Psi^t(I(0))$ *for a map* $\Psi\colon \mathbb{R}^d \times [0, T_{strong}) \to \mathbb{R}^d$*,* $(x,t) \mapsto \Psi(x,t) = \Psi^t(x)$*, subject to the following conditions:*

i) We have $\Psi^0 = \mathrm{Id}$.

ii) For any fixed $t \in [0, T_{strong})$*, the map* $\Psi^t\colon \mathbb{R}^d \to \mathbb{R}^d$ *is a* C^3*-diffeomorphism. Moreover, we assume* $\|\Psi\|_{L_t^\infty W_x^{3,\infty}} < \infty$.

iii) We have $\partial_t\Psi \in C^0([0, T_{strong}); C^2(\mathbb{R}^d;\mathbb{R}^d))$ *and* $\|\partial_t\Psi\|_{L_t^\infty W_x^{2,\infty}} < \infty$.

Moreover, we assume that there exists $r_c \in (0, \frac{1}{2}]$ *with the following property: For all* $t \in [0, T_{strong})$ *and all* $x \in I(t)$ *there exists a function* $g\colon B_1(0) \subset \mathbb{R}^{d-1} \to \mathbb{R}$ *with* $\nabla g(0) = 0$ *such that after a rotation and a translation,* $I(t) \cap B_{2r_c}(x)$ *is given by the graph* $\{(x, g(x))\colon x \in \mathbb{R}^{d-1}\}$*. Furthermore, for any of these functions* g *the pointwise bounds* $|\nabla^m g| \leq r_c^{-(m-1)}$ *hold for all* $1 \leq m \leq 3$.

We have everything in place to give the definition of a strong solution to the free boundary problem for the Navier–Stokes equation for two fluids. For a discussion of the conditions (3.6a)–(3.6c) we refer to Remark 3.36 in the “Appendix”.

Definition 3.6 (Strong solution for the two-phase Navier–Stokes equation)**.** *Let a surface tension constant $\sigma > 0$, the densities and shear viscosities of the two fluids $\rho^{\pm}, \mu^{\pm} > 0$, a finite time horizon $T_{strong} > 0$, a domain Ω_0^+ occupied initially by the first fluid with interface $I_v(0) := \partial\Omega_0^+$, and an initial velocity profile v_0 be given which are subject to the following regularity and compatibility conditions:*

$$v_0 \in W^{2-\frac{2}{q},q}(\mathbb{R}^d \setminus I_v(0)) \text{ for some } q > d+2, \sup_{\mathbb{R}^d \setminus I_v(0)} |v_0| + |\nabla v_0| < \infty, \tag{3.6a}$$

$$[[v_0]] = 0 \text{ on } I_v(0), \quad \nabla \cdot v_0 = 0 \text{ in } \mathbb{R}^d, \tag{3.6b}$$

$$(\mathrm{Id} - \mathrm{n}_{I_v(0)} \otimes \mathrm{n}_{I_v(0)})[[\mu(\chi_0)(\nabla v_0 + \nabla v_0^T)]]\mathrm{n}_{I_v(0)} = 0 \text{ on } I_v(0). \tag{3.6c}$$

Let the initial interface between the fluids $I_v(0)$ be a compact C^3-manifold.

A pair (χ, v) consisting of a velocity field v and an indicator function χ of the volume occupied by the first fluid with

$$\begin{aligned}
&v \in H^1([0,T_{strong}];L^2(\mathbb{R}^d;\mathbb{R}^d)) \cap L^\infty([0,T_{strong}];H^1(\mathbb{R}^d;\mathbb{R}^d)),\\
&\nabla v \in L^1([0,T_{strong}];\mathrm{BV}(\mathbb{R}^d;\mathbb{R}^{d\times d})),\\
&\chi \in L^\infty([0,T_{strong}];\mathrm{BV}(\mathbb{R}^d;\{0,1\})),
\end{aligned}$$

is called a strong solution *to the free boundary problem for the Navier–Stokes equation for two fluids with initial data (χ_0, v_0) if the volume occupied by the first fluid $\Omega_t^+ := \{x \in \mathbb{R}^d : \chi(x,t) = 1\}$ is a smoothly evolving domain and the interface $I_v(t) := \partial\Omega_t^+$ is a smoothly evolving surface in the sense of Definition 3.5, and if additionally the following conditions are satisfied:*

i) The velocity field v has vanishing divergence $\nabla \cdot v = 0$ and the equation for the momentum balance

$$\begin{aligned}
&\int_{\mathbb{R}^d} \rho(\chi(\cdot,T))v(\cdot,T)\cdot\eta(\cdot,T)\,\mathrm{d}x - \int_{\mathbb{R}^d} \rho(\chi_0)v_0\cdot\eta(\cdot,0)\,\mathrm{d}x\\
&\quad = \int_0^T\int_{\mathbb{R}^d} \rho(\chi)v\cdot\partial_t\eta\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_{\mathbb{R}^d} \rho(\chi)v\otimes v : \nabla\eta\,\mathrm{d}x\,\mathrm{d}t\\
&\qquad - \int_0^T\int_{\mathbb{R}^d} \mu(\chi)(\nabla v + \nabla v^T) : \nabla\eta\,\mathrm{d}x\,\mathrm{d}t\\
&\qquad + \sigma\int_0^T\int_{I_v(t)} \mathrm{H}\cdot\eta\,dS\,\mathrm{d}t
\end{aligned} \tag{3.7a}$$

is satisfied for almost every $T \in [0, T_{strong})$ and every smooth vector field $\eta \in C^\infty_{cpt}(\mathbb{R}^d \times [0,T_{strong});\mathbb{R}^d)$ with $\nabla\cdot\eta = 0$. Here, H denotes the mean curvature vector of the interface $I_v(t)$. For the sake of brevity, we have used the abbreviations $\rho(\chi) := \rho^+\chi + \rho^-(1-\chi)$ and $\mu(\chi) := \mu^+\chi + \mu^-(1-\chi)$.

ii) The indicator function χ of the volume occupied by the first fluid satisfies the weak formulation of the transport equation

$$\int_{\mathbb{R}^d} \chi(\cdot,T)\varphi(\cdot,T)\,\mathrm{d}x - \int_{\mathbb{R}^d} \chi_0\varphi(\cdot,0)\,\mathrm{d}x = \int_0^T\int_{\mathbb{R}^d} \chi\,(\partial_t\varphi + (v\cdot\nabla)\varphi)\,\mathrm{d}x\,\mathrm{d}t \tag{3.7b}$$

for almost every $T \in [0,T_{strong}]$ and all $\varphi \in C^\infty_{cpt}(\mathbb{R}^d \times [0,T_{strong}))$.

iii) In the set $\bigcup_{t\in[0,T_{strong}]}(\Omega_t^+\cup\Omega_t^-)\times\{t\}$ *all spatial derivatives up to third order, the time derivative* $\partial_t v$*, as well as the mixed derivative* $\partial_t\nabla v$ *of the velocity field exist, and they satisfy the estimate*

$$\sup_{t\in[0,T_{strong}]}\sup_{x\in\Omega_t^+\cup\Omega_t^-}\sup_{k\in\{0,1,2,3\}}|\nabla^k v(x,t)|+|\partial_t v(x,t)|+|\partial_t\nabla v(x,t)|<\infty. \tag{3.7c}$$

We continue with a remark on the existence of strong solutions in the functional framework of the previous definition.

Remark 3.7. *Local-in-time existence of such strong solutions (starting with smooth initial data subject to the above compatibility conditions) is essentially shown in [89, Theorem 2], up to two details: The authors only consider the system (1.1) in a bounded domain (instead of* $\mathbb{R}^d$*), and they do not state smoothness up to initial time with the corresponding bound (3.7c). The former restriction is just a technicality and the methods extend to unbounded domains, see [126]. The regularity up to initial time with the corresponding bound (3.7c), on the other hand, can be deduced by regularity theory, using the transformed formulation of the problem in [89]; this however requires higher-order regularity and compatibility conditions for the initial data in the following sense. Let* p_0 *be an initial pressure field. Then we assume that*

$$v_0\in W^{4-\frac{2}{q},q}(\mathbb{R}^d\setminus I_v(0)) \text{ for some } q>d+2,\ \sup_{k\in\{0,1,2,3\}}\sup_{\mathbb{R}^d\setminus I_v(0)}|\nabla^k v_0|<\infty, \tag{3.8a}$$

$$\mathrm{n}_{I_v(0)}[[\mu(\chi_0)(\nabla v_0+\nabla v_0^T)-p_0\mathrm{Id}]]\mathrm{n}_{I_v(0)}=\sigma\mathrm{H}(0)\cdot\mathrm{n}_{I_v(0)} \text{ on } I_v(0). \tag{3.8b}$$

$$[[\rho(\chi_0)^{-1}(\mu(\chi_0)\Delta v_0-\nabla p_0)]]=0 \text{ on } I_v(0), \tag{3.8c}$$

$$\nabla\cdot G_0=0 \text{ in } \mathbb{R}^d\setminus I_v(0) \tag{3.8d}$$

$$\text{for } G_0:=\rho(\chi_0)^{-1}\big(\mu(\chi_0)\Delta v_0-\rho(\chi_0)\big((\mathrm{Id}-\mathrm{n}_{I_v(0)}\otimes\mathrm{n}_{I_v(0)})v_0\cdot\nabla\big)v_0-\nabla p_0\big)$$

$$(\mathrm{Id}-\mathrm{n}_{I_v(0)}\otimes\mathrm{n}_{I_v(0)})[[\mu(\chi_0)(\nabla G_0+\nabla G_0^T)]]\mathrm{n}_{I_v(0)}=0 \text{ on } I_v(0). \tag{3.8e}$$

We refer to Remark 3.36 in the "Appendix" for a discussion of these conditions; we also give a brief discussion concerning the existence of strong solutions in the precise functional framework of Definition 3.6 under these additional regularity and compatibility conditions in Remark 3.37 in the "Appendix".

Before we state the main ingredient for the proof of Theorem 3.1, we proceed with two further remarks on the notion of strong solutions. The first concerns the consistency with the notion of varifold solutions due to Abels [1].

Remark 3.8. *Every strong solution* (χ,v) *to the free boundary problem for the incompressible Navier–Stokes equation for two fluids (1.1a)-(1.1c) in the sense of Definition 3.6 canonically defines a varifold solution in the sense of Definition 3.2. Indeed, we can define the varifold* V *by means of* $\mathrm{d}V_t=\delta_{\frac{\nabla\chi}{|\nabla\chi|}}\mathrm{d}|\nabla\chi|$*. Due to the regularity requirements on the family of smoothly evolving surfaces* $I(t)$*, see Definition 3.5, it then follows*

$$\begin{aligned}\int_0^T\int_{I(t)}\mathrm{H}\cdot\varphi\,\mathrm{d}S\,\mathrm{d}t&=-\int_0^T\int_{\mathbb{R}^d}(\mathrm{Id}-n\otimes n):\nabla\varphi\,\mathrm{d}|\nabla\chi(\cdot,t)|\,\mathrm{d}t\\&=-\int_0^T\int_{\mathbb{R}^d\times\mathbb{S}^{d-1}}(\mathrm{Id}-s\otimes s):\nabla\varphi\,\mathrm{d}V_t(x,s)\,\mathrm{d}t,\end{aligned}$$

for almost every $T\in[0,T_{strong})$ *and all* $\varphi\in C^\infty_{cpt}(\mathbb{R}^d\times[0,T_{vari});\mathbb{R}^d)$*, see for instance [3, Lemma 3.4]. Moreover, it follows from the regularity requirements of a strong solution that the velocity field* v *also satisfies the energy dissipation inequality (3.2c). This proves the claim.*

The second remark concerns the validity of the kinematic condition of the interface being transported with the fluid flow in its strong formulation.

Remark 3.9. *Let (χ, v) be a strong solution to the free boundary problem for the incompressible Navier–Stokes equation for two fluids (1.1a)-(1.1c) in the sense of Definition 3.6 on some time interval $[0, T_{strong})$. Let $V_{\mathrm{n}}(x,t)$ denote the normal speed of the interface at $x \in I_v(t)$, i.e., the normal component of $\partial_t \Psi(x,t)$ where $\Psi\colon \mathbb{R}^d \times [0,T_{strong}) \to \mathbb{R}^d$ is the family of diffeomorphisms from the definition of a family of smoothly evolving domains (Definition 3.5). Furthermore, let $\varphi \in C^\infty_{cpt}(\mathbb{R}^d \times (0,T_{strong}))$. Due to the regularity requirements on a family of smoothly evolving domains, see Definition 3.5, we obtain (see for instance [3, Theorem 2.6])*

$$\int_0^{T_{strong}} \int_{\mathbb{R}^d} \chi \partial_t \varphi \,\mathrm{d}x \,\mathrm{d}t = -\int_0^{T_{strong}} \int_{I_v(t)} V_{\mathrm{n}} \varphi \,\mathrm{d}S \,\mathrm{d}t.$$

On the other side, subtracting from the former identity the equation (3.7b) satisfied by the indicator function χ and making use of the incompressibility of the velocity field v we deduce

$$\int_0^{T_{strong}} \int_{I_v(t)} (V_{\mathrm{n}} - \mathrm{n} \cdot v) \varphi \,\mathrm{d}S \,\mathrm{d}t = 0.$$

Since $\varphi \in C^\infty_{cpt}(\mathbb{R}^d \times (0,T_{strong}))$ was arbitrary we recover the identity

$$V_{\mathrm{n}} = \mathrm{n} \cdot v \quad \text{on} \quad \bigcup_{t \in (0,T_{strong})} I_v(t) \times \{t\},$$

that is to say, the kinematic condition of the interface being transported with the flow is satisfied in its strong formulation.

Our weak-strong uniqueness result in Theorem 3.1 relies on the following relative entropy inequality. The regime of equal shear viscosities $\mu_+ = \mu_-$ corresponds to the choice of $w = 0$ in the statement below. Note also that in this case the viscous stress term R_{visc} disappears due to $\mu(\chi_u) - \mu(\chi_v) = 0$.

Proposition 3.10 (Relative entropy inequality)**.** *Let $d \leq 3$. Let (χ_u, u, V) be a varifold solution to the free boundary problem for the incompressible Navier–Stokes equation for two fluids (1.1a)-(1.1c) in the sense of Definition 3.2 on some time interval $[0, T_{vari})$. Let (χ_v, v) be a strong solution to (1.1a)-(1.1c) in the sense of Definition 3.6 on some time interval $[0, T_{strong})$ with $T_{strong} \leq T_{vari}$ and let*

$$w \in L^2([0,T_{strong}); H^1(\mathbb{R}^d;\mathbb{R}^d)) \cap H^1([0,T_{strong}); L^{4/3}(\mathbb{R}^d;\mathbb{R}^d) + L^2(\mathbb{R}^d;\mathbb{R}^d))$$

be a solenoidal vector field with bounded spatial derivative $\|\nabla w\|_{L^\infty} < \infty$. Suppose furthermore that for almost every $t \geq 0$, for every $x \in \mathbb{R}^d$ either x is a Lebesgue point of $\nabla w(\cdot,t)$ or there exists a half-space H_x such that x is a Lebesgue point for both $\nabla w(\cdot,t)|_{H_x}$ and $\nabla w(\cdot,t)|_{\mathbb{R}^d \setminus H_x}$.

For a point (x,t) such that $\mathrm{dist}(x, I_v(t)) < r_c$, denote by $P_{I_v(t)}x$ the projection of x onto the interface $I_v(t)$ of the strong solution. Introduce the extension ξ of the unit normal n_v of the interface of the strong solution defined by

$$\xi(x,t) := \mathrm{n}_v(P_{I_v(t)}x)(1 - \mathrm{dist}(x, I_v(t))^2)\eta(\mathrm{dist}(x, I_v(t)))$$

for some cutoff η with $\eta(s) = 1$ for $s \leq \frac{1}{2} r_c$ and $\eta \equiv 0$ for $s \geq r_c$. Let

$$\bar{V}_{\mathrm{n}}(x,t) := (\mathrm{n}(P_{I_v(t)}x,t) \cdot v(P_{I_v(t)}x,t))\mathrm{n}(P_{I_v(t)}x,t)$$

be an extension of the normal velocity of the interface of the strong solution $I_v(t)$ to an r_c-neighborhood of $I_v(t)$. Let θ be the density $\theta_t = \frac{\mathrm{d}|\nabla\chi_u(\cdot,t)|}{\mathrm{d}|V_t|_{\mathbb{S}^{d-1}}}$ as defined in (3.3) and let $\beta:\mathbb{R}\to\mathbb{R}$ be a truncation of the identity with $\beta(r)=r$ for $|r|\leq\frac{1}{2}$, $|\beta'|\leq 1$, $|\beta''|\leq C$, and $\beta'(r)=0$ for $|r|\geq 1$.

Then the relative entropy

$$
\begin{aligned}
E\big[\chi_u,u,V\big|\chi_v,v\big](T) := \sigma\int_{\mathbb{R}^d} 1-\xi(\cdot,T)\cdot\frac{\nabla\chi_u(\cdot,T)}{|\nabla\chi_u(\cdot,T)|}\,\mathrm{d}|\nabla\chi_u(\cdot,T)| \\
+\int_{\mathbb{R}^d}\frac{1}{2}\rho\big(\chi_u(\cdot,T)\big)\big|u-v-w\big|^2(\cdot,T)\,\mathrm{d}x \\
+\int_{\mathbb{R}^d}\big|\chi_u(\cdot,T)-\chi_v(\cdot,T)\big|\,\Big|\beta\Big(\frac{\mathrm{dist}^\pm(\cdot,I_v(T))}{r_c}\Big)\Big|\,\mathrm{d}x \\
+\sigma\int_{\mathbb{R}^d}1-\theta_T\,\mathrm{d}|V_T|_{\mathbb{S}^{d-1}}
\end{aligned}
\tag{3.9}
$$

is subject to the relative entropy inequality

$$
\begin{aligned}
&E\big[\chi_u,u,V\big|\chi_v,v\big](T)+\int_0^T\int_{\mathbb{R}^d}2\mu(\chi_u)\big|D^{\mathrm{sym}}(u-v-w)\big|^2\,\mathrm{d}x\,\mathrm{d}t\\
&\leq E\big[\chi_u,u,V\big|\chi_v,v\big](0)+R_{surTen}+R_{dt}+R_{visc}+R_{adv}+R_{weightVol}\\
&\quad + A_{visc}+A_{dt}+A_{adv}+A_{surTen}+A_{weightVol}
\end{aligned}
$$

for almost every $T\in(0,T_{strong})$, where we made use of the abbreviations

$$
\begin{aligned}
&R_{surTen} :=\\
&-\sigma\int_0^T\int_{\mathbb{R}^d\times\mathbb{S}^{d-1}}(s-\xi)\cdot\big((s-\xi)\cdot\nabla\big)v\,\mathrm{d}V_t(x,s)\,\mathrm{d}t\\
&+\sigma\int_0^T\int_{\mathbb{R}^d}(1-\theta_t)\,\xi\cdot(\xi\cdot\nabla)v\,\mathrm{d}|V_t|_{\mathbb{S}^{d-1}}\,\mathrm{d}t\\
&+\sigma\int_0^T\int_{\mathbb{R}^d}(\chi_u-\chi_v)\big((u-v-w)\cdot\nabla\big)(\nabla\cdot\xi)\,\mathrm{d}x\,\mathrm{d}t\\
&-\sigma\int_0^T\int_{\mathbb{R}^d}\Big(\xi\cdot\frac{\nabla\chi_u}{|\nabla\chi_u|}\Big)\,\mathrm{n}_v(P_{I_v(t)}x)\cdot\big(\mathrm{n}_v(P_{I_v(t)}x)\cdot\nabla\big)v-\xi\cdot(\xi\cdot\nabla)v\,\mathrm{d}|\nabla\chi_u|\,\mathrm{d}t\\
&+\sigma\int_0^T\int_{\mathbb{R}^d}\frac{\nabla\chi_u}{|\nabla\chi_u|}\cdot\big((\mathrm{Id}-\mathrm{n}_v(P_{I_v(t)}x)\otimes\mathrm{n}_v(P_{I_v(t)}x))(\nabla\bar{V}_\mathrm{n}-\nabla v)^T\cdot\xi\big)\,\mathrm{d}|\nabla\chi_u|\,\mathrm{d}t\\
&+\sigma\int_0^T\int_{\mathbb{R}^d}\frac{\nabla\chi_u}{|\nabla\chi_u|}\cdot\big((\bar{V}_\mathrm{n}-v)\cdot\nabla\big)\xi\,\mathrm{d}|\nabla\chi_u|\,\mathrm{d}t
\end{aligned}
$$

and

$$
\begin{aligned}
R_{dt} &:= -\int_0^T\int_{\mathbb{R}^d}\big(\rho(\chi_u)-\rho(\chi_v)\big)(u-v-w)\cdot\partial_t v\,\mathrm{d}x\,\mathrm{d}t,\\
R_{visc} &:= -\int_0^T\int_{\mathbb{R}^d}2\big(\mu(\chi_u)-\mu(\chi_v)\big)D^{\mathrm{sym}}v : D^{\mathrm{sym}}(u-v)\,\mathrm{d}x\,\mathrm{d}t,\\
R_{adv} &:= -\int_0^T\int_{\mathbb{R}^d}\big(\rho(\chi_u)-\rho(\chi_v)\big)(u-v-w)\cdot(v\cdot\nabla)v\,\mathrm{d}x\,\mathrm{d}t\\
&\quad -\int_0^T\int_{\mathbb{R}^d}\rho(\chi_u)(u-v-w)\cdot\big((u-v-w)\cdot\nabla\big)v\,\mathrm{d}x\,\mathrm{d}t,
\end{aligned}
$$

as well as

$$
\begin{aligned}
&R_{weightVol} := \\
&\quad \int_0^T \int_{\mathbb{R}^d} (\chi_u - \chi_v)\big(\big(\bar{V}_{\mathrm{n}} - (v \cdot \mathrm{n}_v(P_{I_v(t)}x))\mathrm{n}_v(P_{I_v(t)}x)\big) \cdot \nabla\big)\beta\Big(\frac{\mathrm{dist}^{\pm}(\cdot, I_v)}{r_c}\Big) \,\mathrm{d}x \,\mathrm{d}t \\
&\quad + \int_0^T \int_{\mathbb{R}^d} (\chi_u - \chi_v)\big((u - v - w) \cdot \nabla\big)\beta\Big(\frac{\mathrm{dist}^{\pm}(\cdot, I_v)}{r_c}\Big) \,\mathrm{d}x \,\mathrm{d}t.
\end{aligned}
$$

Moreover, we have abbreviated

$$
\begin{aligned}
A_{visc} :=& \int_0^T \int_{\mathbb{R}^d} 2\big(\mu(\chi_u) - \mu(\chi_v)\big) D^{\mathrm{sym}} v : D^{\mathrm{sym}} w \,\mathrm{d}x \,\mathrm{d}t \\
&- \int_0^T \int_{\mathbb{R}^d} 2\mu(\chi_u) D^{\mathrm{sym}} w : D^{\mathrm{sym}}(u - v - w) \,\mathrm{d}x \,\mathrm{d}t,
\end{aligned}
$$

and

$$
\begin{aligned}
A_{dt} :=& - \int_0^T \int_{\mathbb{R}^d} \rho(\chi_u)(u - v - w) \cdot \partial_t w \,\mathrm{d}x \,\mathrm{d}t \\
&- \int_0^T \int_{\mathbb{R}^d} \rho(\chi_u)(u - v - w) \cdot (v \cdot \nabla) w \,\mathrm{d}x \,\mathrm{d}t,
\end{aligned}
$$

$$
\begin{aligned}
A_{adv} :=& - \int_0^T \int_{\mathbb{R}^d} \rho(\chi_u)(u - v - w) \cdot (w \cdot \nabla)(v + w) \,\mathrm{d}x \,\mathrm{d}t \\
&- \int_0^T \int_{\mathbb{R}^d} \rho(\chi_u)(u - v - w) \cdot \big((u - v - w) \cdot \nabla\big) w \,\mathrm{d}x \,\mathrm{d}t, \\
A_{weightVol} :=& \int_0^T \int_{\mathbb{R}^d} (\chi_u - \chi_v)(w \cdot \nabla)\beta\Big(\frac{\mathrm{dist}^{\pm}(\cdot, I_v)}{r_c}\Big) \,\mathrm{d}x \,\mathrm{d}t,
\end{aligned}
$$

as well as

$$
\begin{aligned}
A_{surTen} :=& - \sigma \int_0^T \int_{\mathbb{R}^d \times \mathbb{S}^{d-1}} (s - \xi) \cdot \big((s - \xi) \cdot \nabla\big) w \,\mathrm{d}V_t(x, s) \,\mathrm{d}t \\
&+ \sigma \int_0^T \int_{\mathbb{R}^d} (1 - \theta_t)\, \xi \cdot (\xi \cdot \nabla) w \,\mathrm{d}|V_t|_{\mathbb{S}^{d-1}}(x) \,\mathrm{d}t \\
&+ \sigma \int_0^T \int_{\mathbb{R}^d} (\chi_u - \chi_v)(w \cdot \nabla)(\nabla \cdot \xi) \,\mathrm{d}x \,\mathrm{d}t \\
&+ \sigma \int_0^T \int_{\mathbb{R}^d} (\chi_u - \chi_v)\nabla w : \nabla \xi^T \,\mathrm{d}x \,\mathrm{d}t \\
&- \sigma \int_0^T \int_{\mathbb{R}^d} \xi \cdot \big((\mathrm{n}_u - \xi) \cdot \nabla\big) w \,\mathrm{d}|\nabla \chi_u| \,\mathrm{d}t.
\end{aligned}
$$

If we additionally allow for a density-dependent bulk force $\rho(\chi)f$ acting on the fluid – such as gravity – , only one additional term appears on the right-hand side of the relative entropy inequality of Proposition 3.10, see (3.212). We will comment in Remark 3.35 on the minor changes that occur in the proof of the relative entropy inequality due to the additional bulk force.

Notation

We use $a \wedge b$ (respectively $a \vee b$) as a shorthand notation for the minimum (respectively maximum) of two numbers $a, b \in \mathbb{R}$.

Let $\Omega \subset \mathbb{R}^d$ be open. For a function $u : \Omega \times [0,T] \to \mathbb{R}$, we denote by ∇u its distributional derivative with respect to space and by $\partial_t u$ its derivative with respect to time. For $p \in [1,\infty]$ and an integer $k \in \mathbb{N}_0$, we denote by $L^p(\Omega)$ and $W^{k,p}(\Omega)$ the usual Lebesgue and Sobolev spaces. In the special case $p = 2$ we use as usual $H^k(\Omega) := W^{k,2}(\Omega)$ to denote the Sobolev space. For integration of a function f with respect to the d-dimensional Lebesgue measure respectively the $d-1$-dimensional surface measure, we use the usual notation $\int_\Omega f \mathrm{d}x$ respectively $\int_I f \mathrm{d}S$. For measures other than the natural measure (the Lebesgue measure in case of domains Ω and the surface measure in case of surfaces I), we denote the corresponding Lebesgue spaces by $L^p(\Omega, \mu)$. The space of all compactly supported and infinitely differentiable functions on Ω is denoted by $C^\infty_{cpt}(\Omega)$. The closure of $C^\infty_{cpt}(\Omega)$ with respect to the Sobolev norm $\|\cdot\|_{W^{k,p}(\Omega)}$ is $W^{k,p}_0(\Omega)$, and its dual will be denoted by $W^{-1,p'}(\Omega)$ where $p' \in [0,\infty]$ is the conjugated Hölder exponent of p, i.e. $1/p + 1/p' = 1$. For vector-valued fields, say with range in $\mathbb{R}^d$, we use the notation $L^p(\Omega;\mathbb{R}^d)$, and so on. For a Banach space X, a finite time $T > 0$ and a number $p \in [1,\infty]$ we denote by $L^p([0,T];X)$ the usual Bochner–Lebesgue space. If X itself is a Sobolev space $W^{k,q}$, we denote the norm of $L^p([0,T];X)$ as $\|\cdot\|_{L^p_t W^{k,q}_x}$. When writing $L^\infty_w([0,T];X')$ we refer to the space of bounded and weak-* measurable maps $f\colon [0,T] \to X'$, where X' is the dual space of X. By $L^p(\Omega) + L^q(\Omega)$ we denote the normed space of all functions $u : \Omega \to \mathbb{R}$ which may be written as the sum of two functions $v \in L^p(\Omega)$ and $w \in L^q(\Omega)$. The space $C^k([0,T];X)$ contains all k-times continuously differentiable and X-valued functions on $[0,T]$.

In order to give a suitable weak description of the evolution of the sharp interface, we have to recall the concepts of Caccioppoli sets as well as varifolds. To this end, let $\Omega \subset \mathbb{R}^d$ be open. We denote by $\mathrm{BV}(\Omega)$ the space of functions with bounded variation in Ω. A measurable subset $E \subset \Omega$ is called a set of finite perimeter in Ω (or a Caccioppoli subset of Ω) if its characteristic function χ_E is of bounded variation in Ω. We will write $\partial^* E$ when referring to the reduced boundary of a Caccioppoli subset E of Ω; whereas n denotes the associated measure theoretic (inward pointing) unit normal vector field of $\partial^* E$. For detailed definitions of all these concepts from geometric measure theory, we refer to [61, 39]. In case Ω has a C^2 boundary, we denote by $\mathrm{H}(x)$ the mean curvature vector at $x \in \partial\Omega$. Recall that for a convex function $g : \mathbb{R}^d \to \mathbb{R}$ the *recession function* $g^{rec} : \mathbb{R}^d \to \mathbb{R}$ is defined as $g^{rec}(x) := \lim_{\tau\to\infty} \tau^{-1} g(\tau x)$.

An oriented varifold is simply a non-negative measure $V \in \mathcal{M}(\Omega \times \mathbb{S}^{d-1})$, where $\Omega \subset \mathbb{R}^d$ is open and $\mathbb{S}^{d-1}$ denotes the $(d-1)$-dimensional sphere. For a varifold V, we denote by $|V|_{\mathbb{S}^{d-1}} \in \mathcal{M}(\Omega)$ its local mass density given by $|V|_{\mathbb{S}^{d-1}}(A) := V(A \times \mathbb{S}^{d-1})$ for any Borel set $A \subset \Omega$. For a locally compact separable metric space X we write $\mathcal{M}(X)$ to refer to the space of (signed) finite Radon-measures on X. If $A \subset X$ is a measurable set and $\mu \in \mathcal{M}(X)$, we let $\mu \llcorner A$ be the restriction of μ on A. The k-dimensional Hausdorff measure on $\mathbb{R}^d$ will be denoted by $\mathcal{H}^k$, whereas we write $\mathcal{L}^d(A)$ for the d-dimensional Lebesgue measure of a Lebesgue measurable set $A \subset \mathbb{R}^d$.

Finally, let us fix some tensor notation. First of all, we use $(\nabla v)_{ij} = \partial_j v_i$ as well as $\nabla \cdot v = \sum_i \partial_i v_i$ for a Sobolev vector field $v\colon \mathbb{R}^d \to \mathbb{R}^d$. The symmetric gradient is denoted by $D^{\mathrm{sym}} v := \frac{1}{2}(\nabla v + \nabla v^T)$. For time-dependent fields $v\colon \mathbb{R}^d \times [0,T) \to \mathbb{R}^n$ we denote by $\partial_t v$ the partial derivative with respect to time. Tensor products of vectors $u, v \in \mathbb{R}^d$ will be given by $(u \otimes v)_{ij} = u_i v_j$. For tensors $A = (A_{ij})$ and $B = (B_{ij})$ we write $A : B = \sum_{ij} A_{ij} B_{ij}$.

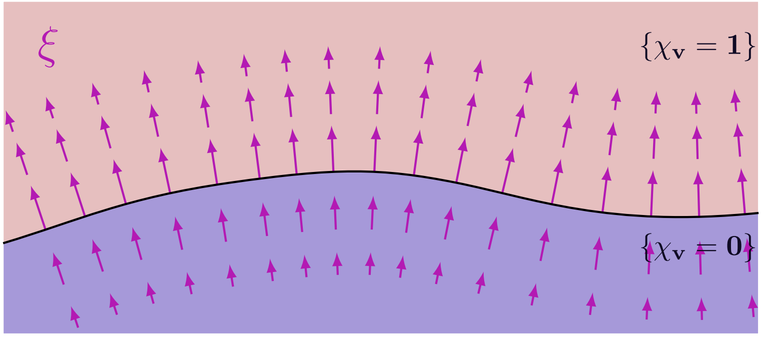

Figure 3.1: An illustration of the vector field ξ.

3.2 Outline of the strategy

3.2.1 The relative entropy

The basic idea of the present work is to measure the "distance" between a varifold solution to the two-phase Navier-Stokes equation (χ_u, u, V) and a strong solution to the two-phase Navier-Stokes equation (χ_v, v) by means of the relative entropy functional

$$\begin{aligned} E\big[\chi_u, u, V\big|\chi_v, v\big](t) :=&\sigma \int_{\mathbb{R}^d} 1 - \xi \cdot \frac{\nabla \chi_u}{|\nabla \chi_u|} \,\mathrm{d}|\nabla \chi_u| + \int_{\mathbb{R}^d} \frac{\rho(\chi_u)}{2} |u - v - w|^2 \,\mathrm{d}x \\ &+ \sigma \int_{\mathbb{R}^d} 1 - \theta_t \,\mathrm{d}|V_t|_{\mathbb{S}^{d-1}} \\ &+ \int_{\mathbb{R}^d} |\chi_u - \chi_v| \left|\beta\Big(\frac{\mathrm{dist}^{\pm}(x, I_v(t))}{r_c}\Big)\right| \,\mathrm{d}x \end{aligned} \tag{3.10}$$

where $\xi : \mathbb{R}^d \times [0, T_{strong}) \to \mathbb{R}^d$ is a suitable extension of the unit normal vector field of the interface of the strong solution and where w is a vector field that will be constructed below and that vanishes in case of equal viscosities $\mu^+ = \mu^-$. More precisely, we choose ξ as

$$\xi(x, t) := \mathrm{n}_v(P_{I_v(t)} x)(1 - \mathrm{dist}(x, I_v(t))^2)\eta(\mathrm{dist}(x, I_v(t)))$$

for some cutoff η with $\eta(s) = 1$ for $s \leq \frac{1}{2} r_c$ and $\eta \equiv 0$ for $s \geq r_c$, where $P_{I_v(t)} x$ denotes for each $t \geq 0$ the projection of x onto the interface $I_v(t)$ of the strong solution and where the unit normal vector field n_v of the interface of the strong solution is oriented to point towards $\{\chi_v(\cdot, t) = 1\}$. For an illustration of the vector field ξ, see Figure 3.1.

Rewriting the relative entropy functional in the form

$$\begin{aligned} &E\big[\chi_u, u, V\big|\chi_v, v\big](t) \\ &= E[\chi_u, u, V](t) + \int_{\mathbb{R}^d} \chi_u \nabla \cdot \xi \,\mathrm{d}x - \int_{\mathbb{R}^d} \rho(\chi_u) u \cdot (v + w) \,\mathrm{d}x \\ &\quad + \int_{\mathbb{R}^d} \frac{1}{2} \rho(\chi_u) |v + w|^2 \,\mathrm{d}x + \int_{\mathbb{R}^d} (\chi_u - \chi_v) \beta\Big(\frac{\mathrm{dist}^{\pm}(x, I_v(t))}{r_c}\Big) \,\mathrm{d}x \end{aligned}$$

with the energy (3.2d), we see that we may estimate the time evolution of the relative entropy $E\big[\chi_u, u, V\big|\chi_v, v\big](t)$ by exploiting the energy dissipation property (3.2c) of the varifold solution and by testing the weak formulation of the two-phase Navier-Stokes equation (3.2a) and (3.2b) against the (sufficiently regular) test functions $v + w$ respectively $\frac{1}{2}|v + w|^2$, $\nabla \cdot \xi$, and $\beta(\frac{\mathrm{dist}^{\pm}(x, I_v(t))}{r_c})$.

As usual in the derivation of weak-strong uniqueness results by the relative entropy method of Dafermos [43] and Di Perna [53], in the case of equal viscosities $\mu^+ = \mu^-$ the goal is the derivation of an estimate of the form

$$\begin{aligned} &E\big[\chi_u, u, V\big|\chi_v, v\big](T) + c\int_0^T \int_{\mathbb{R}^d} |\nabla u - \nabla v|^2 \,\mathrm{d}x\,\mathrm{d}t \\ &\leq C(v, I_v, \mathrm{data}) \int_0^T E\big[\chi_u, u, V\big|\chi_v, v\big](t)\,\mathrm{d}t \end{aligned} \tag{3.11}$$

which implies uniqueness and stability by means of the Gronwall lemma and by the coercivity properties of the relative entropy functional discussed in the next section.

In the case of different viscosities $\mu^+ \neq \mu^-$, we will derive a slightly weaker (but still sufficient) result of roughly speaking the form

$$\begin{aligned} &E\big[\chi_u, u, V\big|\chi_v, v\big](T) + c\int_0^T \int_{\mathbb{R}^d} |\nabla u - \nabla v - \nabla w|^2 \,\mathrm{d}x\,\mathrm{d}t \\ &\leq C(v, I_v, \mathrm{data}) \int_0^T E\big[\chi_u, u, V\big|\chi_v, v\big](t)\ \big|\log E\big[\chi_u, u, V\big|\chi_v, v\big](t)\big|\ \mathrm{d}t, \end{aligned} \tag{3.12}$$

along with estimates on w which include in particular the bound

$$\int_{\mathbb{R}^d} |w(\cdot,T)|^2 \,dx \leq C(v, I_v, \mathrm{data}) E\big[\chi_u, u, V\big|\chi_v, v\big](T).$$

3.2.2 The error control provided by the relative entropy functional

The relative entropy functional (3.10) provides control of the following quantities (up to bounded prefactors):

Velocity error control. The relative entropy $E[\chi_u, u, V|\chi_v, v](t)$ controls the square of the velocity error in the L^2 norm

$$\int_{\mathbb{R}^d} |u(\cdot,t) - v(\cdot,t)|^2 \,\mathrm{d}x$$

at any given time t. In the case of equal viscosities, this is immediate from (3.10) by $w \equiv 0$, while in the case of different viscosities this follows by the estimate $\int_{\mathbb{R}^d} |w|^2 \,\mathrm{d}x \leq C\|\nabla v\|_{L^\infty} \int_{\mathbb{R}^d} 1 - \xi \cdot \frac{\nabla\chi_u}{|\nabla\chi_u|}\,\mathrm{d}|\nabla\chi_u|$ which is a consequence of the construction of w and the choice of ξ, see below.

Interface error control. The relative entropy provides a tilt-excess type control of the error in the interface normal

$$\int_{\mathbb{R}^d} 1 - \xi \cdot \mathrm{n}_u \,\mathrm{d}|\nabla\chi_u|.$$

In particular, it controls the squared error in the interface normal

$$\int_{\mathbb{R}^d} |\mathrm{n}_u - \xi|^2 \,\mathrm{d}|\nabla\chi_u|.$$

The term also controls the total length respectively area (for $d = 2$ respectively $d = 3$) of the part of the interface I_u which is not locally a graph over I_v, see Figure 3.2. For example, in the region around the left purple half-ray the interface of the weak solution is not a graph over the interface of the weak solution. Furthermore, the term controls the length respectively area (for $d = 2$ respectively $d = 3$) of the part of the interface with distance to $I_v(t)$ greater than the cutoff length r_c, as there we have $\xi \equiv 0$.

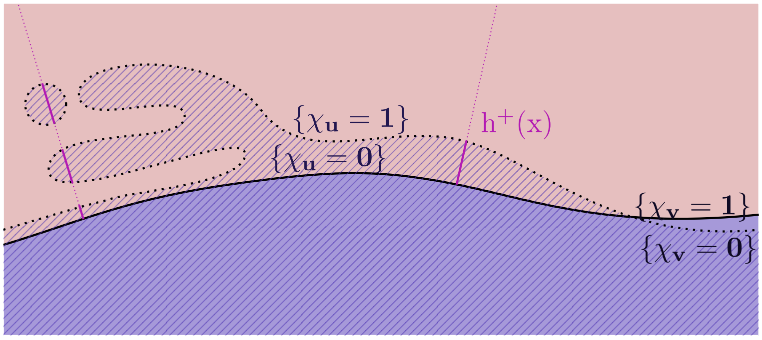

Figure 3.2: An illustration of the interface error. The red and the blue region (separated by the black solid curve) correspond to the regions occupied by the two fluids in the strong solution. The shaded area corresponds to the region occupied by the blue fluid in the varifold solution, the interface in the varifold solution corresponds to the dotted curve.

To give another heuristic explanation of the interface error control and also introduce some notation for subsequent use (for the proof in the case of different viscosities in Section 3.5 and its explanation in Section 3.2.4), we attempt to write the interface of the weak solution as a graph over the interface of the strong solution (at least, to the extent to which this is possible): Denote the local height of the one-sided interface error by $h^+ : I_v(t) \to \mathbb{R}_0^+$ as measured along orthogonal rays originating on $I_v(t)$ (with some cutoff applied away from the interface $I_v(t)$ of the strong solution); denote by h^- the corresponding height of the interface error as measured in the other direction. For example, in Figure 3.2 the quantity $h^+(x)$ for some base point $x \in I_v(t)$ would correspond to the accumulated length of the solid segments in each of the purple rays, the dotted segments not being counted. Note that the rays are orthogonal on $I_v(t)$. Then the tilt-excess type term in the relative entropy also controls the gradient of the one-sided interface error heights

$$\int_{I_v(t)} \min\{|\nabla h^\pm|^2, |\nabla h^\pm|\} \,\mathrm{d}S.$$

Note that wherever $I_u(t)$ is locally a graph over $I_v(t)$ and is not too far away from $I_v(t)$, it must be the graph of the function $h^+ - h^-$. Here, the graph of a function g over the curved interface $I_v(t)$ is defined by the set of points obtained by shifting the points of $I_v(t)$ by the corresponding multiple of the surface normal, i. e. $\{x + g(x)\mathrm{n}_v(x) : x \in I_v(t)\}$.

Varifold multiplicity error control. For varifold solutions, the relative entropy controls the multiplicity error of the varifold

$$\int_{\mathbb{R}^d} 1 - \theta_t(x) \,\mathrm{d}|V_t|_{\mathbb{S}^{d-1}}$$

(note that $\frac{1}{\theta_t(x)}$ corresponds to the multiplicity of the varifold), which in turn by the compatibility condition (3.2e) and the definition of θ_t (see (3.3)) controls the squared error in the normal of the varifold

$$\int_{\mathbb{R}^d} \int_{\mathbb{S}^{d-1}} |s - \mathrm{n}_u|^2 \,\mathrm{d}V_t(s, x).$$

Weighted volume error control. Furthermore, the error in the volume occupied by the two fluids weighted with the distance to the interface of the strong solution

$$\int_{\mathbb{R}^d} |\chi_u - \chi_v| \min\{\mathrm{dist}(x, I_v), 1\} \,\mathrm{d}x$$

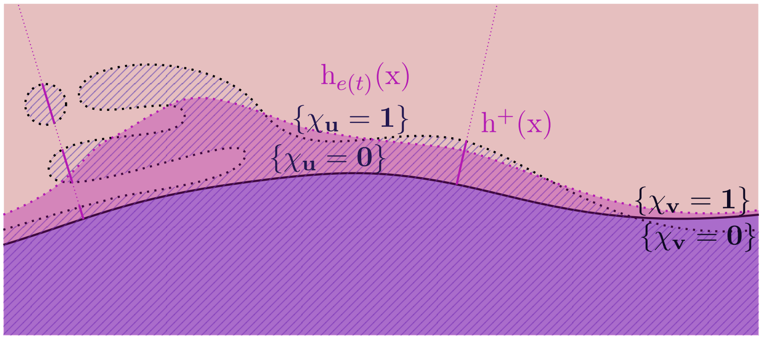

Figure 3.3: An illustration of the approximation of the interface error by the mollified height function $h^+_{e(t)}$.

is controlled. Note that this term is the only term in the relative entropy which is not obtained by the usual relative entropy ansatz $E[x|y] = E[x] - DE[y](x-y) - E[y]$. We have added this lower-order term – as compared to the term $\int_{\mathbb{R}^d} 1 - \xi \cdot \frac{\nabla\chi_u}{|\nabla\chi_u|}\,\mathrm{d}|\nabla\chi_u|$ which provides tilt-excess-type control – to the relative entropy in order to remove the lack of coercivity of the term $\int_{\mathbb{R}^d} 1 - \xi \cdot \frac{\nabla\chi_u}{|\nabla\chi_u|}\,\mathrm{d}|\nabla\chi_u|$ in the limit of vanishing interface length of the varifold solution.

Control of velocity gradient error by dissipation. By means of Korn's inequality, the dissipation term controls the L^2-error in the gradient

$$\int_0^T \int_{\mathbb{R}^d} |\nabla u - \nabla v - \nabla w|^2 \,\mathrm{d}x\,\mathrm{d}t.$$

3.2.3 The case of equal viscosities

For equal viscosities $\mu^+ = \mu^-$, one may choose $w \equiv 0$. As a consequence, the right-hand side in the relative entropy inequality – see Proposition 3.10 above – may be estimated to yield the Gronwall-type inequality (3.11). The details are provided in Section 3.4.

3.2.4 Additional challenges in the case of different viscosities

In the case of different viscosities $\mu_1 \neq \mu_2$ of the two fluids, even for strong solutions the normal derivative of the tangential velocity features a discontinuity at the interface: By the no-slip boundary condition, the velocity is continuous across the interface $[v] = 0$ and the same is true for its tangential derivatives $[(\mathrm{t}\cdot\nabla)v] = 0$. As a consequence of this, the discontinuity of $\mu(\chi_v)$ across the interface and the equilibrium condition for the stresses at the interface

$$[[\mu(\chi)\mathrm{t}\cdot(\mathrm{n}\cdot\nabla)v + \mu(\chi)\mathrm{n}\cdot(\mathrm{t}\cdot\nabla)v]] = 0$$

entail for generic data a discontinuity of the normal derivative of the tangential velocity $\mathrm{t}\cdot(\mathrm{n}\cdot\nabla)v$ across the interface.

As a consequence, it becomes impossible to establish a Gronwall estimate for the standard relative entropy (3.10) with $w \equiv 0$. To see this, consider in the two-dimensional case $d = 2$ two strong solutions u and v with coinciding initial velocities $u(\cdot,0) = v(\cdot,0) = u_0(\cdot)$, but slightly different initial interfaces $\chi_v(\cdot,0) = \chi_{\{|x|\leq 1\}}$ and $\chi_u(\cdot,0) = \chi_{\{|x|\leq 1-\varepsilon\}}$ for some $\varepsilon > 0$. The initial relative entropy is then of the order $\sim \varepsilon^2$. Suppose that (in polar coordinates) the initial velocity u_0 has a profile near the interface like

$$u_0(x,y)=\begin{cases}\mu^-(r-1)e_\phi & \text{for } r=\sqrt{x^2+y^2}<1,\\ \mu^+(r-1)e_\phi & \text{for } r>1.\end{cases}$$

Note that this velocity profile features a kink at the interface. As one verifies readily, as far as the viscosity term is concerned this corresponds to a near-equilibrium profile for the solution (χ_v, v) (in the sense that the viscosity term is bounded). However, in the solution (χ_u, u) the interface is shifted by ε and the profile is no longer an equilibrium profile. By the scaling of the viscosity term, the timescale within which the profile u_0 equilibrates in the annulus of width ε towards a near-affine profile is of the order of ε^2. After this timescale, the velocity u will have changed by about ε in a layer of width $\sim\varepsilon$ around the interface; at the same time, due to the mostly parallel transport at the interface the solution will not have changed much otherwise. As a consequence, the term $\int \frac{1}{2}\rho(\chi_u)|u-v|^2\,\mathrm{d}x$ will be of the order of at least $c\varepsilon^3$ after a time $T\sim\varepsilon^2$, while the other terms in the relative entropy are essentially the same. Thus, the relative entropy has grown by a factor of $1+c\varepsilon$ within a timescale ε^2, which prevents any Gronwall-type estimate.

At the level of the relative entropy inequality (see Proposition 3.10), the derivation of the Gronwall inequality is prevented by the viscosity terms, which read for $w\equiv 0$

$$\begin{aligned}&-\int \frac{\mu(\chi_u)}{2}\big|\nabla u+\nabla u^T-(\nabla v+\nabla v^T)\big|^2\,\mathrm{d}x\\ &+\int (\mu(\chi_v)-\mu(\chi_u))\nabla v:\big(\nabla u+\nabla u^T-(\nabla v+\nabla v^T)\big)\,\mathrm{d}x.\end{aligned}$$

The latter term prevents the derivation of a dissipation estimate: While it is formally quadratic in the difference of the two solutions (χ_u, u) and (χ_v, v), due to the (expected) jump of the velocity gradients ∇v and ∇u at the respective interfaces it is in fact only linear in the interface error.

The key idea for our weak-strong uniqueness result in the case of different viscosities is to construct a vector field w which is small in the L^2 norm but whose gradient compensates for most of the problematic term $(\mu(\chi_v)-\mu(\chi_u))(\nabla v+\nabla v^T)$. To be precise, it is only the normal derivative of the tangential component of v which may be discontinuous at the interface; the tangential derivatives are continuous by the no-slip boundary condition, while the normal derivative of the normal component is continuous by the condition $\nabla\cdot v=0$.

Let us explain our construction of the vector field w at the simple two-dimensional example of a planar interface of the strong solution $I_v=\{(x,0):x\in\mathbb{R}\}$. In this setting, we would like to set for $y>0$

$$w^+(x,y,t):=c(\mu^+,\mu^-)\int_0^{y\wedge h^+(x)}(e_x\cdot\partial_y v)(x,\tilde y)e_x\,\mathrm{d}\tilde y$$

(where e_x just denotes the first vector of the standard basis). Note that due to the bounded integrand, this vector field $w^+(x,y)$ is bounded by $Ch^+(x)$, i. e. it is bounded by the interface error. As we shall see in the proof, the time derivative of w^+ is also bounded in terms of other error terms. The tangential spatial derivative of this vector field $\partial_x w^+(x,y,t)$ is given (up to a constant factor) by $\int_0^{y\wedge h^+(x)}(e_x\cdot\partial_x\partial_y v)(x,\tilde y)e_x\,\mathrm{d}\tilde y+\chi_{y\geq h^+(x)}(e_x\cdot\partial_y v)(x,h^+(x))\partial_x h^+(x)e_x$ which is also a term controlled by $Ch^+(x)+C|\partial_x h^+(x)|$. The normal derivative, on the other hand, is given by $\partial_y w^+(x,y,t)=c(\mu^+,\mu^-)\chi_{\{0\leq y\leq h^+(x)\}}(e_x\cdot\partial_y v)(x,y)e_x$ which (upon choosing $c(\mu^+,\mu^-)$) would precisely compensate the discontinuity of $\partial_y(e_x\cdot v)$ in the region in which the interface of the weak solution is a graph of a function over I_v. Note that our relative entropy functional provides a higher-order control of the size of the region in which the interface of the weak solution is not a graph over the interface of the strong solution.

However, with this choice of vector field $w^+(x,y,t)$, two problems occur: First, the vector field is not solenoidal. For this reason, we introduce an additional Helmholtz projection. Second – and constituting a more severe problem – , the vector field would not necessarily be (spatially) Lipschitz continuous (as the derivative contains a term with $\partial_x h^+(x)$ which is not necessarily bounded), which due to the surface tension terms would be required for the derivation of a Gronwall-type estimate. For this reason, we first regularize the height function h^+ by mollification on a scale of the order of the error. See Proposition 3.26 and Proposition 3.27 for details of our construction of the regularized height function, and Figure 3.3 for an illustration of it. The actual construction of our compensation function w is performed in Proposition 3.28. We then derive an estimate in the spirit of (3.12) in Proposition 3.34.

3.3 Time evolution of geometric quantities and further coercivity properties

3.3.1 Time evolution of the signed distance function

In order to describe the time evolution of various constructions, we need to recall some well-known properties of the signed distance function. Let us start by introducing notation. For a family $(\Omega_t^+)_{t\in[0,T_{strong})}$ of smoothly evolving domains with smoothly evolving interfaces $I(t)$ in the sense of Definition 3.5, the associated signed distance function is given by

$$\operatorname{dist}^\pm(x,I(t)) := \begin{cases} \operatorname{dist}(x,I(t)), & x\in\Omega_t^+, \\ -\operatorname{dist}(x,I(t)), & x\notin\Omega_t^+. \end{cases} \tag{3.13}$$

From Definition 3.5 of a family of smoothly evolving domains it follows that the family of maps $\Phi_t\colon I(t)\times(-r_c,r_c)\to\mathbb{R}^d$ given by $\Phi_t(x,y) := x+y\mathrm{n}(x,t)$ are C^2-diffeomorphisms onto their image $\{x\in\mathbb{R}^d\colon \operatorname{dist}(x,I(t))<r_c\}$ subject to the bounds

$$|\nabla\Phi_t|\leq C, \qquad |\nabla\Phi_t^{-1}|\leq C. \tag{3.14}$$

The signed distance function (resp. its time derivative) to the interface of the strong solution is then of class $C_t^0C_x^3$ (resp. $C_t^0C_x^2$) in the space-time tubular neighborhood $\bigcup_{t\in[0,T_{strong})}\operatorname{im}(\Phi_t)\times\{t\}$ due to the regularity assumptions in Definition 3.5. We also have the bounds

$$|\nabla^{k+1}\operatorname{dist}^\pm(x,I(t))|\leq Cr_c^{-k}, \quad k=1,2, \tag{3.15}$$

and in particular for the mean curvature vector

$$|\mathrm{H}|\leq Cr_c^{-1}. \tag{3.16}$$

Moreover, the projection $P_{I(t)}x$ of a point x onto the nearest point on the manifold $I(t)$ is well-defined and of class $C_t^0C_x^2$ in the same tubular neighborhood.

After having introduced the necessary notation we study the time evolution of the signed distance function to the interface of the strong solution. Because of the kinematic condition that the interface is transported with the flow, we obtain the following statement.

Lemma 3.11. *Let $\chi_v\in L^\infty([0,T_{strong});\mathrm{BV}(\mathbb{R}^d;\{0,1\}))$ be an indicator function such that $\Omega_t^+ := \{x\in\mathbb{R}^d\colon \chi_v(x,t)=1\}$ is a family of smoothly evolving domains and $I_v(t) := \partial\Omega_t^+$ is a family of smoothly evolving surfaces in the sense of Definition 3.5. Let $v\in L^2_{loc}([0,T_{strong}];H^1_{loc}(\mathbb{R}^d;\mathbb{R}^d))$ be a continuous solenoidal vector field such that χ_v solves the equation $\partial_t\chi_v=-\nabla\cdot(\chi_v v)$. The time evolution of the signed distance function to the interface $I_v(t)$ is then given by*

$$\partial_t\operatorname{dist}^\pm(x,I_v(t)) = -\big(\bar{V}_\mathrm{n}(x,t)\cdot\nabla\big)\operatorname{dist}^\pm(x,I_v(t)) \tag{3.17}$$

for any $t \in [0, T_{strong}]$ *and any* $x \in \mathbb{R}^d$ *with* $\operatorname{dist}(x, I_v(t)) \leq r_c$*, where* $\bar{V}_{\mathrm{n}}$ *is the extended normal velocity of the interface given by*

$$\bar{V}_{\mathrm{n}}(x,t) = \big(v(P_{I_v(t)}x, t) \cdot \mathrm{n}_v(P_{I_v(t)}x, t)\big)\mathrm{n}_v(P_{I_v(t)}x, t). \tag{3.18}$$

Moreover, the following formulas hold true

$$\nabla \operatorname{dist}^{\pm}(x, I_v(t)) = \mathrm{n}_v(P_{I_v(t)}x, t), \tag{3.19}$$

$$\nabla \operatorname{dist}^{\pm}(x, I_v(t)) \cdot \partial_t \nabla \operatorname{dist}^{\pm}(x, I_v(t)) = 0, \tag{3.20}$$

$$\nabla \operatorname{dist}^{\pm}(x, I_v(t)) \cdot \partial_j \nabla \operatorname{dist}^{\pm}(x, I_v(t)) = 0, \quad j = 1, \dots, d, \tag{3.21}$$

$$\partial_t \operatorname{dist}^{\pm}(x, I_v(t)) = \partial_t \operatorname{dist}^{\pm}(y, I_v(t))\big|_{y=P_{I_v(t)}x}, \tag{3.22}$$

for all (x,t) *such that* $\operatorname{dist}(x, I_v(t)) \leq r_c$*. The gradient of the projection onto the nearest point on the interface* $I_v(t)$ *is given by*

$$\nabla P_{I_v(t)}x = \mathrm{Id} - \mathrm{n}_v(P_{I_v(t)}x) \otimes \mathrm{n}_v(P_{I_v(t)}x) - \operatorname{dist}^{\pm}(x, I_v(t))\nabla^2 \operatorname{dist}^{\pm}(x, I_v(t)). \tag{3.23}$$

In particular, we have the bound

$$|\nabla P_{I_v(t)}x| \leq C \tag{3.24}$$

for all (x,t) *such that* $\operatorname{dist}(x, I_v(t)) \leq r_c$*.*

Proof. Recall that $\nabla \operatorname{dist}^{\pm}(x, I_v(t))$ for a point $x \in I_v(t)$ on the interface equals the inward pointing normal vector $\mathrm{n}_v(x,t)$ of the interface $I_v(t)$. This also extends away from the interface in the sense that

$$\nabla \operatorname{dist}^{\pm}(y, I_v(t))\big|_{y=P_{I_v(t)}x} = \mathrm{n}_v(P_{I_v(t)}x, t) = \nabla \operatorname{dist}^{\pm}(y, I_v(t))\big|_{y=x} \tag{3.25}$$

for all (x,t) such that $\operatorname{dist}(x, I_v(t)) < r_c$, i. e. (3.19) holds. Hence, we also have the formula $P_{I_v(t)}x = x - \operatorname{dist}^{\pm}(x, I_v(t))\nabla \operatorname{dist}^{\pm}(x, I_v(t))$. Differentiating this representation of the projection onto the interface and using the fact that n_v is a unit vector we obtain using also (3.26)

$$\begin{aligned}
&\nabla \operatorname{dist}^{\pm}(y, I_v(t))\big|_{y=P_{I_v(t)}x} \cdot \partial_t P_{I_v(t)}x \\
&= -\partial_t \operatorname{dist}^{\pm}(x, I_v(t)) - \operatorname{dist}^{\pm}(x, I_v(t))\nabla \operatorname{dist}^{\pm}(P_{I_v(t)}x, I_v(t)) \cdot \partial_t \nabla \operatorname{dist}^{\pm}(x, I_v(t)) \\
&= -\partial_t \operatorname{dist}^{\pm}(x, I_v(t)) - \operatorname{dist}^{\pm}(x, I_v(t))\partial_t \Big(\frac{1}{2}|\nabla \operatorname{dist}^{\pm}(x, I_v(t))|^2\Big) \\
&= -\partial_t \operatorname{dist}^{\pm}(x, I_v(t)).
\end{aligned}$$

Hence, we obtain in addition to (3.25) the formula

$$\partial_t \operatorname{dist}^{\pm}(x, I_v(t)) = \partial_t \operatorname{dist}^{\pm}(y, I_v(t))\big|_{y=P_{I_v(t)}x}.$$

On the other side, on the interface the time derivative of the signed distance function equals up to a sign the normal speed. In our case, the latter is given by the normal component of the given velocity field v evaluated on the interface, see Remark 3.9. This concludes the proof of (3.17). Moreover, (3.20) as well as (3.21) follow immediately from differentiating $|\nabla \operatorname{dist}^{\pm}(x, I_v(t))|^2 = 1$. Finally, (3.23) and (3.24) follow immediately from (3.15) and $P_{I_v(t)}x = x - \operatorname{dist}^{\pm}(x, I_v(t))\mathrm{n}_v(P_{I_v(t)}x)$.

In the above considerations, we have made use of the following result: Consider the auxiliary function $g(x,t) = \operatorname{dist}^{\pm}(P_{I_v(t)}x, I_v(t))$ for (x,t) such that $\operatorname{dist}(x, I_v(t)) < r_c$. Since this function vanishes on the space-time tubular neighborhood of the interface $\bigcup_{t\in(0,T_{strong})}\{x \in \mathbb{R}^d\colon \operatorname{dist}(x,I_v(t)) < r_c\} \times \{t\}$ we compute

$$0 = \frac{\mathrm{d}}{\mathrm{d}t} g(x,t) = \partial_t \operatorname{dist}^\pm(y,I_v(t))\big|_{y=P_{I_v(t)}x} + \nabla \operatorname{dist}^\pm(y,I_v(t))\big|_{y=P_{I_v(t)}x} \cdot \partial_t P_{I_v(t)}x. \quad (3.26)$$

This concludes the proof. □

Remark 3.12. *Consider the situation of Lemma 3.11. We proved that*

$$\partial_t \operatorname{dist}^\pm(x,I_v(t)) = -v(P_{I_v(t)}x,t)\cdot \mathrm{n}_v(P_{I_v(t)}x,t).$$

The right hand side of this identity is of class $L^\infty_t W^{2,\infty}_x$, as the normal component $\mathrm{n}_v(P_{I_v(t)})\cdot\nabla v$ of the velocity gradient ∇v of a strong solution is continuous across the interface $I_v(t)$. To see this, one first observes that the tangential derivatives $((\mathrm{Id} - \mathrm{n}_v(P_{I_v(t)})\otimes\mathrm{n}_v(P_{I_v(t)}))\nabla)v$ are naturally continuous across the interface; one then uses the incompressibility constraint $\nabla\cdot v = 0$ to deduce that $\mathrm{n}_v(P_{I_v(t)})\cdot(\mathrm{n}_v(P_{I_v(t)})\cdot\nabla)v$ is also continuous across the interface.

3.3.2 Properties of the vector field ξ

The vector field ξ – as defined in Proposition 3.10 and illustrated in Figure 3.1 – is an extension of the unit normal vector field n_v associated to the family of smoothly evolving domains occupying the first fluid of the strong solution. We now provide a more detailed account of its definition. The construction in fact consists of two steps. First, we extend the normal vector field n_v to a (space-time) tubular neighborhood of the evolving interfaces $I_v(t)$ by projecting onto the interface. Second, we multiply this construction with a cutoff which decreases quadratically in the distance to the interface of the strong solution (see (3.33)).

Definition 3.13. *Let $\chi_v \in L^\infty([0,T_{strong});\mathrm{BV}(\mathbb{R}^d;\{0,1\}))$ be an indicator function such that $\Omega_t^+ := \{x\in\mathbb{R}^d\colon \chi_v(x,t) = 1\}$ is a family of smoothly evolving domains and $I_v(t) := \partial\Omega_t^+$ is a family of smoothly evolving surfaces in the sense of Definition 3.5. Let η be a smooth cutoff function with $\eta(s) = 1$ for $s \leq \frac{1}{2}$ and $\eta \equiv 0$ for $s \geq 1$. Define another smooth cutoff function $\zeta\colon \mathbb{R}\to[0,\infty)$ as follows:*

$$\zeta(r) = (1-r^2)\eta(r), \quad r\in[-1,1], \quad (3.27)$$

and $\zeta \equiv 0$ for $|r| > 1$. Then, we define a vector field $\xi\colon \mathbb{R}^d\times[0,T_{strong})\to\mathbb{R}^d$ by

$$\xi(x,t) := \begin{cases} \zeta\Big(\frac{\operatorname{dist}^\pm(x,I_v(t))}{r_c}\Big)\mathrm{n}_v(P_{I_v(t)}x,t) & \textit{for } (x,t) \textit{ with } \operatorname{dist}(x,I_v(t)) < r_c, \\ 0 & \textit{else.} \end{cases} \quad (3.28)$$

The definition of ξ has the following consequences.

Remark 3.14. *Observe that the vector field ξ is indeed well-defined in the space-time domain $\mathbb{R}^d\times[0,T_{strong})$ due to the action of the cut-off function ζ; it also satisfies $|\xi|\leq 1$ or, more precisely, the sharper inequality $|\xi| \leq (1-\operatorname{dist}(x,I_v(t))^2)_+$. Furthermore, the extension ξ inherits its regularity from the regularity of the signed distance function to the interface $I_v(t)$. More precisely, it follows that the vector field ξ (resp. its time derivative) is of class $L^\infty_t W^{2,\infty}_x$ (resp. $W^{1,\infty}_t W^{1,\infty}_x$) globally in $\mathbb{R}^d\times[0,T_{strong})$, and the restrictions to the domains $\{\chi_v = 0\}$ and $\{\chi_v = 1\}$ are of class $L^\infty_t C^2_x$. This turns out to be sufficient for our purposes.*

The time derivative of our vector field ξ is given as follows.

Lemma 3.15. *Let $\chi_v \in L^\infty([0,T_{strong});\mathrm{BV}(\mathbb{R}^d;\{0,1\}))$ be an indicator function such that $\Omega_t^+ := \{x \in \mathbb{R}^d \colon \chi_v(x,t) = 1\}$ is a family of smoothly evolving domains and $I_v(t) := \partial\Omega_t^+$ is a family of smoothly evolving surfaces in the sense of Definition 3.5. Let $v \in L^2_{loc}([0,T_{strong}];H^1_{loc}(\mathbb{R}^d;\mathbb{R}^d))$ be a continuous solenoidal vector field such that χ_v solves the equation $\partial_t\chi_v = -\nabla\cdot(\chi_v v)$. Let $\bar{V}_\mathrm{n}$ be the extended normal velocity of the interface (3.18). Then the time evolution of the vector field ξ from Definition 3.13 is given by*

$$\partial_t\xi = -(\bar{V}_\mathrm{n}\cdot\nabla)\xi - \big(\mathrm{Id}-\mathrm{n}_v(P_{I_v(t)}x)\otimes\mathrm{n}_v(P_{I_v(t)}x)\big)(\nabla\bar{V}_\mathrm{n})^T\xi \tag{3.29}$$

in the space-time domain $\mathrm{dist}(x,I_v(t)) < r_c$*, where we abbreviated* $\mathrm{n}_v(P_{I_v(t)}x) = \mathrm{n}_v(P_{I_v(t)}x,t)$.

Proof. We start by deriving a formula for the time evolution of the normal vector field $\mathrm{n}_v(P_{I_v(t)}x,t)$ in the space-time tubular neighborhood $\mathrm{dist}(x,I_v(t)) < r_c$. By (3.19), we may use the formula for the time evolution of the signed distance function from Lemma 3.11. More precisely, due to the regularity of the signed distance function to the interface of the strong solution and the regularity of the vector field $\bar{V}$ (Remark 3.12), we can interchange the differentiation in time and space to obtain

$$\begin{aligned}\partial_t\nabla\,\mathrm{dist}^\pm(x,I_v(t)) &= \nabla\partial_t\,\mathrm{dist}^\pm(x,I_v(t))\\ &\overset{(3.17)}{=} -\nabla\big((\bar{V}_\mathrm{n}\cdot\nabla)\,\mathrm{dist}^\pm(x,I_v(t))\big)\\ &= -(\bar{V}_\mathrm{n}\cdot\nabla)\mathrm{n}_v(P_{I_v(t)}x) - (\nabla\bar{V}_\mathrm{n})^T\cdot\mathrm{n}_v(P_{I_v(t)}x).\end{aligned}$$

Next, we show that the normal-normal component of $\nabla\bar{V}_\mathrm{n}$ vanishes. Observe that by Remark 3.12 and (3.19) it holds

$$\bar{V}_\mathrm{n}(x,t) = -\partial_t\,\mathrm{dist}^\pm(x,I_v(t))\nabla\,\mathrm{dist}^\pm(x,I_v(t)).$$

Hence, by (3.19)–(3.22) and this formula we obtain

$$\begin{aligned}&(\nabla\bar{V}_\mathrm{n})^T(x,t) : \mathrm{n}_v(P_{I_v(t)}x)\otimes\mathrm{n}_v(P_{I_v(t)}x)\\ &\quad= \nabla\bar{V}_\mathrm{n}(x,t)\nabla\,\mathrm{dist}^\pm(x,I_v(t))\cdot\nabla\,\mathrm{dist}^\pm(x,I_v(t))\\ &\quad= -\nabla\,\mathrm{dist}^\pm(x,I_v(t))\cdot\partial_t\nabla\,\mathrm{dist}^\pm(x,I_v(t))\\ &\qquad+ \bar{V}_\mathrm{n}(x,t)\otimes\nabla\,\mathrm{dist}^\pm(x,I_v(t)) : \nabla^2\,\mathrm{dist}^\pm(x,I_v(t))\\ &\quad= 0\end{aligned}$$

as desired. In summary, we have proved so far that

$$\begin{aligned}\partial_t\mathrm{n}_v(P_{I_v(t)}x) &= -(\bar{V}_\mathrm{n}\cdot\nabla)\mathrm{n}_v(P_{I_v(t)}x)\\ &\quad- \big(\mathrm{Id}-\mathrm{n}_v(P_{I_v(t)}x)\otimes\mathrm{n}_v(P_{I_v(t)}x)\big)(\nabla\bar{V}_\mathrm{n})^T\cdot\mathrm{n}_v(P_{I_v(t)}x),\end{aligned} \tag{3.30}$$

which holds in the space-time domain $\mathrm{dist}(x,I_v(t)) < r_c$. However, applying the chain rule to the cut-off function $r \mapsto \zeta(r)$ from (3.27) together with the evolution equation (3.17) for the signed distance to the interface shows that the cut-off away from the interface is also subject to a transport equation:

$$\partial_t\zeta\Big(\frac{\mathrm{dist}^\pm(x,I_v(t))}{r_c}\Big) = -(\bar{V}_\mathrm{n}(x,t)\cdot\nabla)\zeta\Big(\frac{\mathrm{dist}^\pm(x,I_v(t))}{r_c}\Big).$$

By the definition of the vector field ξ, see (3.28), and the product rule, this concludes the proof. □

3.3.3 Properties of the weighted volume term

We next discuss the weighted volume contribution $\int_{\mathbb{R}^d} |\chi_u - \chi_v| \operatorname{dist}(x, I_v(t))\,dx$ to the relative entropy in more detail.

Remark 3.16. *Let β be a truncation of the identity as in Proposition 3.10. Let $\chi_v \in L^\infty([0,T_{strong}); \mathrm{BV}(\mathbb{R}^d;\{0,1\}))$ be an indicator function such that $\Omega_t^+ := \{x \in \mathbb{R}^d\colon \chi_v(x,t) = 1\}$ is a family of smoothly evolving domains, and $I_v(t) := \partial\Omega_t^+$ is a family of smoothly evolving surfaces, in the sense of Definition 3.5. The map*

$$\mathbb{R}^d \times [0,T_{strong}) \ni (x,t) \mapsto \beta\big(\operatorname{dist}^\pm(x, I_v(t))/r_c\big)$$

inherits the regularity of the signed distance function to the interface $I_v(t)$. More precisely, this map (resp. its time derivative) is of class $C_t^0 C_x^3$ (resp. $C_t^0 C_x^2$).

Lemma 3.17. *Let $\chi_v \in L^\infty([0,T_{strong}); \mathrm{BV}(\mathbb{R}^d;\{0,1\}))$ be an indicator function such that $\Omega_t^+ := \{x \in \mathbb{R}^d\colon \chi_v(x,t) = 1\}$ is a family of smoothly evolving domains and $I_v(t) := \partial\Omega_t^+$ is a family of smoothly evolving surfaces in the sense of Definition 3.5. Let $v \in L^2_{loc}([0,T_{strong}]; H^1_{loc}(\mathbb{R}^d;\mathbb{R}^d))$ be a continuous solenoidal vector field such that χ_v solves the equation $\partial_t\chi_v = -\nabla\cdot(\chi_v v)$. Let $\bar{V}_{\mathrm{n}}$ be the extended normal velocity of the interface (3.18). Then the time evolution of the weight function β composed with the signed distance function to the interface $I_v(t)$ is given by the transport equation*

$$\partial_t\beta\Big(\frac{\operatorname{dist}^\pm(\cdot, I_v)}{r_c}\Big) = -(\bar{V}_{\mathrm{n}}\cdot\nabla)\beta\Big(\frac{\operatorname{dist}^\pm(\cdot, I_v)}{r_c}\Big) \tag{3.31}$$

for space-time points (x,t) such that $\operatorname{dist}(x, I_v(t)) < r_c$.

Proof. This is immediate from the chain rule and the time evolution of the signed distance function to the interface of the strong solution, see Lemma 3.11. □

3.3.4 Further coercivity properties of the relative entropy

We collect some further coercivity properties of the relative entropy functional $E\big[\chi_u, u, V\big|\chi_v, v\big]$ as defined in (3.9). These will be of frequent use in the estimation of the terms occurring on the right hand side of the relative entropy inequality from Proposition 3.10. We start for reference purposes with trivial consequences of our choices of the vector field ξ and the weight function β.

Lemma 3.18. *Consider the situation of Proposition 3.10. In particular, let β be the truncation of the identity from Proposition 3.10. By definition, it holds*

$$\min\Big\{\frac{\operatorname{dist}(x, I_v(t))}{r_c}, 1\Big\} \leq \Big|\beta\Big(\frac{\operatorname{dist}^\pm(x, I_v(t))}{r_c}\Big)\Big|. \tag{3.32}$$

Let ξ be the vector field from Definition 3.13 with cutoff multiplier ζ as given in (3.27). By the choice of the cutoff ζ, it holds

$$\frac{|\operatorname{dist}^\pm(x, I_v(t))|^2}{r_c^2} \leq 1 - \zeta\Big(\frac{\operatorname{dist}^\pm(x, I_v(t))}{r_c}\Big). \tag{3.33}$$

We will also make frequent use of the fact that for any unit vector $\mathrm{b} \in \mathbb{R}^d$ we have

$$1 - \zeta\Big(\frac{\operatorname{dist}^\pm(x, I_v(t))}{r_c}\Big) \leq 1 - \mathrm{b}\cdot\xi \quad \textit{and} \quad |\mathrm{b} - \xi|^2 \leq 2(1 - \mathrm{b}\cdot\xi). \tag{3.34}$$

We also want to emphasize that the relative entropy functional controls the squared error in the normal of the varifold.

Lemma 3.19. *Consider the situation of Proposition 3.10. We then have*

$$\int_{\mathbb{R}^d\times\mathbb{S}^{d-1}} \frac{1}{2}|s-\xi|^2\,\mathrm{d}V_t(x,s) \leq E\big[\chi_u,u,V\big|\chi_v,v\big](t) \tag{3.35}$$

for almost every $t\in[0,T_{strong})$.

Proof. Observe first that by means of the compatibility condition (3.2e) we have

$$\begin{aligned}\int_{\mathbb{R}^d\times\mathbb{S}^{d-1}} (1-s\cdot\xi\,)\,\mathrm{d}V_t(x,s) &= \int_{\mathbb{R}^d\times\mathbb{S}^{d-1}} 1\,\mathrm{d}V_t(x,s) - \int_{\mathbb{R}^d} \mathrm{n}_u\cdot\xi\,\mathrm{d}|\nabla\chi_u(\cdot,t)| \\ &= \int_{\mathbb{R}^d} 1\,\mathrm{d}|V_t|_{\mathbb{S}^{d-1}} - \int_{\mathbb{R}^d} \mathrm{n}_u\cdot\xi\,\mathrm{d}|\nabla\chi_u(\cdot,t)|,\end{aligned}$$

which holds for almost every $t\in[0,T_{strong})$. In addition, due to (3.4) one obtains

$$\int_{\mathbb{R}^d} 1-\theta_t\,\mathrm{d}|V_t|_{\mathbb{S}^{d-1}} = \int_{\mathbb{R}^d} 1\,\mathrm{d}|V_t|_{\mathbb{S}^{d-1}} - \int_{\mathbb{R}^d} 1\,\mathrm{d}|\nabla\chi_u(\cdot,t)|$$

for almost every $t\in[0,T_{strong})$. This in turn entails the following identity

$$\begin{aligned}&\int_{\mathbb{R}^d} \big(1-\mathrm{n}_u\cdot\xi\big)\,\mathrm{d}|\nabla\chi_u| + \int_{\mathbb{R}^d} 1-\theta_t\,\mathrm{d}|V_t|_{\mathbb{S}^{d-1}} \\ &= \int_{\mathbb{R}^d\times\mathbb{S}^{d-1}} (1-s\cdot\xi\,)\,\mathrm{d}V_t(x,s),\end{aligned}$$

which holds true for almost every $t\in[0,T_{strong})$. However, the functional on the right hand side controls the squared error in the normal of the varifold: $|s-\xi|^2\leq 2(1-s\cdot\xi)$. This proves the claim. □

We will also refer multiple times to the following bound. In the regime of equal shear viscosities $\mu_+=\mu_-$ we may apply this result with the choice $w=0$. In the general case, we have to include the compensation function w for the velocity gradient discontinuity at the interface.

Lemma 3.20. *Let (χ_u,u,V) be a varifold solution to (1.1a)-(1.1c) in the sense of Definition 3.2 on a time interval $[0,T_{vari})$ with initial data (χ_u^0,u_0). Let (χ_v,v) be a strong solution to (1.1a)-(1.1c) in the sense of Definition 3.6 on a time interval $[0,T_{strong})$ with $T_{strong}\leq T_{vari}$ and initial data (χ_v^0,v_0). Let $w\in L^2([0,T_{strong});H^1(\mathbb{R}^d;\mathbb{R}^d))$ be an arbitrary vector field, and let $F\in L^\infty(\mathbb{R}^d\times[0,T_{strong});\mathbb{R}^d)$ be a bounded vector field. Then*

$$\begin{aligned}&\bigg|\int_0^T\int_{\mathbb{R}^d}(\chi_u-\chi_v)(u-v-w)\cdot F\,\mathrm{d}x\,\mathrm{d}t\bigg| \\ &\leq \delta\int_0^T\int_{\mathbb{R}^d}|\nabla(u-v-w)|^2\,\mathrm{d}x\,\mathrm{d}t + C\frac{1+\|F\|_{L^\infty}^2}{\delta}\int_0^T\int_{\mathbb{R}^d}\rho(\chi_u)|u-v-w|^2\,\mathrm{d}x\,\mathrm{d}t \\ &\quad+\frac{C\|F\|_{L^\infty}}{\delta}\int_0^T\int_{\mathbb{R}^d}|\chi_u-\chi_v|\Big|\beta\Big(\frac{\mathrm{dist}^\pm(\cdot,I_v)}{r_c}\Big)\Big|\,\mathrm{d}x\,\mathrm{d}t\end{aligned}$$

for almost every $T\in[0,T_{strong})$ and all $0<\delta\leq 1$. The absolute constant $C>0$ only depends on the densities $\rho_\pm$.

Proof. We first argue how to control the part away from the interface of the strong solution, i.e., outside of $\{(x,t)\colon \operatorname{dist}(x,I_v(t)) \geq r_c\}$. A straightforward estimate using Hölder's and Young's inequality yields

$$
\begin{aligned}
&\Big|\int_0^T \int_{\{\operatorname{dist}(x,I_v(t))\geq r_c\}} (\chi_u-\chi_v)(u-v-w)\cdot F \,\mathrm{d}x\,\mathrm{d}t\Big| \\
&\leq \frac{\|F\|_{L^\infty}}{2}\int_0^T \int_{\{\operatorname{dist}(x,I_v(t))\geq r_c\}} |\chi_u-\chi_v| \,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \frac{\|F\|_{L^\infty}}{2}\int_0^T \int_{\{\operatorname{dist}(x,I_v(t))\geq r_c\}} |u-v-w|^2 \,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}
$$

Note that by the properties of the truncation of the identity β, see Proposition 3.10, it follows that $|\beta(\operatorname{dist}^\pm(x,I_v(t))/r_c)| \equiv 1$ on $\{(x,t)\colon \operatorname{dist}(x,I_v(t)) \geq r_c\}$. Hence, we obtain

$$
\begin{aligned}
&\Big|\int_0^T \int_{\{\operatorname{dist}(x,I_v(t))\geq r_c\}} (\chi_u-\chi_v)(u-v-w)\cdot F \,\mathrm{d}x\,\mathrm{d}t\Big| \\
&\leq \frac{\|F\|_{L^\infty}}{2}\int_0^T \int_{\mathbb{R}^d} |\chi_u-\chi_v|\cdot\Big|\beta\Big(\frac{\operatorname{dist}^\pm(\cdot,I_v)}{r_c}\Big)\Big| \,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \frac{\|F\|_{L^\infty}}{2(\rho_+\wedge\rho_-)}\int_0^T \int_{\mathbb{R}^d} \rho(\chi_u)|u-v-w|^2 \,\mathrm{d}x\,\mathrm{d}t,
\end{aligned}
\tag{3.36}
$$

which is indeed a bound of required order.

We proceed with the bound for the contribution in the vicinity of the interface of the strong solution. To this end, recall that we are equipped with a family of maps $\Phi_t\colon I_v(t)\times(-r_c,r_c)\to\mathbb{R}^d$ given by $\Phi_t(x,y) := x + y\mathrm{n}_v(x,t)$, which are C^2-diffeomorphisms onto their image $\{x\in\mathbb{R}^d\colon \operatorname{dist}(x,I_v(t)) < r_c\}$. Recall the estimates (3.14). We then move on with a change of variables, the one-dimensional Gagliardo-Nirenberg-Sobolev interpolation inequality

$$
\|g\|_{L^\infty(-r_c,r_c)} \leq C\|g\|_{L^2(-r_c,r_c)}^{\frac12}\|\nabla g\|_{L^2(-r_c,r_c)}^{\frac12} + C\|g\|_{L^2(-r_c,r_c)}
$$

as well as Hölder's and Young's inequality to obtain the bound

$$
\begin{aligned}
&\Big|\int_0^T \int_{\{\operatorname{dist}(x,I_v(t))< r_c\}} (\chi_u-\chi_v)(u-v-w)\cdot F \,\mathrm{d}x\,\mathrm{d}t\Big| \\
&\leq C\|F\|_{L^\infty}\int_0^T \int_{I_v(t)}\int_{-r_c}^{r_c} |(\chi_u-\chi_v)|(\Phi_t(x,y))\,|(u-v-w)|(\Phi_t(x,y))\,\mathrm{d}y\,\mathrm{d}S(x)\,\mathrm{d}t \\
&\leq C\|F\|_{L^\infty}\int_0^T \int_{I_v(t)} \sup_{y\in[-r_c,r_c]} |u-v-w|(x+y\mathrm{n}_v(x,t)) \\
&\qquad\qquad \times\Big(\int_{-r_c}^{r_c} |(\chi_u-\chi_v)|(x+y\mathrm{n}_v(x,t))\,\mathrm{d}y\Big)\,\mathrm{d}S(x)\,\mathrm{d}t \\
&\leq C\frac{\|F\|_{L^\infty}+\|F\|_{L^\infty}^2}{\delta}\int_0^T \int_{\mathbb{R}^d} |u-v-w|^2 \,\mathrm{d}x\,\mathrm{d}t + \delta\int_0^T \int_{\mathbb{R}^d} |\nabla(u-v-w)|^2 \,\mathrm{d}x\,\mathrm{d}t \\
&\quad + C\|F\|_{L^\infty}\int_0^T \int_{I_v(t)} \Big(\int_{-r_c}^{r_c} |(\chi_u-\chi_v)|(x+y\mathrm{n}_v(x,t))\,\mathrm{d}y\Big)^2\,\mathrm{d}S(x)\,\mathrm{d}t.
\end{aligned}
$$

It thus suffices to derive an estimate for the L^2-norm of the local interface error height in normal direction

$$
h(x) = \int_{-r_c}^{r_c} |(\chi_u-\chi_v)|(x+y\mathrm{n}_v(x,t))\,\mathrm{d}y.
$$

The proof of Proposition 3.26 below, where we establish next to the required L^2-bound also several other properties of the local interface error height, shows that (see (3.56))

$$\int_{I_v(t)} |h(x)|^2 \,\mathrm{d}S \leq C \int_{\mathbb{R}^d} |\chi_u - \chi_v| \min\Big\{\frac{\mathrm{dist}(x, I_v(t))}{r_c}, 1\Big\} \,\mathrm{d}x. \tag{3.37}$$

This then concludes the proof. □

We conclude this section with an $L^2_{\mathrm{tan}}L^\infty_{\mathrm{nor}}$-bound for H^1-functions on the tubular neighborhood around the evolving interfaces as well as a bound for the derivatives of the normal velocity of the interface of a strong solution in terms of the associated velocity field v, both of which will be used several times in the estimation of the terms on the right hand side of the relative entropy inequality of Proposition 3.10.

Lemma 3.21. *Consider the situation of Proposition 3.10. We have the estimate*

$$\int_{I_v(t)} \sup_{y \in [-r_c, r_c]} |g(x + y\mathrm{n}_v(x,t))|^2 \,\mathrm{d}S \leq C(\|g\|_{L^2}\|\nabla g\|_{L^2} + \|g\|^2_{L^2}) \tag{3.38}$$

valid for any $g \in H^1(\mathbb{R}^d)$.

Proof. Let $f \in H^1(-r_c, r_c)$. The one-dimensional Gagliardo-Nirenberg-Sobolev inerpolation inequality then implies

$$\|f\|_{L^\infty(-r_c,r_c)} \leq C\|f\|^{\frac{1}{2}}_{L^2(-r_c,r_c)}\|f'\|^{\frac{1}{2}}_{L^2(-r_c,r_c)} + C\|f\|_{L^2(-r_c,r_c)}.$$

From this we obtain together with Hölder's inequality

$$\begin{aligned}
&\int_{I_v(t)} \sup_{y \in [-r_c, r_c]} |g(x + y\mathrm{n}_v(x,t))|^2 \,\mathrm{d}S \\
&\leq C \int_{I_v(t)} \int_{-r_c}^{r_c} |g(x{+}y\mathrm{n}_v(x,t))|^2 \,\mathrm{d}y \,\mathrm{d}S \\
&\quad + C\Big(\int_{I_v(t)} \int_{-r_c}^{r_c} |g(x{+}y\mathrm{n}_v(x,t))|^2 \,\mathrm{d}y \,\mathrm{d}S\Big)^{\frac{1}{2}} \\
&\quad\quad \times \Big(\int_{I_v(t)} \int_{-r_c}^{r_c} |\nabla g(x{+}y\mathrm{n}_v(x,t))|^2 \,\mathrm{d}y \,\mathrm{d}S\Big)^{\frac{1}{2}}.
\end{aligned}$$

This implies (3.38) by making use of the C^2-diffeomorphisms $\Phi_t \colon I_v(t) \times (-r_c, r_c) \to \mathbb{R}^d$ given by $\Phi_t(x,y) = x + y\mathrm{n}_v(x,t)$ and the associated change of variables, using also the bound (3.14). □

Lemma 3.22. *Consider the situation of Proposition 3.10 and define the vector field*

$$V_\mathrm{n}(x,t) := \big(v(x,t) \cdot \mathrm{n}_v(P_{I_v(t)}x, t)\big)\mathrm{n}_v(P_{I_v(t)}x, t),$$

for $(x,t) \in \mathbb{R}^d \times [0, T_{strong})$ such that $\mathrm{dist}(x, I_v(t)) < r_c$. Then

$$\|\nabla V_\mathrm{n}\|_{L^\infty(\mathcal{O})} \leq C r_c^{-1}\|v\|_{L^\infty} + C\|\nabla v\|_{L^\infty}, \tag{3.39}$$

$$\|\nabla^2 V_\mathrm{n}\|_{L^\infty(\mathcal{O})} \leq C r_c^{-2}\|v\|_{L^\infty} + C r_c^{-1}\|\nabla v\|_{L^\infty} + C\|\nabla^2 v\|_{L^\infty_t L^\infty_x(\mathbb{R}^d \setminus I_v(t))}, \tag{3.40}$$

where $\mathcal{O} = \bigcup_{t \in (0, T_{strong})} \{x \in \mathbb{R}^d \colon \mathrm{dist}(x, I_v(t)) < r_c\} \times \{t\}$ denotes the space-time tubular neighborhood of width r_c of the evolving interface of the strong solution.

In particular, we have for $\bar{V}_\mathrm{n}(x,t) := V_\mathrm{n}(P_{I_v(t)}x, t)$ the estimate

$$|\bar{V}_\mathrm{n}(x,t) - V_\mathrm{n}(x,t)| \leq C r_c^{-1}\|v\|_{W^{1,\infty}} \,\mathrm{dist}(x, I_v(t)). \tag{3.41}$$

Proof. The estimates (3.39) and (3.40) are a direct consequence of the regularity requirements on the velocity field v of a strong solution, see Definition 3.6, the pointwise bounds (3.15) and the representation of the normal vector field on the interface in terms of the signed distance function (3.19). □

3.4 Weak-strong uniqueness of varifold solutions: The case of equal viscosities

In this section we provide a proof of the weak-strong uniqueness principle to the free boundary problem for the incompressible Navier-Stokes equation for two fluids (1.1a)-(1.1c) in the case of equal shear viscosities $\mu_+ = \mu_-$. Note that in this case the problematic viscous stress term R_{visc} in the relative entropy inequality (see Proposition 3.10) vanishes because of $\mu(\chi_u) - \mu(\chi_v) = 0$. In this setting, it is possible to choose $w \equiv 0$ which directly implies $A_{visc} = 0$, $A_{adv} = 0$, $A_{dt} = 0$, $A_{weightVol} = 0$, and $A_{surTen} = 0$. It remains to estimate the terms R_{surTen}, R_{adv}, R_{dt}, and $R_{weightVol}$ which are left on the right-hand side of the relative entropy inequality. We directly estimate these terms also for $w \neq 0$ in order to avoid unnecessary repetition, as the estimates for $w \neq 0$ are not more complicated but will be required for the case of different viscosities.

3.4.1 Estimate for the surface tension terms

We start by estimating the terms related to surface tension R_{surTen}.

Lemma 3.23. *Consider the situation of Proposition 3.10. The terms related to surface tension R_{surTen} are estimated by*

$$R_{surTen} \leq \delta \int_0^T \int_{\mathbb{R}^d} |\nabla(u-v-w)|^2 \,\mathrm{d}x \,\mathrm{d}t + C(\delta) r_c^{-4}\big(1 + \|v\|^2_{L_t^\infty W_x^{2,\infty}(\mathbb{R}^d \setminus I_v(t))}\big) \int_0^T E[\chi_u, u, V|\chi_v, v](t) \,\mathrm{d}t \tag{3.42}$$

for any $\delta > 0$.

Proof. We start by using (3.34) and (3.28) to estimate

$$\begin{aligned}
&-\sigma \int_0^T \int_{\mathbb{R}^d} \Big(\xi \cdot \frac{\nabla \chi_u}{|\nabla \chi_u|}\Big) \mathrm{n}_v(P_{I_v(t)}x) \cdot \big(\mathrm{n}_v(P_{I_v(t)}x) \cdot \nabla\big) v - \xi \cdot (\xi \cdot \nabla) v \,\mathrm{d}|\nabla \chi_u| \,\mathrm{d}t \\
&= \sigma \int_0^T \int_{\mathbb{R}^d} \Big(1 - \xi \cdot \frac{\nabla \chi_u}{|\nabla \chi_u|}\Big) \mathrm{n}_v(P_{I_v(t)}x) \cdot \big(\mathrm{n}_v(P_{I_v(t)}x) \cdot \nabla\big) v \,\mathrm{d}|\nabla \chi_u| \,\mathrm{d}t \\
&\quad + \sigma \int_0^T \int_{\mathbb{R}^d} \xi \cdot (\xi \cdot \nabla) v - \mathrm{n}_v(P_{I_v(t)}x) \cdot \big(\mathrm{n}_v(P_{I_v(t)}x) \cdot \nabla\big) v \,\mathrm{d}|\nabla \chi_u| \,\mathrm{d}t \\
&\leq C\|\nabla v\|_{L^\infty} \int_0^T \int_{\mathbb{R}^d} 1 - \xi \cdot \frac{\nabla \chi_u}{|\nabla \chi_u|} \,\mathrm{d}|\nabla \chi_u| \,\mathrm{d}t \\
&\quad + C\|\nabla v\|_{L^\infty} \int_0^T \int_{\mathbb{R}^d} 1 - \zeta\Big(\frac{\mathrm{dist}^\pm(x, I_v(t))}{r_c}\Big) \,\mathrm{d}|\nabla \chi_u| \,\mathrm{d}t \\
&\leq C\|v\|_{L_t^\infty W_x^{1,\infty}} \int_0^T E[\chi_u, u, V|\chi_v, v](t) \,\mathrm{d}t.
\end{aligned} \tag{3.43}$$

Recall from (3.35) that the squared error in the varifold normal is controlled by the relative entropy functional. Together with the bound from Lemma 3.20, (3.15) as well as (3.43) we get an estimate for the first four terms of R_{surTen}

$$
\begin{aligned}
&R_{surTen} \qquad (3.44)\\
&\leq C(\delta) r_c^{-4}(1+\|v\|_{L_t^\infty W_x^{1,\infty}}) \int_0^T E[\chi_u, u, V|\chi_v, v](t)\,\mathrm{d}t\\
&\quad + \frac{\delta}{2}\int_0^T \int_{\mathbb{R}^d} |\nabla(u-v-w)|^2 \,\mathrm{d}x\,\mathrm{d}t\\
&\quad + \sigma \int_0^T \int_{\mathbb{R}^d} \frac{\nabla\chi_u}{|\nabla\chi_u|}\cdot\big((\mathrm{Id}-\mathrm{n}_v(P_{I_v(t)}x)\otimes \mathrm{n}_v(P_{I_v(t)}x))(\nabla\bar{V}_\mathrm{n}-\nabla v)^T\xi\big)\,\mathrm{d}|\nabla\chi_u|\,\mathrm{d}t\\
&\quad + \sigma \int_0^T \int_{\mathbb{R}^d} \frac{\nabla\chi_u}{|\nabla\chi_u|}\cdot\big((\bar{V}_\mathrm{n}-v)\cdot\nabla\big)\xi\,\mathrm{d}|\nabla\chi_u|\,\mathrm{d}t
\end{aligned}
$$

for almost every $T \in [0, T_{strong})$ and all $\delta \in (0,1]$. To estimate the remaining two terms we decompose $\bar{V}_\mathrm{n} - v$ as

$$
\bar{V}_\mathrm{n} - v = (\bar{V}_\mathrm{n} - V_\mathrm{n}) + (V_\mathrm{n} - v), \tag{3.45}
$$

where the vector field V_n is given by

$$
V_\mathrm{n}(x,t) := \big(v(x,t)\cdot \mathrm{n}_v(P_{I_v(t)}x,t)\big)\mathrm{n}_v(P_{I_v(t)}x,t) \tag{3.46}
$$

in the space-time domain $\{\mathrm{dist}(x, I_v(t)) < r_c\}$ (i. e. in contrast to $\bar{V}_\mathrm{n}$, for V_n the velocity v is evaluated not at the projection of x onto the interface, but at x itself). Note that it will not matter as to how V_n and similar quantities are defined outside of the area $\{\mathrm{dist}(x, I_v(t)) < r_c\}$, as the terms will always be multiplied by suitable cutoffs which vanish outside of $\{\mathrm{dist}(x, I_v(t)) < r_c\}$. In the next two steps, we compute and bound the contributions from the two different parts in the decomposition (3.45) of the error $\bar{V}_\mathrm{n} - v$.

First step: Controlling the error $V_\mathrm{n} - v$

By definition of the vector field V_n in (3.46), we may write $V_\mathrm{n} - v = -\big(\mathrm{Id} - \mathrm{n}_v(P_{I_v(t)}x)\otimes \mathrm{n}_v(P_{I_v(t)}x)\big)v$. It is then not clear why the term

$$
\begin{aligned}
&\sigma \int_0^T \int_{\mathbb{R}^d} \frac{\nabla\chi_u}{|\nabla\chi_u|}\cdot\big((\mathrm{Id}-\mathrm{n}_v(P_{I_v(t)}x)\otimes \mathrm{n}_v(P_{I_v(t)}x))(\nabla V_\mathrm{n}-\nabla v)^T\xi\big)\,\mathrm{d}|\nabla\chi_u|\,\mathrm{d}t\\
&\quad + \sigma \int_0^T \int_{\mathbb{R}^d} \frac{\nabla\chi_u}{|\nabla\chi_u|}\cdot\big((V_\mathrm{n}-v)\cdot\nabla\big)\xi\,\mathrm{d}|\nabla\chi_u|\,\mathrm{d}t
\end{aligned}
$$

should be controlled by our relative entropy functional. However, the integrands enjoy a crucial cancellation

$$
(\mathrm{Id}-\mathrm{n}_v(P_{I_v(t)}x)\otimes \mathrm{n}_v(P_{I_v(t)}x))(\nabla V_\mathrm{n}-\nabla v)^T\xi + \big((V_\mathrm{n}-v)\cdot\nabla\big)\xi = 0 \tag{3.47}
$$

in the space-time domain $\{(x,t)\in\mathbb{R}^d\times[0,T_{strong}) : \mathrm{dist}(x, I_v(t)) < r_c\}$. To verify this cancellation, we first recall from (3.19) that $\nabla\,\mathrm{dist}^\pm(x, I_v(t)) = \mathrm{n}_v(P_{I_v(t)}x, t)$. We then start by rewriting

$$
\big((V_\mathrm{n}-v)\cdot\nabla\big)\xi = -\nabla\xi\ (\mathrm{Id}-\nabla\,\mathrm{dist}^\pm(\cdot, I_v)\otimes\nabla\,\mathrm{dist}^\pm(\cdot, I_v))v.
$$

Note that when the derivative hits the cutoff multiplier in the definition of ξ (see (3.28)), the resulting term on the right hand side of the last identity vanishes. Hence, we obtain together with (3.21)

$$
\begin{aligned}
&\big((V_\mathrm{n}-v)\cdot\nabla\big)\xi\\
&= -\zeta\big(r_c^{-1}\,\mathrm{dist}^\pm(\cdot, I_v)\big)\big(\nabla^2\,\mathrm{dist}^\pm(\cdot, I_v)\big)(\mathrm{Id}-\nabla\,\mathrm{dist}^\pm(\cdot, I_v)\otimes\nabla\,\mathrm{dist}^\pm(\cdot, I_v))v\\
&= -\zeta\big(r_c^{-1}\,\mathrm{dist}^\pm(\cdot, I_v)\big)\big(\nabla^2\,\mathrm{dist}^\pm(\cdot, I_v)\big)v.
\end{aligned}
$$

On the other side, another application of (3.21) yields

$$
\begin{aligned}
&(\nabla V_{\mathrm{n}} - \nabla v)^T \xi \\
&= -(\nabla v)^T \big(\mathrm{Id} - \mathrm{n}_v(P_{I_v(t)}x) \otimes \mathrm{n}_v(P_{I_v(t)}x)\big)\xi + \zeta\big(r_c^{-1} \operatorname{dist}^{\pm}(\cdot, I_v)\big)\big(\nabla^2 \operatorname{dist}^{\pm}(\cdot, I_v)\big)v \\
&= \zeta\big(r_c^{-1} \operatorname{dist}^{\pm}(\cdot, I_v)\big)\big(\nabla^2 \operatorname{dist}^{\pm}(\cdot, I_v)\big)v.
\end{aligned}
$$

Therefore, the cancellation (3.47) indeed holds true since by (3.21) the right-hand side of the last computation remains unchanged after projecting via $\mathrm{Id} - \mathrm{n}_v \otimes \mathrm{n}_v$.

Second step: Controlling the error $\bar{V}_{\mathrm{n}} - V_{\mathrm{n}}$

It remains to control the contributions from the following two quantities:

$$
\begin{aligned}
I &:= \int_0^T \int_{\mathbb{R}^d} \mathrm{n}_u \cdot \Big(\big(\mathrm{Id} - \mathrm{n}_v(P_{I_v(t)}x) \otimes \mathrm{n}_v(P_{I_v(t)}x)\big)(\nabla \bar{V}_{\mathrm{n}} - \nabla V_{\mathrm{n}})^T \xi\Big) \,\mathrm{d}|\nabla \chi_u| \,\mathrm{d}t, \\
II &:= \int_0^T \int_{\mathbb{R}^d} \mathrm{n}_u \cdot \big((\bar{V}_{\mathrm{n}} - V_{\mathrm{n}}) \cdot \nabla\big)\xi \,\mathrm{d}|\nabla \chi_u| \,\mathrm{d}t.
\end{aligned}
$$

Note first that we can write

$$
I = \int_0^T \int_{\mathbb{R}^d} (\mathrm{n}_u - \xi) \cdot \Big(\big(\mathrm{Id} - \mathrm{n}_v(P_{I_v(t)}x) \otimes \mathrm{n}_v(P_{I_v(t)}x)\big)(\nabla \bar{V}_{\mathrm{n}} - \nabla V_{\mathrm{n}})^T \xi\Big) \,\mathrm{d}|\nabla \chi_u| \,\mathrm{d}t.
$$

Moreover, recall from (3.23) the formula for the gradient of the projection onto the nearest point on the interface $I_v(t)$. The definition of V_{n} (see (3.46)) and $\bar{V}_{\mathrm{n}}(x) = V_{\mathrm{n}}(P_{I_v(t)}x)$, the product rule, (3.19), (3.15), and (3.21) imply using the definition of ξ and the property $|\xi| \leq 1$

$$
\begin{aligned}
&\Big|\big(\mathrm{Id} - \mathrm{n}_v(P_{I_v(t)}x) \otimes \mathrm{n}_v(P_{I_v(t)}x)\big)(\nabla \bar{V}_{\mathrm{n}} - \nabla V_{\mathrm{n}})^T \xi\Big| \\
&\leq \Big|\big(\mathrm{Id} - \mathrm{n}_v(P_{I_v(t)}x) \otimes \mathrm{n}_v(P_{I_v(t)}x)\big)(\nabla(v(P_{I_v(t)}x)) - \nabla v(x))^T\Big| \\
&\quad + \Big|\big(\mathrm{Id} - \mathrm{n}_v(P_{I_v(t)}x) \otimes \mathrm{n}_v(P_{I_v(t)}x)\big)(\nabla(\mathrm{n}_v(P_{I_v(t)}x))^T(v(P_{I_v(t)}x) - v(x))\Big| \\
&\quad + \|v\|_{L^\infty}\Big|(\nabla(\mathrm{n}_v(P_{I_v(t)}x)))^T \xi\Big| \\
&\leq C r_c^{-1} \|v\|_{W^{2,\infty}(\mathbb{R}^d \setminus I_v(t))} \operatorname{dist}(x, I_v(t))
\end{aligned}
$$

where in the last step we have used also (3.23). Together with Young's inequality and the coercivity properties of the relative entropy (3.33) and (3.34) we then immediately get the estimate

$$
\begin{aligned}
I &\leq C \int_0^T \int_{\mathbb{R}^d} |\mathrm{n}_u - \xi|^2 \,\mathrm{d}|\nabla \chi_u| \,\mathrm{d}t \\
&\quad + C r_c^{-4} \|v\|^2_{L_t^\infty W_x^{2,\infty}(\mathbb{R}^d \setminus I_v(t))} \int_0^T \int_{\mathbb{R}^d} |\operatorname{dist}(x, I_v(t))|^2 \,\mathrm{d}|\nabla \chi_u| \,\mathrm{d}t \\
&\leq C(1 + r_c^{-4} \|v\|^2_{L_t^\infty W_x^{2,\infty}(\mathbb{R}^d \setminus I_v(t))}) \int_0^T E[\chi_u, u, V|\chi_v, v](t) \,\mathrm{d}t. \qquad (3.48)
\end{aligned}
$$

To estimate the second term II, we start by adding zero and then use again $\bar{V}_{\mathrm{n}}(x,t) = V_{\mathrm{n}}(P_{I_v(t)}x,t)$, (3.39), (3.15) as well as (3.33) and (3.34)

$$\begin{aligned} II &= \int_0^T \int_{\mathbb{R}^d} (\mathrm{n}_u - \xi) \cdot \big((\bar{V}_{\mathrm{n}} - V_{\mathrm{n}}) \cdot \nabla\big)\xi \, \mathrm{d}|\nabla\chi_u| \, \mathrm{d}t \\ &\quad + \int_0^T \int_{\mathbb{R}^d} \xi \cdot \big((\bar{V}_{\mathrm{n}} - V_{\mathrm{n}}) \cdot \nabla\big)\xi \, \mathrm{d}|\nabla\chi_u| \, \mathrm{d}t \\ &\leq C(1 + r_c^{-2}\|v\|^2_{L_t^\infty W_x^{1,\infty}}) \int_0^T E[\chi_u, u, V|\chi_v, v](t) \, \mathrm{d}t \\ &\quad + \int_0^T \int_{\mathbb{R}^d} \xi \cdot \big((\bar{V}_{\mathrm{n}} - V_{\mathrm{n}}) \cdot \nabla\big)\xi \, \mathrm{d}|\nabla\chi_u| \, \mathrm{d}t. \end{aligned}$$

Using (3.21), we continue by computing

$$\begin{aligned} &\int_0^T \int_{\mathbb{R}^d} \xi \cdot \big((\bar{V}_{\mathrm{n}} - V_{\mathrm{n}}) \cdot \nabla\big)\xi \, \mathrm{d}|\nabla\chi_u| \, \mathrm{d}t \\ &= r_c^{-1} \int_0^T \int_{\mathbb{R}^d} \zeta'\Big(\frac{\mathrm{dist}^{\pm}(x, I_v(t))}{r_c}\Big)\xi \otimes (\bar{V}_{\mathrm{n}} - V_{\mathrm{n}}) : \mathrm{n}_v(P_{I_v(t)}) \otimes \mathrm{n}_v(P_{I_v(t)}) \, \mathrm{d}|\nabla\chi_u| \, \mathrm{d}t \end{aligned}$$

Hence, it follows from $\zeta'(0) = 0$ and $|\zeta''| \leq C$ as well as (3.41) that

$$\begin{aligned} II &\leq C(1 + r_c^{-2}\|v\|^2_{L_t^\infty W_x^{1,\infty}}) \int_0^T E[\chi_u, u, V|\chi_v, v](t) \, \mathrm{d}t \\ &\quad + C r_c^{-3}\|v\|_{L_t^\infty W_x^{1,\infty}} \int_0^T \int_{\mathbb{R}^d} |\,\mathrm{dist}(x, I_v(t))|^2 \, \mathrm{d}|\nabla\chi_u| \, \mathrm{d}t \\ &\leq C(1 + r_c^{-3}\|v\|^2_{L_t^\infty W_x^{1,\infty}}) \int_0^T E[\chi_u, u, V|\chi_v, v](t) \, \mathrm{d}t. \end{aligned} \tag{3.49}$$

Third step: Summary

Inserting (3.47), (3.48), and (3.49) into (3.44) entails the bound

$$\begin{aligned} &R_{surTen} \\ &\leq \frac{C(\delta)}{r_c^4}(1{+}\|v\|_{L_t^\infty W_x^{2,\infty}(\mathbb{R}^d \setminus I_v(t))} \vee \|v\|^2_{L_t^\infty W_x^{2,\infty}(\mathbb{R}^d \setminus I_v(t))}) \int_0^T E[\chi_u, u, V|\chi_v, v](t) \, \mathrm{d}t \\ &\quad + \delta \int_0^T \int_{\mathbb{R}^d} |\nabla(u - v - w)|^2 \, \mathrm{d}x \, \mathrm{d}t. \end{aligned}$$

This yields the desired estimate. □

3.4.2 Estimate for the remaining terms R_{adv}, R_{dt}, and $R_{weightVol}$

To bound the advection-related terms

$$\begin{aligned} R_{adv} &= -\int_0^T \int_{\mathbb{R}^d} \big(\rho(\chi_u) - \rho(\chi_v)\big)(u - v - w) \cdot (v \cdot \nabla)v \, \mathrm{d}x \, \mathrm{d}t \\ &\quad - \int_0^T \int_{\mathbb{R}^d} \rho(\chi_u)(u - v - w) \cdot \big((u - v - w) \cdot \nabla\big)v \, \mathrm{d}x \, \mathrm{d}t \end{aligned}$$

from the relative entropy inequality, the time-derivative related terms R_{dt}, and the terms resulting from the weighted volume control term in the relative entropy

$$R_{weightVol} := \int_0^T \int_{\mathbb{R}^d} (\chi_u - \chi_v)\big((\bar{V}_{\mathrm{n}} - V_{\mathrm{n}}) \cdot \nabla\big)\beta\Big(\frac{\mathrm{dist}^{\pm}(\cdot, I_v)}{r_c}\Big)\,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_{\mathbb{R}^d} (\chi_u - \chi_v)\big((u - v - w) \cdot \nabla\big)\beta\Big(\frac{\mathrm{dist}^{\pm}(\cdot, I_v)}{r_c}\Big)\,\mathrm{d}x\,\mathrm{d}t$$

(with $V_{\mathrm{n}}(x,t) := (\mathrm{n}_v(P_{I_v(t)}x,t) \otimes \mathrm{n}_v(P_{I_v(t)}x,t))v(x,t)$), we use mostly straightforward estimates.

Lemma 3.24. *Consider the situation of Proposition 3.10. The terms R_{adv}, R_{dt}, and $R_{weightVol}$ are subject to the bounds*

$$R_{adv} \le C(\delta)(1 + \|v\|^4_{L^\infty_t W^{1,\infty}_x}) \int_0^T E[\chi_u, u, V|\chi_v, v](t)\,\mathrm{d}t + \delta \int_0^T \int_{\mathbb{R}^d} |\nabla(u - v - w)|^2\,\mathrm{d}x\,\mathrm{d}t, \tag{3.50}$$

$$R_{dt} \le \delta \int_0^T \int_{\mathbb{R}^d} |\nabla(u - v - w)|^2\,\mathrm{d}x\,\mathrm{d}t + C(\delta)\|\partial_t v\|_{L^\infty_{x,t}(\mathbb{R}^d \setminus I_v(t))} \int_0^T E[\chi_u, u, V|\chi_v, v](t)\,\mathrm{d}t, \tag{3.51}$$

and

$$R_{weightVol} \le \delta \int_0^T \int_{\mathbb{R}^d} |\nabla(u - v - w)|^2\,\mathrm{d}x\,\mathrm{d}t + C(\delta)r_c^{-2}(1 + \|v\|_{L^\infty_t W^{1,\infty}_x}) \int_0^T E[\chi_u, u, V|\chi_v, v](t)\,\mathrm{d}t \tag{3.52}$$

for any $\delta > 0$.

Proof. To derive (3.50), we use a direct estimate for the second term in R_{adv} as well as Lemma 3.20 for the first term.

The bound (3.51) is derived similarly.

Finally, we show estimate (3.52). Note that by definition we have $\bar{V}_{\mathrm{n}}(x,t) = V_{\mathrm{n}}(P_{I_v(t)}x,t)$. Hence, we obtain using the bound (3.41) as well as (3.32) and $|\beta'| \le C$

$$R_{weightVol} \le C\|v\|_{L^\infty_t W^{1,\infty}_x} \int_0^T \int_{\mathbb{R}^d} |\chi_u - \chi_v|\,\Big|\beta\Big(\frac{\mathrm{dist}^{\pm}(x, I_v(t))}{r_c}\Big)\Big|\,\mathrm{d}x\,\mathrm{d}t + Cr_c^{-1} \int_0^T \int_{\{\mathrm{dist}(x,I_v(t)) \le r_c\}} |\chi_u - \chi_v||u - v - w|\,\mathrm{d}x\,\mathrm{d}t.$$

An application of Lemma 3.20 yields (3.52). □

3.4.3 The weak-strong uniqueness principle in the case of equal viscosities

We conclude our discussion of the case of equal shear viscosities $\mu_+ = \mu_-$ for the free boundary problem for the incompressible Navier-Stokes equation for two fluids (1.1a)-(1.1c) with the proof of the weak-strong uniqueness principle.

Proposition 3.25. *Let $d \leq 3$. Let (χ_u, u, V) be a varifold solution to the free boundary problem for the incompressible Navier-Stokes equation for two fluids (1.1a)-(1.1c) in the sense of Definition 3.2 on some time interval $[0, T_{vari})$ with initial data (χ_u^0, u_0). Let (χ_v, v) be a strong solution to (1.1a)-(1.1c) in the sense of Definition 3.6 on some time interval $[0, T_{strong})$ with $T_{strong} \leq T_{vari}$ and initial data (χ_v^0, v_0). We assume that the shear viscosities of the two fluids coincide, i.e., $\mu^+ = \mu^-$.*

Then, there exists a constant $C > 0$ which only depends on the data of the strong solution such that the stability estimate

$$E[\chi_u, u, V|\chi_v, v](T) \leq E[\chi_u, u, V|\chi_v, v](0)e^{CT}$$

holds. In particular, if the initial data of the varifold solution and the strong solution coincide, the varifold solution must be equal to the strong solution in the sense

$$\chi_u(\cdot, t) = \chi_v(\cdot, t) \qquad \text{and} \qquad u(\cdot, t) = v(\cdot, t)$$

almost everywhere for almost every $t \in [0, T_{strong})$. Furthermore, in this case the varifold is given by

$$\mathrm{d}V_t = \delta_{\frac{\nabla \chi_v}{|\nabla \chi_v|}} \mathrm{d}|\nabla \chi_v|$$

for almost every $t \in [0, T_{strong})$.

Proof. Applying the relative entropy inequality from Proposition 3.10 with $w = 0$, using the fact that the problematic term R_{visc} vanishes in the case of equal shear viscosities $\mu_+ = \mu_-$, as well as using the bounds from (3.42), (3.50), (3.51) and (3.52), we observe that we established the following bound

$$\begin{aligned}
&E[\chi_u, u, V|\chi_v, v](T) + c\int_0^T \int_{\mathbb{R}^d} |\nabla u - \nabla v|^2 \,\mathrm{d}x \,\mathrm{d}t \qquad (3.53)\\
&\leq E[\chi_u, u, V|\chi_v, v](0) + \delta\int_0^T \int_{\mathbb{R}^d} |\nabla u - \nabla v|^2 \,\mathrm{d}x \,\mathrm{d}t \\
&\quad + \frac{C(\delta)}{r_c^4}\big(1 + \|\partial_t v\|_{L^\infty_{x,t}(\mathbb{R}^d \setminus I_v(t))} + \|v\|_{L^\infty_t W^{2,\infty}_x(\mathbb{R}^d \setminus I_v(t))} \vee \|v\|^2_{L^\infty_t W^{2,\infty}_x(\mathbb{R}^d \setminus I_v(t))}\big) \\
&\quad \times \int_0^T E[\chi_u, u, V|\chi_v, v](t) \,\mathrm{d}t
\end{aligned}$$

for almost every $T \in [0, T_{strong})$. An absorption argument along with a subsequent application of Gronwall's lemma then immediately yields the asserted stability estimate.

Consider the case of coinciding initial conditions, i.e., $E[\chi_u, u, V|\chi_v, v](0) = 0$. In this case, we deduce from the stability estimate that the relative entropy vanishes for almost every $t \in [0, T_{strong})$. From this it immediately follows that $u(\cdot, t) = v(\cdot, t)$ as well as $\chi_u(\cdot, t) = \chi_v(\cdot, t)$ almost everywhere for almost every $t \in [0, T_{strong})$.

The asserted representation of the varifold V of the varifold solution follows from the following considerations. First, we deduce $|\nabla \chi_u(\cdot, t)| = |V_t|_{\mathbb{S}^{d-1}}$ for almost every $t \in [0, T_{strong})$ as a consequence of the fact that the density of the varifold satisfies $\theta_t = \frac{\mathrm{d}|\nabla \chi_u(\cdot,t)|}{\mathrm{d}|V_t|_{\mathbb{S}^{d-1}}} \equiv 1$ almost everywhere for almost every $t \in [0, T_{strong})$. The remaining fact that the measure on $\mathbb{S}^{d-1}$ is given by $\delta_{\mathrm{n}_u(x,t)}$ for $|V_t|_{\mathbb{S}^{d-1}}$-almost every $x \in \mathbb{R}^d$ for almost every $t \in [0, T_{strong})$ then follows from the control of the squared error in the normal of the varifold by the relative entropy functional, see (3.35). This concludes the proof. ∎

3.5 Weak-strong uniqueness of varifold solutions: The case of different viscosities

We turn to the derivation of the weak-strong uniqueness principle in the case of different shear viscosities of the two fluids. In this regime, we cannot anymore ignore the viscous stress term $(\mu(\chi_v)-\mu(\chi_u))(\nabla v+\nabla v^T)$. The key idea is to construct a solenoidal vector field w which is small in the L^2-norm but whose gradient compensates for most of this problematic term, and then use the relative entropy inequality from Proposition 3.10 with this function. The precise definition as well as a list of all the relevant properties of this vector field are the content of Proposition 3.28.

A main ingredient for the construction of w are the local interface error heights as measured in orthogonal direction from the interface of the strong solution (see Figure 3.2). For this reason, we first prove the relevant properties of the local heights of the interface error in Proposition 3.26. However, in order to control certain surface-tension terms in the relative entropy inequality, we actually need the vector field w to have bounded spatial derivatives. To this aim, we perform an additional regularization of the height functions. This will be carried out in detail in Proposition 3.27 by a (time-dependent) mollification. After all these preparations, in Section 3.5.4–3.5.8 we then further estimate the additional terms A_{visc}, A_{dt}, A_{adv}, and A_{surTen} in the relative entropy inequality from Proposition 3.10. Based on these bounds, in Section 3.5.9 we finally provide the proof of the stability estimate and the weak-strong uniqueness principle for varifold solutions to the free boundary problem for the incompressible Navier-Stokes equation for two fluids (1.1a)-(1.1c) from Theorem 3.1.

3.5.1 The evolution of the local height of the interface error

Consider a strong solution (χ_v, v) to the free boundary problem for the incompressible Navier-Stokes equation for two fluids (1.1a)-(1.1c) in the sense of Definition 3.6 on some time interval $[0, T_{strong})$. For the sake of better readability, let us recall some definitions and constructions related to the associated family of evolving interfaces $I_v(t)$ of the strong solution.

For the family $(\Omega_t^+)_{t\in[0,T_{strong})}$ of smoothly evolving domains of the strong solution, the associated signed distance function is given by

$$\mathrm{dist}^\pm(x, I_v(t)) = \begin{cases} \mathrm{dist}(x, I_v(t)), & x \in \Omega_t^+, \\ -\mathrm{dist}(x, I_v(t)), & x \notin \Omega_t^+. \end{cases}$$

From Definition 3.5 of a family of smoothly evolving domains it follows that the family of maps $\Phi_t\colon I_v(t)\times(-r_c, r_c)\to\mathbb{R}^d$ given by $\Phi_t(x,y) := x + y\mathrm{n}_v(x,t)$ are C^2-diffeomorphisms onto their image $\{x\in\mathbb{R}^d\colon \mathrm{dist}(x, I_v(t)) < r_c\}$. Here, $\mathrm{n}_v(\cdot,t)$ denotes the normal vector field of the interface $I_v(t)$ pointing inwards $\{x\in\mathbb{R}^d\colon \chi_v(x,t)=1\}$. The signed distance function (resp. its time derivative) to the interface $I_v(t)$ of the strong solution is then of class $C_t^0C_x^3$ (resp. $C_t^0C_x^2$) in the space-time tubular neighborhood $\bigcup_{t\in[0,T_{strong})}\mathrm{im}(\Phi_t)\times\{t\}$ due to the regularity assumptions in Definition 3.5. Moreover, the projection $P_{I_v(t)}x$ of a point x onto the nearest point on the manifold $I_v(t)$ is well-defined and of class $C_t^0C_x^2$ in the same tubular neighborhood. Observe that the inverse of Φ_t is given by $\Phi_t^{-1}(x) = (P_{I_v(t)}x, \mathrm{dist}^\pm(x, I_v(t)))$ for all $x\in\mathbb{R}^d$ such that $\mathrm{dist}(x, I_v(t)) < r_c$.

In Lemma 3.11, we computed the time evolution of the signed distance function to the interface $I_v(t)$ of a strong solution. Recall also the various relations for the projected inner unit normal vector field $\mathrm{n}_v(P_{I_v(t)}x, t)$ from Lemma 3.11, which will be of frequent use in subsequent computations. Finally, we remind the reader of the definition of the vector field ξ from Definition 3.13, which is a global extension of the inner unit normal vector field of the interface $I_v(t)$. For an illustration of the vector field ξ, we recall Figure 3.1; for an illustration of h^+, we refer to Figure 3.2.

Proposition 3.26. *Let $\chi_v \in L^\infty([0,T_{strong});\mathrm{BV}(\mathbb{R}^d;\{0,1\}))$ be an indicator function such that $\Omega_t^+ := \{x \in \mathbb{R}^d \colon \chi_v(x,t)=1\}$ is a family of smoothly evolving domains and $I_v(t) := \partial\Omega_t^+$ is a family of smoothly evolving surfaces in the sense of Definition 3.5. Let ξ be the extension of the unit normal vector field n_v from Definition 3.13.*

Let $\theta : [0,\infty) \to [0,1]$ be a smooth cutoff with $\theta \equiv 0$ outside of $[0,\frac{1}{2}]$ and $\theta \equiv 1$ in $[0,\frac{1}{4}]$. For an indicator function $\chi_u \in L^\infty([0,T_{strong}];\mathrm{BV}(\mathbb{R}^d;\{0,1\}))$ and $t \geq 0$, we define the local height of the one-sided interface error $h^+(\cdot,t) : I_v(t) \to \mathbb{R}_0^+$ as

$$h^+(x,t) := \int_0^\infty (1-\chi_u)(x+y\mathrm{n}_v(x,t),t)\,\theta\Big(\frac{y}{r_c}\Big)\,\mathrm{d}y. \tag{3.54}$$

Similarly, we introduce the local height of the interface error in the other direction

$$h^-(x,t) := \int_0^\infty \chi_u(x-y\mathrm{n}_v(x,t),t)\theta\Big(\frac{y}{r_c}\Big)\,\mathrm{d}y.$$

Then h^+ and h^- have the following properties:
***a) (L^2-bound)** We have the estimates $|h^\pm(x,t)| \leq \frac{r_c}{2}$ and*

$$\int_{I_v(t)} |h^\pm(x,t)|^2\,\mathrm{d}S(x) \leq C\int_{\mathbb{R}^d} |\chi_u - \chi_v| \min\Big\{\frac{\mathrm{dist}(x,I_v(t))}{r_c},1\Big\}\,\mathrm{d}x. \tag{3.55a}$$

***b) (H^1-bound)** Moreover, the estimate holds*

$$\begin{aligned}&\int_{I_v(t)} \min\{|\nabla^{\tan} h^\pm(x,t)|^2, |\nabla^{\tan} h^\pm(x,t)|\}\,\mathrm{d}S + |D^s h^\pm|(I_v(t)) \\ &\leq C\int_{\mathbb{R}^d} 1-\xi\cdot\frac{\nabla\chi_u}{|\nabla\chi_u|}\,\mathrm{d}|\nabla\chi_u| + \frac{C}{r_c^2}\int_{\mathbb{R}^d}|\chi_u-\chi_v|\min\Big\{\frac{\mathrm{dist}(x,I_v(t))}{r_c},1\Big\}\,\mathrm{d}x.\end{aligned} \tag{3.55b}$$

***c) (Approximation property)** The functions h^+ and h^- provide an approximation of the set $\{\chi_u = 1\}$ in terms of a subgraph over the set $I_v(t)$ by setting*

$$\chi_{v,h^+,h^-} := \chi_v - \chi_{0\leq \mathrm{dist}^\pm(x,I_v(t))\leq h^+(P_{I_v(t)}x,t)} + \chi_{-h^-(P_{I_v(t)}x,t)\leq \mathrm{dist}^\pm(x,I_v(t))\leq 0},$$

up to an error of

$$\begin{aligned}&\int_{\mathbb{R}^d} \big|\chi_u - \chi_{v,h^+,h^-}\big|\,\mathrm{d}x \\ &\leq C\int_{\mathbb{R}^d} 1-\xi\cdot\frac{\nabla\chi_u}{|\nabla\chi_u|}\,\mathrm{d}|\nabla\chi_u| + C\int_{\mathbb{R}^d}|\chi_u-\chi_v|\min\Big\{\frac{\mathrm{dist}(x,I_v(t))}{r_c},1\Big\}\,\mathrm{d}x.\end{aligned} \tag{3.55c}$$

***d) (Time evolution)** Let v be a solenoidal vector field*

$$v \in L^2([0,T_{strong}];H^1(\mathbb{R}^d;\mathbb{R}^d)) \cap L^\infty([0,T_{strong}];W^{1,\infty}(\mathbb{R}^d;\mathbb{R}^d))$$

such that in the domain $\bigcup_{t\in[0,T_{strong})}(\Omega_t^+\cup\Omega_t^-)\times\{t\}$ the second spatial derivatives of the vector field v exist and satisfy $\sup_{t\in[0,T_{strong})}\sup_{x\in\Omega_t^+\cup\Omega_t^-}|\nabla^2 v(x,t)| < \infty$. Assume that χ_v solves the equation $\partial_t\chi_v = -\nabla\cdot(\chi_v v)$. If χ_u solves the equation $\partial_t\chi_u = -\nabla\cdot(\chi_u u)$ for another solenoidal vector field $u \in L^2([0,T_{strong}];H^1(\mathbb{R}^d;\mathbb{R}^d))$, we have the following estimate on the *time derivative of the local interface error heights* $h^\pm$*:*

$$\begin{aligned}
&\bigg|\frac{\mathrm{d}}{\mathrm{d}t}\int_{I_v(t)}\eta(x)h^\pm(x,t)\,\mathrm{d}S(x)-\int_{I_v(t)}h^\pm(x,t)(\mathrm{Id}-\mathrm{n}_v\otimes\mathrm{n}_v)v(x,t)\cdot\nabla\eta(x)\,\mathrm{d}S(x)\bigg| \qquad (3.55\mathrm{d})\\
&\leq\frac{C}{r_c^2}\|\eta\|_{W^{1,4}(I_v(t))}\bigg(\int_{I_v(t)}|\bar h^\pm|^4\,\mathrm{d}S\bigg)^{1/4}\\
&\qquad\times\bigg(\int_{I_v(t)}\sup_{y\in[-r_c,r_c]}|u-v|^2(x+y\mathrm{n}_v(x,t),t)\,\mathrm{d}S(x)\bigg)^{1/2}\\
&\quad+C\frac{1+\|v\|_{W^{2,\infty}(\mathbb{R}^d\setminus I_v(t))}}{r_c^3}\|\eta\|_{L^2(I_v(t))}\\
&\qquad\times\bigg(\int_{\mathbb{R}^d}|\chi_u(x,t)-\chi_v(x,t)|\min\Big\{\frac{\mathrm{dist}(x,I_v(t))}{r_c},1\Big\}\,\mathrm{d}x\bigg)^{\frac12}\\
&\quad+\frac{C(1+\|v\|_{W^{1,\infty}})}{r_c^2}\max_{p\in\{2,4\}}\|\eta\|_{W^{1,p}(I_v(t))}\int_{\mathbb{R}^d}1-\xi\cdot\frac{\nabla\chi_u}{|\nabla\chi_u|}\,\mathrm{d}|\nabla\chi_u|\\
&\quad+C\|\eta\|_{L^2(I_v(t))}\bigg(\int_{I_v(t)}|u-v|^2\,\mathrm{d}S\bigg)^{1/2}
\end{aligned}$$

for any test function $\eta\in C^\infty_{cpt}(\mathbb{R}^d)$ *with* $\mathrm{n}_v\cdot\nabla\eta=0$ *on the interface* $I_v(t)$*, and where* $\bar h^\pm$ *is defined as* $h^\pm$ *but now with respect to the modified cut-off* $\bar\theta(\cdot)=\theta\big(\frac{\cdot}{2}\big)$*.*

Proof. **Step 1: Proof of the estimate on the L^2-norm.** The trivial estimate $|h^\pm(x,t)|\leq\frac{r_c}{2}$ follows directly from the definition of $h^\pm$. To establish the L^2-estimate, let $\ell^+(x):=\int_0^{r_c}(1-\chi_u)(x+y\mathrm{n}_v(x,t),t)\,\mathrm{d}y$. A straighforward estimate then gives

$$|\ell^+(x)|^2=2\int_0^{\ell^+(x)}y\,\mathrm{d}y\leq C\int_0^{r_c}|\chi_u(\Phi_t(x,y),t)-\chi_v(\Phi_t(x,y),t)|\frac{y}{r_c}\,\mathrm{d}y. \qquad (3.56)$$

Note that the term on the left hand side dominates $|h^+|^2$ since we dropped the cutoff function. Hence, the desired estimate on the L^2-norm of h^+ follows at once by a change of variables and recalling the fact that $\mathrm{dist}(\Phi_t(x,y),I_v(t))=y$. The corresponding bound for h^- then follows along the same lines.

Step 2: Proof of the estimate on the spatial derivative (3.55b)**.** The definition (3.54) is equivalent to

$$h^+(\Phi_t(x,0),t)=\int_0^\infty(1-\chi_u)(\Phi_t(x,y))\,\theta\Big(\frac{y}{r_c}\Big)\,\mathrm{d}y.$$

We compute for any smooth vector field $\eta\in C^\infty_{cpt}(\mathbb{R}^d;\mathbb{R}^d)$ (recall that $\Phi_t(x,0)=x$ and $\mathrm{dist}(\Phi_t(x,y),I_v(t))=y$ for any $x\in I_v(t)$ and any y with $|y|\leq r_c$)

$$\begin{aligned}
&\int_{I_v(t)}\eta(x)\cdot\mathrm{d}(D_x^{\tan}h^+(\cdot,t))(x)\\
&=-\int_{I_v(t)}h^+(x,t)\nabla^{\tan}\cdot\eta(x)\,\mathrm{d}S(x)-\int_{I_v(t)}h^+(x,t)\eta(x)\cdot\mathrm{H}(x,t)\,\mathrm{d}S(x)\\
&=-\int_0^{r_c}\int_{I_v(t)}(1-\chi_u)(\Phi_t(x,y),t)\theta\Big(\frac{y}{r_c}\Big)\nabla^{\tan}\cdot\eta(x)\,\mathrm{d}S(x)\,\mathrm{d}y\\
&\quad-\int_0^{r_c}\int_{I_v(t)}(1-\chi_u)(\Phi_t(x,y),t)\theta\Big(\frac{y}{r_c}\Big)\eta(x)\cdot\mathrm{H}(\Phi_t(x,0),t)\,\mathrm{d}S(x)\,\mathrm{d}y
\end{aligned}$$

$$
\begin{aligned}
&= -\int_{\mathbb{R}^d} (1-\chi_u)(x,t)\theta\Big(\frac{\operatorname{dist}(x,I_v(t))}{r_c}\Big)|\det \nabla\Phi_t^{-1}(x)| \\
&\qquad\qquad \times (\mathrm{Id} - \mathrm{n}_v(P_{I_v(t)}x)\otimes \mathrm{n}_v(P_{I_v(t)}x)) : \nabla\eta(P_{I_v(t)}x)\,\mathrm{d}x \\
&\quad - \int_{\mathbb{R}^d} (1-\chi_u)(x,t)\theta\Big(\frac{\operatorname{dist}(x,I_v(t))}{r_c}\Big)\eta(P_{I_v(t)}x)\cdot \mathrm{H}(P_{I_v(t)}x)|\det\nabla\Phi_t^{-1}(x)|\,\mathrm{d}x \\
&= -\int_{\mathbb{R}^d} \theta\Big(\frac{\operatorname{dist}(x,I_v(t))}{r_c}\Big)|\det\nabla\Phi_t^{-1}(x)|\eta(P_{I_v(t)}x)(\mathrm{Id} - \mathrm{n}_v(P_{I_v(t)}x)\otimes\mathrm{n}_v(P_{I_v(t)}x))\cdot \mathrm{d}\nabla\chi_u \\
&\quad + \int_{\mathbb{R}^d} (1-\chi_u)(x,t)\theta\Big(\frac{\operatorname{dist}(x,I_v(t))}{r_c}\Big)\eta(P_{I_v(t)}x) \\
&\qquad \cdot\Big(\nabla\cdot\big((\mathrm{Id} - \mathrm{n}_v(P_{I_v(t)}x)\otimes\mathrm{n}_v(P_{I_v(t)}x))|\det\nabla\Phi_t^{-1}|\big) - \mathrm{H}(P_{I_v(t)}x)|\det\nabla\Phi_t^{-1}|\Big)\,\mathrm{d}x,
\end{aligned}
$$

where in the last step we have used $\nabla\operatorname{dist}^{\pm}(x,I_v(t)) = \mathrm{n}_v(P_{I_v(t)}x)$. This yields by another change of variables in the second integral, the fact that $\chi_v(\Phi_t(x,y),t) = 1$ for any $y>0$, (3.15), (3.16), $|\det\nabla\Phi_t^{-1}|\le C$ as well as by abbreviating $\mathrm{n}_u = \frac{\nabla\chi_u}{|\nabla\chi_u|}$

$$
\begin{aligned}
&\int_{U\cap I_v(t)} 1\,\mathrm{d}|D_x^{\tan}h^+(\cdot,t)| \\
&\le C\int_{\{x+y\mathrm{n}_v(x,t)\colon x\in U\cap I_v(t),y\in(-r_c,r_c)\}} \big|\mathrm{n}_v(P_{I_v(t)}x) - \mathrm{n}_u\big|\,\mathrm{d}|\nabla\chi_u(\cdot,t)| \\
&\quad + \frac{C}{r_c}\int_{U\cap I_v(t)}\int_0^{r_c} |\chi_u(\Phi_t(x,y),t) - \chi_v(\Phi_t(x,y),t)|\,\mathrm{d}y\,\mathrm{d}S(x)
\end{aligned}
$$

for any Borel set $U\subset\mathbb{R}^d$. Recall that the indicator function $\chi_u(\cdot,t)$ of the varifold solution is of bounded variation in $I := \{x\in\mathbb{R}^d\colon \operatorname{dist}^{\pm}(x,I_v(t))\in(-r_c,r_c)\}$. In particular, $E^+ := \{x\in\mathbb{R}^d\colon \chi_u>0\}\cap I$ is a set of finite perimeter in I. Applying Theorem 3.39 in local coordinates the sections

$$E_x^+ = \{y\in(-r_c,r_c)\colon \chi_u(x+y\mathrm{n}_v(x,t))>0\}$$

are guaranteed to be one-dimensional Caccioppoli sets in $(-r_c,r_c)$ for $\mathcal{H}^{d-1}$-almost every $x\in I_v(t)$. Note that whenever $|\mathrm{n}_v\cdot\mathrm{n}_u|\le\frac12$ then $1-\mathrm{n}_v\cdot\mathrm{n}_u\ge\frac12$, and therefore using also the co-area formula for rectifiable sets (see [12, (2.72)])

$$
\begin{aligned}
&\int_{U\cap I_v(t)} 1\,\mathrm{d}|D_x^{\tan}h^+(\cdot,t)| \\
&\le \frac{C}{r_c}\int_{U\cap I_v(t)}\int_0^{r_c} |\chi_u(\Phi_t(x,y),t) - \chi_v(\Phi_t(x,y),t)|\,\mathrm{d}y\,\mathrm{d}S(x) \\
&\quad + C\int_{U\cap I_v(t)}\int_{\partial^*E_x^+\cap\{\mathrm{n}_v(x)\cdot\mathrm{n}_u(x+y\mathrm{n}_v(x,t))\ge\frac12\}\cap(-r_c,r_c)} \frac{|\mathrm{n}_v(x)-\mathrm{n}_u|}{|\mathrm{n}_v(x)\cdot\mathrm{n}_u|}\,\mathrm{d}\mathcal{H}^0(y)\,\mathrm{d}S(x) \\
&\quad + C\int_{\{x+y\mathrm{n}_v(x,t)\colon x\in U\cap I_v(t),y\in(-r_c,r_c),\mathrm{n}_v(x)\cdot\mathrm{n}_u(x+y\mathrm{n}_v(x,t))\le\frac12\}} \big(1-\mathrm{n}_v(P_{I_v(t)}x)\cdot\mathrm{n}_u\big)\,\mathrm{d}|\nabla\chi_u(\cdot,t)|.
\end{aligned} \tag{3.57}
$$

We now distinguish between different cases depending on $x\in I_v(t)$ up to $\mathcal{H}^{d-1}$-measure zero. We start with the set of points $x\in A_1\subset I_v(t)$ such that

$$
\begin{aligned}
&\int_0^{r_c} |\chi_u(\Phi_t(x,y),t) - \chi_v(\Phi_t(x,y),t)|\,\mathrm{d}y \\
&+ \int_{\partial^*E_x^+\cap\{\mathrm{n}_v(x)\cdot\mathrm{n}_u(x+y\mathrm{n}_v(x,t))\ge\frac12\}\cap(-r_c,r_c)} \frac{|\mathrm{n}_v(x)-\mathrm{n}_u|}{|\mathrm{n}_v(x)\cdot\mathrm{n}_u|}\,\mathrm{d}\mathcal{H}^0(y) \\
&+ \sup_{y\in\{\tilde y\in(-r_c,r_c)\cap\partial^*E_x^+\colon \mathrm{n}_v(x)\cdot\mathrm{n}_u(x+\tilde y\mathrm{n}_v(x,t))\le\frac12\}} 1-\mathrm{n}_v(P_{I_v(t)}x)\cdot\mathrm{n}_u(x+y\mathrm{n}_v(x,t))
\end{aligned} \tag{3.58}
$$

$$\leq \frac{1}{4}.$$

By splitting the measure $D_x^{\tan} h^+$ into a part which is absolutely continuous with respect to the surface measure on $I_v(t)$, for which we denote the density by $\nabla^{\tan} h^+$, as well as a singular part $D^s h^+$, we obtain from (3.57) (note that the third integral in (3.57) does not contribute to this estimate by the definition of the set $A_1 \subset I_v(t)$)

$$\begin{aligned}
&\int_{U\cap I_v(t)\cap A_1} |\nabla^{\tan} h^+|(x)\,\mathrm{d}S(x)\\
&\leq \int_{U\cap I_v(t)\cap A_1} \frac{C}{r_c}\int_0^{r_c} |\chi_u(\Phi_t(x,y),t)-\chi_v(\Phi_t(x,y),t)|\,\mathrm{d}y\,\mathrm{d}S(x)\\
&\quad+\int_{U\cap I_v(t)\cap A_1} C\int_{\partial^* E_x^+\cap\{\mathrm{n}_v(x)\cdot\mathrm{n}_u(x+y\mathrm{n}_v(x,t))\geq\frac12\}\cap(-r_c,r_c)} \frac{|\mathrm{n}_v(x)-\mathrm{n}_u|}{|\mathrm{n}_v(x)\cdot\mathrm{n}_u|}\,\mathrm{d}\mathcal{H}^0(y)\,\mathrm{d}S(x)
\end{aligned}$$

for every Borel set $U \subset \mathbb{R}^d$. Since U was arbitrary, we deduce that $|\nabla^{\tan} h^+|$ is bounded on A_1 by the two integrands on the right hand side of the last inequality. Hence, we obtain

$$\begin{aligned}
&\int_{A_1} |\nabla^{\tan} h^+|^2(x)\,\mathrm{d}S(x) + |D^s h^+|(A_1)\\
&\leq C r_c^{-2}\int_{I_v(t)}\bigg|\int_0^{r_c} |\chi_u(\Phi_t(x,y),t)-\chi_v(\Phi_t(x,y),t)|\,\mathrm{d}y\bigg|^2\,\mathrm{d}S(x)\\
&\quad+C\int_{I_v(t)\cap A_1}\bigg|\int_{\partial^* E_x^+\cap\{\mathrm{n}_v(x)\cdot\mathrm{n}_u(x+y\mathrm{n}_v(x,t))\geq\frac12\}\cap(-r_c,r_c)} |\mathrm{n}_v-\mathrm{n}_u|\,\mathrm{d}\mathcal{H}^0(y)\bigg|^2\,\mathrm{d}S(x).
\end{aligned}$$

The first term on the right hand side can be estimated as in the proof of the L^2-bound for $h^\pm$. To bound the second term, we make the following observation. First, we may represent the one-dimensional Caccioppoli sets E_x^+ as a finite union of disjoint intervals (see [12, Proposition 3.52]). It then follows from property iv) in Theorem 3.39 that $\partial^* E_x^+\cap(-r_c,r_c)$ can only contain at most one point. Indeed, otherwise we would find at least one point $y \in \partial^* E_x^+ \cap (-r_c, r_c)$ such that $\mathrm{n}_v(x)\cdot\mathrm{n}_u(x+y\mathrm{n}_v(x,t)) < 0$ which is a contradiction to the definition of A_1. By another application of the co-area formula for rectifiable sets (see [12, (2.72)]) we therefore get

$$\begin{aligned}
&\int_{A_1} |\nabla^{\tan} h^+|^2(x)\,\mathrm{d}S(x) + |D^s h^+|(A_1)\\
&\leq \frac{C}{r_c^2}\int_{\mathbb{R}^d} |\chi_u-\chi_v|\min\Big\{\frac{\mathrm{dist}(x,I_v(t))}{r_c},1\Big\}\,\mathrm{d}x \qquad (3.59)\\
&\quad+C\int_{\{\mathrm{dist}(x,I_v(t))<r_c\}} 1-\mathrm{n}_v(P_{I_v(t)}x)\cdot\frac{\nabla\chi_u}{|\nabla\chi_u|}\,\mathrm{d}|\nabla\chi_u|(x).
\end{aligned}$$

We now turn to the second case, namely the set of points $A_2 := I_v(t)\setminus A_1$. We begin with a preliminary computation. When splitting E_x^+ into a finite family of disjoint open intervals as before, it again follows from property iv) in Theorem 3.39 that every second point $y \in \partial^* E_x^+ \cap (-r_c, r_c)$ has to have the property that $\mathrm{n}_v(x)\cdot\mathrm{n}_u(x+y\mathrm{n}_v(x,t)) < 0$, i.e., $|\mathrm{n}_v(x)-\mathrm{n}_u| \leq 2 \leq 2(1-\mathrm{n}_v(x)\cdot\mathrm{n}_u)$. In particular, by another application of the co-area formula for rectifiable sets (see [12, (2.72)]) we obtain the bound

$$\begin{aligned}
&\int_{A_2}\int_{\partial^* E_x^+\cap\{\mathrm{n}_v(x)\cdot\mathrm{n}_u(x+y\mathrm{n}_v(x,t))\geq\frac12\}\cap(-r_c,r_c)} \frac{|\mathrm{n}_v(x)-\mathrm{n}_u|}{|\mathrm{n}_v(x)\cdot\mathrm{n}_u|}\,\mathrm{d}\mathcal{H}^0(y)\,\mathrm{d}S(x)\\
&\leq 8\int_{\{\mathrm{dist}(x,I_v(t))<r_c\}} 1-\mathrm{n}_v(P_{I_v(t)}x)\cdot\frac{\nabla\chi_u}{|\nabla\chi_u|}\,\mathrm{d}|\nabla\chi_u|(x). \qquad (3.60)
\end{aligned}$$

Now, we proceed as follows. By definition of A_2, either one of the three summands in (3.58) has to be $\geq \frac{1}{12}$. We distinguish between two cases. If the third one is not, then this actually means that the set $\{\tilde{y} \in (-r_c, r_c) \cap \partial^* E_x^+ \colon \mathrm{n}_v(x) \cdot \mathrm{n}_u(x+\tilde{y}\mathrm{n}_v(x,t)) \leq \frac{1}{2}\}$ is empty, i.e., the third summand has to vanish. Hence, either one of the first two summands in (3.58) has to be $\geq \frac{1}{8}$. If the first one is not, we use that $\int_0^{r_c} |\chi_u(\Phi_t(x,y),t) - \chi_v(\Phi_t(x,y),t)| \,\mathrm{d}y \leq r_c$ and bound this by the second term and then (3.60). If the second one is not, then

$$
\begin{aligned}
\ell^+(x) &:= \int_0^{r_c} |\chi_u(\Phi_t(x,y),t) - \chi_v(\Phi_t(x,y),t)| \,\mathrm{d}y \leq r_c \\
&\leq \frac{C}{r_c} \int_0^{\ell^+(x)} y \,\mathrm{d}y \leq C \int_0^{r_c} |\chi_u(\Phi_t(x,y),t) - \chi_v(\Phi_t(x,y),t)| \frac{y}{r_c} \,\mathrm{d}y.
\end{aligned} \tag{3.61}
$$

Now, we move on with the remaining case, i.e., that the third summand in (3.58) does not vanish. In other words, $\{\tilde{y} \in (-r_c, r_c) \cap \partial^* E_x^+ \colon \mathrm{n}_v(x) \cdot \mathrm{n}_u(x+\tilde{y}\mathrm{n}_v(x,t)) \leq \frac{1}{2}\}$ is non-empty. We then estimate

$$
\begin{aligned}
&\int_0^{r_c} |\chi_u(\Phi_t(x,y),t) - \chi_v(\Phi_t(x,y),t)| \,\mathrm{d}y \\
&\leq r_c \leq 2r_c \int_{\partial^* E_x^+ \cap (-r_c, r_c)} 1 - \mathrm{n}_v(x) \cdot \mathrm{n}_u(x+y\mathrm{n}_v(x,t)) \,\mathrm{d}\mathcal{H}^0(y).
\end{aligned} \tag{3.62}
$$

Taking finally $U = A_2$ in (3.57), the conclusions of the above case study together with the three estimates (3.60), (3.61) and (3.62) followed by another application of the co-area formula for rectifiable sets (see [12, (2.72)]) to further estimate the latter, then imply that

$$
\begin{aligned}
&\int_{A_2} |\nabla^{\tan} h^+|(x) \,\mathrm{d}S(x) + |D^s h^+|(A_2) \\
&\leq \frac{C}{r_c} \int_{\mathbb{R}^d} |\chi_u - \chi_v| \min\Big\{ \frac{\mathrm{dist}(x, I_v(t))}{r_c}, 1 \Big\} \,\mathrm{d}x \\
&\quad + C \int_{\{\mathrm{dist}(x, I_v(t)) < r_c\}} 1 - \mathrm{n}_v(P_{I_v(t)}x) \cdot \frac{\nabla \chi_u}{|\nabla \chi_u|} \,\mathrm{d}|\nabla \chi_u|(x).
\end{aligned} \tag{3.63}
$$

The two estimates (3.59) and (3.63) thus entail the desired upper bound (3.55b) for the (tangential) gradient of $h^\pm$ with ξ replaced by $\mathrm{n}_v(P_{I_v(t)}x)$. However, one may replace $\mathrm{n}_v(P_{I_v(t)}x)$ by ξ because of (3.34).

Step 3: Proof of the approximation property for the interface (3.55c). In order to establish (3.55c), we rewrite using the coordinate transform Φ_t (recall that it holds $\mathrm{dist}^\pm(\Phi_t(x,y), I_v(t)) = y$ and that $|h^\pm| \leq r_c$)

$$
\begin{aligned}
&\int_{\mathbb{R}^d} |\chi_u - \chi_{v,h^+,h^-}| \,\mathrm{d}x \\
&= \int_{I_v(t)} \int_0^{r_c} \det \nabla \Phi_t(x,y) |\chi_u(\Phi_t(x,y)) - 1 + \chi_{\{y \leq h^+(x)\}}| \,\mathrm{d}y \,\mathrm{d}S(x) \\
&\quad + \int_{I_v(t)} \int_{-r_c}^0 \det \nabla \Phi_t(x,y) |\chi_u(\Phi_t(x,y)) - \chi_{\{y \geq -h^-(x)\}}| \,\mathrm{d}y \,\mathrm{d}S(x) \\
&\quad + \int_{\{\mathrm{dist}(x, I_v(t)) \geq r_c\}} |\chi_u - \chi_v| \,\mathrm{d}x.
\end{aligned} \tag{3.64}
$$

In order to derive a bound for the first term on the right-hand side of (3.64), we distinguish between different cases depending on $x \in I_v(t)$ up to $\mathcal{H}^{d-1}$-measure zero. We first distinguish between $h^+(x) \geq \frac{r_c}{4}$ and $h^+(x) < \frac{r_c}{4}$. In the former case, a straightforward estimate yields (recall (3.14))

$$\begin{aligned}&\bigg|\int_0^{r_c} \det\nabla\Phi_t(x,y)|\chi_u(\Phi_t(x,y)) - 1 + \chi_{\{y\leq h^+(x)\}}|\,\mathrm{d}y\bigg|\\&\quad\leq Cr_c \leq \frac{C}{r_c}\int_0^{h^+(x)} y\,\mathrm{d}y \leq C\int_0^{r_c} |\chi_u(\Phi_t(x,y)) - \chi_v(\Phi_t(x,y))|\frac{y}{r_c}\,\mathrm{d}y,\end{aligned} \tag{3.65}$$

which is indeed of required order after a change of variables. We now consider the other case, i.e., $h^+(x) < \frac{r_c}{4}$. Recall that the indicator function $\chi_u(\cdot,t)$ of the varifold solution is of bounded variation in $I^+ := \{x \in \mathbb{R}^d\colon \operatorname{dist}^\pm(x, I_v(t)) \in (0, r_c)\}$. In particular, $E^+ := \{x \in \mathbb{R}^d\colon 1 - \chi_u > 0\} \cap I_+$ is a set of finite perimeter in I^+. Recall also that $E^+ = I^+ \cap \{x \in \mathbb{R}^d\colon (\chi_v - \chi_u)_+ > 0\}$ since $\chi_v \equiv 1$ in I^+. Applying Theorem 3.39 in local coordinates, the sections

$$E^+_x = \{y \in (0, r_c)\colon 1 - \chi_u(x + y\mathrm{n}_v(x,t)) > 0\}$$

are guaranteed to be one-dimensional Caccioppoli sets in $(0, r_c)$ for $\mathcal{H}^{d-1}$-almost every $x \in I_v(t)$. Hence, we may represent the one-dimensional section E^+_x for such $x \in I_v(t)$ as a finite union of disjoint intervals (see [12, Proposition 3.52])

$$E^+_x \cap (0, r_c) = \bigcup_{m=1}^{K(x)} (a_m, b_m).$$

If $K(x) = 0$ then $h^+(x) = 0$, and the inner integral in the first term on the right hand side of (3.64) vanishes for this x. If $K(x) = 1$ and $a_1 = 0$, then by definition of $h^+(x)$ we have $(a_1, b_1) = (0, h^+(x))$ (recall that we now consider the case $h^+(x) \leq \frac{r_c}{4}$). Thus, again the inner integral in the first term on the right hand side of (3.64) vanishes for this x. Hence, it remains to discuss the case that there is at least one non-empty interval in the decomposition of E^+_x, say (a, b), such that $a \in (0, r_c)$. From property iv) in Theorem 3.39 it then follows that

$$\mathrm{n}_v(x,t) \cdot \frac{-\nabla\chi_{E^+}}{|\nabla\chi_{E^+}|}(x + a\mathrm{n}_v(x,t)) \leq 0.$$

Hence, we may bound

$$\begin{aligned}&\bigg|\int_0^{r_c} \det\nabla\Phi_t(x,y)|\chi_u(\Phi_t(x,y)) - 1 + \chi_{\{y\leq h^+(x)\}}|\,\mathrm{d}y\bigg|\\&\quad\leq Cr_c \leq C\int_{(0,r_c)\cap(\partial^*E^+)_x} 1 - \mathrm{n}_v(x,t)\cdot\frac{-\nabla\chi_{E^+}}{|\nabla\chi_{E^+}|}(x + y\mathrm{n}_v(x,t))\,\mathrm{d}\mathcal{H}^0(y)\end{aligned}$$

Gathering the bounds from the different cases together with the estimate in (3.65), we therefore obtain by the co-area formula for rectifiable sets (see [12, (2.72)]) together with the change of variables $\Phi_t(x,y)$

$$\begin{aligned}&\bigg|\int_{I_v(t)}\int_0^{r_c} \det\nabla\Phi_t(x,y)|\chi_u(\Phi_t(x,y)) - 1 + \chi_{\{y\leq h^+(x)\}}|\,\mathrm{d}y\,\mathrm{d}S(x)\bigg|\\&\quad\leq C\int_{I_v(t)}\int_{(0,r_c)\cap(\partial^*E^+)_x} 1 - \mathrm{n}_v(x,t)\cdot\frac{-\nabla\chi_{E^+}}{|\nabla\chi_{E^+}|}(x + y\mathrm{n}_v(x,t))\,\mathrm{d}\mathcal{H}^0(y)\,\mathrm{d}S(x)\\&\qquad+ C\int_{\mathbb{R}^d}\int_{-r_c}^{r_c} |\chi_u(\Phi_t(x,y)) - \chi_v(\Phi_t(x,y))|\frac{y}{r_c}\,dy\,\mathrm{d}x\\&\quad\leq C\int_{\{\operatorname{dist}(x,I_v(t))<r_c\}} 1 - \mathrm{n}_v(P_{I_v(t)}x)\cdot\frac{\nabla\chi_u}{|\nabla\chi_u|}\,\mathrm{d}|\nabla\chi_u|(x)\\&\qquad+ C\int_{\mathbb{R}^d} |\chi_u - \chi_v|\min\Big\{\frac{\operatorname{dist}(x, I_v(t))}{r_c}, 1\Big\}\,\mathrm{d}x,\end{aligned}$$

which is by (3.34) as well as (3.33) indeed a bound of desired order. Moreover, performing analogous estimates for the second term on the right-hand side of (3.64) and estimating the third term on the right-hand side of (3.64) trivially, we then get

$$\begin{aligned}&\int_{\mathbb{R}^d} |\chi_u - \chi_{v,h^+,h^-}| \,\mathrm{d}x \\ &\qquad \leq C \int_{\mathbb{R}^d} 1 - \xi \cdot \frac{\nabla \chi_u}{|\nabla \chi_u|} \,\mathrm{d}|\nabla \chi_u| + C \int_{\mathbb{R}^d} |\chi_u - \chi_v| \min\Big\{\frac{\operatorname{dist}(x, I_v(t))}{r_c}, 1\Big\} \,\mathrm{d}x\end{aligned}$$

which is precisely the desired estimate (3.55c).

Step 4: Proof of estimate on the time derivative (3.55d)**.** To bound the time derivative, we compute using the weak formulation of the continuity equation $\partial_t \chi_u = -\nabla \cdot (\chi_u u)$ and abbreviating $I^+(t) := \{x \in \mathbb{R}^d : \operatorname{dist}^{\pm}(x, I_v(t)) \in [0, r_c)\}$ (recall that the boundary $\partial I^+(t) = I_v(t)$ moves with normal speed $\mathrm{n}_v \cdot v$)

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t} \int_{I_v(t)} \eta(x) h^+(x,t) \,\mathrm{d}S(x) \\ &= \frac{\mathrm{d}}{\mathrm{d}t} \int_{I_v(t)} \int_0^\infty \eta(x)(1-\chi_u)(x + y\mathrm{n}_v(x,t), t)\, \theta\Big(\frac{y}{r_c}\Big) \,\mathrm{d}y \,\mathrm{d}S(x) \\ &= \frac{\mathrm{d}}{\mathrm{d}t} \int_{I^+(t)} \eta(P_{I_v(t)}x) |\det \nabla \Phi_t^{-1}|(x) (1-\chi_u)(x,t)\, \theta\Big(\frac{\operatorname{dist}(x, I_v(t))}{r_c}\Big) \,\mathrm{d}x \\ &= \int_{I^+(t)} (1-\chi_u)(x,t) u \cdot \nabla \Big(\eta(P_{I_v(t)}x) |\det \nabla \Phi_t^{-1}|(x)\, \theta\Big(\frac{\operatorname{dist}(x, I_v(t))}{r_c}\Big)\Big) \,\mathrm{d}x \\ &\quad + \int_{I_v(t)} (\mathrm{n}_v \cdot u)(x,t)(1-\chi_u)(x,t)\eta(P_{I_v(t)}x) |\det \nabla \Phi_t^{-1}|(x) \theta\Big(\frac{\operatorname{dist}(x, I_v(t))}{r_c}\Big) \,\mathrm{d}S(x) \\ &\quad + \int_{I^+(t)} (1-\chi_u)(x,t) \frac{\mathrm{d}}{\mathrm{d}t}\Big(\eta(P_{I_v(t)}x) |\det \nabla \Phi_t^{-1}|(x) \theta\Big(\frac{\operatorname{dist}(x, I_v(t))}{r_c}\Big)\Big) \,\mathrm{d}x \\ &\quad - \int_{I_v(t)} (\mathrm{n}_v \cdot v)(x,t)(1-\chi_u)(x,t)\eta(P_{I_v(t)}x) |\det \nabla \Phi_t^{-1}|(x) \theta\Big(\frac{\operatorname{dist}(x, I_v(t))}{r_c}\Big) \,\mathrm{d}S(x).\end{aligned}$$

Recall from (3.23) the formula for the gradient of the projection onto the nearest point on the interface $I_v(t)$. Recalling also the definitions of the extended normal velocity $V_\mathrm{n}(x,t) := \big(v(x,t) \cdot \mathrm{n}_v(P_{I_v(t)}x, t)\big)\, \mathrm{n}_v(P_{I_v(t)}x, t)$ and its projection $\bar{V}_\mathrm{n}(x,t) := V_\mathrm{n}(P_{I_v(t)}x, t)$ from (3.46) respectively (3.18), we also have

$$\begin{aligned}&- \int_{I^+} (1-\chi_u(x,t)) |\det \nabla \Phi_t^{-1}|(x) \theta\Big(\frac{\operatorname{dist}(x, I_v(t))}{r_c}\Big) (\nabla \eta)(P_{I_v(t)}x) \\ &\qquad\qquad \cdot \Big(\big(v(P_{I_v(t)}x, t) - \bar{V}_\mathrm{n}(x,t)\big) \cdot \nabla\Big) P_{I_v(t)}x \,\mathrm{d}x \\ &= - \int_{I_v(t)} \int_0^{r_c} (1 - \chi_u(\Phi_t(x,y), t)) \theta\Big(\frac{y}{r_c}\Big) \nabla \eta(x) \\ &\qquad\qquad \cdot \Big(\big(v(x,t) - V_\mathrm{n}(x,t)\big) \cdot \nabla\Big) P_{I_v(t)}(\Phi_t(x,y)) \,\mathrm{d}y \,\mathrm{d}S(x) \\ &= - \int_{I_v(t)} h^+(x,t)(\mathrm{Id} - \mathrm{n}_v(x) \otimes \mathrm{n}_v(x)) v(x,t) \cdot \nabla \eta(x) \,\mathrm{d}S(x) \\ &\quad + \int_{I^+(t)} (1-\chi_u(x,t)) |\det \nabla \Phi_t^{-1}|(x) \theta\Big(\frac{\operatorname{dist}(x, I_v(t))}{r_c}\Big) \operatorname{dist}(x, I_v(t)) (\nabla \eta)(P_{I_v(t)}x) \\ &\qquad\qquad \cdot \Big(\big(v(P_{I_v(t)}x, t) - \bar{V}_\mathrm{n}(x,t)\big) \cdot \nabla\Big) \mathrm{n}_v(P_{I_v(t)}x) \,\mathrm{d}x.\end{aligned}$$

Adding this formula to the above formula for $\frac{\mathrm{d}}{\mathrm{d}t} \int_{I_v(t)} \eta(x) h^+(x,t) \,\mathrm{d}S(x)$, introducing the abbreviation $f := |\det \nabla \Phi_t^{-1}|(x)\, \theta(\frac{\operatorname{dist}(x, I_v(t))}{r_c})$, and using the fact that $\chi_v = 1$ in $I^+(t)$, we obtain

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{I_v(t)} \eta(x)h^+(x,t)\,\mathrm{d}x - \int_{I_v(t)} h^+(x,t)(\mathrm{Id}-\mathrm{n}_v\otimes\mathrm{n}_v)v(x,t)\cdot\nabla\eta(x)\,\mathrm{d}S(x)\\
&= \int_{I^+(t)} (\chi_u(x,t)-\chi_v(x,t))f(x)\operatorname{dist}(x,I_v(t))(\nabla\eta)(P_{I_v(t)}x)\\
&\qquad\qquad \cdot\big((v(P_{I_v(t)}x,t)-\bar{V}_\mathrm{n}(x,t))\cdot\nabla\big)\mathrm{n}_v(P_{I_v(t)}x)\,\mathrm{d}x\\
&\quad - \int_{I^+(t)} (\chi_u(x,t)-\chi_v(x,t))\eta(P_{I_v(t)}x)(u-v)\cdot\nabla f\,\mathrm{d}x\\
&\quad - \int_{I^+(t)} (\chi_u(x,t)-\chi_v(x,t))f(x)(\nabla\eta)(P_{I_v(t)}x)\cdot((u-v)\cdot\nabla)P_{I_v(t)}x\,\mathrm{d}x\\
&\quad - \int_{I^+(t)} (\chi_u(x,t)-\chi_v(x,t))f(x)(\nabla\eta)(P_{I_v(t)}x)\\
&\qquad\qquad \cdot\big((v(x,t)-(v(P_{I_v(t)}x,t)-\bar{V}_\mathrm{n}(x,t)))\cdot\nabla\big)P_{I_v(t)}x\,\mathrm{d}x\\
&\quad - \int_{I^+(t)} (\chi_u(x,t)-\chi_v(x,t))f(x)(\nabla\eta)(P_{I_v(t)}x)\cdot\frac{\mathrm{d}}{\mathrm{d}t}P_{I_v(t)}x\,\mathrm{d}x\\
&\quad - \int_{I^+(t)} (\chi_u(x,t)-\chi_v(x,t))\,\eta(P_{I_v(t)}x)\Big(\frac{\mathrm{d}}{\mathrm{d}t}f+v\cdot\nabla f\Big)\,\mathrm{d}x\\
&\quad + \int_{I_v(t)} \mathrm{n}_v\cdot(u-v)(1-\chi_u)\eta\,\mathrm{d}S.
\end{aligned}
\tag{3.66}
$$

Note that $f(x)=|\det\nabla\Phi_t^{-1}|(x)\,\theta(\frac{\operatorname{dist}(x,I_v(t))}{r_c})=1$ for any t and any $x\in I_v(t)$. Thus, we have $\frac{\mathrm{d}}{\mathrm{d}t}f+v\cdot\nabla f=0$ on $I_v(t)$. Furthermore, we have $|\nabla\bar{V}_\mathrm{n}|\le\frac{C}{r_c^2}\|v\|_{W^{1,\infty}}$ and $|\nabla^2\bar{V}_\mathrm{n}|\le\frac{C}{r_c^3}\|v\|_{W^{2,\infty}(\mathbb{R}^d\setminus I_v(t))}$ because of $\bar{V}_\mathrm{n}(x)=V_\mathrm{n}(P_{I_v(t)}x)$, (3.15), the corresponding estimate (3.39) for the gradient of V_n as well as the formula (3.23) for the gradient of $P_{I_v(t)}$. Because of (3.19) and the equation (3.30) for the time evolution of the normal vector, we thus get the bounds $|\frac{\mathrm{d}}{\mathrm{d}t}\nabla\operatorname{dist}^\pm(\cdot,I_v(t))|\le\frac{C}{r_c^2}\|v\|_{W^{1,\infty}}$ and $|\nabla\frac{\mathrm{d}}{\mathrm{d}t}\nabla\operatorname{dist}^\pm(\cdot,I_v(t))|\le\frac{C}{r_c^3}\|v\|_{W^{2,\infty}(\mathbb{R}^d\setminus I_v(t))}$. Taking all of these bounds together, we obtain $|f|\le\frac{C}{r_c}$, $|\nabla f|\le\frac{C}{r_c^2}$ and $|\nabla^2 f|+|\nabla\frac{\mathrm{d}}{\mathrm{d}t}f|\le\frac{C}{r_c^3}(1+\|v\|_{W^{2,\infty}(\mathbb{R}^d\setminus I_v(t))})$. As a consequence, we get

$$
\Big|\frac{\mathrm{d}}{\mathrm{d}t}f+v\cdot\nabla f\Big|\le\frac{C}{r_c^3}(1+\|v\|_{W^{2,\infty}(\mathbb{R}^d\setminus I_v(t))})\operatorname{dist}(\cdot,I_v(t)).
\tag{3.67}
$$

Moreover, we may compute

$$
\frac{\mathrm{d}}{\mathrm{d}t}P_{I_v(t)}x=-\mathrm{n}_v(P_{I_v(t)}x)\frac{\mathrm{d}}{\mathrm{d}t}\operatorname{dist}^\pm(x,I_v(t))-\operatorname{dist}^\pm(x,I_v(t))\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{n}_v(P_{I_v(t)}x)).
\tag{3.68}
$$

Since $\mathrm{n}_v\cdot\nabla\eta=0$ holds on the interface $I_v(t)$ by assumption, we obtain from (3.68)

$$
\begin{aligned}
&-\int_{I^+(t)} (\chi_u(x,t)-\chi_v(x,t))f(x)(\nabla\eta)(P_{I_v(t)}x)\cdot\frac{\mathrm{d}}{\mathrm{d}t}P_{I_v(t)}x\,\mathrm{d}x\\
&=\int_{I^+(t)} (\chi_u(x,t)-\chi_v(x,t))\operatorname{dist}^\pm(x,I_v(t))f(x)(\nabla\eta)(P_{I_v(t)}x)\cdot\frac{d}{dt}(\mathrm{n}_v(P_{I_v(t)}x))\,\mathrm{d}x.
\end{aligned}
$$

In what follows, we will by slight abuse of notation use $\nabla^{\tan}g(x)$ as a shorthand for $(\mathrm{Id}-\mathrm{n}_v(P_{I_v(t)}x)\otimes\mathrm{n}_v(P_{I_v(t)}x))\nabla g(x)$ for scalar fields as well as $(\nabla^{\tan}\cdot g)(x)$ instead of $(\mathrm{Id}-\mathrm{n}_v(P_{I_v(t)}x)\otimes\mathrm{n}_v(P_{I_v(t)}x)):\nabla g(x)$ for vector fields. Let us also abbreviate $P^{\tan}x:=(\mathrm{Id}-\mathrm{n}_v(P_{I_v(t)}x)\otimes\mathrm{n}_v(P_{I_v(t)}x))$. Note that by assumption $(\nabla\eta)(P_{I_v(t)}x)=(\nabla^{\tan}\eta)(P_{I_v(t)}x)$. Moreover, it follows from (3.20), (3.21) and (3.19) that $\mathrm{n}_v(P_{I_v(t)}x)\cdot\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{n}_v(P_{I_v(t)}x))=0$. Hence, we may rewrite with an integration by parts (recall the notation $P^{\tan}(x) = (\mathrm{Id} - \mathrm{n}_v \otimes \mathrm{n}_v)(P_{I_v(t)}x, t))$

$$\begin{aligned}
&\int_{I^+(t)} (\chi_u(x,t) - \chi_v(x,t)) \operatorname{dist}^\pm(x, I_v(t)) f(x) (\nabla^{\tan}\eta)(P_{I_v(t)}x) \cdot \frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{n}_v(P_{I_v(t)}x)) \,\mathrm{d}x \qquad (3.69)\\
&= -\int_{I^+(t)} (\chi_u - \chi_v)(x,t) \operatorname{dist}^\pm(x, I_v(t)) \eta(P_{I_v(t)}x) \\
&\qquad \times \Big(\frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{n}_v(P_{I_v(t)}x)) \otimes \nabla\Big) : f(x) P^{\tan}(x) \,\mathrm{d}x \\
&\quad - \int_{I^+(t)} (\chi_u - \chi_v)(x,t) \operatorname{dist}^\pm(x, I_v(t)) f(x) \eta(P_{I_v(t)}x) \nabla^{\tan} \cdot \frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{n}_v(P_{I_v(t)}x)) \,\mathrm{d}x \\
&\quad - \int_{\mathbb{R}^d} \operatorname{dist}^\pm(x, I_v(t)) f(x) \eta(P_{I_v(t)}x) \Big(\frac{\nabla\chi_u}{|\nabla\chi_u|} - \mathrm{n}_v(P_{I_v(t)}x)\Big) \cdot \frac{\mathrm{d}}{\mathrm{d}t}(\mathrm{n}_v(P_{I_v(t)}x)) \,\mathrm{d}|\nabla\chi_u|.
\end{aligned}$$

Using from (3.21) and (3.19) that the spatial partial derivatives of the extended normal vector field are orthogonal to the gradient of the signed distance function, the same argument also shows that

$$\begin{aligned}
&\int_{I^+(t)} (\chi_u(x,t) - \chi_v(x,t)) f(x) \operatorname{dist}(x, I_v(t)) (\nabla^{\tan}\eta)(P_{I_v(t)}x) \qquad (3.70)\\
&\qquad \cdot \big((v(P_{I_v(t)}x, t) - \bar{V}_{\mathrm{n}}(x,t)) \cdot \nabla\big) \mathrm{n}_v(P_{I_v(t)}x) \,\mathrm{d}x \\
&= -\int_{I^+(t)} (\chi_u(x,t) - \chi_v(x,t)) \operatorname{dist}(x, I_v(t)) \eta(P_{I_v(t)}x) \\
&\qquad \times \big(((v(P_{I_v(t)}x, t) - \bar{V}_{\mathrm{n}}(x,t)) \cdot \nabla) \mathrm{n}_v(P_{I_v(t)}x) \otimes \nabla\big) : f(x) P^{\tan}(x) \,\mathrm{d}x \\
&\quad - \int_{I^+(t)} (\chi_u(x,t) - \chi_v(x,t)) \operatorname{dist}(x, I_v(t)) f(x) \eta(P_{I_v(t)}x) \\
&\qquad \times \nabla^{\tan} \cdot \big(((v(P_{I_v(t)}x, t) - \bar{V}_{\mathrm{n}}(x,t)) \cdot \nabla) \mathrm{n}_v(P_{I_v(t)}x)\big) \,\mathrm{d}x \\
&\quad - \int_{\mathbb{R}^d} \operatorname{dist}^\pm(x, I_v(t)) f(x) \eta(P_{I_v(t)}x) \Big(\frac{\nabla\chi_u}{|\nabla\chi_u|} - \mathrm{n}_v(P_{I_v(t)}x)\Big) \\
&\qquad \cdot \big((v(P_{I_v(t)}x, t) - \bar{V}_{\mathrm{n}}(x,t)) \cdot \nabla\big) \mathrm{n}_v(P_{I_v(t)}x) \,\mathrm{d}|\nabla\chi_u|.
\end{aligned}$$

It follows from (3.23) as well as (3.21) and (3.19) that $(\mathrm{n}_v(P_{I_v(t)}x) \cdot \nabla) P_{I_v(t)}x = 0$. Hence, we obtain

$$\begin{aligned}
&\int_{I^+(t)} (\chi_u(x,t) - \chi_v(x,t)) f(x) (\nabla\eta)(P_{I_v(t)}x) \qquad (3.71)\\
&\qquad \cdot \big((v(x,t) - (v(P_{I_v(t)}x, t) - \bar{V}_{\mathrm{n}}(x,t))) \cdot \nabla\big) P_{I_v(t)}(x) \,\mathrm{d}x \\
&= \int_{I^+(t)} (\chi_u - \chi_v)(x,t) f(x) (\nabla\eta)(P_{I_v(t)}x) \cdot \big((v(x,t) - v(P_{I_v(t)}x, t)) \cdot \nabla\big) P_{I_v(t)}x \,\mathrm{d}x.
\end{aligned}$$

Since the domain of integration is $I^+(t)$, we may write

$$\begin{aligned}
&v(x,t) - v(P_{I_v(t)}x, t) \\
&= \operatorname{dist}^\pm(x, I_v(t)) \int_{(0,1]} \nabla v\big(P_{I_v(t)}x + \lambda \operatorname{dist}^\pm(x, I_v(t)) \mathrm{n}_v(P_{I_v(t)}x)\big) \, d\lambda \cdot \mathrm{n}_v(P_{I_v(t)}x).
\end{aligned}$$

From this and the fact $\mathrm{n}_v(P_{I_v(t)}) \cdot \nabla P_{I_v(t)}(x) = 0$, we deduce by another integration by parts that (where $|F| \leq r_c^{-1}\|v\|_{W^{2,\infty}(\mathbb{R}^d\setminus I_v(t))}$)

$$
\begin{aligned}
&\int_{I^+(t)} (\chi_u(x,t)-\chi_v(x,t)) f(x) (\nabla^{\tan}\eta)(P_{I_v(t)}x)\cdot((v(x,t)-v(P_{I_v(t)}x,t))\cdot\nabla)P_{I_v(t)}x \,\mathrm{d}x \qquad (3.72)\\
&= -\int_{I^+(t)} (\chi_u(x,t)-\chi_v(x,t))\eta(P_{I_v(t)}x)\\
&\qquad\qquad \times \big(((v(x,t)-v(P_{I_v(t)}x,t))\cdot\nabla)P_{I_v(t)}x\otimes\nabla\big) : f(x)P^{\tan}x \,\mathrm{d}x\\
&\quad -\int_{I^+(t)} (\chi_u(x,t)-\chi_v(x,t)) f(x)\eta(P_{I_v(t)}x)((v(x,t)-v(P_{I_v(t)}x,t))\cdot\nabla(\nabla^{\tan}\cdot P_{I_v(t)}x) \,\mathrm{d}x\\
&\quad -\int_{I^+(t)} (\chi_u(x,t)-\chi_v(x,t))\,\mathrm{dist}(x,I_v(t)) f(x)\eta(P_{I_v(t)}x)F(x,t) : \nabla P_{I_v(t)}x \,\mathrm{d}x\\
&\quad -\int_{\mathbb{R}^d} f(x)\eta(P_{I_v(t)}x)\Big(\frac{\nabla\chi_u}{|\nabla\chi_u|}-\mathrm{n}_v(P_{I_v(t)}x)\Big)\cdot\big((v(x,t)-v(P_{I_v(t)}x,t))\cdot\nabla\big)P_{I_v(t)}x \,\mathrm{d}|\nabla\chi_u|.
\end{aligned}
$$

Hence, plugging in (3.71), (3.70) and (3.72), (3.69) into (3.66) and using the estimates $|\nabla\bar{V}_{\mathrm{n}}| \leq \frac{C}{r_c^2}\|v\|_{W^{1,\infty}}$, $|\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{n}_v(P_{I_v(t)}x)| \leq \frac{C}{r_c^2}\|v\|_{W^{1,\infty}}$, $|\nabla\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{n}_v(P_{I_v(t)}x)| \leq \frac{C}{r_c^3}\|v\|_{W^{2,\infty}(\mathbb{R}^d\setminus I_v(t))}$, and $|\nabla f| \leq \frac{C}{r_c^2}$, we obtain

$$
\begin{aligned}
&\left|\frac{\mathrm{d}}{\mathrm{d}t}\int_{I_v(t)}\eta(x)h^+(x,t)\,\mathrm{d}x - \int_{I_v(t)} h^+(x,t)(\mathrm{Id}-\mathrm{n}_v\otimes\mathrm{n}_v)v(x,t)\cdot\nabla\eta(x)\,dS(x)\right|\\
&\leq \frac{C}{r_c^2}\int_{\{\mathrm{dist}(x,I_v(t))\leq r_c\}} |\chi_u(x,t)-\chi_v(x,t)||u(x,t)-v(x,t)||\eta(P_{I_v(t)}x)| \,\mathrm{d}x\\
&\quad + \frac{C}{r_c}\int_{\{\mathrm{dist}(x,I_v(t))\leq r_c\}} |\chi_u(x,t)-\chi_v(x,t)||u(x,t)-v(x,t)||\nabla\eta(P_{I_v(t)}x)| \,\mathrm{d}x\\
&\quad + \frac{C(1+\|v\|_{W^{1,\infty}})}{r_c}\int_{\{\mathrm{dist}(x,I_v(t))\leq r_c\}} \left|\frac{\nabla\chi_u}{|\nabla\chi_u|}-\mathrm{n}_v(P_{I_v(t)}x)\right|\frac{|\,\mathrm{dist}^\pm(x,I_v(t))|}{r_c}|\eta(P_{I_v(t)}x)| \,\mathrm{d}|\nabla\chi_u|(x)\\
&\quad + \frac{C(1+\|v\|_{W^{2,\infty}(\mathbb{R}^d\setminus I_v(t))})}{r_c^3}\int_{\{\mathrm{dist}(x,I_v(t))\leq r_c\}} |\chi_u(x,t)-\chi_v(x,t)|\frac{|\,\mathrm{dist}^\pm(x,I_v(t))|}{r_c}|\eta(P_{I_v(t)}x)| \,\mathrm{d}x\\
&\quad + C\int_{I_v(t)} |u-v||\eta| \,\mathrm{d}S.
\end{aligned}
$$

This yields by the change of variables $\Phi_t(x,y)$ and a straightforward estimate

$$
\begin{aligned}
&\left|\frac{\mathrm{d}}{\mathrm{d}t}\int_{I_v(t)}\eta(x)h^+(x,t)\,dx - \int_{I_v(t)} h^+(x,t)(\mathrm{Id}-\mathrm{n}_v\otimes\mathrm{n}_v)v(x,t)\cdot\nabla\eta(x)\,\mathrm{d}S(x)\right|\\
&\leq \frac{C}{r_c^2}\|\eta\|_{W^{1,4}(I_v(t))}\bigg(\int_{I_v(t)}\Big(\int_0^{\frac{r_c}{2}} |\chi_u-\chi_v|(x+y\mathrm{n}_v(x,t),t)\,\mathrm{d}y\Big)^4\,\mathrm{d}S\bigg)^{1/4}\\
&\qquad \times\bigg(\int_{I_v(t)}\sup_{y\in[-r_c,r_c]}|u-v|^2(x+y\mathrm{n}_v(x,t),t)\,\mathrm{d}S(x)\bigg)^{1/2}\\
&\quad + \frac{C(1+\|v\|_{W^{2,\infty}(\mathbb{R}^d\setminus I_v(t))})}{r_c^3}\|\eta\|_{L^2(I_v(t))}\\
&\qquad \times\bigg(\int_{\mathbb{R}^d}|\chi_u(x,t)-\chi_v(x,t)|\min\Big\{\frac{\mathrm{dist}(x,I_v(t))}{r_c},1\Big\}\,\mathrm{d}x\bigg)^{\frac12}\\
&\quad + \frac{C(1+\|v\|_{W^{1,\infty}})}{r_c}\|\eta\|_{L^\infty(I_v(t))}\bigg(\int_{\{\mathrm{dist}(x,I_v(t))\leq r_c\}}\left|\frac{\nabla\chi_u}{|\nabla\chi_u|}-\mathrm{n}_v(P_{I_v(t)}x)\right|^2\mathrm{d}|\nabla\chi_u|\bigg)^{\frac12}
\end{aligned}
$$

$$\times \bigg(\int_{\{\operatorname{dist}(x,I_v(t))\leq r_c\}} \frac{|\operatorname{dist}^\pm(x,I_v(t))|^2}{r_c^2} \,\mathrm{d}|\nabla\chi_u|\bigg)^{\frac{1}{2}}$$

$$+ C\bigg(\int_{I_v(t)} |u-v|^2 \,dS\bigg)^{1/2} \|\eta\|_{L^2(I_v(t))}.$$

Using finally the Sobolev embedding to bound the L^∞-norm of η on the interface (which is either one- or two-dimensional; note that the constant in the Sobolev embedding may be bounded by Cr_c^{-1} for our geometry), we infer from this estimate the desired bound (3.55d), using also (3.34) and (3.33). This concludes the proof. □

3.5.2 A regularization of the local height of the interface error

In order to modify our relative entropy to compensate for the velocity gradient discontinuity at the interface, we need regularized versions of the local heights of the interface error h^+ and h^- which in particular have Lipschitz regularity. To this aim, we fix some function $e(t) > 0$ and basically apply a mollifier on scale $e(t)$ to the local interface error heights h^+ and h^- at each time. An illustration of h^+ and its mollification $h^+_{e(t)}$ is provided in Figure 3.2 and Figure 3.3, respectively. These regularized versions $h^+_{e(t)}$ and $h^-_{e(t)}$ of the local interface error heights then have the following properties:

Proposition 3.27. *Let $\chi_v \in L^\infty([0,T_{strong});\mathrm{BV}(\mathbb{R}^d;\{0,1\}))$ be an indicator function such that $\Omega_t^+ := \{x\in\mathbb{R}^d\colon \chi_v(x,t)=1\}$ is a family of smoothly evolving domains and $I_v(t) := \partial\Omega_t^+$ is a family of smoothly evolving surfaces in the sense of Definition 3.5. Let ξ be the extension of the unit normal vector field n_v from Definition 3.13.*

Let $\chi_u \in L^\infty([0,T_{strong});\mathrm{BV}(\mathbb{R}^d;\{0,1\}))$ be another indicator function and let then h^+ resp. h^- be as defined in Proposition 3.26. Let $\theta\colon \mathbb{R}^+\to[0,1]$ be a smooth cutoff with $\theta(s)=1$ for $s\in[0,\frac14]$ and $\theta(s)=0$ for $s\geq\frac12$. Let $e\colon[0,T_{strong})\to(0,r_c]$ be a C^1-function and define the regularized height of the local interface error

$$h^\pm_{e(t)}(x,t) := \frac{\int_{I_v(t)} \theta\big(\frac{|\tilde x - x|}{e(t)}\big) h^\pm(\tilde x,t)\,\mathrm{d}S(\tilde x)}{\int_{I_v(t)} \theta\big(\frac{|\tilde x - x|}{e(t)}\big)\,\mathrm{d}S(\tilde x)}. \tag{3.73}$$

Then $h^+_{e(t)}$ and $h^-_{e(t)}$ have the following properties:

a) (H^1-bound) *If the interface error terms from the relative entropy are bounded by*

$$\int_{\mathbb{R}^d} 1-\xi(\cdot,t)\cdot\frac{\nabla\chi_u(\cdot,t)}{|\nabla\chi_u(\cdot,t)|}\,\mathrm{d}|\nabla\chi_u(\cdot,t)|$$
$$+\int_{\mathbb{R}^d} \big|\chi_u(\cdot,t)-\chi_v(\cdot,t)\big|\,\Big|\beta\Big(\frac{\operatorname{dist}^\pm(\cdot,I_v(t))}{r_c}\Big)\Big|\,\mathrm{d}x \leq e(t)^2,$$

we have the Lipschitz estimate $|\nabla h^\pm_{e(t)}(\cdot,t)|\leq Cr_c^{-2}$, the global bound $|\nabla^2 h^\pm_{e(t)}(\cdot,t)|\leq Ce(t)^{-1}r_c^{-4}$, and the bound

$$\int_{I_v(t)} |\nabla h^\pm_{e(t)}|^2 + |h^\pm_{e(t)}|^2\,\mathrm{d}S \leq \frac{C}{r_c^2}\int_{\mathbb{R}^d} 1-\xi\cdot\frac{\nabla\chi_u}{|\nabla\chi_u|}\,\mathrm{d}|\nabla\chi_u| \tag{3.74a}$$
$$+\frac{C}{r_c^4}\int_{\mathbb{R}^d}|\chi_u-\chi_v|\min\Big\{\frac{\operatorname{dist}(x,I_v(t))}{r_c},1\Big\}\,\mathrm{d}x.$$

b) (Improved approximation property) *The functions $h^+_{e(t)}$ and $h^-_{e(t)}$ provide an approximation for the interface of the weak solution*

$$\chi_{v,h^+_{e(t)},h^-_{e(t)}} := \chi_v - \chi_{0\leq\operatorname{dist}^\pm(x,I_v(t))\leq h^+_{e(t)}(P_{I_v(t)}x,t)} \tag{3.74b}$$
$$+\chi_{-h^-_{e(t)}(P_{I_v(t)}x,t)\leq\operatorname{dist}^\pm(x,I_v(t))\leq 0},$$

up to an error of

$$
\begin{aligned}
&\int_{\mathbb{R}^d} |\chi_u - \chi_{v,h^+_{e(t)},h^-_{e(t)}}| \,\mathrm{d}x \\
&\leq C\int_{\mathbb{R}^d} 1-\xi\cdot\frac{\nabla\chi_u}{|\nabla\chi_u|}\,\mathrm{d}|\nabla\chi_u| + C\int_{\mathbb{R}^d} |\chi_u-\chi_v|\min\Big\{\frac{\mathrm{dist}(x,I_v(t))}{r_c},1\Big\}\,\mathrm{d}x \qquad (3.74c)\\
&\quad + Ce(t)\Big(\int_{\mathbb{R}^d} 1-\xi\cdot\frac{\nabla\chi_u}{|\nabla\chi_u|}\,\mathrm{d}|\nabla\chi_u|\Big)^{1/2}\mathcal{H}^{d-1}(I_v(t))^{1/2}\\
&\quad + C\frac{e(t)}{r_c}\Big(\int_{\mathbb{R}^d} |\chi_u-\chi_v|\min\Big\{\frac{\mathrm{dist}(x,I_v(t))}{r_c},1\Big\}\,\mathrm{d}x\Big)^{1/2}\mathcal{H}^{d-1}(I_v(t))^{1/2}.
\end{aligned}
$$

c) (Time evolution) *Let v be a solenoidal vector field*

$$
v\in L^2([0,T_{strong}];H^1(\mathbb{R}^d;\mathbb{R}^d))\cap L^\infty([0,T_{strong}];W^{1,\infty}(\mathbb{R}^d;\mathbb{R}^d))
$$

such that in the domain $\bigcup_{t\in[0,T_{strong})}(\Omega_t^+\cup\Omega_t^-)\times\{t\}$ the second spatial derivatives of the vector field v exist and satisfy $\sup_{t\in[0,T_{strong})}\sup_{x\in\Omega_t^+\cup\Omega_t^-}|\nabla^2 v(x,t)|<\infty$. Assume that χ_v solves the equation $\partial_t\chi_v=-\nabla\cdot(\chi_v v)$. If χ_u solves the equation $\partial_t\chi_u=-\nabla\cdot(\chi_u u)$ for another solenoidal vector field $u\in L^2([0,T_{strong}];H^1(\mathbb{R}^d;\mathbb{R}^d))$, we have the following estimate on the time derivative of $h^\pm_{e(t)}$:

$$
\begin{aligned}
&\Big|\frac{\mathrm{d}}{\mathrm{d}t}\int_{I_v(t)}\eta(x)h^\pm_{e(t)}(x,t)\,\mathrm{d}x - \int_{I_v(t)} h^\pm_{e(t)}(x,t)(\mathrm{Id}-\mathrm{n}_v\otimes\mathrm{n}_v)v(x,t)\cdot\nabla\eta(x)\,\mathrm{d}S(x)\Big| \qquad (3.74d)\\
&\leq \frac{C}{e(t)r_c^2}\|\eta\|_{L^4(I_v(t))}\Big(\int_{I_v(t)}|\bar h^\pm|^4\,\mathrm{d}S\Big)^{1/4}\\
&\quad\times\Big(\int_{I_v(t)}\sup_{y\in[-r_c,r_c]}|u-v|^2(x+y\mathrm{n}_v(x,t),t)\,\mathrm{d}S(x)\Big)^{1/2}\\
&\quad + C\frac{(1+\|v\|_{W^{1,\infty}})}{e(t)r_c}\max_{p\in\{2,4\}}\|\eta\|_{L^p(I_v(t))}\int_{\mathbb{R}^d}1-\xi\cdot\frac{\nabla\chi_u}{|\nabla\chi_u|}\,\mathrm{d}|\nabla\chi_u|\\
&\quad + Cr_c^{-4}\|v\|_{W^{1,\infty}}(1+e'(t))\Big(\int_{\mathbb{R}^d}1-\xi\cdot\frac{\nabla\chi_u}{|\nabla\chi_u|}\,\mathrm{d}|\nabla\chi_u|\Big)^{1/2}||\eta||_{L^2(I_v(t))}\\
&\quad + C\Big(\frac{1+\|v\|_{W^{2,\infty}(\mathbb{R}^d\setminus I_v(t))}}{r_c}+\frac{\|v\|_{W^{1,\infty}}}{r_c^6}(1+e'(t))\Big)\|\eta\|_{L^2(I_v(t))}\\
&\qquad\times\Big(\int_{\mathbb{R}^d}|\chi_u(x,t)-\chi_v(x,t)|\,\min\Big\{\frac{\mathrm{dist}(x,I_v(t))}{r_c},1\Big\}\,\mathrm{d}x\Big)^{\frac12}\\
&\quad + C\|\eta\|_{L^2(I_v(t))}\Big(\int_{I_v(t)}|u-v|^2\,\mathrm{d}S\Big)^{\frac12}
\end{aligned}
$$

for any smooth test function $\eta\in C^\infty_{cpt}(\mathbb{R}^d)$ with $\mathrm{n}_v\cdot\nabla\eta=0$ on the interface $I_v(t)$, and where $\bar h^\pm$ is defined as $h^\pm$ but now with respect to the modified cut-off function $\bar\theta(\cdot)=\theta(\frac{\cdot}{2})$.

Proof. **Proof of a).** In order to estimate the spatial derivative $\nabla h^\pm_{e(t)}$, we compute using the fact that $\nabla_x\theta(\frac{|x-\tilde x|}{e(t)})=-\nabla_{\tilde x}\theta(\frac{|x-\tilde x|}{e(t)})$ (note that all of the subsequent gradients are to be understood in the tangential sense on the manifold $I_v(t)$)

$$\begin{aligned}
\nabla h^{\pm}_{e(t)}(x,t) &= -\frac{\int_{I_v(t)} \nabla_{\tilde x}\theta\big(\frac{|\tilde x - x|}{e(t)}\big) h^{\pm}(\tilde x,t)\,\mathrm{d}S(\tilde x)}{\int_{I_v(t)} \theta\big(\frac{|\tilde x - x|}{e(t)}\big)\,\mathrm{d}S(\tilde x)} \\
&\quad + \frac{\int_{I_v(t)} \theta\big(\frac{|\tilde x - x|}{e(t)}\big) h^{\pm}(\tilde x,t)\,\mathrm{d}S(\tilde x) \int_{I_v(t)} \nabla_{\tilde x}\theta\big(\frac{|\tilde x - x|}{e(t)}\big)\,\mathrm{d}S(\tilde x)}{\big(\int_{I_v(t)} \theta\big(\frac{|\tilde x - x|}{e(t)}\big)\,\mathrm{d}S(\tilde x)\big)^2} \\
&= \frac{\int_{I_v(t)} \theta\big(\frac{|\tilde x - x|}{e(t)}\big) \nabla h^{\pm}(\tilde x,t)\,\mathrm{d}S(\tilde x)}{\int_{I_v(t)} \theta\big(\frac{|\tilde x - x|}{e(t)}\big)\,\mathrm{d}S(\tilde x)} + \frac{\int_{I_v(t)} \theta\big(\frac{|\tilde x - x|}{e(t)}\big)\,\mathrm{d}D^s h^{\pm}(\tilde x)}{\int_{I_v(t)} \theta\big(\frac{|\tilde x - x|}{e(t)}\big)\,\mathrm{d}S(\tilde x)} \\
&\quad + \frac{\int_{I_v(t)} \theta\big(\frac{|\tilde x - x|}{e(t)}\big) h^{\pm}(\tilde x,t)\mathrm{H}(\tilde x,t)\,\mathrm{d}S(\tilde x)}{\int_{I_v(t)} \theta\big(\frac{|\tilde x - x|}{e(t)}\big)\,\mathrm{d}S(\tilde x)} \\
&\quad - \frac{\int_{I_v(t)} \theta\big(\frac{|\tilde x - x|}{e(t)}\big) h^{\pm}(\tilde x,t)\,\mathrm{d}S(\tilde x) \int_{I_v(t)} \theta\big(\frac{|\tilde x - x|}{e(t)}\big)\mathrm{H}(\tilde x,t)\,\mathrm{d}S(\tilde x)}{\big(\int_{I_v(t)} \theta\big(\frac{|\tilde x - x|}{e(t)}\big)\,\mathrm{d}S(\tilde x)\big)^2}.
\end{aligned}$$

Introduce the convex function

$$G(p) := \begin{cases} |p|^2 & \text{for } |p| \le 1, \\ 2|p| - 1 & \text{for } |p| \ge 1. \end{cases} \tag{3.75}$$

Using the estimate (3.16), the obvious bounds $G(p+\tilde p) \le CG(p) + CG(\tilde p)$ and $G(\lambda p) \le C(\lambda + \lambda^2)G(p)$ for any p, $\tilde p$, and $\lambda > 0$, and Jensen's inequality, we obtain (as the recession function of G is given by $2|p|$)

$$\begin{aligned}
G(|\nabla h^{\pm}_{e(t)}(x,t)|) &\le C\frac{\int_{I_v(t)} \theta\big(\frac{|\tilde x - x|}{e(t)}\big)\big(G(|\nabla h^{\pm}(\tilde x,t)|) + G(r_c^{-1}|h^{\pm}(\tilde x,t)|)\big)\,\mathrm{d}S(\tilde x)}{\int_{I_v(t)} \theta\big(\frac{|\tilde x - x|}{e(t)}\big)\,\mathrm{d}S(\tilde x)} \\
&\quad + C\frac{\int_{I_v(t)} \theta\big(\frac{|\tilde x - x|}{e(t)}\big)\,\mathrm{d}|D^s h^{\pm}|(\tilde x,t)}{\int_{I_v(t)} \theta\big(\frac{|\tilde x - x|}{e(t)}\big)\,\mathrm{d}S(\tilde x)}.
\end{aligned} \tag{3.76}$$

Consider $x \in I_v(t)$. By the assumption from Definition 3.5, there is a C^3-function $g\colon B_1(0) \subset \mathbb{R}^{d-1} \to \mathbb{R}$ with $\|\nabla g\|_{L^\infty} \le 1$, $g(0) = 0$, and $\nabla g(0) = 0$, and such that $I_v(t) \cap B_{2r_c}(x)$ is after rotation and translation given as the graph $\{(x, g(x)) : x \in \mathbb{R}^{d-1}\}$. Using the fact that $\theta \equiv 0$ on $\mathbb{R} \setminus [0, \frac{1}{2}]$ and $e(t) < r_c \le 1$, i.e., the map $I_v(t) \ni \tilde x \mapsto \theta(\frac{|\tilde x - x|}{e(t)})$ is supported in a coordinate patch given by the graph of g, we then may bound

$$\begin{aligned}
\int_{I_v(t)} \theta\Big(\frac{|\tilde x - x|}{e(t)}\Big)\,\mathrm{d}S(\tilde x) &\le \int_{I_v(t) \cap B_{\frac{e(t)}{2}}(x)} 1\,\mathrm{d}S(\tilde x) \le C\int_{\{\tilde x \in \mathbb{R}^{d-1}\colon |\tilde x| < \frac{e(t)}{2}\}} 1\,\mathrm{d}\tilde x \\
&\le Ce(t)^{d-1}.
\end{aligned}$$

We also obtain a lower bound using that $\theta \equiv 1$ on $[0, \frac{1}{4}]$ and again $e(t) < r_c \le 1$

$$\begin{aligned}
\int_{I_v(t)} \theta\Big(\frac{|\tilde x - x|}{e(t)}\Big)\,\mathrm{d}S(\tilde x) &\ge \int_{I_v(t) \cap B_{\frac{e(t)}{4}}(x)} 1\,\mathrm{d}S(\tilde x) \ge c\int_{\{\tilde x \in \mathbb{R}^{d-1}\colon |\tilde x| < ce(t)\}} 1\,\mathrm{d}\tilde x \\
&\ge ce(t)^{d-1}.
\end{aligned}$$

In summary, we infer that

$$ce(t)^{d-1} \le \int_{I_v(t)} \theta\Big(\frac{|\tilde x - x|}{e(t)}\Big)\,\mathrm{d}S(\tilde x) \le Ce(t)^{d-1}. \tag{3.77}$$

Making use of (3.77), the assumptions $\int_{\mathbb{R}^d} 1-\xi\cdot \mathrm{n}_u \,d|\nabla\chi_u| \leq e(t)^2 \leq r_c^2 < 1$ and $d\leq 3$, the upper bounds $|\theta|\leq 1$ and $G(\lambda p)\leq C(\lambda+\lambda^2)G(p)$, as well as the already established L^2-resp. H^1-bound for the local interface error heights $h^\pm$ from (3.55a) resp. (3.55b) we deduce

$$G(|\nabla h^\pm_{e(t)}(x,t)|) \leq Cr_c^{-2},$$

which is precisely the first assertion in a). Similarly, one derives the other desired estimate $G(e(t)|\nabla^2 h^\pm_{e(t)}(x,t)|)\leq Cr_c^{-4}$.

Integrating (3.76) over $I_v(t)$ and employing the global upper bound $|\nabla h^\pm_{e(t)}(\cdot,t)|\leq Cr_c^{-2}$, which in turn entails $G(|\nabla h^\pm_{e(t)}(\cdot,t)|)\geq cr_c^2|\nabla h^\pm_{e(t)}(\cdot,t)|^2$, we get

$$\begin{aligned}
&\int_{I_v(t)} |\nabla h^\pm_{e(t)}(x,t)|^2 \,\mathrm{d}S(x)\\
&\leq Cr_c^{-2}\int_{I_v(t)} \frac{\int_{I_v(t)}\theta\big(\frac{|\tilde x-x|}{e(t)}\big)G(|\nabla h^\pm(\tilde x,t)|)+G(r_c^{-1}|h^\pm(\tilde x,t)|)\,\mathrm{d}S(\tilde x)}{\int_{I_v(t)}\theta\big(\frac{|\tilde x-x|}{e(t)}\big)\,\mathrm{d}S(\tilde x)}\,\mathrm{d}S(x)\\
&\quad + Cr_c^{-2}\int_{I_v(t)} \frac{\int_{I_v(t)}\theta\big(\frac{|\tilde x-x|}{e(t)}\big)\,\mathrm{d}|D^s h^\pm|(\tilde x,t)}{\int_{I_v(t)}\theta\big(\frac{|\tilde x-x|}{e(t)}\big)\,\mathrm{d}S(\tilde x)}\,\mathrm{d}S(x).
\end{aligned} \tag{3.78}$$

Applying Fubini's theorem and using the bounds (3.77), $G(\lambda p)\leq C(\lambda+\lambda^2)G(p)$, as well as (3.55a) and (3.55b) we deduce the estimate on $\int_{I_v(t)}|\nabla h^\pm_{e(t)}|^2\,\mathrm{d}S$ stated in a). The estimate on $\int_{I_v(t)}|h^\pm_{e(t)}|^2\,\mathrm{d}S$ follows by an analogous argument, first squaring (3.73) and applying Jensen's inequality, then integrating over $I_v(t)$, and finally using (3.77), Fubini as well as (3.55a) and (3.55b).

Proof of b). We start with a change of variables to estimate (recall (3.14))

$$\begin{aligned}
&\int_{\mathbb{R}^d} |\chi_{v,h^+_{e(t)},h^-_{e(t)}} - \chi_{v,h^+,h^-}|\,\mathrm{d}x\\
&\leq C\int_{I_v(t)}\int_0^{r_c} |\chi_{0\leq \mathrm{dist}^\pm(x,I_v(t))\leq h^+_{e(t)}(P_{I_v(t)}x,t)} - \chi_{0\leq \mathrm{dist}^\pm(x,I_v(t))\leq h^+(P_{I_v(t)}x,t)}|\,\mathrm{d}y\,\mathrm{d}S\\
&\quad + C\int_{I_v(t)}\int_0^{r_c} |\chi_{-h^-_{e(t)}(P_{I_v(t)}x,t)\leq \mathrm{dist}^\pm(x,I_v(t))\leq 0} - \chi_{-h^-(P_{I_v(t)}x,t)\leq \mathrm{dist}^\pm(x,I_v(t))\leq 0}|\,\mathrm{d}y\,\mathrm{d}S\\
&\leq C\int_{I_v(t)} |h^+_{e(t)}(x,t)-h^+(x,t)| + |h^-_{e(t)}(x,t)-h^-(x,t)|\,\mathrm{d}S(x).
\end{aligned}$$

By adding zero and using (3.55c) we therefore obtain

$$\begin{aligned}
&\int_{\mathbb{R}^d} |\chi_u - \chi_{v,h^+_{e(t)},h^-_{e(t)}}|\,\mathrm{d}x\\
&\quad\leq \int_{\mathbb{R}^d}|\chi_u-\chi_{v,h^+,h^-}|\,\mathrm{d}x + \int_{\mathbb{R}^d}|\chi_{v,h^+_{e(t)},h^-_{e(t)}} - \chi_{v,h^+,h^-}|\,\mathrm{d}x\\
&\quad\leq C\int_{\mathbb{R}^d} 1-\xi\cdot\frac{\nabla\chi_u}{|\nabla\chi_u|}\,d|\nabla\chi_u|\\
&\quad\quad + C\int_{\mathbb{R}^d}|\chi_u-\chi_v|\min\Big\{\frac{\mathrm{dist}(x,I_v(t))}{r_c},1\Big\}\,\mathrm{d}x\\
&\quad\quad + C\int_{I_v(t)} |h^+_{e(t)}(x,t)-h^+(x,t)| + |h^-_{e(t)}(x,t)-h^-(x,t)|\,\mathrm{d}S(x).
\end{aligned}$$

Observe that one can decompose

$$h^\pm(x,t) = h^\pm_{e(t)}(x,t) + \sum_{k=0}^\infty \big(h^\pm_{2^{-k-1}e(t)}(x,t) - h^\pm_{2^{-k}e(t)}(x,t)\big).$$

A straightforward estimate in local coordinates then yields

$$\begin{aligned}
&\int_{I_v(t)} \big|h^{\pm}_{2^{-k}e(t)} - h^{\pm}_{2^{-k-1}e(t)}\big| \,\mathrm{d}S \\
&\leq C2^{-k}e(t) \int_{I_v(t)} 1 \,\mathrm{d}|D^{\tan}h^{\pm}| \\
&\leq C2^{-k}e(t) \int_{I_v(t)} 1 \,\mathrm{d}|D^{s}h^{\pm}| + C2^{-k}e(t) \int_{I_v(t)} |\nabla h^{+}|\chi_{\{|\nabla h^+|\geq 1\}} \,\mathrm{d}S \\
&\quad + C2^{-k}e(t)\bigg(\int_{I_v(t)} |\nabla h^{+}|^2\chi_{\{|\nabla h^+|\leq 1\}} \,\mathrm{d}S\bigg)^{1/2} \mathcal{H}^{d-1}(I_v(t))^{1/2}.
\end{aligned}$$

Using (3.55b) and summing with respect to $k \in \mathbb{N}$, we get the desired estimate (3.74c).

Proof of c). Note that

$$\int_{I_v(t)} \eta(x)h^{\pm}_{e(t)}(x,t) \,\mathrm{d}S = \int_{I_v(t)} h^{\pm}(\tilde{x},t) \int_{I_v(t)} \frac{\theta\big(\frac{|\tilde{x}-x|}{e(t)}\big)\eta(x)}{\int_{I_v(t)} \theta\big(\frac{|\hat{x}-x|}{e(t)}\big) \,\mathrm{d}S(\hat{x})} \,\mathrm{d}S(x) \,\mathrm{d}S(\tilde{x}).$$

Abbreviating

$$\eta_e(\tilde{x},t) := \int_{I_v(t)} \frac{\theta\big(\frac{|\tilde{x}-x|}{e(t)}\big)\eta(x)}{\int_{I_v(t)} \theta\big(\frac{|\hat{x}-x|}{e(t)}\big) \,\mathrm{d}S(\hat{x})} \,\mathrm{d}S(x),$$

we compute

$$\begin{aligned}
|\nabla^{\tan}_{\tilde{x}}\eta_e(\tilde{x},t)| &= \bigg|\int_{I_v(t)} \frac{\nabla^{\tan}_{\tilde{x}}\theta\big(\frac{|\tilde{x}-x|}{e(t)}\big)\eta(x)}{\int_{I_v(t)} \theta\big(\frac{|\hat{x}-x|}{e(t)}\big) \,\mathrm{d}S(\hat{x})} \,\mathrm{d}S(x)\bigg| \\
&\leq \int_{I_v(t)} \bigg(\frac{|\theta'|\big(\frac{|\tilde{x}-x|}{e(t)}\big)}{\int_{I_v(t)} \theta\big(\frac{|\hat{x}-x|}{e(t)}\big) \,\mathrm{d}S(\hat{x})}\bigg) \frac{\eta(x)}{e(t)} \,\mathrm{d}S(x).
\end{aligned}$$

As in the argument for (3.77), one checks that $\int_{I_v(t)} |\theta'|(\frac{|\tilde{x}-x|}{e(t)}) \,\mathrm{d}S(x) \leq Ce(t)^{d-1}$. Using the lower bound from (3.77), the proof for the standard L^p-inequality for convolutions carries over and we obtain $\|\eta_e\|_{L^p(I_v(t))} \leq C\|\eta\|_{L^p(I_v(t))}$ as well as

$$\int_{I_v(t)} |\nabla\eta_e(x,t)|^p \,\mathrm{d}S(x) \leq \frac{C}{e(t)^p} \int_{I_v(t)} |\eta(x,t)|^p \,\mathrm{d}S(x)$$

for any $p \geq 1$. As a consequence of (3.55d) and these considerations, we deduce

$$\begin{aligned}
&\bigg|\frac{\mathrm{d}}{\mathrm{d}t}\int_{I_v(t)} \eta(x)h^{\pm}_{e(t)}(x,t) \,\mathrm{d}x - \int_{I_v(t)} h^{\pm}(\tilde{x},t)\frac{\mathrm{d}}{\mathrm{d}t}\eta_e(\tilde{x},t) \,\mathrm{d}S(\tilde{x}) \\
&\qquad - \int_{I_v(t)} h^{\pm}(\tilde{x},t)(\mathrm{Id}-\mathrm{n}_v\otimes\mathrm{n}_v)v(\tilde{x},t)\cdot\nabla_{\tilde{x}}\eta_e(\tilde{x},t) \,\mathrm{d}S(\tilde{x})\bigg| \\
&\leq \frac{C}{e(t)r_c^2}\|\eta\|_{L^4(I_v(t))}\bigg(\int_{I_v(t)} |\bar{h}^{\pm}|^4 \,\mathrm{d}S\bigg)^{1/4} \\
&\quad \times\bigg(\int_{I_v(t)} \sup_{y\in[-r_c,r_c]} |u-v|^2(x+y\mathrm{n}_v(x,t),t) \,\mathrm{d}S(x)\bigg)^{1/2}
\end{aligned} \tag{3.79}$$

$$
\begin{aligned}
&+ C\frac{1+\|v\|_{W^{2,\infty}(\mathbb{R}^d\setminus I_v(t))}}{r_c}\|\eta\|_{L^2(I_v(t))} \\
&\quad\times\bigg(\int_{\mathbb{R}^d}|\chi_u(x,t)-\chi_v(x,t)|\,\min\Big\{\frac{\operatorname{dist}(x,I_v(t))}{r_c},1\Big\}\,\mathrm{d}x\bigg)^{\frac12} \\
&+ C\frac{(1+\|v\|_{W^{1,\infty}})}{r_c e(t)}\max_{p\in\{2,4\}}\|\eta\|_{L^p(I_v(t))}\int_{\mathbb{R}^d}1-\xi\cdot\frac{\nabla\chi_u}{|\nabla\chi_u|}\,\mathrm{d}|\nabla\chi_u| \\
&+ C\|\eta\|_{L^2(I_v(t))}\bigg(\int_{I_v(t)}|u-v|^2\,\mathrm{d}S\bigg)^{1/2}.
\end{aligned}
$$

Using the estimate $|v(x,t)-v(\tilde{x},t)|\le C|x-\tilde{x}|\|\nabla v\|_{L^\infty}$, we infer

$$
\begin{aligned}
&\bigg|\int_{I_v(t)}h^\pm(\tilde{x},t)v(\tilde{x},t)\cdot\nabla_{\tilde{x}}\int_{I_v(t)}\frac{\theta\big(\frac{|\tilde{x}-x|}{e(t)}\big)\eta(x)}{\int_{I_v(t)}\theta\big(\frac{|\hat{x}-x|}{e(t)}\big)\,\mathrm{d}S(\hat{x})}\,\mathrm{d}S(x)\,\mathrm{d}S(\tilde{x}) \qquad (3.80)\\
&\quad+\int_{I_v(t)}\eta(x)(v(x,t)\cdot\nabla)h^\pm_{e(t)}(x,t)\,\mathrm{d}S(x)\bigg| \\
&=\bigg|\int_{I_v(t)}\int_{I_v(t)}\eta(x)h^\pm(\tilde{x},t)v(\tilde{x},t)\cdot\nabla_{\tilde{x}}\frac{\theta\big(\frac{|\tilde{x}-x|}{e(t)}\big)}{\int_{I_v(t)}\theta\big(\frac{|\hat{x}-x|}{e(t)}\big)\,\mathrm{d}S(\hat{x})}\,\mathrm{d}S(x)\,\mathrm{d}S(\tilde{x}) \\
&\quad+\int_{I_v(t)}\int_{I_v(t)}\eta(x)h^\pm(\tilde{x},t)v(x,t)\cdot\nabla_x\frac{\theta\big(\frac{|\tilde{x}-x|}{e(t)}\big)}{\int_{I_v(t)}\theta\big(\frac{|\hat{x}-x|}{e(t)}\big)\,\mathrm{d}S(\hat{x})}\,\mathrm{d}S(\tilde{x})\,\mathrm{d}S(x)\bigg| \\
&\le\int_{I_v(t)}\int_{I_v(t)}h^\pm(\tilde{x},t)\|\nabla v\|_{L^\infty}\frac{|\theta'|\big(\frac{|\tilde{x}-x|}{e(t)}\big)|\tilde{x}-x||\eta(x)|}{e(t)\int_{I_v(t)}\theta\big(\frac{|\hat{x}-x|}{e(t)}\big)\,\mathrm{d}S(\hat{x})}\,\mathrm{d}S(x)\,\mathrm{d}S(\tilde{x}) \\
&\quad+\int_{I_v(t)}\int_{I_v(t)}h^\pm(\tilde{x},t)\|v\|_{L^\infty}\frac{\theta\big(\frac{|\tilde{x}-x|}{e(t)}\big)|\eta(x)|\Big|\nabla_x\int_{I_v(t)}\theta\big(\frac{|\hat{x}-x|}{e(t)}\big)\,\mathrm{d}S(\hat{x})\Big|}{\Big(\int_{I_v(t)}\theta\big(\frac{|\hat{x}-x|}{e(t)}\big)\,\mathrm{d}S(\hat{x})\Big)^2}\,\mathrm{d}S(x)\,\mathrm{d}S(\tilde{x}) \\
&\le Cr_c^{-1}\|v\|_{W^{1,\infty}}\bigg(\int_{I_v(t)}|h^\pm(x,t)|^2\,\mathrm{d}S(x)\bigg)^{1/2}\bigg(\int_{I_v(t)}|\eta(x)|^2\,\mathrm{d}S(x)\bigg)^{1/2}
\end{aligned}
$$

where in the last step we have used the simple equality

$$
\begin{aligned}
\nabla_x^{\tan}\int_{I_v(t)}\theta\Big(\frac{|\hat{x}-x|}{e(t)}\Big)\,\mathrm{d}S(\hat{x})&=-\int_{I_v(t)}\nabla_{\hat{x}}^{\tan}\theta\Big(\frac{|\hat{x}-x|}{e(t)}\Big)\,\mathrm{d}S(\hat{x}) \qquad (3.81)\\
&=\int_{I_v(t)}\theta\Big(\frac{|\hat{x}-x|}{e(t)}\Big)\mathrm{H}(\hat{x})\,\mathrm{d}S(\hat{x})
\end{aligned}
$$

and the bounds (3.16) and (3.77). Recall from the transport theorem for moving hypersurfaces (see [128]) that we have for any $f\in C^1(\mathbb{R}^d\times[0,T_{strong}))$

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_{I_v(t)}f(x,t)\,\mathrm{d}S(x)=\int_{I_v(t)}\partial_t f(x,t)\,\mathrm{d}S(x)+\int_{I_v(t)}V_{\mathrm{n}}\cdot\nabla f(x,t)\,\mathrm{d}S(x) \qquad (3.82)\\
+\int_{I_v(t)}f(x,t)\,\mathrm{H}\cdot V_{\mathrm{n}}\,\mathrm{d}S(x)
\end{aligned}
$$

with the normal velocity $V_{\mathrm{n}}(x,t)=(v(x,t)\cdot\mathrm{n}_v(P_{I_v(t)}x,t))\mathrm{n}_v(P_{I_v(t)}x,t)$. Making use of (3.82) and $\frac{\mathrm{d}}{\mathrm{d}t}P_{I_v(t)}\tilde{x} = -V_{\mathrm{n}}(\tilde{x},t)$ for $\tilde{x}\in I_v(t)$ (see (3.68)), we then compute for every $\tilde{x}\in I_v(t)$

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_{I_v(t)}\theta\Big(\frac{|\hat{x}-x|}{e(t)}\Big)\,\mathrm{d}S(\hat{x}) = \frac{\mathrm{d}}{\mathrm{d}t}\int_{I_v(t)}\theta\Big(\frac{|P_{I_v(t)}\hat{x}-P_{I_v(t)}x|}{e(t)}\Big)\,\mathrm{d}S(\hat{x})\\
&= -\frac{e'(t)}{e(t)^2}\int_{I_v(t)}\theta'\Big(\frac{|\hat{x}-x|}{e(t)}\Big)|\hat{x}-x|\,\mathrm{d}S(\hat{x})\\
&\quad+\frac{1}{e(t)}\int_{I_v(t)}\theta'\Big(\frac{|\hat{x}-x|}{e(t)}\Big)\frac{(\hat{x}-x)\cdot(V_{\mathrm{n}}(\hat{x},t)-V_{\mathrm{n}}(x,t))}{e(t)|\hat{x}-x|}\,\mathrm{d}S(\hat{x})\\
&\quad+\int_{I_v(t)}\theta\Big(\frac{|\hat{x}-x|}{e(t)}\Big)V_{\mathrm{n}}(\hat{x})\cdot\mathrm{H}(\hat{x})\,\mathrm{d}S(\hat{x}).
\end{aligned}
$$

This together with another application of (3.82) and the fact that $\mathrm{n}_v\cdot\nabla\eta=0$ on the interface $I_v(t)$ implies for $\tilde{x}\in I_v(t)$

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\eta_e(\tilde{x},t) = \frac{\mathrm{d}}{\mathrm{d}t}\int_{I_v(t)}\frac{\theta\big(\frac{|P_{I_v(t)}\tilde{x}-P_{I_v(t)}x|}{e(t)}\big)\eta(x)}{\int_{I_v(t)}\theta\big(\frac{|P_{I_v(t)}\hat{x}-P_{I_v(t)}x|}{e(t)}\big)\,\mathrm{d}S(\hat{x})}\,\mathrm{d}S(x) \qquad (3.83)\\
&= \int_{I_v(t)}\Bigg(\frac{\theta\big(\frac{|\tilde{x}-x|}{e(t)}\big)\eta(x)}{\int_{I_v(t)}\theta\big(\frac{|\hat{x}-x|}{e(t)}\big)\,\mathrm{d}S(\hat{x})}\Bigg)V_{\mathrm{n}}(x)\cdot\mathrm{H}(x)\,\mathrm{d}S(x)\\
&\quad-\int_{I_v(t)}\frac{\theta\big(\frac{|\tilde{x}-x|}{e(t)}\big)\eta(x)\big(\int_{I_v(t)}\theta\big(\frac{|\hat{x}-x|}{e(t)}\big)V_{\mathrm{n}}(\hat{x})\cdot\mathrm{H}(\hat{x})\,\mathrm{d}S(\hat{x})\big)}{\big(\int_{I_v(t)}\theta\big(\frac{|\hat{x}-x|}{e(t)}\big)\,\mathrm{d}S(\hat{x})\big)^2}\,\mathrm{d}S(x)\\
&\quad+\int_{I_v(t)}\frac{\eta(x)\theta'\big(\frac{|\tilde{x}-x|}{e(t)}\big)\frac{(\tilde{x}-x)\cdot(V_{\mathrm{n}}(\tilde{x})-V_{\mathrm{n}}(x))}{e(t)|\tilde{x}-x|}}{\int_{I_v(t)}\theta\big(\frac{|\hat{x}-x|}{e(t)}\big)\,\mathrm{d}S(\hat{x})}\,\mathrm{d}S(x)\\
&\quad-\int_{I_v(t)}\frac{\theta\big(\frac{|\tilde{x}-x|}{e(t)}\big)\eta(x)\int_{I_v(t)}\theta'\big(\frac{|\hat{x}-x|}{e(t)}\big)\frac{(\hat{x}-x)\cdot(V_{\mathrm{n}}(\hat{x},t)-V_{\mathrm{n}}(x,t))}{e(t)|\hat{x}-x|}\,\mathrm{d}S(\hat{x})}{\big(\int_{I_v(t)}\theta\big(\frac{|\hat{x}-x|}{e(t)}\big)\,\mathrm{d}S(\hat{x})\big)^2}\,\mathrm{d}S(x)\\
&\quad-\frac{e'(t)}{e(t)}\int_{I_v(t)}\frac{F'_{e,\theta}(\tilde{x},x)\eta(x)}{\int_{I_v(t)}\theta\big(\frac{|\hat{x}-x|}{e(t)}\big)\,\mathrm{d}S(\hat{x})}\,\mathrm{d}S(x)
\end{aligned}
$$

where $F'_{e,\theta}(t)\colon I_v(t)\times I_v(t)\to\mathbb{R}$ is the kernel

$$
\begin{aligned}
F'_{e,\theta}(t)(\tilde{x},x) :=\ &\theta'\Big(\frac{|\tilde{x}-x|}{e(t)}\Big)\frac{|P_{I_v(t)}\tilde{x}-P_{I_v(t)}x|}{e(t)} \qquad (3.84)\\
&-\theta\Big(\frac{|\tilde{x}-x|}{e(t)}\Big)\frac{\int_{I_v(t)}\theta'\big(\frac{|\hat{x}-x|}{e(t)}\big)\frac{|P_{I_v(t)}\hat{x}-P_{I_v(t)}x|}{e(t)}\,\mathrm{d}S(\hat{x})}{\int_{I_v(t)}\theta\big(\frac{|\hat{x}-x|}{e(t)}\big)\,\mathrm{d}S(\hat{x})}.
\end{aligned}
$$

Observe that we have

$$
\int_{I_v(t)}F'_{e,\theta}(t)(\tilde{x},x)\,\mathrm{d}S(\tilde{x}) = 0. \qquad (3.85)
$$

By the choice of the cutoff θ, we see that for every given $x\in I_v(t)$ the kernel $F'_{e,\theta}(t)$ is supported in $B_{e(t)/2}(x)\cap I_v(t)$. Moreover, the exact same argumentation which led to the upper bound in (3.77) (we only used the support and upper bound for θ as well as $e(t)\leq r_c$) shows that the kernel $F'_{e,\theta}$ satisfies the upper bound

$$
\int_{I_v(t)}|F'_{e,\theta}(\tilde{x},x)|^p\,\mathrm{d}S(\tilde{x}) \leq C(p)e(t)^{d-1} \qquad (3.86)
$$

for any $1 \leq p < \infty$. We next intend to rewrite the function $F'_{e,\theta}(\tilde{x}, x)$ for fixed x as the divergence of a vector field. By the property (3.85), we may consider Neumann problem for the (tangential) Laplacian with right hand side $F'_{e,\theta}(\cdot, x)$ in some neighborhood (of scale $e(t)$) of the point x. To do this we first rescale the setup, i.e., we consider the kernel $F'_1(\tilde{x}, x) := F'_{e,\theta}(e(t)\tilde{x}, e(t)x)$ for $\tilde{x}, x \in e(t)^{-1}I_v(t)$. By scaling and the fact that $F'_{e,\theta}$ is supported on scale $e(t)/2$, it follows that $F'_1(\cdot, x)$ has zero average on $e(t)^{-1}I_v(t) \cap B_1(x)$ for every point $x \in e(t)^{-1}I_v(t)$ and that

$$\int_{e(t)^{-1}I_v(t)} |F'_1(\tilde{x}, x)|^p \, \mathrm{d}S(\tilde{x}) \leq C(p). \tag{3.87}$$

We fix $x \in e(t)^{-1}I_v(t)$ and solve on $e(t)^{-1}I_v(t) \cap B_1(x)$ the weak formulation of the equation $-\Delta^{\tan}_{\tilde{x}} \hat{F}_1(\cdot, x) = F'_1(\cdot, x)$ with vanishing Neumann boundary condition. More precisely, we require $\hat{F}_1(\cdot, x)$ to have vanishing average on $e(t)^{-1}I_v(t) \cap B_1(x)$ (note that in the weak formulation the curvature term does not appear because it gets contracted with the tangential derivative of the test function). By elliptic regularity and (3.87), it follows

$$||\nabla^{\tan} \hat{F}_1(\tilde{x}, x)||_{L^\infty} \leq C. \tag{3.88}$$

We now rescale back to $I_v(t)$ and define $\hat{F}_{e,\theta}(\tilde{x}, x) := e(t)^2 \hat{F}_1(e(t)^{-1}\tilde{x}, e(t)^{-1}x)$ for $x \in I_v(t)$ and $\tilde{x} \in I_v(t) \cap B_{e(t)}(x)$. For fixed $x \in I_v(t)$, $\hat{F}_{e,\theta}(\cdot, x)$ has vanishing average on $I_v(t) \cap B_{e(t)}(x)$ and solves $-\Delta^{\tan}_{\tilde{x}} \hat{F}_{e,\theta}(\cdot, x) = F'_{e,\theta}(\cdot, x)$ on $I_v(t) \cap B_{e(t)}(x)$ with vanishing Neumann boundary condition. We finally introduce $F_{e,\theta}(\tilde{x}, x) := \nabla^{\tan}_{\tilde{x}} \hat{F}_{e,\theta}(\tilde{x}, x)$ for $x \in I_v(t)$ and $\tilde{x} \in I_v(t) \cap B_{e(t)}(x)$. It then follows from scaling, (3.88) as well as $e(t) < r_c$ that $\nabla_{\tilde{x}} \cdot F_{e,\theta}(\tilde{x}, x) = F'_{e,\theta}$ and

$$||e^{-1}(t) F_{e,\theta}(\tilde{x}, x)||_{L^\infty} \leq C. \tag{3.89}$$

We now have everything in place to proceed with estimating the term

$$\Big| \int_{I_v(t)} h^{\pm}(\tilde{x}, t) \frac{\mathrm{d}}{\mathrm{d}t} \eta_e(\tilde{x}, t) \, \mathrm{d}S(\tilde{x}) \Big|.$$

To this end, we will make use of (3.83) and estimate term by term. Because of (3.16), (3.77), $\|\eta_e\|_{L^p(I_v(t))} \leq C\|\eta\|_{L^p(I_v(t))}$, the estimate

$$\int_{I_v(t)} |\theta'| \Big(\frac{|\tilde{x} - x|}{e(t)} \Big) \, \mathrm{d}S(\tilde{x}) \leq C e(t)^{d-1},$$

the Lipschitz property $|V_{\mathrm{n}}(x) - V_{\mathrm{n}}(\tilde{x})| \leq ||\nabla v||_{L^\infty} |x - \tilde{x}|$, and the fact that $\theta(s) = 0$ for $s \geq 1$, the first four terms on the right-hand side of (3.83) are straightforward to estimate and result in the bound

$$C r_c^{-1} \|v\|_{W^{1,\infty}} \|h^{\pm}(\cdot, t)\|_{L^2(I_v(t))} \|\eta\|_{L^2(I_v(t))}. \tag{3.90}$$

To estimate the fifth term, we first apply Fubini's theorem and then perform an integration by parts (recall that we imposed vanishing Neumann boundary conditions) which entails because of the above considerations

$$
\begin{aligned}
&\frac{1}{e(t)}\int_{I_v(t)} h^{\pm}(\tilde{x},t)\int_{I_v(t)} \frac{F'_{e,\theta}(\tilde{x},x)}{\int_{I_v(t)}\theta\big(\frac{|\hat{x}-x|}{e(t)}\big)\,\mathrm{d}S(\hat{x})}\eta(x)\,\mathrm{d}S(x)\,\mathrm{d}S(\tilde{x})\\
&=\int_{I_v(t)}\bigg(\int_{I_v(t)\cap B_{\frac{3}{4}e(t)}(x)} h^{\pm}(\tilde{x},t)\frac{e(t)^{-1}F'_{e,\theta}(\tilde{x},x)}{\int_{I_v(t)}\theta\big(\frac{|\hat{x}-x|}{e(t)}\big)\,\mathrm{d}S(\hat{x})}\,\mathrm{d}S(\tilde{x})\bigg)\eta(x)\,\mathrm{d}S(x)\\
&=-\int_{I_v(t)}\bigg(\int_{I_v(t)\cap B_{\frac{3}{4}e(t)}} \nabla_{\tilde{x}}h^{\pm}(\tilde{x},t)\cdot\frac{e(t)^{-1}F_{e,\theta}(\tilde{x},x)}{\int_{I_v(t)}\theta\big(\frac{|\hat{x}-x|}{e(t)}\big)\,\mathrm{d}S(\hat{x})}\,\mathrm{d}S(\tilde{x})\bigg)\eta(x)\,\mathrm{d}S(x)\\
&\quad-\int_{I_v(t)} h^{\pm}(\tilde{x},t)\mathrm{H}(\tilde{x},t)\cdot\bigg(\int_{I_v(t)}\frac{e(t)^{-1}F_{e,\theta}(\tilde{x},x)}{\int_{I_v(t)}\theta\big(\frac{|\hat{x}-x|}{e(t)}\big)\,\mathrm{d}S(\hat{x})}\eta(x)\,\mathrm{d}S(x)\bigg)\,\mathrm{d}S(\tilde{x}).
\end{aligned}
$$

Using (3.89) as well as the lower bound from (3.77) we see that the second term can be estimated by a term of the form (3.90). For the first term, note that by the properties of $F_{e,\theta}$ we may interpret the integral in brackets as the mollification of $\nabla h^{\pm}$ on scale $e(t)$. Applying the argument which led to (3.78) (for this, we only need the upper bound (3.89) for $F_{e,\theta}$, a lower bound as in (3.77) is only required for θ) we observe that one can bound this term similar to $\|\nabla h^{\pm}_{e(t)}(\cdot,t)\|_{L^2(I_v(t))}$. We therefore obtain the bound

$$
\begin{aligned}
&\bigg|\int_{I_v(t)} h^{\pm}(\tilde{x},t)\frac{\mathrm{d}}{\mathrm{d}t}\eta_e(\tilde{x},t)\,\mathrm{d}S(\tilde{x})\bigg|\\
&\leq Cr_c^{-4}\|v\|_{W^{1,\infty}}(1+e'(t))\bigg(\int_{\mathbb{R}^d} 1-\xi\cdot\frac{\nabla\chi_u}{|\nabla\chi_u|}\,\mathrm{d}|\nabla\chi_u|\bigg)^{1/2}||\eta||_{L^2(I_v(t))}\\
&\quad+Cr_c^{-6}\|v\|_{W^{1,\infty}}(1+e'(t))\bigg(\int_{\mathbb{R}^d}|\chi_u-\chi_v|\min\Big\{\frac{\mathrm{dist}(x,I_v(t))}{r_c},1\Big\}\,\mathrm{d}x\bigg)^{1/2}||\eta||_{L^2(I_v(t))}.
\end{aligned}
$$

Hence, combining (3.79) with these estimates for the fourth term from (3.83) as well as (3.90) and (3.80), we obtain the desired estimate on the time derivative. This concludes the proof. □

3.5.3 Construction of the compensation function w for the velocity gradient discontinuity

We turn to the construction of a compensating vector field, which shall be small in the L^2-norm but whose associated viscous stress $\mu(\chi_u)D^{\mathrm{sym}}w$ shall compensate for (most of) the problematic viscous term $(\mu(\chi_u)-\mu(\chi_v))D^{\mathrm{sym}}v$ appearing on the right hand side of the relative entropy inequality from Proposition 3.10 in the case of different shear viscosities.

Before we state the main result of this section, we introduce some further notation. Let $h^+_{e(t)}$ be defined as in Proposition 3.27. We then denote by $P_{h^+_{e(t)}}$ the downward projection onto the graph of $h^+_{e(t)}$, i.e.,

$$
P_{h^+_{e(t)}}(x,t) := P_{I_v(t)}x + h^+_{e(t)}(P_{I_v(t)}x,t)\mathrm{n}_v(P_{I_v(t)}x,t).
$$

for all (x,t) such that $\mathrm{dist}(x,I_v(t))<r_c$. Note that this map does not define an orthogonal projection. Analogously, one introduces the projection $P_{h^-_{e(t)}}$ onto the graph of $h^-_{e(t)}$.

Proposition 3.28. *Let (χ_u,u,V) be a varifold solution to the free boundary problem for the incompressible Navier-Stokes equation for two fluids (1.1a)-(1.1c) in the sense of Definition 3.2 on some time interval $[0,T_{vari})$. Let (χ_v,v) be a strong solution to (1.1a)-(1.1c) in the sense of Definition 3.6 on some time interval $[0,T_{strong})$ with $T_{strong}\leq T_{vari}$. Let ξ be the* *extension of the inner unit normal vector field* n_v *of the interface* $I_v(t)$ *from Definition 3.13. Let* $e\colon [0,T_{strong}) \to (0,r_c]$ *be a* C^1*-function and assume that the relative entropy is bounded by* $E[\chi_u,u,V|\chi_v,v](t) \le e(t)^2$*. Let the regularized local interface error heights* $h^+_{e(t)}$ *and* $h^-_{e(t)}$ *be defined as in Proposition 3.27.*

Then there exists a solenoidal vector field $w \in L^2([0,T_{strong}];H^1(\mathbb{R}^d))$ *such that* w *is subject to the estimates*

$$\begin{aligned}\int_{\mathbb{R}^d} |w|^2 \,\mathrm{d}x \le C(r_c^{-4}R^2\|v\|^2_{W^{2,\infty}(\mathbb{R}^d\setminus I_v(t))}+1)& \\ \times \int_{I_v(t)} |h^+_{e(t)}|^2+|\nabla h^+_{e(t)}|^2+|h^-_{e(t)}|^2+|\nabla h^-_{e(t)}|^2 \,\mathrm{d}S,&\end{aligned} \tag{3.91}$$

where $R>0$ *is such that* $I_v(t)+B_{r_c}\subset B_R(0)$*, and*

$$\begin{aligned}&\int_{\{\mathrm{dist}^\pm(x,I_v(t))\ge 0\}} \big|\nabla w - \chi_{0\le \mathrm{dist}^\pm(x,I_v(t))\le h^+_{e(t)}(P_{I_v(t)}x)} W\otimes \mathrm{n}_v(P_{I_v(t)}x,t)\big|^2 \,\mathrm{d}x \\ &+\int_{\{\mathrm{dist}^\pm(x,I_v(t))\le 0\}} \big|\nabla w - \chi_{-h^-_{e(t)}(P_{I_v(t)}x)\le \mathrm{dist}^\pm(x,I_v(t))\le 0} W\otimes \mathrm{n}_v(P_{I_v(t)}x,t)\big|^2 \,\mathrm{d}x \\ &+\int_{\mathbb{R}^d} \chi_{\mathrm{dist}^\pm(x,I_v(t))\notin[-h^-_{e(t)}(P_{I_v(t)}x),h^+_{e(t)}(P_{I_v(t)}x)]}|\nabla w|^2 \,\mathrm{d}x \\ &\le Cr_c^{-4}\|v\|^2_{W^{2,\infty}(\mathbb{R}^d\setminus I_v(t))}\int_{I_v(t)} |h^+_{e(t)}|^2+|\nabla h^+_{e(t)}|^2+|h^-_{e(t)}|^2+|\nabla h^-_{e(t)}|^2 \,\mathrm{d}S,\end{aligned} \tag{3.92}$$

where the vector field W *is given by*

$$W(x,t) := \frac{2(\mu_+-\mu_-)}{\mu_+(1-\chi_v)+\mu_-\chi_v}\big(\mathrm{Id}-\mathrm{n}_v\otimes\mathrm{n}_v\big)(P_{I_v(t)}x)\big(D^{\mathrm{sym}}v\cdot\mathrm{n}_v(P_{I_v(t)}x)\big), \tag{3.93}$$

with the symmetric gradient defined by $D^{\mathrm{sym}}v := \frac{1}{2}(\nabla v+\nabla v^T)$*, as well as the estimates*

$$\begin{aligned}&\int_{I_v(t)} \sup_{y\in(-r_c,r_c)} |w(x+y\mathrm{n}_v(x,t))|^2 \,\mathrm{d}S(x) \\ &\le Cr_c^{-4}\|v\|^2_{W^{2,\infty}(\mathbb{R}^d\setminus I_v(t))}\int_{I_v(t)} |h^+_{e(t)}|^2+|\nabla h^+_{e(t)}|^2+|h^-_{e(t)}|^2+|\nabla h^-_{e(t)}|^2 \,\mathrm{d}S,\end{aligned} \tag{3.94}$$

$$\begin{aligned}\|\nabla w\|_{L^\infty} \le Cr_c^{-4}|\log e(t)|\|v\|_{W^{2,\infty}(\mathbb{R}^d\setminus I_v(t))}+Cr_c^{-3}\|\nabla^3 v\|_{L^\infty(\mathbb{R}^d\setminus I_v(t))}& \\ +Cr_c^{-9}\big(1+\mathcal{H}^{d-1}(I_v(t))\big)\|v\|_{W^{2,\infty}(\mathbb{R}^d\setminus I_v(t))},&\end{aligned} \tag{3.95}$$

$$\begin{aligned}&\bigg(\int_{I_v(t)} \sup_{y\in[-r_c,r_c]} |(\nabla w)^T(x+y\mathrm{n}_v(x,t))\mathrm{n}_v(x,t)|^2 \,\mathrm{d}S(x)\bigg)^{\frac{1}{2}} \\ &\le Cr_c^{-9}(1+\mathcal{H}^{d-1}(I_v(t)))\|v\|_{W^{2,\infty}(\mathbb{R}^d\setminus I_v(t))}e(t)+Cr_c^{-2}\|v\|_{W^{3,\infty}(\mathbb{R}^d\setminus I_v(t))}e(t) \\ &\quad +Cr_c^{-1}\|v\|_{W^{2,\infty}(\mathbb{R}^d\setminus I_v(t))}|\log e(t)|^{\frac{1}{2}}e(t)\end{aligned} \tag{3.96}$$

and

$$\partial_t w(\cdot,t) = -\big(v(\cdot,t)\cdot\nabla\big)w(\cdot,t)+g+\hat{g}, \tag{3.97}$$

where the vector fields g and $\hat{g}$ are subject to the bounds

$$\begin{aligned}
&\|\hat{g}\|_{L^{\frac{4}{3}}(\mathbb{R}^d)} \\
&\leq C\frac{\|v\|_{W^{1,\infty}}\|v\|_{W^{2,\infty}(\mathbb{R}^d\setminus I_v(t))}}{e(t)r_c^3}\bigg(\int_{I_v(t)}|\bar{h}^{\pm}|^4\,\mathrm{d}S\bigg)^{\frac{1}{4}} \\
&\qquad\qquad\times\bigg(\int_{I_v(t)}|h^+_{e(t)}|^2+|\nabla h^+_{e(t)}|^2+|h^-_{e(t)}|^2+|\nabla h^-_{e(t)}|^2\,\mathrm{d}S\bigg)^{\frac{1}{2}} \\
&\quad+C\frac{\|v\|_{W^{1,\infty}}}{e(t)r_c^2}\bigg(\int_{I_v(t)}|\bar{h}^{\pm}|^4\,\mathrm{d}S\bigg)^{\frac{1}{4}}(\|u-v-w\|_{L^2}^{\frac{1}{2}}\|\nabla(u-v-w)\|_{L^2}^{\frac{1}{2}}+\|u-v-w\|_{L^2}) \\
&\quad+C\frac{\|v\|_{W^{1,\infty}}(1+\|v\|_{W^{1,\infty}})}{e(t)}\int_{\mathbb{R}^d}1-\xi\cdot\frac{\nabla\chi_u}{|\nabla\chi_u|}\,\mathrm{d}|\nabla\chi_u|,
\end{aligned}\tag{3.98}$$

and

$$\begin{aligned}
&\|g\|_{L^2(\mathbb{R}^d)} \\
&\leq C\frac{1+\|v\|_{W^{1,\infty}}}{r_c^2}(\|\partial_t\nabla v\|_{L^\infty(\mathbb{R}^d\setminus I_v(t))}+(R^2+1)\|v\|_{W^{2,\infty}(\mathbb{R}^d\setminus I_v(t))}) \\
&\qquad\qquad\times\bigg(\int_{I_v(t)}|h^+_{e(t)}|^2+|\nabla h^+_{e(t)}|^2+|h^-_{e(t)}|^2+|\nabla h^-_{e(t)}|^2\,\mathrm{d}S\bigg)^{\frac{1}{2}} \\
&\quad+C\frac{\|v\|_{W^{1,\infty}}(1+\|v\|_{W^{1,\infty}})}{e(t)r_c}\int_{\mathbb{R}^d}1-\xi\cdot\frac{\nabla\chi_u}{|\nabla\chi_u|}\,\mathrm{d}|\nabla\chi_u| \\
&\quad+Cr_c^{-2}(1+e'(t))\|v\|_{W^{1,\infty}}^2\bigg(\int_{I_v(t)}|h^{\pm}|^2\,\mathrm{d}S\bigg)^{\frac{1}{2}} \\
&\quad+C\frac{\|v\|_{W^{1,\infty}}(1+\|v\|_{W^{2,\infty}(\mathbb{R}^d\setminus I_v(t))})}{r_c}\bigg(\int_{\mathbb{R}^d}|\chi_u-\chi_v|\min\Big\{\frac{\mathrm{dist}(x,I_v(t))}{r_c},1\Big\}\,\mathrm{d}x\bigg)^{\frac{1}{2}} \\
&\quad+C\|v\|_{W^{1,\infty}}(\|u-v-w\|_{L^2}^{\frac{1}{2}}\|\nabla(u-v-w)\|_{L^2}^{\frac{1}{2}}+\|u-v-w\|_{L^2}),
\end{aligned}\tag{3.99}$$

where $\bar{h}^{\pm}$ is defined as $h^{\pm}$ but now with respect to the modified cut-off function $\bar{\theta}(\cdot)=\theta(\frac{\cdot}{2})$, see Proposition 3.26. Furthermore, w may be taken to have the regularity $\nabla w(\cdot,t)\in W^{1,\infty}(\mathbb{R}^d\setminus(I_v(t)\cup I_{h_e^+}(t)\cup I_{h_e^+}(t)))$ for almost every t, where $I_{h_e^{\pm}}(t)$ denotes the C^3-manifold $\{x\pm h^{\pm}_{e(t)}(x)\mathrm{n}_v(x):x\in I_v(t)\}$.

Proof. **Step 1: Definition of w.** Let η be a cutoff supported at each $t\in[0,T_{strong})$ in the set $I_v(t)+B_{r_c/2}$ with $\eta\equiv 1$ in $I_v(t)+B_{r_c/4}$ and $|\nabla\eta|\leq Cr_c^{-1}$, $|\nabla^2\eta|\leq Cr_c^{-2}$ as well as $|\partial_t\eta|\leq Cr_c^{-1}\|v\|_{L^\infty}$ and $|\partial_t\nabla\eta|\leq Cr_c^{-2}\|v\|_{W^{1,\infty}}$. For example, one may choose $\eta(x,t):=\theta(\frac{\mathrm{dist}(x,I_v(t))}{r_c})$ where $\theta\colon\mathbb{R}^+\to[0,1]$ is the smooth cutoff already used in the definition of the regularized local interface error heights in Proposition 3.27.

Define the vector field W as given in (3.93) and set (making use of the notation $a\wedge b=\min\{a,b\}$ and $a\vee b=\max\{a,b\}$)

$$w^+(x,t):=\eta\int_0^{(\mathrm{dist}^{\pm}(x,I_v(t))\vee 0)\wedge h^+_{e(t)}(P_{I_v(t)}x)}W(P_{I_v(t)}x+y\mathrm{n}_v(P_{I_v(t)}x,t))\,\mathrm{d}y\tag{3.100}$$

as well as

$$w^-(x,t):=\eta\int_{(\mathrm{dist}^{\pm}(x,I_v(t))\wedge 0)\vee -h^-_{e(t)}(P_{I_v(t)}x)}^0W(P_{I_v(t)}x+y\mathrm{n}_v(P_{I_v(t)}x,t))\,\mathrm{d}y.\tag{3.101}$$

For this choice, we have

$$\begin{aligned}
&\nabla w^+(x,t) \qquad\qquad (3.102)\\
&= \chi_{0\leq \mathrm{dist}^\pm(x,I_v(t))\leq h^+_{e(t)}(P_{I_v(t)}x)} W(x)\otimes \mathrm{n}_v(P_{I_v(t)}x)\\
&\quad + \eta\,\chi_{\mathrm{dist}^\pm(x,I_v(t))>h^+_{e(t)}(P_{I_v(t)}x)} W(P_{h^+_{e(t)}}x)\otimes \nabla h^+_{e(t)}(P_{I_v(t)}x)\nabla P_{I_v(t)}(x)\\
&\quad + \eta\int_0^{(\mathrm{dist}^\pm(x,I_v(t))\vee 0)\wedge h^+_{e(t)}(P_{I_v(t)}x)} \nabla W(P_{I_v(t)}x+y\mathrm{n}_v(P_{I_v(t)}x))(\nabla P_{I_v(t)}x+y\nabla \mathrm{n}_v(P_{I_v(t)}x))\,\mathrm{d}y\\
&\quad + \nabla\eta\int_0^{(\mathrm{dist}^\pm(x,I_v(t))\vee 0)\wedge h^+_{e(t)}(P_{I_v(t)}x)} W(P_{I_v(t)}x+y\mathrm{n}_v(P_{I_v(t)}x))\,\mathrm{d}y
\end{aligned}$$

(note that this directly implies the last claim about the regularity of w, namely $\nabla w(\cdot,t)\in W^{1,\infty}(\mathbb{R}^d\setminus(I_v(t)\cup I_{h_e^+}(t)\cup I_{h_e^+}(t)))$ for almost every t) as well as

$$\begin{aligned}
&\partial_t w^+(x,t) \qquad\qquad (3.103)\\
&= \chi_{0\leq \mathrm{dist}^\pm(x,I_v(t))\leq h^+_{e(t)}(P_{I_v(t)}x)} W(x)\partial_t\,\mathrm{dist}^\pm(x,I_v(t))\\
&\quad + \eta\,\chi_{\mathrm{dist}^\pm(x,I_v(t))>h^+_{e(t)}(P_{I_v(t)}x)} W(P_{h^+_{e(t)}}x)\big(\partial_t h^+_{e(t)}(P_{I_v(t)}x)+\partial_t P_{I_v(t)}x\cdot\nabla h^+_{e(t)}(P_{I_v(t)}x)\big)\\
&\quad + \eta\int_0^{(\mathrm{dist}^\pm(x,I_v(t))\vee 0)\wedge h^+_{e(t)}(P_{I_v(t)}x)} \partial_t W(P_{I_v(t)}x+y\mathrm{n}_v(P_{I_v(t)}x))\,\mathrm{d}y\\
&\quad + \eta\int_0^{(\mathrm{dist}^\pm(x,I_v(t))\vee 0)\wedge h^+_{e(t)}(P_{I_v(t)}x)} \nabla W(P_{I_v(t)}x+y\mathrm{n}_v(P_{I_v(t)}x))(\partial_t P_{I_v(t)}x+y\partial_t \mathrm{n}_v(P_{I_v(t)}x))\,\mathrm{d}y\\
&\quad + \partial_t\eta\int_0^{(\mathrm{dist}^\pm(x,I_v(t))\vee 0)\wedge h^+_{e(t)}(P_{I_v(t)}x)} W(P_{I_v(t)}x+y\mathrm{n}_v(P_{I_v(t)}x))\,\mathrm{d}y.
\end{aligned}$$

Moreover, note that (3.102) entails by the definition of the vector field W

$$\begin{aligned}
&\nabla\cdot w^+(x,t) \qquad\qquad (3.104)\\
&= \eta\,\chi_{\mathrm{dist}^\pm(x,I_v(t))>h^+_{e(t)}(P_{I_v(t)}x)} W(P_{h^+_{e(t)}}x)\cdot\nabla h^+_{e(t)}(P_{I_v(t)}x)\nabla P_{I_v(t)}(x)\\
&\quad + \eta\int_0^{(\mathrm{dist}^\pm(x,I_v(t))\vee 0)\wedge h^+_{e(t)}(P_{I_v(t)}x)} \mathrm{tr}\,\nabla W(P_{I_v(t)}x+y\mathrm{n}_v(P_{I_v(t)}x))(\nabla P_{I_v(t)}x+y\nabla \mathrm{n}_v(P_{I_v(t)}x))\,\mathrm{d}y\\
&\quad + \nabla\eta\cdot\int_0^{(\mathrm{dist}^\pm(x,I_v(t))\vee 0)\wedge h^+_{e(t)}(P_{I_v(t)}x)} W(P_{I_v(t)}x+y\mathrm{n}_v(P_{I_v(t)}x))\,\mathrm{d}y.
\end{aligned}$$

Analogous formulas and properties can be derived for w^-. The function $w^+ + w^-$ would then satisfy our conditions, with the exception of the solenoidality $\nabla\cdot w = 0$. For this reason, we introduce the (usual) kernel

$$\theta(x) := \frac{1}{\mathcal{H}^{d-1}(\mathbb{S}^{d-1})}\frac{x}{|x|^d}$$

and set

$$w(x,t) := w^+(x,t) - (\theta * \nabla\cdot w^+)(x,t) + w^-(x,t) - (\theta*\nabla\cdot w^-)(x,t). \qquad (3.105)$$

It is immediate that $\nabla\cdot w = 0$.

Step 2: Estimates on w and ∇w. From (3.102), $|\nabla\eta| \leq Cr_c^{-1}$ as well as the bounds (3.15) and (3.24) we deduce the pointwise bound

$$\begin{aligned}
&\big|\nabla w^+ - \chi_{0\leq \mathrm{dist}^\pm(x,I_v(t))\leq h^+_{e(t)}(P_{I_v(t)}x)} W(x)\otimes \mathrm{n}_v(P_{I_v(t)}x)\big| \\
&\leq C\chi_{\operatorname{supp}\eta} r_c^{-1}\|\nabla v\|_{L^\infty}|\nabla h^+_{e(t)}(P_{I_v(t)}x)| \\
&\quad + C\chi_{\operatorname{supp}\eta}\big(r_c^{-2}\|\nabla v\|_{L^\infty} + r_c^{-1}\|\nabla^2 v\|_{L^\infty(\mathbb{R}^d\setminus I_v(t))}\big)|h^+_{e(t)}(P_{I_v(t)}x)| \\
&\quad + Cr_c^{-1}\chi_{\operatorname{supp}\eta}\|\nabla v\|_{L^\infty}|h^+_{e(t)}(P_{I_v(t)}x)|
\end{aligned} \tag{3.106}$$

and therefore by integration and a change of variables Φ_t

$$\begin{aligned}
&\int_{\mathbb{R}^d}\Big|\nabla w^+ - \chi_{0\leq \mathrm{dist}^\pm(x,I_v(t))\leq h^+_{e(t)}(P_{I_v(t)}x)} W(x)\otimes \mathrm{n}_v(P_{I_v(t)}x)\Big|^2 \,\mathrm{d}x \\
&\leq C(r_c^{-4}\|\nabla v\|^2_{L^\infty} + r_c^{-2}\|\nabla^2 v\|^2_{L^\infty(\mathbb{R}^d\setminus I_v(t))})\int_{\mathbb{R}^d}\chi_{\operatorname{supp}\eta}(|h^+_{e(t)}|^2 + |\nabla h^+_{e(t)}|^2)(P_{I_v(t)}x)\,\mathrm{d}x \\
&\leq Cr_c^{-4}\|v\|^2_{W^{2,\infty}(\mathbb{R}^d\setminus I_v(t))}\int_{I_v(t)}|h^+_{e(t)}|^2 + |\nabla h^+_{e(t)}|^2\,\mathrm{d}S.
\end{aligned} \tag{3.107}$$

Observe that this also implies by (3.93)

$$\int_{\mathbb{R}^d}|\nabla\cdot w^+|^2\,\mathrm{d}x \leq Cr_c^{-4}\|v\|^2_{W^{2,\infty}(\mathbb{R}^d\setminus I_v(t))}\int_{I_v(t)}|h^+_{e(t)}|^2 + |\nabla h^+_{e(t)}|^2\,\mathrm{d}S. \tag{3.108}$$

From this, Theorem 3.38, and the fact that $\nabla\theta$ is a singular integral kernel subject to the assumptions of Theorem 3.38, we deduce

$$\int_{\mathbb{R}^d}\big|\nabla(\theta * (\nabla\cdot w^+))\big|^2\,\mathrm{d}x \leq Cr_c^{-4}\|v\|^2_{W^{2,\infty}(\mathbb{R}^d\setminus I_v(t))}\int_{I_v(t)}|h^+_{e(t)}|^2 + |\nabla h^+_{e(t)}|^2\,\mathrm{d}S. \tag{3.109}$$

Combining the estimates (3.107) and (3.109) with the corresponding inequalities for w^- and $\theta * \nabla\cdot w^-$, we deduce our estimate (3.92).

The trivial estimate $|w^+(x,t)| \leq \chi_{\operatorname{supp}\eta}(x,t)\|\nabla v\|_{L^\infty} h^+_{e(t)}(P_{I_v(t)}x)$ gives by the change of variables Φ_t

$$\int_{\mathbb{R}^d}|w^+|^2\,\mathrm{d}x \leq Cr_c\int_{I_v(t)}|h^+_{e(t)}|^2\,\mathrm{d}S. \tag{3.110}$$

Now, let $R > 1$ be big enough such that $I_v(t) + B_{r_c} \subset B_R(0)$ for all $t \in [0, T_{strong})$. We then estimate with an integration by parts and Theorem 3.38 applied to the singular integral operator $\nabla\theta$

$$\begin{aligned}
\int_{\mathbb{R}^d\setminus B_{3R}(0)}\big|\theta * (\nabla\cdot w^+)\big|^2\,\mathrm{d}x &= \int_{\mathbb{R}^d\setminus B_{3R}(0)}\bigg|\int_{B_R(0)}\theta(x-\tilde{x})(\nabla\cdot w^+(\tilde{x}))\,\mathrm{d}\tilde{x}\bigg|^2\,\mathrm{d}x \\
&\leq \int_{\mathbb{R}^d}\bigg|\int_{B_R(0)}\nabla\theta(x-\tilde{x})w^+(\tilde{x})\,\mathrm{d}\tilde{x}\bigg|^2\,\mathrm{d}x \\
&\leq C\int_{B_R(0)}|w^+|^2\,\mathrm{d}x.
\end{aligned} \tag{3.111}$$

By Young's inequality for convolutions, (3.108), (3.110) and (3.111) we then obtain

$$\begin{aligned}
&\int_{\mathbb{R}^d} \big|\theta * (\nabla\cdot w^+)\big|^2 \,\mathrm{d}x \\
&= \int_{B_{3R}(0)} \big|\theta * (\nabla\cdot w^+)\big|^2 \,\mathrm{d}x + \int_{\mathbb{R}^d\setminus B_{3R}(0)} \big|\theta * (\nabla\cdot w^+)\big|^2 \,\mathrm{d}x \\
&\leq C\Big(\int_{B_{3R}(0)} \frac{1}{|x|^{d-1}}\,\mathrm{d}x\Big)^2 \int_{\mathbb{R}^d} |\nabla\cdot w^+|^2\,\mathrm{d}x + C\int_{\mathbb{R}^d} |w^+|^2\,\mathrm{d}x \qquad (3.112)\\
&\leq C(r_c^{-4}R^2\|v\|^2_{W^{2,\infty}(\mathbb{R}^d\setminus I_v(t))}+1)\int_{I_v(t)} |h^+_{e(t)}|^2 + |\nabla h^+_{e(t)}|^2\,\mathrm{d}S.
\end{aligned}$$

Together with the respective estimates for w^- and $\theta * (\nabla\cdot w^-)$, this implies (3.91). The estimate (3.94) follows directly from (3.100) and the estimates (3.109) and (3.112) on the H^1-norm of $\theta * (\nabla\cdot w^+)$ as well as the definition of w^- and the analogous estimates for $\theta * (\nabla\cdot w^-)$.

Step 3: L^∞-estimates for ∇w. Regarding the estimate (3.95) on $\|\nabla w\|_{L^\infty}$ we have by (3.106) and the estimates $|\nabla h^+_{e(t)}| \leq Cr_c^{-2}$ and $|h^+_{e(t)}| \leq r_c \leq 1$ from Proposition 3.27

$$\|\nabla w^+\|_{L^\infty} \leq Cr_c^{-4}\|v\|_{W^{2,\infty}(\mathbb{R}^d\setminus I_v(t))}. \qquad (3.113)$$

To estimate $|\nabla(\theta * (\nabla\cdot w^+))|$, we first compute starting with (3.104)

$$\begin{aligned}
&\nabla(\nabla\cdot w^+)(x,t) \qquad (3.114)\\
&= \eta\,\chi_{\mathrm{dist}^\pm(x,I_v)>h^+_{e(t)}(P_{I_v(t)}x)} W(P_{h^+_{e(t)}}x)\cdot\nabla^2 h^+_{e(t)}(P_{I_v(t)}x)\nabla P_{I_v(t)}(x)\nabla P_{I_v(t)}(x)\\
&\quad + \big(W(P_{h^+_{e(t)}}x)\cdot\nabla h^+_{e(t)}(P_{I_v(t)}x)\nabla P_{I_v(t)}(x)\big)\,\nabla\chi_{\mathrm{dist}^\pm(x,I_v)>h^+_{e(t)}(P_{I_v(t)}x)}\\
&\quad + F(x,t),
\end{aligned}$$

where $F(x,t)$ is subject to a bound of the form $|F(x,t)| \leq Cr_c^{-5}\|v\|_{W^{3,\infty}(\mathbb{R}^d\setminus I_v(t))})$ and supported in $I_v(t)+B_{r_c}$. Next, we decompose the kernel θ as $\theta = \sum_{k=-\infty}^{\infty}\theta_k$ with smooth functions θ_k with $\operatorname{supp}\theta_k \subset B_{2^{k+1}}\setminus B_{2^{k-1}}$. More precisely, we first choose a smooth function $\varphi\colon \mathbb{R}_+\to[0,1]$ such that $\varphi(s)=0$ whenever $s\notin[-1/2,2]$ and such that $\sum_{k\in\mathbb{Z}}\varphi(2^k s)=1$ for all $s>0$. Such a function indeed exists, see for instance [15]. We then let $\theta_k(x) := \varphi(2^k|x|)\theta(x)$. Note that $\|\theta_k\|_{L^1(\mathbb{R}^d)} \leq C2^k$, $\|\nabla\theta_k\|_{L^1(\mathbb{R}^d)} \leq C$ as well as $|\nabla\theta_k| \leq C(2^k)^{-d}$. We estimate

$$\begin{aligned}
|\nabla(\theta * (\nabla\cdot w^+))| \leq &\sum_{k=\lfloor\log e^2(t)\rfloor}^{0} |\nabla(\theta_k * (\nabla\cdot w^+))| + \sum_{k=1}^{\infty} |\nabla(\theta_k * (\nabla\cdot w^+))| \qquad (3.115)\\
&+ \sum_{k=-\infty}^{\lfloor\log e^2(t)\rfloor-1} |\theta_k * \nabla(\nabla\cdot w^+)|.
\end{aligned}$$

Using Young's inequality for convolutions as well as the estimate $\|\nabla\theta_k\|_{L^1(\mathbb{R}^d)} \leq C$ we obtain

$$\sum_{k=\lfloor\log e^2(t)\rfloor}^{0} |\nabla(\theta_k * (\nabla\cdot w^+))| \leq 2C|\log e(t)|\|\nabla\cdot w^+\|_{L^\infty}. \qquad (3.116)$$

Moreover, it follows from $|\nabla\theta_k| \leq C(2^k)^{-d}$, the precise formula for $\nabla\cdot w^+$ in (3.104), (3.15), (3.24), a change of variables and Hölder's inequality that

$$\begin{aligned}
&\sum_{k=1}^{\infty} |\nabla(\theta_k * (\nabla\cdot w^+))| && (3.117)\\
&\leq Cr_c^{-2}\|v\|_{W^{2,\infty}(\mathbb{R}^d\setminus I_v(t))} \sum_{k=1}^{\infty}(2^k)^{-d}\int_{I_v(t)+B_{r_c/2}} |\nabla h^+_{e(t)}(P_{I_v(t)}x)| + |h^+_{e(t)}(P_{I_v(t)}x)|\,\mathrm{d}x\\
&\leq Cr_c^{-2}\|v\|_{W^{2,\infty}(\mathbb{R}^d\setminus I_v(t))}\sqrt{\mathcal{H}^{d-1}(I_v(t))}\bigg(\int_{I_v(t)} |\nabla h^+_{e(t)}|^2 + |h^+_{e(t)}|^2\,\mathrm{d}S\bigg)^{\frac{1}{2}}.
\end{aligned}$$

Using (3.114), the estimate $|\nabla^2 h^{\pm}_{e(t)}(\cdot,t)| \leq Cr_c^{-4}e(t)^{-1}$ from Proposition 3.27, (3.15), (3.24) and again Young's inequality for convolutions (recall that $\|\theta_k\|_{L^1(\mathbb{R}^d)} \leq C2^k$), we get

$$\sum_{k=-\infty}^{\lfloor \log e^2(t)\rfloor-1} |\theta_k * \nabla(\nabla\cdot w^+)|(\tilde{x},t) \leq I + II + III \tag{3.118}$$

where the three terms on the right hand side are given by

$$I := \sum_{k=-\infty}^{\lfloor \log e^2(t)\rfloor-1} 2^k Cr_c^{-5}\|v\|_{W^{3,\infty}(\mathbb{R}^d\setminus I_v(t))} \leq Cr_c^{-5}\|v\|_{W^{3,\infty}(\mathbb{R}^d\setminus I_v(t))}e^2(t) \tag{3.119}$$

and

$$II := Cr_c^{-5}\|v\|_{W^{1,\infty}}e(t)^{-1}\sum_{k=-\infty}^{\lfloor \log e^2(t)\rfloor-1} 2^k \leq Cr_c^{-5}\|v\|_{W^{1,\infty}}e(t) \tag{3.120}$$

as well as

$$\begin{aligned}
III := \sum_{k=-\infty}^{\lfloor \log e^2(t)\rfloor-1}\bigg|\int_{\mathbb{R}^d}\theta_k(x-\tilde{x})\otimes\big(W(P_{h^+_{e(t)}}x)\cdot(\nabla P_{I_v(t)})^T(x)\nabla h^+_{e(t)}(P_{I_v(t)}x)\big) && (3.121)\\
\mathrm{d}\nabla\chi_{\mathrm{dist}^{\pm}(x,I_v(t))>h^+_{e(t)}(P_{I_v(t)}x)}(x)\bigg|.
\end{aligned}$$

To estimate the latter term, we proceed as follows. First of all, note that by the definition of $h^+_{e(t)}$ in (3.73) as well as the trivial bound $|h^+| \leq r_c$ it holds $|h^+_{e(t)}| \leq r_c$. Then for all $\tilde{x} \in I_v(t) + \{|x| > r_c + 2^{\lfloor \log e^2(t)\rfloor}\}$ and all $k \leq \lfloor \log e^2(t)\rfloor - 1$ we observe that $\chi_{\{\mathrm{dist}^{\pm}(x,I_v(t))>h^+_{e(t)}(P_{I_v(t)}x)\}}(x) = 1$ for all $x \in \mathbb{R}^d$ such that $|x-\tilde{x}| \leq 2^{k+1}$. In particular, for such $\tilde{x}$ the third term on the right hand side of (3.118) vanishes since the corresponding second term in the formula for $\nabla(\nabla\cdot w^+)$ (see (3.114)) does not appear anymore.

Hence, let $\tilde{x} \in I_v(t) + \{|x| \leq r_c + 2^{\lfloor \log e^2(t)\rfloor}\}$ and denote by F the tangent plane to the manifold $\{\mathrm{dist}^{\pm}(x,I_v(t)) = h^+_{e(t)}(P_{I_v(t)}x)\}$ at the nearest point to $\tilde{x}$. We then have for any $\psi \in C^\infty_{cpt}(\mathbb{R}^d)$

$$\begin{aligned}
&\int_{\mathbb{R}^d}\psi(x)\,\mathrm{d}\nabla\chi_{\{\mathrm{dist}^{\pm}(x,I_v(t))>h^+_{e(t)}(P_{I_v(t)}x)\}}(x) - \int_{\mathbb{R}^d}\psi(x)\,\mathrm{d}\nabla\chi_{\{\mathrm{dist}^{\pm}(x,F)>0\}}(x)\\
&= \int_{\{\mathrm{dist}^{\pm}(x,I_v(t))>h^+_{e(t)}(P_{I_v(t)}x)\}}\nabla\psi(x)\,\mathrm{d}x - \int_{\{\mathrm{dist}^{\pm}(x,F)>0\}}\nabla\psi(x)\,\mathrm{d}x
\end{aligned}$$

and as a consequence

$$\begin{aligned}
&\int_{\mathbb{R}^d} \theta_k(x-\tilde{x}) \otimes \big(W(P_{h^+_{e(t)}}x)\cdot(\nabla P_{I_v(t)})^T(x)\nabla h^+_{e(t)}(P_{I_v(t)}x)\big)\,\mathrm{d}\nabla\chi_{\{\mathrm{dist}^\pm(x,I_v(t))>h^+_{e(t)}(P_{I_v(t)}x)\}}(x)\\
&= \int_F \theta_k(x-\tilde{x}) \otimes \big(W(P_{h^+_{e(t)}}x)\cdot(\nabla P_{I_v(t)})^T(x)\nabla h^+_{e(t)}(P_{I_v(t)}x)\big)\mathrm{n}_F\,\mathrm{d}S(x)\\
&\quad + \int_{\mathbb{R}^d} \big(\chi_{\{\mathrm{dist}^\pm(x,I_v(t))>h^+_{e(t)}(P_{I_v(t)}x)\}} - \chi_{\{\mathrm{dist}^\pm(x,F)>0\}}\big)\\
&\qquad\qquad \nabla\Big(\theta_k(x-\tilde{x}) \otimes \big(W(P_{h^+_{e(t)}}x)\cdot(\nabla P_{I_v(t)})^T(x)\nabla h^+_{e(t)}(P_{I_v(t)}x)\big)\Big)\,\mathrm{d}x.
\end{aligned}$$

Recall that we defined $\theta_k(x) := \varphi(2^k|x|)\theta(x)$ where $\varphi\colon \mathbb{R}_+ \to [0,1]$ is a smooth function such that $\varphi(s)=0$ whenever $s\notin[-1/2,2]$ and such that $\sum_{k\in\mathbb{Z}}\varphi(2^k s)=1$ for all $s>0$. Hence, $|\mathrm{n}_F\cdot\theta_k(x-\tilde{x})| \leq C\frac{|\mathrm{n}_F\cdot(x-\cdot\tilde{x})|}{|x-\tilde{x}|^d} \leq C\frac{\mathrm{dist}(\tilde{x},F)}{|x-\tilde{x}|^d}$ for all $x\in F$. It also follows from the definition of θ that $\int_F(\mathrm{Id}-\mathrm{n}_F\otimes\mathrm{n}_F)\theta_k(x-\tilde{x})\,\mathrm{d}S(x)=0$. Hence we may solve $(\mathrm{Id}-\mathrm{n}_F\otimes\mathrm{n}_F)\theta_k(\cdot-\tilde{x}) = \Delta^{\tan}_x\tilde{\theta}_k(\cdot,\tilde{x})$ on $B_{2^{k+2}}(\tilde{x})\cap F$ with vanishing Neumann boundary conditions. In particular, for $\hat{\theta}_k(x,\tilde{x}) := \nabla^{\tan}_x\tilde{\theta}_k(x,\tilde{x})$ we obtain $(\mathrm{Id}-\mathrm{n}_F\otimes\mathrm{n}_F)\theta_k(x-\tilde{x}) = \nabla^{\tan}_x\cdot\nabla_x\hat{\theta}_k(x,\tilde{x})$. It follows from elliptic regularity that $\hat{\theta}(\cdot,\tilde{x})$ is C^∞. Moreover, since we could have rescaled θ_k first to unit scale, then solved the associated problem on that scale, and finally rescaled the solution back to the dyadic scale k we see that $|\hat{\theta}_k(x,\tilde{x})| \leq C(2^k)^{2-d}$. We then have by an integration by parts

$$\begin{aligned}
\bigg|\int_F (\mathrm{Id}-\mathrm{n}_F\otimes\mathrm{n}_F)\theta_k(x-\tilde{x})\otimes\psi\,\mathrm{d}S(x)\bigg| &\leq \int_{F\cap B_{2^{k+1}}(\tilde{x})} |\hat{\theta}_k(x,\tilde{x})||\nabla^{\tan}\psi|\,\mathrm{d}S(x)\\
&\leq C(2^k)^{2-d}\int_{F\cap B_{2^{k+1}}(\tilde{x})} |\nabla^{\tan}\psi|\,\mathrm{d}S(x)
\end{aligned}$$

for any $\psi\in C^1_{cpt}(\mathbb{R}^d;\mathbb{R}^d)$. Furthermore, it holds

$$\int_{B_{2^k}(\tilde{x})} |\chi_{\{\mathrm{dist}^\pm(x,I_v(t))>h^+_{e(t)}(P_{I_v(t)}x)\}} - \chi_{\{\mathrm{dist}^\pm(x,F)>0\}}|\,\mathrm{d}x \leq C\|\nabla^2 h^+_{e(t)}\|_{L^\infty}(2^k)^{d+1}.$$

Using these considerations in the previous formula, we obtain

$$\begin{aligned}
&\bigg|\int_{\mathbb{R}^d} \theta_k(x-\tilde{x}) \otimes \big(W(P_{h^+_{e(t)}}x)\cdot(\nabla P_{I_v(t)})^T(x)\nabla h^+_{e(t)}(P_{I_v(t)}x)\big)\,\mathrm{d}\nabla\chi_{\{\mathrm{dist}^\pm(x,I_v(t))>h^+_{e(t)}(P_{I_v(t)}x)\}}(x)\bigg| \qquad (3.122)\\
&\leq \int_{F\cap B_{2^{k+1}}(\tilde{x})\setminus B_{2^{k-1}}(\tilde{x})} \frac{\mathrm{dist}(\tilde{x},F)}{|\tilde{x}-x|^d}|W(P_{h^+_{e(t)}}x)\cdot(\nabla P_{I_v(t)})^T(x)\nabla h^+_{e(t)}(P_{I_v(t)}x)|\,\mathrm{d}S(x)\\
&\quad + \int_{F\cap B_{2^{k+1}}(\tilde{x})} C(2^k)^{2-d}|\nabla(W(P_{h^+_{e(t)}}x)\cdot(\nabla P_{I_v(t)})^T(x)\nabla h^+_{e(t)}(P_{I_v(t)}x))|\,\mathrm{d}S(x)\\
&\quad + C\|\nabla^2 h^+_{e(t)}\|_{L^\infty}(2^k)^{d+1}\Big\|\nabla\Big(\theta_k(x-\tilde{x})\otimes\big(W(P_{h^+_{e(t)}}x)\cdot(\nabla P_{I_v(t)})^T(x)\nabla h^+_{e(t)}(P_{I_v(t)}x)\big)\Big)\Big\|_{L^\infty}.
\end{aligned}$$

Making use of the fact that the integral vanishes for $\mathrm{dist}(\tilde{x},F)\geq 2^{k+1}$ and the bounds (3.15) and (3.24) we obtain

$$\begin{aligned}
&\int_{F\cap B_{2^{k+1}}(\tilde{x})\setminus B_{2^{k-1}}(\tilde{x})} \frac{\mathrm{dist}(\tilde{x},F)}{|\tilde{x}-x|^d}|W(P_{h^+_{e(t)}}x)\cdot(\nabla P_{I_v(t)})^T(x)\nabla h^+_{e(t)}(P_{I_v(t)}x)|\,\mathrm{d}S(x) \qquad (3.123)\\
&\leq \chi_{\{\mathrm{dist}(\tilde{x},F)<2^k\}}Cr_c^{-3}\|v\|_{W^{1,\infty}}\frac{\mathrm{dist}(\tilde{x},F)}{2^k}\int_{F\cap B_{2^{k+1}}(\tilde{x})\setminus B_{2^{k-1}}(\tilde{x})} \frac{|\nabla h^+_{e(t)}(P_{I_v(t)}x)|}{|\tilde{x}-x|^{d-1}}\,\mathrm{d}S(x).
\end{aligned}$$

Using also $|\nabla h^+_{e(t)}| \leq Cr_c^{-2}$ and $|\nabla^2 h^+_{e(t)}| \leq Cr_c^{-4}e(t)^{-1}$ from Proposition 3.27, we get

$$\begin{aligned}&\int_{F\cap B_{2^{k+1}}(\tilde{x})} C(2^k)^{2-d}|\nabla(W(P_{h^+_{e(t)}}x)\cdot(\nabla P_{I_v(t)})^T(x)\nabla h^+_{e(t)}(P_{I_v(t)}x))|\,\mathrm{d}S(x) \\ &\leq C2^k\big(e(t)^{-1}r_c^{-5}\|v\|_{W^{1,\infty}}+r_c^{-4}\|v\|_{W^{2,\infty}(\mathbb{R}^d\setminus I_v(t))}\big)\end{aligned} \tag{3.124}$$

and

$$\begin{aligned}&C\|\nabla^2 h^+_{e(t)}\|_{L^\infty}(2^k)^{d+1}\big\|\nabla\big(\theta_k(x-\tilde{x})\otimes\big(W(P_{h^+_{e(t)}}x)\cdot(\nabla P_{I_v(t)})^T(x)\nabla h^+_{e(t)}(P_{I_v(t)}x)\big)\big)\big\|_{L^\infty} \\ &\leq Cr_c^{-4}e(t)^{-1}2^kr_c^{-3}\|v\|_{W^{1,\infty}} \\ &\quad + Cr_c^{-4}e(t)^{-1}(2^k)^2\big(e(t)^{-1}r_c^{-5}\|v\|_{W^{1,\infty}}+r_c^{-4}\|v\|_{W^{2,\infty}(\mathbb{R}^d\setminus I_v(t))}\big).\end{aligned} \tag{3.125}$$

Using (3.122), (3.123), (3.124) and (3.125) to estimate the term in (3.121), we get

$$\begin{aligned}III &\leq C\frac{\|v\|_{W^{1,\infty}}}{r_c^3}\sum_{k=-\infty}^{\lfloor\log e^2(t)\rfloor-1}\chi_{\{\mathrm{dist}(\tilde{x},F)<2^k\}}\frac{\mathrm{dist}(\tilde{x},F)}{2^k}\int_{F\cap B_{2^{k+1}}(\tilde{x})\setminus B_{2^{k-1}}(\tilde{x})}\frac{|\nabla h^+_{e(t)}(P_{I_v(t)}x)|}{|\tilde{x}-x|^{d-1}}\,\mathrm{d}S(x) \\ &\quad + Cr_c^{-9}\|v\|_{W^{2,\infty}(\mathbb{R}^d\setminus I_v(t))}e(t).\end{aligned} \tag{3.126}$$

In turn, combining this with (3.119) and (3.120) and gathering also (3.116), (3.117), (3.113) as well as the corresponding bounds for ∇w^- and $\nabla(\theta*\nabla\cdot w^-)$, we then finally deduce (3.95).

Step 4: L^2L^∞-estimate for ∇w. By making use of the precise formula (3.102) for ∇w^+ and the definition of the vector field W in (3.93), we immediately get

$$\begin{aligned}&\int_{I_v(t)}\sup_{y\in[-r_c,r_c]}|(\nabla w^+)^T(x+y\mathrm{n}_v(x,t))\cdot\mathrm{n}_v(x,t)|^2\,\mathrm{d}S(x) \\ &\leq Cr_c^{-2}\|v\|_{W^{2,\infty}(\mathbb{R}^d\setminus I_v(t))}\int_{I_v(t)}|h^+_{e(t)}|^2+|\nabla h^+_{e(t)}|^2\,\mathrm{d}S.\end{aligned} \tag{3.127}$$

To estimate the contribution from $|\nabla(\theta*(\nabla\cdot w^+))|$ we use the same dyadic decomposition as in (3.115). We start with the terms in the range $k=\lfloor\log e^2(t)\rfloor,\ldots,0$.

Let $x\in I_v(t)$ and $y\in(-r_c,r_c)$ be fixed. We abbreviate $\bar{x}:=x+y\mathrm{n}_v(x,t)$. Denote by F_x the tangent plane of the interface $I_v(t)$ at the point x. Let $\Phi_{F_x}\colon F_x\times\mathbb{R}\to\mathbb{R}^d$ be the diffeomorphism given by $\Phi_{F_x}(\hat{x},\hat{y}):=\hat{x}+\hat{y}\mathrm{n}_{F_x}(\hat{x})$. We start estimating using the change of variables Φ_{F_x}, the bound $|\nabla\theta_k(x)|\leq C\chi_{2^{k-1}\leq|x|\leq 2^{k+1}}|x|^{-d}$, as well as the fact that $\hat{x}+y\mathrm{n}_{F_x}(\hat{x})=\hat{x}+y\mathrm{n}_v(x,t)$ is exactly the point on the ray originating from $\hat{x}\in F_x$ in normal direction which is closest to $\bar{x}$

$$\begin{aligned}&|\big(\nabla(\theta_k*(\nabla\cdot w^+))\big)^T(x+y\mathrm{n}_v(x,t))| \\ &\leq\int_{(B_{2^{k+1}}(\bar{x})\setminus B_{2^{k-1}}(\bar{x}))\cap(I_v(t)+B_{r_c/2})}|\nabla\theta_k(\bar{x}-\tilde{x})||(\nabla\cdot w^+)(\tilde{x})|\,\mathrm{d}\tilde{x} \\ &\leq C\int_{F_x\cap(B_{2^{k+1}}(x)\setminus B_{2^{k-1}}(x))}\sup_{\hat{y}\in[-r_c,r_c]}\frac{|(\nabla\cdot w^+)(\hat{x}+\hat{y}\mathrm{n}_{F_x}(\hat{x}))|}{|x-\hat{x}|^{d-1}}\,\mathrm{d}S(\hat{x}).\end{aligned}$$

Note that the right hand side is independent of y. Hence, we may estimate with Minkowski's inequality

$$\begin{aligned}&\bigg(\int_{I_v(t)}\sup_{y\in[-r_c,r_c]}\bigg|\sum_{k=\lfloor\log e^2(t)\rfloor-1}^{0}\nabla(\theta_k*(\nabla\cdot w^+))(x+y\mathrm{n}_v(x,t))\bigg|^2\,\mathrm{d}S(x)\bigg)^{\frac12} \\ &\leq C|\log e(t)|\bigg(\int_{I_v(t)}\bigg|\int_{F_x}\sup_{\hat{y}\in[-r_c,r_c]}\frac{|(\nabla\cdot w^+)(\hat{x}+\hat{y}\mathrm{n}_{F_x}(\hat{x}))|}{|x-\hat{x}|^{d-1}}\,\mathrm{d}S(\hat{x})\bigg|^2\,\mathrm{d}S(x)\bigg)^{\frac12}\end{aligned}$$

The inner integral is to be understood in the Cauchy principal value sense. To proceed we use the L^2-theory for singular operators of convolution type, the precise formula (3.104) for $\nabla\cdot w^+$ as well as (3.15) and (3.24) which entails

$$\begin{aligned}
&\bigg(\int_{I_v(t)}\bigg|\int_{F_x}\sup_{\hat y\in[-r_c,r_c]}\frac{|(\nabla\cdot w^+)(\hat x+\hat y\mathrm{n}_{F_x}(\hat x))|}{|x-\hat x|^{d-1}}\,\mathrm{d}S(\hat x)\bigg|^2\,\mathrm{d}S(x)\bigg)^{\frac12}\\
&\leq C\bigg(\int_{I_v(t)}\sup_{y\in[-r_c,r_c]}|(\nabla\cdot w^+)(x+y\mathrm{n}_v(x,t))|^2\,\mathrm{d}S(x)\bigg)^{\frac12}\\
&\leq Cr_c^{-1}\|v\|^{\frac12}_{W^{2,\infty}(\mathbb{R}^d\setminus I_v(t))}\bigg(\int_{I_v(t)}|h^+_{e(t)}|^2+|\nabla h^+_{e(t)}|^2\,\mathrm{d}S\bigg)^{\frac12}.
\end{aligned}$$

An application of (3.74a) and the assumption $E[\chi_u,u,V|\chi_v,v](t)\leq e^2(t)$ finally yields

$$\begin{aligned}
&\bigg(\int_{I_v(t)}\sup_{y\in[-r_c,r_c]}\bigg|\sum_{k=\lfloor\log e^2(t)\rfloor-1}^{0}\nabla(\theta_k*(\nabla\cdot w^+))(x+y\mathrm{n}_v(x,t))\bigg|^2\,\mathrm{d}S(x)\bigg)^{1/2}\\
&\leq Cr_c^{-5}\|v\|_{W^{2,\infty}(\mathbb{R}^d\setminus I_v(t))}|\log e(t)|e(t).
\end{aligned}\tag{3.128}$$

We move on with the contributions in the range $k=1,\dots,\infty$. Note that by (3.117) we may directly infer from (3.74a) and the assumption $E[\chi_u,u,V|\chi_v,v](t)\leq e^2(t)$

$$\begin{aligned}
&\int_{I_v(t)}\sup_{y\in[-r_c,r_c]}\bigg|\sum_{k=1}^{\infty}\big(\nabla(\theta_k*(\nabla\cdot w^+))\big)^T(x+y\mathrm{n}_v(x,t))\cdot\mathrm{n}_v(x,t)\bigg|^2\,\mathrm{d}S(x)\\
&\leq Cr_c^{-8}\|v\|^2_{W^{2,\infty}(\mathbb{R}^d\setminus I_v(t))}\mathcal{H}^{d-1}(I_v(t))^2e^2(t).
\end{aligned}\tag{3.129}$$

Moreover, the contributions estimated in (3.119) and (3.120) result in a bound of the form (recall that $e(t)<r_c$)

$$Cr_c^{-4}\|v\|^2_{W^{3,\infty}(\mathbb{R}^d\setminus I_v(t))}e^2(t)+Cr_c^{-8}\|v\|^2_{W^{1,\infty}}e^2(t).\tag{3.130}$$

Note that when summing the respective bounds from (3.124) and (3.125) over the relevant range $k=-\infty,\dots,\lfloor\log e^2(t)\rfloor-1$, we actually gain a factor $e(t)$, i.e., the contributions estimated in (3.124) and (3.125) then directly yield a bound of the form

$$Cr_c^{-18}\|v\|^2_{W^{2,\infty}(\mathbb{R}^d\setminus I_v(t))}e^2(t).\tag{3.131}$$

Finally, the contribution from (3.123) may be estimated as follows. Let $x\in I_v(t)$, $y\in[-r_c,r_c]$ and denote by $F_{\bar x}$ the tangent plane to the manifold $\{\mathrm{dist}^\pm(x,I_v(t))=h^+_{e(t)}(P_{I_v(t)}x)\}$ at the nearest point to $\bar x=x+y\mathrm{n}_v(x,t)$. In light of (3.123), we start estimating for $k\leq\lfloor\log e^2(t)\rfloor-1$ by using Jensen's inequality, the bound $|\nabla h^+_{e(t)}|\leq Cr_c^{-2}$ from Proposition 3.27, as well as the fact that $|\bar x-\tilde x|\geq|x-\tilde x|$ for all $\tilde x\in I_v(t)$ (since $x=P_{I_v(t)}\bar x$ is the closest point to $\bar x$ on the interface $I_v(t)$)

$$\begin{aligned}
&\bigg|\int_{F_{\bar x}\cap B_{2^{k+1}}(\bar x)\setminus B_{2^{k-1}}(\bar x)}\frac{|\nabla h^+_{e(t)}(P_{I_v(t)}\tilde x)|}{|\bar x-\tilde x|^{d-1}}\,\mathrm{d}S(\tilde x)\bigg|^2\\
&\leq\int_{F_{\bar x}\cap B_{2^{k+1}}(\bar x)\setminus B_{2^{k-1}}(\bar x)}\frac{|\nabla h^+_{e(t)}(P_{I_v(t)}\tilde x)|^2}{|\bar x-\tilde x|^{d-1}}\,\mathrm{d}S(\tilde x)\\
&\leq Cr_c^{-2(d-1)}\int_{I_v(t)\cap B_{Cr_c^{-2}2^{k+1}}(x)}\frac{|\nabla h^+_{e(t)}(\tilde x)|^2}{|x-\tilde x|^{d-1}}\,\mathrm{d}S(\tilde x).
\end{aligned}$$

Since this bound does not depend anymore on $y \in [-r_c, r_c]$, we may estimate the contributions from (3.123) using Minkowski's inequality as well as once more the L^2-theory for singular operators of convolution type to reduce everything to the H^1-bound (3.74a) for the local interface error heights. All in all, the contributions from (3.123) are therefore bounded by

$$C r_c^{-14} \|v\|_{W^{1,\infty}}^2 e^2(t). \tag{3.132}$$

The asserted bound (3.96) then finally follows from collecting the estimates (3.127), (3.128), (3.129), (3.130), (3.131) and (3.132) together with the analogous bounds for ∇w^- and $\nabla(\theta * \nabla \cdot w^-)$.

Step 5: Estimate on the time derivative $\partial_t w$. To estimate $\partial_t w^+$, we first deduce using (3.103), $|\partial_t \eta| \leq C r_c^{-1} \|v\|_{L^\infty}$, $|\frac{\mathrm{d}}{\mathrm{d}t} \mathrm{n}_v(P_{I_v(t)} x)| \leq \frac{C}{r_c^2} \|v\|_{W^{1,\infty}}$ (which follows from (3.30)), (3.17) and finally (3.68) that

$$\begin{aligned}
&\partial_t w^+(x,t) \\
&= \chi_{0 \leq \mathrm{dist}^\pm(x, I_v) \leq h^+_{e(t)}(P_{I_v(t)} x)} W(x) \partial_t \,\mathrm{dist}^\pm(x, I_v(t)) \\
&\quad + \eta\, \chi_{\mathrm{dist}^\pm(x, I_v) > h^+_{e(t)}(P_{I_v(t)} x)} W(P_{h^+_{e(t)}} x) \big(\partial_t h^+_{e(t)}(P_{I_v(t)} x) + \partial_t P_{I_v(t)} x \cdot \nabla h^+_{e(t)}(P_{I_v(t)} x)\big) \\
&\quad + \tilde g^+
\end{aligned}$$

for some vector field $\tilde g^+$ subject to $\|\tilde g^+(\cdot,t)\|_{L^2} \leq C r_c^{-2} (1 + \|v\|_{W^{1,\infty}}) (\|v\|_{W^{1,\infty}} + \|\partial_t \nabla v\|_{L^\infty(\mathbb{R}^d \setminus I_v(t))} + \|v\|_{W^{2,\infty}(\mathbb{R}^d \setminus I_v(t))}) (\int_{I_v(t)} |h^+_{e(t)}(\cdot,t)|^2 \,\mathrm{d}S)^{1/2}$. Using (3.102), (3.17) as well as (3.68) we may compute

$$\begin{aligned}
&(v(x) \cdot \nabla) w^+(x,t) \\
&+ \chi_{0 \leq \mathrm{dist}^\pm(x, I_v(t)) \leq h^+_{e(t)}(P_{I_v(t)} x)} W(x) \partial_t \,\mathrm{dist}^\pm(x, I_v(t)) \\
&+ \eta\, \chi_{\mathrm{dist}^\pm(x, I_v(t)) > h^+_{e(t)}(P_{I_v(t)} x)} W(P_{h^+_{e(t)}} x) \partial_t P_{I_v(t)} x \cdot \nabla h^+_{e(t)}(P_{I_v(t)} x) \\
&= \eta\, \chi_{\mathrm{dist}^\pm(x, I_v(t)) > h^+_{e(t)}(P_{I_v(t)} x)} W(P_{h^+_{e(t)}} x) (\mathrm{Id} - \mathrm{n}_v \otimes \mathrm{n}_v) v(P_{I_v(t)} x) \cdot \nabla h^+_{e(t)}(P_{I_v(t)} x) \\
&\quad + \chi_{0 \leq \mathrm{dist}^\pm(x, I_v(t)) \leq h^+_{e(t)}(P_{I_v(t)} x)} W(x) \big((v(x) - v(P_{I_v(t)} x)\big) \cdot \mathrm{n}_v(P_{I_v(t)}) \\
&\quad + \eta\, \chi_{\mathrm{dist}^\pm(x, I_v(t)) > h^+_{e(t)}(P_{I_v(t)} x)} W(P_{h^+_{e(t)}} x) \big(\nabla P_{I_v(t)}(x) v(x) - v(P_{I_v(t)} x)\big) \cdot \nabla h^+_{e(t)}(P_{I_v(t)} x) \\
&\quad + \tilde g^+_1,
\end{aligned}$$

for some $\|\tilde g^+_1\|_{L^2} \leq C r_c^{-2} \|v\|_{W^{2,\infty}(\mathbb{R}^d \setminus I_v(t))} (\int_{I_v(t)} |h^+_{e(t)}(\cdot,t)|^2 + |\nabla h^+_{e(t)}(\cdot,t)|^2 \,\mathrm{d}S)^{\frac{1}{2}}$. This computation in turn implies

$$\begin{aligned}
&\partial_t w^+(x,t) \\
&= -(v(x) \cdot \nabla) w^+(x,t) \\
&\quad + \eta\, \chi_{\mathrm{dist}^\pm(x, I_v(t)) > h^+_{e(t)}(P_{I_v(t)} x)} W(P_{h^+_{e(t)}} x) \big(\partial_t h^+_{e(t)}(P_{I_v(t)} x) + (\mathrm{Id} - \mathrm{n}_v \otimes \mathrm{n}_v) v(P_{I_v(t)} x) \cdot \nabla h^+_{e(t)}(P_{I_v(t)} x)\big) \\
&\quad + g^+
\end{aligned} \tag{3.133}$$

for some g^+ with

$$\begin{aligned}
&\|g^+\|_{L^2} \\
&\leq C r_c^{-2} (1 + \|v\|_{W^{1,\infty}}) (\|\partial_t \nabla v\|_{L^\infty(\mathbb{R}^d \setminus I_v(t))} + \|v\|_{W^{2,\infty}(\mathbb{R}^d \setminus I_v(t))}) \bigg(\int_{I_v(t)} |h^+_{e(t)}|^2 + |\nabla h^+_{e(t)}|^2 \,\mathrm{d}S\bigg)^{\frac{1}{2}}.
\end{aligned}$$

We now aim to make use of (3.74d) to further estimate the second term in the right hand side of (3.133). To establish the corresponding L^2- resp. $L^{\frac{4}{3}}$-contributions, we first need to perform an integration by parts in order to use (3.74d). The resulting curvature term as well as all other terms which do not appear in the third term of (3.133) can be directly bounded by a term whose associated L^2-norm is controlled by

$$Cr_c^{-1}\|v\|_{W^{1,\infty}}\|v\|_{W^{2,\infty}(\mathbb{R}^d\setminus I_v(t))}\Big(\int_{I_v(t)}|h^+_{e(t)}(\cdot,t)|^2+|\nabla h^+_{e(t)}(\cdot,t)|^2\,\mathrm{d}S\Big)^{\frac12}.$$

Hence, using (3.74d) in (3.133) implies

$$\partial_t w^+(x,t) = -(v\cdot\nabla)w^+(x,t)+\bar g^+ + \hat g^+ \tag{3.134}$$

with the corresponding L^2-bound

$$\begin{aligned}
&\|\bar g^+\|_{L^2(\mathbb{R}^d)} \qquad (3.135)\\
&\leq C\frac{1+\|v\|_{W^{1,\infty}}}{r_c^2}\big(\|\partial_t\nabla v\|_{L^\infty(\mathbb{R}^d\setminus I_v(t))}+\|v\|_{W^{2,\infty}(\mathbb{R}^d\setminus I_v(t))}\big)\Big(\int_{I_v(t)}|h^+_{e(t)}|^2+|\nabla h^+_{e(t)}|^2\,\mathrm{d}S\Big)^{\frac12}\\
&\quad + C\frac{\|v\|_{W^{1,\infty}}(1+\|v\|_{W^{1,\infty}})}{e(t)r_c}\int_{\mathbb{R}^d}1-\xi\cdot\frac{\nabla\chi_u}{|\nabla\chi_u|}\,\mathrm{d}|\nabla\chi_u|\\
&\quad + Cr_c^{-2}\|v\|^2_{W^{1,\infty}}(1+e'(t))\big(\|h^\pm(\cdot,t)\|_{L^2(I_v(t))}+\|\nabla h^\pm_{e(t)}(\cdot,t)\|_{L^2(I_v(t))}\big)\\
&\quad + C\frac{\|v\|_{W^{1,\infty}}(1+\|v\|_{W^{2,\infty}(\mathbb{R}^d\setminus I_v(t))})}{r_c}\Big(\int_{\mathbb{R}^d}|\chi_u-\chi_v|\min\Big\{\frac{\mathrm{dist}(x,I_v(t))}{r_c},1\Big\}\,\mathrm{d}x\Big)^{\frac12}\\
&\quad + C\|v\|_{W^{1,\infty}}\Big(\int_{I_v(t)}|u-v|^2\,\mathrm{d}S\Big)^{\frac12}
\end{aligned}$$

and $L^{\frac43}$-estimate

$$\begin{aligned}
&\|\hat g^+\|_{L^{\frac43}(\mathbb{R}^d)} \qquad (3.136)\\
&\leq C\frac{\|v\|_{W^{1,\infty}}}{e(t)r_c^2}\Big(\int_{I_v(t)}|\bar h^\pm|^4\,\mathrm{d}S\Big)^{\frac14}\Big(\int_{I_v(t)}\sup_{y\in[-r_c,r_c]}|u-v|^2(x+y\mathrm{n}_v(x,t),t)\,\mathrm{d}S(x)\Big)^{\frac12}\\
&\quad + C\frac{\|v\|_{W^{1,\infty}}(1+\|v\|_{W^{1,\infty}})}{e(t)}\int_{\mathbb{R}^d}1-\xi\cdot\frac{\nabla\chi_u}{|\nabla\chi_u|}\,\mathrm{d}|\nabla\chi_u|.
\end{aligned}$$

In both bounds, we add and subtract the compensation function w and therefore obtain together with (3.94) and (3.38)

$$\begin{aligned}
\int_{I_v(t)}|u-v|^2\,\mathrm{d}S &\leq \int_{I_v(t)}\sup_{y\in[-r_c,r_c]}|u-v|^2(x+y\mathrm{n}_v(x,t),t)\,\mathrm{d}S(x)\\
&\leq \int_{I_v(t)}\sup_{y\in[-r_c,r_c]}|u-v-w|^2(x+y\mathrm{n}_v(x,t),t)\,\mathrm{d}S(x)\\
&\quad + \int_{I_v(t)}\sup_{y\in[-r_c,r_c]}|w(x+y\mathrm{n}_v(x,t),t)|^2\,\mathrm{d}S(x)\\
&\leq Cr_c^{-4}\|v\|^2_{W^{2,\infty}(\mathbb{R}^d\setminus I_v(t))}\int_{I_v(t)}|h^\pm_{e(t)}|^2+|\nabla h^\pm_{e(t)}|^2\,\mathrm{d}S \qquad (3.137)\\
&\quad + C(\|u-v-w\|_{L^2}\|\nabla(u-v-w)\|_{L^2}+\|u-v-w\|^2_{L^2}).
\end{aligned}$$

Analogous estimates may be derived for w^-. We therefore proceed with the terms related to $\theta*\nabla\cdot w^\pm$. First of all, note that the singular integral operator $(\theta*\nabla\cdot)$ satisfies (see Theorem 3.38)

$$\|\theta*\nabla\cdot\hat g\|_{L^{\frac43}(\mathbb{R}^d)}\leq C\|\hat g\|_{L^{\frac43}(\mathbb{R}^d)},\quad \|\theta*\nabla\cdot\bar g\|_{L^2(\mathbb{R}^d)}\leq C\|\bar g\|_{L^2(\mathbb{R}^d)}. \tag{3.138}$$

Furthermore, to estimate $\|\theta * \nabla \cdot ((v\cdot\nabla)w^+) - (v\cdot\nabla)(\theta * \nabla\cdot w^+)\|_{L^2(\mathbb{R}^d)}$ we first replace v with its normal velocity $V_{\mathrm{n}}(x) := (v(x)\cdot \mathrm{n}_v(P_{I_v(t)}x))\mathrm{n}_v(P_{I_v(t)}x)$. We want to exploit the fact that the vector field V_{n} has bounded derivatives up to second order, see (3.39) and (3.40). Moreover, the kernel $\nabla^2\theta(x-\tilde{x})\otimes(\tilde{x}-x)$ gives rise to a singular integral operator of convolution type, as does $\nabla\theta$. To see this, we need to check whether its average over $\mathbb{S}^{d-1}$ vanishes. We write $x\otimes\nabla^2\theta(x) = \nabla F(x) - \delta_{ij}e_i\otimes\nabla\theta\otimes e_j$, where $F(x) = x\otimes\nabla\theta(x)$. Now, since $\nabla\theta$ is homogeneous of degree $-d$, F itself is homogeneous of degree $-(d-1)$. Hence, we compute $\int_{B_1\setminus B_r}\nabla F\,\mathrm{d}x = \int_{\mathbb{S}^{d-1}}\mathrm{n}\otimes F\,\mathrm{d}S - \int_{r\mathbb{S}^{d-1}}\mathrm{n}\otimes F\,\mathrm{d}S = 0$ for every $0<r<1$. Passing to the limit $r\to 1$ shows that ∇F, and therefore also $\nabla^2\theta(x)\otimes x$, have vanishing average on $\mathbb{S}^{d-1}$. We may now compute (where the integrals are well defined in the Cauchy principal value sense due to the above considerations) for almost every $x\in\mathbb{R}^d$

$$
\begin{aligned}
&\int_{\mathbb{R}^d}\nabla\theta(x-\tilde{x})\cdot(V_{\mathrm{n}}(\tilde{x},t)\cdot\nabla_{\tilde{x}})w^+(\tilde{x},t) - (V_{\mathrm{n}}(x,t)\cdot\nabla_x)\nabla\theta(x-\tilde{x})\cdot w^+(\tilde{x},t)\,\mathrm{d}\tilde{x}\\
&= \int_{\mathbb{R}^d}\nabla\theta(x-\tilde{x})((V_{\mathrm{n}}(\tilde{x},t)-V_{\mathrm{n}}(x,t))\cdot\nabla_{\tilde{x}})w^+(\tilde{x},t)\,\mathrm{d}\tilde{x}\\
&= \int_{\mathbb{R}^d}\nabla^2\theta(x-\tilde{x}) : (V_{\mathrm{n}}(\tilde{x},t)-V_{\mathrm{n}}(x,t)-(\tilde{x}-x)\cdot\nabla V_{\mathrm{n}}(\tilde{x},t))\otimes w^+(\tilde{x},t)\,\mathrm{d}\tilde{x}\\
&\quad - \int_{\mathbb{R}^d}\nabla\theta(x-\tilde{x})\cdot(\nabla\cdot V_{\mathrm{n}})(\tilde{x},t)w^+(\tilde{x},t)\,\mathrm{d}\tilde{x}\\
&\quad + \int_{\mathbb{R}^d}\nabla^2\theta(x-\tilde{x}) : ((\tilde{x}-x)\cdot\nabla)V_{\mathrm{n}}(\tilde{x},t)\otimes\, w^+(\tilde{x},t)\,\mathrm{d}\tilde{x}.
\end{aligned}
$$

Note that we have $|V_{\mathrm{n}}(\tilde{x},t)-V_{\mathrm{n}}(x,t)-(\tilde{x}-x)\cdot\nabla V_{\mathrm{n}}(x,t)| \leq \|\nabla^2 V_{\mathrm{n}}\|_{L^\infty}|\tilde{x}-x|^2$ and $|V_{\mathrm{n}}(\tilde{x},t)-V_{\mathrm{n}}(x,t)-(\tilde{x}-x)\cdot\nabla V_{\mathrm{n}}(x,t)| \leq ||\nabla V_{\mathrm{n}}||_{L^\infty}|\tilde{x}-x|$. We then estimate using Young's inequality for convolutions and $|\nabla^2\theta(x)|\leq|x|^{-d-1}$

$$
\int_{\mathbb{R}^d\setminus B_{3R}(0)}\left|\int_{B_R(0)}\nabla^2\theta(x-\tilde{x}) : (V_{\mathrm{n}}(\tilde{x})-V_{\mathrm{n}}(x)-(\tilde{x}-x)\cdot\nabla V_{\mathrm{n}}(\tilde{x}))\otimes w^+(\tilde{x})\,\mathrm{d}\tilde{x}\right|^2\mathrm{d}x \tag{3.139}
$$
$$
\begin{aligned}
&\leq C\|\nabla V_{\mathrm{n}}\|^2_{L^\infty}\int_{\mathbb{R}^d\setminus B_{3R}(0)}\left|\int_{B_R(0)}\frac{1}{|x-\tilde{x}|^d}|w^+(\tilde{x})|\,\mathrm{d}\tilde{x}\right|^2\mathrm{d}x\\
&\leq C\|\nabla V_{\mathrm{n}}\|^2_{L^\infty}||\,|\cdot|^{-d}\,||^2_{L^2(\mathbb{R}^d\setminus B_R)}\left|\int_{B_R(0)}|w^+|\,\mathrm{d}x\right|^2\\
&\leq CR^{-d}R^d\int_{B_R(0)}|w^+|^2\,\mathrm{d}x.
\end{aligned}
$$

As a consequence, we obtain from (3.139), Young's inequality for convolutions, (3.110) as well as (3.40)

$$
\int_{\mathbb{R}^d}\left|\int_{\mathbb{R}^d}\nabla^2\theta(x-\tilde{x}) : (V_{\mathrm{n}}(\tilde{x})-V_{\mathrm{n}}(x)-(\tilde{x}-x)\cdot\nabla V_{\mathrm{n}}(\tilde{x}))\otimes w^+(\tilde{x})\,\mathrm{d}\tilde{x}\right|^2\mathrm{d}x \tag{3.140}
$$
$$
\begin{aligned}
&\leq C\|\nabla^2 V_{\mathrm{n}}\|^2_{L^\infty}\int_{B_{3R}(0)}\left|\int_{\mathbb{R}^d}\frac{|w^+(\tilde{x})|}{|x-\tilde{x}|^{d-1}}\,\mathrm{d}\tilde{x}\right|^2\mathrm{d}x + C\|\nabla V_{\mathrm{n}}\|^2_{L^\infty}\int_{B_R(0)}|w^+|^2\,\mathrm{d}x\\
&\leq Cr_c^{-4}\|v\|^2_{W^{2,\infty}(\mathbb{R}^d\setminus I_v(t))}(1+R^2)\int_{I_v(t)}|h^+_{e(t)}|^2\,\mathrm{d}S.
\end{aligned}
$$

Applying Theorem 3.38 to the singular integral operators $\nabla\theta$ resp. $\nabla^2\theta\otimes x$ as well as making use of (3.39), (3.110) and (3.140) we then obtain the estimate

$$\begin{aligned}&\int_{\mathbb{R}^d} |\theta * \nabla\cdot((V_{\mathrm{n}}\cdot\nabla)w^+) - (V_{\mathrm{n}}\cdot\nabla)(\theta*\nabla\cdot w^+)|^2 \,\mathrm{d}x \\ &\leq Cr_c^{-4}\|v\|^2_{W^{2,\infty}(\mathbb{R}^d\setminus I_v(t))}(1+R^2)\int_{I_v(t)} |h^+_{e(t)}|^2 \,\mathrm{d}S \\ &\quad + C\|\nabla V_{\mathrm{n}}\|^2_{L^\infty}\int_{\mathbb{R}^d}|w^+|^2\,\mathrm{d}x \\ &\leq Cr_c^{-4}\|v\|^2_{W^{2,\infty}(\mathbb{R}^d\setminus I_v(t))}(1+R^2)\int_{I_v(t)} |h^+_{e(t)}|^2 \,\mathrm{d}S.\end{aligned} \tag{3.141}$$

It remains to estimate $\|\theta*\nabla\cdot((V_{\tan}\cdot\nabla)w^+) - (V_{\tan}\cdot\nabla)(\theta*\nabla\cdot w^+)\|_{L^2(\mathbb{R}^d)}$ with $V_{\tan}(x) = (\mathrm{Id} - \mathrm{n}_v(P_{I_v(t)}x)\otimes\mathrm{n}_v(P_{I_v(t)}x))v(x)$ denoting the tangential velocity of v. To this end, note that we may rewrite

$$\begin{aligned}&\int_{\mathbb{R}^d}\nabla\theta(x-\tilde{x})\cdot(V_{\tan}(\tilde{x},t)\cdot\nabla_{\tilde{x}})w^+(\tilde{x},t) - (\nabla\cdot w^+(\tilde{x},t))(V_{\tan}(x,t)\cdot\nabla_x)\theta(x-\tilde{x})\,\mathrm{d}\tilde{x} \\ &= \int_{\mathbb{R}^d}\nabla\theta(x-\tilde{x})\big(\nabla w^+(\tilde{x}) - \chi_{0\leq \mathrm{dist}^\pm(\tilde{x},I_v(t))\leq h^+_{e(t)}(P_{I_v(t)}\tilde{x})}W(\tilde{x})\otimes\mathrm{n}_v(P_{I_v(t)}\tilde{x})\big)V_{\tan}(\tilde{x},t)\,\mathrm{d}\tilde{x} \\ &\quad - \int_{\mathbb{R}^d}(\nabla\cdot w^+(\tilde{x},t))(V_{\tan}(x,t)\cdot\nabla_x)\theta(x-\tilde{x})\,\mathrm{d}\tilde{x}.\end{aligned}$$

Using Theorem 3.38, (3.107) as well as (3.108) we then obtain

$$\begin{aligned}&\|\theta*\nabla\cdot((V_{\tan}\cdot\nabla)w^+) - (V_{\tan}\cdot\nabla)(\theta*\nabla\cdot w^+)\|^2_{L^2(\mathbb{R}^d)} \\ &\leq Cr_c^{-4}\|v\|^2_{L^\infty}\|v\|^2_{W^{2,\infty}(\mathbb{R}^d\setminus I_v(t))}\int_{I_v(t)}|h^+_{e(t)}|^2 + |\nabla h^+_{e(t)}|^2\,\mathrm{d}S.\end{aligned} \tag{3.142}$$

Putting all the estimates (3.135), (3.136), (3.137), (3.138), (3.141) and (3.142) together, we get

$$\partial_t w(x,t) + (v\cdot\nabla)w(x,t) = g + \hat{g}$$

with the asserted bounds. This concludes the proof. □

3.5.4 Estimate for the additional surface tension terms

Having established all the relevant properties of the compensating vector field w in Proposition 3.28, we can now estimate the additional terms in the relative entropy inequality from Proposition 3.10. To this end, we start with the additional surface tension terms given by

$$\begin{aligned} A_{surTen} &= -\sigma\int_0^T\int_{\mathbb{R}^d\times\mathbb{S}^{d-1}}(s-\xi)\cdot\big((s-\xi)\cdot\nabla\big)w\,\mathrm{d}V_t(x,s)\,\mathrm{d}t \\ &\quad + \sigma\int_0^T\int_{\mathbb{R}^d}(1-\theta_t)\,\xi\cdot(\xi\cdot\nabla)w\,\mathrm{d}|V_t|_{\mathbb{S}^{d-1}}(x)\,\mathrm{d}t \\ &\quad + \sigma\int_0^T\int_{\mathbb{R}^d}(\chi_u-\chi_v)(w\cdot\nabla)(\nabla\cdot\xi)\,\mathrm{d}x\,\mathrm{d}t \\ &\quad + \sigma\int_0^T\int_{\mathbb{R}^d}(\chi_u-\chi_v)\nabla w : \nabla\xi^T\,\mathrm{d}x\,\mathrm{d}t \\ &\quad - \sigma\int_0^T\int_{\mathbb{R}^d}\xi\cdot\big((\mathrm{n}_u-\xi)\cdot\nabla\big)w\,\mathrm{d}|\nabla\chi_u|\,\mathrm{d}t \\ &=: I + II + III + IV + V.\end{aligned} \tag{3.143}$$

A precise estimate for these terms is the content of the following result.

Lemma 3.29. *Let the assumptions and notation of Proposition 3.28 be in place. In particular, we assume that there exists a C^1-function $e\colon [0,T_{strong}) \to [0,r_c)$ such that the relative entropy is bounded by $E[\chi_u,u,V,|\chi_v,v](t) \le e^2(t)$. Then the additional surface tension terms A_{surTen} are bounded by a Gronwall-type term*

$$\begin{aligned}A_{surTen} \le{}& \frac{C}{r_c^{10}}(1+\|v\|^2_{L_t^\infty W_x^{2,\infty}(\mathbb{R}^d\setminus I_v(t))}+\|v\|_{L_t^\infty W_x^{3,\infty}(\mathbb{R}^d\setminus I_v(t))})\\&\int_0^T(1+|\log e(t)|)E[\chi_u,u,V|\chi_v,v](t)\,\mathrm{d}t\\&+\frac{C}{r_c^{10}}(1+\|v\|^2_{L_t^\infty W_x^{2,\infty}(\mathbb{R}^d\setminus I_v(t))}+\|v\|_{L_t^\infty W_x^{3,\infty}(\mathbb{R}^d\setminus I_v(t))})\\&\int_0^T(1+|\log e(t)|)e(t)E[\chi_u,u,V|\chi_v,v]^{\frac12}(t)\,\mathrm{d}t.\end{aligned}\tag{3.144}$$

Proof. We estimate term by term in (3.143). A straightforward estimate for the first two terms using also the coercivity property (3.35) yields

$$\begin{aligned}I+II \le{}& C\int_0^T\|\nabla w(t)\|_{L_x^\infty}\int_{\mathbb{R}^d\times\mathbb{S}^{d-1}}|s-\xi|^2\,\mathrm{d}V_t(x,s)\ \mathrm{d}t\\&+C\int_0^T\|\nabla w(t)\|_{L_x^\infty}\int_{\mathbb{R}^d}(1-\theta_t)\,\mathrm{d}|V_t|_{\mathbb{S}^{d-1}}(x)\ \mathrm{d}t\\\le{}& C\int_0^T\|\nabla w(t)\|_{L_x^\infty}E[\chi_u,u,V|\chi_v,v](t)\,\mathrm{d}t.\end{aligned}\tag{3.145}$$

Making use of (3.15), a change of variables Φ_t, Hölder's and Young's inequality, (3.94), (3.37), (3.74a) as well as the coercivity property (3.32) the term III may be bounded by

$$\begin{aligned}III \le{}& \frac{C}{r_c^2}\int_0^T\int_{I_v(t)}\sup_{y\in[-r_c,r_c]}|w(x+y\mathrm{n}_v(x,t))|\int_{-r_c}^{r_c}|\chi_u-\chi_v|(x+y\mathrm{n}_v(x,t))\ \mathrm{d}y\ \mathrm{d}S\ \mathrm{d}t\\\le{}& \frac{C}{r_c^2}\int_0^T\int_{I_v(t)}\sup_{y\in[-r_c,r_c]}|w(x+y\mathrm{n}_v(x,t))|^2\ \mathrm{d}S\ \mathrm{d}t\\&+\frac{C}{r_c^2}\int_0^T\int_{I_v(t)}\left|\int_{-r_c}^{r_c}|\chi_u-\chi_v|(x+y\mathrm{n}_v(x,t))\ \mathrm{d}y\right|^2\ \mathrm{d}S\ \mathrm{d}t\\\le{}& \frac{C}{r_c^6}\|v\|^2_{L_t^\infty W_x^{2,\infty}(\mathbb{R}^d\setminus I_v(t))}\int_0^T\int_{I_v(t)}|h^\pm_{e(t)}|^2+|\nabla h^\pm_{e(t)}|^2\ \mathrm{d}S\ \mathrm{d}t\\&+\frac{C}{r_c^2}\int_0^T\int_{\mathbb{R}^d}|\chi_u-\chi_v|\min\left\{\frac{\mathrm{dist}(x,I_v(t))}{r_c},1\right\}\ \mathrm{d}x\ \mathrm{d}t\\\le{}& \frac{C}{r_c^{10}}\|v\|^2_{L_t^\infty W_x^{2,\infty}(\mathbb{R}^d\setminus I_v(t))}\int_0^T\int_{\mathbb{R}^d}1-\xi\cdot\frac{\nabla\chi_u}{|\nabla\chi_u|}\ \mathrm{d}|\nabla\chi_u|\ \mathrm{d}t\\&+\frac{C}{r_c^{10}}(1+\|v\|^2_{L_t^\infty W_x^{2,\infty}(\mathbb{R}^d\setminus I_v(t))})\int_0^T\int_{\mathbb{R}^d}|\chi_u-\chi_v|\min\left\{\frac{\mathrm{dist}(x,I_v(t))}{r_c},1\right\}\ \mathrm{d}x\ \mathrm{d}t\\\le{}& \frac{C}{r_c^{10}}(1+\|v\|^2_{L_t^\infty W_x^{2,\infty}(\mathbb{R}^d\setminus I_v(t))})\int_0^T E[\chi_u,u,V|\chi_v,v](t)\,\mathrm{d}t.\end{aligned}\tag{3.146}$$

For the term IV, we first add zero, then perform an integration by parts which is followed by an application of Hölder's inequality to obtain

$$
\begin{aligned}
IV &\leq C\int_0^T \bigg(\int_{\mathbb{R}^d} |\chi_u - \chi_{v,h^+_{e(t)},h^-_{e(t)}}|\,\mathrm{d}x\bigg)^{\frac{1}{2}} \bigg(\int_{\mathbb{R}^d} |(\nabla w)^T : \nabla\xi|^2\,\mathrm{d}x\bigg)^{\frac{1}{2}}\,\mathrm{d}t \\
&\quad + C\int_0^T \bigg|\int_{\mathbb{R}^d} (\chi_v - \chi_{v,h^+_{e(t)},h^-_{e(t)}})(w\cdot\nabla)(\nabla\cdot\xi)\,\mathrm{d}x\bigg|\,\mathrm{d}t \\
&\quad + C\int_0^T \bigg|\int_{\mathbb{R}^d} ((w\cdot\nabla)\xi)\cdot\mathrm{d}\nabla(\chi_v - \chi_{v,h^+_{e(t)},h^-_{e(t)}})\bigg|\,\mathrm{d}t \\
&=: (IV)_a + (IV)_b + (IV)_c.
\end{aligned}
\tag{3.147}
$$

By definition of ξ, see (3.28), recall that

$$
\nabla\xi = \frac{\zeta'\big(\frac{\mathrm{dist}^\pm(x,I_v(t))}{r_c}\big)}{r_c}\mathrm{n}_v(P_{I_v(t)}x)\otimes\mathrm{n}_v(P_{I_v(t)}x) + \zeta\Big(\frac{\mathrm{dist}^\pm(x,I_v(t))}{r_c}\Big)\nabla^2\,\mathrm{dist}^\pm(x,I_v(t)).
$$

Recalling also (3.92), (3.93) and (3.109) as well as making use of (3.74c), (3.15), (3.24), (3.74a) and finally the coercivity property (3.32) the term $(IV)_a$ from (3.147) is estimated by

$$
\begin{aligned}
(IV)_a &\leq \frac{C}{r_c}\int_0^T E[\chi_u,u,V|\chi_v,v](t) + e(t)E[\chi_u,u,V|\chi_v,v]^{\frac{1}{2}}(t)\,\mathrm{d}t \\
&\quad + \frac{C}{r_c^4}\|v\|^2_{L^\infty_t W^{2,\infty}_x(\mathbb{R}^d\setminus I_v(t))}\int_0^T\int_{I_v(t)} |h^\pm_{e(t)}|^2 + |\nabla h^\pm_{e(t)}|^2\,\mathrm{d}S\,\mathrm{d}t \\
&\leq \frac{C}{r_c^8}(1+\|v\|^2_{L^\infty_t W^{2,\infty}_x(\mathbb{R}^d\setminus I_v(t))})\int_0^T E[\chi_u,u,V|\chi_v,v](t) + e(t)E[\chi_u,u,V|\chi_v,v]^{\frac{1}{2}}(t)\,\mathrm{d}t.
\end{aligned}
\tag{3.148}
$$

Recalling from (3.74b) the definition of $\chi_{v,h^+_{e(t)},h^-_{e(t)}}$, we may estimate the term $(IV)_b$ from (3.147) by a change of variables Φ_t, (3.15), Hölder's and Young's inequality, (3.94) as well as (3.74a)

$$
\begin{aligned}
(IV)_b &\leq \frac{C}{r_c^2}\int_0^T\int_{I_v(t)} |h^\pm_{e(t)}|^2\,\mathrm{d}S\,\mathrm{d}t \\
&\quad + \frac{C}{r_c^2}\int_0^T\int_{I_v(t)} \sup_{y\in[-r_c,r_c]} |w(x+y\mathrm{n}_v(x,t))|^2\,\mathrm{d}S\,\mathrm{d}t \\
&\leq \frac{C}{r_c^{10}}\|v\|^2_{L^\infty_t W^{2,\infty}_x(\mathbb{R}^d\setminus I_v(t))}\int_0^T E[\chi_u,u,V|\chi_v,v](t)\,\mathrm{d}t.
\end{aligned}
\tag{3.149}
$$

To estimate the term $(IV)_c$ from (3.147), we again make use of the definition of $\chi_{v,h^+_{e(t)},h^-_{e(t)}}$, (3.15), Hölder's and Young's inequality, (3.94) as well as (3.74a) which yields the following bound

$$
\begin{aligned}
(IV)_c &\leq \frac{C}{r_c}\int_0^T\int_{I_v(t)} |\nabla h^\pm_{e(t)}| \sup_{y\in[-r_c,r_c]} |w(x+y\mathrm{n}_v(x,t))|\,\mathrm{d}S\,\mathrm{d}t \\
&\leq \frac{C}{r_c^9}\|v\|_{L^\infty_t W^{2,\infty}_x(\mathbb{R}^d\setminus I_v(t))}\int_0^T E[\chi_u,u,V|\chi_v,v](t)\,\mathrm{d}t.
\end{aligned}
\tag{3.150}
$$

Hence, taking together the bounds from (3.148), (3.149) and (3.150) we obtain

$$
\begin{aligned}
IV &\leq \frac{C}{r_c^{10}}(1+\|v\|^2_{L^\infty_t W^{2,\infty}_x(\mathbb{R}^d\setminus I_v(t))})\int_0^T E[\chi_u,u,V|\chi_v,v](t)\,\mathrm{d}t \\
&\quad + \frac{C}{r_c^{10}}(1+\|v\|^2_{L^\infty_t W^{2,\infty}_x(\mathbb{R}^d\setminus I_v(t))})\int_0^T e(t)E^{\frac{1}{2}}[\chi_u,u,V|\chi_v,v](t)\,\mathrm{d}t.
\end{aligned}
\tag{3.151}
$$

In order to estimate the term V, we argue as follows. In a first step, we split $\mathbb{R}^d$ into the region $I_v(t)+B_{r_c}$ near to and the region $\mathbb{R}^d\setminus(I_v(t)+B_{r_c})$ away from the interface of the strong solution. Recall then that the indicator function $\chi_u(\cdot,t)$ of the varifold solution is of bounded variation in $I_v(t)+B_{r_c}$. In particular, $E^+:=\{x\in\mathbb{R}^d\colon \chi_u>0\}\cap(I_v(t)+B_{r_c})$ is a set of finite perimeter in $I_v(t)+B_{r_c}$. Applying Theorem 3.39 in local coordinates, the sections

$$E^+_x=\{y\in(-r_c,r_c)\colon \chi_u(x+y\mathrm{n}_v(x,t))>0\}$$

are guaranteed to be one-dimensional Caccioppoli sets in $(-r_c,r_c)$, and such that all of the four properties listed in Theorem 3.39 hold true for $\mathcal{H}^{d-1}$-almost every $x\in I_v(t)$. Recall from [12, Proposition 3.52] that one-dimensional Caccioppoli sets are in fact finite unions of disjoint intervals. We then distinguish for $\mathcal{H}^{d-1}$-almost every $x\in I_v(t)$ between the cases that $\mathcal{H}^0(\partial^*E^+_x)\leq 2$ or $\mathcal{H}^0(\partial^*E^+_x)>2$. In other words, we distinguish between those sections which consist of at most one interval and those which consist of at least two intervals. It also turns out to be useful to further keep track of whether $\mathrm{n}_v\cdot\mathrm{n}_u\leq\frac{1}{2}$ or $\mathrm{n}_v\cdot\mathrm{n}_u\geq\frac{1}{2}$ holds.

We then obtain by Young's and Hölder's inequality as well as the fact that due to Definition 3.13 the vector field ξ is supported in $I_v(t)+B_{r_c}$

$$\begin{aligned}
V\leq&\int_0^T\bigg(\int_{\{x+y\mathrm{n}_v(x,t)\in\partial^*E^+\colon x\in I_v(t),\,|y|<r_c,\,\mathcal{H}^0(\partial^*E^+_x)\leq 2,\,\mathrm{n}_v(x)\cdot\mathrm{n}_u(x+y\mathrm{n}_v(x,t))\geq\frac{1}{2}\}}|(\nabla w)^T\xi|^2\,\mathrm{d}\mathcal{H}^{d-1}\bigg)^{1/2} \qquad (3.152)\\
&\qquad\times\bigg(\int_{\mathbb{R}^d}|\mathrm{n}_u-\xi|^2\,\mathrm{d}|\nabla\chi_u|\bigg)^{1/2}\,\mathrm{d}t\\
&+C\int_0^T\|\nabla w(t)\|_{L^\infty_x}\bigg(\int_{\{x+y\mathrm{n}_v(x,t)\in\partial^*E^+\colon x\in I_v(t),\,|y|<r_c,\,\mathcal{H}^0(\partial^*E^+_x)>2,\,\mathrm{n}_v(x)\cdot\mathrm{n}_u(x+y\mathrm{n}_v(x,t))\geq\frac{1}{2}\}}1\,\mathrm{d}\mathcal{H}^{d-1}\bigg)\,\mathrm{d}t\\
&+C\int_0^T\|\nabla w(t)\|_{L^\infty_x}\bigg(\int_{\{x+y\mathrm{n}_v(x,t)\in\partial^*E^+\colon x\in I_v(t),\,|y|<r_c,\,\mathrm{n}_v(x)\cdot\mathrm{n}_u(x+y\mathrm{n}_v(x,t))\leq\frac{1}{2}\}}1\,\mathrm{d}\mathcal{H}^{d-1}\bigg)\,\mathrm{d}t\\
&+C\int_0^T\|\nabla w(t)\|_{L^\infty_x}\bigg(\int_{\mathbb{R}^d\setminus(I_v(t)+B_{r_c})}1\,\mathrm{d}|\nabla\chi_u|\bigg)\,\mathrm{d}t\\
\leq&\,C\int_0^T\|\nabla w(t)\|_{L^\infty_x}E[\chi_u,u,V|\chi_v,v](t)\,\mathrm{d}t\\
&+C\int_0^T\bigg(\int_{\{x+y\mathrm{n}_v(x,t)\in\partial^*E^+\colon x\in I_v(t),\,|y|<r_c,\,\mathcal{H}^0(\partial^*E^+_x)\leq 2,\,\mathrm{n}_v(x)\cdot\mathrm{n}_u(x+y\mathrm{n}_v(x,t))\geq\frac{1}{2}\}}|(\nabla w)^T\xi|^2\,\mathrm{d}\mathcal{H}^{d-1}\bigg)^{\frac{1}{2}}\\
&\qquad\times\bigg(\int_{\mathbb{R}^d}|\mathrm{n}_u-\xi|^2\,\mathrm{d}|\nabla\chi_u|\bigg)^{1/2}\,\mathrm{d}t\\
&+C\int_0^T\|\nabla w(t)\|_{L^\infty_x}\bigg(\int_{\{x+y\mathrm{n}_v(x,t)\in\partial^*E^+\colon x\in I_v(t),\,|y|<r_c,\,\mathcal{H}^0(\partial^*E^+_x)>2,\,\mathrm{n}_v(x)\cdot\mathrm{n}_u(x+y\mathrm{n}_v(x,t))\geq\frac{1}{2}\}}1\,\mathrm{d}\mathcal{H}^{d-1}\bigg)\,\mathrm{d}t\\
=:&\,C\int_0^T\|\nabla w(t)\|_{L^\infty_x}E[\chi_u,u,V|\chi_v,v](t)\,\mathrm{d}t+V_a+V_b.
\end{aligned}$$

To estimate V_a from (3.152), we use the co-area formula for rectifiable sets (see [12, (2.72)]), (3.96), Hölder's inequality and the coercivity property (3.34) which together yield (we abbreviate in the first line $F(x,y,t):=(\nabla w)^T(x{+}y\mathrm{n}_v(x,t))\mathrm{n}_v(x,t)$)

$$V_a\leq C\int_0^T\bigg(\int_{\{x\in I_v(t)\colon \mathcal{H}^0(\partial^*E^+_x)\leq 2\}}\int_{\{y\in\partial^*E^+_x\colon \mathrm{n}_v(x)\cdot\mathrm{n}_u(x+y\mathrm{n}_v(x,t))\geq\frac{1}{2}\}}|F(x,y,t)|^2\,\mathrm{d}\mathcal{H}^0(y)\,\mathrm{d}S(x)\bigg)^{\frac{1}{2}} \qquad (3.153)$$

$$\times\bigg(\int_{\mathbb{R}^d}|\mathrm{n}_u-\xi|^2\,\mathrm{d}|\nabla\chi_u|\bigg)^{1/2}\,\mathrm{d}t$$

$$\leq C\int_0^T\bigg(\int_{I_v(t)}\sup_{y\in[-r_c,r_c]}|(\nabla w)^T(x{+}y\mathrm{n}_v(x,t))\cdot\mathrm{n}_v(x,t)|^2\,\mathrm{d}S(x)\bigg)^{\frac12}$$

$$\times\bigg(\int_{\mathbb{R}^d}|\mathrm{n}_u-\xi|^2\,\mathrm{d}|\nabla\chi_u|\bigg)^{1/2}\,\mathrm{d}t$$

$$\leq\frac{C}{r_c^9}\|v\|_{L_t^\infty W_x^{3,\infty}(\mathbb{R}^d\setminus I_v(t))}\int_0^T(1+|\log e(t)|)e(t)E[\chi_u,u,V|\chi_v,v]^{\frac12}(t)\,\mathrm{d}t.$$

It remains to bound the term V_b from (3.152). To this end, we make use of the fact that it follows from property iv) in Theorem 3.39 that every second point $y\in\partial^*E_x^+\cap(-r_c,r_c)$ has to have the property that $\mathrm{n}_v(x)\cdot\mathrm{n}_u(x{+}y\mathrm{n}_v(x,t))<0$, i.e., $1\leq 1-\mathrm{n}_v(x)\cdot\mathrm{n}_u(x{+}y\mathrm{n}_v(x,t))$. We may therefore estimate with the help of the co-area formula for rectifiable sets (see [12, (2.72)]) and the bound (3.95)

$$V_b\leq C\int_0^T\|\nabla w(t)\|_{L_x^\infty}\int_{\{x\in I_v(t)\colon\mathcal{H}^0(\partial^*E_x^+)>2\}}\int_{\{y\in\partial^*E_x^+\,:\,\mathrm{n}_v(x)\cdot\mathrm{n}_u(x+y\mathrm{n}_v(x,t))\geq\frac12\}}1\,\mathrm{d}\mathcal{H}^0(y)\,\mathrm{d}S(x)\,\mathrm{d}t \tag{3.154}$$

$$\leq C\int_0^T\|\nabla w(t)\|_{L_x^\infty}\int_{I_v(t)}\int_{\partial^*E_x^+}1-\mathrm{n}_v(x,t)\cdot\mathrm{n}_u(x{+}y\mathrm{n}_v(x,t))\,\mathrm{d}\mathcal{H}^0(y)\,\mathrm{d}S(x)\,\mathrm{d}t$$

$$\leq\frac{C}{r_c^9}|\log e(t)|\|v\|_{L_t^\infty W_x^{3,\infty}(\mathbb{R}^d\setminus I_v(t))}\int_0^T E[\chi_u,u,V|\chi_v,v](t)\,\mathrm{d}t.$$

All in all, we obtain from the assumption $E[\chi_u,u,V|\chi_v,v](t)\leq e^2(t)$ as well as (3.152), (3.153), (3.154) and (3.95)

$$V\leq\frac{C}{r_c^9}\|v\|_{L_t^\infty W_x^{3,\infty}(\mathbb{R}^d\setminus I_v(t))}\int_0^T(1+|\log e(t)|)e(t)E[\chi_u,u,V|\chi_v,v]^{\frac12}(t)\,\mathrm{d}t. \tag{3.155}$$

Hence, we deduce from the bounds (3.145), (3.146), (3.151), (3.155) as well as (3.95) the asserted estimate for the additional surface tension terms. □

3.5.5 Estimate for the viscosity terms

In contrast to the case of equal shear viscosities $\mu_+=\mu_-$, we have to deal with the problematic viscous stress term given by $(\mu(\chi_v)-\mu(\chi_u))(\nabla v+\nabla v^T)$. We now show that the choice of w indeed compensates for (most of) this term in the sense that the viscosity terms from Proposition 3.10

$$\begin{aligned}R_{visc}+A_{visc}=&-\int_0^T\int_{\mathbb{R}^d}2\big(\mu(\chi_u)-\mu(\chi_v)\big)D^{\mathrm{sym}}v:D^{\mathrm{sym}}(u-v)\,\mathrm{d}x\,\mathrm{d}t\\&+\int_0^T\int_{\mathbb{R}^d}2\big(\mu(\chi_u)-\mu(\chi_v)\big)D^{\mathrm{sym}}v:D^{\mathrm{sym}}w\,\mathrm{d}x\,\mathrm{d}t\\&-\int_0^T\int_{\mathbb{R}^d}2\mu(\chi_u)D^{\mathrm{sym}}w:D^{\mathrm{sym}}(u-v-w)\,\mathrm{d}x\,\mathrm{d}t\end{aligned} \tag{3.156}$$

may be bounded by a Gronwall-type term.

Lemma 3.30. *Let the assumptions and notation of Proposition 3.28 be in place. In particular, we assume that there exists a C^1-function $e\colon[0,T_{strong})\to[0,r_c)$ such that the relative entropy is bounded by $E[\chi_u,u,V,|\chi_v,v](t)\leq e^2(t)$.*

Then, for any $\delta > 0$ there exists a constant $C > 0$ such that the viscosity terms $R_{visc}+A_{visc}$ may be estimated by

$$\begin{aligned} R_{visc} + A_{visc} \leq \frac{C}{r_c^8}\|v\|^2_{L^\infty_t W^{2,\infty}_x(\mathbb{R}^d\setminus I_v(t))} \int_0^T E[\chi_u, u, V|\chi_v, v](t)\,\mathrm{d}t \\ + \frac{C}{r_c}\|v\|^2_{L^\infty_t W^{1,\infty}_x} \int_0^T e(t) E[\chi_u, u, V|\chi_v, v]^{\frac{1}{2}}(t)\,\mathrm{d}t \\ + \delta \int_0^T \int_{\mathbb{R}^d} |D^{\mathrm{sym}}(u-v-w)|^2\,\mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{3.157}$$

Proof. We argue pointwise for the time variable and start by adding zero

$$\begin{aligned} &R_{visc} + A_{visc} \\ &= -2\int_{\mathbb{R}^d} (\mu(\chi_u) - \mu(\chi_v)) D^{\mathrm{sym}} v : D^{\mathrm{sym}}(u-v-w)\,\mathrm{d}x \\ &\quad - 2\int_{\mathbb{R}^d} \mu(\chi_u) D^{\mathrm{sym}} w : D^{\mathrm{sym}}(u-v-w)\,\mathrm{d}x \\ &= -2\int_{\mathbb{R}^d} \Big(\mu(\chi_u) - \mu(\chi_v) - (\mu^- - \mu^+)\chi_{0\leq \mathrm{dist}^\pm(x,I_v(t))\leq h^+_{e(t)}(P_{I_v(t)}x)} \\ &\qquad - (\mu^+ - \mu^-)\chi_{-h^-_{e(t)}(P_{I_v(t)}x)\leq \mathrm{dist}^\pm(x,I_v(t))\leq 0}\Big) D^{\mathrm{sym}} v : D^{\mathrm{sym}}(u-v-w)\,\mathrm{d}x \\ &\quad - 2\int_{\mathbb{R}^d} \chi_{\mathrm{dist}^\pm(x,I_v(t))\notin[-h^-_{e(t)}(P_{I_v(t)}x),h^+_{e(t)}(P_{I_v(t)}x)]} \mu(\chi_u) D^{\mathrm{sym}} w : D^{\mathrm{sym}}(u-v-w)\,\mathrm{d}x \\ &\quad - 2\int_{\mathbb{R}^d} \chi_{0\leq \mathrm{dist}^\pm(x,I_v(t))\leq h^+_{e(t)}(P_{I_v(t)}x)} (\mu(\chi_u) - \mu^-) D^{\mathrm{sym}} w : D^{\mathrm{sym}}(u-v-w)\,\mathrm{d}x \\ &\quad - 2\int_{\mathbb{R}^d} \chi_{-h^-_{e(t)}(P_{I_v(t)}x)\leq \mathrm{dist}^\pm(x,I_v(t))\leq 0} (\mu(\chi_u) - \mu^+) D^{\mathrm{sym}} w : D^{\mathrm{sym}}(u-v-w)\,\mathrm{d}x \\ &\quad - 2\int_{\mathbb{R}^d} \chi_{0\leq \mathrm{dist}^\pm(x,I_v(t))\leq h^+_{e(t)}(P_{I_v(t)}x)} ((\mu^- - \mu^+) D^{\mathrm{sym}} v + \mu^- D^{\mathrm{sym}} w) : \nabla(u-v-w)\,\mathrm{d}x \\ &\quad - 2\int_{\mathbb{R}^d} \chi_{-h^-_{e(t)}(P_{I_v(t)}x)\leq \mathrm{dist}^\pm(x,I_v(t))\leq 0} ((\mu^+ - \mu^-) D^{\mathrm{sym}} v + \mu^+ D^{\mathrm{sym}} w) : \nabla(u-v-w)\,\mathrm{d}x \\ &=: I + II + III + IV + V + VI. \end{aligned} \tag{3.158}$$

We start by estimating the first four terms. Note that $\mu(\chi_u) - \mu^- = (\mu_+ - \mu_-)\chi_u$. Recalling the definition of $\chi_{v,h^+_{e(t)},h^-_{e(t)}}$ from (3.74b) we see that

$$\chi_{0\leq \mathrm{dist}^\pm(x,I_v(t))\leq h^+_{e(t)}(P_{I_v(t)}x)}\chi_u = \chi_{0\leq \mathrm{dist}^\pm(x,I_v(t))\leq h^+_{e(t)}(P_{I_v(t)}x)}(\chi_u - \chi_{v,h^+_{e(t)},h^-_{e(t)}}).$$

Hence, we may rewrite

$$\begin{aligned} III = -2\int_{\mathbb{R}^d} \chi_{0\leq \mathrm{dist}^\pm(x,I_v(t))\leq h^+_{e(t)}(P_{I_v(t)}x)} (\mu_+ - \mu_-)(\chi_u - \chi_{v,h^+_{e(t)},h^-_{e(t)}}) \\ \times (W\otimes \mathrm{n}_v(P_{I_v(t)}x)) : D^{\mathrm{sym}}(u-v-w)\,\mathrm{d}x \\ - 2\int_{\mathbb{R}^d} \chi_{0\leq \mathrm{dist}^\pm(x,I_v(t))\leq h^+_{e(t)}(P_{I_v(t)}x)} (\mu_+ - \mu_-) \\ \times (\nabla w - W\otimes \mathrm{n}_v(P_{I_v(t)}x)) : D^{\mathrm{sym}}(u-v-w)\,\mathrm{d}x. \end{aligned}$$

Carrying out an analogous computation for IV, using again the definition of the smoothed approximation $\chi_{v,h^+_{e(t)},h^-_{e(t)}}$ for χ_u from (3.74b) and using (3.92) as well as (3.93), we then get the bound

$$
\begin{aligned}
&I+II+III+IV \\
&\leq C\|v\|_{W^{1,\infty}}\bigg(\int_{\mathbb{R}^d}|\chi_u-\chi_{v,h^+_{e(t)},h^-_{e(t)}}|\,\mathrm{d}x\bigg)^{1/2}\bigg(\int_{\mathbb{R}^d}|D^{\mathrm{sym}}(u-v-w)|^2\,\mathrm{d}x\bigg)^{1/2} \\
&\quad+\frac{C}{r_c^2}\|v\|_{W^{2,\infty}(\mathbb{R}^d\setminus I_v(t))}\bigg(\int_{I_v(t)}|h^\pm_{e(t)}|^2+|\nabla h^\pm_{e(t)}|^2\,\mathrm{d}S\bigg)^{1/2}\bigg(\int_{\mathbb{R}^d}|D^{\mathrm{sym}}(u-v-w)|^2\,\mathrm{d}x\bigg)^{1/2}.
\end{aligned}
$$

Plugging in the estimates (3.74a) and (3.74c), we obtain by Young's inequality

$$
\begin{aligned}
I+II+III+IV \leq{}& \frac{C\delta^{-1}}{r_c^8}\|v\|^2_{W^{2,\infty}(\mathbb{R}^d\setminus I_v(t))}E[\chi_u,u,V|\chi_v,v](t) \\
&+\frac{C\delta^{-1}}{r_c}\|v\|^2_{W^{1,\infty}}e(t)E[\chi_u,u,V|\chi_v,v]^{\frac{1}{2}}(t) \\
&+C\delta^{-1}\|v\|^2_{W^{1,\infty}}E[\chi_u,u,V|\chi_v,v](t) \\
&+\delta\|D^{\mathrm{sym}}(u-v-w)\|_{L^2}
\end{aligned}
\tag{3.159}
$$

for every $\delta\in(0,1)$. To estimate the last two terms V and VI in (3.158), we may rewrite making use of the definition (3.93) of the vector field W and abbreviating $\mathrm{n}_v=\mathrm{n}_v(P_{I_v(t)}x)$, $\mathrm{dist}^\pm=\mathrm{dist}^\pm(x,I_v(t))$ as well as $h^+_{e(t)}=h^+_{e(t)}(P_{I_v(t)}x)$

$$
\begin{aligned}
&-\int_{\mathbb{R}^d}\chi_{0\leq\mathrm{dist}^\pm\leq h^+_{e(t)}}((\mu^--\mu^+)D^{\mathrm{sym}}v+\mu^-D^{\mathrm{sym}}w):\nabla(u-v-w)\,\mathrm{d}x \\
&=-\int_{\mathbb{R}^d}\chi_{0\leq\mathrm{dist}^\pm\leq h^+_{e(t)}}((\mu^--\mu^+)(\mathrm{Id}-\mathrm{n}_v\otimes\mathrm{n}_v)(D^{\mathrm{sym}}v\cdot\mathrm{n}_v)\otimes\mathrm{n}_v+\mu^-D^{\mathrm{sym}}w) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad:\nabla(u-v-w)\,\mathrm{d}x \\
&\quad-\int_{\mathbb{R}^d}\chi_{0\leq\mathrm{dist}^\pm\leq h^+_{e(t)}}(\mu^--\mu^+)D^{\mathrm{sym}}v\,(\mathrm{Id}-\mathrm{n}_v\otimes\mathrm{n}_v):\nabla(u-v-w)\,\mathrm{d}x \\
&\quad-\int_{\mathbb{R}^d}\chi_{0\leq\mathrm{dist}^\pm\leq h^+_{e(t)}}(\mu^--\mu^+)(\mathrm{n}_v\cdot D^{\mathrm{sym}}v\cdot\mathrm{n}_v)(\mathrm{n}_v\otimes\mathrm{n}_v):\nabla(u-v-w)\,\mathrm{d}x \\
&=-\int_{\mathbb{R}^d}\chi_{0\leq\mathrm{dist}^\pm\leq h^+_{e(t)}}((\mu^--\mu^+)(\mathrm{Id}-\mathrm{n}_v\otimes\mathrm{n}_v)(D^{\mathrm{sym}}v\cdot\mathrm{n}_v)\otimes\mathrm{n}_v+\mu^-D^{\mathrm{sym}}w) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad:\nabla(u-v-w)\,\mathrm{d}x \\
&\quad-\int_{\mathbb{R}^d}\chi_{0\leq\mathrm{dist}^\pm\leq h^+_{e(t)}}(\mu^--\mu^+)D^{\mathrm{sym}}v\,(\mathrm{Id}-\mathrm{n}_v\otimes\mathrm{n}_v):\nabla(u-v-w)\,\mathrm{d}x \\
&\quad+\int_{\mathbb{R}^d}\chi_{0\leq\mathrm{dist}^\pm\leq h^+_{e(t)}}(\mu^--\mu^+)(\mathrm{n}_v\cdot D^{\mathrm{sym}}v\cdot\mathrm{n}_v)(\mathrm{Id}-\mathrm{n}_v\otimes\mathrm{n}_v):\nabla(u-v-w)\,\mathrm{d}x, \\
&=\frac{1}{2}\int_{\mathbb{R}^d}\chi_{0\leq\mathrm{dist}^\pm\leq h^+_{e(t)}}((W\otimes\mathrm{n}_v-\nabla w)+(W\otimes\mathrm{n}_v-\nabla w)^T):\nabla(u-v-w)\,\mathrm{d}x \\
&\quad+(\mu^--\mu^+)\int_{\mathbb{R}^d}\chi_{0\leq\mathrm{dist}^\pm\leq h^+_{e(t)}}((\mathrm{Id}-\mathrm{n}_v\otimes\mathrm{n}_v)(D^{\mathrm{sym}}v\cdot\mathrm{n}_v)\otimes\mathrm{n}_v):\nabla(u-v-w)\,\mathrm{d}x \\
&\quad-\int_{\mathbb{R}^d}\chi_{0\leq\mathrm{dist}^\pm\leq h^+_{e(t)}}(\mu^--\mu^+)D^{\mathrm{sym}}v\,(\mathrm{Id}-\mathrm{n}_v\otimes\mathrm{n}_v):\nabla(u-v-w)\,\mathrm{d}x \\
&\quad+\int_{\mathbb{R}^d}\chi_{0\leq\mathrm{dist}^\pm\leq h^+_{e(t)}}(\mu^--\mu^+)(\mathrm{n}_v\cdot D^{\mathrm{sym}}v\cdot\mathrm{n}_v)(\mathrm{Id}-\mathrm{n}_v\otimes\mathrm{n}_v):\nabla(u-v-w)\,\mathrm{d}x,
\end{aligned}
$$

where in the penultimate step we have used the fact that $\nabla\cdot(u-v-w)=0$, and in the last step we added zero. This yields after an integration by parts

$$
\begin{aligned}
&-\int_{\mathbb{R}^d} \chi_{0\leq \mathrm{dist}^\pm\leq h^+_{e(t)}}((\mu^- - \mu^+)D^{\mathrm{sym}}v + \mu^- D^{\mathrm{sym}}w) : \nabla(u-v-w)\,\mathrm{d}x \\
&= \frac{1}{2}\int_{\mathbb{R}^d} \chi_{0\leq \mathrm{dist}^\pm\leq h^+_{e(t)}}((W\otimes \mathrm{n}_v - \nabla w) + (W\otimes \mathrm{n}_v - \nabla w)^T) : \nabla(u-v-w)\,\mathrm{d}x \\
&\quad - (\mu^- - \mu^+)\int_{\mathbb{R}^d} \chi_{0\leq \mathrm{dist}^\pm\leq h^+_{e(t)}} \nabla\cdot(\mathrm{n}_v\otimes(\mathrm{Id}-\mathrm{n}_v\otimes\mathrm{n}_v)(D^{\mathrm{sym}}v\cdot \mathrm{n}_v))\cdot(u-v-w)\,\mathrm{d}x \\
&\quad + (\mu^- - \mu^+)\int_{\mathbb{R}^d} (\mathrm{n}_v\cdot(u-v-w))(\mathrm{Id}-\mathrm{n}_v\otimes\mathrm{n}_v)(D^{\mathrm{sym}}v\cdot\mathrm{n}_v)\cdot \mathrm{d}\nabla\chi_{0\leq \mathrm{dist}^\pm\leq h^+_{e(t)}} \\
&\quad + (\mu^- - \mu^+)\int_{\mathbb{R}^d} \chi_{0\leq \mathrm{dist}^\pm\leq h^+_{e(t)}} \nabla\cdot\big((D^{\mathrm{sym}}v - (\mathrm{n}_v\cdot D^{\mathrm{sym}}v\cdot\mathrm{n}_v)\,\mathrm{Id})(\mathrm{Id}-\mathrm{n}_v\otimes\mathrm{n}_v)\big) \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \cdot(u-v-w)\,\mathrm{d}x \\
&\quad + (\mu^- - \mu^+)\int_{\mathbb{R}^d} (u-v-w) \\
&\qquad\qquad\qquad \cdot(D^{\mathrm{sym}}v - (\mathrm{n}_v\cdot D^{\mathrm{sym}}v\cdot\mathrm{n}_v)\,\mathrm{Id})(\mathrm{Id}-\mathrm{n}_v\otimes\mathrm{n}_v)\,\mathrm{d}\nabla\chi_{0\leq \mathrm{dist}^\pm\leq h^+_{e(t)}}.
\end{aligned}
$$

As a consequence of (3.92), (3.74a), (3.15) and the global Lipschitz estimate $|\nabla h^\pm_e(\cdot,t)| \leq Cr_c^{-2}$ from Proposition 3.27, we obtain

$$
\begin{aligned}
&\bigg|\int_{\mathbb{R}^d} \chi_{0\leq \mathrm{dist}^\pm(x,I_v(t))\leq h^+_{e(t)}(P_{I_v(t)}x)}((\mu^- - \mu^+)D^{\mathrm{sym}}v + \mu^- D^{\mathrm{sym}}w) : \nabla(u-v-w)\,\mathrm{d}x\bigg| \\
&\leq \frac{C}{r_c^{7/2}}\|v\|_{W^{2,\infty}(\mathbb{R}^d\setminus I_v(t))} E\big[\chi_u,u,V\big|\chi_v,v\big]^{1/2}\|\nabla(u-v-v)\|_{L^2} \\
&\quad + \frac{C}{r_c}\|v\|_{W^{2,\infty}(\mathbb{R}^d\setminus I_v(t))}\int_{\mathbb{R}^d} \chi_{0\leq \mathrm{dist}^\pm(x,I_v(t))\leq h^+_{e(t)}(P_{I_v(t)}x)}|u-v-w|\,\mathrm{d}x \\
&\quad + \frac{C}{r_c^2}\|v\|_{W^{1,\infty}}\int_{I_v(t)} \sup_{y\in(-r_c,r_c)} |u-v-w|(x+y\mathrm{n}_v(x,t))|\nabla h^+_{e(t)}(x)|\,\mathrm{d}S(x).
\end{aligned}
$$

By a change of variables Φ_t, (3.14), (3.38), (3.74a) and an application of Young's and Korn's inequality, the latter two terms may be further estimated by

$$
\begin{aligned}
&\frac{C}{r_c^2}\|v\|_{W^{2,\infty}(\mathbb{R}^d\setminus I_v(t))}\bigg(\int_{I_v(t)} \sup_{y\in(-r_c,r_c)} |u-v-w|^2(x+y\mathrm{n}_v(x,t))\,\mathrm{d}S\bigg)^{\frac{1}{2}} \\
&\qquad\qquad \times\bigg(\int_{I_v(t)} |h^+_{e(t)}|^2 + |\nabla h^+_{e(t)}|^2\,\mathrm{d}S\bigg)^{\frac{1}{2}} \\
&\leq \frac{C}{r_c^3}\|v\|_{W^{2,\infty}(\mathbb{R}^d\setminus I_v(t))} E[\chi_u,u,V|\chi_v,v]^{\frac{1}{2}}(t)\|u-v-w\|_{L^2} \\
&\quad + \frac{C}{r_c^2}\|v\|_{W^{2,\infty}(\mathbb{R}^d\setminus I_v(t))} E[\chi_u,u,V|\chi_v,v]^{\frac{1}{2}}(t)\|\nabla(u-v-w)\|_{L^2} \\
&\leq \frac{C\delta^{-1}}{r_c^4}\|v\|^2_{W^{2,\infty}(\mathbb{R}^d\setminus I_v(t))} E[\chi_u,u,V|\chi_v,v](t) + \delta\|D^{\mathrm{sym}}(u-v-w)\|_{L^2}
\end{aligned}
$$

for every $\delta\in(0,1]$. In total, we obtain the bound

$$
V \leq \frac{C\delta^{-1}}{r_c^4}\|v\|^2_{W^{2,\infty}(\mathbb{R}^d\setminus I_v(t))} E[\chi_u,u,V|\chi_v,v](t) + \delta\|D^{\mathrm{sym}}(u-v-w)\|_{L^2} \tag{3.160}
$$

where $\delta\in(0,1)$ is again arbitrary. Analogously, one can derive a bound of the same form for the last term VI in (3.158). Together with the bounds from (3.159) as well as (3.160) this concludes the proof. □

3.5.6 Estimate for terms with the time derivative of the compensation function

We proceed with the estimate for the terms from the relative entropy inequality of Proposition 3.10

$$\begin{aligned} A_{dt} := &- \int_0^T \int_{\mathbb{R}^d} \rho(\chi_u)(u - v - w) \cdot \partial_t w \,\mathrm{d}x\,\mathrm{d}t \\ &- \int_0^T \int_{\mathbb{R}^d} \rho(\chi_u)(u - v - w) \cdot (v \cdot \nabla) w \,\mathrm{d}x\,\mathrm{d}t, \end{aligned} \tag{3.161}$$

which are related to the time derivative of the compensation function w.

Lemma 3.31. *Let the assumptions and notation of Proposition 3.28 be in place. In particular, we assume that there exists a C^1-function $e\colon [0, T_{strong}) \to [0, r_c)$ such that the relative entropy is bounded by $E[\chi_u, u, V, |\chi_v, v](t) \le e^2(t)$.*

Then, for any $\delta > 0$ there exists a constant $C > 0$ such that A_{dt} may be estimated by

$$\begin{aligned} A_{dt} \le &\frac{C}{r_c^{22}} \|v\|^2_{L_t^\infty W_x^{1,\infty}} (1+\|v\|_{L_t^\infty W_x^{2,\infty}(\mathbb{R}^d \setminus I_v(t))})^2 \int_0^T (1+|\log e(t)|) E[\chi_u, u, V|\chi_v, v](t) \,\mathrm{d}t \\ &+ \frac{C}{r_c^{11}} \|v\|_{L_t^\infty W_x^{1,\infty}} (1+\|v\|_{L_t^\infty W_x^{2,\infty}(\mathbb{R}^d \setminus I_v(t))}) \int_0^T (1+|\log e(t)|) E[\chi_u, u, V|\chi_v, v](t) \,\mathrm{d}t \\ &+ \frac{C}{r_c^8} (1+\|v\|_{L_t^\infty W_x^{1,\infty}})(\|\partial_t \nabla v\|_{L_{x,t}^\infty} + (R^2+1)\|v\|_{L_t^\infty W_x^{2,\infty}(\mathbb{R}^d \setminus I_v(t))}) \\ &\qquad \times \int_0^T E[\chi_u, u, V|\chi_v, v](t) \,\mathrm{d}t \\ &+ \frac{C}{r_c^2} \|v\|^2_{L_t^\infty W_x^{1,\infty}} \int_0^T (1 + e'(t)) E[\chi_u, u, V|\chi_v, v](t) \,\mathrm{d}t \\ &+ \delta \int_0^T \int_{\mathbb{R}^d} |D^{\mathrm{sym}}(u - v - w)|^2 \,\mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{3.162}$$

Proof. To estimate the terms involving the time derivative of w we make use of the decomposition of $\partial_t w + (v \cdot \nabla) w$ from (3.97):

$$\begin{aligned} &\left| - \int_0^T \int_{\mathbb{R}^d} \rho(\chi_u)(u-v-w) \cdot \partial_t w \,\mathrm{d}x\,\mathrm{d}t - \int_0^T \int_{\mathbb{R}^d} \rho(\chi_u)(u-v-w) \cdot (v \cdot \nabla) w \,\mathrm{d}x\,\mathrm{d}t \right| \\ &\le \int_0^T \|g\|_{L^2} \|u - v - w\|_{L^2} \,\mathrm{d}t + \int_0^T \|\hat{g}\|_{L^{\frac{4}{3}}} \|u - v - w\|_{L^4} \,\mathrm{d}t. \end{aligned}$$

Employing the bounds (3.55a), (3.55b) and the assumption $E[\chi_u, u, V|\chi_v, v](t) \le e(t)^2$ together with the Orlicz-Sobolev embedding (3.224) from Proposition 3.41 or (3.227) from Lemma 3.42 depending on the dimension, we obtain

$$\left(\int_{I_v(t)} |\bar{h}^\pm|^4 \,\mathrm{d}S \right)^{\frac{1}{4}} \le \frac{C}{r_c^6} e(t) \Big(1 + \log \frac{1}{e(t)} \Big)^{\frac{1}{4}}. \tag{3.163}$$

Making use of (3.74a), the bound for the vector field $\hat{g}$ from (3.98), the Gagliardo-Nirenberg-Sobolev embedding $\|u-v-w\|_{L^4} \le C\|\nabla(u-v-w)\|^{1-\alpha}_{L^2} \|u-v-w\|^{\alpha}_{L^2}$, with $\alpha = \frac{1}{2}$ for $d = 2$

and $\alpha = \frac{1}{4}$ for $d = 3$, as well as the assumption $E[\chi_u, u, V|\chi_v, v](t) \leq e(t)^2$ we obtain

$$\begin{aligned} &\|\hat{g}\|_{L^{\frac{4}{3}}}\|u-v-w\|_{L^4} \qquad (3.164)\\ &\leq C\frac{\|v\|_{W^{1,\infty}}\|v\|_{W^{2,\infty}(\mathbb{R}^d\setminus I_v(t))}}{r_c^{11}}\Big(1+\log\frac{1}{e(t)}\Big)^{\frac{1}{4}}\\ &\qquad\qquad \times (\|\nabla(u-v-w)\|_{L^2}+\|u-v-w\|_{L^2})E[\chi_u,u,V|\chi_v,v]^{\frac{1}{2}}(t)\\ &\quad + C\frac{\|v\|_{W^{1,\infty}}}{r_c^8}\Big(1+\log\frac{1}{e(t)}\Big)^{\frac{1}{4}}(\|\nabla(u-v-w)\|_{L^2}+\|u-v-w\|_{L^2})\|u-v-w\|_{L^2}\\ &\quad + C\frac{\|v\|_{W^{1,\infty}}}{r_c^8}\Big(1+\log\frac{1}{e(t)}\Big)^{\frac{1}{4}}\|\nabla(u-v-w)\|_{L^2}^{\frac{3}{2}-\alpha}\|u-v-w\|_{L^2}^{\frac{1}{2}+\alpha}\\ &\quad + C\|v\|_{W^{1,\infty}}(1+\|v\|_{W^{1,\infty}})E[\chi_u,u,V|\chi_v,v]^{\frac{1}{2}}(t)(\|\nabla(u-v-w)\|_{L^2}+\|u-v-w\|_{L^2}). \end{aligned}$$

Now, by an application of Young's and Korn's inequality for all the terms on the right hand side of (3.164) which include an L^2-norm of the gradient of $u - v - w$ (in the case $d = 3$ we use $a^{\frac{5}{4}}b^{\frac{3}{4}} = (a(8\delta/5)^{\frac{1}{2}})^{\frac{5}{4}}(b(8\delta/5)^{-\frac{5}{6}})^{\frac{3}{4}} \leq \delta a^2 + \frac{3}{8}\big(\frac{8}{5}\big)^{-\frac{5}{3}}\delta^{-\frac{5}{3}}b^2$, which follows from Young's inequality with exponents $p = \frac{8}{5}$ and $q = \frac{8}{3}$) we obtain

$$\begin{aligned} &\|\hat{g}\|_{L^{\frac{4}{3}}}\|u-v-w\|_{L^4}\\ &\leq \frac{C}{\delta^{\frac{5}{3}}r_c^{22}}\|v\|_{W^{1,\infty}}^2(1+\|v\|_{W^{2,\infty}(\mathbb{R}^d\setminus I_v(t))})^2(1+|\log e(t)|)E[\chi_u,u,V|\chi_v,v](t) \qquad (3.165)\\ &\quad + \frac{C}{r_c^{11}}\|v\|_{W^{1,\infty}}(1+\|v\|_{W^{2,\infty}(\mathbb{R}^d\setminus I_v(t))})(1+|\log e(t)|)E[\chi_u,u,V|\chi_v,v](t)\\ &\quad + \delta\|D^{\mathrm{sym}}(u-v-w)\|_{L^2}^2, \end{aligned}$$

where $\delta \in (0,1)$ is arbitrary. This gives the desired bound for the $L^{\frac{4}{3}}$-contribution of $\partial_t w + (v\cdot\nabla)w$. Concerning the L^2-contribution, we estimate using (3.55a), (3.74a), the bound for $\|g\|_{L^2}$ from (3.99) as well as the assumption $E[\chi_u, u, V|\chi_v, v](t) \leq e(t)^2$

$$\begin{aligned} &\|g\|_{L^2}\|u-v-w\|_{L^2} \qquad (3.166)\\ &\leq C\frac{1+\|v\|_{W^{1,\infty}}}{r_c^8}(\|\partial_t\nabla v\|_{L^\infty(\mathbb{R}^d\setminus I_v(t))}+(R^2+1)\|v\|_{W^{2,\infty}(\mathbb{R}^d\setminus I_v(t))})E[\chi_u,u,V|\chi_v,v]^{\frac{1}{2}}(t)\|u-v-w\|_{L^2}\\ &\quad + C\|v\|_{W^{1,\infty}}(1+\|v\|_{W^{1,\infty}})E[\chi_u,u,V|\chi_v,v]^{\frac{1}{2}}(t)\|u-v-w\|_{L^2}\\ &\quad + \frac{C}{r_c^2}(1+e'(t))\|v\|_{W^{1,\infty}}^2E[\chi_u,u,V|\chi_v,v]^{\frac{1}{2}}(t)\|u-v-w\|_{L^2}\\ &\quad + C\frac{\|v\|_{W^{1,\infty}}(1+\|v\|_{W^{2,\infty}(\mathbb{R}^d\setminus I_v(t))})}{r_c}E[\chi_u,u,V|\chi_v,v]^{\frac{1}{2}}(t)\|u-v-w\|_{L^2}\\ &\quad + C\|v\|_{W^{1,\infty}}(\|\nabla(u-v-w)\|_{L^2}+\|u-v-w\|_{L^2})\|u-v-w\|_{L^2}. \end{aligned}$$

Hence, by another application of Young's and Korn's inequality, we may bound

$$\begin{aligned} &\|g\|_{L^2}\|u-v-w\|_{L^2} \qquad (3.167)\\ &\leq \frac{C}{r_c^8}(1+\|v\|_{W^{1,\infty}})(\|\partial_t\nabla v\|_{L^\infty(\mathbb{R}^d\setminus I_v(t))}+(R^2+1)\|v\|_{W^{2,\infty}(\mathbb{R}^d\setminus I_v(t))})E[\chi_u,u,V|\chi_v,v](t)\\ &\quad + \frac{C}{r_c^2}\|v\|_{W^{1,\infty}}^2(1+e'(t))E[\chi_u,u,V|\chi_v,v](t)\\ &\quad + C\delta^{-1}\|v\|_{W^{1,\infty}}^2E[\chi_u,u,V|\chi_v,v](t)\\ &\quad + \delta\|D^{\mathrm{sym}}(u-v-w)\|_{L^2}^2 \end{aligned}$$

where $\delta \in (0,1]$ is again arbitrary. All in all, (3.165) and (3.167) therefore imply the desired bound. □

3.5.7 Estimate for the additional advection terms

We move on with the additional advection terms from the relative entropy inequality of Proposition 3.10

$$\begin{aligned} A_{adv} = &-\int_0^T \int_{\mathbb{R}^d} \rho(\chi_u)(u-v-w)\cdot(w\cdot\nabla)(v+w)\,\mathrm{d}x\,\mathrm{d}t \\ &-\int_0^T \int_{\mathbb{R}^d} \rho(\chi_u)(u-v-w)\cdot\big((u-v-w)\cdot\nabla\big)w\,\mathrm{d}x\,\mathrm{d}t. \end{aligned} \tag{3.168}$$

A precise estimate is the content of the following result.

Lemma 3.32. *Let the assumptions and notation of Proposition 3.28 be in place. In particular, we assume that there exists a C^1-function $e\colon[0,T_{strong})\to[0,r_c)$ such that the relative entropy is bounded by $E[\chi_u,u,V,|\chi_v,v](t)\le e^2(t)$. Then the additional advection terms A_{adv} may be bounded by a Gronwall-type term*

$$A_{adv} \le \frac{C}{r_c^{14}}(1+R)\|v\|^2_{L_t^\infty W_x^{3,\infty}(\mathbb{R}^d\setminus I_v(t))}\int_0^T (1+|\log e(t)|)E[\chi_u,u,V|\chi_v,v](t)\,\mathrm{d}t. \tag{3.169}$$

Proof. A straightforward estimate yields

$$\begin{aligned} A_{adv} \le\, & C(\|v\|_{L_t^\infty W_x^{1,\infty}}+\|\nabla w\|_{L^\infty_{x,t}})\|u-v-w\|_{L^2_{x,t}}\bigg(\int_0^T\int_{\mathbb{R}^d}|w|^2\,\mathrm{d}x\,\mathrm{d}t\bigg)^{\frac{1}{2}} \\ &+C\|\nabla w\|_{L^\infty_{x,t}}\|u-v-w\|^2_{L^2_{x,t}}. \end{aligned}$$

Making use of (3.91), (3.95) as well as (3.74a) immediately shows that the desired bound holds true. □

3.5.8 Estimate for the additional weighted volume term

It finally remains to state the estimate for the additional weighted volume term from the relative entropy inequality of Proposition 3.10

$$A_{weightVol} := \int_0^T\int_{\mathbb{R}^d}(\chi_u-\chi_v)(w\cdot\nabla)\beta\Big(\frac{\mathrm{dist}^{\pm}(\cdot,I_v)}{r_c}\Big)\,\mathrm{d}x\,\mathrm{d}t. \tag{3.170}$$

Lemma 3.33. *Let the assumptions and notation of Proposition 3.28 be in place. In particular, we assume that there exists a C^1-function $e\colon[0,T_{strong})\to[0,r_c)$ such that the relative entropy is bounded by $E[\chi_u,u,V,|\chi_v,v](t)\le e^2(t)$. Then the additional weighted volume term $A_{weightVol}$ may be bounded by a Gronwall term*

$$A_{weightVol} \le \frac{C}{r_c^{10}}\big(1+\|v\|^2_{L_t^\infty W_x^{2,\infty}(\mathbb{R}^d\setminus I_v(t))}\big)\int_0^T E[\chi_u,u,V|\chi_v,v](t)\,\mathrm{d}t. \tag{3.171}$$

Proof. We may use the exact same argument as in the derivation of the estimate for the term III from the additional surface tension terms A_{surTen}, see (3.146). □

3.5.9 The weak-strong uniqueness principle with different viscosities

Before we proceed with the proof of Theorem 3.1, let us summarize the estimates from the previous sections in the form of a post-processed relative entropy inequality. The proof is a direct consequence of the relative entropy inequality from Proposition 3.10 and the bounds (3.42), (3.50), (3.51), (3.52), (3.144), (3.157), (3.162), (3.169) and (3.171).

Proposition 3.34 (Post-processed relative entropy inequality)**.** *Let $d \leq 3$. Let (χ_u, u, V) be a varifold solution to the free boundary problem for the incompressible Navier–Stokes equation for two fluids (1.1a)-(1.1c) in the sense of Definition 3.2 on some time interval $[0, T_{vari})$. Let (χ_v, v) be a strong solution to (1.1a)-(1.1c) in the sense of Definition 3.6 on some time interval $[0, T_{strong})$ with $T_{strong} \leq T_{vari}$.*

Let ξ be the extension of the inner unit normal vector field n_v of the interface $I_v(t)$ from Definition 3.13. Let w be the vector field contructed in Proposition 3.28. Let β be the truncation of the identity from Proposition 3.10, and let θ be the density $\theta_t = \frac{\mathrm{d}|\nabla\chi_u(\cdot,t)|}{\mathrm{d}|V_t|_{\mathbb{S}^{d-1}}}$. Let $e\colon [0, T_{strong}) \to (0, r_c]$ be a C^1-function and assume that the relative entropy

$$
\begin{aligned}
E\big[\chi_u, u, V\big|\chi_v, v\big](T) := \sigma \int_{\mathbb{R}^d} 1 - \xi(\cdot,T)\cdot\frac{\nabla\chi_u(\cdot,T)}{|\nabla\chi_u(\cdot,T)|}\,\mathrm{d}|\nabla\chi_u(\cdot,T)| \\
+ \int_{\mathbb{R}^d} \frac{1}{2}\rho\big(\chi_u(\cdot,T)\big)\big|u-v-w\big|^2(\cdot,T)\,\mathrm{d}x \\
+ \int_{\mathbb{R}^d} \big|\chi_u(\cdot,T)-\chi_v(\cdot,T)\big|\,\Big|\beta\Big(\frac{\mathrm{dist}^\pm(\cdot, I_v(T))}{r_c}\Big)\Big|\,\mathrm{d}x \\
+ \sigma\int_{\mathbb{R}^d} 1-\theta_T\,\mathrm{d}|V_T|_{\mathbb{S}^{d-1}}
\end{aligned}
$$

is bounded by $E[\chi_u, u, V|\chi_v, v](t) \leq e(t)^2$.

Then the relative entropy is subject to the estimate

$$
\begin{aligned}
&E[\chi_u, u, V|\chi_v, v](T) + c\int_0^T\int_{\mathbb{R}^d} |\nabla(u-v-w)|^2\,\mathrm{d}x\;\,\mathrm{d}t \qquad (3.172)\\
&\leq E[\chi_u, u, V|\chi_v, v](0)\\
&\quad + C\int_0^T (1+|\log e(t)|)\,E[\chi_u, u, V|\chi_v, v](t)\,\mathrm{d}t\\
&\quad + C\int_0^T (1+|\log e(t)|)\,e(t)\sqrt{E[\chi_u, u, V|\chi_v, v](t)}\,\mathrm{d}t\\
&\quad + C\int_0^T \Big(\frac{\mathrm{d}}{\mathrm{d}t}e(t)\Big)E[\chi_u, u, V|\chi_v, v](t)\,\mathrm{d}t
\end{aligned}
$$

for almost every $T \in [0, T_{strong})$. Here, $C > 0$ is a constant which is structurally of the form $C = \widetilde{C} r_c^{-22}$ with a constant $\tilde{C} = \tilde{C}(r_c, \|v\|_{L_t^\infty W_x^{3,\infty}}, \|\partial_t v\|_{L_t^\infty W_x^{1,\infty}})$, depending on the various norms of the velocity field of the strong solution, the regularity parameter r_c of the interface of the strong solution, and the physical parameters $\rho^\pm$, $\mu^\pm$, and σ.

We have everything in place to to prove the main result of this work.

Proof of Theorem 3.1. The proof of Theorem 3.1 is based on the post-processed relative entropy inequality of Proposition 3.34. It amounts to nothing but a more technical version of the upper bound

$$
E(t) \leq e^{e^{-Ct}\log E(0)}
$$

valid for all solutions of the differential inequality $\frac{d}{dt}E(t) \leq CE(t)|\log E(t)|$. However, it is made more technical by the more complex right-hand side (3.34) in the relative entropy inequality (which involves the anticipated upper bound $e(t)^2$) and the smallness assumption on the relative entropy $E[\chi_u, u, V|\chi_v, v](t)$ needed for the validity of the relative entropy inequality.

We start the proof with the precise choice of the function $e(t)$ as well as the necessary smallness assumptions on the initial relative entropy. We then want to exploit the post-processed form of the relative entropy inequality from Proposition 3.34 to compare $E[\chi_u, u, V|\chi_v, v](t)$ with $e(t)$.

Let $C > 0$ be the constant from Proposition 3.34 and choose $\delta > 0$ such that $\delta < \frac{1}{6(C+1)}$. Let $\varepsilon > 0$ (to be chosen in a moment, but finally we will let $\varepsilon \to 0$) and consider the strictly increasing function

$$e(t) := e^{\frac{1}{2}e^{-\frac{t}{\delta}}\log(E[\chi_u,u,V|\chi_v,v](0)+\varepsilon)}. \tag{3.173}$$

Note that $e^2(0) = E[\chi_u, u, V|\chi_v, v](0) + \varepsilon$ which strictly dominates the relative entropy at the initial time. To ensure the smallness of this function, let us choose $c > 0$ small enough such that whenever we have $E[\chi_u, u, V|\chi_v, v](0) < c$ and $\varepsilon < c$, it holds that

$$e(t) < \frac{1}{3C} \wedge r_c \tag{3.174}$$

for all $t \in [0, T_{strong})$. This is indeed possible since the condition in (3.174) is equivalent to $\frac{1}{2}\log(E[\chi_u, u, V|\chi_v, v](0) + \varepsilon) < e^{\frac{T_{strong}}{\delta}} \log(\frac{1}{3C} \wedge r_c)$. For technical reasons to be seen later, we will also require $c > 0$ be small enough such that

$$e^{-\frac{T_{strong}}{\delta}}\frac{1}{6\delta}\big|\log(E[\chi_u, u, V|\chi_v, v](0) + \varepsilon)\big| > C \tag{3.175}$$

whenever $E[\chi_u, u, V|\chi_v, v](0) < c$ and $\varepsilon < c$. We proceed with some further computations. We start with

$$\frac{\mathrm{d}}{\mathrm{d}t}e(t) = \frac{1}{2\delta}|\log(E[\chi_u, u, V|\chi_v, v](0) + \varepsilon)|e(t)e^{-\frac{t}{\delta}} = \frac{1}{\delta}|\log e(t)|e(t). \tag{3.176}$$

This in particular entails

$$\begin{aligned} e^2(T) - e^2(\tau) &= \int_\tau^T \frac{\mathrm{d}}{\mathrm{d}t}e^2(t)\,\mathrm{d}t \\ &= \frac{1}{\delta}|\log(E[\chi_u, u, V|\chi_v, v](0) + \varepsilon)|\int_\tau^T e^2(t)e^{-\frac{t}{\delta}}\,\mathrm{d}t. \end{aligned} \tag{3.177}$$

After these preliminary considerations, let us consider the relative entropy inequality from Proposition 3.10. Arguing similarly to the derivation of the relative entropy inequality in Proposition 3.10 but using the energy dissipation inequality in its weaker form

$$E[\chi_u, u, V|\chi_v, v](T) \le E[\chi_u, u, V|\chi_v, v](\tau)$$

for a. e. $\tau \in [0, T]$, we may deduce (upon modifying the solution on a subset of $[0, T_{strong})$ of vanishing measure)

$$\limsup_{T\downarrow\tau} E[\chi_u, u, V|\chi_v, v](T) \le E[\chi_u, u, V|\chi_v, v](\tau) \tag{3.178}$$

for all $\tau \in [0, T_{strong})$. Now, consider the set $\mathcal{T} \subset [0, T_{strong})$ which contains all $\tau \in [0, T_{strong})$ such that $\limsup_{T\downarrow\tau} E[\chi_u, u, V|\chi_v, v](T) > e^2(\tau)$. Arguing by contradiction, we assume $\mathcal{T} \neq \emptyset$ and define

$$T^* := \inf \mathcal{T}.$$

Since $E[\chi_u, u, V|\chi_v, v](0) < e^2(0)$ and e^2 is strictly increasing, we deduce by the same argument which established (3.178) that $T^* > 0$. Hence, we can apply Proposition 3.34 at least for times $T < T^*$ (with $\tau = 0$). However, by the same argument as before the relative entropy inequality from Proposition 3.10 shows that $E[\chi_u, u, V|\chi_v, v](T^*) \leq E[\chi_u, u, V|\chi_v, v](T) + C(T^* - T)$ for all $T < T^*$, whereas $E[\chi_u, u, V|\chi_v, v](T)$ may be bounded by means of the post-processed relative entropy inequality. Hence, we obtain using also (3.173) and (3.176)

$$\begin{aligned} E[\chi_u, u, V|\chi_v, v](T^*) \leq E[\chi_u, u, V|\chi_v, v](0) \\ + C\int_0^{T^*} e^2(t)\,\mathrm{d}t \\ + C\frac{1}{2\delta}\big|\log(E[\chi_u, u, V|\chi_v, v](0)+\varepsilon)\big|\int_0^{T^*} e^3(t)e^{-\frac{t}{\delta}}\,\mathrm{d}t \\ + C\frac{1}{2}\big|\log(E[\chi_u, u, V|\chi_v, v](0)+\varepsilon)\big|\int_0^{T^*} e^2(t)e^{-\frac{t}{\delta}}\,\mathrm{d}t. \end{aligned} \tag{3.179}$$

We compare this to the equation (3.177) for $e^2(t)$ (with $\tau = 0$ and $T = T^*$). Recall that $e^2(0)$ strictly dominates the relative entropy at the initial time. Because of (3.175), the second term on the right hand side of (3.179) is dominated by one third of the right hand side of (3.177). Because of (3.174) and the choice $\delta < \frac{1}{6(C+1)}$ the same is true for the other two terms on the right hand side of (3.179). In particular, we obtain using also (3.178)

$$\limsup_{T\downarrow T^*} E[\chi_u, u, V|\chi_v, v](T) - e^2(T^*) \leq E[\chi_u, u, V|\chi_v, v](T^*) - e^2(T^*) < 0,$$

which contradicts the definition of T^*. This concludes the proof since the asserted stability estimate as well as the weak-strong uniqueness principle is now a consequence of letting $\varepsilon \to 0$. □

3.6 Derivation of the relative entropy inequality

Proof of Proposition 3.10. We start with the following observation. Since the phase-dependent density $\rho(\chi_v)$ depends linearly on the indicator function χ_v of the volume occupied by the first fluid, it consequently satisfies

$$\begin{aligned} \int_{\mathbb{R}^d} \rho(\chi_v(\cdot,T))\varphi(\cdot,T)\,\mathrm{d}x - \int_{\mathbb{R}^d} \rho(\chi_v^0)\varphi(\cdot,0)\,\mathrm{d}x \\ = \int_0^T \int_{\mathbb{R}^d} \rho(\chi_v)(\partial_t\varphi + (v\cdot\nabla)\varphi)\,\mathrm{d}x\,\mathrm{d}t \end{aligned} \tag{3.180}$$

for almost every $T \in [0, T_{strong})$ and all $\varphi \in C^\infty_{cpt}(\mathbb{R}^d \times [0, T_{strong}))$. By approximation, the equation holds for all $\varphi \in W^{1,\infty}(\mathbb{R}^d \times [0, T_{strong}))$. Testing this equation with $v \cdot \eta$, where $\eta \in C^\infty_{cpt}(\mathbb{R}^d \times [0, T_{strong}); \mathbb{R}^d)$ is a smooth vector field, we then obtain

$$\begin{aligned} \int_{\mathbb{R}^d} \rho(\chi_v(\cdot,T))v(\cdot,T)\cdot\eta(\cdot,T)\,\mathrm{d}x - \int_{\mathbb{R}^d} \rho(\chi_v^0)v_0\cdot\eta(\cdot,0)\,\mathrm{d}x \\ = \int_0^T \int_{\mathbb{R}^d} \rho(\chi_v)(v\cdot\partial_t\eta + \eta\cdot\partial_t v)\,\mathrm{d}x\,\mathrm{d}t \\ + \int_0^T \int_{\mathbb{R}^d} \rho(\chi_v)(\eta\cdot(v\cdot\nabla)v + v\cdot(v\cdot\nabla)\eta)\,\mathrm{d}x\,\mathrm{d}t \end{aligned} \tag{3.181}$$

for almost every $T \in [0, T_{strong})$. Note that the velocity field v of a strong solution has the required regularity to justify the preceding step. Next, we subtract from (3.181) the equation for the momentum balance (3.7a) of the strong solution evaluated with a test function $\eta \in C^\infty_{cpt}(\mathbb{R}^d \times [0, T_{strong}); \mathbb{R}^d)$ such that $\nabla \cdot \eta = 0$. This shows that the velocity field v of the strong solution satisfies

$$\begin{aligned} 0 = &\int_0^T \int_{\mathbb{R}^d} \rho(\chi_v)\eta \cdot (v \cdot \nabla)v \,\mathrm{d}x \,\mathrm{d}t + \int_0^T \int_{\mathbb{R}^d} \mu(\chi_v)(\nabla v + \nabla v^T) : \nabla\eta \,\mathrm{d}x \,\mathrm{d}t \\ &+ \int_0^T \int_{\mathbb{R}^d} \rho(\chi_v)\eta \cdot \partial_t v \,\mathrm{d}x \,\mathrm{d}t - \sigma \int_0^T \int_{I_v(t)} \mathrm{H} \cdot \eta \,\mathrm{d}S \,\mathrm{d}t \end{aligned} \tag{3.182}$$

which holds for almost every $T \in [0, T_{strong})$ and all $\eta \in C^\infty_{cpt}(\mathbb{R}^d \times [0, T_{strong}); \mathbb{R}^d)$ such that $\nabla \cdot \eta = 0$. The aim is now to test the latter equation with the field $u - v - w$. To this end, we fix a radial mollifier $\phi\colon \mathbb{R}^d \to [0, \infty)$ such that ϕ is smooth, supported in the unit ball and $\int_{\mathbb{R}^d} \phi \,\mathrm{d}x = 1$. For $n \in \mathbb{N}$ we define $\phi_n(\cdot) := n^d\phi(n\,\cdot)$ as well as $u_n := \phi_n * u$ and analogously v_n and w_n. We then test (3.182) with the test function $u_n - v_n - w_n$ and let $n \to \infty$. Since the traces of u_n, v_n and w_n on $I_v(t)$ converge pointwise almost everywhere to the respective traces of u, v and w, we indeed may pass to the limit in the surface tension term of (3.182). Hence, we obtain the identity

$$\begin{aligned} &- \int_0^T \int_{\mathbb{R}^d} \mu(\chi_v)(\nabla v + \nabla v^T) : \nabla(u - v - w) \,\mathrm{d}x \,\mathrm{d}t \\ &= \int_0^T \int_{\mathbb{R}^d} \rho(\chi_v)(u - v - w) \cdot (v \cdot \nabla)v \,\mathrm{d}x \,\mathrm{d}t \\ &\quad + \int_0^T \int_{\mathbb{R}^d} \rho(\chi_v)(u - v - w) \cdot \partial_t v \,\mathrm{d}x \,\mathrm{d}t \\ &\quad - \sigma \int_0^T \int_{I_v(t)} \mathrm{H} \cdot (u - v - w) \,\mathrm{d}S \,\mathrm{d}t, \end{aligned} \tag{3.183}$$

which holds true for almost every $T \in [0, T_{strong})$.

In the next step, we test the analogue of (3.180) for the phase-dependent density $\rho(\chi_u)$ of the varifold solution with the test function $\frac{1}{2}|v + w|^2$ and obtain

$$\begin{aligned} &\int_{\mathbb{R}^d} \frac{1}{2}\rho(\chi_u(\cdot, T))|v + w|^2(\cdot, T) \,\mathrm{d}x - \int_{\mathbb{R}^d} \frac{1}{2}\rho(\chi_u^0)|v_0 + w(\cdot, 0)|^2 \,\mathrm{d}x \\ &= \int_0^T \int_{\mathbb{R}^d} \rho(\chi_u)(v + w) \cdot \partial_t(v + w) \,\mathrm{d}x \,\mathrm{d}t \\ &\quad + \int_0^T \int_{\mathbb{R}^d} \rho(\chi_u)(v + w) \cdot (u \cdot \nabla)(v + w) \,\mathrm{d}x \,\mathrm{d}t \end{aligned} \tag{3.184}$$

for almost every $T \in [0, T_{strong})$. Recall also from the definition of a varifold solution that we are equipped with the energy dissipation inequality

$$\begin{aligned} &\int_{\mathbb{R}^d} \frac{1}{2}\rho(\chi_u(\cdot, T))|u(\cdot, T)|^2 \,\mathrm{d}x + \sigma|V_T|(\mathbb{R}^d \times \mathbb{S}^{d-1}) \\ &+ \int_0^T \int_{\mathbb{R}^d} \frac{\mu(\chi_u)}{2}\left|\nabla u + \nabla u^T\right|^2 \,\mathrm{d}x \,\mathrm{d}t \\ &\leq \int_{\mathbb{R}^d} \frac{1}{2}\rho(\chi_u^0)|u_0|^2 \,\mathrm{d}x + \sigma|\nabla\chi_u^0|(\mathbb{R}^d), \end{aligned} \tag{3.185}$$

which holds for almost every $T \in [0, T_{strong})$.

Finally, we want to test the equation for the momentum balance (3.2a) of the varifold solution with the test function $v + w$. Since the normal derivative of the tangential velocity of a strong solution may feature a discontinuity at the interface, we have to proceed by an approximation argument, i.e., we use the mollified version $v_n + w_n$ as a test function. Note that v_n resp. w_n are elements of $L^\infty([0, T_{strong}); C^0(\mathbb{R}^d))$. Hence, we may indeed use $v_n + w_n$ as a test function in the surface tension term of the equation for the momentum balance (3.2a) of the varifold solution. However, it is not clear a priori why one may pass to the limit $n \to \infty$ in this term.

To argue that this is actually possible, we choose a precise representative for ∇v resp. ∇w on the interface $I_v(t)$. This is indeed necessary also for the velocity field of the strong solution since the normal derivative of the tangential component of v may feature a jump discontinuity at the interface. However, by the regularity assumptions on v, see Definition 3.6 of a strong solution, and the assumptions on the compensating vector field w, for almost every $t \in [0, T_{strong})$ every point $x \in \mathbb{R}^d$ is either a Lebesgue point of ∇v (respectively ∇w) or there exist two half spaces H_1 and H_2 passing through x such that x is a Lebesgue point for both $\nabla v|_{H_1}$ and $\nabla v|_{H_2}$ (respectively $\nabla w|_{H_1}$ and $\nabla w|_{H_2}$). In particular, by the L^∞ bounds on ∇v and ∇w the limit of the mollifications ∇v_n respectively ∇w_n exist at every point $x \in \mathbb{R}^d$ and we may define ∇v respectively ∇w at every point $x \in \mathbb{R}^d$ as this limit.

Recall then that we have chosen the mollifiers ϕ_n to be radially symmetric. Hence, the approximating sequences ∇v_n resp. ∇w_n converge pointwise everywhere to the precise representation as chosen before. Since both limits are bounded, we may pass to the limit $n \to \infty$ in every term appearing from testing the equation for the momentum balance (3.2a) of the varifold solution with the test function $v_n + w_n$. This entails

$$\begin{aligned}
&- \int_{\mathbb{R}^d} \rho(\chi_u(\cdot,T)) u(\cdot,T) \cdot (v+w)(\cdot,T) \,\mathrm{d}x + \int_{\mathbb{R}^d} \rho(\chi_u^0) u_0 \cdot (v+w)(\cdot,0) \,\mathrm{d}x \\
&- \int_0^T \int_{\mathbb{R}^d} \mu(\chi_u)(\nabla u + \nabla u^T) : \nabla(v+w) \,\mathrm{d}x \,\mathrm{d}t \\
&= - \int_0^T \int_{\mathbb{R}^d} \rho(\chi_u) u \cdot \partial_t (v+w) \,\mathrm{d}x \,\mathrm{d}t - \int_0^T \int_{\mathbb{R}^d} \rho(\chi_u) u \cdot (u \cdot \nabla)(v+w) \,\mathrm{d}x \,\mathrm{d}t \\
&\quad + \sigma \int_0^T \int_{\mathbb{R}^d \times \mathbb{S}^{d-1}} (\mathrm{Id} - s \otimes s) : \nabla(v+w) \,\mathrm{d}V_t(x,s) \,\mathrm{d}t
\end{aligned} \tag{3.186}$$

for almost every $T \in [0, T_{strong})$. The next step consists of summing (3.183), (3.184), (3.185) and (3.186). We represent this sum as follows:

$$\begin{aligned}
&LHS_{kin}(T) + LHS_{visc} + LHS_{surEn}(T) \\
&\leq RHS_{kin}(0) + RHS_{surEn}(0) + RHS_{dt} + RHS_{adv} + RHS_{surTen},
\end{aligned} \tag{3.187}$$

where each individual term is obtained in the following way. The terms related to kinetic energy at time T on the left hand side of (3.184), (3.185) and (3.186) in total yield the contribution

$$LHS_{kin}(T) = \int_{\mathbb{R}^d} \frac{1}{2} \rho(\chi_u(\cdot,T)) |u - v - w|^2 (\cdot,T) \,\mathrm{d}x. \tag{3.188}$$

The same computation may be carried out for the initial kinetic energy terms

$$RHS_{kin}(0) = \int_{\mathbb{R}^d} \frac{1}{2} \rho(\chi_u^0) |u_0 - v_0 - w(\cdot,0)|^2 \,\mathrm{d}x. \tag{3.189}$$

Note that because of (3.4) it holds

$$\sigma |V_T|(\mathbb{R}^d \times \mathbb{S}^{d-1}) = \sigma |\nabla \chi_u(\cdot,T)|(\mathbb{R}^d) + \sigma \int_{\mathbb{R}^d} 1 - \theta_T \,\mathrm{d}|V_T|_{\mathbb{S}^{d-1}}.$$

The terms in the energy dissipation inequality related to surface energy are therefore given by

$$LHS_{surEn}(T) = \sigma|\nabla\chi_u(\cdot,T)|(\mathbb{R}^d) + \sigma\int_{\mathbb{R}^d} 1-\theta_T \,\mathrm{d}|V_T|_{\mathbb{S}^{d-1}} \tag{3.190}$$

as well as

$$RHS_{surEn}(0) = \sigma|\nabla\chi_u^0|(\mathbb{R}^d). \tag{3.191}$$

Moreover, collecting all advection terms on the right hand side of (3.183), (3.184), and (3.186) as well as adding zero gives the contribution

$$\begin{aligned}
RHS_{adv} &= -\int_0^T\int_{\mathbb{R}^d} \rho(\chi_u)(u-v-w)\cdot(v\cdot\nabla)w \,\mathrm{d}x\,\mathrm{d}t \\
&\quad -\int_0^T\int_{\mathbb{R}^d} \big(\rho(\chi_u)-\rho(\chi_v)\big)(u-v-w)\cdot(v\cdot\nabla)v \,\mathrm{d}x\,\mathrm{d}t \\
&\quad -\int_0^T\int_{\mathbb{R}^d} \rho(\chi_u)(u-v-w)\cdot\big((u-v)\cdot\nabla\big)(v+w) \,\mathrm{d}x\,\mathrm{d}t \\
&= -\int_0^T\int_{\mathbb{R}^d} \rho(\chi_u)(u-v-w)\cdot(v\cdot\nabla)w \,\mathrm{d}x\,\mathrm{d}t \\
&\quad -\int_0^T\int_{\mathbb{R}^d} \big(\rho(\chi_u)-\rho(\chi_v)\big)(u-v-w)\cdot(v\cdot\nabla)v \,\mathrm{d}x\,\mathrm{d}t \\
&\quad -\int_0^T\int_{\mathbb{R}^d} \rho(\chi_u)(u-v-w)\cdot\big((u-v-w)\cdot\nabla\big)v \,\mathrm{d}x\,\mathrm{d}t \\
&\quad -\int_0^T\int_{\mathbb{R}^d} \rho(\chi_u)(u-v-w)\cdot(w\cdot\nabla)(v+w) \,\mathrm{d}x\,\mathrm{d}t \\
&\quad -\int_0^T\int_{\mathbb{R}^d} \rho(\chi_u)(u-v-w)\cdot\big((u-v-w)\cdot\nabla\big)w \,\mathrm{d}x\,\mathrm{d}t.
\end{aligned} \tag{3.192}$$

Next, we may rewrite those terms on the right hand side of (3.183), (3.184), and (3.186) which contain a time derivative as follows

$$\begin{aligned}
RHS_{dt} &= -\int_0^T\int_{\mathbb{R}^d} \big(\rho(\chi_u)-\rho(\chi_v)\big)(u-v-w)\cdot\partial_t v \,\mathrm{d}x\,\mathrm{d}t \\
&\quad -\int_0^T\int_{\mathbb{R}^d} \rho(\chi_u)(u-v-w)\cdot\partial_t w \,\mathrm{d}x\,\mathrm{d}t.
\end{aligned} \tag{3.193}$$

Furthermore, the terms related to surface tension on the right hand side of (3.183) and (3.186) are given by

$$\begin{aligned}
RHS_{surTen} &= \sigma\int_0^T\int_{\mathbb{R}^d\times\mathbb{S}^{d-1}} (\mathrm{Id}-s\otimes s):\nabla v \,\mathrm{d}V_t(x,s)\,\mathrm{d}t - \sigma\int_0^T\int_{I_v(t)} \mathrm{H}\cdot(u-v)\,\mathrm{d}S\,\mathrm{d}t \\
&\quad + \sigma\int_0^T\int_{\mathbb{R}^d\times\mathbb{S}^{d-1}} (\mathrm{Id}-s\otimes s):\nabla w \,\mathrm{d}V_t(x,s)\,\mathrm{d}t + \sigma\int_0^T\int_{I_v(t)} \mathrm{H}\cdot w\,\mathrm{d}S\,\mathrm{d}t.
\end{aligned} \tag{3.194}$$

We proceed by rewriting the surface tension terms. For the sake of brevity, let us abbreviate from now on $\mathrm{n}_u = \frac{\nabla\chi_u}{|\nabla\chi_u|}$. Using the incompressibility of v and adding zero, we start by rewriting

$$
\begin{aligned}
&\sigma\int_0^T\int_{\mathbb{R}^d\times\mathbb{S}^{d-1}}(\mathrm{Id}{-}s\otimes s):\nabla v\,\mathrm{d}V_t(x,s)\;\mathrm{d}t\\
&=\sigma\int_0^T\int_{\mathbb{R}^d}\mathrm{n}_u\cdot(\mathrm{n}_u\cdot\nabla)v\,\mathrm{d}|\nabla\chi_u|\;\mathrm{d}t-\sigma\int_0^T\int_{\mathbb{R}^d\times\mathbb{S}^{d-1}}s\cdot(s\cdot\nabla)v\,\mathrm{d}V_t(x,s)\;\mathrm{d}t\\
&\quad-\sigma\int_0^T\int_{\mathbb{R}^d}\mathrm{n}_u\cdot(\mathrm{n}_u\cdot\nabla)v\,\mathrm{d}|\nabla\chi_u|\;\mathrm{d}t.
\end{aligned}
$$

Next, by means of the compatibility condition (3.2e) we can write

$$
\begin{aligned}
&\sigma\int_0^T\int_{I_v(t)}\mathrm{n}_u\cdot(\mathrm{n}_u\cdot\nabla)v\,\mathrm{d}S\;\mathrm{d}t-\sigma\int_0^T\int_{\mathbb{R}^d\times\mathbb{S}^{d-1}}s\cdot(s\cdot\nabla)v\,\mathrm{d}V_t(x,s)\;\mathrm{d}t\\
&=-\sigma\int_0^T\int_{\mathbb{R}^d\times\mathbb{S}^{d-1}}(s{-}\xi)\cdot\big((s{-}\xi)\cdot\nabla\big)v\,\mathrm{d}V_t(x,s)\;\mathrm{d}t\\
&\quad-\sigma\int_0^T\int_{\mathbb{R}^d\times\mathbb{S}^{d-1}}\xi\cdot\big((s{-}\xi)\cdot\nabla\big)v\,\mathrm{d}V_t(x,s)\;\mathrm{d}t\\
&\quad+\sigma\int_0^T\int_{\mathbb{R}^d}\mathrm{n}_u\cdot\big((\mathrm{n}_u-\xi)\cdot\nabla\big)v\,\mathrm{d}|\nabla\chi_u|\,\mathrm{d}t.
\end{aligned}
$$

Moreover, the compatibility condition (3.2e) also ensures that

$$
-\sigma\int_0^T\int_{\mathbb{R}^d\times\mathbb{S}^{d-1}}\xi\cdot(s\cdot\nabla)v\,\mathrm{d}V_t(x,s)\,\mathrm{d}t=-\sigma\int_0^T\int_{\mathbb{R}^d}\xi\cdot(\mathrm{n}_u\cdot\nabla)v\,\mathrm{d}|\nabla\chi_u|\,\mathrm{d}t,
$$

whereas it follows from (3.4)

$$
\begin{aligned}
&\sigma\int_0^T\int_{\mathbb{R}^d\times\mathbb{S}^{d-1}}\xi\cdot(\xi\cdot\nabla)v\,\mathrm{d}V_t(x,s)\,\mathrm{d}t\\
&=\sigma\int_0^T\int_{\mathbb{R}^d}(1-\theta_t)\,\xi\cdot(\xi\cdot\nabla)v\,\mathrm{d}|V_t|_{\mathbb{S}^{d-1}}(x)\;\mathrm{d}t+\sigma\int_0^T\int_{\mathbb{R}^d}\xi\cdot(\xi\cdot\nabla)v\,\mathrm{d}|\nabla\chi_u|\,\mathrm{d}t.
\end{aligned}
$$

Using that the divergence of ξ equals the divergence of $\mathrm{n}_v(P_{I_v(t)}x)$ *on* the interface of the strong solution (i. e. $\mathrm{H}=-(\nabla\cdot\xi)\mathrm{n}_v$; see Definition 3.13, i.e., the cutoff function does not contribute to the divergence *on* the interface), that the latter quantity equals the scalar mean curvature (recall that $\mathrm{n}_v=\frac{\nabla\chi_v}{|\nabla\chi_v|}$ points inward) as well as once more the incompressibility of the velocity fields v resp. u we may also rewrite

$$
-\sigma\int_0^T\int_{I_v(t)}\mathrm{H}\cdot(u-v)\,\mathrm{d}S\;\mathrm{d}t=-\sigma\int_0^T\int_{\mathbb{R}^d}\chi_v\big((u-v)\cdot\nabla\big)(\nabla\cdot\xi)\;\mathrm{d}x\;\mathrm{d}t.
$$

The preceding five identities together then imply that

$$
\begin{aligned}
&\sigma\int_0^T\int_{\mathbb{R}^d\times\mathbb{S}^{d-1}}(\mathrm{Id}{-}s\otimes s):\nabla v\,\mathrm{d}V_t(x,s)\;\mathrm{d}t-\sigma\int_0^T\int_{I_v(t)}\mathrm{H}\cdot(u{-}v)\,\mathrm{d}S\;\mathrm{d}t\\
&=-\sigma\int_0^T\int_{\mathbb{R}^d}\mathrm{n}_u\cdot(\mathrm{n}_u\cdot\nabla)v\,\mathrm{d}|\nabla\chi_u|\;\mathrm{d}t\qquad(3.195)\\
&\quad-\sigma\int_0^T\int_{\mathbb{R}^d}\chi_v\big((u-v)\cdot\nabla\big)(\nabla\cdot\xi)\;\mathrm{d}x\;\mathrm{d}t\\
&\quad-\sigma\int_0^T\int_{\mathbb{R}^d\times\mathbb{S}^{d-1}}(s{-}\xi)\cdot\big((s{-}\xi)\cdot\nabla\big)v\,\mathrm{d}V_t(x,s)\;\mathrm{d}t
\end{aligned}
$$

$$+\sigma\int_0^T\int_{\mathbb{R}^d}(1-\theta_t)\,\xi\cdot(\xi\cdot\nabla)v\,\mathrm{d}|V_t|_{\mathbb{S}^{d-1}}(x)\,\mathrm{d}t$$
$$+\sigma\int_0^T\int_{\mathbb{R}^d}(\mathrm{n}_u-\xi)\cdot\big((\mathrm{n}_u-\xi)\cdot\nabla\big)v\,\mathrm{d}|\nabla\chi_u|\,\mathrm{d}t.$$

Following the computation which led to (3.195) we also obtain the identity

$$\begin{aligned}
&\sigma\int_0^T\int_{\mathbb{R}^d\times\mathbb{S}^{d-1}}(\mathrm{Id}-s\otimes s):\nabla w\,\mathrm{d}V_t(x,s)\,\mathrm{d}t\\
&=-\sigma\int_0^T\int_{\mathbb{R}^d}\mathrm{n}_u\cdot(\mathrm{n}_u\cdot\nabla)w\,\mathrm{d}|\nabla\chi_u|\,\mathrm{d}t\\
&\quad-\sigma\int_0^T\int_{\mathbb{R}^d\times\mathbb{S}^{d-1}}(s-\xi)\cdot\big((s-\xi)\cdot\nabla\big)w\,\mathrm{d}V_t(x,s)\,\mathrm{d}t\\
&\quad+\sigma\int_0^T\int_{\mathbb{R}^d}(1-\theta_t)\,\xi\cdot(\xi\cdot\nabla)w\,\mathrm{d}|V_t|_{\mathbb{S}^{d-1}}(x)\,\mathrm{d}t\\
&\quad+\sigma\int_0^T\int_{\mathbb{R}^d}(\mathrm{n}_u-\xi)\cdot\big((\mathrm{n}_u-\xi)\cdot\nabla\big)w\,\mathrm{d}|\nabla\chi_u|\,\mathrm{d}t.
\end{aligned}$$

Using the fact that w is divergence-free, we may also rewrite

$$\begin{aligned}
&-\sigma\int_0^T\int_{\mathbb{R}^d}\mathrm{n}_u\cdot(\mathrm{n}_u\cdot\nabla)w\,\mathrm{d}|\nabla\chi_u|\,\mathrm{d}t\\
&=-\sigma\int_0^T\int_{\mathbb{R}^d}\mathrm{n}_u\cdot\big((\mathrm{n}_u-\xi)\cdot\nabla\big)w\,\mathrm{d}|\nabla\chi_u|\,\mathrm{d}t+\sigma\int_0^T\int_{\mathbb{R}^d}\chi_u\nabla\cdot\big((\xi\cdot\nabla)w\big)\,\mathrm{d}x\,\mathrm{d}t\\
&=-\sigma\int_0^T\int_{\mathbb{R}^d}\xi\cdot\big((\mathrm{n}_u-\xi)\cdot\nabla\big)w\,\mathrm{d}|\nabla\chi_u|\,\mathrm{d}t\\
&\quad-\sigma\int_0^T\int_{\mathbb{R}^d}(\mathrm{n}_u-\xi)\cdot\big((\mathrm{n}_u-\xi)\cdot\nabla\big)w\,\mathrm{d}|\nabla\chi_u|\,\mathrm{d}t\\
&\quad+\sigma\int_0^T\int_{\mathbb{R}^d}\chi_u\nabla w:\nabla\xi^T\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

Appealing once more to the fact that $\xi=\mathrm{n}_v$ on the interface I_v of the strong solution (see Definition 3.13) and $\nabla\cdot w=0$, we obtain

$$\begin{aligned}
&\sigma\int_0^T\int_{I_v(t)}\mathrm{H}\cdot w\,\mathrm{d}S\,\mathrm{d}t\\
&=-\sigma\int_0^T\int_{\mathbb{R}^d}(\mathrm{Id}-\mathrm{n}_v\otimes\mathrm{n}_v):\nabla w\,\mathrm{d}S\,\mathrm{d}t=\sigma\int_0^T\int_{\mathbb{R}^d}\mathrm{n}_v\cdot(\xi\cdot\nabla)w\,\mathrm{d}S\,\mathrm{d}t\\
&=-\sigma\int_0^T\int_{\mathbb{R}^d}\chi_v\nabla\cdot\big((\xi\cdot\nabla)w\big)\,\mathrm{d}x\,\mathrm{d}t=-\sigma\int_0^T\int_{\mathbb{R}^d}\chi_v\nabla w:\nabla\xi^T\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

The last three identities together with (3.195) and (3.194) in total finally yield the following representation of the surface tension terms on the right hand side of (3.183) and (3.186)

$$\begin{aligned}
RHS_{surTen}&=-\sigma\int_0^T\int_{\mathbb{R}^d}\mathrm{n}_u\cdot(\mathrm{n}_u\cdot\nabla)v\,\mathrm{d}|\nabla\chi_u|\,\mathrm{d}t\qquad(3.196)\\
&\quad+\sigma\int_0^T\int_{\mathbb{R}^d}(\mathrm{n}_u-\xi)\cdot\big((\mathrm{n}_u-\xi)\cdot\nabla\big)v\,\mathrm{d}|\nabla\chi_u|\,\mathrm{d}t\\
&\quad-\sigma\int_0^T\int_{\mathbb{R}^d}\chi_v\big((u-v)\cdot\nabla\big)(\nabla\cdot\xi)\,\mathrm{d}x\,\mathrm{d}t
\end{aligned}$$

$$
\begin{aligned}
&- \sigma \int_0^T \int_{\mathbb{R}^d\times\mathbb{S}^{d-1}} (s-\xi)\cdot\big((s-\xi)\cdot\nabla\big) v \,\mathrm{d}V_t(x,s)\,\mathrm{d}t \\
&+ \sigma \int_0^T \int_{\mathbb{R}^d} (1-\theta_t)\,\xi\cdot(\xi\cdot\nabla)v \,\mathrm{d}|V_t|_{\mathbb{S}^{d-1}}(x)\,\mathrm{d}t \\
&- \sigma \int_0^T \int_{\mathbb{R}^d\times\mathbb{S}^{d-1}} (s-\xi)\cdot\big((s-\xi)\cdot\nabla\big) w \,\mathrm{d}V_t(x,s)\,\mathrm{d}t \\
&+ \sigma \int_0^T \int_{\mathbb{R}^d} (1-\theta_t)\,\xi\cdot(\xi\cdot\nabla)w \,\mathrm{d}|V_t|_{\mathbb{S}^{d-1}}(x)\,\mathrm{d}t \\
&- \sigma \int_0^T \int_{\mathbb{R}^d} \xi\cdot\big((\mathrm{n}_u-\xi)\cdot\nabla\big) w \,\mathrm{d}|\nabla\chi_u|\,\mathrm{d}t \\
&+ \sigma \int_0^T \int_{\mathbb{R}^d} (\chi_u-\chi_v)\nabla w : \nabla\xi^T \,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}
$$

It remains to collect the viscosity terms from the left hand side of (3.183), (3.185) and (3.186). Adding also zero, we obtain

$$
\begin{aligned}
LHS_{visc} &= \int_0^T \int_{\mathbb{R}^d} 2\big(\mu(\chi_u)-\mu(\chi_v)\big) D^{\mathrm{sym}}v : D^{\mathrm{sym}}(u-v-w)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad - \int_0^T \int_{\mathbb{R}^d} 2\mu(\chi_u) D^{\mathrm{sym}}v : D^{\mathrm{sym}}(u-v-w)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \int_0^T \int_{\mathbb{R}^d} 2\mu(\chi_u) D^{\mathrm{sym}}u : D^{\mathrm{sym}}u\,\mathrm{d}x\,\mathrm{d}t \\
&\quad - \int_0^T \int_{\mathbb{R}^d} 2\mu(\chi_u) D^{\mathrm{sym}}u : D^{\mathrm{sym}}(v+w)\,\mathrm{d}x\,\mathrm{d}t \\
&= \int_0^T \int_{\mathbb{R}^d} 2\mu(\chi_u) |D^{\mathrm{sym}}(u-v-w)|^2\,\mathrm{d}x\,\mathrm{d}t \qquad (3.197)\\
&\quad + \int_0^T \int_{\mathbb{R}^d} 2\big(\mu(\chi_u)-\mu(\chi_v)\big) D^{\mathrm{sym}}v : D^{\mathrm{sym}}(u-v-w)\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \int_0^T \int_{\mathbb{R}^d} 2\mu(\chi_u) D^{\mathrm{sym}}w : D^{\mathrm{sym}}(u-v-w)\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}
$$

In particular, as an intermediate summary we obtain the following bound making already use of the notation of Proposition 3.10: Taking the bound (3.187) together with the identities from (3.188) to (3.193) as well as (3.196) and (3.197) yields

$$
\begin{aligned}
&\int_{\mathbb{R}^d} \frac{1}{2}\rho(\chi_u(\cdot,T))|u-v-w|^2(\cdot,T)\,\mathrm{d}x + \int_0^T \int_{\mathbb{R}^d} 2\mu(\chi_u)|D^{\mathrm{sym}}(u-v-w)|^2\,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \sigma|\nabla\chi_u(\cdot,T)|(\mathbb{R}^d) + \sigma\int_{\mathbb{R}^d} 1-\theta_T\,\mathrm{d}|V_T|_{\mathbb{S}^{d-1}} \\
&\quad \le \int_{\mathbb{R}^d} \frac{1}{2}\rho(\chi_u^0)|u_0-v_0-w(\cdot,0)|^2\,\mathrm{d}x + \sigma|\nabla\chi_u^0|(\mathbb{R}^d) \qquad (3.198)\\
&\quad + R_{dt} + R_{visc} + R_{adv} + A_{visc} + A_{dt} + A_{adv} + A_{surTen} \\
&\quad - \sigma\int_0^T \int_{\mathbb{R}^d} \mathrm{n}_u\cdot(\mathrm{n}_u\cdot\nabla)v\,\mathrm{d}|\nabla\chi_u|\,\mathrm{d}t \\
&\quad + \sigma\int_0^T \int_{\mathbb{R}^d} (\mathrm{n}_u-\xi)\cdot\big((\mathrm{n}_u-\xi)\cdot\nabla\big)v\,\mathrm{d}|\nabla\chi_u|\,\mathrm{d}t \\
&\quad - \sigma\int_0^T \int_{\mathbb{R}^d} \chi_v\big((u-v-w)\cdot\nabla\big)(\nabla\cdot\xi)\,\mathrm{d}x\,\mathrm{d}t
\end{aligned}
$$

$$
\begin{aligned}
&- \sigma \int_0^T \int_{\mathbb{R}^d} \chi_u \big(w \cdot \nabla\big)(\nabla \cdot \xi) \,\mathrm{d}x \,\mathrm{d}t \\
&- \sigma \int_0^T \int_{\mathbb{R}^d \times \mathbb{S}^{d-1}} (s-\xi) \cdot \big((s-\xi) \cdot \nabla\big) v \,\mathrm{d}V_t(x,s) \,\mathrm{d}t \\
&+ \sigma \int_0^T \int_{\mathbb{R}^d} (1-\theta_t)\, \xi \cdot (\xi \cdot \nabla) v \,\mathrm{d}|V_t|_{\mathbb{S}^{d-1}}(x) \,\mathrm{d}t.
\end{aligned}
$$

The aim of the next step is to use $\sigma(\nabla \cdot \xi)$ (see Definition 3.13) as a test function in the transport equation (3.2b) for the indicator function χ_u of the varifold solution. For the sake of brevity, we will write again $\mathrm{n}_u = \frac{\nabla \chi_u}{|\nabla \chi_u|}$. Plugging in $\sigma(\nabla \cdot \xi)$ and integrating by parts yields

$$
\begin{aligned}
-\sigma &\int_{\mathbb{R}^d} \mathrm{n}_u(\cdot,T) \cdot \xi(\cdot,T) \,\mathrm{d}|\nabla \chi_u(\cdot,T)| + \sigma \int_{\mathbb{R}^d} \mathrm{n}_u^0 \cdot \xi(\cdot,0) \,\mathrm{d}|\nabla \chi_u^0| \\
&= -\sigma \int_0^T \int_{\mathbb{R}^d} \mathrm{n}_u \cdot \partial_t \xi \,\mathrm{d}|\nabla \chi_u| \,\mathrm{d}t + \sigma \int_0^T \int_{\mathbb{R}^d} \chi_u (u \cdot \nabla)(\nabla \cdot \xi\,) \,\mathrm{d}x \,\mathrm{d}t
\end{aligned}
$$

for almost every $T \in [0, T_{strong})$. Making use of the evolution equation (3.29) for ξ and the fact that ξ is supported in the space-time domain $\{\mathrm{dist}(x, I_v(t)) < r_c\}$, we get by adding zero

$$
\begin{aligned}
&- \sigma \int_{\mathbb{R}^d} \mathrm{n}_u(\cdot,T) \cdot \xi(\cdot,T) \,\mathrm{d}|\nabla \chi_u(\cdot,T)| + \sigma \int_{\mathbb{R}^d} \mathrm{n}_u^0 \cdot \xi(\cdot,0) \,\mathrm{d}|\nabla \chi_u^0| \qquad (3.199)\\
&= \sigma \int_0^T \int_{\mathbb{R}^d} \mathrm{n}_u \cdot \Big(\big(\mathrm{Id} - \mathrm{n}_v(P_{I_v(t)}x) \otimes \mathrm{n}_v(P_{I_v(t)}x)\big)(\nabla v)^T \xi\Big) \,\mathrm{d}|\nabla \chi_u| \,\mathrm{d}t \\
&\quad + \sigma \int_0^T \int_{\mathbb{R}^d} \mathrm{n}_u \cdot (v \cdot \nabla) \xi \,\mathrm{d}|\nabla \chi_u| \,\mathrm{d}t \\
&\quad + \sigma \int_0^T \int_{\mathbb{R}^d} \chi_u (u \cdot \nabla)(\nabla \cdot \xi\,) \,\mathrm{d}x \,\mathrm{d}t \\
&\quad + \sigma \int_0^T \int_{\mathbb{R}^d} \mathrm{n}_u \cdot \Big(\big(\mathrm{Id} - \mathrm{n}_v(P_{I_v(t)}x) \otimes \mathrm{n}_v(P_{I_v(t)}x)\big)(\nabla \bar{V}_{\mathrm{n}} - \nabla v)^T \xi\Big) \,\mathrm{d}|\nabla \chi_u| \,\mathrm{d}t \\
&\quad + \sigma \int_0^T \int_{\mathbb{R}^d} \mathrm{n}_u \cdot \big((\bar{V}_{\mathrm{n}} - v) \cdot \nabla\big) \xi \,\mathrm{d}|\nabla \chi_u| \,\mathrm{d}t
\end{aligned}
$$

which holds for almost every $T \in [0, T_{strong})$. Next, we study the quantity

$$
\begin{aligned}
RHS_{tilt} := \sigma &\int_0^T \int_{\mathbb{R}^d} \mathrm{n}_u \cdot (v \cdot \nabla) \xi \,\mathrm{d}|\nabla \chi_u| \,\mathrm{d}t \\
&+ \sigma \int_0^T \int_{\mathbb{R}^d} \chi_u (u \cdot \nabla)(\nabla \cdot \xi\,) \,\mathrm{d}x \,\mathrm{d}t \qquad (3.200)\\
&+ \sigma \int_0^T \int_{\mathbb{R}^d} \mathrm{n}_u \cdot \Big(\big(\mathrm{Id} - \mathrm{n}_v(P_{I_v(t)}x) \otimes \mathrm{n}_v(P_{I_v(t)}x)\big)(\nabla v)^T \cdot \xi\Big) \,\mathrm{d}|\nabla \chi_u| \,\mathrm{d}t.
\end{aligned}
$$

Due to the regularity of v resp. ξ as well as the incompressibility of the velocity field v we get

$$
\begin{aligned}
\sigma \int_0^T \int_{\mathbb{R}^d} \mathrm{n}_u \cdot (v \cdot \nabla) \xi \,\mathrm{d}|\nabla \chi_u| \,\mathrm{d}t &= -\sigma \int_0^T \int_{\mathbb{R}^d} \chi_u \nabla \cdot (v \cdot \nabla) \xi \,\mathrm{d}x \,\mathrm{d}t \\
&= -\sigma \int_0^T \int_{\mathbb{R}^d} \chi_u \nabla^2 : v \otimes \xi \,\mathrm{d}x \,\mathrm{d}t \\
&= -\sigma \int_0^T \int_{\mathbb{R}^d} \chi_u \nabla \cdot \big((\xi \cdot \nabla) v\big) \,\mathrm{d}x \,\mathrm{d}t
\end{aligned}
$$

$$
\begin{aligned}
&- \sigma \int_0^T \int_{\mathbb{R}^d} \chi_u \nabla \cdot \big(v(\nabla \cdot \xi)\big) \,\mathrm{d}x \,\mathrm{d}t \\
&= -\sigma \int_0^T \int_{\mathbb{R}^d} \chi_u (v \cdot \nabla)(\nabla \cdot \xi\,) \,\mathrm{d}x \,\mathrm{d}t \\
&\quad + \sigma \int_0^T \int_{\mathbb{R}^d} \mathrm{n}_u \cdot (\xi \cdot \nabla) v \,\mathrm{d}|\nabla \chi_u| \,\mathrm{d}t.
\end{aligned} \tag{3.201}
$$

Exploiting the fact that $\xi(x) = \mathrm{n}_v(P_{I_v(t)}x)\zeta(x)$ and $\mathrm{n}_v(P_{I_v(t)}x)$ only differ by a scalar prefactor, namely the cut-off multiplier $\zeta(x)$ which one can shift around, it turns out to be helpful to rewrite

$$
\begin{aligned}
&\sigma \int_0^T \int_{\mathbb{R}^d} \mathrm{n}_u \cdot \Big(\big(\mathrm{Id} - \mathrm{n}_v(P_{I_v(t)}x) \otimes \mathrm{n}_v(P_{I_v(t)}x)\big)(\nabla v)^T \cdot \xi\Big) \,\mathrm{d}|\nabla \chi_u| \,\mathrm{d}t \\
&= \sigma \int_0^T \int_{\mathbb{R}^d} \xi \cdot \big((\mathrm{Id} - \mathrm{n}_v(P_{I_v(t)}x) \otimes \mathrm{n}_v(P_{I_v(t)}x))\mathrm{n}_u \cdot \nabla\big) v \,\mathrm{d}|\nabla \chi_u| \,\mathrm{d}t \\
&= \sigma \int_0^T \int_{\mathbb{R}^d} \xi \cdot \Big(\big(\mathrm{n}_u - (\mathrm{n}_v(P_{I_v(t)}x) \cdot \mathrm{n}_u)\mathrm{n}_v(P_{I_v(t)}x)\big) \cdot \nabla\Big) v \,\mathrm{d}|\nabla \chi_u| \,\mathrm{d}t \\
&= \sigma \int_0^T \int_{\mathbb{R}^d} \xi \cdot \big((\mathrm{n}_u - \xi) \cdot \nabla\big) v \,\mathrm{d}|\nabla \chi_u| \,\mathrm{d}t \\
&\quad - \sigma \int_0^T \int_{\mathbb{R}^d} (\xi \cdot \mathrm{n}_u)\, \mathrm{n}_v(P_{I_v(t)}x) \cdot \big(\mathrm{n}_v(P_{I_v(t)}x) \cdot \nabla\big) v \,\mathrm{d}|\nabla \chi_u| \,\mathrm{d}t \\
&\quad + \sigma \int_0^T \int_{\mathbb{R}^d} \xi \cdot (\xi \cdot \nabla) v \,\mathrm{d}|\nabla \chi_u| \,\mathrm{d}t.
\end{aligned} \tag{3.202}
$$

Hence, by using (3.201) and (3.202) we obtain

$$
\begin{aligned}
RHS_{tilt} &= \sigma \int_0^T \int_{\mathbb{R}^d} \mathrm{n}_u \cdot (\mathrm{n}_u \cdot \nabla) v \,\mathrm{d}|\nabla \chi_u| \,\mathrm{d}t \\
&\quad - \sigma \int_0^T \int_{\mathbb{R}^d} (\mathrm{n}_u - \xi) \cdot \big((\mathrm{n}_u - \xi) \cdot \nabla\big) v \,\mathrm{d}|\nabla \chi_u| \,\mathrm{d}t \\
&\quad + \sigma \int_0^T \int_{\mathbb{R}^d} \chi_u \big((u - v) \cdot \nabla\big)(\nabla \cdot \xi) \,\mathrm{d}x \,\mathrm{d}t \\
&\quad - \sigma \int_0^T \int_{\mathbb{R}^d} (\xi \cdot \mathrm{n}_u)\, \mathrm{n}_v(P_{I_v(t)}x) \cdot \big(\mathrm{n}_v(P_{I_v(t)}x) \cdot \nabla\big) v \,\mathrm{d}|\nabla \chi_u| \,\mathrm{d}t \\
&\quad + \sigma \int_0^T \int_{\mathbb{R}^d} \xi \cdot (\xi \cdot \nabla) v \,\mathrm{d}|\nabla \chi_u| \,\mathrm{d}t.
\end{aligned} \tag{3.203}
$$

This in turn finally entails

$$
\begin{aligned}
&- \sigma \int_{\mathbb{R}^d} \mathrm{n}_u(\cdot, T) \cdot \xi(\cdot, T) \,\mathrm{d}|\nabla \chi_u(\cdot, T)| + \sigma \int_{\mathbb{R}^d} \mathrm{n}_u^0 \cdot \xi(\cdot, 0) \,\mathrm{d}|\nabla \chi_u^0| \\
&= \sigma \int_0^T \int_{\mathbb{R}^d} \mathrm{n}_u \cdot (\mathrm{n}_u \cdot \nabla) v \,\mathrm{d}|\nabla \chi_u| \,\mathrm{d}t \\
&\quad - \sigma \int_0^T \int_{\mathbb{R}^d} (\mathrm{n}_u - \xi) \cdot \big((\mathrm{n}_u - \xi) \cdot \nabla\big) v \,\mathrm{d}|\nabla \chi_u| \,\mathrm{d}t \\
&\quad + \sigma \int_0^T \int_{\mathbb{R}^d} \chi_u \big((u - v) \cdot \nabla\big)(\nabla \cdot \xi) \,\mathrm{d}x \,\mathrm{d}t \\
&\quad - \sigma \int_0^T \int_{\mathbb{R}^d} (\xi \cdot \mathrm{n}_u)\, \mathrm{n}_v(P_{I_v(t)}x) \cdot \big(\mathrm{n}_v(P_{I_v(t)}x) \cdot \nabla\big) v - \xi \cdot (\xi \cdot \nabla) v \,\mathrm{d}|\nabla \chi_u| \,\mathrm{d}t
\end{aligned} \tag{3.204}
$$

$$+\sigma\int_0^T\int_{\mathbb{R}^d}\mathrm{n}_u\cdot\big(\big(\mathrm{Id}-\mathrm{n}_v(P_{I_v(t)}x)\otimes\mathrm{n}_v(P_{I_v(t)}x)\big)(\nabla\bar{V}_\mathrm{n}-\nabla v)^T\xi\big)\,\mathrm{d}|\nabla\chi_u|\,\mathrm{d}t$$
$$+\sigma\int_0^T\int_{\mathbb{R}^d}\mathrm{n}_u\cdot\big((\bar{V}_\mathrm{n}-v)\cdot\nabla\big)\xi\,\mathrm{d}|\nabla\chi_u|\,\mathrm{d}t,$$

which holds for almost every $T\in[0,T_{strong})$.

In a last step, we use the truncation of the identity β from Proposition 3.10 composed with the signed distance to the interface of the strong solution as a test function in the transport equations (3.2b) resp. (3.7b) for the indicator functions χ_v resp. χ_u of the two solutions. However, observe first that by the precise choice of the weight function β it holds

$$(\chi_u-\chi_v)\beta\Big(\frac{\mathrm{dist}^\pm(\cdot,I_v)}{r_c}\Big)=|\chi_u-\chi_v|\Big|\beta\Big(\frac{\mathrm{dist}^\pm(\cdot,I_v)}{r_c}\Big)\Big|.$$

Hence, when testing the equation (3.2b) for the indicator function of the varifold solution and then subtracting the corresponding result from testing the equation (3.7b) for the indicator function of the strong solution, we obtain

$$\begin{aligned}&\int_{\mathbb{R}^d}\big|\chi_u(\cdot,T)-\chi_v(\cdot,T)\big|\Big|\beta\Big(\frac{\mathrm{dist}^\pm(\cdot,I_v(T))}{r_c}\Big)\Big|\,\mathrm{d}x\\&=\int_{\mathbb{R}^d}|\chi_u^0-\chi_v^0|\Big|\beta\Big(\frac{\mathrm{dist}^\pm(\cdot,I_v(0))}{r_c}\Big)\Big|\,\mathrm{d}x\\&\quad+\int_0^T\int_{\mathbb{R}^d}\chi_u\Big(\partial_t\beta\Big(\frac{\mathrm{dist}^\pm(\cdot,I_v)}{r_c}\Big)+(u\cdot\nabla)\beta\Big(\frac{\mathrm{dist}^\pm(\cdot,I_v)}{r_c}\Big)\Big)\,\mathrm{d}x\,\mathrm{d}t\\&\quad-\int_0^T\int_{\mathbb{R}^d}\chi_v\Big(\partial_t\beta\Big(\frac{\mathrm{dist}^\pm(\cdot,I_v)}{r_c}\Big)+(v\cdot\nabla)\beta\Big(\frac{\mathrm{dist}^\pm(\cdot,I_v)}{r_c}\Big)\Big)\,\mathrm{d}x\,\mathrm{d}t,\end{aligned}\tag{3.205}$$

which holds for almost every $T\in[0,T_{strong})$. Note that testing with the function $\beta(\frac{\mathrm{dist}^\pm(x,I_v(t))}{r_c})$ is admissible due to the bound $\chi_u,\chi_v\in L^\infty([0,T_{strong});L^1(\mathbb{R}^d))$ (recall that we assume $\chi_u,\chi_v\in L^\infty([0,T_{strong});\mathrm{BV}(\mathbb{R}^d))$ in our definition of solutions) and due to the fact that $\beta(\frac{\mathrm{dist}^\pm(x,I_v(t))}{r_c})$ is of class C^1. Indeed, one first multiplies β by a cutoff $\theta_{\tilde{R}}\in C^\infty_{cpt}(\mathbb{R}^d)$ on a scale $\tilde{R}$, i.e. $\theta\equiv 1$ on $\{x\in\mathbb{R}^d\colon|x|\leq\tilde{R}\}$, $\theta\equiv 0$ outside of $\{x\in\mathbb{R}^d\colon|x|\geq 2\tilde{R}\}$ and $\|\nabla\theta_R\|_{L^\infty(\mathbb{R}^d)}\leq C\tilde{R}^{-1}$ for some universal constant $C>0$. Then, one can use $\theta_{\tilde{R}}\beta$ in the transport equations as test functions and pass to the limit $\tilde{R}\to\infty$ because of the integrability of χ_v and χ_u. From this, one obtains the above equation.

Since the weight β vanishes at $r=0$, we may infer from the incompressibility of the velocity fields that

$$\begin{aligned}&\int_0^T\int_{\mathbb{R}^d}\chi_v\big((u-v)\cdot\nabla\big)\beta\Big(\frac{\mathrm{dist}^\pm(\cdot,I_v)}{r_c}\Big)\,\mathrm{d}x\,\mathrm{d}t\\&\quad=-\int_0^T\int_{I_v(t)}\big(\mathrm{n}_v\cdot(u-v)\big)\beta(0)\,\mathrm{d}S\,\mathrm{d}t=0.\end{aligned}$$

Hence, we can rewrite (3.205) as

$$\begin{aligned}&\int_{\mathbb{R}^d}\big|\chi_u(\cdot,T)-\chi_v(\cdot,T)\big|\Big|\beta\Big(\frac{\mathrm{dist}^\pm(\cdot,I_v(T))}{r_c}\Big)\Big|\,\mathrm{d}x\\&=\int_{\mathbb{R}^d}|\chi_u^0-\chi_v^0|\Big|\beta\Big(\frac{\mathrm{dist}^\pm(\cdot,I_v(0))}{r_c}\Big)\Big|\,\mathrm{d}x\\&\quad+\int_0^T\int_{\mathbb{R}^d}(\chi_u-\chi_v)\Big(\partial_t\beta\Big(\frac{\mathrm{dist}^\pm(\cdot,I_v)}{r_c}\Big)+\big((u-v)\cdot\nabla\big)\beta\Big(\frac{\mathrm{dist}^\pm(\cdot,I_v)}{r_c}\Big)\Big)\,\mathrm{d}x\,\mathrm{d}t\\&\quad+\int_0^T\int_{\mathbb{R}^d}(\chi_u-\chi_v)(v\cdot\nabla)\beta\Big(\frac{\mathrm{dist}^\pm(\cdot,I_v)}{r_c}\Big)\,\mathrm{d}x\,\mathrm{d}t\end{aligned}$$

for almost every $T \in [0, T_{strong})$. It remains to make use of the evolution equation for β composed with the signed distance function to the interface of the strong solution. But before we do so, let us remark that because of (3.19)

$$(v \cdot \nabla)\beta\Big(\frac{\mathrm{dist}^\pm(\cdot, I_v)}{r_c}\Big) = (V_\mathrm{n} \cdot \nabla)\beta\Big(\frac{\mathrm{dist}^\pm(\cdot, I_v)}{r_c}\Big),$$

where the vector field V_n is the projection of the velocity field v of the strong solution onto the subspace spanned by the unit normal $\mathrm{n}_v(P_{I_v(t)}x)$:

$$V_\mathrm{n}(x,t) := \big(v(x,t) \cdot \mathrm{n}_v(P_{I_v(t)}x, t)\big)\mathrm{n}_v(P_{I_v(t)}x, t)$$

for all (x,t) such that $\mathrm{dist}(x, I_v(t)) < r_c$. Thus, using the evolution equation (3.31) we finally obtain the identity

$$\begin{aligned}
&\int_{\mathbb{R}^d} \big|\chi_u(\cdot,T) - \chi_v(\cdot,T)\big|\Big|\beta\Big(\frac{\mathrm{dist}^\pm(\cdot, I_v(T))}{r_c}\Big)\Big| \,\mathrm{d}x \\
&= \int_{\mathbb{R}^d} \big|\chi_u^0 - \chi_v^0\big|\Big|\beta\Big(\frac{\mathrm{dist}^\pm(\cdot, I_v(0))}{r_c}\Big)\Big| \,\mathrm{d}x \\
&\quad + \int_0^T \int_{\mathbb{R}^d} (\chi_u - \chi_v)\big((u-v) \cdot \nabla\big)\beta\Big(\frac{\mathrm{dist}^\pm(\cdot, I_v)}{r_c}\Big) \,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \int_0^T \int_{\mathbb{R}^d} (\chi_u - \chi_v)\big((V_\mathrm{n} - \bar{V}_\mathrm{n}) \cdot \nabla\big)\beta\Big(\frac{\mathrm{dist}^\pm(\cdot, I_v)}{r_c}\Big) \,\mathrm{d}x\,\mathrm{d}t,
\end{aligned} \tag{3.206}$$

which holds true for almost every $T \in [0, T_{strong})$.

The asserted relative entropy inequality now follows from a combination of the bounds (3.198), (3.204) as well as (3.206). This concludes the proof. □

Remark 3.35. *Let us comment on the minor changes that occur in the proof of Proposition 3.10 when allowing for a bulk force $\rho(\chi)f$ such as gravity in Definition 3.2 of a varifold solution (resp. Definition 3.6 of a strong solution), where*

$$f \in W^{1,\infty}([0, T_{vari}]; H^1(\mathbb{R}^d;\mathbb{R}^d)) \cap W^{1,\infty}([0, T_{vari}]; W^{1,\infty}(\mathbb{R}^d;\mathbb{R}^d)).$$

In this case, the right hand side of the equation for the momentum balance (3.2a) for the varifold solution (χ_u, u, V) has to be amended by the term

$$+\int_0^T \int_{\mathbb{R}^d} \rho(\chi_u) f \cdot \eta \,\mathrm{d}x\,\mathrm{d}t, \tag{3.207}$$

whereas the right hand side of (3.7a) for the strong solution (χ_v, v) in addition includes

$$+\int_0^T \int_{\mathbb{R}^d} \rho(\chi_v) f \cdot \eta \,\mathrm{d}x\,\mathrm{d}t. \tag{3.208}$$

Moreover, the energy dissipation inequality (3.2c) of the varifold solution (χ_u, u, V) now also features on the right hand side the term

$$+\int_0^T \int_{\mathbb{R}^d} \rho(\chi_u) f \cdot u \,\mathrm{d}x\,\mathrm{d}t. \tag{3.209}$$

Hence, as a consequence of including a bulk force it is clear that an additional term $RHS_{bulkForce}$ has to appear in the inequality (3.187), and therefore also in the relative entropy inequality of Proposition 3.10. We derive the term $RHS_{bulkForce}$ by a quick review of the changes to be made for the argument from (3.181) to (3.186) which are the basis for (3.187).

First, the identity (3.182) was obtained from (3.181) (which itself remains unchanged) by subtracting the equation for the momentum balance of the strong solution. Due to the additional term (3.208), this means that we pick up in (3.183) after essentially testing with $\eta = u - v - w$ *an extra term of the form*

$$-\int_0^T \int_{\mathbb{R}^d} \rho(\chi_v) f \cdot (u - v - w) \,\mathrm{d}x \,\mathrm{d}t. \tag{3.210}$$

Second, we note that (3.184) remains unchanged under the inclusion of bulk forces, whereas (3.185) now includes on the right hand side the term (3.209) as it is merely a reminder of the energy dissipation inequality for the varifold solution. Third, since (3.186) arose essentially from testing (3.2a) with $\eta = v + w$ *and multiplying the resulting identity by* -1*, we pick up in (3.186) due to (3.207) the additional term*

$$-\int_0^T \int_{\mathbb{R}^d} \rho(\chi_u) f \cdot (v + w) \,\mathrm{d}x \,\mathrm{d}t. \tag{3.211}$$

Finally, as (3.187) was obtained by summing (3.183), (3.184), (3.185) and (3.186), it thus follows from (3.209), (3.210) and (3.211) that

$$RHS_{bulkForce} = \int_0^T \int_{\mathbb{R}^d} (\rho(\chi_u) - \rho(\chi_v))(u - v - w) \cdot f \,\mathrm{d}x \,\mathrm{d}t. \tag{3.212}$$

Since the whole argument after the derivation of (3.187) is unaffected from the inclusion of bulk forces, we deduce that the only additional term appearing in the relative entropy inequality from Proposition 3.10 is given by (3.212). Because of the simple computation $\rho(\chi_u) - \rho(\chi_v) = (\rho^+ - \rho^-)(\chi_u - \chi_v)$ *and* $f \in L^\infty(\mathbb{R}^d \times [0, T_{strong}); \mathbb{R}^d)$ *it follows from an application of Lemma 3.20 and an absorption argument that the weak-strong uniqueness principle as well as the stability estimate of Theorem 3.1 are still valid.*

3.7 Appendix

We begin with a remark on the higher order compatibility conditions (3.6b)–(3.8c) for the initial data in Definition 3.6 of a strong solution.

Remark 3.36. *The conditions (3.6b)–(3.8b) are standard in the literature on strong solutions for the two-phase Navier-Stokes problem with surface tension, see, for example, the works [125] and [127]. Denoting by* $v^\pm \colon \bigcup_{t \in [0, T_{strong}]} \Omega_t^\pm \times \{t\} \to \mathbb{R}^d$ *the velocity fields of the two respective fluids, then (3.6b)–(3.8b) are necessary to have continuity up to the initial time and up to the interface for the velocity fields* $v^\pm$ *and their spatial gradients* $\nabla v^\pm$*.*

The condition in (3.8c) is necessary for $\partial_t v^\pm$ *being continuous up to the initial time and the interface. Indeed, writing* $\bar{v}^\pm(x,t) := v^\pm(\Psi^t x, t)$ *by making use of the diffeomorphisms from Definition 3.5 we compute*

$$\partial_t \bar{v}^\pm|_{t=0} = \partial_t v^\pm|_{t=0} + (\partial_t \Psi^t|_{t=0} \cdot \nabla) v^\pm|_{t=0}$$

since $\Psi^0 = \mathrm{Id}$*. Moreover, we have* $[[\bar{v}^\pm(t)]] = 0$ *on* $I_v(0)$ *for all* $t \in [0, T_{strong}]$*, and therefore in particular* $[[\partial_t \bar{v}^\pm|_{t=0}]] = 0$*. Hence, it follows from Remark 3.9 and the fact that the tangential derivatives of* $v^\pm$ *naturally coincide on the interface that (3.8c) has to hold. One then verifies similarly that the conditions (3.8d)–(3.8e) are necessary for* $\partial_t \nabla v^\pm$ *being continuous up to the initial time and the interface.*

One may also allow here for sufficiently regular, density-dependent bulk forces like gravity. The only difference concerns the compatibility conditions (3.8c)–(3.8e) for which one has to include $\rho(\chi_0) f(\cdot, 0)$ *in the obvious way.*

We proceed with a remark on the existence of strong solutions in the precise functional framework of Definition 3.6 based on the assumption of the higher order regularity and compatibility conditions for the initial data (3.6a)–(3.8c).

Remark 3.37. *We start by making precise what one can infer from the existing literature about the existence of strong solutions to the two-phase Navier-Stokes problem with surface tension. Note that all what is said until (3.216) also holds true if one considers gravity, see for instance the works of Prüss and Simonett [125], [127] and [128, p. 581]. The remaining claims hold true after a suitable adaptation of the higher order compatibility conditions for the initial data, see the end of Remark 3.36.*

It follows from [89, Theorem 2]—up to the technicality that the authors consider the problem in a bounded domain and not the full space $\mathbb{R}^d$, in which case one may also consult [126]—that under the assumptions on the initial data in Definition 3.6 there exists a uniformly continuous, bounded velocity field $v \in C(\mathbb{R}^d \times [0, T_{strong}]; \mathbb{R}^d)$ which is of Sobolev regularity at least (where $q > d + 2$ is arbitrary):

$$v \in L^q([0, T_{strong}]; W^{2,q}(\mathbb{R}^d \setminus I_v(t); \mathbb{R}^d)) \cap H^1([0, T_{strong}]; L^q(\mathbb{R}^d; \mathbb{R}^d)). \tag{3.213}$$

This regularity directly implies by interpolation

$$\sup_{t\in[0,T_{strong}]} \sup_{\mathbb{R}^d \setminus I_v(t)} |\nabla v| < \infty. \tag{3.214}$$

Furthermore, it entails the existence of a pressure field p with $\nabla p \in L^q([0, T_{strong}]; L^q(\mathbb{R}^d))$, as well as a family of smoothly evolving sets $(\Omega_t^+)_{t\in[0,T_{strong}]}$ with smoothly evolving surfaces $(I_v(t))_{t\in[0,T_{strong}]}$ with indicator function χ in the sense of Definition 3.5. More precisely, the diffeomorphisms in Definition 3.5 inherit the regularity of the height function h constructed in [89, Theorem 2] and are thus, for the time being, short of one degree of spatial regularity to what is called for in Definition 3.5.

Moreover, it is proved in [89] that in the time interval $(0, T_{strong}]$ the interface is actually real analytic and that the velocity field v and the pressure p are real analytic as well; at least for positive times and away from the interface. Hence, the triple (v, p, χ) is for positive times a classical solution to the free boundary problem for the incompressible Navier–Stokes equation for two fluids (1.1a)-(1.1c). Since (3.213) also entails that

$$v \in H^1([0, T_{strong}]; L^2(\mathbb{R}^d; \mathbb{R}^d)) \cap L^\infty([0, T_{strong}]; H^1(\mathbb{R}^d; \mathbb{R}^d)), \tag{3.215}$$

$$\nabla v \in L^1([0, T_{strong}]; \mathrm{BV}(\mathbb{R}^d; \mathbb{R}^{d\times d})), \tag{3.216}$$

it remains to establish the estimate (3.7c) for spatial derivatives of order $k \in \{2, 3\}$, for the time derivative $\partial_t v$, and the mixed derivative $\partial_t \nabla v$, as well as that the diffeomorphisms from Definition 3.5 have one additional order of spatial regularity. For this, one relies on the higher order regularity and compatibility conditions for the initial data as given in Definition 3.6. Let us sketch how this works.

The argument uses the transformed formulation of the problem, see [89, (2.2)], stating it on a fixed domain $\mathbb{R}^d \setminus \Sigma$ with a real analytic reference interface Σ. At least for short times, the evolving interface $I_v(t)$ is then described by means of the graph of a height function h over this reference surface Σ. Moreover, the evolving domains occupied by the two fluids are described by means of the associated Hanzawa transform (see [89, p. 740] for the definition of the diffeomorphisms Θ_h). Defining the transformed velocity field $\bar{v} := v \circ \Theta_h$ and the *transformed pressure* $\bar{p} := p \circ \Theta_h$ *one obtains a quasilinear problem for* $(\bar{v}, \bar{p}, h)$ *of the type*

$$\begin{aligned} \rho\partial_t\bar{v} - \mu\Delta\bar{v} + \nabla\bar{p} &= F_1(h, \nabla^{\tan}h, \partial_t h, \nabla\bar{v}, \nabla^2\bar{v}, \nabla\bar{p}) && \text{in } \mathbb{R}^d \setminus \Sigma, \\ \nabla\cdot\bar{v} &= F_2(h, \nabla^{\tan}h, \nabla\bar{v}) && \text{in } \mathbb{R}^d \setminus \Sigma, \\ -[[\mu(\nabla\bar{v}+\nabla\bar{v}^T) - \bar{p}\mathrm{Id}]]\mathrm{n}_\Sigma &= F_3(h, \nabla^{\tan}h, \nabla\bar{v}) && \text{on } \Sigma, \\ [[\bar{v}]] &= 0 && \text{on } \Sigma, \\ \partial_t h - \mathrm{n}_\Sigma\cdot\bar{v} &= F_4(h, \nabla^{\tan}h, \bar{v}) && \text{on } \Sigma, \end{aligned} \tag{3.217}$$

where we abbreviated with $\nabla^{\tan}$ *the surface gradient on* Σ*. It is crucial that the non-linearities on the right hand side are at least quadratic, and that each term which is of the same order as the principal linear part on the left hand side comes with a factor of* h *or its derivative. Let us denote in the following by* $\bar{v}^{\pm}$ *resp.* $\bar{p}^{\pm}$ *the transformed velocity fields resp. the transformed pressures of the two fluids in their respective domains* $\Omega_h^{\pm}$*.*

All regularity properties stated before hold naturally for the transformed data $(\bar{v}, \bar{p}, h)$*. Moreover, regularity up to the interface is established in [128, Section 9.4] in the sense that we have for the respective one-sided traces*

$$\bar{v}^{\pm} \in C^\infty(\Sigma \times (0, T_{strong}]; \mathbb{R}^d), \tag{3.218}$$

$$((\mathrm{Id}{-}\mathrm{n}_\Sigma\otimes\mathrm{n}_\Sigma)\nabla)\bar{v}^{\pm} \in C^\infty(\Sigma \times (0, T_{strong}]; \mathbb{R}^{d\times d}), \tag{3.219}$$

$$\bar{p}^{\pm} \in C^\infty(\Sigma \times (0, T_{strong}]). \tag{3.220}$$

Let us only focus on how to establish the estimate (3.7c) in the vicinity of the interface; in the bulk one may proceed more directly without having to distinguish between tangential and normal directions. The first step then consists of taking the derivative with respect to a tangential vector field t_Σ *in the transformed problem (3.217); this shows that the tangential derivatives* $(\mathrm{t}_\Sigma\cdot\nabla)\bar{v}$*,* $(\mathrm{t}_\Sigma\cdot\nabla)\bar{p}$ *and* $(\mathrm{t}_\Sigma\cdot\nabla)h$ *satisfy a system analogous to (3.217). Recalling that we have assumed the higher regularity conditions (3.8a), we conclude that the theory of [89] applies to the tangential derivatives, yielding the regularity* $(\mathrm{t}_\Sigma\cdot\nabla)\bar{v} \in L^q([0, T_{strong}]; W^{2,q}(\mathbb{R}^d \setminus I_v(t); \mathbb{R}^d)) \cap H^1([0, T_{strong}]; L^q(\mathbb{R}^d; \mathbb{R}^d))$*. Since this holds for all tangential vector fields* t_Σ*, we conclude that*

$$\sup_{t\in[0,T_{strong}]}\ \sup_{x\in\Omega_h^{\pm}(t)} |\nabla((\mathrm{Id}{-}\mathrm{n}_\Sigma\otimes\mathrm{n}_\Sigma)\nabla)\bar{v}^{\pm}(x,t)| < \infty. \tag{3.221}$$

By transforming back to the original variables, we deduce a corresponding estimate for the term $\nabla((\mathrm{Id}{-}\mathrm{n}_{I_v}\otimes\mathrm{n}_{I_v})\nabla)v^{\pm}$*. Differentiating the constraint* $\nabla\cdot v^{\pm} = 0$ *in the bulk and using Schwarz's theorem to change the order of differentiation (which is admissible by the smoothness of the velocity in the bulk), one infers that actually all components of the second derivative* $\nabla^2 v^{\pm}$ *except for the normal-normal second derivative of the tangential velocity satisfy an analogous bound to (3.221). To establish the regularity for the last missing component, the idea is to extract from the Laplacian the normal-normal second derivative and to use the equation for* $v^{\pm}$*. For this, however, we first need to establish regularity for the time derivative* $\partial_t v^{\pm}$*.*

This is basically done by differentiating the transformed problem (3.217) in time, from which one derives an analogous problem for the time derivative time derivatives $\partial_t\bar{v}$*,* $\partial_t\bar{p}$*, and* $\partial_t h$ *of the transformed velocity* $\bar{v}$*, pressure* $\bar{p}$*, and height* h*. Arguing as before and using the compatibility conditions (3.8c)–(3.8d), we infer that*

$$\sup_{t\in[0,T_{strong}]}\ \sup_{x\in\Omega_h^{\pm}(t)} |\partial_t\bar{v}^{\pm}(x,t)| < \infty \tag{3.222}$$

and thus that also $\partial_t v$ satisfies a corresponding estimate, since we already know that $\partial_t h$ is continuous up to the initial time $t = 0$. From this one may then infer that (3.7c) holds true for $k = 2$ by using the equation for $v^\pm$ as already explained before.

Up until now, we only know that $h \in C([0, T_{strong}]; C^2(\Sigma)) \cap C^1([0, T_{strong}]; C^1(\Sigma))$ such that $\sup_{(x,t)\in\Sigma\times[0,T_{strong}]} |\nabla^{\tan}\nabla^{\tan}h(x,t)| + |\partial_t\nabla^{\tan}h(x,t)| < \infty$. However, taking tangential derivatives of the equation for h in the transformed problem (3.217) together with the one order higher regularity for the velocity field shows that $h \in C([0, T_{strong}]; C^3(\Sigma)) \cap C^1([0, T_{strong}]; C^2(\Sigma))$ with a corresponding bound for the highest derivatives. In particular, the diffeomorphisms Θ_h share the same properties from which we conclude that the diffeomorphisms from Definition 3.5 satisfy the required regularity and bounds.

Finally, one may follow the above argument to verify that (3.7c) also holds true for $k = 3$ and the mixed derivative $\partial_t\nabla v$. To this end, one relies on the higher regularity condition (3.8a) as well as the higher compatibility conditions (3.8d)–(3.8e). First, one differentiates the equation for the tangential derivatives of $v^\pm$ another time in the tangential direction to obtain boundedness of the gradient of the tangential-tangential second derivatives of the velocity fields $v^\pm$. Differentiating the constraint $\nabla \cdot v^\pm = 0$ in the bulk twice, using Schwarz's theorem to change the order of differentiation, and differentiating in time the equation for ∇v (leading again to a similar system) yields the bound for $\partial_t\nabla v^\pm$ and all third spatial derivatives of $v^\pm$ except for the normal-normal-normal third derivative.

For this, one differentiates in the bulk the equation for $v^\pm$ in normal direction concluding that the missing third derivative can be expressed by terms which are already controlled. This concludes the remark on the existence of strong solutions in the precise functional framework of Definition 3.6.

We rely several times in this work on the following standard result for singular integral operators of convolution type.

Theorem 3.38 (Boundedness of singular integral operators of convolution type in L^p)**.** *Let $d \geq 2$, $p \in (1, \infty)$, and let $K : \mathbb{S}^{d-1} \to \mathbb{R}$ be a function of class C^1 with vanishing average. Let $f \in L^p(\mathbb{R}^d)$ and define*

$$\mathcal{K}f(x) := \int_{\mathbb{R}^d} \frac{K\big(\frac{x-\tilde{x}}{|x-\tilde{x}|}\big)}{|x-\tilde{x}|^d} f(\tilde{x})\, d\tilde{x},$$

where the integral is understood in the Cauchy principal value sense. Then there exists a constant $C > 0$ depending only on d, p, and K such that

$$\|\mathcal{K}f\|_{L^p(\mathbb{R}^d)} \leq C\|f\|_{L^p(\mathbb{R}^d)}.$$

We also state a non-trivial result from geometric measure theory on properties of one-dimensional sections of Caccioppoli sets.

Theorem 3.39 ([39, Theorem G])**.** *Consider a set G of finite perimeter in $\mathbb{R}^d$, denote by $\nu^G = (\nu^G_{x_1}, \ldots, \nu^G_{x_{d-1}}, \nu^G_y) \in \mathbb{R}^d$ the associated measure theoretic inner unit normal vector field of the reduced boundary $\partial^* G$, and let χ^*_G be the precise representative of the bounded variation function χ_G. Then for Lebesgue almost every $x \in \mathbb{R}^{d-1}$ the one-dimensional sections $G_x := \{y \in \mathbb{R} \colon (x, y) \in G\}$ satisfy the following properties:*

i) G_x is a set of finite perimeter in $\mathbb{R}$, $\chi_G(x, \cdot) = \chi^*_G(x, \cdot)$ Lebesgue almost everywhere in G_x,

ii) $(\partial^* G)_x = \partial^* G_x$,

iii) $\nu^G_y(x, y) \neq 0$ for all $y \in \mathbb{R}$ such that $(x, y) \in \partial^* G$, and *iv)* $\lim_{y\to y_0^+}\chi_G^*(x,y)=1$ *and* $\lim_{y\to y_0^-}\chi_G^*(x,y)=0$ *whenever* $\nu_y^G(x,y_0)>0$*, and vice versa if* $\nu_y^G(x,y_0)<0$.

In particular, for every Lebesgue measurable set $M\subset\mathbb{R}^{d-1}$ *there exists a Borel measurable subset* $M_G\subset M$ *such that* $\mathcal{L}^{d-1}(M\setminus M_G)=0$ *and the four properties stated above are satisfied for all* $y\in M_G$.

To bound the L^4-norm of the interface error heights $h^\pm$ in the case of a two-dimensional interface, we employ the following optimal Orlicz–Sobolev embedding.

Theorem 3.40 (Optimal Orlicz-Sobolev embedding, [40, Theorem 1])**.** *For every* $d\geq 2$*, there exists a constant* K *depending only on* d *such that the following holds true: Let* $A:[0,\infty)\to[0,\infty)$ *be a convex function with* $A(0)=0$*,* $A(t)\to\infty$ *for* $t\to\infty$*, and*

$$\int_0^1\Big(\frac{t}{A(t)}\Big)^{1/(d-1)}dt<\infty.$$

Define

$$H(r):=\bigg(\int_0^r\Big(\frac{t}{A(t)}\Big)^{1/(d-1)}dt\bigg)^{(d-1)/d}$$

and

$$B(s):=A(H^{-1}(s)).$$

Then for any weakly differentiable function u *decaying to* 0 *at infinity in the sense* $\{|u(x)|>s\}<\infty$ *for all* $s>0$*, the following estimate holds true:*

$$\int_{\mathbb{R}^d}B\bigg(\frac{|u(x)|}{K\big(\int_{\mathbb{R}^d}A(|\nabla u(x)|)\,dx\big)^{1/d}}\bigg)\,dx\leq\int_{\mathbb{R}^d}A(|\nabla u(x)|)\,dx. \tag{3.223}$$

The application of the optimal Orlicz-Sobolev embedding to our setting is stated and proved next.

Proposition 3.41. *Let* $T>0$ *and* $(I(t))_{t\in[0,T]}$ *be a family of smoothly evolving surfaces in* $\mathbb{R}^3$ *in the sense of Definition 3.5. Consider* $u\in L^\infty([0,T];\mathrm{BV}(I(t)))$ *such that* $|u|\leq 1$*. Let* $e\colon[0,T]\to(0,\infty)$ *be a measurable function. We define*

$$A_{e(t)}(s):=\begin{cases}e(t)s & \text{for } s\leq e(t),\\ s^2 & \text{for } e(t)\leq s\leq 1,\\ 2s-1 & \text{for } s\geq 1.\end{cases}$$

We also set $A_{e(t)}(Du(t)):=\int_{I(t)}A_{e(t)}(|\nabla u(t)|)\,\mathrm{d}S+|D^su(t)|(I(t))$*. Then the following estimate holds true*

$$\begin{aligned}\int_{I(t)}|u(x,t)|^4\,\mathrm{d}S&\leq\frac{C}{r_c^{12}}\Big(1+\log\frac{1}{e(t)}\Big)\\&\quad\times\Big(e(t)^4+\frac{1}{e(t)^2}\big(\|u(t)\|_{L^2(I(t))}^6+A_{e(t)}^3(Du(t))\big)+\|u(t)\|_{L^2(I(t))}^4+A_{e(t)}^2(Du(t))\Big)\end{aligned}\tag{3.224}$$

for almost every $t\in[0,T]$ *and a constant* $C>0$.

Proof. Let $U \subset \mathbb{R}^2$ be an open and bounded set and consider $u \in C^1_{cpt}(U)$ such that $\|u\|_{L^\infty} \leq 1$. For the sake of brevity, let us suppress for the moment the dependence on the variable $t \in [0, T)$. The idea is to apply the optimal Orlicz–Sobolev embedding provided by the preceding theorem with respect to the convex function A_e. Observe first that A_e indeed satisfies all the assumptions. Moreover, since $d = 2$ we compute

$$(H(r))^2 = \int_0^r \frac{s}{A_e(s)}\,\mathrm{d}s = \begin{cases} \frac{r}{e} & \text{for } r \leq e, \\ 1 + \log\frac{r}{e} & \text{for } e \leq r \leq 1, \\ 1 + \log\frac{1}{e} + \frac{r-1}{2} + \frac{1}{4}\log(2r-1) & \text{for } r \geq 1. \end{cases}$$

As a consequence, we get

$$H^{-1}(y) = \begin{cases} = ey^2 & \text{for } y \leq 1, \\ = e\exp(y^2 - 1) & \text{for } 1 \leq y \leq \sqrt{1 + \log\frac{1}{e}}, \\ \geq (y^2 - 1 - \log\frac{1}{e}) + 1 & \text{for } y \geq \sqrt{1 + \log\frac{1}{e}}, \\ \leq 2(y^2 - 1 - \log\frac{1}{e}) + 1 & \text{for } y \geq \sqrt{1 + \log\frac{1}{e}}. \end{cases}$$

This in turn entails

$$B(s) = A_e(H^{-1}(s)) = \begin{cases} = e^2 s^2 & \text{for } s \leq 1, \\ = e^2\exp(2s^2 - 2) & \text{for } 1 \leq s \leq \sqrt{1 + \log\frac{1}{e}}, \\ \geq s^2 - \log\frac{1}{e} & \text{for } s \geq \sqrt{1 + \log\frac{1}{e}}. \end{cases} \tag{3.225}$$

We then deduce from Theorem 3.40, $d = 2$, $\|u\|_{L^\infty} \leq 1$, the bound $\exp(s^2) \geq \frac{1}{2}s^4$ for all $s \geq 0$ as well as the bound $s^2 - \log\frac{1}{e} \geq \frac{s^2}{1+\log\frac{1}{e}}$ for all $s \geq \sqrt{1 + \log\frac{1}{e}}$

$$\begin{aligned}
&\int_U |u(x)|^4\,\mathrm{d}x \\
&= \int_{U\cap\left\{|u|\leq K\sqrt{A_e(Du)}\right\}} |u(x)|^4\,\mathrm{d}x \\
&\quad + \int_{U\cap\left\{K\sqrt{A_e(Du)}\leq|u|\leq K\sqrt{A_e(Du)}\sqrt{1+\log\frac{1}{e}}\right\}} |u(x)|^4\,\mathrm{d}x \\
&\quad + \int_{U\cap\left\{|u|\geq K\sqrt{A_e(Du)}\sqrt{1+\log\frac{1}{e}}\right\}} |u(x)|^4\,\mathrm{d}x \\
&\leq K^4\frac{A_e^2(Du)}{e^2}\int_{U\cap\left\{|u|\leq K\sqrt{A_e(Du)}\right\}} e^2\frac{|u(x)|^2}{K^2 A_e(Du)}\,\mathrm{d}x \\
&\quad + K^4\frac{A_e^2(Du)}{e^2}\int_{U\cap\left\{K\sqrt{A_e(Du)}\leq|u|\leq K\sqrt{A_e(Du)}\sqrt{1+\log\frac{1}{e}}\right\}} e^2\frac{|u(x)|^4}{K^4 A_e^2(Du)}\,\mathrm{d}x \\
&\quad + K^2\Big(1 + \log\frac{1}{e}\Big)A_e(Du)\int_{U\cap\left\{|u|\geq K\sqrt{A_e(Du)}\sqrt{1+\log\frac{1}{e}}\right\}} \frac{|u(x)|^4}{K^2\big(1+\log\frac{1}{e}\big)A_e(Du)}\,\mathrm{d}x \\
&\leq C\Big(1 + \log\frac{1}{e}\Big)\Big(\frac{1}{e^2}A_e^3(Du) + A_e^2(Du)\Big),
\end{aligned}$$

which is precisely what is claimed. Note that since u is continuously differentiable, the singular part in the definition of $A_e(Du)$ vanishes.

In a next step, we want to extend to smooth functions u on the manifold $I(t)$. By assumption, we may cover $I(t)$ with a finite family of open sets of the form $U(x_i) := I(t) \cap$

$B_{2r_c}(x_i)$, $x_i \in I(t)$, such that $U(x_i)$ can be represented as the graph of a function $g\colon B_1(0) \subset \mathbb{R}^2 \to \mathbb{R}$ with $|\nabla g| \leq 1$ and $|\nabla^2 g| \leq r_c^{-1}$. We fix a partition of unity $\{\varphi_i\}_i$ subordinate to this finite cover of $I(t)$. Note that $|\nabla \varphi_i| \leq C r_c^{-1}$. Note also that the cardinality of the open cover is uniformly bounded in t. Hence, we proceed with deriving the desired bound only for one $u\varphi$, where $\varphi = \varphi_i$ is supported in $U = U(x_i)$. Abbreviating $\tilde{u} = u \circ g$ and $\tilde{\varphi} = \varphi \circ g$, we obtain from the previous step

$$\begin{aligned}\int_U |u\varphi|^4 \,\mathrm{d}S &= \int_{B_1(0)} |(u\varphi)(g(x))|^4 \sqrt{1+|\nabla g(x)|^2} \,\mathrm{d}x \\ &\leq \sqrt{2}C\Big(1+\log\frac{1}{e}\Big)\Big(\frac{1}{e^2}A_e^3\big(D(\tilde{u}\tilde{\varphi})\big) + A_e^2\big(D(\tilde{u}\tilde{\varphi})\big)\Big).\end{aligned}$$

Using the bounds $A_e(t+\tilde{t}) \leq CA_e(t) + CA_e(\tilde{t})$ and $A_e(\lambda t) \leq C(\lambda+\lambda^2)A_e(t)$, which hold for all $\lambda > 0$ and all $t, \tilde{t} \geq 0$, as well as the product and chain rule we compute

$$A_e(D(\tilde{u}\tilde{\varphi})) \leq Cr_c^{-2} \int_{B_1(0)} A_e\big(|u|(g(x))\big) \,\mathrm{d}x + C\int_{B_1(0)} A_e\big(|\nabla u|(g(x))\big) \,\mathrm{d}x.$$

By definition of A_e we can further estimate

$$\int_{B_1(0)} A_e\big(|u|(g(x))\big) \,\mathrm{d}x \leq Ce^2 + \int_{B_1(0)} |u|^2(g(x)) \,\mathrm{d}x.$$

Changing back to the local coordinates on the manifold $I(t)$ we deduce

$$\begin{aligned}\int_U |u|^4 \,\mathrm{d}S \leq{}& \frac{C}{r_c^6}\Big(1+\log\frac{1}{e}\Big) \\ &\times \Big(e^4 + \frac{1}{e^2}\big(\|u\|_{L^2(I(t))}^6 + A_e^3(Du)\big) + \|u\|_{L^2(I(t))}^4 + A_e^2(Du)\Big).\end{aligned} \tag{3.226}$$

This yields the claim in the case of a smooth function $u\colon I(t) \to \mathbb{R}$.

In a last step, we extend this estimate by mollification to $u \in \mathrm{BV}(I(t))$ with $\|u\|_{L^\infty} \leq 1$. To this end, let $\theta\colon \mathbb{R}^+ \to [0,1]$ be a smooth cutoff with $\theta(s) = 1$ for $s \in [0, \frac{1}{4}]$ and $\theta(s) = 0$ for $s \geq \frac{1}{2}$. We then define for each $n \in \mathbb{N}$

$$u_n(x,t) := \frac{\int_{I(t)} \theta(n|\tilde{x}-x|)u(\tilde{x},t) \,\mathrm{d}S(\tilde{x})}{\int_{I(t)} \theta(n|\tilde{x}-x|) \,\mathrm{d}S(\tilde{x})}.$$

Since the analogous bound to (3.77) holds true, we infer $\|u_n\|_{L^\infty} \leq 1$ as well as $\|u_n - u\|_{L^1(I(t))} \to 0$ as $n \to \infty$. In particular, we have pointwise almost everywhere convergence at least for a subsequence. This in turn implies by Lebesgue's dominated convergence theorem that $\|u_n - u\|_{L^4(I(t))} \to 0$ as $n \to \infty$ at least for a subsequence. Moreover, the exact same computation which led to (3.76) shows

$$\begin{aligned}A_{e(t)}(|\nabla u_n(x,t)|) \leq{}& C\frac{\int_{I(t)} \theta(n|\tilde{x}-x|)\big(A_{e(t)}(|\nabla u(\tilde{x},t)|) + A_{e(t)}(r_c^{-1}|u(\tilde{x},t)|)\big) \,\mathrm{d}S(\tilde{x})}{\int_{I(t)} \theta(n|\tilde{x}-x|) \,\mathrm{d}S(\tilde{x})} \\ &+ C\frac{\int_{I(t)} \theta(n|\tilde{x}-x|) \,\mathrm{d}|D^s u|(\tilde{x},t)}{\int_{I(t)} \theta(n|\tilde{x}-x|) \,\mathrm{d}S(\tilde{x})}.\end{aligned}$$

Integrating this bound over the manifold and then using Fubini shows that

$$A_{e(t)}(Du_n(t)) \leq Cr_c^{-2} A_{e(t)}(Du(t))$$

holds true uniformly over all $n \in \mathbb{N}$. By applying the bound (3.226) from the second step, we may conclude the proof. □

In the case where the interface I_v is a curve in $\mathbb{R}^2$, a much more elementary argument yields the following bound.

Lemma 3.42. *Let $T > 0$ and let $(I(t))_{t\in[0,T]}$ be a family of smoothly evolving curves in $\mathbb{R}^2$ in the sense of Definition 3.5. Let $u \in L^\infty([0,T];\mathrm{BV}(I(t)))$ such that $|u| \leq 1$. Consider the convex function*

$$G(s) := \begin{cases} s^2, & |s| \leq 1, \\ 2s-1, & |s| > 1. \end{cases}$$

We also define $|Du(t)|_G := \int_{I(t)} G(|\nabla u(x,t)|)\,\mathrm{d}S + |D^s u(t)|(\Gamma)$. Then,

$$\int_{I(t)} |u(x,t)|^4 \,\mathrm{d}S \leq \frac{C(1+\mathcal{H}^1(I(t)))^3}{r_c^4}\big(|Du(t)|_G^2 + |Du(t)|_G^4 + ||u||_{L^2(I(t))}^4\big) \tag{3.227}$$

holds true for almost every $t \in [0,T]$ with some universal constant $C > 0$.

Proof. Fix $t > 0$. First, observe that $I(t)$ essentially consists of a finite number of nonintersecting curves. By approximation, we may assume $u(t) \in W^{1,1}(I(t))$.

Let η_i be a partition of unity on $I(t)$ with $|\nabla^{\tan}\eta_i(x)| \leq Cr_c^{-1}$ such that the support of each η_i is isometrically equivalent to a bounded interval (note that the Definition 3.5 implies a lower bound of cr_c for the length of any connected component of $I(t)$) and such that at any point $x \in I(t)$ there are at most two i with $\eta_i(x) > 0$.

Treating by abuse of notation the function $\eta_i u$ as if defined on a real interval $I = (a,b)$, we then write

$$\begin{aligned}
\eta_i(x)u(x) &= \int_a^x \eta_i(y)u'(y) + \eta_i'(y)u(y)\,\mathrm{d}y \\
&= \int_a^x \eta_i'(y)u(y)\,\mathrm{d}y + \int_a^x \eta_i(y)\big(\max\big\{\min\{u'(y),1\},-1\big\}\big)\,\mathrm{d}y \\
&\quad + \int_a^x \eta_i(y)\big(\big(u'(y)-1\big)_+ - \big(u'(y)-(-1)\big)_-\big)\,\mathrm{d}y.
\end{aligned}$$

Hence, we may estimate using Jensen's inequality

$$\begin{aligned}
\eta_i(x)|u(x)| &\leq |I(t)|^{1/2}\bigg(\int_{I(t)} \eta_i|\max\big\{\min\{|\nabla^{\tan}u|,1\},-1\big\}|^2\,\mathrm{d}S\bigg)^{1/2} \\
&\quad + \int_{I(t)} \eta_i\,(|\nabla^{\tan}u|-1)_+\,\mathrm{d}S + Cr_c^{-1}\int_{I(t)\cap\operatorname{supp}\eta_i} |u|\,\mathrm{d}S
\end{aligned}$$

for any $x \in I(t)$. Taking the fourth power, integrating over x, and summing over i, we deduce

$$\begin{aligned}
\int_{I(t)} |u(x)|^4\,\mathrm{d}S &\leq C|I(t)|^3\bigg(\int_{I(t)} |\max\big\{\min\{|\nabla^{\tan}u|,1\},-1\big\}|^2\,\mathrm{d}y\bigg)^2 \\
&\quad + C|I(t)|\bigg(\int_{I(t)} (|\nabla^{\tan}u|-1)_+\,\mathrm{d}S\bigg)^4 \\
&\quad + Cr_c^{-4}|I(t)|^3\bigg(\int_{I(t)} |u|^2\,\mathrm{d}y\bigg)^2.
\end{aligned}$$

From this we infer the desired estimate by approximation. □

CHAPTER 4

Weak-strong uniqueness for planar multiphase mean curvature flow

Abstract. We prove that in the absence of topological changes, the notion of BV solutions to planar multiphase mean curvature flow does not allow for a mechanism for (unphysical) non-uniqueness. Our approach is based on the local structure of the energy landscape near a classical evolution by mean curvature. Mean curvature flow being the gradient flow of the surface energy functional, we develop a gradient-flow analogue of the notion of calibrations. Just like the existence of a calibration guarantees that one has reached a global minimum in the energy landscape, the existence of a "gradient flow calibration" ensures that the route of steepest descent in the energy landscape is unique and stable.

4.1 Main results & definitions

In the following, we present our weak-strong uniqueness principle for BV solutions of multiphase mean curvature flow in the plane. In addition, we provide a quantitative stability estimate, i. e., as long as a strong solution exists, any solution to the BV formulation of multiphase mean curvature flow with slightly perturbed initial data remains close to it. Our results are valid under minimal assumptions on the surface tensions, see Definition 4.8.

Theorem 4.1 (Weak-strong uniqueness and quantitative stability)**.** *Let $d = 2$ and $P \in \mathbb{N}$, $P \geq 2$. Let $\chi = (\chi_1, \ldots, \chi_P)$ be a* BV *solution of multiphase mean curvature flow in the sense of Definition 4.11 on some time interval $[0, T_{\mathrm{BV}})$. Let $\bar\chi = (\bar\chi_1, \ldots, \bar\chi_P)$ be a strong solution of multiphase mean curvature flow on $\mathbb{R}^d$ in the sense of Definition 4.14 on some time interval $[0, T_{\mathrm{strong}})$ with $T_{\mathrm{strong}} \leq T_{\mathrm{BV}}$.*

Then, the BV solution χ must coincide with the strong solution $\bar\chi$ for almost all $0 \leq t < T_{\mathrm{strong}}$, provided that it starts from the same initial data.

Furthermore, the evolution by multiphase mean curvature is stable with respect to perturbations in the initial data in the sense that for every $T \in (0, T_{\mathrm{strong}})$ the stability estimates

$$E[\chi|\xi](t) \leq e^{Ct} E[\chi|\xi](0)$$
$$E_{\mathrm{volume}}[\chi|\bar\chi](t) \leq e^{Ct}\big(E_{\mathrm{volume}}[\chi|\bar\chi](0) + E[\chi|\xi](0)\big)$$

hold true for almost every $t \in [0,T]$, where the constant $C > 0$ only depends on $\bar\chi$ and T through certain higher derivatives of functions associated to $\bar\chi$. The interface error functional $E[\chi|\xi]$ is defined in (4.2), with $\xi_{i,j}$ denoting the gradient flow calibration for the classical solution $\bar\chi$ as constructed in Proposition 4.6, and the bulk error functional $E_{\mathrm{volume}}[\chi|\bar\chi]$ is defined in (4.5).

Proof. This is an immediate consequence of the conditional weak-strong uniqueness principle of Proposition 4.5, and the existence results of Proposition 4.6 and Lemma 4.7 realizing the assumptions of Proposition 4.5 in the planar setting. □

4.1.1 Calibrations and inclusion principle

The key ingredient for our uniqueness result prior to topology changes is the following gradient flow analogue of the notion of calibrations for minimal partitions. Our main result, Theorem 4.1, is then an immediate consequence of two implications: First, the existence of a gradient flow calibration guarantees uniqueness of the BV solution (see Proposition 4.3 and Proposition 4.5) in arbitrary ambient dimension $d \geq 2$; second, classical solutions to planar multiphase mean curvature flow are calibrated in the sense that a gradient flow calibration exists (see Proposition 4.6 and Lemma 4.7).

Definition 4.2 (Calibrations for the gradient flow and calibrated flows). *Let $d \geq 2$, $P \geq 2$ be integers and let $\sigma \in \mathbb{R}^{P\times P}$ be an admissible matrix of surface tensions in the sense of Definition 4.8. Let $T > 0$, and for all $i \in \{1,\dots,P\}$ let $\bar\Omega_i := \bigcup_{t\in[0,T]} \bar\Omega_i(t)\times\{t\}$ such that for all $t \in [0,T]$ the family $(\bar\Omega_1(t),\dots,\bar\Omega_P(t))$ is a partition of finite surface energy of $\mathbb{R}^d$ in the sense of Definition 4.10. For each $i,j \in \{1,\dots,P\}$ with $i \neq j$ and all $t \in [0,T]$, let $\bar I_{i,j}(t) := \partial^*\bar\Omega_i(t) \cap \partial^*\bar\Omega_j(t)$ be the interface between the phases i and j at time t.*

A pair $(\xi = (\xi_i)_{i\in\{1,\dots,P\}}, B)$ consisting of vector fields

$$\xi_i \in C^1([0,T]; C^0_{\mathrm{cpt}}(\mathbb{R}^d;\mathbb{R}^d)) \cap C^0([0,T]; C^1_{\mathrm{cpt}}(\mathbb{R}^d;\mathbb{R}^d)), \quad i \in \{1,\dots,P\},$$
$$B \in C^0([0,T]; C^1_{\mathrm{cpt}}(\mathbb{R}^d;\mathbb{R}^d))$$

is called a calibration for the gradient flow *for the* calibrated flow $(\bar\Omega_1,\dots,\bar\Omega_P)$ on $[0,T]$ *if the following conditions are satisfied:*

- *For each pair of phases $i,j \in \{1,\dots,P\}$ and all $t \in [0,T]$, the vector field*

$$\xi_{i,j}(\cdot,t) := \frac{1}{\sigma_{i,j}}(\xi_i - \xi_j)(\cdot,t) \tag{4.1a}$$

  coincides on $\bar I_{i,j}(t)$ with the associated unit normal vector field $\bar{\mathrm{n}}_{i,j}(\cdot,t)$ (with the convention that $\bar{\mathrm{n}}_{i,j}(\cdot,t)$ points from phase i into phase j), and it satisfies an estimate of the form

$$|\xi_{i,j}(x,t)| \leq 1 - c\min\{\mathrm{dist}^2(x,\bar I_{i,j}(t)),\, 1\} \tag{4.1b}$$

  for some $c \in (0,1)$ and all $(x,t) \in \mathbb{R}^d \times [0,T]$.

- *The evolution of the vector fields $\xi_{i,j}$ is approximately transported by the velocity field B in the sense*

$$\big|\partial_t\xi_{i,j} + (B\cdot\nabla)\xi_{i,j} + (\nabla B)^{\mathsf{T}}\xi_{i,j}\big|(x,t) \leq C\big(\mathrm{dist}(x,\bar I_{i,j}(t)) \wedge 1\big) \tag{4.1c}$$

  and

$$\big|\partial_t|\xi_{i,j}|^2 + (B\cdot\nabla)|\xi_{i,j}|^2\big|(x,t) \leq C\big(\mathrm{dist}^2(x,\bar I_{i,j}(t)) \wedge 1\big) \tag{4.1d}$$

  for some $C > 0$ and all $(x,t) \in \mathbb{R}^d \times [0,T]$.

- *For each $t \in [0,T]$, the normal component of the velocity field $B(\cdot,t)$ near the interface $\bar{I}_{i,j}(t)$ is approximately given by the mean curvature of $\bar{I}_{i,j}(t)$ in the sense that*

$$\big|\xi_{i,j} \cdot B + \nabla \cdot \xi_{i,j}\big|(x,t) \le C\big(\operatorname{dist}(x, \bar{I}_{i,j}(t)) \wedge 1\big) \tag{4.1e}$$

for some $C > 0$ and all $(x,t) \in \mathbb{R}^d \times [0,T]$.

Note that, at least heuristically, such a calibrated flow is a solution to mean curvature flow as on $\bar{I}_{i,j}$ the normal velocity $\bar{\mathrm{n}}_{i,j} \cdot B$ coincides with the mean curvature due to (4.1e).

The next proposition states that for general $d \ge 2$ the existence of a gradient flow calibration for a given time-evolving partition of $\mathbb{R}^d$ into P domains $(\bar{\Omega}_1, \ldots, \bar{\Omega}_P)$ constrains the possible locations of the interfaces in weak (BV) solutions to mean curvature flow to the corresponding interfaces of the partition $(\bar{\Omega}_1, \ldots, \bar{\Omega}_P)$. This assertion may be seen as a multi-phase analogue of the varifold comparison principle by Ilmanen [82, Theorem 10.7], which for two-phase mean curvature flow provides a corresponding inclusion given any Brakke solution and a level set solution. Note that such an inclusion does not yet yield uniqueness of BV solutions, as it does not exclude the sudden vanishing of all phases except one.

Proposition 4.3 (Quantitative inclusion principle)**.** *Let $d \ge 2$ and $P \ge 2$ be integers and let $\sigma \in \mathbb{R}^{P\times P}$ be an admissible matrix of surface tensions, see Definition 4.8. Let $T > 0$, and let $(\bar{\Omega}_1, \ldots, \bar{\Omega}_P)$ be a calibrated flow on $[0,T]$ in the sense of Definition 4.2.*

Then the interfaces $I_{i,j}(t) := \partial^*\{\chi_i(t) = 1\} \cap \partial^*\{\chi_j(t) = 1\}$ of any BV *solution $(\chi_1, \ldots, \chi_P)$ to mean curvature flow on $[0,T]$ in the sense of Definition 4.11 with the same initial data as the calibrated flow must be contained in the corresponding interfaces $\bar{I}_{i,j}(t) := \partial^*\bar{\Omega}_i(t) \cap \partial^*\bar{\Omega}_j(t)$ for a. e. $0 < t < T$, i.e., it holds $I_{i,j}(t) \subset \bar{I}_{i,j}(t)$ for all i,j with $i \neq j$ up to $\mathcal{H}^{d-1}$ null sets.*

Furthermore, the existence of a gradient flow calibration also implies a stability estimate: Introducing the interface error functional

$$E[\chi|\xi](t) := \sum_{i,j=1, i\neq j}^{P} \sigma_{i,j} \int_{I_{i,j}(t)} 1 - \xi_{i,j}(\cdot,t) \cdot \mathrm{n}_{i,j}(\cdot,t) \,\mathrm{d}\mathcal{H}^{d-1}, \tag{4.2}$$

there exists a constant $C > 0$ depending on the calibrated flow such that we have the stability estimate

$$E[\chi|\xi](t) \le e^{Ct} E[\chi|\xi](0) \tag{4.3}$$

for general BV solutions $\chi = (\chi_1, \ldots \chi_P)$ and almost every $t \in [0,T]$.

As already discussed, the interface error control provided by the functional (4.2) suffers from a lack of coercivity concerning the vanishing of interface length in a BV solution. For this reason, we also have to consider a lower-order term $E_{\mathrm{volume}}[\chi|\bar{\chi}]$, see (4.5) below, which controls bulk deviations from the grains of the strong solution $\bar{\Omega}$. The main input for the bulk error functional is captured in the following notion of transported weights.

Definition 4.4 (Transported weights)**.** *Let $d \ge 2$, $P \ge 2$ be integers and denote by $T \in (0,\infty)$ a finite time horizon. For all $i \in \{1, \ldots, P\}$ let $\bar{\Omega}_i := \bigcup_{t\in[0,T]} \bar{\Omega}_i(t) \times \{t\}$ such that for all $t \in [0,T]$ the family $(\bar{\Omega}_1(t), \ldots, \bar{\Omega}_P(t))$ is a partition of finite surface energy of $\mathbb{R}^d$ in the sense of Definition 4.10. Denote by $\bar{\chi} = (\bar{\chi}_1, \ldots, \bar{\chi}_P)$ the associated family of indicator functions for $\bar{\Omega} = (\bar{\Omega}_1, \ldots, \bar{\Omega}_P)$. Assume that for all $i \in \{1, \ldots, P\}$ the measure $\partial_t \bar{\chi}_i$ is absolutely continuous with respect to the measure $|\nabla \bar{\chi}_i|$, and that the boundary $\partial\bar{\Omega}_i(\cdot,t)$ is Lipschitz at all times $t \in [0,T]$. Let finally $B \in C^0([0,T]; C^1_{\mathrm{cpt}}(\mathbb{R}^d; \mathbb{R}^d))$.*

In this setting, a family of measurable maps

$$\vartheta_i \colon \mathbb{R}^d \times [0,T] \to [-1,1], \quad i \in \{1,\ldots,P\},$$

is called a family of transported weights with respect to $(\bar\Omega, B)$ on [0,T] *if the following conditions are satisfied:*

- *(Regularity) For all phases $i \in \{1,\ldots,P\}$ it holds*

$$\vartheta_i \in W^{1,1}(\mathbb{R}^d \times [0,T]) \cap W^{1,\infty}(\mathbb{R}^d \times [0,T]).$$

- *(Coercivity) For all phases $i \in \{1,\ldots,P\}$ and all $t \in [0,T]$, we have $\vartheta_i(\cdot,t) < 0$ in the essential interior of $\bar\Omega_i(t)$, $\vartheta_i(\cdot,t) > 0$ in the essential exterior of $\bar\Omega_i(t)$, and $\vartheta_i(\cdot,t) = 0$ on $\partial\bar\Omega_i(t)$.*

- *(Advection equation) The weights are transported by the vector field B in the sense that*

$$|\partial_t \vartheta_i + (B\cdot\nabla)\vartheta_i| \le C|\vartheta_i| \tag{4.4}$$

  holds true in $\mathbb{R}^d \times [0,T]$ for all phases $i \in \{1,\ldots,P\}$.

The merit of the previous definition is that it allows to sharpen the quantitative inclusion principle of Proposition 4.3 to a conditional weak-strong uniqueness principle (with an associated conditional stability estimate) for BV solutions of multiphase mean curvature flow; see Proposition 4.5 below for the precise statement. The result is conditional in the sense that in addition to the existence of a gradient flow calibration (see Definition 4.2), the existence of a family of transported weights (see Definition 4.4) is assumed. However, the crucial point is that it already holds in arbitrary ambient dimension $d \ge 2$.

Proposition 4.5 (Conditional weak-strong uniqueness and quantitative stability)**.** *Let $d \ge 2$, $P \ge 2$ be integers and $\sigma \in \mathbb{R}^{P\times P}$ be an admissible matrix of surface tensions in the sense of Definition 4.8. Let $\chi = (\chi_1,\ldots,\chi_P)$ be a BV solution of multiphase mean curvature flow in the sense of Definition 4.11 on $[0,T]$. Let moreover $\bar\Omega = (\bar\Omega_1,\ldots,\bar\Omega_P)$ be as in Definition 4.4 on $[0,T]$. The associated family of indicator functions is denoted by $\bar\chi = (\bar\chi_1,\ldots,\bar\chi_P)$.*

Assume also that there exists a gradient flow calibration $((\xi_i)_{i\in\{1,\ldots,P\}}, B)$ with respect to $\bar\Omega$ on $[0,T]$ in the sense of Definition 4.2, and that there exists a family of transported weights $(\vartheta_i)_{i\in\{1,\ldots,P\}}$ with respect to $(\bar\Omega, B)$ on $[0,T]$ in the sense of Definition 4.4. Recall the definition (4.2) of the interface error functional, and define a bulk error functional by

$$E_{\mathrm{volume}}[\chi|\bar\chi](t) := \sum_{i=1}^P \int_{\mathbb{R}^d} |\chi_i(\cdot,t) - \bar\chi_i(\cdot,t)||\vartheta_i(\cdot,t)|\,\mathrm{d}x, \quad t \in [0,T]. \tag{4.5}$$

Then it holds

$$\chi(\cdot,0) = \bar\chi(\cdot,0) \text{ a.e. in } \mathbb{R}^d \Rightarrow \chi(\cdot,t) = \bar\chi(\cdot,t) \text{ a.e. in } \mathbb{R}^d \text{ for a.e. } t \in [0,T].$$

Moreover, the interface error functional $E[\chi|\xi]$ from (4.2) and the bulk error functional $E_{\mathrm{volume}}[\chi|\bar\chi]$ from (4.5) satisfy quantitative stability estimates of the form

$$E[\chi|\xi](t) \le e^{Ct} E[\chi|\xi](0) \tag{4.6}$$

$$E_{\mathrm{volume}}[\chi|\bar\chi](t) \le e^{Ct}\big(E_{\mathrm{volume}}[\chi|\bar\chi](0) + E[\chi|\xi](0)\big) \tag{4.7}$$

for almost every $t \in [0,T]$.

4.1.2 Gradient flow calibrations for regular networks

In view of Proposition 4.5 above, the question of weak-strong uniqueness for BV solutions of multiphase mean curvature flow is reduced to the task of constructing a gradient flow calibration and a family of transported weights. As it turns out, in the planar case the existence of a classical solution to mean curvature flow — in the sense of a smooth evolution of curves meeting at triple junctions with the correct contact angle, see Definition 4.14 — entails the existence of a calibration for the gradient flow:

Proposition 4.6. *Let $d = 2$ and $P \in \mathbb{N}$, $P \geq 2$. Let $(\bar{\Omega}_1, \ldots, \bar{\Omega}_P)$ be a strong solution to multiphase mean curvature flow on $[0, T]$ in the sense of Definition 4.14. Then there exists an associated gradient flow calibration on $[0, T]$ in the sense of Definition 4.2.*

In the same setting as above, one can in addition establish the existence of a family of transported weights.

Lemma 4.7. *Let $d = 2$ and $P \in \mathbb{N}$, $P \geq 2$. Let $(\bar{\Omega}_1, \ldots, \bar{\Omega}_P)$ be a strong solution to multiphase mean curvature flow on $[0, T]$ in the sense of Definition 4.14. Let B denote the velocity field from Proposition 4.6. Then there exists a family of transported weights on $[0, T]$ with respect to $(\bar{\Omega}, B)$ in the sense of Definition 4.4.*

4.1.3 Basic definitions

In the following, we recall the precise definitions of the solution concepts for multiphase mean curvature flow which our main results are concerned with. We begin with the notion of admissible surface tensions.

Definition 4.8 (Admissible matrix of surface tensions)**.** *Let $P \geq 2$ be an integer and $\sigma = (\sigma_{i,j})_{i,j=1,\ldots,P} \in \mathbb{R}^{P\times P}$. The matrix σ is called an* admissible matrix of surface tensions *if the following conditions are satisfied:*

i) (Symmetry) It holds that $\sigma_{i,j} = \sigma_{j,i}$ and $\sigma_{i,i} = 0$ for every $i, j \in \{1, \ldots, P\}$.

ii) (Positivity) We have $\sigma_{\min} := \min\{\sigma_{i,j} : i, j \in \{1, \ldots, P\}, i \neq j\} > 0$.

iii) (Coercivity) The matrix of surface tensions σ is non-degenerately ℓ^2-embeddable into $\mathbb{R}^{P-1}$, i.e., there exists a non-degenerate $(P-1)$-simplex $(q_1, \ldots, q_P)$ in $\mathbb{R}^{P-1}$ such that $\sigma_{i,j} = |q_i - q_j|$ for all $i, j \in \{1, \ldots, P\}$, see Figure 4.1b.

We briefly comment on the previous definition.

Remark 4.9. *The above conditions on the matrix of surface tensions are natural, which is clear for the first two items, while condition iii) already appeared in [102] as being necessary for the existence of calibrations in the static case. It implies another coercivity condition in the form of the strict triangle inequality*

$$\sigma_{i,j} < \sigma_{i,k} + \sigma_{k,j} \tag{4.8}$$

for all choices of pairwise distinct $i, j, k \in \{1, \ldots, P\}$.

We call condition iii) of Definition 4.8 and condition (4.8) coercivity properties for the following reasons: First, the strict triangle inequality (4.8) will ensure that our relative entropy functional provides control on wetting, i.e., the nucleation of a thin layer of a third phase along the smooth part of an interface between two phases. Second, the embeddability condition iii) will prevent the nucleation of a fourth phase (or clusters of phases) at a triple junction.

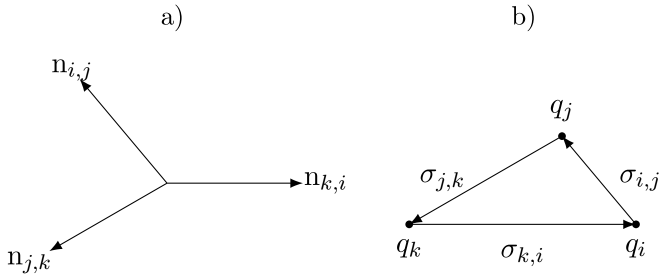

Figure 4.1: a) Normals $n_{i,j}$, $n_{j,k}$ and $n_{k,i}$ satisfying the balance-of-forces condition $\sigma_{i,j}n_{i,j} + \sigma_{j,k}n_{j,k} + \sigma_{k,i}n_{k,i} = 0$. b) Sketch of the points q_i, q_j and q_k of the l^2-embedding of σ.

It is well known, see [137, Section 3], that condition iii) of Definition 4.8 may be equivalently phrased as follows: The symmetric $(P \times P)$-matrix $Q = (\sigma_{i,j}^2)_{i,j=1,\dots,P}$ is strictly conditionally negative definite in the sense that

$$z \cdot Qz < 0 \quad \textit{for all } z \in \mathbb{R}^P \setminus \{0\} \textit{ with } \sum_{i=1}^{P} z_i = 0. \tag{4.9}$$

In a second step, we turn to the notion of partitions with finite interface energy.

Definition 4.10 (Partitions with finite interface energy, cf. [12])**.** *Let $d \geq 2$, let $P \geq 2$ be an integer and let $\sigma \in \mathbb{R}^{P\times P}$ be an admissible matrix of surface tensions in the sense of Definition 4.8. Let $(\Omega_1, \dots, \Omega_P)$ be a partition of $\mathbb{R}^d$ in the sense that for $i, j = 1, \dots, P$ with $i \neq j$ we have $\Omega_i \subset \mathbb{R}^d$ and the sets $\Omega_i \cap \Omega_j$ and $\mathbb{R}^d \setminus \bigcup_{i=1}^P \Omega_i$ have $\mathcal{L}^d$-measure zero. Let $\chi_i := \chi_{\Omega_i}$ denote the characteristic function of the $\mathcal{L}^d$-measurable set Ω_i for $i = 1, \dots, P$.*

We call $\chi = (\chi_1, \dots, \chi_P)$, or equivalently $(\Omega_1, \dots, \Omega_P)$, a partition of $\mathbb{R}^d$ with finite interface energy *if the energy*

$$E[\chi] := \sum_{i,j=1, i\neq j}^{P} \sigma_{i,j} \int_{\mathbb{R}^d} \frac{1}{2}\big(\mathrm{d}|\nabla\chi_i| + \mathrm{d}|\nabla\chi_j| - \mathrm{d}|\nabla(\chi_i{+}\chi_j)|\big) \tag{4.10}$$

is finite.

Note that for a partition of $\mathbb{R}^d$ with finite interface energy, each Ω_i is a set of finite perimeter. By introducing the interfaces $I_{i,j} := \partial^*\Omega_i \cap \partial^*\Omega_j$ as the intersection of the respective reduced boundaries, the energy of a partition χ can be rewritten in the equivalent form

$$E[\chi] = \sum_{i,j=1, i\neq j}^{P} \sigma_{i,j} \int_{I_{i,j}} 1 \,\mathrm{d}\mathcal{H}^{d-1}. \tag{4.11}$$

We next recall the notion of BV solutions to multiphase mean curvature flow, cf. [98, 99].

Definition 4.11 (BV solutions for multiphase mean curvature flow)**.** *Let $d \geq 2$ and $P \geq 2$ be integers. Let $\sigma \in \mathbb{R}^{P\times P}$ be an admissible matrix of surface tensions in the sense of Definition 4.8, and let $T_{\mathrm{BV}} > 0$ be a finite time horizon. Let $\chi_0 = (\chi_{0,1}, \dots, \chi_{0,P})$ be an initial partition of $\mathbb{R}^d$ with finite interface energy in the sense of Definition 4.10.*

A measurable map

$$\chi = (\chi_1, \dots, \chi_P)\colon \mathbb{R}^d \times [0, T_{\mathrm{BV}}) \to \{0,1\}^P,$$

or the corresponding tuple of sets $\Omega_i := \bigcup_{t\in[0,T_{\mathrm{BV}})} \Omega_i(t)\times\{t\}$, $\Omega_i(t) := \{\chi_i(t){=}1\}$ for $i \in \{1, \dots, P\}$ and $t \in [0, T_{\mathrm{BV}})$, is called a BV solution for multiphase mean curvature flow with initial data χ_0 *if the following conditions are satisfied:*

i) (Partition with finite interface energy) For almost every $t \in [0, T_{\mathrm{BV}})$, $\chi(\cdot,t)$ is a partition of $\mathbb{R}^d$ with finite interface energy in the sense of Definition 4.10 and

$$\operatorname*{ess\,sup}_{t\in[0,T_{\mathrm{BV}})} E[\chi(\cdot,t)] = \operatorname*{ess\,sup}_{t\in[0,T_{\mathrm{BV}})} \sum_{i,j=1,i\neq j}^{P} \sigma_{i,j} \int_{I_{i,j}(t)} 1 \,\mathrm{d}\mathcal{H}^{d-1} < \infty, \tag{4.12a}$$

where for all $t \in [0,T]$ we denote by $I_{i,j}(t) = \partial^*\Omega_i(t) \cap \partial^*\Omega_j(t)$ for $i \neq j$ the interface between the phases $\Omega_i(t)$ and $\Omega_j(t)$.

ii) (Evolution equation) For all $i \in \{1,\ldots,P\}$, there exist normal velocities $V_i \in L^2(\mathbb{R}^d \times [0,T_{\mathrm{BV}}), |\nabla\chi_i| \otimes \mathcal{L}^1)$ in the sense that each χ_i satisfies the evolution equation

$$\begin{aligned} &\int_{\mathbb{R}^d} \chi_i(\cdot,T)\varphi(\cdot,T)\,\mathrm{d}x - \int_{\mathbb{R}^d} \chi_{0,i}\varphi(\cdot,0)\,\mathrm{d}x \\ &= \int_0^T \int_{\mathbb{R}^d} V_i\varphi \,\mathrm{d}|\nabla\chi_i|\,\mathrm{d}t + \int_0^T \int_{\mathbb{R}^d} \chi_i \partial_t\varphi \,\mathrm{d}x\,\mathrm{d}t \end{aligned} \tag{4.12b}$$

for almost every $T \in [0,T_{\mathrm{BV}})$ and all $\varphi \in C^\infty_{\mathrm{cpt}}(\mathbb{R}^d \times [0,T_{\mathrm{BV}}))$. Moreover, the (reflection) symmetry condition $V_i\frac{\nabla\chi_i}{|\nabla\chi_i|} = V_j\frac{\nabla\chi_j}{|\nabla\chi_j|}$ shall hold $\mathcal{H}^{d-1}\otimes\mathcal{L}^1$-almost everywhere on $\bigcup_{t\in[0,T_{\mathrm{BV}})} I_{i,j}(t)\times\{t\}$, $i \neq j$.

iii) (BV formulation of mean curvature) The normal velocities satisfy the equation

$$\begin{aligned} &\sum_{i,j=1,i\neq j}^{P} \sigma_{i,j} \int_0^T \int_{I_{i,j}(t)} V_i \frac{\nabla\chi_i}{|\nabla\chi_i|}\cdot \mathrm{B}\,\mathrm{d}\mathcal{H}^{d-1}\,\mathrm{d}t \\ &= \sum_{i,j=1,i\neq j}^{P} \sigma_{i,j} \int_0^T \int_{I_{i,j}(t)} \Big(\mathrm{Id} - \frac{\nabla\chi_i}{|\nabla\chi_i|}\otimes\frac{\nabla\chi_i}{|\nabla\chi_i|}\Big) : \nabla\mathrm{B}\,\mathrm{d}\mathcal{H}^{d-1}\,\mathrm{d}t \end{aligned} \tag{4.12c}$$

for almost every $T \in [0,T_{\mathrm{BV}})$ and all $B \in C^\infty_{\mathrm{cpt}}(\mathbb{R}^d \times [0,T_{\mathrm{BV}});\mathbb{R}^d)$.

iv) (Energy dissipation inequality) The sharp energy dissipation inequality

$$E[\chi(\cdot,T)] + \sum_{i,j=1,i\neq j}^{P} \sigma_{i,j} \int_0^T \int_{I_{i,j}(t)} |V_i|^2 \,\mathrm{d}\mathcal{H}^{d-1}\,\mathrm{d}t \leq E[\chi_0] \tag{4.12d}$$

holds true for almost every $T \in [0,T_{\mathrm{BV}})$.

The same definition can be used to define a BV solution for multiphase mean curvature flow on $[0,T_{\mathrm{BV}}]$ for maps $\chi = (\chi_1,\ldots,\chi_P)\colon \mathbb{R}^d \times [0,T_{\mathrm{BV}}] \to \{0,1\}^P$.

Next, we give the definition of strong solutions to multiphase mean curvature flow. To this end, we first define a notion of regular partitions and regular networks of interfaces (cf. [110, Definitions 2.1, 2.7 and 4.7]).

Definition 4.12 (Regular partitions and networks of interfaces)**.** *Let $d = 2$, let $P \geq 2$ be an integer, and let $(\bar\Omega_1,\ldots,\bar\Omega_P)$ be a partition with finite interface energy of open subsets of $\mathbb{R}^2$ such that the closure of $\partial^*\bar\Omega_i$ is given by $\partial\bar\Omega_i$. Moreover, let $\bar\chi_i := \chi_{\bar\Omega_i}$ denote the characteristic function of the $\mathcal{L}^d$-measurable set $\bar\Omega_i$, and let $\bar I_{i,j} := \partial\bar\Omega_i \cap \partial\bar\Omega_j$ denote the respective interfaces for $i \neq j$.*

We call $\bar\chi = (\bar\chi_1,\ldots,\bar\chi_P)$, or equivalently $(\bar\Omega_1,\ldots,\bar\Omega_P)$, a regular partition of $\mathbb{R}^2$ *and $\mathcal{I} := \bigcup_{i\neq j} \bar I_{i,j}$* a regular network of interfaces in $\mathbb{R}^2$ *if the following properties are satisfied:*

i) (Regularity) Each interface $\bar{I}_{i,j}$ is a one-dimensional manifold with boundary of class C^5. The interior of each interface is embedded. Moreover, each interface $\bar{I}_{i,j}$ is compact and consists of finitely many connected components.

ii) (Multi-points are triple junctions) Only different interfaces may intersect, and if this is the case then only at their boundary. Moreover, at each intersection point exactly three interfaces meet. In other words, all multi-points of the network of interfaces are triple junctions.

iii) (Balance-of-forces condition) Let $p \in \mathbb{R}^2$ be a triple junction present in the network. Assume for notational concreteness that at the triple junction p, the three phases $\bar{\Omega}_i$, $\bar{\Omega}_j$ and $\bar{\Omega}_k$ meet. Then, the balance-of-forces condition.

$$\sigma_{i,j}\bar{\mathrm{n}}_{i,j}(p) + \sigma_{j,k}\bar{\mathrm{n}}_{j,k}(p) + \sigma_{k,i}\bar{\mathrm{n}}_{k,i}(p) = 0 \tag{4.13}$$

has to be satisfied, see Figure 4.1a. Here, $\bar{\mathrm{n}}_{i,j}(x)$ denotes the unit normal vector of the interface $\bar{I}_{i,j}$ at $x \in \bar{I}_{i,j}$ pointing from phase $\bar{\Omega}_i$ towards phase $\bar{\Omega}_j$.

iv) (Second- and third-order compatibility) We additionally have the second-order compatibility condition

$$\sigma_{i,j}H_{i,j}(p) + \sigma_{j,k}H_{j,k}(p) + \sigma_{k,i}H_{k,i}(p) = 0 \tag{4.14}$$

for the scalar mean curvatures $H_{i,j} := -\nabla^{\tan}\cdot\bar{\mathrm{n}}_{i,j}$, which is equivalent to the existence of a "velocity" vector $B(p) \in \mathbb{R}^2$ with $H_{l,m}(p) = \bar{\mathrm{n}}_{l,m}(p)\cdot B(p)$ for all distinct $l, m \in \{i, j, k\}$. For the choice of tangent vectors $\bar{\tau}_{i,j} := J^{-1}\bar{\mathrm{n}}_{i,j}$ with $J := \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, we furthermore have the third-order condition

$$\begin{aligned} \bar{\tau}_{i,j}(p)\cdot\left(H_{i,j}B + \nabla H_{i,j}\right)(p) &= \bar{\tau}_{j,k}(p)\cdot\left(H_{j,k}B + \nabla H_{j,k}\right)(p) \\ &= \bar{\tau}_{k,i}(p)\cdot\left(H_{k,i}B + \nabla H_{k,i}\right)(p). \end{aligned} \tag{4.15}$$

Here, we slightly abuse notation by denoting the tangential derivative of $H_{i,j}$ in direction $\bar{\tau}_{i,j}$ by $\bar{\tau}_{i,j}\cdot\nabla H_{i,j}$.

Let $\sigma \in \mathbb{R}^{P\times P}$ be an admissible matrix of surface tensions in the sense of Definition 4.8. We call $\bar{\chi} = (\bar{\chi}_1, \ldots, \bar{\chi}_P)$, or equivalently $(\bar{\Omega}_1, \ldots, \bar{\Omega}_P)$, a regular partition of $\mathbb{R}^2$ with finite interface energy, *if $\bar{\chi}$ satisfies*

$$E[\bar{\chi}] := \sum_{i,j=1, i\neq j}^{P} \sigma_{i,j}\int_{\bar{I}_{i,j}} 1\,\mathrm{d}S < \infty \tag{4.16}$$

in addition to the previous requirements.

Interpreting the triple junction as a free boundary of the interfaces, the identities (4.14) and (4.15) can be viewed as parabolic compatibility conditions: They arise from differentiating in time the zero-th order condition of p being the common endpoint of $\bar{I}_{i,j}$, $\bar{I}_{j,k}$, and $\bar{I}_{k,i}$; and the first-order condition (4.13) in time, respectively. Keeping in mind parabolic scaling, the condition (4.14) is indeed second order, while (4.15) is third order.

We say that a regular partition along with its associated regular network of interfaces evolves smoothly if no topological changes occur in the sense of the following definition:

Definition 4.13 (Smoothly evolving partitions and smoothly evolving networks of interfaces)**.** *Let $d = 2$, let $P \geq 2$ be an integer and let $\bar{\chi}_0 = (\bar{\chi}_1^0, \ldots, \bar{\chi}_P^0)$ be a regular partition of $\mathbb{R}^2$ with a regular network of interfaces $\mathcal{I}_0 = \bigcup_{i\neq j}\bar{I}_{i,j}^0$ in the sense of Definition 4.12. Let $T > 0$, and consider $\bar{\Omega}_i := \bigcup_{t\in[0,T]}\bar{\Omega}_i(t)\times\{t\}$, $i \in \{1, \ldots, P\}$, so that for*

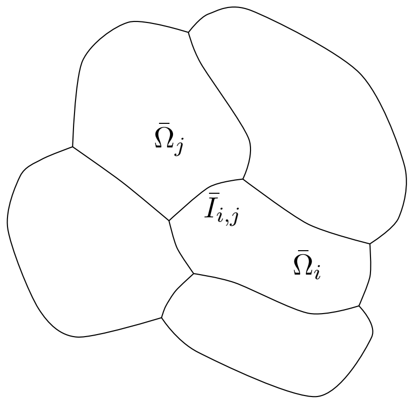

Figure 4.2: Sketch of a regular partition of the plane and the corresponding regular network.

all $t \in [0,T]$ *the family* $(\bar\Omega_1(t),\ldots,\bar\Omega_P(t))$ *is a regular partition of* $\mathbb{R}^2$ *in the sense of Definition 4.12. For each* $i \in \{1,\ldots,P\}$ *let* $\bar\chi_i\colon \mathbb{R}^2\times[0,T] \to \mathbb{R}^2$ *be the characteristic function of* $\bar\Omega_i$*, and for each pair* $i \neq j$ *with* $i,j \in \{1,\ldots,P\}$ *and all* $t \in [0,T]$ *define the interfaces* $\bar I_{i,j}(t) := \partial\bar\Omega_i(t) \cap \partial\bar\Omega_j(t)$*.*

We say that $\bar\chi = (\bar\chi_1,\ldots,\bar\chi_P)$*, or equivalently* $(\bar\Omega_1,\ldots,\bar\Omega_P)$*, is a* smoothly evolving regular partition of $\mathbb{R}^2\times[0,T]$ *and* $\mathcal{I} := \bigcup_{i,j\in\{1,\ldots,P\},i\neq j} \bar I_{i,j}$ *is a* smoothly evolving regular network of interfaces in $\mathbb{R}^2\times[0,T]$*, where* $\bar I_{i,j} := \bigcup_{t\in[0,T]} \bar I_{i,j}(t)\times\{t\}$ *for all* $i,j \in \{1,\ldots,P\}$ *with* $i \neq j$*, if there exists a time-dependent family of diffeomorphisms*

$$\psi^t\colon \mathbb{R}^2 \to \mathbb{R}^2, \quad t \in [0,T],$$

with the following properties:

i) $\psi^0 = \mathrm{Id}$, $\bar\chi_i(t) = \bar\chi_i^0 \circ \left(\psi^t\right)^{-1}$ *and* $\bar I_{i,j}(t) = \psi^t(\bar I_{i,j}^0)$ *for all* $i,j \in \{1,\ldots,P\}$ *with* $i \neq j$ *and all* $t \in [0,T]$*,*

ii) for all $i,j \in \{1,\ldots,P\}$ *with* $i \neq j$*, the map*

$$\psi_{i,j}\colon \bar I_{i,j}^0\times[0,T] \to \bar I_{i,j}, \quad (x,t) \to (\psi^t(x),t)$$

is a diffeomorphism of class $(C_t^0C_x^5 \cap C_t^1C_x^3)(\bar I_{i,j}^0\times[0,T])$*.*

We have everything in place to proceed with the definition of strong solutions for multiphase mean curvature flow.

Definition 4.14 (Strong solution for multiphase mean curvature flow)**.** *Let* $d = 2$*,* $P \geq 2$ *be an integer,* $\sigma \in \mathbb{R}^{P\times P}$ *be an admissible matrix of surface tensions in the sense of Definition 4.8, and let* $T_{\mathrm{strong}} > 0$ *be a finite time horizon. Let* $\bar\chi_0 = (\bar\chi_1^0,\ldots,\bar\chi_P^0)$ *be an initial regular partition of* $\mathbb{R}^2$ *with finite interface energy in the sense of Definition 4.12.*

A measurable map

$$\bar\chi = (\bar\chi_1,\ldots,\bar\chi_P)\colon \mathbb{R}^d \times [0,T_{\mathrm{strong}}) \to \{0,1\}^P,$$

or the corresponding tuple of sets $\bar\Omega_i := \bigcup_{t\in[0,T_{\mathrm{strong}})} \bar\Omega_i(t)\times\{t\}$*,* $\bar\Omega_i(t) := \{\bar\chi_i(t){=}1\}$ *for* $i \in \{1,\ldots,P\}$ *and* $t \in [0,T_{\mathrm{strong}})$*, is called a* strong solution for multiphase mean curvature flow with initial data $\bar\chi_0$ *if for all* $T \in [0,T_{\mathrm{strong}})$ *it is a* strong solution for multiphase mean curvature flow on $[0,T]$ *in the following sense:*

i) (Smoothly evolving regular partition with finite interface energy) The map $\bar\chi$ *is a smoothly evolving regular partition of* $\mathbb{R}^2\times[0,T]$ *and* $\mathcal{I} := \bigcup_{i,j\in\{1,\ldots,P\},i\neq j} \bar I_{i,j}$ *is a smoothly evolving regular network of interfaces in* $\mathbb{R}^2\times[0,T]$ *in the sense of Definition 4.13. In particular, for every* $t\in[0,T]$, $\bar\chi(\cdot,t)$ *is a regular partition of* $\mathbb{R}^2$ *and* $\bigcup_{i\neq j}\bar I_{i,j}(t)$ *is a regular network of interfaces in* $\mathbb{R}^2$ *in the sense of Definition 4.12 such that*

$$\sup_{t\in[0,T]} E[\bar\chi(\cdot,t)] = \sup_{t\in[0,T]} \sum_{i,j=1,i\neq j}^{P} \sigma_{i,j}\int_{\bar I_{i,j}(t)} 1\,\mathrm{d}S < \infty. \tag{4.17a}$$

ii) (Evolution by mean curvature) For $i,j=1,\ldots,P$ *with* $i\neq j$ *and* $(x,t)\in\bar I_{i,j}$ *let* $\bar V_{i,j}(x,t)$ *denote the normal speed of the interface at the point* $x\in\bar I_{i,j}(t)$, *i.e.,* $\bar V_{i,j}(x,t) := (\bar{\mathrm{n}}_{i,j}(x,t),0)\cdot\partial_t\psi_{i,j}(y,t)$ *at* $y=(\psi^t)^{-1}(x)\in\bar I_{i,j}(0)$, *where* $\psi_{i,j}$ *and* ψ^t *are the maps from Definition 4.13. Denoting by* $\mathrm{H}_{i,j}(x,t)$ *the mean curvature vector of* $\bar I_{i,j}(t)$ *at* $x\in\bar I_{i,j}(t)$, *we then assume that*

$$\bar V_{i,j}(x,t)\bar{\mathrm{n}}_{i,j}(x,t) = \mathrm{H}_{i,j}(x,t), \quad \text{for all } t\in[0,T],\ x\in\bar I_{i,j}(t). \tag{4.17b}$$

iii) (Initial conditions) We have $\bar\chi_i(x,0)=\bar\chi_{0,i}(x)$ *for all points* $x\in\mathbb{R}^d$ *and each phase* $i\in\{1,\ldots,P\}$.

4.1.4 Relative entropy inequality

The key ingredient for the proof of Proposition 4.3 is the derivation of a Gronwall-type inequality for the tilt-excess-like error functional (4.2): We aim to derive an estimate of the form

$$E[\chi|\xi](T) \le E[\chi|\xi](0) + C(\xi)\int_0^T E[\chi|\xi](t)\,\mathrm{d}t \tag{4.18}$$

for almost all admissible times $T\ge 0$ from which one may infer the desired stability estimate (4.3) by an application of Gronwall's lemma. The weak-strong uniqueness principle then follows by means of the coercivity properties of the relative entropy error functional (4.2) and a subsequent estimate for $E_{\mathrm{volume}}[\chi|\bar\chi]$, see Proposition 4.5. The following result contains the first key step in the derivation of the Gronwall-type inequality (4.18); it is valid for general vector fields ξ_i and B with sufficient smoothness (not just for gradient flow calibrations).

Proposition 4.15 (Relative entropy inequality)**.** *Let* $d\ge 2$, $P\ge 2$ *be integers, and let* $\sigma\in\mathbb{R}^{P\times P}$ *be an admissible matrix of surface tensions in the sense of Definition 4.8. Let* $\chi=(\chi_1,\ldots,\chi_P)$ *be a BV solution of multiphase mean curvature flow in the sense of Definition 4.11 on some time interval* $[0,T']$ *with* $T'>0$. *For* $i,j=1,\ldots,P$ *with* $i\neq j$ *we denote by*

$$\mathrm{n}_{i,j} := \frac{\nabla\chi_j}{|\nabla\chi_j|} = -\frac{\nabla\chi_i}{|\nabla\chi_i|}, \quad \mathcal{H}^{d-1}\text{-a.e. on } I_{i,j}, \tag{4.19}$$

the (measure-theoretic) unit normal vector of the interface $I_{i,j}$ *pointing from the* i*-th to the* j*-th phase of the* BV *solution. Moreover, let*

$$V_{i,j} := V_i = -V_j, \quad \mathcal{H}^{d-1}\text{-a.e. on } I_{i,j}. \tag{4.20}$$

Let $(\xi_{i,j})_{i\neq j\in\{1,\ldots,P\}}$ *and* $(\xi_i)_{i=1,\ldots,P}$ *be families of compactly supported vector fields such that*

$$\xi_{i,j},\xi_i \in C^1([0,T'];C^0_{\mathrm{cpt}}(\mathbb{R}^d;\mathbb{R}^d))\cap C^0([0,T'];C^1_{\mathrm{cpt}}(\mathbb{R}^d;\mathbb{R}^d))$$

as well as $\sigma_{i,j}\xi_{i,j} = \xi_i - \xi_j$ *for all* $i \neq j$*. Let*

$$B \in C^0([0,T'];C^1_{\mathrm{cpt}}(\mathbb{R}^d;\mathbb{R}^d))$$

be an arbitrary compactly supported vector field. Consistently with (4.2)*, define the interface error control*

$$E[\chi|\xi](t) := \sum_{i,j=1,i\neq j}^{P} \sigma_{i,j} \int_{I_{i,j}(t)} 1 - \xi_{i,j}(\cdot,t)\cdot \mathrm{n}_{i,j}(\cdot,t)\,\mathrm{d}\mathcal{H}^{d-1}. \tag{4.21}$$

Then the interface error control is subject to the estimate

$$\begin{aligned} &E[\chi|\xi](T) \\ &+ \sum_{i,j=1,i\neq j}^{P} \frac{\sigma_{i,j}}{2} \int_0^T \int_{I_{i,j}(t)} |V_{i,j}+\nabla\cdot\xi_{i,j}|^2 + |V_{i,j}\mathrm{n}_{i,j}-(B\cdot\xi_{i,j})\xi_{i,j}|^2\,\mathrm{d}\mathcal{H}^{d-1}\,\mathrm{d}t \\ &\leq E[\chi|\xi](0) + R_{\mathrm{dt}} + R_{\mathrm{dissip}} \end{aligned} \tag{4.22}$$

for almost every $T \in [0,T']$*. Here, we made use of the abbreviations*

$$\begin{aligned} R_{\mathrm{dt}} :=& - \sum_{i,j=1,i\neq j}^{P} \sigma_{i,j} \int_0^T \int_{I_{i,j}(t)} \frac{1}{2}\big(\partial_t|\xi_{i,j}|^2+(B\cdot\nabla)|\xi_{i,j}|^2\big)\,\mathrm{d}\mathcal{H}^{d-1}\,\mathrm{d}t \\ &- \sum_{i,j=1,i\neq j}^{P} \sigma_{i,j} \int_0^T \int_{I_{i,j}(t)} \big(\partial_t\xi_{i,j}+(B\cdot\nabla)\xi_{i,j}+(\nabla B)^{\mathsf{T}}\xi_{i,j}\big)\cdot(\mathrm{n}_{i,j}-\xi_{i,j})\,\mathrm{d}\mathcal{H}^{d-1}\,\mathrm{d}t, \\ R_{\mathrm{dissip}} :=& \sum_{i,j=1,i\neq j}^{P} \sigma_{i,j} \int_0^T \int_{I_{i,j}(t)} \frac{1}{2}|(\nabla\cdot\xi_{i,j}) + B\cdot\xi_{i,j}|^2\,\mathrm{d}\mathcal{H}^{d-1}\,\mathrm{d}t \\ &- \sum_{i,j=1,i\neq j}^{P} \sigma_{i,j} \int_0^T \int_{I_{i,j}(t)} \frac{1}{2}|B\cdot\xi_{i,j}|^2(1-|\xi_{i,j}|^2)\,\mathrm{d}\mathcal{H}^{d-1}\,\mathrm{d}t \\ &- \sum_{i,j=1,i\neq j}^{P} \sigma_{i,j} \int_0^T \int_{I_{i,j}(t)} (1-\mathrm{n}_{i,j}\cdot\xi_{i,j})(\nabla\cdot\xi_{i,j})(B\cdot\xi_{i,j})\,\mathrm{d}\mathcal{H}^{d-1}\,\mathrm{d}t \\ &+ \sum_{i,j=1,i\neq j}^{P} \sigma_{i,j} \int_0^T \int_{I_{i,j}(t)} \Big((\mathrm{Id}-\xi_{i,j}\otimes\xi_{i,j})B\Big)\cdot(V_{i,j}+\nabla\cdot\xi_{i,j})\mathrm{n}_{i,j}\,\mathrm{d}\mathcal{H}^{d-1}\,\mathrm{d}t \\ &+ \sum_{i,j=1,i\neq j}^{P} \sigma_{i,j} \int_0^T \int_{I_{i,j}(t)} (1-\mathrm{n}_{i,j}\cdot\xi_{i,j})(\nabla\cdot B)\,\mathrm{d}\mathcal{H}^{d-1}\,\mathrm{d}t \\ &- \sum_{i,j=1,i\neq j}^{P} \sigma_{i,j} \int_0^T \int_{I_{i,j}(t)} (\mathrm{n}_{i,j}-\xi_{i,j})\otimes(\mathrm{n}_{i,j}-\xi_{i,j}) : \nabla B\,\mathrm{d}\mathcal{H}^{d-1}\,\mathrm{d}t. \end{aligned}$$

4.1.5 Structure of the paper

The remaining part of the paper is organized as follows. Section 4.2 illustrates our strategy at the two most important examples, a smooth interface and a triple junction.

In Section 4.3, we prove the stability of calibrated flows and exploit all properties of our gradient flow calibrations and the *weak solution*: In Subsection 4.3.1 we derive the relative entropy inequality in its full generality of Proposition 4.15; and in Subsection 4.3.2, we prove the quantitative inclusion principle, Proposition 4.3. The latter is lifted to the conditional weak-strong uniqueness principle of Proposition 4.5 in Subsection 4.3.3.

The next three sections of the manuscript are devoted to the construction of our gradient flow calibrations given a *strong solution*. First, we provide explicit constructions at a smooth interface (Section 4.4) and at a triple junction (Section 4.5). These cases do not only serve as model examples but they also form the building blocks for our general construction in Section 4.6. Therein, we glue together these local constructions to obtain a gradient flow calibration for regular networks, which establishes Proposition 4.6.

Section 4.7 finally provides the construction of a family of transported weights given a strong solution.

4.2 Outline of the strategy

4.2.1 Idea of proof for a smooth interface

Let us give a brief idea of the proof, ignoring technical difficulties in the simplest case of two phases sharing one single interface with $\sigma = 1$. In that case, it is sufficient to describe the weak solution and the calibrated flow by a single phase $\Omega(t) \subset \mathbb{R}^d$, resp. $\bar\Omega(t) \subset \mathbb{R}^d$ for $t \in [0,T]$, the first being a set of finite perimeter and the second being a simply connected, smooth set. The relative entropy is then simply given by

$$E[\chi|\xi](t) = \int_{\partial^*\Omega(t)} (1 - \mathrm{n}\cdot\xi)\,\mathrm{d}\mathcal{H}^{d-1},$$

which has the interpretation of an oriented excess of the weak solution with respect to the strong one. Here $\chi = \chi(x,t)$ denotes the characteristic function of $\Omega = \Omega(t)$ and $\mathrm{n} = -\frac{\mathrm{d}\nabla\chi}{\mathrm{d}|\nabla\chi|}$ denotes the (measure theoretic) exterior unit normal of $\partial^*\Omega(t)$. Furthermore, the vector field $\xi(\cdot,t)$ is an extension of the exterior unit normal $\bar{\mathrm{n}}(\cdot,t)$ of the calibrated, smooth interface $\bar I(t) := \partial\bar\Omega$ necessitated by the fact that we evaluate it on the weak solution.

In order to relate the extension ξ to the evolution, we require it to be transported along an extension B of the velocity field of $\bar I$ in the sense that

$$\partial_t\xi = -(B\cdot\nabla)\,\xi - (\nabla B)^{\mathsf{T}}\xi + O\big(\operatorname{dist}(\cdot,\bar I\,)\big), \tag{4.23}$$

which will help make the second term of R_{dt} small (see Proposition 4.15 for the definition). The extension for B will be done such that it is constant in the "normal" ξ-direction, meaning we have $(\xi\cdot\nabla)B = 0$, and such that the motion law $\bar{\mathrm{n}}\cdot B = \bar V = H = -\nabla^{\tan}\cdot\bar{\mathrm{n}}$ is still approximately true away from the interface in the sense that

$$\xi\cdot B = -\nabla\cdot\xi + O\big(\operatorname{dist}(\cdot,\bar I\,)\big), \tag{4.24}$$

helping with the first term of R_{dissip}.

As we also want the functional $E[\chi|\xi]$ to ensure that χ cannot be too far away from $\bar\chi$, we allow for ξ to be short, i.e., we have $|\xi| \le 1$, and we ask this effect to be transported by B up to quadratic error

$$\partial_t|\xi|^2 + (B\cdot\nabla)|\xi|^2 = O\big(\operatorname{dist}^2(\cdot,\bar I\,)\big), \tag{4.25}$$

keeping the first term of R_{dt} small.

In the present case of a single interface, the construction of these vector fields is straightforward using the signed distance function $s = s(x,t)$ to the smooth interface $\bar I$: We set

$$\xi := \zeta(s)\nabla s \quad\text{and}\quad B := -(\Delta s)\xi,$$

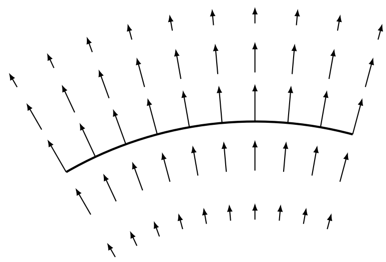

Figure 4.3: Illustration of the vector field ξ at a smooth interface $\bar{I}(t)$. The vector field ξ extends the unit normal vector field of $\bar{I}(t)$ by projection onto $\bar{I}(t)$ and multiplication with a cutoff function.

where ζ is a suitable cut-off function such that $\zeta(\tilde{s}) = 1 - \tilde{s}^2$ close to $\tilde{s} = 0$. Note that since $|\nabla s| = 1$, this implies

$$s^2 = 1 - \zeta(s) \leq 1 - \zeta(s)\,\mathrm{n}\cdot\nabla s = 1 - \mathrm{n}\cdot\xi \tag{4.26}$$

in the region where s is small, so that the relative entropy controls the (truncated) L^2 distance of the weak solution and the calibrated flow.

In the following heuristic derivation of the relative entropy inequality (from Proposition 4.15) in the case of a single interface, we will use the abbreviation $\int_{\partial^*\Omega}\cdot := \int_{\partial^*\Omega(t)}\cdot\,\mathrm{d}\mathcal{H}^{d-1}$ for the integral along a time slice $\partial^*\Omega(t)$, $t\in[0,T]$, of the weak solution. Recall that V denotes the normal velocity of the weak solution characterized by the distributional equation $\partial_t\chi = V\,|\nabla\chi|$, see (4.12b), so that the sign convention is $V>0$ for expanding Ω.

The optimal energy dissipation rate (4.12d) and the definition (4.12b) of V imply

$$\frac{\mathrm{d}}{\mathrm{d}t}E[\chi|\xi] = \frac{\mathrm{d}}{\mathrm{d}t}|\partial^*\Omega| - \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega(\nabla\cdot\xi)\,\mathrm{d}x \leq -\int_{\partial^*\Omega}V^2 - \int_{\partial^*\Omega}V\,(\nabla\cdot\xi) - \int_{\partial^*\Omega}\partial_t\xi\cdot\mathrm{n}.$$

Testing the distributional mean curvature flow equation (4.12c) with the extended velocity field B gives

$$0 = \int_{\partial^*\Omega}V\,(\mathrm{n}\cdot B) + \int_{\partial^*\Omega}(\mathrm{Id}-\mathrm{n}\otimes\mathrm{n}) : \nabla B.$$

Adding these terms to the right-hand side of the previous inequality yields

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}E[\chi|\xi] \leq &-\int_{\partial^*\Omega}\big(V^2 + V\,(\nabla\cdot\xi) - V\,(\mathrm{n}\cdot B)\big) + \int_{\partial^*\Omega}(\nabla\cdot B) - \int_{\partial^*\Omega}\mathrm{n}\otimes\mathrm{n}\colon\nabla B\\ &-\int_{\partial^*\Omega}\partial_t\xi\cdot\mathrm{n}.\end{aligned}$$

We now write $B = (\xi\cdot B)\,\xi + (\mathrm{Id}-\xi\otimes\xi)\,B$, which we interpret as a decomposition of B into "normal" and "tangential" parts. Then we complete the squares, and add and subtract $(B\cdot\nabla)\,\xi + (\nabla B)^{\mathsf{T}}\xi$ to make the transport equation for ξ appear. We obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}E[\chi|\xi] \leq &-\frac{1}{2}\int_{\partial^*\Omega}\Big((V+\nabla\cdot\xi)^2 + |V\mathrm{n} - (\xi\cdot B)\,\xi|^2\Big)\\ &+\frac{1}{2}\int_{\partial^*\Omega}\Big((\nabla\cdot\xi)^2 + |\xi|^2\,(\xi\cdot B)^2\Big) + \int_{\partial^*\Omega}V\mathrm{n}\cdot(\mathrm{Id}-\xi\otimes\xi)\,B\\ &+\int_{\partial^*\Omega}(\nabla\cdot B) - \int_{\partial^*\Omega}\mathrm{n}\otimes\mathrm{n}\colon\nabla B\\ &+\int_{\partial^*\Omega}\mathrm{n}\cdot(B\cdot\nabla)\,\xi + \int_{\partial^*\Omega}\xi\cdot(\mathrm{n}\cdot\nabla)\,B\\ &-\int_{\partial^*\Omega}\Big(\partial_t\xi + (B\cdot\nabla)\,\xi + (\nabla B)^{\mathsf{T}}\xi\Big)\cdot\mathrm{n},\end{aligned} \tag{4.27}$$

where the second line collects precisely the terms left after completing the squares.

By symmetry considerations, we have

$$
\begin{aligned}
0 &= \int_{\Omega} \nabla \cdot [\nabla \cdot (B \otimes \xi - \xi \otimes B)] \,\mathrm{d}x = \int_{\partial^*\Omega} [\nabla \cdot (B \otimes \xi - \xi \otimes B)] \cdot \mathrm{n} \\
&= \int_{\partial^*\Omega} [(\nabla \cdot \xi)\, \mathrm{n} \cdot B + \mathrm{n} \cdot (\xi \cdot \nabla)\, B - (\nabla \cdot B)\, \mathrm{n} \cdot \xi - \mathrm{n} \cdot (B \cdot \nabla)\, \xi]\,,
\end{aligned}
$$

where for the second line we used $(\xi \cdot \nabla)B = 0$. Now we use $|\xi| \leq 1$ to drop the prefactor $|\xi|^2$ of $(\xi \cdot B)^2$ in the second right-hand side integral in inequality (4.27), complete the square, add the above identity to obtain, and collect all terms involving ∇B

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} E[\chi|\xi] \leq &-\frac{1}{2} \int_{\partial^*\Omega} \Big((V + \nabla \cdot \xi)^2 + |V \mathrm{n} - (\xi \cdot B)\, \xi|^2 \Big) \\
&+ \frac{1}{2} \int_{\partial^*\Omega} (\nabla \cdot \xi + \xi \cdot B)^2 + \int_{\partial^*\Omega} (\nabla \cdot \xi)\, (\mathrm{n} - \xi) \cdot B \\
&+ \int_{\partial^*\Omega} V \mathrm{n} \cdot (\mathrm{Id} - \xi \otimes \xi)\, B + \int_{\partial^*\Omega} (1 - \mathrm{n} \cdot \xi)\, (\nabla \cdot B) \\
&- \int_{\partial^*\Omega} (\mathrm{n} - \xi) \otimes (\mathrm{n} - \xi) : \nabla B + \int_{\partial^*\Omega} \xi \otimes \xi \colon \nabla B \\
&- \int_{\partial^*\Omega} \Big(\partial_t \xi + (B \cdot \nabla)\, \xi + (\nabla B)^{\mathsf{T}} \xi \Big) \cdot \mathrm{n}.
\end{aligned}
$$

Once more, we decompose B into "tangential" and "normal" components with respect to ξ and manipulate the last integral to finally arrive at the entropy dissipation inequality

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} E[\chi|\xi] \leq &-\frac{1}{2} \int_{\partial^*\Omega} \Big((V + \nabla \cdot \xi)^2 + |V \mathrm{n} - (\xi \cdot B)\, \xi|^2 \Big) \\
&+ \frac{1}{2} \int_{\partial^*\Omega} (\nabla \cdot \xi + \xi \cdot B)^2 + \int_{\partial^*\Omega} (\nabla \cdot \xi)\, (\mathrm{n} \cdot \xi - 1)\, (\xi \cdot B) \\
&+ \int_{\partial^*\Omega} (\nabla \cdot \xi + V)\, \mathrm{n} \cdot (\mathrm{Id} - \xi \otimes \xi)\, B \\
&+ \int_{\partial^*\Omega} (1 - \mathrm{n} \cdot \xi)\, (\nabla \cdot B) - \int_{\partial^*\Omega} (\mathrm{n} - \xi) \otimes (\mathrm{n} - \xi) : \nabla B \\
&- \int_{\partial^*\Omega} \Big(\partial_t \xi + (B \cdot \nabla)\, \xi + (\nabla B)^{\mathsf{T}} \xi \Big) \cdot (\mathrm{n} - \xi) \\
&- \int_{\partial^*\Omega} (\partial_t \xi + (B \cdot \nabla)\, \xi) \cdot \xi.
\end{aligned}
$$

Now let us briefly argue term-by-term that the right-hand side can be controlled by the relative entropy $E[\chi|\xi]$, which together with a Gronwall argument and a subsequent estimate (4.7) of the bulk error would yield Theorem 4.1 for $P = 2$. Thanks to (4.24), the first term of the second line is quadratic in $\operatorname{dist}(\cdot, \bar{I})$ and therefore controlled by the relative entropy due to (4.26). The second integral of the second line is controlled by the relative entropy since $\nabla \cdot \xi$ and $\xi \cdot B$ are uniformly bounded; for the second term one can use (4.26). To handle the third line, we use Cauchy-Schwarz and Young, and absorb $\int (\nabla \cdot \xi + V)^2$ in the first integral. The remaining integral of $|(\mathrm{Id} - \xi \otimes \xi)\mathrm{n}|^2 = |\mathrm{n} - (\xi \cdot \mathrm{n})\xi|^2 \lesssim |\mathrm{n} - \xi|^2 + (1 - \mathrm{n} \cdot \xi)^2$ is controlled by the relative entropy. Clearly, both terms in the fourth line are controlled by the relative entropy. Finally, the integrals in the fifth and sixth lines are quadratic due to (4.23) and the factor $\mathrm{n} - \xi$, and (4.25), respectively.

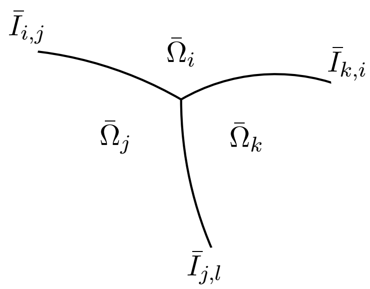

Figure 4.4: Sketch of a triple junction.

4.2.2 Idea of proof for a triple junction

The second model case is given by a triple junction, say, with equal surface tensions. To illustrate the additional difficulties, we also present the idea of our proof in this case. However, we restrict ourselves to the case $d = 2$.

We denote the phases of the *weak* solution by Ω_1, Ω_2, and Ω_3 with characteristic functions χ_1, χ_2, and χ_3. To simplify notation, we identify indices if they are equivalent mod 3, i. e., we define $\chi_4 := \chi_1$, $\chi_5 := \chi_2$, $\chi_0 := \chi_3$, and so on. Following the notation of Proposition 4.15, we denote the normal vector of the interface $I_{i,i+1} = \partial^*\Omega_i \cap \partial^*\Omega_{i+1}$ between phases i and $i+1$ for $i = 1, 2, 3$ in the weak solution by

$$\mathrm{n}_{i,i+1} := \frac{\mathrm{d}\nabla\chi_{i+1}}{\mathrm{d}|\nabla\chi_{i+1}|} = -\frac{\mathrm{d}\nabla\chi_i}{\mathrm{d}|\nabla\chi_i|} \qquad \mathcal{H}^1\text{-a. e. on } \partial^*\Omega_i \cap \partial^*\Omega_{i+1}.$$

The normal velocity of $I_{i,i+1}$, denoted by V_i, is characterized by the distributional identity $\partial_t\chi_i = V_i|\nabla\chi_i|$. Furthermore, we will consider its restriction $V_{i,i+1} := V_i|_{I_{i,i+1}}$ to the interface $I_{i,i+1}$ together with the symmetry condition $V_{i+1,i} := -V_{i,i+1}$. As before, the corresponding quantities in the *calibrated* solution will be indicated by an additional bar on top of the quantity, i.e., for example $\bar{\chi}_i$ for the indicator function of the corresponding phases, $\bar{\mathrm{n}}_{i,i+1}$ for the corresponding normal, and so on.

The first key step is to construct extensions $\xi_{i,i+1}$, $i = 1, 2, 3$, of the unit normal vector field $\bar{\mathrm{n}}_{i,i+1}$ of the *calibrated* interfaces $\bar{I}_{i,i+1}$. As in the case of a single interface, the extensions $\xi_{i,i+1}$ and the velocity field B are constructed to have the following properties:

- The time evolution of the vector fields $\xi_{i,i+1}$ is approximately described by transport along the flow of the velocity field B. More precisely, for the vector field B we have for $i = 1, 2, 3$ that

$$\partial_t\xi_{i,i+1} = -(B\cdot\nabla)\xi_{i,i+1} - (\nabla B)^{\mathsf{T}}\xi_{i,i+1} + O(\mathrm{dist}(\cdot, \bar{I}_{i,i+1})).$$

- On each interface $\bar{I}_{i,i+1}$, $i = 1, 2, 3$, of the calibrated solution, the normal part of the velocity field B must satisfy $\bar{\mathrm{n}}_{i,i+1}\cdot B = \bar{H}_{i,i+1} := -\nabla^{\tan}\cdot\bar{\mathrm{n}}_{i,i+1}$, where $\bar{H}_{i,i+1}$ is the scalar mean curvature of $\bar{I}_{i,i+1}$. We strengthen this identity to approximately hold even away from the interface, in form of

$$\xi_{i,i+1}\cdot B = -\nabla\cdot\xi_{i,i+1} + O(\mathrm{dist}(\cdot, \bar{I}_{i,i+1})) \qquad \text{for } i = 1, 2, 3.$$

- The vector fields $\xi_{i,i+1}$ have at most unit length $|\xi_{i,i+1}| \leq 1$.

- The length of the vector fields $\xi_{i,i+1}$ is advected with the flow of B to higher order

$$\partial_t|\xi_{i,i+1}|^2 = -(B\cdot\nabla)|\xi_{i,i+1}|^2 + O\big(\mathrm{dist}^2(\cdot, \bar{I}_{i,i+1})\big) \qquad \text{for } i = 1, 2, 3.$$

The new aspect of a triple junction as opposed to a single interface is that one also has to extend the normal of an interface to locations where a different interface may be closer. To this end, we turn to Herring's angle condition (4.13), which in our case of equal surface tensions says that the three interfaces must meet at the triple junction to form equal angles of 120° each, and require it to hold throughout the domain in the sense that

$$\sum_{i=1}^{3} \xi_{i,i+1}(x,t) = 0 \qquad \text{for all } x, t. \tag{4.28}$$

Furthermore, note carefully that we only define a single extension B of the velocity field, and that B is not necessarily a normal vector field on each interface $\bar{I}_{i,i+1}$: Indeed, we expect the triple junction $p(t)$ to move according to $\frac{\mathrm{d}}{\mathrm{d}t}p = B(p(t),t)$, so that not allowing for tangential components would pin the triple junction in space. It turns out that in addition to Herring's angle condition, which we take to be of first order, we require higher-order compatibility conditions of the interfaces at the triple junction. For instance, in part iv) of Definition 4.12 we have already seen that the second-order condition $H_{1,2}(p(t),t)+H_{2,3}(p(t),t)+H_{3,1}(p(t),t) = 0$ is equivalent to the existence of the vector $B(p(t),t)$.

To construct the extensions $\xi_{i,i+1}$ of the normal vector fields $\bar{\mathrm{n}}_{i,i+1}$, $i = 1,2,3$, we first partition space into six wedge-shaped sets around the triple junction: Three contain one strong interface each, while the remaining three wedges lie entirely within a single phase, see Figure 4.5a. On the mixed phase wedges, we first extend the corresponding normal by an expansion ansatz, see Figure 4.5b, and then define the remaining vector fields to satisfy the identity (4.28) by 120° rotations of the ansatz, see Figure 4.5c. On the single phase wedges, we will interpolate between the competing definitions of the two adjacent mixed phase wedges.

All rigorous discussions of compatibility will be deferred to Section 4.5, and we will only describe the initial extension procedure here. Let us fix $i = 1,2,3$. In fact, it is more instructive to first extend the velocity field B in the wedge-shaped neighborhood of the interface $\bar{I}_{i,i+1}$. To this end, we recall $\bar{\tau}_{i,i+1} = J^{-1}\bar{\mathrm{n}}_{i,i+1}$ on $\bar{I}_{i,i+1}$ with $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ from Definition 4.12 and use the extension ansatz

$$B := \bar{H}_{i,i+1}\bar{\mathrm{n}}_{i,i+1} + \alpha_{i,i+1}\bar{\tau}_{i,i+1} + \beta_{i,i+1}s_{i,i+1}\bar{\tau}_{i,i+1},$$

where $\bar{\mathrm{n}}_{i,i+1}$ and $\bar{\tau}_{i,i+1}$ are extended to be constant in the $\bar{\mathrm{n}}_{i,i+1}$-direction, $s_{i,i+1}$ is the signed distance function to $\bar{I}_{i,i+1}$ with the sign convention $\nabla s_{i,i+1} = \bar{\mathrm{n}}_{i,i+1}$, and $\alpha_{i,i+1}$ and $\beta_{i,i+1}$ are still to be determined. As $\frac{\mathrm{d}}{\mathrm{d}t}p(t) = B(p(t),t)$, it is reasonable that $\alpha_{i,i+1}(p(t),t) := \bar{\tau}_{i,i+1}(p(t),t) \cdot \frac{\mathrm{d}}{\mathrm{d}t}p(t)$ should be the tangential velocity of p at the triple junction. It turns out to be convenient to extend $\alpha_{i,i+1}$ along the interface $\bar{I}_{i,i+1}$ by means of the *ordinary* differential equation $(\bar{\tau}_{i,i+1} \cdot \nabla)\alpha_{i,i+1} = H^2_{i,i+1}$. In view of the third-order compatibility condition 4.15, the choice $\beta_{i,i+1}(x,t) := (\bar{\tau}_{i,i+1} \cdot \nabla)H_{i,i+1} + \alpha_{i,i+1}H_{i,i+1}$ for $x \in \bar{I}_{i,i+1}(t)$ is a good candidate to make B independent of i. To define $\alpha_{i,i+1}$ and $\beta_{i,i+1}$ away from the interface, we once again require them to be constant in $\bar{\mathrm{n}}_{i,i+1}$-direction.

It turns out that as the extension $\xi = \xi_{i,i+1}(x,t)$ of $\bar{\mathrm{n}}_{i,i+1}$ one should take

$$\xi = \bar{\mathrm{n}} + \alpha s\bar{\tau} - \tfrac{1}{2}\alpha^2 s^2\bar{\mathrm{n}} \tag{4.29}$$

where the functions $\alpha = \alpha_{i,i+1}(x,t)$ are as above and we dropped the indices $i, i+1$ for ease of notation. Note that in particular $\xi_{i,i+1} = \bar{\mathrm{n}}_{i,i+1}$ on the interface $\bar{I}_{i,i+1}$ and that we allow for linear corrections of the tangential component as we move away from the interface, but only for quadratic corrections of the normal component of ξ.

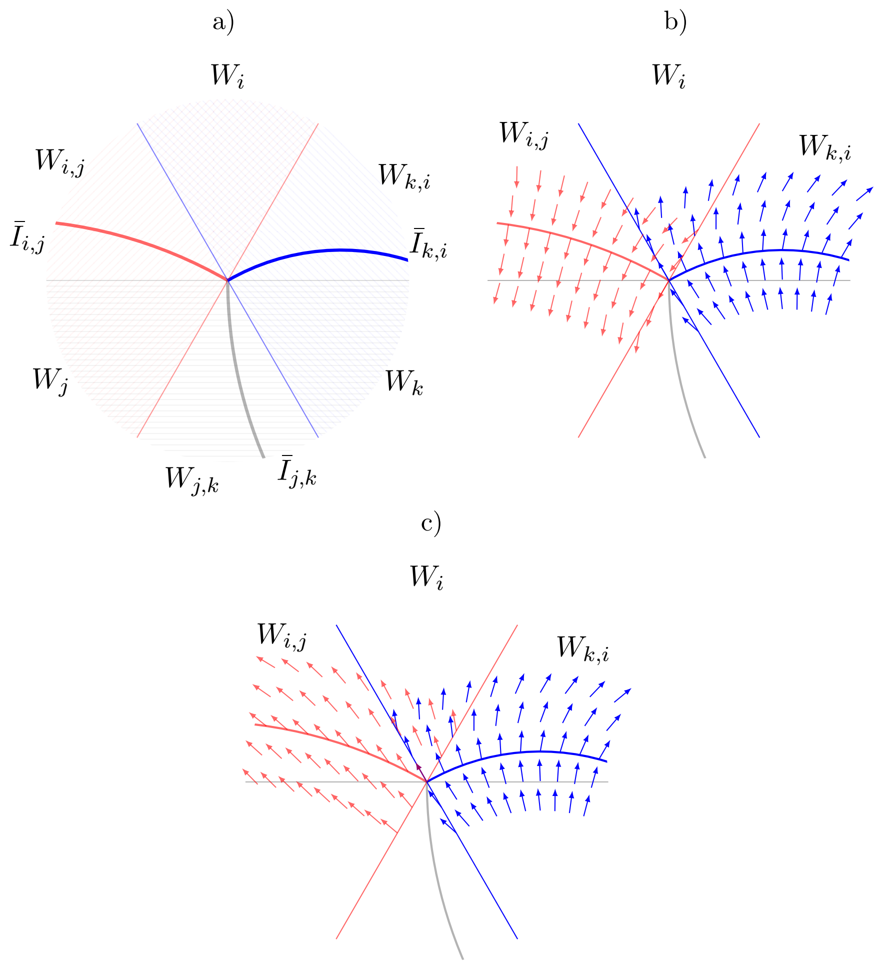

Figure 4.5: a) The gray, horizontally hatched domain is $\mathbb{H}_{j,k}$, the region hatched in red from the bottom left to the top right is $\mathbb{H}_{i,j}$, and $\mathbb{H}_{k,i}$ is shown hatched in blue from the top left to the bottom right. The simply hatched regions indicate the wedges $W_{i,j}$, $W_{j,k}$ and $W_{k,i}$ containing the interfaces $\bar{I}_{i,j}$, $\bar{I}_{j,k}$ and $\bar{I}_{k,i}$. The interpolation wedges W_i, W_j and W_k are shown as doubly hatched regions. b) Sketch of the initial extensions of $\bar{\mathrm{n}}_{k,i}$ in blue on the right and $\bar{\mathrm{n}}_{i,j}$ in red on the left, defined on $W_{k,i}$ and $W_{i,j}$, as well as the two respective neighboring interpolation wedges. c) The image shows the vector field $\bar{\mathrm{n}}_{k,i}$ (in blue on the right) and the rotated vector field $R\bar{\mathrm{n}}_{i,j}$ (in red on the left), where R is the clockwise rotation by 120°.

We then measure the error between the weak solution χ and the calibrated solution $\bar{\chi}$ by means of the relative entropy functional

$$E[\chi|\xi](t) := \sum_{i=1}^{3} \int_{I_{i,i+1}(t)} (1 - \mathrm{n}_{i,i+1} \cdot \xi_{i,i+1}) \, \mathrm{d}\mathcal{H}^1.$$

Let us use the abbreviation $\sum_i = \sum_{i=1}^3$ for the summation over the three relevant indices.

As in the two-phase case, we only use two ingredients to evaluate the time evolution of the relative entropy: the energy dissipation inequality for the weak solution in the sharp form

$$\frac{\mathrm{d}}{\mathrm{d}t} \sum_i \int_{I_{i,i+1}} 1 \, \mathrm{d}\mathcal{H}^1 \leq - \sum_{i=1}^{3} \int_{I_{i,i+1}} V_{i,i+1}^2 \, \mathrm{d}\mathcal{H}^1,$$

and the weak formulation of the evolution equation of the indicator functions χ_i

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^d}\chi_i\varphi\,\mathrm{d}x=\int_{\partial^*\Omega_i}V_i\varphi\;\mathrm{d}\mathcal{H}^1+\int_{\mathbb{R}^d}\chi_i\partial_t\varphi\,\mathrm{d}x$$

for compactly supported, smooth φ. In order to make use of the latter equation, we have to rewrite the contributions $\int_{I_{i,i+1}}\mathrm{n}_{i,i+1}\cdot\xi_{i,i+1}(x,t)$ as a volume integral. It turns out that the annihilation condition $\sum_i\xi_{i,i+1}(x,t)=0$ enables us to rewrite $\xi_{i,i+1}$ as

$$\xi_{i,i+1}=\xi_i-\xi_{i+1}\tag{4.30}$$

by defining the vector field ξ_i as $\xi_i:=\frac{1}{3}(\xi_{i,i+1}-\xi_{i-1,i})$. Combining (4.30) with the symmetry $\mathrm{n}_{i,i+1}=-\frac{\mathrm{d}\nabla\chi_i}{\mathrm{d}|\nabla\chi_i|}=\frac{\mathrm{d}\nabla\chi_{i+1}}{\mathrm{d}|\nabla\chi_{i+1}|}$ and the decomposition $\partial^*\Omega_i=I_{i-1,i}\cup I_{i,i+1}$, we rewrite the second term in the relative entropy as

$$\begin{aligned}-\sum_i\int_{I_{i,i+1}}\mathrm{n}_{i,i+1}\cdot\xi_{i,i+1}\,\mathrm{d}\mathcal{H}^1&=\sum_i\left(\int_{I_{i,i+1}}\xi_i\cdot\mathrm{d}\nabla\chi_i+\int_{I_{i,i+1}}\xi_{i+1}\cdot\mathrm{d}\nabla\chi_{i+1}\right)\\&=\sum_i\int_{\partial^*\Omega_i}\xi_i\cdot\mathrm{d}\nabla\chi_i\\&=-\sum_i\int_{\mathbb{R}^d}\chi_i(\nabla\cdot\xi_i)\,\mathrm{d}x.\end{aligned}$$

This indeed enables us to evaluate the time evolution of the relative entropy as

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}E[\chi|\xi]\le&-\sum_i\int_{I_{i,i+1}}V_{i,i+1}^2\,\mathrm{d}\mathcal{H}^1\\&-\sum_i\int_{\partial^*\Omega_i}V_i(\nabla\cdot\xi_i)\,\mathrm{d}\mathcal{H}^1+\sum_i\int_{\partial^*\Omega_i}\partial_t\xi_i\cdot\mathrm{d}\nabla\chi_i\,\mathrm{d}\mathcal{H}^1.\end{aligned}$$

Arguing analogously to the previous computation in reverse order—that is, splitting the integrals into contributions $\partial^*\Omega_i\cap\partial^*\Omega_{i+1}=I_{i,i+1}$, using (4.30) and the definitions of $\mathrm{n}_{i,i+1}$ and $V_{i,i+1}$—we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}E[\chi|\xi]\le-\sum_i\int_{I_{i,i+1}}V_{i,i+1}^2\,\mathrm{d}\mathcal{H}^1&-\sum_i\int_{I_{i,i+1}}V_{i,i+1}(\nabla\cdot\xi_{i,i+1})\,\mathrm{d}\mathcal{H}^1\\&-\sum_i\int_{I_{i,i+1}}\partial_t\xi_{i,i+1}\cdot\mathrm{n}_{i,i+1}\,\mathrm{d}\mathcal{H}^1.\end{aligned}$$

Now we proceed as in the two-phase case in the previous section: The BV formulation of mean curvature flow in this three-phase setting reads

$$\sum_i\int_{I_{i,i+1}}V_{i,i+1}\mathrm{n}_{i,i+1}\cdot B\,\mathrm{d}\mathcal{H}^1=-\sum_i\int_{I_{i,i+1}}(\mathrm{Id}-\mathrm{n}_{i,i+1}\otimes\mathrm{n}_{i,i+1}):\nabla B\,\mathrm{d}\mathcal{H}^1.$$

Following precisely the same algebraic manipulations as in the two-phase case we obtain

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}E[\chi|\xi]\\
&\leq -\frac{1}{2}\sum_i \int_{I_{i,i+1}} \Big((V_{i,i+1}+\nabla\cdot\xi_{i,i+1})^2 + |V_{i,i+1}\mathrm{n}_{i,i+1} - (\xi_{i,i+1}\cdot B)\,\xi_{i,i+1}|^2\Big)\,\mathrm{d}\mathcal{H}^1\\
&\quad + \frac{1}{2}\sum_i \int_{I_{i,i+1}} (\nabla\cdot\xi_{i,i+1}+\xi_{i,i+1}\cdot B)^2\,\mathrm{d}\mathcal{H}^1\\
&\quad + \sum_i \int_{I_{i,i+1}} (\nabla\cdot\xi_{i,i+1})\,(\mathrm{n}_{i,i+1}\cdot\xi_{i,i+1}-1)\,(\xi_{i,i+1}\cdot B)\,\mathrm{d}\mathcal{H}^1\\
&\quad + \sum_i \int_{I_{i,i+1}} (\nabla\cdot\xi_{i,i+1}+V_{i,i+1})\,\mathrm{n}_{i,i+1}\cdot(\mathrm{Id}-\xi_{i,i+1}\otimes\xi_{i,i+1})\,B\,\mathrm{d}\mathcal{H}^1\\
&\quad + \sum_i \int_{I_{i,i+1}} (1-\mathrm{n}_{i,i+1}\cdot\xi_{i,i+1})\,(\nabla\cdot B)\,\mathrm{d}\mathcal{H}^1\\
&\quad - \sum_i \int_{I_{i,i+1}} (\mathrm{n}_{i,i+1}-\xi_{i,i+1})\otimes(\mathrm{n}_{i,i+1}-\xi_{i,i+1}) : \nabla B\,\mathrm{d}\mathcal{H}^1\\
&\quad - \sum_i \int_{I_{i,i+1}} \Big(\partial_t\xi_{i,i+1}+(B\cdot\nabla)\,\xi_{i,i+1}+(\nabla B)^{\mathsf{T}}\xi_{i,i+1}\Big)\cdot(\mathrm{n}_{i,i+1}-\xi_{i,i+1})\,\mathrm{d}\mathcal{H}^1\\
&\quad - \sum_i \int_{I_{i,i+1}} (\partial_t\xi_{i,i+1}+(B\cdot\nabla)\,\xi_{i,i+1})\cdot\xi_{i,i+1}\,\mathrm{d}\mathcal{H}^1.
\end{aligned}
$$

With this inequality at our disposal we can conclude as in the two-phase case.

4.3 Stability of calibrated flows

This section is devoted to the proof of the stability properties of calibrated flows. In the next three subsections, we derive the relative entropy inequality Proposition 4.15 and the quantitative inclusion principle Proposition 4.3.

4.3.1 Relative entropy inequality: Proof of Proposition 4.15

We start with the proof of the relative entropy inequality for a BV solution $\chi = (\chi_1, \ldots, \chi_P)$ of multiphase mean curvature flow in the sense of Definition 4.11. The definition of the relative entropy functional $E[\chi|\xi]$ can be found in (4.21).

Proof of Proposition 4.15. In order to make use of the evolution equations (4.12b) for the indicator functions χ_i of the BV solution, we start by rewriting the interface error control of our relative entropy. Using $\sigma_{i,j}\xi_{i,j} = \xi_i - \xi_j$ from Definition 4.2 of a gradient flow calibration, the symmetry relation $\mathrm{n}_{i,j} = -\mathrm{n}_{j,i}$, the definition (4.19) of the measure theoretic normal as well as the representation of the energy (4.11), we obtain by an application of the generalized divergence theorem

$$
\begin{aligned}
&\sum_{i,j=1,i\neq j}^{P} \sigma_{i,j}\int_{I_{i,j}(T)} 1-\xi_{i,j}(\cdot,T)\cdot\mathrm{n}_{i,j}(\cdot,T)\,\mathrm{d}\mathcal{H}^{d-1}\\
&= E[\chi(\cdot,T)] - \sum_{i,j=1,i\neq j}^{P}\int_{I_{i,j}(T)} (\xi_i(\cdot,T)-\xi_j(\cdot,T))\cdot\mathrm{n}_{i,j}(\cdot,T)\,\mathrm{d}\mathcal{H}^{d-1}
\end{aligned}
$$

$$
\begin{aligned}
&= E[\chi(\cdot,T)] + \sum_{i=1}^{P}\sum_{j=1,j\neq i}^{P}\int_{I_{i,j}(T)} \xi_i(\cdot,T)\cdot\frac{\nabla\chi_i(\cdot,T)}{|\nabla\chi_i(\cdot,T)|}\,\mathrm{d}\mathcal{H}^{d-1}\\
&\quad+\sum_{j=1}^{P}\sum_{i=1,i\neq j}^{P}\int_{I_{i,j}(T)} \xi_j(\cdot,T)\cdot\frac{\nabla\chi_j(\cdot,T)}{|\nabla\chi_j(\cdot,T)|}\,\mathrm{d}\mathcal{H}^{d-1}\\
&= E[\chi(\cdot,T)] + 2\sum_{i=1}^{P}\int_{\mathbb{R}^d} \xi_i(\cdot,T)\cdot\frac{\nabla\chi_i(\cdot,T)}{|\nabla\chi_i(\cdot,T)|}\,\mathrm{d}|\nabla\chi_i(\cdot,T)|\\
&= E[\chi(\cdot,T)] - 2\sum_{i=1}^{P}\int_{\mathbb{R}^d}\chi_i(\cdot,T)(\nabla\cdot\xi_i(\cdot,T))\,\mathrm{d}x.
\end{aligned}
\tag{4.31}
$$

This enables us to compute by the sharp energy dissipation inequality (4.12d), the evolution equations (4.12b) for the indicator functions χ_i of the BV solution, and definition (4.20) of the velocities $V_{i,j}$ for almost every $T \in [0, T']$

$$
\begin{aligned}
&E[\chi|\xi](T)\\
&\leq E[\chi(\cdot,0)] - 2\sum_{i=1}^{P}\int_{\mathbb{R}^d}\chi_{0,i}(\nabla\cdot\xi_i(\cdot,0))\,\mathrm{d}x - \sum_{i,j=1,i\neq j}^{P}\sigma_{i,j}\int_0^T\int_{I_{i,j}(t)}|V_{i,j}|^2\,\mathrm{d}\mathcal{H}^{d-1}\,\mathrm{d}t\\
&\quad - 2\sum_{i=1}^{P}\int_0^T\int_{\mathbb{R}^d}\chi_i\partial_t(\nabla\cdot\xi_i)\,\mathrm{d}x\,\mathrm{d}t - 2\sum_{i=1}^{P}\int_0^T\int_{\mathbb{R}^d}V_i(\nabla\cdot\xi_i)\,\mathrm{d}|\nabla\chi_i|\,\mathrm{d}t.
\end{aligned}
$$

The first two terms combine to $E_{\mathrm{interface}}[\chi|\bar{\chi}](0)$ using (4.31) in reverse order. We aim to rewrite the latter two terms back to surface integrals over the interfaces as well. To this end, we argue analogously to the computation in (4.31) but now in reverse order. Using first the generalized divergence theorem, then splitting the integrals over the reduced boundaries of the phases into contributions over the interfaces $I_{i,j} = \partial^*\Omega_i \cap \partial^*\Omega_j$ by means of $\sigma_{i,j}\xi_{i,j} = \xi_i - \xi_j$ from Definition 4.2 of a gradient flow calibration we obtain

$$
\begin{aligned}
-2\sum_{i=1}^{P}\int_0^T\int_{\mathbb{R}^d}\chi_i\partial_t(\nabla\cdot\xi_i)\,\mathrm{d}x\,\mathrm{d}t &= 2\sum_{i=1}^{P}\int_0^T\int_{\mathbb{R}^d}\frac{\nabla\chi_i}{|\nabla\chi_i|}\cdot\partial_t\xi_i\,\mathrm{d}|\nabla\chi_i|\,\mathrm{d}t\\
&= \sum_{i=1}^{P}\sum_{j=1,j\neq i}^{P}\int_0^T\int_{I_{i,j}(t)}\frac{\nabla\chi_i}{|\nabla\chi_i|}\cdot\partial_t\xi_i\,\mathrm{d}\mathcal{H}^{d-1}\,\mathrm{d}t\\
&\quad+\sum_{j=1}^{P}\sum_{i=1,i\neq j}^{P}\int_0^T\int_{I_{i,j}(t)}\frac{\nabla\chi_j}{|\nabla\chi_j|}\cdot\partial_t\xi_j\,\mathrm{d}\mathcal{H}^{d-1}\,\mathrm{d}t\\
&\overset{(4.19)}{=} -\sum_{i,j=1,i\neq j}^{P}\int_0^T\int_{I_{i,j}(t)}\mathrm{n}_{i,j}\cdot\partial_t(\xi_i-\xi_j)\,\mathrm{d}\mathcal{H}^{d-1}\,\mathrm{d}t\\
&= -\sum_{i,j=1,i\neq j}^{P}\sigma_{i,j}\int_0^T\int_{I_{i,j}(t)}\mathrm{n}_{i,j}\cdot\partial_t\xi_{i,j}\,\mathrm{d}\mathcal{H}^{d-1}\,\mathrm{d}t.
\end{aligned}
$$

The term incorporating the normal velocities is treated similarly. In addition to the above ingredients, i.e., $\sigma_{i,j}\xi_{i,j} = \xi_i - \xi_j$ from Definition 4.2 of a gradient flow calibration and splitting the integrals over the reduced boundaries of the phases into contributions over the interfaces $I_{i,j} = \partial^*\Omega_i \cap \partial^*\Omega_j$, we also use that $V_{i,j} = -V_{j,i}$ on $\bar{I}_{i,j}$ together with definition (4.20) to compute

$$
\begin{aligned}
-2\sum_{i=1}^{P}\int_0^T\int_{\mathbb{R}^d} V_i(\nabla\cdot\xi_i)\,\mathrm{d}|\nabla\chi_i|\,\mathrm{d}t &= -\sum_{i=1}^{P}\sum_{j=1,j\neq i}^{P}\int_0^T\int_{I_{i,j}(t)} V_{i,j}(\nabla\cdot\xi_i)\,\mathrm{d}\mathcal{H}^{d-1}\,\mathrm{d}t \\
&\quad+\sum_{j=1}^{P}\sum_{i=1,i\neq j}^{P}\int_0^T\int_{I_{i,j}(t)} V_{i,j}(\nabla\cdot\xi_j)\,\mathrm{d}\mathcal{H}^{d-1}\,\mathrm{d}t \\
&= -\sum_{i,j=1,i\neq j}^{P}\sigma_{i,j}\int_0^T\int_{I_{i,j}(t)} V_{i,j}(\nabla\cdot\xi_{i,j})\,\mathrm{d}\mathcal{H}^{d-1}\,\mathrm{d}t.
\end{aligned}
$$

Combining the last two identities, we obtain for almost every $T\in[0,T']$

$$
\begin{aligned}
&E[\chi|\xi](T) \\
&\leq E[\chi|\xi](0) - \sum_{i,j=1,i\neq j}^{P}\sigma_{i,j}\int_0^T\int_{I_{i,j}(t)} |V_{i,j}|^2\,\mathrm{d}\mathcal{H}^{d-1}\,\mathrm{d}t \\
&\quad- \sum_{i,j=1,i\neq j}^{P}\sigma_{i,j}\int_0^T\int_{I_{i,j}(t)} \mathrm{n}_{i,j}\cdot\partial_t\xi_{i,j}\,\mathrm{d}\mathcal{H}^{d-1}\,\mathrm{d}t \\
&\quad- \sum_{i,j=1,i\neq j}^{P}\sigma_{i,j}\int_0^T\int_{I_{i,j}(t)} V_{i,j}(\nabla\cdot\xi_{i,j})\,\mathrm{d}\mathcal{H}^{d-1}\,\mathrm{d}t.
\end{aligned}
$$

For the next step, we use the vector field B as a test function in the BV formulation of mean curvature flow (4.12c). Adding the resulting equation to the previous inequality, observing in the process that $V_i\frac{\nabla\chi_i}{|\nabla\chi_i|} = -V_{i,j}\mathrm{n}_{i,j}$ on $I_{i,j}$ due to (4.19) and (4.20), as well as decomposing $B = (\mathrm{Id}-\xi_{i,j}\otimes\xi_{i,j})B + (B\cdot\xi_{i,j})\xi_{i,j}$, we obtain

$$
\begin{aligned}
&E[\chi|\xi](T) \\
&\leq E[\chi|\xi](0) - \sum_{i,j=1,i\neq j}^{P}\sigma_{i,j}\int_0^T\int_{I_{i,j}(t)} |V_{i,j}|^2\,\mathrm{d}\mathcal{H}^{d-1}\,\mathrm{d}t \qquad (4.32)\\
&\quad+ \sum_{i,j=1,i\neq j}^{P}\sigma_{i,j}\int_0^T\int_{I_{i,j}(t)} (B\cdot\xi_{i,j})\xi_{i,j}\cdot V_{i,j}\mathrm{n}_{i,j}\,\mathrm{d}\mathcal{H}^{d-1}\,\mathrm{d}t \\
&\quad- \sum_{i,j=1,i\neq j}^{P}\sigma_{i,j}\int_0^T\int_{I_{i,j}(t)} V_{i,j}(\nabla\cdot\xi_{i,j})\,\mathrm{d}\mathcal{H}^{d-1}\,\mathrm{d}t \\
&\quad+ \sum_{i,j=1,i\neq j}^{P}\sigma_{i,j}\int_0^T\int_{I_{i,j}(t)} (\mathrm{Id}-\xi_{i,j}\otimes\xi_{i,j})B\cdot V_{i,j}\mathrm{n}_{i,j}\,\mathrm{d}\mathcal{H}^{d-1}\,\mathrm{d}t \\
&\quad+ \sum_{i,j=1,i\neq j}^{P}\sigma_{i,j}\int_0^T\int_{I_{i,j}(t)} (\nabla\cdot B)\,\mathrm{d}\mathcal{H}^{d-1}\,\mathrm{d}t \\
&\quad- \sum_{i,j=1,i\neq j}^{P}\sigma_{i,j}\int_0^T\int_{I_{i,j}(t)} \mathrm{n}_{i,j}\otimes\mathrm{n}_{i,j} : \nabla B\,\mathrm{d}\mathcal{H}^{d-1}\,\mathrm{d}t \\
&\quad- \sum_{i,j=1,i\neq j}^{P}\sigma_{i,j}\int_0^T\int_{I_{i,j}(t)} \mathrm{n}_{i,j}\cdot\partial_t\xi_{i,j}\,\mathrm{d}\mathcal{H}^{d-1}\,\mathrm{d}t,
\end{aligned}
$$

which holds for almost every $T \in [0, T']$. In order to obtain the dissipation term on the left hand side of the relative entropy inequality (4.22), we complete the squares yielding for almost every $T \in [0, T']$

$$
\begin{aligned}
&- \sum_{i,j=1, i\neq j}^{P} \sigma_{i,j} \int_0^T \int_{I_{i,j}(t)} |V_{i,j}|^2 \,\mathrm{d}\mathcal{H}^{d-1}\,\mathrm{d}t \\
&+ \sum_{i,j=1, i\neq j}^{P} \sigma_{i,j} \int_0^T \int_{I_{i,j}(t)} (B\cdot\xi_{i,j})\xi_{i,j}\cdot V_{i,j}\mathrm{n}_{i,j} \,\mathrm{d}\mathcal{H}^{d-1}\,\mathrm{d}t \\
&- \sum_{i,j=1, i\neq j}^{P} \sigma_{i,j} \int_0^T \int_{I_{i,j}(t)} V_{i,j}(\nabla\cdot\xi_{i,j}) \,\mathrm{d}\mathcal{H}^{d-1}\,\mathrm{d}t \\
&= - \sum_{i,j=1, i\neq j}^{P} \sigma_{i,j} \int_0^T \int_{I_{i,j}(t)} \Big(\frac{1}{2}|V_{i,j}+\nabla\cdot\xi_{i,j}|^2 + \frac{1}{2}|V_{i,j}\mathrm{n}_{i,j}-(B\cdot\xi_{i,j})\xi_{i,j}|^2\Big) \,\mathrm{d}\mathcal{H}^{d-1}\,\mathrm{d}t \\
&\quad + \sum_{i,j=1, i\neq j}^{P} \sigma_{i,j} \int_0^T \int_{I_{i,j}(t)} \Big(\frac{1}{2}|\nabla\cdot\xi_{i,j}|^2 + \frac{1}{2}|(B\cdot\xi_{i,j})\xi_{i,j}|^2\Big) \,\mathrm{d}\mathcal{H}^{d-1}\,\mathrm{d}t.
\end{aligned} \tag{4.33}
$$

Furthermore, on the one hand, adding and subtracting $(B\cdot\nabla)\xi_{i,j} + (\nabla B)^{\mathsf{T}}\xi_{i,j}$ yields

$$
\begin{aligned}
&\sum_{i,j=1, i\neq j}^{P} \sigma_{i,j} \int_0^T \int_{I_{i,j}(t)} (\nabla\cdot B) \,\mathrm{d}\mathcal{H}^{d-1}\,\mathrm{d}t \\
&- \sum_{i,j=1, i\neq j}^{P} \sigma_{i,j} \int_0^T \int_{I_{i,j}(t)} \mathrm{n}_{i,j}\otimes\mathrm{n}_{i,j} : \nabla B \,\mathrm{d}\mathcal{H}^{d-1}\,\mathrm{d}t \\
&- \sum_{i,j=1, i\neq j}^{P} \sigma_{i,j} \int_0^T \int_{I_{i,j}(t)} \mathrm{n}_{i,j}\cdot\partial_t\xi_{i,j} \,\mathrm{d}\mathcal{H}^{d-1}\,\mathrm{d}t \\
&= \sum_{i,j=1, i\neq j}^{P} \sigma_{i,j} \int_0^T \int_{I_{i,j}(t)} (\nabla\cdot B) \,\mathrm{d}\mathcal{H}^{d-1}\,\mathrm{d}t \\
&\quad - \sum_{i,j=1, i\neq j}^{P} \sigma_{i,j} \int_0^T \int_{I_{i,j}(t)} (\mathrm{n}_{i,j}-\xi_{i,j})\cdot(\mathrm{n}_{i,j}\cdot\nabla)B \,\mathrm{d}\mathcal{H}^{d-1}\,\mathrm{d}t \\
&\quad + \sum_{i,j=1, i\neq j}^{P} \sigma_{i,j} \int_0^T \int_{I_{i,j}(t)} \big((B\cdot\nabla)\xi_{i,j}\big)\cdot\mathrm{n}_{i,j} \,\mathrm{d}\mathcal{H}^{d-1}\,\mathrm{d}t \\
&\quad - \sum_{i,j=1, i\neq j}^{P} \sigma_{i,j} \int_0^T \int_{I_{i,j}(t)} \big(\partial_t\xi_{i,j}+(B\cdot\nabla)\xi_{i,j}+(\nabla B)^{\mathsf{T}}\xi_{i,j}\big)\cdot\mathrm{n}_{i,j} \,\mathrm{d}\mathcal{H}^{d-1}\,\mathrm{d}t
\end{aligned} \tag{4.34}
$$

for almost every $T \in [0, T']$. On the other hand, we may exploit symmetry to obtain (relying again on the by now routine fact that one can switch back and forth between certain volume integrals and surface integrals over the individual interfaces by means of $\sigma_{i,j}\xi_{i,j} = \xi_i - \xi_j$ from Definition 4.2 of a gradient flow calibration, the symmetry relation $\mathrm{n}_{i,j} = -\mathrm{n}_{j,i}$ and the definition (4.19))

$$
\sum_{i,j=1, i\neq j}^{P} \sigma_{i,j} \int_0^T \int_{I_{i,j}(t)} \mathrm{n}_{i,j}\cdot\big(\nabla\cdot(B\otimes\xi_{i,j})\big) \,\mathrm{d}\mathcal{H}^{d-1}\,\mathrm{d}t
$$

$$
\begin{aligned}
&= \sum_{i,j=1,i\neq j}^{P} \int_0^T \int_{I_{i,j}(t)} \mathrm{n}_{i,j} \cdot \big(\nabla \cdot (B \otimes (\xi_i - \xi_j))\big) \,\mathrm{d}\mathcal{H}^{d-1} \,\mathrm{d}t \\
&= -2 \sum_{i=1}^{P} \int_0^T \int_{\mathbb{R}^d} \frac{\nabla \chi_i}{|\nabla \chi_i|} \cdot \big(\nabla \cdot (B \otimes \xi_i)\big) \,\mathrm{d}\mathcal{H}^{d-1} \,\mathrm{d}t \\
&= 2 \sum_{i=1}^{P} \int_0^T \int_{\mathbb{R}^d} \chi_i \nabla \cdot \big(\nabla \cdot (B \otimes \xi_i)\big) \,\mathrm{d}x \,\mathrm{d}t \\
&= 2 \sum_{i=1}^{P} \int_0^T \int_{\mathbb{R}^d} \chi_i \nabla \cdot \big(\nabla \cdot (\xi_i \otimes B)\big) \,\mathrm{d}x \,\mathrm{d}t \\
&= \sum_{i,j=1,i\neq j}^{P} \sigma_{i,j} \int_0^T \int_{I_{i,j}(t)} \mathrm{n}_{i,j} \cdot \big(\nabla \cdot (\xi_{i,j} \otimes B)\big) \,\mathrm{d}\mathcal{H}^{d-1} \,\mathrm{d}t.
\end{aligned}
$$

Because of this identity, we can now compute

$$
\begin{aligned}
0 &= \sum_{i,j=1,i\neq j}^{P} \sigma_{i,j} \int_0^T \int_{I_{i,j}(t)} \mathrm{n}_{i,j} \cdot \big(\nabla \cdot (B \otimes \xi_{i,j} - \xi_{i,j} \otimes B)\big) \,\mathrm{d}\mathcal{H}^{d-1} \,\mathrm{d}t \\
&= \sum_{i,j=1,i\neq j}^{P} \sigma_{i,j} \int_0^T \int_{I_{i,j}(t)} (\nabla \cdot \xi_{i,j}) B \cdot \mathrm{n}_{i,j} \,\mathrm{d}\mathcal{H}^{d-1} \,\mathrm{d}t \\
&\quad + \sum_{i,j=1,i\neq j}^{P} \sigma_{i,j} \int_0^T \int_{I_{i,j}(t)} \mathrm{n}_{i,j} \cdot (\xi_{i,j} \cdot \nabla) B \,\mathrm{d}\mathcal{H}^{d-1} \,\mathrm{d}t \\
&\quad - \sum_{i,j=1,i\neq j}^{P} \sigma_{i,j} \int_0^T \int_{I_{i,j}(t)} \mathrm{n}_{i,j} \cdot (B \cdot \nabla) \xi_{i,j} \,\mathrm{d}\mathcal{H}^{d-1} \,\mathrm{d}t \\
&\quad - \sum_{i,j=1,i\neq j}^{P} \sigma_{i,j} \int_0^T \int_{I_{i,j}(t)} (\nabla \cdot B) \xi_{i,j} \cdot \mathrm{n}_{i,j} \,\mathrm{d}\mathcal{H}^{d-1} \,\mathrm{d}t.
\end{aligned}
\tag{4.35}
$$

Making use of the identities (4.33) and (4.34) in the inequality (4.32) as well as adding (4.35) to the right hand side of (4.32), we arrive at the following bound for the time evolution of the interface error control of our relative entropy functional

$$
\begin{aligned}
&E[\chi|\xi](T) \\
&+ \sum_{i,j=1,i\neq j}^{P} \sigma_{i,j} \int_0^T \int_{I_{i,j}(t)} \Big(\frac{1}{2}|V_{i,j} + \nabla \cdot \xi_{i,j}|^2 + \frac{1}{2}|V_{i,j}\mathrm{n}_{i,j} - (B \cdot \xi_{i,j})\xi_{i,j}|^2\Big) \,\mathrm{d}\mathcal{H}^{d-1} \,\mathrm{d}t \\
&\leq E[\chi|\xi](0) \\
&\quad + \sum_{i,j=1,i\neq j}^{P} \sigma_{i,j} \int_0^T \int_{I_{i,j}(t)} \Big(\frac{1}{2}|\nabla \cdot \xi_{i,j}|^2 + \frac{1}{2}|(B \cdot \xi_{i,j})\xi_{i,j}|^2\Big) \,\mathrm{d}\mathcal{H}^{d-1} \,\mathrm{d}t \\
&\quad + \sum_{i,j=1,i\neq j}^{P} \sigma_{i,j} \int_0^T \int_{I_{i,j}(t)} (\nabla \cdot \xi_{i,j}) B \cdot \mathrm{n}_{i,j} \,\mathrm{d}\mathcal{H}^{d-1} \,\mathrm{d}t \\
&\quad + \sum_{i,j=1,i\neq j}^{P} \sigma_{i,j} \int_0^T \int_{I_{i,j}(t)} (\mathrm{Id} - \xi_{i,j} \otimes \xi_{i,j}) B \cdot V_{i,j}\mathrm{n}_{i,j} \,\mathrm{d}\mathcal{H}^{d-1} \,\mathrm{d}t \\
&\quad + \sum_{i,j=1,i\neq j}^{P} \sigma_{i,j} \int_0^T \int_{I_{i,j}(t)} (\nabla \cdot B)(1 - \xi_{i,j} \cdot \mathrm{n}_{i,j}) \,\mathrm{d}\mathcal{H}^{d-1} \,\mathrm{d}t
\end{aligned}
\tag{4.36}
$$

$$
\begin{aligned}
&- \sum_{i,j=1,i\neq j}^{P} \sigma_{i,j} \int_0^T \int_{I_{i,j}(t)} (\mathrm{n}_{i,j} - \xi_{i,j}) \otimes \mathrm{n}_{i,j} : \nabla B \,\mathrm{d}\mathcal{H}^{d-1}\,\mathrm{d}t \\
&+ \sum_{i,j=1,i\neq j}^{P} \sigma_{i,j} \int_0^T \int_{I_{i,j}(t)} \mathrm{n}_{i,j} \otimes \xi_{i,j} : \nabla B \,\mathrm{d}\mathcal{H}^{d-1}\,\mathrm{d}t \\
&- \sum_{i,j=1,i\neq j}^{P} \sigma_{i,j} \int_0^T \int_{I_{i,j}(t)} \big(\partial_t \xi_{i,j} + (B\cdot\nabla)\xi_{i,j} + (\nabla B)^\mathsf{T} \xi_{i,j}\big) \cdot \mathrm{n}_{i,j} \,\mathrm{d}\mathcal{H}^{d-1}\,\mathrm{d}t,
\end{aligned}
$$

which is valid for almost every $T \in [0, T']$. Completing squares and adding zero yields for the second, third and fourth term on the right hand side of (4.36)

$$
\begin{aligned}
&\sum_{i,j=1,i\neq j}^{P} \sigma_{i,j} \int_0^T \int_{I_{i,j}(t)} \Big(\frac{1}{2}|\nabla\cdot\xi_{i,j}|^2 + \frac{1}{2}|(B\cdot\xi_{i,j})\xi_{i,j}|^2\Big) \,\mathrm{d}\mathcal{H}^{d-1}\,\mathrm{d}t \\
&+ \sum_{i,j=1,i\neq j}^{P} \sigma_{i,j} \int_0^T \int_{I_{i,j}(t)} (\nabla\cdot\xi_{i,j}) B\cdot \mathrm{n}_{i,j} \,\mathrm{d}\mathcal{H}^{d-1}\,\mathrm{d}t \\
&+ \sum_{i,j=1,i\neq j}^{P} \sigma_{i,j} \int_0^T \int_{I_{i,j}(t)} (\mathrm{Id} - \xi_{i,j}\otimes\xi_{i,j}) B \cdot V_{i,j}\mathrm{n}_{i,j} \,\mathrm{d}\mathcal{H}^{d-1}\,\mathrm{d}t \\
&= \sum_{i,j=1,i\neq j}^{P} \sigma_{i,j} \int_0^T \int_{I_{i,j}(t)} \frac{1}{2}|(\nabla\cdot\xi_{i,j}) + B\cdot\xi_{i,j}|^2 \,\mathrm{d}\mathcal{H}^{d-1}\,\mathrm{d}t \qquad (4.37)\\
&\quad - \sum_{i,j=1,i\neq j}^{P} \sigma_{i,j} \int_0^T \int_{I_{i,j}(t)} \frac{1}{2}|B\cdot\xi_{i,j}|^2 (1 - |\xi_{i,j}|^2) \,\mathrm{d}\mathcal{H}^{d-1}\,\mathrm{d}t \\
&\quad + \sum_{i,j=1,i\neq j}^{P} \sigma_{i,j} \int_0^T \int_{I_{i,j}(t)} (\mathrm{Id} - \xi_{i,j}\otimes\xi_{i,j}) B \cdot (V_{i,j} + \nabla\cdot\xi_{i,j})\mathrm{n}_{i,j} \,\mathrm{d}\mathcal{H}^{d-1}\,\mathrm{d}t \\
&\quad - \sum_{i,j=1,i\neq j}^{P} \sigma_{i,j} \int_0^T \int_{I_{i,j}(t)} (1 - \mathrm{n}_{i,j}\cdot\xi_{i,j})(\nabla\cdot\xi_{i,j})(B\cdot\xi_{i,j}) \,\mathrm{d}\mathcal{H}^{d-1}\,\mathrm{d}t.
\end{aligned}
$$

Adding zero in the last term on the right hand side of (4.36) in order to generate the transport equation for the length of the vector fields $\xi_{i,j}$, we observe that the last three terms on the right hand side of (4.36) combine to

$$
\begin{aligned}
&- \sum_{i,j=1,i\neq j}^{P} \sigma_{i,j} \int_0^T \int_{I_{i,j}(t)} (\mathrm{n}_{i,j} - \xi_{i,j}) \otimes \mathrm{n}_{i,j} : \nabla B \,\mathrm{d}\mathcal{H}^{d-1}\,\mathrm{d}t \\
&+ \sum_{i,j=1,i\neq j}^{P} \sigma_{i,j} \int_0^T \int_{I_{i,j}(t)} \mathrm{n}_{i,j} \otimes \xi_{i,j} : \nabla B \,\mathrm{d}\mathcal{H}^{d-1}\,\mathrm{d}t \\
&- \sum_{i,j=1,i\neq j}^{P} \sigma_{i,j} \int_0^T \int_{I_{i,j}(t)} \big(\partial_t \xi_{i,j} + (B\cdot\nabla)\xi_{i,j} + (\nabla B)^\mathsf{T} \xi_{i,j}\big) \cdot \mathrm{n}_{i,j} \,\mathrm{d}\mathcal{H}^{d-1}\,\mathrm{d}t \\
&= - \sum_{i,j=1,i\neq j}^{P} \sigma_{i,j} \int_0^T \int_{I_{i,j}(t)} (\mathrm{n}_{i,j} - \xi_{i,j}) \otimes (\mathrm{n}_{i,j} - \xi_{i,j}) : \nabla B \,\mathrm{d}\mathcal{H}^{d-1}\,\mathrm{d}t \qquad (4.38)\\
&\quad - \sum_{i,j=1,i\neq j}^{P} \sigma_{i,j} \int_0^T \int_{I_{i,j}(t)} \big(\partial_t \xi_{i,j} + (B\cdot\nabla)\xi_{i,j} + (\nabla B)^\mathsf{T} \xi_{i,j}\big) \cdot (\mathrm{n}_{i,j} - \xi_{i,j}) \,\mathrm{d}\mathcal{H}^{d-1}\,\mathrm{d}t
\end{aligned}
$$

$$- \sum_{i,j=1,i\neq j}^{P} \sigma_{i,j} \int_0^T \int_{I_{i,j}(t)} \frac{1}{2}\big(\partial_t|\xi_{i,j}|^2 + (B\cdot\nabla)|\xi_{i,j}|^2\big)\,\mathrm{d}\mathcal{H}^{d-1}\,\mathrm{d}t.$$

Employing the notation of Proposition 4.15 as well as using (4.37) and (4.38) in (4.36), we deduce that the right hand side of (4.36) indeed reduces to

$$\begin{aligned}
&E[\chi|\xi](T)\\
&+ \sum_{i,j=1,i\neq j}^{P} \sigma_{i,j} \int_0^T \int_{I_{i,j}(t)} \Big(\frac{1}{2}|V_{i,j} + \nabla\cdot\xi_{i,j}|^2 + \frac{1}{2}|V_{i,j}\mathrm{n}_{i,j} - (B\cdot\xi_{i,j})\xi_{i,j}|^2\Big)\,\mathrm{d}\mathcal{H}^{d-1}\,\mathrm{d}t\\
&\leq E[\chi|\xi](0) + R_{\mathrm{dt}} + R_{\mathrm{dissip}},
\end{aligned}$$

which is valid for almost every $T \in [0, T']$. This concludes the proof of (4.22). □

4.3.2 Quantitative inclusion principle: Proof of Proposition 4.3

We now prove the inclusion principle stating that interfaces of any BV solution must be contained in the corresponding interfaces of a calibrated flow, provided both start with the same initial data.

Proof of Proposition 4.3. Step 1: The stability estimate (4.3). The starting point is the estimate on the evolution of the interface error functional (4.2) from Proposition 4.15. In the following, we estimate the terms appearing on the right hand side one-by-one. Let $T \in [0, T']$.

Due to (4.1c), (4.1d), as well as the trivial relation

$$|\mathrm{n}_{i,j} - \xi_{i,j}|^2 \leq 2(1 - \mathrm{n}_{i,j}\cdot\xi_{i,j}) \tag{4.39}$$

(which follows by $|\xi_{i,j}| \leq 1$), we immediately deduce

$$|R_{\mathrm{dt}}| \leq C \int_0^T E[\chi|\xi](t)\,\mathrm{d}t. \tag{4.40}$$

Making use of the simple estimate $1 - |\xi_{i,j}|^2 \leq 2(1 - |\xi_{i,j}|) \leq 2(1 - \mathrm{n}_{i,j}\cdot\xi_{i,j})$ and again the bound (4.39), we also obtain

$$\begin{aligned}
|R_{\mathrm{dissip}}| &\leq \sum_{i,j=1,i\neq j}^{P} \sigma_{i,j} \int_0^T \int_{I_{i,j}(t)} \frac{1}{2}|(\nabla\cdot\xi_{i,j}) + B\cdot\xi_{i,j}|^2\,\mathrm{d}\mathcal{H}^{d-1}\,\mathrm{d}t\\
&\quad + \sum_{i,j=1,i\neq j}^{P} \sigma_{i,j} \int_0^T \int_{I_{i,j}(t)} (\mathrm{Id} - \xi_{i,j}\otimes\xi_{i,j})B\cdot(V_{i,j} + \nabla\cdot\xi_{i,j})\mathrm{n}_{i,j}\,\mathrm{d}\mathcal{H}^{d-1}\,\mathrm{d}t\\
&\quad + C\int_0^T E[\chi|\xi](t)\,\mathrm{d}t\\
&=: I + II + C\int_0^T E[\chi|\xi](t)\,\mathrm{d}t.
\end{aligned}$$

By means of (4.1e), we may directly estimate

$$|I| \leq C \int_0^T E[\chi|\xi](t)\,\mathrm{d}t.$$

Furthermore, by an application of Hölder's and Young's inequality we deduce

$$
\begin{aligned}
|II| &= \Big| \sum_{i,j=1,i\neq j}^{P} \sigma_{i,j} \int_0^T \int_{I_{i,j}(t)} (\mathrm{Id}-\xi_{i,j}\otimes\xi_{i,j})B\cdot(V_{i,j}+\nabla\cdot\xi_{i,j})(\mathrm{n}_{i,j}-\xi_{i,j})\,\mathrm{d}\mathcal{H}^{d-1}\,\mathrm{d}t \Big| \\
&\leq \delta \sum_{i,j=1,i\neq j}^{P} \sigma_{i,j} \int_0^T \int_{I_{i,j}(t)} \frac{1}{2}(V_{i,j}+\nabla\cdot\xi_{i,j})^2\,\mathrm{d}\mathcal{H}^{d-1}\,\mathrm{d}t \\
&\quad + C\delta^{-1} \int_0^T E[\chi|\xi](t)\,\mathrm{d}t,
\end{aligned}
$$

uniformly over all $\delta\in(0,1)$. Hence, we get the bound

$$
\begin{aligned}
|R_{\mathrm{dissip}}| \leq \delta \sum_{i,j=1,i\neq j}^{P} \sigma_{i,j} \int_0^T \int_{I_{i,j}(t)} \frac{1}{2}(V_{i,j}+\nabla\cdot\xi_{i,j})^2\,\mathrm{d}\mathcal{H}^{d-1}\,\mathrm{d}t \\
+ C\delta^{-1} \int_0^T E[\chi|\xi](t)\,\mathrm{d}t.
\end{aligned}
\tag{4.41}
$$

Plugging in the bounds from (4.40) and (4.41) into the relative entropy inequality from Proposition 4.15, and then choosing $\delta\in(0,1)$ sufficiently small in order to absorb the first right-hand side term, we therefore get constants $C_1, C_2 > 0$ such that the estimate

$$
\begin{aligned}
&E[\chi|\xi](T) \\
&+ C_1 \sum_{i,j=1,i\neq j}^{P} \int_0^T \int_{I_{i,j}(t)} \Big(\frac{1}{2}(V_{i,j}+\nabla\cdot\xi_{i,j})^2 + \frac{1}{2}|V_{i,j}\mathrm{n}_{i,j}-(B\cdot\xi_{i,j})\xi_{i,j}|^2\Big)\,\mathrm{d}\mathcal{H}^{d-1}\,\mathrm{d}t \\
&\leq C_2 \int_0^T E[\chi|\xi](t)\,\mathrm{d}t
\end{aligned}
\tag{4.42}
$$

holds true for almost every $T\in[0,T']$. By an application of Gronwall's lemma, the asserted stability estimate (4.3) from Proposition 4.3 follows.

Step 3: Weak-strong comparison. For coinciding initial conditions $E[\chi|\xi](0) = 0$, the stability estimate (4.3) entails $E[\chi|\xi] = 0$ for almost every $t\in[0,T']$. From this and (4.1b), it immediately follows that $I_{i,j}(t)\subset \bar{I}_{i,j}(t)$ holds up to an $\mathcal{H}^{d-1}$-null set for almost every $t\in[0,T']$. This proves the quantitative inclusion principle for BV solutions of multiphase mean curvature flow. □

4.3.3 Conditional weak-strong uniqueness: Proof of Proposition 4.5

We start with an analogue of the relative entropy inequality of Proposition 4.15 in terms of the bulk error functional $E_{\mathrm{volume}}[\chi|\bar{\chi}]$ from (4.5).

Lemma 4.16. *Let $d\geq 2$, $P\geq 2$ be integers and $\sigma\in\mathbb{R}^{P\times P}$ be an admissible matrix of surface tensions in the sense of Definition 4.8. Let $\chi = (\chi_1,\ldots,\chi_P)$ be a BV solution of multiphase mean curvature flow in the sense of Definition 4.11 on some time interval $[0,T']$. Recall from (4.19) resp. (4.20) the definitions of the (measure-theoretic) unit normal vectors $\mathrm{n}_{i,j}$ resp. of the normal velocities $V_{i,j}$ of a BV solution. Let moreover $\bar{\Omega} = (\bar{\Omega}_1,\ldots,\bar{\Omega}_P)$ be a time-dependent partition of $\mathbb{R}^d$ with finite interface energy on $[0,T']$ as in Definition 4.4, and assume that there exists an associated family of transported weights $(\vartheta_i)_{i\in\{1,\ldots,P\}}$ with velocity field B. Finally, let $(\xi_{i,j})_{i\neq j\in\{1,\ldots,P\}}$ be a family of compactly supported vector fields such that*

$$\xi_{i,j}\in C^0([0,T'];C^1_{\mathrm{cpt}}(\mathbb{R}^d;\mathbb{R}^d)).$$

Then, the bulk error functional $E_{\mathrm{volume}}[\chi|\bar\chi]$ *from* (4.5) *is subject to the identity*

$$E_{\mathrm{volume}}[\chi|\bar\chi](T) = E_{\mathrm{volume}}[\chi|\bar\chi](0) + R_{\mathrm{volume}} \tag{4.43}$$

for almost every $T \in [0,T']$*. Here, we made use of the abbreviation*

$$\begin{aligned} R_{\mathrm{volume}} := &- \sum_{i,j=1,i\neq j}^{P} \int_0^T \int_{I_{i,j}(t)} \vartheta_i (B\cdot\xi_{i,j} - V_{i,j}) \,\mathrm{d}\mathcal{H}^{d-1}\,\mathrm{d}t \\ &- \sum_{i,j=1,i\neq j}^{P} \int_0^T \int_{I_{i,j}(t)} \vartheta_i B\cdot(\mathrm{n}_{i,j} - \xi_{i,j}) \,\mathrm{d}\mathcal{H}^{d-1}\,\mathrm{d}t \\ &+ \sum_{i=1}^{P} \int_0^T \int_{\mathbb{R}^d} (\chi_i - \bar\chi_i)\vartheta_i (\nabla\cdot B)\,\mathrm{d}x\,\mathrm{d}t \\ &+ \sum_{i=1}^{P} \int_0^T \int_{\mathbb{R}^d} (\chi_i - \bar\chi_i)(\partial_t\vartheta_i + (B\cdot\nabla)\vartheta_i)\,\mathrm{d}x\,\mathrm{d}t. \end{aligned}$$

Denote for $i,j \in \{1,\ldots,P\}$ *with* $i \neq j$ *and* $t \in [0,T']$ *by* $\bar I_{i,j}(t) := \partial\bar\Omega_i(t) \cap \partial\bar\Omega_j(t)$ *the interfaces associated with* $\bar\Omega$*. Then, the identity* (4.43) *may be upgraded to the estimate*

$$\begin{aligned} &E_{\mathrm{volume}}[\chi|\bar\chi](T) \\ &\leq E_{\mathrm{volume}}[\chi|\bar\chi](0) + \delta \sum_{i,j=1,i\neq j}^{P} \int_0^T \int_{I_{i,j}(t)} |B\cdot\xi_{i,j} - V_{i,j}|^2 \,\mathrm{d}\mathcal{H}^{d-1}\,\mathrm{d}t \qquad (4.44) \\ &\quad + \frac{C}{\delta} \sum_{i,j=1,i\neq j}^{P} \int_0^T \int_{I_{i,j}(t)} \mathrm{dist}^2(\cdot,\bar I_{i,j}) \wedge 1 \,\mathrm{d}\mathcal{H}^{d-1}\,\mathrm{d}t \\ &\quad + \frac{C}{\delta} \sum_{i,j=1,i\neq j}^{P} \int_0^T \int_{I_{i,j}(t)} 1 - \mathrm{n}_{i,j}\cdot\xi_{i,j} \,\mathrm{d}\mathcal{H}^{d-1}\,\mathrm{d}t \\ &\quad + C\sum_{i=1}^{P} \int_0^T E_{\mathrm{volume}}[\chi|\bar\chi](t)\,\mathrm{d}t \end{aligned}$$

valid for almost every $T \in [0,T']$*, all* $\delta \in (0,1]$ *and a constant* $C > 0$ *being independent of* δ*.*

Proof. We split the proof into two steps.

Proof of (4.43)*.* To compute the time evolution, note that the sign conditions on ϑ_i from Definition 4.4 of a family of transported weights is precisely what is needed to have

$$E_{\mathrm{volume}}[\chi|\bar\chi](T) = \sum_{i=1}^{P} \int_{\mathbb{R}^d} (\chi_i(\cdot,T) - \bar\chi_i(\cdot,T))\vartheta_i(\cdot,T)\,\mathrm{d}x.$$

Hence, we may make use of the evolution equations (4.12b) for the indicator functions χ_i of the BV solution which together with $\partial_t\bar\chi_i \ll |\nabla\bar\chi_i|$ and $\vartheta_i = 0$ on $\mathrm{supp}\,|\nabla\bar\chi_i|$ (see Definition 4.4) yields for almost every $T \in [0,T']$

$$\begin{aligned} &E_{\mathrm{volume}}[\chi|\bar\chi](T) \\ &= E_{\mathrm{volume}}[\chi|\bar\chi](0) + \sum_{i=1}^{P} \int_0^T \int_{\mathbb{R}^d} (\chi_i - \bar\chi_i)\partial_t\vartheta_i\,\mathrm{d}x\,\mathrm{d}t + \sum_{i=1}^{P} \int_0^T \int_{\mathbb{R}^d} V_i\vartheta_i\,\mathrm{d}|\nabla\chi_i|\,\mathrm{d}t. \end{aligned}$$

We next use the convention (4.20) and rewrite

$$\sum_{i=1}^{P}\int_0^T\int_{\mathbb{R}^d} V_i\vartheta_i \,\mathrm{d}|\nabla\chi_i|\,\mathrm{d}t = \sum_{i,j=1,i\neq j}^{P}\int_0^T\int_{I_{i,j}(t)} \vartheta_i V_{i,j}\,\mathrm{d}\mathcal{H}^1\,\mathrm{d}t.$$

Furthermore, by adding and subtracting $(B\cdot\nabla)\vartheta_i$, an integration by parts, $\vartheta_i = 0$ on $\operatorname{supp}|\nabla\bar\chi_i|$ (see Definition 4.4), and the definition (4.19) of the measure theoretic unit normal, we obtain

$$\begin{aligned}
&\sum_{i=1}^{P}\int_0^T\int_{\mathbb{R}^d}(\chi_i-\bar\chi_i)\partial_t\vartheta_i\,\mathrm{d}x\,\mathrm{d}t\\
&= -\sum_{i=1}^{P}\int_0^T\int_{\mathbb{R}^d}(\chi_i-\bar\chi_i)(B\cdot\nabla)\vartheta_i\,\mathrm{d}x\,\mathrm{d}t + \sum_{i=1}^{P}\int_0^T\int_{\mathbb{R}^d}(\chi_i-\bar\chi_i)(\partial_t\vartheta_i+(B\cdot\nabla)\vartheta_i)\,\mathrm{d}x\,\mathrm{d}t\\
&= -\sum_{i=1}^{P}\int_0^T\int_{\mathbb{R}^d}(\chi_i-\bar\chi_i)\nabla\cdot(\vartheta_i B)\,\mathrm{d}x\,\mathrm{d}t + \sum_{i=1}^{P}\int_0^T\int_{\mathbb{R}^d}(\chi_i-\bar\chi_i)\vartheta_i(\nabla\cdot B)\,\mathrm{d}x\,\mathrm{d}t\\
&\quad+\sum_{i=1}^{P}\int_0^T\int_{\mathbb{R}^d}(\chi_i-\bar\chi_i)(\partial_t\vartheta_i+(B\cdot\nabla)\vartheta_i)\,\mathrm{d}x\,\mathrm{d}t\\
&= \sum_{i=1}^{P}\int_0^T\int_{\mathbb{R}^d}\frac{\nabla\chi_i}{|\nabla\chi_i|}\cdot\vartheta_i B\,\mathrm{d}|\nabla\chi_i|\,\mathrm{d}t + \sum_{i=1}^{P}\int_0^T\int_{\mathbb{R}^d}(\chi_i-\bar\chi_i)\vartheta_i(\nabla\cdot B)\,\mathrm{d}x\,\mathrm{d}t\\
&\quad+\sum_{i=1}^{P}\int_0^T\int_{\mathbb{R}^d}(\chi_i-\bar\chi_i)(\partial_t\vartheta_i+(B\cdot\nabla)\vartheta_i)\,\mathrm{d}x\,\mathrm{d}t\\
&= -\sum_{i,j=1,i\neq j}^{P}\int_0^T\int_{I_{i,j}(t)}\vartheta_i B\cdot\xi_{i,j}\,\mathrm{d}\mathcal{H}^1\,\mathrm{d}t\\
&\quad-\sum_{i,j=1,i\neq j}^{P}\int_0^T\int_{I_{i,j}(t)}\vartheta_i B\cdot(\mathrm{n}_{i,j}-\xi_{i,j})\,\mathrm{d}\mathcal{H}^1\,\mathrm{d}t\\
&\quad+\sum_{i=1}^{P}\int_0^T\int_{\mathbb{R}^d}(\chi_i-\bar\chi_i)\vartheta_i(\nabla\cdot B)\,\mathrm{d}x\,\mathrm{d}t\\
&\quad+\sum_{i=1}^{P}\int_0^T\int_{\mathbb{R}^d}(\chi_i-\bar\chi_i)(\partial_t\vartheta_i+(B\cdot\nabla)\vartheta_i)\,\mathrm{d}x\,\mathrm{d}t
\end{aligned}$$

for almost every $T\in[0,T']$. The combination of the previous three displays thus proves (4.43) as asserted.

Step 2: Proof of (4.44). Starting point is of course (4.43) meaning that we need to estimate the term R_{volume}. First, we may infer based on the bound (4.4) on the advective derivative of the transported weights ϑ_i, the bound $|B|\leq C$ (see Definition 4.4), Hölder's and Young's inequality as well as the bound (4.39) that the estimate

$$\begin{aligned}
|R_{\mathrm{volume}}| &\leq \delta\sum_{i,j=1,i\neq j}^{P}\int_0^T\int_{I_{i,j}(t)}|B\cdot\xi_{i,j}-V_{i,j}|^2\,\mathrm{d}\mathcal{H}^{d-1}\,\mathrm{d}t\\
&\quad+\frac{C}{\delta}\sum_{i,j=1,i\neq j}^{P}\int_0^T\int_{I_{i,j}(t)}\vartheta_i^2\,\mathrm{d}\mathcal{H}^{d-1}\,\mathrm{d}t
\end{aligned}$$

$$+\frac{C}{\delta}\sum_{i,j=1,i\neq j}^{P}\int_0^T\int_{I_{i,j}(t)}1-\mathrm{n}_{i,j}\cdot\xi_{i,j}\,\mathrm{d}\mathcal{H}^{d-1}\,\mathrm{d}t$$
$$+C\sum_{i=1}^{P}\int_0^T E_{\mathrm{volume}}[\chi|\bar\chi](t)\,\mathrm{d}t$$

holds true, uniformly over all $\delta\in(0,1)$. As $\vartheta_i=0$ on $\operatorname{supp}|\nabla\bar\chi_i|$, $\vartheta_i\in W^{1,\infty}_{x,t}(\mathbb{R}^d\times[0,T'];[-1,1])$ and $\partial\bar\Omega_i$ is Lipschitz (see Definition 4.4), we may further estimate

$$\vartheta_i^2\leq C(\mathrm{dist}^2(\cdot,\partial\bar\Omega_i)\wedge 1)\leq C(\mathrm{dist}^2(\cdot,\bar I_{i,j})\wedge 1)$$

for all phases $i,j\in\{1,\ldots,P\}$ with $i\neq j$. This, however, concludes the proof. □

We have everything in place to lift the quantitative inclusion principle from Proposition 4.3 to the conditional weak-strong uniqueness principle of Proposition 4.5 (with an associated conditional stability estimate).

Proof of Proposition 4.5. The stability estimate (4.6) concerning the interface error is already a consequence of Proposition 4.3 (for which one only needs to assume the existence of a gradient flow calibration $((\xi_i)_{i\in\{1,\ldots,P\}},B)$ with respect to $\bar\Omega$). Recall from (4.1b) that $\mathrm{dist}^2(\cdot,\bar I_{i,j})\wedge 1\leq C(1-|\xi_{i,j}|)$ for all $i,j\in\{1,\ldots,P\}$ with $i\neq j$. Inserting this into the corresponding right hand side term of (4.44), adding the estimate (4.42) from the proof of Proposition 4.3 to the estimate (4.44), and choosing $\delta\in(0,1]$ in (4.44) sufficiently small then entails

$$E[\chi|\xi](T)+E_{\mathrm{volume}}[\chi|\bar\chi](T)\leq C\int_0^T E[\chi|\xi](t)+E_{\mathrm{volume}}[\chi|\bar\chi](t)\,\mathrm{d}t$$

for almost every $T\in[0,T']$. The stability estimate (4.7) for the bulk error is now a direct consequence of Gronwall's lemma.

It remains to prove the conditional weak-strong uniqueness statement. To this end, note first that $\chi(\cdot,0)=\bar\chi(\cdot,0)$ almost everywhere in $\mathbb{R}^d$ entails $E[\chi|\xi](0)=0$ and $E_{\mathrm{volume}}[\chi|\bar\chi](0)=0$ as a consequence of the respective definitions (4.2) and (4.5). In view of the stability estimate (4.7), this directly implies $E_{\mathrm{volume}}[\chi|\bar\chi](T)=0$ for almost every $T\in[0,T']$. It then follows from the coercivity properties of a family of transported weights (see Definition 4.4) that $\chi(\cdot,T)=\bar\chi(\cdot,T)$ almost everywhere in $\mathbb{R}^d$ for almost every $T\in[0,T']$. This, however, is the desired weak-strong uniqueness principle. □

4.4 Gradient flow calibrations at a smooth manifold

The aim of this section is to construct a gradient flow calibration in the simple situation of one single connected manifold (with or without boundary) evolving by mean curvature, see Lemma 4.18 for the main result of this section. For the sake of simplicity, we stick to the case $d=2$, but the construction in this section immediately carries over to arbitrary dimensions.

In terms of a gradient flow calibration for a whole network of interfaces in the sense of Definition 4.2, the vector fields constructed in Lemma 4.18 provide the local building block at a smooth two-phase interface of the network. These vector fields therefore only live in a small tubular neighborhood of the evolving interface, so that in the case of general networks a suitable localization in terms of a family of cutoff functions will be necessary. We defer these considerations to Section 4.6.1.

First, we provide the precise setting of this section by giving a suitable notion of neighborhood for a single space-time connected component of the evolving network of interfaces.

Definition 4.17. *Let $d = 2$ and $P \in \mathbb{N}$, $P \geq 2$. Let $(\bar\Omega_1, \ldots, \bar\Omega_P)$ be a strong solution to multiphase mean curvature flow in the sense of Definition 4.14. Fix phases $i, j \in \{1, \ldots, P\}$ with $i \neq j$ such that $\bar I_{i,j} = \bigcup_{t\in[0,T]} \bar I_{i,j}(t) \times \{t\}$ is a non-trivial interface (possibly with boundary). A scale $r_{i,j} \in (0, 1]$ is called an* admissible localization radius *for the interface $\bar I_{i,j}$ if for all $t \in [0, T]$ the following two ball conditions are satisfied:*

i) For each interior point $x \in \bar I_{i,j}(t)$ it holds $\overline{B_{r_{i,j}}(x \pm r_{i,j}\bar{\mathrm{n}}_{i,j}(x, t))} \cap \bar I_{i,j}(t) = \{x\}$.

ii) In addition, for a boundary point $x \in \partial \bar I_{i,j}(t)$ (i.e., a triple junction) denote by $\bar{\mathrm{t}}_{i,j}(x, t)$ the tangent at x pointing away from the curve $\bar I_{i,j}(t)$, and by $\mathbb{H}_{\bar{\mathrm{t}}_{i,j}}(x, t)$ the half-space $\{y \in \mathbb{R}^2 : (y - x) \cdot \bar{\mathrm{t}}_{i,j}(x, t) > 0\}$. We then require that $\overline{B_{r_{i,j}}(y)} \cap \bar I_{i,j}(t) = \{x\}$ for all $y \in \partial B_{r_{i,j}}(x) \cap \overline{\mathbb{H}_{\bar{\mathrm{t}}_{i,j}}(x, t)}$.

It follows from our regularity requirements in Definition 4.14 that an admissible localization radius always exists. Moreover,

$$\Psi_{i,j} : \bar I_{i,j} \times (-r_{i,j}, r_{i,j}) \to \mathbb{R}^2 \times [0, T], \quad (x, t, s) \mapsto (x + s\bar{\mathrm{n}}_{i,j}(x, t), t) \tag{4.45}$$

defines a bijective map onto its image

$$\begin{aligned} \mathrm{im}(\Psi_{i,j}) &:= \Psi_{i,j}(\bar I_{i,j} \times (-r_{i,j}, r_{i,j})) \\ &= \bigcup_{t\in[0,T]} \bigg(\big\{ \mathrm{dist}(\cdot, \bar I_{i,j}(t)) < r_{i,j} \big\} \setminus \bigcup_{x\in\partial \bar I_{i,j}(t)} \big(\mathbb{H}_{\bar{\mathrm{t}}_{i,j}}(x, t) \cap B_{r_{i,j}}(x)\big) \bigg) \times \{t\}, \end{aligned} \tag{4.46}$$

and the inverse map is a diffeomorphism of class $(C^0_t C^4_x \cap C^1_t C^2_x)(\overline{\mathrm{im}(\Psi_{i,j})})$. We may further split the inverse of the diffeomorphism (4.45) as follows:

$$\Psi^{-1}_{i,j} : \mathrm{im}(\Psi_{i,j}) \to \bar I_{i,j} \times (-r_{i,j}, r_{i,j}), \quad (x, t) \mapsto \big(P_{i,j}(x, t), t, s_{i,j}(x, t)\big)$$

where the map $s_{i,j} : \mathrm{im}(\Psi_{i,j}) \to (-r_{i,j}, r_{i,j})$ represents a signed distance function

$$s_{i,j}(x, t) := \begin{cases} \mathrm{dist}(x, \bar I_{i,j}(t)), & (x, t) \in \Psi_{i,j}\big(\bar I_{i,j} \times [0, r_{i,j})\big), \\ -\mathrm{dist}(x, \bar I_{i,j}(t)), & (x, t) \in \Psi_{i,j}\big(\bar I_{i,j} \times (-r_{i,j}, 0)\big), \end{cases} \tag{4.47}$$

and the map $P_{i,j} : \mathrm{im}(\Psi_{i,j}) \to \bigcup_{t\in[0,T]} \bar I_{i,j}(t)$ represents in each time slice the projection onto the nearest point on the interface in the sense that

$$P_{i,j}(x, t) := P_{\bar I_{i,j}(t)}(x) = \operatorname*{arg\,min}_{y\in \bar I_{i,j}(t)} |y - x|, \quad (x, t) \in \mathrm{im}(\Psi_{\bar I_{i,j}}). \tag{4.48}$$

Note that we have the identity

$$P_{i,j}(x, t) = x - s_{i,j}(x, t)\bar{\mathrm{n}}_{i,j}\big(P_{i,j}(x, t), t\big) \in \bar I_{i,j}(t), \quad (x, t) \in \mathrm{im}(\Psi_{i,j}). \tag{4.49}$$

As a consequence of our regularity assumptions on $\bar I_{i,j}$, see again Definition 4.14, we also know that (for the former, one may consult Lemma 4.19 below)

$$s_{i,j} \in (C^0_t C^5_x \cap C^1_t C^3_x)(\overline{\mathrm{im}(\Psi_{i,j})}), \quad P_{i,j} \in (C^0_t C^4_x \cap C^1_t C^2_x)(\overline{\mathrm{im}(\Psi_{i,j})}). \tag{4.50}$$

We may now introduce extensions of the unit normal $\bar{\mathrm{n}}_{i,j}$ and the scalar mean curvature $H_{i,j}$ (oriented with respect to $\bar{\mathrm{n}}_{i,j}$) of the interface $\bar I_{i,j}$ to the space-time domain $\mathrm{im}(\Psi_{i,j})$. Slightly abusing notation, we define

$$\bar{\mathrm{n}}_{i,j} : \mathrm{im}(\Psi_{i,j}) \to \mathbb{S}^1, \quad (x, t) \mapsto \nabla s_{i,j}(x, t), \tag{4.51}$$

$$H_{i,j} : \mathrm{im}(\Psi_{i,j}) \to \mathbb{R}, \quad (x, t) \mapsto (-\Delta s_{i,j})(P_{i,j}(x, t), t). \tag{4.52}$$

We register as a consequence of the definitions that

$$\bar{\mathrm{n}}_{i,j} \in (C_t^0 C_x^4 \cap C_t^1 C_x^2)(\overline{\mathrm{im}(\Psi_{i,j})}), \quad H_{i,j} \in (C_t^0 C_x^3 \cap C_t^1 C_x^1)(\overline{\mathrm{im}(\Psi_{i,j})}). \tag{4.53}$$

The following result provides a (two-phase version of a) gradient flow calibration for a single *connected* interface. Note that the velocity field B can accomodate arbitrary tangential components, a fact we will exploit when constructing a velocity field for general networks in Section 4.6.

Lemma 4.18. *Let $d = 2$ and $P \in \mathbb{N}$, $P \geq 2$. Let $(\bar{\Omega}_1, \ldots, \bar{\Omega}_P)$ be a strong solution to multiphase mean curvature flow in the sense of Definition 4.14. Fix $i, j \in \{1, \ldots, P\}$ with $i \neq j$ such that $\bar{I}_{i,j} = \bigcup_{t\in[0,T]} \bar{I}_{i,j}(t)\times\{t\}$ is a non-trivial interface. Let $r_{i,j} \in (0,1]$ be an admissible localization radius for $\bar{I}_{i,j}$ in the sense of Definition 4.17. Fix a space-time connected component (of which there are finitely many) $\mathcal{T} = \bigcup_{t\in[0,T]} \mathcal{T}(t)\times\{t\} \subset \bar{I}_{i,j}$ of the interface $\bar{I}_{i,j}$. Denote by $\Psi_{\mathcal{T}}$ the restriction of the diffeomorphism (4.45) to $\mathcal{T}\times(-r_{i,j}, r_{i,j})$, and its image by $\mathrm{im}(\Psi_{\mathcal{T}}) := \Psi_{\mathcal{T}}(\mathcal{T}\times(-r_{i,j}, r_{i,j}))$.*

Let $\alpha \in C_t^0 C_x^2(\overline{\mathrm{im}(\Psi_{\mathcal{T}})})$ be an arbitrary map, and define the tangent vector field

$$\bar{\tau}_{i,j} := J^{\mathsf{T}} \bar{\mathrm{n}}_{i,j} \colon \mathrm{im}(\Psi_{i,j}) \to \mathbb{S}^1 \in (C_t^0 C_x^4 \cap C_t^1 C_x^2)(\overline{\mathrm{im}(\Psi_{i,j})}) \tag{4.54}$$

where J denotes the counter-clockwise rotation by $90°$. Then the vector fields $\xi_{i,j} \colon \mathrm{im}(\Psi_{\mathcal{T}}) \to \mathbb{S}^1$ and $B \colon \mathrm{im}(\Psi_{\mathcal{T}}) \to \mathbb{R}^2$ given by

$$\xi_{i,j} := \bar{\mathrm{n}}_{i,j}, \tag{4.55}$$

$$B := H_{i,j}\bar{\mathrm{n}}_{i,j} + \alpha\bar{\tau}_{i,j} \tag{4.56}$$

satisfy $\xi_{i,j} \in (C_t^0 C_x^4 \cap C_t^1 C_x^2)(\overline{\mathrm{im}(\Psi_{\mathcal{T}})})$, $B \in C_t^0 C_x^2(\overline{\mathrm{im}(\Psi_{\mathcal{T}})})$, with corresponding quantitative estimates

$$r_{i,j}^k |\nabla^k \xi_{i,j}| \leq C, \qquad k \in \{0, 1, \ldots, 4\}, \tag{4.57}$$

$$r_{i,j}^{k+2} |\partial_t \nabla^k \xi_{i,j}| \leq C, \qquad k \in \{0, 1, 2\}, \tag{4.58}$$

$$r_{i,j}^k |\nabla^k B| \leq C r_{i,j}^{-1} + C \sum_{l=0}^{k} r_{i,j}^l |\nabla^l \alpha|, \qquad k \in \{0, 1, 2\}, \tag{4.59}$$

throughout the space-time domain $\mathrm{im}(\Psi_{\mathcal{T}})$. Moreover, it holds

$$\partial_t s_{i,j} + (B \cdot \nabla) s_{i,j} = 0, \tag{4.60}$$

$$\partial_t \xi_{i,j} + (B \cdot \nabla)\xi_{i,j} + (\nabla B)^{\mathsf{T}} \xi_{i,j} = 0, \tag{4.61}$$

$$\xi_{i,j} \cdot \partial_t \xi_{i,j} + \xi_{i,j} \cdot (B \cdot \nabla)\xi_{i,j} = 0, \tag{4.62}$$

$$|B \cdot \xi_{i,j} + \nabla \cdot \xi_{i,j}| \leq C r_{i,j}^{-2} \operatorname{dist}(\cdot, \bar{I}_{i,j}) \tag{4.63}$$

throughout the space-time domain $\mathrm{im}(\Psi_{\mathcal{T}})$. The constant in the estimates (4.57), (4.59) and (4.63) is independent of $r_{i,j}$.

Proof. For ease of notation, we omit all indices, superscripts, and arguments for the rest of the proof unless specifically required otherwise. Since Ψ represents in each time slice a tubular neighborhood diffeomorphism on scale $r \in (0,1]$, we have $\max_{k=0,\ldots,5} r^k |\nabla^k s| \leq Cr$ throughout $\mathrm{im}(\Psi)$. From the definitions (4.51), (4.54), (4.52) and (4.49), we then deduce $\max_{k=0,\ldots,4} r^k(|\nabla^k \bar{\mathrm{n}}| + |\nabla^k \bar{\tau}| + |\nabla^k P|) \leq C$ and $\max_{k=0,\ldots,3} r^k |\nabla^k H| \leq Cr^{-1}$. Due to (4.64) and (4.66), it holds $\partial_t s = -H$. Hence, we obtain the bounds $\max_{k=0,\ldots,3} r^{k+2} |\partial_t \nabla^k s| \leq Cr$, $\max_{k=0,1,2} r^k(|\partial_t \nabla^k \bar{\mathrm{n}}| + |\partial_t \nabla^k \bar{\tau}| + |\partial_t \nabla^k P|) \leq C$ and finally $\max_{k=0,1} r^{k+2} |\partial_t \nabla^k H| \leq Cr^{-1}$. The estimates (4.57)–(4.59) now directly follow from the definitions (4.55)–(4.56).

It follows from (4.64) and (4.66) below, as well as from the orthogonality $\bar\tau\cdot\bar{\mathrm{n}}=0$ that the tangential term in the definition of B does not have an effect on the transport equation (4.64) for the signed distance s, i.e., we have

$$\partial_t s = -\big(H\bar{\mathrm{n}}\cdot\nabla\big)s = -\big(B\cdot\nabla\big)s.$$

We may take the gradient of this identity so that by definition of ξ we have

$$\partial_t \xi = \nabla\partial_t s = -\big(B\cdot\nabla\big)\xi - \big(\nabla B\big)^{\mathsf{T}}\xi,$$

which proves (4.61). The validity of (4.62) is evident from the fact that $|\xi|^2\equiv 1$. For the identity (4.63), note first that $B\cdot\xi=\bar{\mathrm{n}}\cdot\xi=H$ as a consequence of the orthogonality $\bar\tau\cdot\bar{\mathrm{n}}=0$. By definition (4.51) and definition (4.55), it holds $\nabla\cdot\xi=\Delta s$. Hence, $B\cdot\xi=H=-\nabla\cdot\xi+O(r^{-2}\operatorname{dist}(\cdot,\bar I))$ in view of the definition (4.53) and the regularity estimates for the signed distance. This concludes the proof. ☐

The preceding result relies on a number of well-known properties of the signed distance and the nearest point projection. For further reference, we present them here in a separate statement.

Lemma 4.19. *Let $d=2$ and $P\in\mathbb{N}$, $P\geq 2$. Let $(\bar\Omega_1,\ldots,\bar\Omega_P)$ be a strong solution to multiphase mean curvature flow in the sense of Definition 4.14. Fix $i,j\in\{1,\ldots,P\}$ with $i\neq j$ such that $\bar I_{i,j}=\bigcup_{t\in[0,T]}\bar I_{i,j}(t)\times\{t\}$ is a non-trivial interface. Let $r_{i,j}\in(0,1]$ be an admissible localization radius for $\bar I_{i,j}$ in the sense of Definition 4.17.*

Then $s_{i,j}\in(C^0_tC^5_x\cap C^1_tC^3_x)(\overline{\operatorname{im}(\Psi_{i,j})})$. The time evolution of the signed distance $s_{i,j}$ is moreover given by transport along the flow of the mean curvature vector field in the sense that we have

$$\partial_t s_{i,j} = -\big(H_{i,j}\bar{\mathrm{n}}_{i,j}\cdot\nabla\big)s_{i,j} \quad \textit{throughout } \operatorname{im}(\Psi_{i,j}). \tag{4.64}$$

The gradient of the projection map (4.49) is given by

$$\nabla P_{i,j} = \bar\tau_{i,j}\otimes\bar\tau_{i,j} - s_{i,j}\nabla\bar{\mathrm{n}}_{i,j} \quad \textit{throughout } \operatorname{im}(\Psi_{i,j}). \tag{4.65}$$

Finally, for all $(x,t)\in\operatorname{im}(\Psi_{i,j})$ the derivatives of the signed distance $s_{i,j}$ are subject to the relations

$$\nabla s_{i,j}(x,t) = \nabla s_{i,j}(y,t)|_{y=P_{i,j}(x,t)} = \bar{\mathrm{n}}_{i,j}(x,t), \tag{4.66}$$
$$\nabla s_{i,j}(x,t)\cdot\partial_t\nabla s_{i,j}(x,t) = 0, \tag{4.67}$$
$$\big(\nabla s_{i,j}(x,t)\cdot\nabla\big)\nabla s_{i,j}(x,t) = 0, \tag{4.68}$$
$$\partial_t s_{i,j}(x,t) = \partial_t s_{i,j}(y,t)|_{y=P_{i,j}(x,t)}. \tag{4.69}$$

Proof. The representation of $s_{i,j}$ as a component of the inverse of $\Psi_{i,j}$ initially gives the regularity $s_{i,j}\in(C^0_tC^4_x\cap C^1_tC^2_x)(\overline{\operatorname{im}(\Psi_{i,j})})$. A proof of the well-known identities (4.64)–(4.69) was given for instance in [64, Lemma 10] with the only difference being the precise form of the normal velocity of the evolving family of interfaces. The higher regularity for the signed distance $s_{i,j}$ and its time derivative $\partial_t s_{i,j}$ finally follows from (4.53) and the identity (4.66). ☐

4.5 Gradient flow calibrations at a triple junction

The aim of this section is to construct a gradient flow calibration in the model case of three regular interfaces meeting at a single triple junction. The space-time trajectory of such a triple junction will be denoted by $\mathcal{T} = \bigcup_{t\in[0,T]} \mathcal{T}(t)\times\{t\}$ where $\mathcal{T}(t) \subset \mathbb{R}^2$ is a singleton for all $t \in [0,T]$. For simplicity, we assume throughout the section that the triple junction consists of interfaces between the phases $1, 2$ and 3. We will also use cyclical indices $i = 1, 2, 3$ throughout the section.

Similar to the previous one, the constructions provided in this section are local in the sense that they are restricted to a sufficiently small space-time neighborhood of the evolving triple junction $\mathcal{T}$. We first formalize this by introducing the notion of an *admissible localization radius* $r = r_{\mathcal{T}} \in (0,1]$ *for the triple junction* $\mathcal{T}$ in Definition 4.20. We then state the main result of this section, Proposition 4.22, which provides all relevant properties of the constructed calibrations.

The construction of a calibration $\xi_{i,j}$ for $i, j \in \{1, 2, 3\}$ with $i \neq j$ along with an associated velocity field B proceeds in three steps. First, we extend the normal of the interface $\bar{I}_{i,j}$ of the strong solution to auxiliary vector fields $\widetilde{\xi}_{i,j}$ defined on the natural domain $\mathbb{H}_{i,j} := \operatorname{im}(\Psi_{i,j}) \cap \bigcup_{t\in[0,T]} B_r(\mathcal{T}(t))\times\{t\}$, see Figure 4.6a, on which the nearest point-projection onto $\bar{I}_{i,j}$ is well-defined and regular; see Definition 4.17 and the subsequent discussion. One should think of $\widetilde{\xi}_{i,j}$ as the main building block for the vector field $\xi_{i,j}$ on the domain $\mathbb{H}_{i,j}$ containing the corresponding interface $\bar{I}_{i,j}$. Similarly, we also construct auxiliary velocity fields $B_{i,j}$ on $\mathbb{H}_{i,j}$ by choosing its normal component as an extension of the scalar mean curvature $H_{i,j}$ of the interface $\bar{I}_{i,j}$.

In the second step, we aim to identify a candidate vector field for the definition of $\xi_{i,j}$ outside of its natural domain of definition $\mathbb{H}_{i,j}$. The guiding principle is to make sure that the Herring angle condition at the triple junction

$$\sigma_{1,2}\bar{\mathrm{n}}_{1,2} + \sigma_{2,3}\bar{\mathrm{n}}_{2,3} + \sigma_{3,1}\bar{\mathrm{n}}_{3,1} = 0, \tag{4.70}$$

is satisfied by the calibrations $(\xi_{1,2}, \xi_{2,3}, \xi_{3,1})$ in the whole neighborhood of the triple junction:

$$\sigma_{1,2}\xi_{1,2} + \sigma_{2,3}\xi_{2,3} + \sigma_{3,1}\xi_{3,1} = 0. \tag{4.71}$$

This allows us to define vector fields (ξ_1, ξ_2, ξ_3) such that $\sigma_{i,i+1}\xi_{i,i+1} = \xi_i - \xi_{i+1}$ holds true for all cyclical indices $i = 1, 2, 3$. The latter identity in turn is precisely the property of gradient flow calibrations necessary to differentiate the relative entropy functional in time.

In order to achieve (4.71) we note that it represents an angle condition. As the union of the domains $\mathbb{H}_{i,i+1}$ for $i = 1, 2, 3$ covers a neighborhood of the triple junction, see Figure 4.5a, we would like to define $\xi_{i+1,i-1}$ and $\xi_{i-1,i}$ on $\mathbb{H}_{i,i+1}$ by simply rotating $\widetilde{\xi}_{i,i+1}$, see Figure 4.5c.

However, as these domains overlap, see Figure 4.6a, we will have to interpolate between the competing definitions of the calibrations and velocities. To this end, we partition the neighborhood of the triple junction into six wedges centered at the triple junction as indicated in Figure 4.6b, three of which are denoted by $W_{i,j} = W_{j,i}$ and the remaining three by W_i. We require that $B_r(\mathcal{T}(t)) \cap \bar{I}_{i,j} \subset W_{i,j} \cup \mathcal{T}(t) \subset \mathbb{H}_{i,j}$, see Figure 4.6b, the first inclusion corresponding to a geometric smallness condition for the interfaces away from the triple junction. For the remaining three wedges it is required that $W_i \subset \bigcap_{j\neq i} \mathbb{H}_{i,j}$, see again Figure 4.6b. We will refer to these wedges as *interpolation wedges* since on them we will interpolate between the two competing calibrations and velocities.

Definition 4.20. *Let $d = 2$ and $P \in \mathbb{N}$, $P \geq 2$. Let $(\bar{\Omega}_1, \dots, \bar{\Omega}_P)$ be a strong solution to multiphase mean curvature flow in the sense of Definition 4.14. Let $\mathcal{T} = \bigcup_{t\in[0,T]} \mathcal{T}(t)\times\{t\}$ be an evolving triple junction present in the network of interfaces of $\bar{\Omega}$, and assume for*

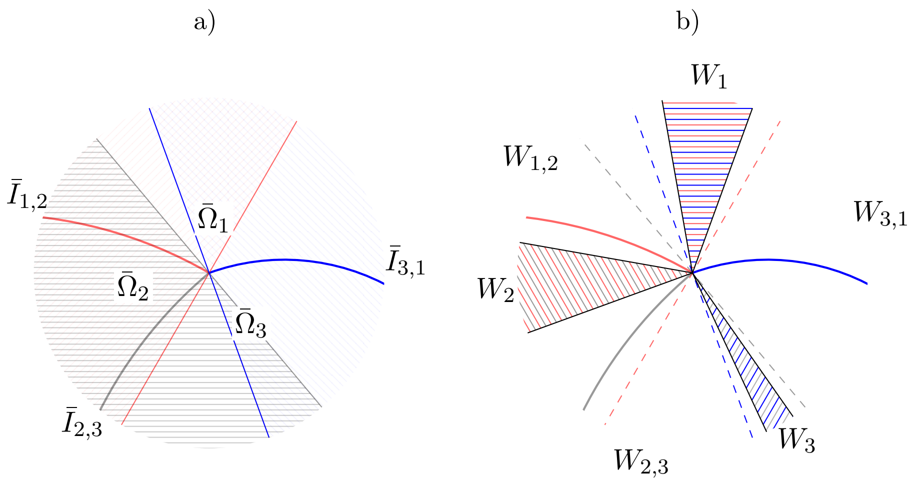

Figure 4.6: a) Sketch of a triple junction with phases $\bar\Omega_1$, $\bar\Omega_2$, and $\bar\Omega_3$; and the corresponding interfaces. The bottom left to top right hatched region is the domain $\mathbb{H}_{1,2}$, the horizontally hatched region is $\mathbb{H}_{2,3}$, and the top left to bottom right hatching represents $\mathbb{H}_{3,1}$. b) The interpolation wedges, shown as hatched, are given by W_1, W_2 and W_3. The remaining wedges $W_{1,2}$, $W_{2,3}$ and $W_{3,1}$ contain the corresponding interfaces.

simplicity that it is formed by the phases $1,2$ *and* 3. *For each* $i \in \{1,2,3\}$, *denote by* $\mathcal{T}_{i,i+1} = \bigcup_{t\in[0,T]} \mathcal{T}_{i,i+1}(t)\times\{t\}$ *the unique space-time connected component of* $\bar I_{i,i+1}$ *with an endpoint at the triple junction, and let* $r_{i,i+1} \in (0,1]$ *be an admissible localization radius for the interface* $\bar I_{i,i+1}$ *in the sense of Definition 4.17.*

We call a scale $r = r_{\mathcal{T}} \in (0, r_{1,2} \wedge r_{2,3} \wedge r_{3,1}]$ an admissible localization radius for the triple junction $\mathcal{T}$ *if there exists a wedge decomposition of the space-time neighborhood* $\mathcal{U}_r := \bigcup_{t\in[0,T]} B_r(\mathcal{T}(t))\times\{t\}$ *of the triple junction in the following precise sense:*

i) For each $i \in \{1,2,3\}$ *there exist space-time domains* $W_{i,i+1} := \bigcup_{t\in[0,T]} W_{i,i+1}(t)\times\{t\}$ *and* $W_i := \bigcup_{t\in[0,T]} W_i(t)\times\{t\}$ *(in order to not rely on cyclical notation in later sections, we also define* $W_{i+1,i} := W_{i,i+1}$ *for all* $i \in \{1,2,3\}$*) subject to the following requirements:*

First, for each $t \in [0,T]$ *the six sets* $(W_{i,i+1}(t))_{i\in\{1,2,3\}}$ *and* $(W_i(t))_{i\in\{1,2,3\}}$ *are pairwise disjoint, non-empty open subsets of* $B_r(\mathcal{T}(t))$ *such that*

$$\bigcup_{i\in\{1,2,3\}} \overline{W_{i,i+1}(t)} \cup \overline{W_i(t)} = \overline{B_r(\mathcal{T}(t))}. \tag{4.72}$$

Second, there exist six time-dependent unit vectors $(X^i_{i,i+1}, X^{i+1}_{i,i+1})_{i\in\{1,2,3\}}$ *of class* $C^1([0,T])$ *such that for all* $i \in \{1,2,3\}$ *and all* $t \in [0,T]$ *we have*

$$W_{i,i+1}(t) = \big(\mathcal{T}(t) + \big\{\gamma_1 X^i_{i,i+1}(t) + \gamma_2 X^{i+1}_{i,i+1}(t) \colon \gamma_1,\gamma_2 \in (0,\infty)\big\}\big) \cap B_r(\mathcal{T}(t)), \tag{4.73}$$

$$W_i(t) = \big(\mathcal{T}(t) + \big\{\gamma_1 X^i_{i,i+1}(t) + \gamma_2 X^i_{i-1,i}(t) \colon \gamma_1,\gamma_2 \in (0,\infty)\big\}\big) \cap B_r(\mathcal{T}(t)). \tag{4.74}$$

For all $i \in \{1,2,3\}$, *the scalar products* $X^i_{i,i+1} \cdot X^{i+1}_{i,i+1} \in (0,1)$ *and* $X^i_{i,i+1} \cdot X^i_{i-1,i}$ *are constant in time, and their values only depend on the surface tensions.*

Third, we require that for all $i \in \{1,2,3\}$ *and all* $t \in [0,T]$ *it holds*

$$B_r(\mathcal{T}(t)) \cap \mathcal{T}_{i,i+1}(t) \subset W_{i,i+1}(t) \cup \mathcal{T}(t) \subset \mathbb{H}_{i,i+1}(t), \tag{4.75}$$

$$W_i(t) \subset \mathbb{H}_{i,i+1}(t) \cap \mathbb{H}_{i,i-1}(t), \tag{4.76}$$

with the space-time domains $\mathbb{H}_{i,i+1} := \bigcup_{t\in[0,T]} \mathbb{H}_{i,i+1}(t)\times\{t\}$ *being defined by* $\mathbb{H}_{i,i+1}(t) := \{x \in \mathbb{R}^2 \colon (x,t) \in \operatorname{im}(\Psi_{i,i+1})\} \cap B_r(\mathcal{T}(t))$, $t \in [0,T]$.

ii) There exists a constant $C = C(\sigma) > 0$ depending only on the surface tensions such that for all $i \in \{1,2,3\}$

$$\max\{\mathrm{dist}(\cdot,\mathcal{T}),\mathrm{dist}(\cdot,\bar I_{i,i+1}),\mathrm{dist}(\cdot,\bar I_{i-1,i})\} \leq C \min_{j=1,2,3} \mathrm{dist}(\cdot,\bar I_{j,j+1}) \quad \text{in } W_i, \tag{4.77}$$

$$\mathrm{dist}(\cdot,\bar I_{i,i+1}) \leq C \min_{j=1,2,3} \mathrm{dist}(\cdot,\bar I_{j,j+1}) \quad \text{in } W_{i,i+1}, \tag{4.78}$$

$$\mathrm{dist}(x,\mathcal{T}) \leq C\,\mathrm{dist}(\cdot,\bar I_{i,i+1}) \quad \text{in } W_{i-1,i} \cup W_{i+1,i-1}. \tag{4.79}$$

In view of the properties (4.73)–(4.76), we call each $W_{i,i+1}$ an interface wedge*, and each W_i an* interpolation wedge.

The following lemma ensures the existence of an admissible localization radius for a triple junction; in particular, that we can indeed find wedges with the desired properties. Its proof is deferred to the end of Subsection 4.5.2.

Lemma 4.21. *Let the assumptions of Definition 4.20 be in place. Then there exists an admissible localization radius for the triple junction $\mathcal{T}$. In fact, one may choose $r = \frac{1}{C}(r_{1,2} \wedge r_{2,3} \wedge r_{3,1})$ for a constant $C = C(\sigma) \geq 1$ depending only on the surface tensions at the triple junction.*

As a final remark concerning the construction of the calibrations and the velocity, one has to make sure that they have sufficiently high regularity at the triple junction. Naively, one might choose the auxiliary vector fields $\widetilde{\xi}_{i,j}$ as in the case of a single connected interface from the previous section, i.e., $\widetilde{\xi}_{i,j} := \bar{\mathrm{n}}_{i,j}$ on $\mathbb{H}_{i,j}$. However, this ansatz after the rotation and interpolation steps only provides continuous vector fields $\xi_{i,j}$ which in general already fail to be Lipschitz at the triple junction, as we will see later. Hence, in the first step we will employ a more careful expansion ansatz in terms of the signed distance function to $\bar I_{i,j}$, see (4.90).

We are now in a position to state the main result of this section, namely the existence of a gradient flow calibration in the vicinity of an evolving triple junction.

Proposition 4.22. *Let $d = 2$ and $P \in \mathbb{N}$, $P \geq 2$. Let $(\bar\Omega_1,\ldots,\bar\Omega_P)$ be a strong solution to multiphase mean curvature flow in the sense of Definition 4.14. Let $\mathcal{T} = \bigcup_{t\in[0,T]} \mathcal{T}(t)\times\{t\}$ be an evolving triple junction present in the network of interfaces of the strong solution, and assume for simplicity that it is formed by the phases 1, 2 and 3. Let $r = r_{\mathcal{T}} \in (0,1]$ be an associated admissible localization radius for the triple junction $\mathcal{T}$ as given by Lemma 4.21. In particular, for all distinct $i,j \in \{1,2,3\}$, let $r_{i,j}$ be an admissible localization radius for $\bar I_{i,j}$ in the sense of Definition 4.17.*

Then there exists a constant $\hat C = \hat C(\bar\Omega) \geq 1$, depending only on $\bar\Omega$ but independent of $(r_{i,j})_{i,j\in\{1,2,3\},i\neq j}$, so that the radius $\hat r := \hat C^{-1} r$ has the following properties: Define $\mathcal{U}_{\hat r} := \bigcup_{t\in[0,T]} B_{\hat r}(\mathcal{T}(t)) \times \{t\}$. For all $i,j \in \{1,2,3\}$ with $i \neq j$, there exist continuous extensions of the unit-normal vector fields and a continuous velocity field

$$\xi_{i,j} \colon \mathcal{U}_{\hat r} \to \mathbb{R}^2, \quad B \colon \mathcal{U}_{\hat r} \to \mathbb{R}^2,$$

which are of regularity $\xi_{i,j} \in (C_t^0 C_x^2 \cap C_t^1 C_x^0)(\overline{\mathcal{U}_{\hat r}} \setminus \mathcal{T})$ resp. $B \in C_t^0 C_x^2(\overline{\mathcal{U}_{\hat r}} \setminus \mathcal{T})$, and which are furthermore subject to the following properties:

i) It holds $\xi_{i,j}(x,t) = \bar{\mathrm{n}}_{i,j}(x,t)$ for all $t \in [0,T]$ and for all $x \in \mathcal{T}_{i,j}(t) \cap B_{\hat r}(\mathcal{T}(t))$, where $\mathcal{T}_{i,j}$ is the unique space-time connected component of $\bar I_{i,j}$ with an endpoint at the triple junction $\mathcal{T}$. We also have $|\xi_{i,j}(x,t)| = 1$ for all $(x,t) \in \mathcal{U}_{\hat r}$. Expressing the triple junction in form of $\mathcal{T}(t) = \{p(t)\}$, it holds $B(p(t),t) = \frac{\mathrm{d}}{\mathrm{d}t}p(t)$ for all $t \in [0,T]$.

ii) We have the skew-symmetry relation $\xi_{i,j} = -\xi_{j,i}$.

iii) The family of vector fields $(\xi_{i,j})_{i\neq j}$ *satisfies the Herring angle condition* (4.70) *in the entire neighborhood of the triple junction, i.e., it holds for all* $(x,t) \in \mathcal{U}_{\hat{r}}$

$$\sigma_{1,2}\xi_{1,2}(x,t) + \sigma_{2,3}\xi_{2,3}(x,t) + \sigma_{3,1}\xi_{3,1}(x,t) = 0. \tag{4.80}$$

iv) There exists a constant $C = C(\bar{\Omega}) > 0$, *depending only on the strong solution* $\bar{\Omega}$ *but independent of* $\hat{r}$, *such that throughout* $\mathcal{U}_{\hat{r}} \setminus \mathcal{T}$ *and for all* $i,j \in \{1,2,3\}$ *with* $i \neq j$, *we have the bounds*

$$|\partial_t\xi_{i,j} + (B\cdot\nabla)\xi_{i,j} + (\nabla B)^{\mathsf{T}}\xi_{i,j}| \leq C\hat{r}^{-3}\operatorname{dist}(\cdot,\bar{I}_{i,j}), \tag{4.81}$$

$$|(\nabla\cdot\xi_{i,j}) + B\cdot\xi_{i,j}| \leq C\hat{r}^{-2}\operatorname{dist}(\cdot,\bar{I}_{i,j}), \tag{4.82}$$

$$\xi_{i,j}\cdot\partial_t\xi_{i,j} + \xi_{i,j}\cdot(B\cdot\nabla)\xi_{i,j} = 0. \tag{4.83}$$

v) Finally, there exists a constant $C = C(\bar{\Omega}) > 0$, *depending only on the strong solution* $\bar{\Omega}$ *but independent of* $\hat{r}$, *such that*

$$\hat{r}^2|\partial_t\xi_{i,j}| \leq C, \quad \hat{r}^k|\nabla^k\xi_{i,j}| \leq C, \qquad k \in \{0,1,2\}, \tag{4.84}$$

$$\hat{r}^k|\nabla^k B| \leq C\hat{r}^{-1}, \qquad k \in \{0,1,2\} \tag{4.85}$$

throughout the space-time domain $\mathcal{U}_{\hat{r}} \setminus \mathcal{T}$.

4.5.1 Construction close to individual interfaces

For all what follows in this subsection, let the assumptions of Proposition 4.22 and the notation of Section 4.4 and Definition 4.20 be in place. In this subsection, we for $i = 1,2,3$ first introduce the previously discussed auxiliary vector fields $\widetilde{\xi}_{i,i+1}$ as extensions of the normal $\bar{\mathrm{n}}_{i,i+1}$ to the domains $\mathbb{H}_{i,i+1}$.

We would like to define $\widetilde{\xi}_{i,i+1}$, and later also the velocity field B, by an expansion ansatz in terms of the signed distance function $s_{i,i+1}$ to the interface $\bar{I}_{i,i+1}$, see (4.47). To this end, two sets of coefficient functions will be of importance. First, for every $i \in \{1,2,3\}$ we introduce a function

$$\alpha_{i,i+1}\colon \mathbb{H}_{i,i+1} \to \mathbb{R}, \quad (x,t) \mapsto \widehat{\alpha}_{i,i+1}(P_{i,i+1}(x,t),t) \tag{4.86}$$

being defined by projection onto $\bar{I}_{i,i+1}$ in terms of the solution

$$\widehat{\alpha}_{i,i+1}\colon \bigcup_{t\in[0,T]} \mathcal{T}_{i,i+1}(t) \times \{t\} \to \mathbb{R} \tag{4.87}$$

to the following ODE posed on the space-time connected component $\mathcal{T}_{i,i+1}$ of the interface $\bar{I}_{i,i+1}$ with initial condition at the triple junction $\mathcal{T}(t) = \{p(t)\}$:

$$\begin{cases} \widehat{\alpha}_{i,i+1}(p(t),t) &= \bar{\tau}_{i,i+1}(p(t),t)\cdot\frac{\mathrm{d}}{\mathrm{d}t}p(t) \\ (\bar{\tau}_{i,i+1}(x,t)\cdot\nabla)\,\widehat{\alpha}_{i,i+1}(x,t) &= H^2_{i,i+1}(x,t), \qquad x \in \mathcal{T}_{i,i+1}(t). \end{cases} \tag{4.88}$$

Second, we define for each $i \in \{1,2,3\}$ a function $\beta_{i,i+1}\colon \mathbb{H}_{i,i+1} \to \mathbb{R}$ by means of

$$\beta_{i,i+1} := -\alpha_{i,i+1}H_{i,i+1} - (\bar{\tau}_{i,i+1}\cdot\nabla)H_{i,i+1}. \tag{4.89}$$

Finally recalling the definitions (4.49), (4.51), (4.52) and (4.54), the ansatz for the extension $\widetilde{\xi}_{i,i+1}$ of the normal vector field $\bar{\mathrm{n}}_{i,i+1}|_{\bar{I}_{i,i+1}}$ then is

$$\begin{aligned}\widetilde{\xi}_{i,i+1}(x,t) := \bar{\mathrm{n}}_{i,i+1}(x,t)&\\ + \alpha_{i,i+1}(x,t)s_{i,i+1}(x,t)\bar{\tau}_{i,i+1}(x,t)&\\ - \frac{1}{2}\alpha_{i,i+1}^2(x,t)s_{i,i+1}^2(x,t)\bar{\mathrm{n}}_{i,i+1}(x,t)&\end{aligned} \tag{4.90}$$

and $\widetilde{\xi}_{i+1,i} := -\widetilde{\xi}_{i,i+1}$ for $t \in [0,T]$, $x \in \mathbb{H}_{i,i+1}(t)$, and $i \in \{1,2,3\}$.

We briefly present the regularity properties of $\widetilde{\xi}_{i,i+1}$.

Lemma 4.23. *Let the assumptions of Proposition 4.22 be in place, in particular the notation of Definition 4.20. For all phases $i \in \{1,2,3\}$, the auxiliary vector field $\widetilde{\xi}_{i,i+1}$ is of class $(C_t^0C_x^2 \cap C_t^1C_x^0)(\overline{\mathbb{H}_{i,i+1}})$. More precisely, we have the estimates*

$$|\widetilde{\xi}_{i,i+1}| + r_{i,i+1}|\nabla\widetilde{\xi}_{i,i+1}| + r_{i,i+1}^2\big(|\nabla^2\widetilde{\xi}_{i,i+1}|+|\partial_t\widetilde{\xi}_{i,i+1}|\big) \leq C \tag{4.91}$$

for some $C = C(\bar{\Omega}) > 0$ only depending on $\bar{\Omega}$ but independent of $(r_{i,j})_{i,j\in\{1,2,3\},i\neq j}$.

Proof. Step 1 (Qualitative differentiability): In view of the expansion ansatz (4.90), the regularity (4.50) of the signed distance $s_{i,i+1}$, the regularity (4.53) of the normal $\bar{\mathrm{n}}_{i,i+1}$, and the regularity (4.54) of the tangent $\bar{\tau}_{i,i+1}$, it suffices to prove that $\alpha_{i,i+1} \in (C_t^0C_x^2 \cap C_t^1C_x^0)(\overline{\mathbb{H}_{i,i+1}})$ to conclude $\widetilde{\xi}_{i,i+1} \in (C_t^0C_x^2 \cap C_t^1C_x^0)(\overline{\mathbb{H}_{i,i+1}})$.

We start with the time regularity of the initial value of the ODE (4.88). Using the evolution equation $\frac{\mathrm{d}}{\mathrm{d}t}p(t)\cdot\bar{\mathrm{n}}_{i,i+1}(p(t),t) = H_{i,i+1}(p(t),t)$ at the triple junction we get

$$\frac{\mathrm{d}}{\mathrm{d}t}p(t) = H_{i,i+1}(p(t),t)\bar{\mathrm{n}}_{i,i+1}(p(t),t) + \Big(\bar{\tau}_{i,i+1}(p(t),t)\cdot\frac{\mathrm{d}}{\mathrm{d}t}p(t)\Big)\bar{\tau}_{i,i+1}(p(t),t) \tag{4.92}$$

for $i \in \{1,2,3\}$. Note that this identity is equivalent to the second-order compatibility condition (4.14). We can now identify the term in the parenthesis as $\alpha_{i,i+1}(p(t),t)$ due to the intial value of the ODE (4.88) and multiply the above equation with the rotation matrix J in order to deduce

$$-H_{1,2}\,\bar{\tau}_{1,2} + \alpha_{1,2}\,\bar{\mathrm{n}}_{1,2} = -H_{2,3}\,\bar{\tau}_{2,3} + \alpha_{2,3}\,\bar{\mathrm{n}}_{2,3} = -H_{3,1}\,\bar{\tau}_{3,1} + \alpha_{3,1}\,\bar{\mathrm{n}}_{3,1} \tag{4.93}$$

at the triple junction.

For $i \neq j$, we then define $c_{i,j} := \bar{\mathrm{n}}_{i,i+1}(p(t),t)\cdot\bar{\mathrm{n}}_{j,j+1}(p(t),t)$ and $d_{i,j} := \bar{\mathrm{n}}_{i,i+1}(p(t),t)\cdot\bar{\tau}_{j,j+1}(p(t),t)$ and notice that they are indeed constant in time due to only depending on the angles between interfaces determined by the surface tensions. Furthermore, note $|c_{i,j}| < 1$ as the surface tensions satisfy the triangle inequality. Multiplying (4.93) with the normal $\bar{\mathrm{n}}_{i,i+1}(p(t),t)$ thus yields

$$\alpha_{i,i+1}(p(t),t) = -H_{j,j+1}(p(t),t)d_{i,j} + \alpha_{j,j+1}(p(t),t)c_{i,j}$$

for all $i \neq j$ and all $t \in [0,T]$. Switching the roles of i and j in the previous formula entails

$$\alpha_{i,i+1}(p(t),t) = -(1-c_{i,j}^2)^{-1}\big(H_{j,j+1}(p(t),t)d_{i,j} + H_{i,i+1}(p(t),t)d_{i,j}c_{i,j}\big) \tag{4.94}$$

for all $i \neq j$ and all $t \in [0,T]$. Hence, we deduce $t \mapsto \alpha_{i,i+1}(p(t),t) \in C^1([0,T])$.

We procedd by explicitly integrating the ODE (4.88), and exploiting the regularity (4.53) of the extended scalar mean curvature $H_{i,i+1}$, as well as the regularity of the space-time curve $\mathcal{T}_{i,i+1}$. Let us make this argument explicit. To this end, we first choose a C^5 diffeomorphic parametrization $\gamma_0\colon [0,1] \to \mathcal{T}_{i,i+1}(0)$ of the initial curve $\mathcal{T}_{i,i+1}(0)$ such that

$\gamma_0(0) = p(0)$, and then define $\gamma_t(s) := \psi^t(\gamma_0(s))$ for all $(s,t) \in [0,1]\times[0,T]$ by means of the flow maps from Definition 4.13. Capturing orientation by means of the constant $c_\pm = \bar\tau_{i,i+1}(\gamma_t(s),t) \cdot \frac{\partial_s\gamma_t(s)}{|\partial_s\gamma_t(s)|} \in \{\pm 1\}$, we set

$$\widetilde{\alpha}_{i,i+1}(s,t) := \bar\tau_{i,i+1}(p(t),t) \cdot \frac{\mathrm{d}}{\mathrm{d}t}p(t) + c_\pm \int_0^s H^2_{i,i+1}(\gamma_t(\ell),t)|\partial_s\gamma_t(\ell)|\,\mathrm{d}\ell \tag{4.95}$$

for all $(s,t) \in [0,1]\times[0,T]$, and then have

$$\widehat{\alpha}_{i,i+1}(x,t) = \widetilde{\alpha}_{i,i+1}\big((\gamma_t)^{-1}(x),t\big) \tag{4.96}$$

for all $t \in [0,T]$ and all $x \in \mathcal{T}_{i,i+1}(t)$. The validity of (4.88) is indeed a simple consequence of the ansatz (4.95), the definition (4.96) and the chain rule. The required regularity $\alpha_{i,i+1} \in (C^0_tC^2_x \cap C^1_tC^0_x)(\overline{\mathbb{H}_{i,i+1}})$ in turn follows from the regularity (4.50) of the projection, the regularity (4.54) of the tangent, the regularity (4.53) of the curvature, and the regularity condition *ii)* of Definition 4.13.

Step 2 (Quantitative estimates): Since in each time slice the map $\Psi_{i,i+1}$ from (4.45) represents a tubular neighborhood diffeomorphism on scale $r_{i,i+1} \in (0,1]$, we deduce

$$r^k_{i,i+1}|\nabla^k s_{i,i+1}| \leq Cr_{i,i+1}, \quad k \in \{0,1,2,3,4,5\}, \tag{4.97}$$

and thus from the definitions (4.51) and (4.54) that

$$r^k_{i,i+1}|\nabla^k \bar{\mathrm{n}}_{i,i+1}| + r^k_{i,i+1}|\nabla^k\bar\tau_{i,i+1}| \leq C, \quad k \in \{0,1,2,3,4\}. \tag{4.98}$$

The previous estimates in addition entail the following bounds for the nearest-point projections due to (4.49) (in form of $P_{i,i+1}(x,t) = x - s_{i,i+1}(x,t)\nabla s_{i,i+1}(x,t)$) and the (extensions of the) scalar mean curvatures due to (4.52)

$$r^k_{i,i+1}|\nabla^k P_{i,i+1}| \leq Cr_{i,i+1}, \qquad k \in \{1,2,3,4\}, \tag{4.99}$$

$$r^k_{i,i+1}|\nabla^k H_{i,i+1}| \leq Cr^{-1}_{i,i+1}, \qquad k \in \{0,1,2,3\}. \tag{4.100}$$

As a consequence of the evolution equation (4.64) for the signed distance, we also obtain the following estimate on the time derivatives

$$\begin{aligned} r_{i,i+1}|\partial_t s_{i,i+1}| + r^2_{i,i+1}|\partial_t\bar{\mathrm{n}}_{i,i+1}| + r^2_{i,i+1}|\partial_t\bar\tau_{i,i+1}| \\ + r_{i,i+1}|\partial_t P_{i,i+1}| + r^3_{i,i+1}|\partial_t H_{i,i+1}| \leq C. \end{aligned} \tag{4.101}$$

It then follows from the representations (4.94) and (4.92) that

$$r_{i,i+1}|\alpha_{i,i+1}(p(t),t)| + r_{i,i+1}\Big|\frac{\mathrm{d}}{\mathrm{d}t}p(t)\Big| + r^3_{i,i+1}\Big|\frac{\mathrm{d}}{\mathrm{d}t}\alpha_{i,i+1}(p(t),t)\Big| \leq C \tag{4.102}$$

for all $t \in [0,T]$.

We next claim that

$$\max_{k=0,1,2} r^k_{i,i+1}|\nabla^k\alpha_{i,i+1}| + r^2_{i,i+1}|\partial_t\alpha_{i,i+1}| \leq Cr^{-1}_{i,i+1}. \tag{4.103}$$

Once this is established, the asserted bound (4.91) for the derivatives of the vector fields $\widetilde{\xi}_{i,i+1}$ can then be directly inferred from the ansatz (4.90) and the above regularity estimates. The estimate (4.103), however, is a consequence of the regularity estimates (4.98)–(4.102) and the representations (4.86)–(4.88) in form of $\nabla\alpha_{i,i+1} = H^2_{i,i+1}(\bar\tau_{i,i+1}\otimes\bar\tau_{i,i+1} : \nabla P_{i,i+1})\bar\tau_{i,i+1}$. For later reference, we note that

$$(\bar\tau_{i,i+1}\cdot\nabla)\alpha_{i,i+1} = H^2_{i,i+1} + O(r^{-3}_{i,i+1}|s_{i,i+1}|) \tag{4.104}$$

due to (4.65), (4.97) and (4.100). □

Ultimately, the point of the ansatz (4.90) is to ensure both (4.80) throughout $B_r(\mathcal{T}(t))$ and sufficiently high regularity of $\xi_{i,j}$ at the triple junction. Moreover, the relations (4.88) and (4.89) also holding true away from the triple junction turns out to be crucial to obtain the estimates (4.81) and (4.82) on the whole space-time domain. The first step towards these goals are the following relations, which in particular yield that—after rotation $R_{(i,j)}$—the vector fields are compatible to second order at the triple junction:

Lemma 4.24. *Let the assumptions of Proposition 4.22 be in place. For each pair $i, j \in \{1, 2, 3\}$ there exist uniquely determined rotations $R_{(i,j)} \in SO(2)$, only depending on the restriction $(\sigma_{i,j})_{i,j=1,2,3}$ of the admissible matrix of surface tensions for the given strong solution $\bar{\Omega}$, such that*

$$\bar{n}_{i,i+1}(\cdot, t) = R_{(i,j)}\bar{n}_{j,j+1}(\cdot, t) \quad \text{at } \mathcal{T}(t) \tag{4.105}$$

for all $t \in [0, T]$, and

$$R_{(i,j)}R_{(j,i)} = \mathrm{Id}, \tag{4.106}$$

$$R_{(i,i-1)}R_{(i-1,i+1)}R_{(i+1,i)} = \mathrm{Id}\,. \tag{4.107}$$

Furthermore, the ansatz (4.90) satisfies the first-order compatibility conditions at the triple junction:

$$\widetilde{\xi}_{i,i+1}(\cdot, t) = R_{(i,j)}\widetilde{\xi}_{j,j+1}(\cdot, t) \qquad \text{at } \mathcal{T}(t), \tag{4.108}$$

$$\nabla\widetilde{\xi}_{i,i+1}(\cdot, t) = \nabla\big(R_{(i,j)}\widetilde{\xi}_{j,j+1}\big)(\cdot, t) \qquad \text{at } \mathcal{T}(t), \tag{4.109}$$

for all $t \in [0, T]$.

Proof. Identity (4.105) uniquely defines $R_{(i,j)}$. It is immediate from the ansatz (4.90) and (4.105) that the zero-order condition (4.108) is satisfied. The two properties (4.106) and (4.107) follow from

$$R_{(i,j)}R_{(j,i)}\bar{n}_{i,i+1} = \bar{n}_{i,i+1}, \tag{4.110}$$

$$R_{(i,i-1)}R_{(i-1,i+1)}R_{(i+1,i)}\bar{n}_{i,i+1} = \bar{n}_{i,i+1}, \tag{4.111}$$

which follow straightforwardly from iterating (4.105). Therefore, it is sufficient to prove the remaining statement (4.109) for $j = i + 1$, as it then follows automatically for $j = i - 1$ by (4.106) and (4.107) that at $\mathcal{T}(t)$ it holds

$$\begin{aligned}
\nabla\big(R_{(i,i-1)}\widetilde{\xi}_{i-1,i}\big)(\cdot, t) &= R_{(i,i+1)}\nabla\big(R_{(i+1,i-1)}\widetilde{\xi}_{i-1,i}\big)(\cdot, t) \\
&= R_{(i,i+1)}\nabla\big(\widetilde{\xi}_{i+1,i-1}\big)(\cdot, t) \\
&= \nabla\big(R_{(i,i+1)}\widetilde{\xi}_{i+1,i-1}\big)(\cdot, t) = \nabla\widetilde{\xi}_{i,i+1}(\cdot, t).
\end{aligned}$$

For ease of notation, we also fix the index i and omit all indices, superscripts, and arguments for the rest of the proof unless specifically required otherwise. The ansatz (4.90) then reads

$$\widetilde{\xi} = \bar{n} + \alpha s\bar{\tau} - \frac{1}{2}\alpha^2 s^2\bar{n}. \tag{4.112}$$

By definition (4.51), $\nabla^2 s$ being symmetric, the identity (4.68), and the orthogonality relation $\bar{\tau} \cdot \bar{n} = 0$ we have $\nabla\bar{n} = \Delta s\, \bar{\tau} \otimes \bar{\tau}$. Hence, by the definitions (4.54) and (4.52) as well as the estimate (4.97), we then get

$$\nabla\bar{n} = -H\bar{\tau} \otimes \bar{\tau} + O(r^{-2}|s|), \tag{4.113}$$

$$\nabla\bar{\tau} = \quad H\bar{n} \otimes \bar{\tau} + O(r^{-2}|s|). \tag{4.114}$$

As a result we infer from this and (4.103)

$$\nabla\widetilde{\xi} = -H\,\bar{\tau}\otimes\bar{\tau} + \alpha\,\bar{\tau}\otimes\bar{\mathrm{n}} + O(r^{-2}|s|). \tag{4.115}$$

This in turn yields

$$\nabla\widetilde{\xi} = \bar{\tau}\otimes(-H\,\bar{\tau} + \alpha\,\bar{\mathrm{n}}) \quad \text{at the triple junction } \mathcal{T}. \tag{4.116}$$

Now we are in a position to prove the compatibility condition (4.109). By (4.105) and $J\bar{\tau} = \bar{\mathrm{n}}$, see (4.54), we obtain

$$\bar{\tau}_{i,i+1} = R_{(i,j)}\bar{\tau}_{j,j+1} \quad \text{at the triple junction } \mathcal{T}. \tag{4.117}$$

Moreover, expressing the evolving triple junction in form of $\mathcal{T}(t) = \{p(t)\}$ for all $t \in [0,T]$, it follows from the evolution equation $\frac{\mathrm{d}}{\mathrm{d}t}p\cdot\bar{\mathrm{n}} = H$ and the choice of the initial value in the ODE (4.88) that

$$\frac{\mathrm{d}}{\mathrm{d}t}p = \quad H_{1,2}\bar{\mathrm{n}}_{1,2} + \alpha_{1,2}\bar{\tau}_{1,2} = \quad H_{2,3}\bar{\mathrm{n}}_{2,3} + \alpha_{2,3}\bar{\tau}_{2,3} = \quad H_{3,1}\bar{\mathrm{n}}_{3,1} + \alpha_{3,1}\bar{\tau}_{3,1}, \tag{4.118}$$

$$-H_{1,2}\bar{\tau}_{1,2} + \alpha_{1,2}\bar{\mathrm{n}}_{1,2} = -H_{2,3}\bar{\tau}_{1,2} + \alpha_{2,3}\bar{\mathrm{n}}_{2,3} = -H_{3,1}\bar{\tau}_{1,2} + \alpha_{3,1}\bar{\mathrm{n}}_{3,1} \tag{4.119}$$

at the triple junction $\mathcal{T}$ (the latter follows from the former by multiplication with J). Therefore, by (4.116), (4.117) and (4.119) we indeed at $\mathcal{T}$ get

$$\begin{aligned}\nabla\big(R_{(i,j)}\widetilde{\xi}_{j,j+1}\big) &= R_{(i,j)}\bar{\tau}_{j,j+1}\otimes\big(-H_{j,j+1}\bar{\tau}_{j,j+1} + \alpha_{j,j+1}\bar{\mathrm{n}}_{j,j+1}\big)\\ &= \bar{\tau}_{i,i+1}\otimes\big(-H_{i,i+1}\bar{\tau}_{i,i+1} + \alpha_{i,i+1}\bar{\mathrm{n}}_{i,i+1}\big)\\ &= \nabla\widetilde{\xi}_{i,i+1}.\end{aligned}$$

This concludes the proof of Lemma 4.24. ☐

Recall that apart from the family of vector fields $(\xi_{i,j})_{i\neq j}$, the notion of gradient flow calibrations also requires a suitably defined velocity field B. For its construction in the vicinity of a triple junction, we introduce in a first step certain auxiliary symmetric velocity fields $\widetilde{B}_{(i,j)} = \widetilde{B}_{(j,i)}$. To this end, we start for every $i \in \{1,2,3\}$, $t\in[0,T]$ and $x \in \mathbb{H}_{i,i+1}(t)$ with an expansion ansatz of the form

$$\begin{aligned}\widetilde{B}_{(i,i+1)}(x,t) := H_{i,i+1}(x,t)&\bar{\mathrm{n}}_{i,i+1}(x,t)\\ &+ \alpha_{i,i+1}(x,t)\bar{\tau}_{i,i+1}(x,t)\\ &+ \beta_{i,i+1}(x,t)s_{i,i+1}(x,t)\bar{\tau}_{i,i+1}(x,t)\end{aligned} \tag{4.120}$$

and also set $\widetilde{B}_{(i+1,i)} := \widetilde{B}_{(i,i+1)}$.

Lemma 4.25. *Let the assumptions of Proposition 4.22 be in place, in particular the notation of Definition 4.20. For all phases $i \in \{1,2,3\}$, the auxiliary velocity field $\widetilde{B}_{(i,i+1)}$ is of class $C^0_tC^2_x(\overline{\mathbb{H}_{i,i+1}})$. More precisely, we have the estimates*

$$|\widetilde{B}_{(i,i+1)}| + r_{i,i+1}|\nabla\widetilde{B}_{(i,i+1)}| + r^2_{i,i+1}|\nabla^2\widetilde{B}_{(i,i+1)}| \leq Cr^{-1}_{i,i+1} \tag{4.121}$$

for some $C = C(\bar{\Omega}) > 0$, depending only on $\bar{\Omega}$ but independent of $(r_{i,j})_{i,j\in\{1,2,3\},i\neq j}$.

Proof. In view of the expansion ansatz (4.120) and the ingredients of the proof of Lemma 4.23, it suffices to prove that $\beta_{i,i+1} \in C^0_tC^2_x(\overline{\mathbb{H}_{i,i+1}})$ with corresponding estimate

$$|\nabla^k\beta_{i,i+1}| \leq Cr^{-k-2}_{i,i+1}, \qquad k\in\{0,1,2\}. \tag{4.122}$$

Recalling the definition (4.89) of the coefficients $\beta_{i,i+1}$, the bound (4.122) is immediate from (4.103), (4.98), (4.100), and (4.101). ☐

We again have to make sure that our ansatz (4.120) for the auxiliary velocity fields satisfies a first-order compatibility condition at the triple junction.

Lemma 4.26. *Let the assumptions of Proposition 4.22 be in place. Expressing the evolving triple junction in form of $\mathcal{T}(t) = \{p(t)\}$ for all $t \in [0,T]$, for every $i,j \in \{1,2,3\}$ the ansatz (4.120) then satisfies*

$$\widetilde{B}_{(i,i+1)}(p(t),t) = \widetilde{B}_{(j,j+1)}(p(t),t) = \frac{\mathrm{d}}{\mathrm{d}t}p(t), \tag{4.123}$$

$$\nabla\widetilde{B}_{(i,i+1)}(p(t),t) = \nabla\widetilde{B}_{(j,j+1)}(p(t),t) \tag{4.124}$$

for all $t \in [0,T]$.

Proof. We again fix the index i and omit all indices, superscripts, and arguments unless specifically required. At the triple junction, we have

$$\widetilde{B}(p(t),t) = \frac{\mathrm{d}}{\mathrm{d}t}p(t) \tag{4.125}$$

by (4.118) and the ansatz (4.120). This of course proves (4.123).

An explicit computation making use of the ansatz (4.120), the estimates (4.113) and (4.114), the choices of the coefficients (4.88) and (4.89)—in particular (4.104)—as well as the estimates (4.103) and (4.122) moreover gives

$$\begin{aligned}\nabla\widetilde{B} &= \big(-H^2 + (\bar\tau\cdot\nabla\alpha)\big)\,\bar\tau\otimes\bar\tau \\ &\quad + ((\bar\tau\cdot\nabla)H + \alpha H)\bar{\mathrm{n}}\otimes\bar\tau \\ &\quad + \beta\bar\tau\otimes\bar{\mathrm{n}} + O(r^{-3}|s|) \\ &= \beta\,(\bar\tau\otimes\bar{\mathrm{n}} - \bar{\mathrm{n}}\otimes\bar\tau) + O(r^{-3}|s|).\end{aligned} \tag{4.126}$$

As we have $(\bar\tau\otimes\bar{\mathrm{n}} - \bar{\mathrm{n}}\otimes\bar\tau)\,\bar{\mathrm{n}} = \bar\tau = -J\bar{\mathrm{n}}$ and $(\bar\tau\otimes\bar{\mathrm{n}} - \bar{\mathrm{n}}\otimes\bar\tau)\,\bar\tau = -\bar{\mathrm{n}} = -J\bar\tau$ it follows that $(\bar\tau\otimes\bar{\mathrm{n}} - \bar{\mathrm{n}}\otimes\bar\tau) = -J$, where we recall that J denotes the counter-clockwise rotation by 90°. Therefore we get

$$\nabla\widetilde{B} = -\beta J + O(r^{-3}|s|). \tag{4.127}$$

Hence, (4.124) holds true once we established that $\beta_{1,2} = \beta_{2,3} = \beta_{3,1}$ at the triple junction. This, however, follows from a combination of the definition (4.89), the choice of the initial value in the ODE (4.88), and the third-order compatibility condition (4.15). □

In a preparatory step towards the proof of (4.81) and (4.82), we now present the corresponding estimates for the (rotated) auxiliary vector fields $\widetilde{\xi}_{i,i+1}$ and the auxiliary velocity fields $\widetilde{B}_{(i,i+1)}$ on their respective domains of definition.

Lemma 4.27. *Let the assumptions of Proposition 4.22 be in place, in particular the notation of Definition 4.20. Then there exists a constant $C = C(\bar\Omega) > 0$, depending only on $\bar\Omega$ but independent of $(r_{i,j})_{i,j\in\{1,2,3\},i\neq j}$, such that the following holds: For every $i,j \in \{1,2,3\}$ and throughout the space-time domain $\mathbb{H}_{j,j+1}$ we have*

$$\begin{aligned}&\big|\partial_t R_{(i,j)}\widetilde{\xi}_{j,j+1} + (\widetilde{B}_{(j,j+1)}\cdot\nabla)R_{(i,j)}\widetilde{\xi}_{j,j+1} + (\nabla\widetilde{B}_{(j,j+1)})^{\mathsf{T}}R_{(i,j)}\widetilde{\xi}_{j,j+1}\big| \\ &\leq Cr_{j,j+1}^{-3}\,\mathrm{dist}(\cdot,\bar I_{j,j+1}),\end{aligned} \tag{4.128}$$

as well as

$$\big|\nabla\cdot R_{(i,j)}\widetilde{\xi}_{j,j+1}+\widetilde{B}_{(j,j+1)}\cdot R_{(i,j)}\widetilde{\xi}_{j,j+1}\big|\leq Cr_{j,j+1}^{-2}\operatorname{dist}(\cdot,\bar{I}_{j,j+1}), \tag{4.129}$$

$$\big|1-|R_{(i,j)}\widetilde{\xi}_{j,j+1}|^2\big|\leq Cr_{j,j+1}^{-4}\operatorname{dist}^4(\cdot,\bar{I}_{j,j+1}), \tag{4.130}$$

$$\big|\nabla|R_{(i,j)}\widetilde{\xi}_{j,j+1}|^2\big|\leq Cr_{j,j+1}^{-4}\operatorname{dist}^3(\cdot,\bar{I}_{j,j+1}), \tag{4.131}$$

$$\big|\partial_t|R_{(i,j)}\widetilde{\xi}_{j,j+1}|^2\big|\leq Cr_{j,j+1}^{-5}\operatorname{dist}^3(\cdot,\bar{I}_{j,j+1}), \tag{4.132}$$

$$\big|\partial_t|R_{(i,j)}\widetilde{\xi}_{j,j+1}|^2+(\widetilde{B}_{(j,j+1)}\cdot\nabla)|R_{(i,j)}\widetilde{\xi}_{j,j+1}|^2\big|\leq Cr_{j,j+1}^{-6}\operatorname{dist}^4(\cdot,\bar{I}_{j,j+1}). \tag{4.133}$$

We also have for all pairs $i,j\in\{1,2,3\}$ with $i\neq j$ throughout the intersection $\mathbb{H}_{i,i+1}\cap\mathbb{H}_{j,j+1}$ that (with $r_{\min}:=r_{1,2}\wedge r_{2,3}\wedge r_{3,1}$)

$$|R_{(i,j)}\widetilde{\xi}_{j,j+1}-R_{(i,j-1)}\widetilde{\xi}_{j-1,j}|\leq Cr_{\min}^{-2}\operatorname{dist}^2(\cdot,\mathcal{T}), \tag{4.134}$$

$$|\nabla R_{(i,j)}\widetilde{\xi}_{j,j+1}-\nabla R_{(i,j-1)}\widetilde{\xi}_{j-1,j}|\leq Cr_{\min}^{-2}\operatorname{dist}(\cdot,\mathcal{T}), \tag{4.135}$$

$$|\widetilde{B}_{(i,i+1)}-\widetilde{B}_{(j,j+1)}|\leq Cr_{\min}^{-3}\operatorname{dist}^2(\cdot,\mathcal{T}), \tag{4.136}$$

$$|\nabla\widetilde{B}_{(i,i+1)}-\nabla\widetilde{B}_{(j,j+1)}|\leq Cr_{\min}^{-3}\operatorname{dist}(\cdot,\mathcal{T}). \tag{4.137}$$

Proof. By the ansatz (4.90) and $R_{(i,j)}\in SO(2)$ we have

$$\begin{aligned}|R_{(i,j)}\widetilde{\xi}_{j,j+1}|^2&=\Big(1-\frac{1}{2}\alpha_{j,j+1}^2s_{j,j+1}^2\Big)^2+\alpha_{j,j+1}^2s_{j,j+1}^2\\&=1+\frac{1}{4}\alpha_{j,j+1}^4s_{j,j+1}^4\end{aligned} \tag{4.138}$$

from which together with (4.97), (4.103), (4.101), and (4.121) the estimates (4.130)–(4.133) immediately follow.

To prove the estimates (4.128)–(4.129), let $i,j\in\{1,2,3\}$ be fixed. For what follows, we omit all indices and arguments unless specifically required. Plugging in the ansatz (4.90) for $\widetilde{\xi}$ and introducing the commutator $[C,D]:=CD-DC$ for matrices $C,D\in\mathbb{R}^{d\times d}$, we obtain

$$\begin{aligned}\partial_tR\widetilde{\xi}+(\widetilde{B}\cdot\nabla)R\widetilde{\xi}+(\nabla\widetilde{B})^\mathsf{T}R\widetilde{\xi}&=\Big(1-\frac{1}{2}\alpha^2s^2\Big)R\big(\partial_t\bar{\mathrm{n}}+(\widetilde{B}\cdot\nabla)\bar{\mathrm{n}}+(\nabla\widetilde{B})^\mathsf{T}\bar{\mathrm{n}}\big)\\&\quad+\alpha sR\big(\partial_t\bar{\tau}+(\widetilde{B}\cdot\nabla)\bar{\tau}+(\nabla\widetilde{B})^\mathsf{T}\bar{\tau}\big)\\&\quad+\alpha\big(\partial_ts+(\widetilde{B}\cdot\nabla)s\big)\big(R\bar{\tau}-\alpha sR\bar{\mathrm{n}}\big)\\&\quad+[(\nabla\widetilde{B})^\mathsf{T},R]\tilde{\xi}\\&\quad+\big(\partial_t\alpha+(\widetilde{B}\cdot\nabla)\alpha\big)s\big(R\bar{\tau}-\alpha sR\bar{\mathrm{n}}\big).\end{aligned}$$

By the ansatz (4.120), the auxiliary velocity $\widetilde{B}$ only corrects $H\bar{\mathrm{n}}$ in tangential direction. Hence, the identities (4.61) and (4.60) are applicable and we obtain

$$\partial_t\bar{\mathrm{n}}+(\widetilde{B}\cdot\nabla)\bar{\mathrm{n}}+(\nabla\widetilde{B})^\mathsf{T}\bar{\mathrm{n}}=0,\quad\partial_ts+(\widetilde{B}\cdot\nabla)s=0$$

throughout $\mathbb{H}_{j,j+1}$. Recalling the definition $\bar{\tau}=J\bar{\mathrm{n}}$, cf. (4.54), we deduce from the previous display

$$\partial_t\bar{\tau}+(\widetilde{B}\cdot\nabla)\bar{\tau}+(\nabla\widetilde{B})^\mathsf{T}\bar{\tau}=[(\nabla\widetilde{B})^\mathsf{T},J]\bar{\mathrm{n}}$$

throughout $\mathbb{H}_{j,j+1}$. Hence, recalling (4.127) and using the fact that $[J^\mathsf{T},R]=0$ on account of both matrices being rotations in the plane we get

$$[(\nabla\widetilde{B})^\mathsf{T},R]=O(r^{-3}|s|),\quad[(\nabla\widetilde{B})^\mathsf{T},J]=O(r^{-3}|s|)$$

throughout $\mathbb{H}_{j,j+1}$. Together with the estimate (4.103), the previous four displays in combination imply (4.128).

We turn to the proof of (4.129). Due to the computation (4.115) of $\nabla\widetilde{\xi}$ we have on the one side

$$\nabla \cdot R\widetilde{\xi} = -H(R\bar{\tau} \cdot \bar{\tau}) + \alpha(R\bar{\tau} \cdot \bar{\mathrm{n}}) + O(r^{-2}|s|). \tag{4.139}$$

On the other side, making use of the definitions (4.90) and (4.120) of $\widetilde{\xi}$ and $\widetilde{B}$ we obtain

$$\widetilde{B} \cdot R\widetilde{\xi} = H\bar{\mathrm{n}} \cdot R\bar{\mathrm{n}} + \alpha(\bar{\tau} \cdot R\bar{\mathrm{n}}) + O(r^{-2}|s|). \tag{4.140}$$

Furthermore, recalling $J\bar{\tau} = \bar{\mathrm{n}}$, $J^{\mathsf{T}} = J^{-1} = -J$, and $[J^{\mathsf{T}}, R] = 0$ gives

$$\begin{aligned} R\bar{\tau} \cdot \bar{\tau} &= RJ^{-1}\bar{\mathrm{n}} \cdot \bar{\tau} = R\bar{\mathrm{n}} \cdot J\bar{\tau} = R\bar{\mathrm{n}} \cdot \bar{\mathrm{n}}, \\ R\bar{\tau} \cdot \bar{\mathrm{n}} &= RJ^{-1}\bar{\mathrm{n}} \cdot \bar{\mathrm{n}} = R\bar{\mathrm{n}} \cdot J\bar{\mathrm{n}} = -R\bar{\mathrm{n}} \cdot \bar{\tau}. \end{aligned}$$

Therefore, we can combine (4.139) and (4.140) to yield the estimate (4.129).

We proceed with the verification of the bounds (4.134) and (4.135). As by (4.108) and (4.109) the Taylor polynomials at the triple junction of the functions $R_{(i,j)}\widetilde{\xi}_{j,j+1}$ and $R_{(i,j-1)}\widetilde{\xi}_{j-1,j}$ agree up to first order, the estimate (4.134) follows by bounding the remainders using (4.91). One can argue similarly for the estimate (4.135). On the basis of (4.123), (4.124) and (4.121), the estimates (4.136) and (4.137) follow by the same argument. □

4.5.2 Gluing construction by interpolation

Throughout this subsection, let again the assumptions of Proposition 4.22 and the notation of Section 4.4 and Definition 4.20 be in place. As we discussed in the previous subsection, the auxiliary vector fields $\widetilde{\xi}_{i,i+1}$ and the auxiliary velocity fields $\widetilde{B}_{(i,i+1)}$ serve as the definition of the vector fields $\xi_{i,i+1}$ and the velocity field B on the interface wedge $W_{i,i+1}$, see Figure 4.6b for the partition of the neighborhood of the triple junction.

The next step is to extend $\xi_{i,i+1}$ and B to the entirety of the space-time domain. As we want Herring's angle condition (4.71) to hold throughout the ball $B_r(\mathcal{T}(t))$ we are essentially forced to set $\xi_{i,i+1} = R_{(i,j)}\xi_{j,j+1}$ for all $i, j \in \{1, 2, 3\}$ wherever the latter is defined, and where $R_{(i,j)}$ is given in Lemma 4.24. As their domains of definition $\mathbb{H}_{i,i+1}$ overlap, we resort to an interpolation procedure on the interpolation wedges W_i, see again Figure 4.6b. We similarly deal with the issue of combining the velocity fields $\widetilde{B}_{(i,i+1)}$ into a single field. To this end, we first define suitable interpolation functions which move and rotate with the evolving triple junction.

Lemma 4.28. *Let the assumptions of Proposition 4.22 be in place, in particular the notation of Definition 4.20. Then there exists a constant $C = C(\bar{\Omega}) > 0$, depending only on $\bar{\Omega}$ but independent of $(r_{i,j})_{i,j\in\{1,2,3\},i\neq j}$, and interpolation functions*

$$\lambda_i \colon \bigcup_{t\in[0,T]} \big(B_r(\mathcal{T}(t)) \cap \overline{W}_i(t)\big) \setminus \mathcal{T}(t) \times \{t\} \to [0,1]$$

for every $i \in \{1, 2, 3\}$ which satisfy the following properties:

i) It holds for all $t \in [0, T]$ that

$$\lambda_i(x,t) = 0 \quad \text{for} \quad x \in \big(\partial W_i(t) \cap \partial W_{i,i+1}(t)\big) \setminus \mathcal{T}(t), \tag{4.141}$$
$$\lambda_i(x,t) = 1 \quad \text{for} \quad x \in \big(\partial W_i(t) \cap \partial W_{i-1,i}(t)\big) \setminus \mathcal{T}(t). \tag{4.142}$$

ii) We have the estimates ($r_{\min} := r_{1,2} \wedge r_{2,3} \wedge r_{3,1}$)

$$|\nabla\lambda_i(x,t)| \leq C\,\mathrm{dist}(x,\mathcal{T}(t))^{-1}, \quad |\partial_t\lambda_i(x,t)| \leq Cr_{\min}^{-1}\,\mathrm{dist}(x,\mathcal{T}(t))^{-1}, \tag{4.143}$$

$$|\nabla^2\lambda_i(x,t)| \leq C\,\mathrm{dist}(x,\mathcal{T}(t))^{-2} \tag{4.144}$$

for all $t \in [0,T]$ *and all* $x \in \big(B_r(\mathcal{T}(t)) \cap \overline{W}_i(t)\big) \setminus \mathcal{T}(t)$*. Furthermore, it holds*

$$\nabla\lambda_i(x,t) = 0, \quad \partial_t\lambda_i(x,t) = 0, \tag{4.145}$$

$$\nabla^2\lambda_i(x,t) = 0 \tag{4.146}$$

for all $t \in [0,T]$ *and all* $x \in \big(B_r(\mathcal{T}(t)) \cap \partial W_i(t)\big) \setminus \mathcal{T}(t)$*.*

iii) Expressing the evolving triple junction via $\mathcal{T}(t) = \{p(t)\}$ *for all* $t \in [0,T]$*, we have a bound on the advective derivative*

$$\Big|\partial_t\lambda_i(x,t) + \Big(\frac{\mathrm{d}}{\mathrm{d}t}p(t)\cdot\nabla\Big)\lambda_i(x,t)\Big| \leq Cr_{\min}^{-2} \tag{4.147}$$

for all $t \in [0,T]$ *and all* $x \in \big(B_r(\mathcal{T}(t)) \cap \overline{W}_i(t)\big) \setminus \mathcal{T}(t)$*.*

Proof. Due to (4.74), the interpolation wedge $W_i(t)$ is the restriction to $B_r(\mathcal{T}(t))$ of the interior of the conical hull spanned by two unit vectors $X^i_{i,i+1}(t)$ and $X^i_{i-1,i}(t)$, whereas $W_{i,i+1}(t)$ is the restriction to $B_r(\mathcal{T}(t))$ of the interior of the conical hull spanned by unit vectors $X^i_{i,i+1}(t)$ and $X^{i+1}_{i,i+1}(t)$ due to (4.73). In particular, we can represent $\partial W_i(t) \cap \partial W_{i,i+1}(t) = \{\gamma X^i_{i,i+1}(t)\colon \gamma \geq 0\}$ and $\partial W_i(t) \cap \partial W_{i-1,i}(t) = \{\gamma X^i_{i-1,i}(t)\colon \gamma \geq 0\}$. As the vectors $X^i_{i,i+1}(t)$ and $X^i_{i-1,i}(t)$ can be expressed as a (fixed-in-time) linear combination of the unit-normals $\bar{\mathrm{n}}_{i,j}(p(t),t)$ at the triple junction, we have due to (4.102), (4.98) and (4.101) the bounds

$$\Big|\frac{\mathrm{d}}{\mathrm{d}t}X^i_{i,i+1}(t)\Big| + \Big|\frac{\mathrm{d}}{\mathrm{d}t}X^i_{i-1,i}(t)\Big| \leq Cr_{\min}^{-2} \leq Cr_{\min}^{-1}\,\mathrm{dist}(x,\mathcal{T}(t))^{-1} \tag{4.148}$$

for all $t \in [0,T]$, all $x \in B_r(\mathcal{T}(t))$, and all $i \in \{1,2,3\}$.

By Definition 4.20, the opening angle θ_i of the interpolation wedge W_i, defined by $\cos(\theta_i) = X^i_{i,i+1}(t)\cdot X^i_{i-1,i}(t) \in (0,1)$, is time-independent and satisfies $\theta_i \in (0,\frac{\pi}{2})$. (The angles only depend on $\bar{\Omega}$ through the surface tensions.) Let $\widetilde{\lambda}\colon \mathbb{R} \to [0,1]$ be any smooth function such that $\widetilde{\lambda} \equiv 0$ on $(-\infty,\frac{1}{3}]$ and $\widetilde{\lambda} \equiv 1$ on $[\frac{2}{3},\infty)$. We define

$$\lambda_i(x,t) := \widetilde{\lambda}\left(\frac{1 - X^i_{i,i+1}(t)\cdot\frac{x-p(t)}{|x-p(t)|}}{1-\cos\theta_i}\right).$$

Then the properties (4.141)–(4.146) are immediate consequences of the definitions and the bounds (4.148) and (4.102); cf. also the subsequent computation.

It remains to check the bound (4.147) on the advective derivative. To this end, we abbreviate $\lambda_i(x,t) = \widehat{\lambda}_i\big(X^i_{i,i+1}(t)\cdot\frac{x-p(t)}{|x-p(t)|}\big)$ for the obvious choice of function $\widehat{\lambda}_i\colon \mathbb{R} \to [0,1]$ and simply compute

$$\begin{aligned}
&\partial_t\lambda_i(x,t)\\
&= -\widehat{\lambda}'_i\frac{X^i_{i,i+1}(t)}{|x-p(t)|}\cdot\Big(\mathrm{Id} - \frac{x-p(t)}{|x-p(t)|}\otimes\frac{x-p(t)}{|x-p(t)|}\Big)\frac{\mathrm{d}}{\mathrm{d}t}p(t) + \widehat{\lambda}'_i\frac{x-p(t)}{|x-p(t)|}\cdot\frac{\mathrm{d}}{\mathrm{d}t}X^i_{i,i+1}(t)\\
&= -\Big(\frac{\mathrm{d}}{\mathrm{d}t}p(t)\cdot\nabla\Big)\lambda_i(x,t) + \widehat{\lambda}'_i\frac{x-p(t)}{|x-p(t)|}\cdot\frac{\mathrm{d}}{\mathrm{d}t}X^i_{i,i+1}(t)
\end{aligned}$$

where $\widehat{\lambda}'_i$ is evaluated at $X^i_{i,i+1}(t)\cdot\frac{x-p(t)}{|x-p(t)|}$. From this, the last remaining claim (4.147) immediately follows due to the estimate (4.148). ☐

Equipped with these interpolating functions we are finally in the position to prove the main result of this section.

Proof of Proposition 4.22. Step 1: Interpolation of the vector fields. We define (not yet normalized) extensions of the normal vector fields $\bar{\mathrm{n}}_{i,j}|_{\bar{I}_{i,j}}$ on the space-time neighborhood of the triple junction $\mathcal{U}_r := \bigcup_{t\in[0,T]} B_r(\mathcal{T}(t)) \times \{t\}$ as follows:

$$\widehat{\xi}_{i,i+1}(x,t) := \begin{cases} R_{(i,j)}\widetilde{\xi}_{j,j+1}(x,t) & \text{if } x \in W_{j,j+1}(t), \\ (1-\lambda_j(x,t))R_{(i,j)}\widetilde{\xi}_{j,j+1}(x,t) \\ \quad + \lambda_j(x,t)R_{(i,j-1)}\widetilde{\xi}_{j-1,j}(x,t) & \text{if } x \in \overline{W}_j(t), \end{cases} \tag{4.149}$$

and $\widehat{\xi}_{i+1,i} := -\widehat{\xi}_{i,i+1}$ for $i \in \{1,2,3\}$. The velocity field is given by

$$B(x,t) := \begin{cases} \widetilde{B}_{(j,j+1)}(x,t) & \text{if } x \in W_{j,j+1}(t), \\ (1-\lambda_j(x,t))\widetilde{B}_{(j,j+1)}(x,t) \\ \quad + \lambda_j(x,t)\widetilde{B}_{(j-1,j)}(x,t) & \text{if } x \in \overline{W}_j(t). \end{cases} \tag{4.150}$$

In the subsequent steps of the proof, we first establish all required properties in terms of the vector fields $\widehat{\xi}_{i,j}$ and B. Only in the penultimate step we will choose the radius $\hat{r} = \hat{r}(\bar{\Omega}) \leq r$ and define unit-length vector fields $\xi_{i,j}$ by normalization of the vector fields $\widehat{\xi}_{i,j}$ defined in (4.149) above. The last step is then devoted to verify the required properties for the normalized vector fields $\xi_{i,j}$.

Step 2: Regularity of $\widehat{\xi}_{i,j}$ and B, the estimates (4.84) and (4.85), and properties i)–iii). We first remark that the above definitions make sense due to the second inclusion in (4.75) and the inclusion in (4.76). Indeed, these inclusions are precisely what is needed so that the building blocks $\widetilde{\xi}_{i,i+1}$ and $\widetilde{B}_{(i,i+1)}$ are only evaluated on their domains of definition.

For every $i \in \{1,2,3\}$, we obtain $\widehat{\xi}_{i,i+1}(x,t) = \widetilde{\xi}_{i,i+1}(x,t) = \bar{\mathrm{n}}_{i,i+1}(x,t)$ for all $t \in [0,T]$ and all $x \in \mathcal{T}_{i,i+1}(t) \cap B_r(\mathcal{T}(t))$ from the first inclusion in (4.75) and the ansatz (4.90), taking care of property i); obviously except for the normalization condition away from the interfaces. The second property $\widehat{\xi}_{i,j} = -\widehat{\xi}_{j,i}$ for $i,j \in \{1,2,3\}$ with $i \neq j$ holds by definition. For every $j \in \{1,2,3\}$ we moreover have

$$\sigma_{1,2}\widehat{\xi}_{1,2} + \sigma_{2,3}\widehat{\xi}_{2,3} + \sigma_{3,1}\widehat{\xi}_{3,1} \equiv \big(\sigma_{1,2}R_{(1,j)} + \sigma_{2,3}R_{(2,j)} + \sigma_{3,1}R_{(3,j)}\big)\,\widetilde{\xi}_{j,j+1} = 0$$

on $W_{j,j+1}(t)$ by the defining property (4.105) of the rotations $R_{(i,j)}$. A similar argument ensures validity of (4.80) on the interpolation wedges $\overline{W}_j(t)$.

By the compatibility condition (4.108) for the auxiliary vector fields $\widetilde{\xi}_{j,j+1}$ at the triple junction, as well as the conditions (4.141) and (4.142) for the interpolation functions, the vector fields $\widehat{\xi}_{i,j}$ are continuous. Similarly, their first and second derivatives are continuous across the boundaries of the interpolation wedges $\bigcup_{t\in[0,T]} \big(\big(B_r(\mathcal{T}(t)) \cap \partial W_i(t)\big) \setminus \mathcal{T}(t)\big) \times \{t\}$ by the properties (4.145) and (4.146) of the interpolation functions.

Moreover, all spatial derivatives up to second order are bounded in $\mathcal{U}_r \setminus \mathcal{T}$ with the asserted estimate given by (4.84). Indeed, in the interface wedges $W_{j,j+1}$ this follows from the estimates (4.91) and the definition (4.149). On the closure of the interpolation wedges W_j, we first compute using the definition (4.149)

$$\begin{aligned} \nabla\widehat{\xi}_{i,i+1} &= (1-\lambda_j)\nabla R_{(i,j)}\widetilde{\xi}_{j,j+1} + \lambda_j \nabla R_{(i,j-1)}\widetilde{\xi}_{j-1,j} \\ &\quad - (R_{(i,j)}\widetilde{\xi}_{j,j+1} - R_{(i,j-1)}\widetilde{\xi}_{j-1,j})\nabla\lambda_j, \end{aligned} \tag{4.151}$$

$$\begin{aligned} \nabla^2\widehat{\xi}_{i,i+1} &= (1-\lambda_j)\nabla^2 R_{(i,j)}\widetilde{\xi}_{j,j+1} + \lambda_j \nabla^2 R_{(i,j-1)}\widetilde{\xi}_{j-1,j} \\ &\quad - (\nabla R_{(i,j)}\widetilde{\xi}_{j,j+1} - \nabla R_{(i,j-1)}\widetilde{\xi}_{j-1,j})\nabla\lambda_j \\ &\quad - (R_{(i,j)}\widetilde{\xi}_{j,j+1} - R_{(i,j-1)}\widetilde{\xi}_{j-1,j})\nabla^2\lambda_j. \end{aligned} \tag{4.152}$$

Now, the bound (4.84) with respect to spatial derivatives follows from the blowup (4.143) and (4.144) of the interpolation functions, the estimates (4.91), (4.134) and (4.135) for the auxiliary vector fields $\widetilde{\xi}_{j,j+1}$, as well as the estimate (4.77). In total, this proves $\widehat{\xi}_{i,j} \in C_t^0 C_x^2(\overline{\mathcal{U}_r} \setminus \mathcal{T})$. The other property $\widehat{\xi}_{i,j} \in C_t^1 C_x^0(\overline{\mathcal{U}_r} \setminus \mathcal{T})$ together with the asserted bound (4.84) in terms of the time derivative follows similarly making use of Lemma 4.23, (4.134), (4.143), (4.77) and the computation on the closure of W_j

$$\partial_t \widehat{\xi}_{i,i+1} = (1-\lambda_j)\partial_t R_{(i,j)} \widetilde{\xi}_{j,j+1} + \lambda_j \partial_t R_{(i,j-1)} \widetilde{\xi}_{j-1,j} - (R_{(i,j)} \widetilde{\xi}_{j,j+1} - R_{(i,j-1)} \widetilde{\xi}_{j-1,j}) \partial_t \lambda_j.$$

We proceed with the regularity of the velocity field B. First, by the compatibility condition (4.123) for the auxiliary velocity fields $\widetilde{B}_{(j,j+1)}$ at the triple junction, as well as the conditions (4.141) and (4.142) for the interpolation functions, the velocity field B is continuous. The asserted bound (4.85) is a consequence of the definition (4.150), the estimates (4.121), (4.136) and (4.137) for the auxiliary velocity fields, the controlled blowup (4.143) of the interpolation functions, the estimate (4.77) as well as the computation

$$\nabla B = (1-\lambda_j)\nabla \widetilde{B}_{(j,j+1)} + \lambda_j \nabla \widetilde{B}_{(j-1,j)} + (\widetilde{B}_{(j-1,j)} - \widetilde{B}_{(j,j+1)}) \nabla \lambda_j, \tag{4.153}$$

$$\nabla^2 B = (1-\lambda_j)\nabla^2 \widetilde{B}_{(j,j+1)} + \lambda_j \nabla^2 \widetilde{B}_{(j-1,j)} + (\nabla \widetilde{B}_{(j-1,j)} - \nabla \widetilde{B}_{(j,j+1)}) \nabla \lambda_j + (\widetilde{B}_{(j-1,j)} - \widetilde{B}_{(j,j+1)}) \nabla^2 \lambda_j \tag{4.154}$$

on the closure of W_j. This proves $B \in C_t^0 C_x^2(\overline{\mathcal{U}_r} \setminus \mathcal{T})$.

Step 3: Proof of the estimate ($r_{\min} := r_{1,2} \wedge r_{2,3} \wedge r_{3,1}$)

$$|\partial_t \widehat{\xi}_{i,j} + (B \cdot \nabla) \widehat{\xi}_{i,j} + (\nabla B)^{\mathsf{T}} \widehat{\xi}_{i,j}| \le C r_{\min}^{-3} \operatorname{dist}(\cdot, \bar{I}_{i,j}) \quad \text{in } \mathcal{U}_r. \tag{4.155}$$

By the skew-symmetry $\widehat{\xi}_{i,j} = -\widehat{\xi}_{j,i}$, we only have to prove (4.155) for $j = i+1$. Let $i \in \{1,2,3\}$. First, we remark that the validity of (4.81) for the vector field $\widehat{\xi}_{i,i+1}$ on the interface wedges $W_{j,j+1}$ for all $j = 1,2,3$ follows from the estimate (4.128), the definitions (4.149) and (4.150), and the estimate (4.78). Hence, it remains to prove the bound (4.155) for $\widehat{\xi}_{i,i+1}$ on each interpolation wedge W_j, $j \in \{1,2,3\}$.

To this end, let us fix $j \in \{1,2,3\}$. For the sake of readability, let us introduce the abbreviations, $\lambda = \lambda_j$, $R = R_{(i,j)}$, $R' = R_{(i,j-1)}$, $\widetilde{\xi} = \widetilde{\xi}_{j,j+1}$, $\widetilde{\xi}' = \widetilde{\xi}_{j-1,j}$, $\widetilde{B} = \widetilde{B}_{(j,j+1)}$ and $\widetilde{B}' = \widetilde{B}_{(j-1,j)}$. Using the product rule and the definition (4.149) of $\widehat{\xi}_{i,i+1}$ on the closure of the interpolation wedge W_j, we have

$$\begin{aligned}\Big(\partial_t + (B\cdot\nabla) + (\nabla B)^{\mathsf{T}}\Big)\widehat{\xi}_{i,i+1} = (1-\lambda)\Big(\partial_t + (B\cdot\nabla) + (\nabla B)^{\mathsf{T}}\Big)R\widetilde{\xi} \\ + \lambda\Big(\partial_t + (B\cdot\nabla) + (\nabla B)^{\mathsf{T}}\Big)R'\widetilde{\xi}' \\ + \big(\partial_t\lambda + (B\cdot\nabla)\lambda\big)(R'\widetilde{\xi}' - R\widetilde{\xi}).\end{aligned} \tag{4.156}$$

We want to manipulate the first two right-hand side terms to make the advection equations (4.128) appear. To this end, we write $B = \widetilde{B} + \lambda(\widetilde{B}' - \widetilde{B})$ and obtain

$$\begin{aligned}\Big(\partial_t + (B\cdot\nabla) + (\nabla B)^{\mathsf{T}}\Big)R\widetilde{\xi} =& \big(\partial_t + (\widetilde{B}\cdot\nabla) + (\nabla\widetilde{B})^{\mathsf{T}}\big)R\widetilde{\xi} \\ &+ \big(\lambda(\widetilde{B}' - \widetilde{B})\cdot\nabla\big)R\widetilde{\xi} + \lambda\big(\nabla\widetilde{B}' - \nabla\widetilde{B}\big)^{\mathsf{T}}R\widetilde{\xi} \\ &+ \big((\widetilde{B}' - \widetilde{B})\cdot R\widetilde{\xi}\big)\nabla\lambda.\end{aligned}$$

Using the compatibility conditions (4.136)–(4.137) for the auxiliary velocity fields alongside with the bounds (4.91), (4.143), and the estimate (4.77) one shows that the last three right-hand side terms are of order $O(r_{\min}^{-3}\operatorname{dist}(\cdot,\bar{I}_{i,i+1}))$. By (4.128) and (4.77) the first term on the right-hand side is also of order $O(r_{\min}^{-3}\operatorname{dist}(\cdot,\bar{I}_{i,i+1}))$.

Consequently, the first term on the right-hand side of equation (4.156) is of required order. A similar argument shows that the second one is, too. Finally, also the third term is of the desired order by the bounds (4.143) on λ, the second-order compatibility (4.134), and the estimate (4.77), concluding the proof of (4.155).

Step 4: Proof of the estimate ($r_{\min} := r_{1,2}\wedge r_{2,3}\wedge r_{3,1}$)

$$|\nabla\cdot\widehat{\xi}_{i,j} + B\cdot\widehat{\xi}_{i,j}| \leq Cr_{\min}^{-2}\operatorname{dist}(\cdot,\bar{I}_{i,j}) \quad \text{in } \mathcal{U}_r. \tag{4.157}$$

Let $i\in\{1,2,3\}$, and by the skew-symmetry $\widehat{\xi}_{i,j}=-\xi_{j,i}$, it again suffices to prove (4.157) in terms of $\widehat{\xi}_{i,i+1}$. Note that because of (4.149)–(4.150), (4.129), and (4.78) it only remains to prove (4.157) for the vector field $\widehat{\xi}_{i,i+1}$ in the closure of the interpolation wedges W_j, $j\in\{1,2,3\}$. We again fix $j\in\{1,2,3\}$ and use the same abbreviations as in the previous step.

We proceed similarly as in the proof of (4.155). Making use of (4.149) we get

$$\nabla\cdot\widehat{\xi}_{i,i+1} = (1-\lambda)\nabla\cdot R\widetilde{\xi} + \lambda\nabla\cdot R'\widetilde{\xi}' + \big((R'\widetilde{\xi}'-R\widetilde{\xi})\cdot\nabla\big)\lambda.$$

By the controlled blowup (4.143) of the interpolation functions, the compatibility estimate (4.134), the approximate mean curvature flow equation (4.129) and the estimate (4.77) it then follows

$$\nabla\cdot\widehat{\xi}_{i,i+1} = -(1-\lambda)\widetilde{B}\cdot R\widetilde{\xi} - \lambda\widetilde{B}'\cdot R'\widetilde{\xi}' + O(r_{\min}^{-2}\operatorname{dist}(\cdot,\bar{I}_{i,i+1})).$$

Finally, the compatibility estimates (4.135) and (4.136) in conjunction with definitions (4.149)–(4.150) and the estimate (4.77) imply the desired bound (4.157).

Step 5: Proof of the estimates ($r_{\min} := r_{1,2}\wedge r_{2,3}\wedge r_{3,1}$)

$$\big|1-|\widehat{\xi}_{i,j}|^2\big| \leq Cr_{\min}^{-2}\operatorname{dist}^2(\cdot,\bar{I}_{i,j}) \qquad \text{in } \mathcal{U}_r, \tag{4.158}$$

$$r_{\min}^2\big|\partial_t|\widehat{\xi}_{i,j}|^2\big| + r_{\min}\big|\nabla|\widehat{\xi}_{i,j}|^2\big| \leq Cr_{\min}^{-1}\operatorname{dist}(\cdot,\bar{I}_{i,j}) \qquad \text{in } \mathcal{U}_r. \tag{4.159}$$

Let $i\in\{1,2,3\}$. The validity of (4.158) resp. (4.159) for the vector field $\widehat{\xi}_{i,i+1}$ in interface wedges $W_{j,j+1}$, $j\in\{1,2,3\}$, is directly implied by the definition (4.149), the bound (4.78), as well as the estimates (4.130) resp. (4.131)–(4.132).

For all $j\in\{1,2,3\}$, we then may compute on the closure of the interpolation wedge W_j by (4.149) and adding zero several times

$$\begin{aligned}|\widehat{\xi}_{i,i+1}|^2 &= \lambda^2|R\widetilde{\xi}|^2 + (1-\lambda)^2|R'\widetilde{\xi}'|^2 + 2\lambda(1-\lambda)(R\widetilde{\xi}\cdot R'\widetilde{\xi}') \\ &= 1-\lambda(1-\lambda)|R\widetilde{\xi}-R'\widetilde{\xi}'|^2 + \lambda(|R\widetilde{\xi}|^2-1) + (1-\lambda)(|R'\widetilde{\xi}'|^2-1).\end{aligned} \tag{4.160}$$

Hence, the estimates (4.158) and (4.159) are the result of the estimates (4.91), (4.134), (4.130)–(4.132), (4.143) and (4.77).

Step 6: Choice of $\hat{r}=\hat{r}(\bar{\Omega})\leq r$ and definition of normalized vector fields $\xi_{i,j}$. We first define $\hat{r} := r\wedge\frac{1}{\sqrt{2C}}(r_{1,2}\wedge r_{2,3}\wedge r_{3,1})$ with $C>0$ being the constant of (4.158). Note then that (4.158) implies

$$\frac{1}{2}\leq|\widehat{\xi}_{i,j}|^2\leq\frac{3}{2} \qquad \text{in } \mathcal{U}_{\hat{r}} = \bigcup_{t\in[0,T]} B_{\hat{r}}(\mathcal{T}(t))\times\{t\} \tag{4.161}$$

for all $i,j \in \{1,2,3\}$ with $i \neq j$. We may then define

$$\xi_{i,j}(x,t) := \frac{\widehat{\xi}_{i,j}(x,t)}{|\widehat{\xi}_{i,j}(x,t)|} \qquad \text{for all } (x,t) \in \mathcal{U}_{\hat{r}} \tag{4.162}$$

and all $i,j \in \{1,2,3\}$ with $i \neq j$. It remains to verify the asserted properties in terms of the vector fields $\xi_{i,j}$ and B on the restricted space-time domain $\mathcal{U}_{\hat{r}}$.

Step 7: Conclusion. Since $\xi_{i,j}(x,t) = \widehat{\xi}_{i,j}(x,t)$ for all $t \in [0,T]$ and all $x \in \mathcal{T}_{i,j}(t) \cap B_{\hat{r}}(\mathcal{T}(t))$, property *i)* is an immediate consequence of definition (4.162). Note that (4.83) trivially follows. Obviously, the skew-symmetry relation in property *ii)* carries over from $\widehat{\xi}_{i,j}$ to $\xi_{i,j}$. Validity of the Herring angle condition (4.80) in terms of the vector fields $\xi_{i,j}$ also follows immediately from their definition (4.162) and the fact that the vector fields $\widehat{\xi}_{i,j}$ already satisfy (4.80). Indeed, recall that the vector fields $\widehat{\xi}_{1,2}$, $\widehat{\xi}_{2,3}$ resp. $\widehat{\xi}_{3,1}$ can be obtained from each of the other ones by a rotation due to Lemma 4.24.

For a proof of (4.84) (recall that the estimate (4.85) is already part of *Step 2*), we simply compute

$$(\partial_t, \nabla)\xi_{i,j} = \frac{1}{|\widehat{\xi}_{i,j}|}\Big(\mathrm{Id} - \frac{\widehat{\xi}_{i,j}}{|\widehat{\xi}_{i,j}|} \otimes \frac{\widehat{\xi}_{i,j}}{|\widehat{\xi}_{i,j}|}\Big)(\partial_t, \nabla)\widehat{\xi}_{i,j}. \tag{4.163}$$

Because of (4.161), the estimate $\hat{r}|\nabla\xi_{i,j}| + \hat{r}^2|\partial_t\xi_{i,j}| \leq C$ throughout $\mathcal{U}_{\hat{r}} \setminus \mathcal{T}$ thus follows from the corresponding estimate in terms of $\widehat{\xi}_{i,j}$ from *Step 2* of this proof. One proceeds similarly for the required estimate on the second-order spatial derivative.

It therefore remains to argue that the estimates (4.81) and (4.82) hold true. Using the product rule and the choice of $\hat{r}$ in the previous step, we may on $\mathcal{U}_{\hat{r}}$ compute

$$\begin{aligned}
&\Big(\partial_t + (B\cdot\nabla) + (\nabla B)^\mathsf{T}\Big)\frac{\widehat{\xi}_{i,j}}{|\widehat{\xi}_{i,j}|} \\
&= \frac{1}{|\widehat{\xi}_{i,j}|}\Big(\partial_t + (B\cdot\nabla) + (\nabla B)^\mathsf{T}\Big)\widehat{\xi}_{i,j} - \frac{1}{2|\widehat{\xi}_{i,j}|^3}\widehat{\xi}_{i,j}\,(\partial_t + (B\cdot\nabla))\,|\widehat{\xi}_{i,j}|^2
\end{aligned}$$

By (4.155) and (4.161), the first right-hand side term is of the order $O(\hat{r}^{-3}\operatorname{dist}(\cdot,\bar{I}_{i,j}))$. To handle the second term, it suffices to apply the estimate (4.159), the estimate on the magnitude of the velocity $|B| \leq C\hat{r}^{-1}$ from *Step 2*, and the estimate (4.161). This proves the estimate (4.81).

We now turn to the proof of (4.82). Here, we compute on $\mathcal{U}_{\hat{r}}$ by means of the choice of $\hat{r}$ in the previous step

$$\nabla\cdot\frac{\widehat{\xi}_{i,j}}{|\widehat{\xi}_{i,j}|} = \frac{\nabla\cdot\widehat{\xi}_{i,j}}{|\widehat{\xi}_{i,j}|} - \frac{(\widehat{\xi}_{i,j}\cdot\nabla)|\widehat{\xi}_{i,j}|^2}{2|\widehat{\xi}_{i,j}|^3}.$$

It is immediate from the estimates (4.161) and (4.159) to estimate the second term as being of order $O(\hat{r}^{-2}\operatorname{dist}(\cdot,\bar{I}_{i,j}))$. Using the approximate mean curvature flow equation (4.157) for the first term and the definition (4.162) of $\xi_{i,j}$ then yields

$$\nabla\cdot\frac{\widehat{\xi}_{i,j}}{|\widehat{\xi}_{i,j}|} = -B\cdot\frac{\widehat{\xi}_{i,j}}{|\widehat{\xi}_{i,j}|} + O\big(\hat{r}^{-2}\operatorname{dist}(\cdot,\bar{I}_{i,j})\big) = -B\cdot\xi_{i,j} + O\big(\hat{r}^{-2}\operatorname{dist}(\cdot,\bar{I}_{i,j})\big).$$

In total, this gives (4.82). □

Finally, we provide the elementary-geometric proof for the existence of wedges with the desired properties.

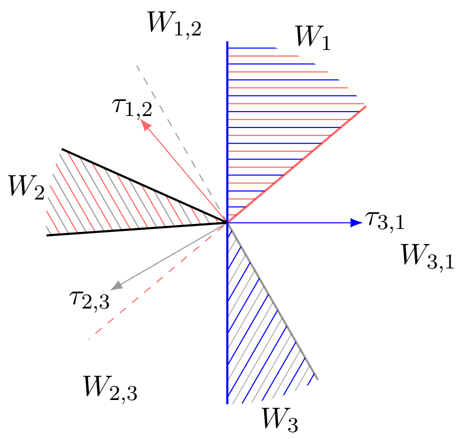

Figure 4.7: If the angle between two tangent vectors is less than 90°, we trisect it to obtain the desired interpolation wedge, see for example W_2. Otherwise, we take the corresponding intersection of the half-spaces, as is done for W_1 and W_3. The wedges $W_{1,2}$, $W_{2,3}$ and $W_{3,1}$ lie inbetween.

Proof of Lemma 4.21. We recall some notation in conjunction with Definition 4.17. For each (cyclic) $i \in \{1,2,3\}$ and all $t \in [0,T]$, the unit vector $\bar{\mathrm{t}}_{i,i+1}(p(t),t)$ denotes the tangent of $\bar{I}_{i,i+1}(t)$ at the triple junction $\mathcal{T}(t) = \{p(t)\}$, with the orientation chosen such that it "points away" from the curve $\bar{I}_{i,i+1}(t)$. Define then $\bar{\tau}_{i,i+1}(t) := -\bar{\mathrm{t}}_{i,i+1}(p(t),t)$ and $\mathbb{H}_{\bar{\tau}_{i,i+1}}(t) = \{x \in \mathbb{R}^2 \colon (x - p(t)) \cdot \bar{\tau}_{i,i+1}(t) > 0\}$. Note that

$$\sigma_{1,2}\bar{\tau}_{1,2}(t) + \sigma_{2,3}\bar{\tau}_{2,3}(t) + \sigma_{3,1}\bar{\tau}_{3,1}(t) = 0, \quad t \in [0,T]. \tag{4.164}$$

Using the balance of forces condition (4.164) together with the strict triangle inequality (4.8) we see that there exist constant-in-time angles $\theta_i \in (0,\pi)$ such that $\cos(\theta_i) = \bar{\tau}_{i,i+1}(t)\cdot\bar{\tau}_{i-1,i}(t)$ for $i = 1,2,3$ and $t \in [0,T]$. For the following argument, see also Figure 4.7.

If $\theta_i > \frac{\pi}{2}$ we may define $X^i_{i,i+1}(t), X^i_{i-1,i}(t) \in \mathbb{S}^1$ such that the cone $C_i(t) := \mathcal{T}(t) + \{\gamma_1 X^i_{i,i+1}(t) + \gamma_2 X^i_{i-1,i}(t) \colon \gamma_1, \gamma_2 \in (0,\infty)\}$ satisfies $C_i(t) = \mathbb{H}_{\bar{\tau}_{i,i+1}}(t) \cap \mathbb{H}_{\bar{\tau}_{i-1,i}}(t)$. Otherwise, we choose $X^i_{i,i+1}(t), X^i_{i-1,i}(t) \in \mathbb{S}^1$ such that the cone $C_i(t) := \mathcal{T}(t) + \{\gamma_1 X^i_{i,i+1}(t) + \gamma_2 X^i_{i-1,i}(t) \colon \gamma_1, \gamma_2 \in (0,\infty)\}$ is the middle third of the cone $\{\gamma_1\bar{\tau}_{i,i+1}(t) + \gamma_2\bar{\tau}_{i-1,i}(t) \colon \gamma_1, \gamma_2 \in (0,\infty)\}$. In both cases, defining for $i \in \{1,2,3\}$ and $t \in [0,T]$ the cone $C_{i,i+1}(t) := \mathcal{T}(t) + \{\gamma_1 X^i_{i,i+1}(t) + \gamma_2 X^{i+1}_{i,i+1}(t) \colon \gamma_1, \gamma_2 \in (0,\infty)\}$ we then have

$$C_i(t) \subset \mathbb{H}_{\bar{\tau}_{i,i+1}}(t) \cap \mathbb{H}_{\bar{\tau}_{i-1,i}}(t), \tag{4.165}$$

$$C_{i,i+1}(t) \subset \mathbb{H}_{\bar{\tau}_{i,i+1}}(t), \tag{4.166}$$

$$\bigcup_{i=1,2,3} \overline{C_i(t)} \cup \overline{C_{i,i+1}(t)} = \mathbb{R}^2, \tag{4.167}$$

$$p(t) + \tau_{i,i+1}(t) \in C_{i,i+1}(t) \tag{4.168}$$

for all $i \in \{1,2,3\}$ and all $t \in [0,T]$.

Let $r \in (0, r_{1,2} \wedge r_{2,3} \wedge r_{3,1}]$, and for $i \in \{1,2,3\}$ and $t \in [0,T]$ define $W_i(t) := C_i(t) \cap B_r(\mathcal{T}(t))$ and $W_{i,i+1}(t) := C_{i,i+1}(t) \cap B_r(\mathcal{T}(t))$. As (4.72) follows immediately from (4.167) it suffices to argue that there exists a constant $C = C(\sigma) \geq 1$, depending only on the surface tensions at the triple junction, such that $r := \frac{1}{C}(r_{1,2} \wedge r_{2,3} \wedge r_{3,1})$ gives rise to the inclusions (4.75)–(4.76) and the comparability of distances in form of (4.77)–(4.78).

First, (4.76) follows from (4.165) and the fact that $\mathbb{H}_{\bar{\tau}_{i,i+1}}(t) \cap B_r(\mathcal{T}(t))$ is included in the t-time slice of the image of the diffeomorphism from (4.45), see (4.46). Analogously, one derives the second inclusion of (4.75) from (4.166). For the first inclusion of (4.75), i.e., the curve trapping condition, one may argue as follows. On one side, it follows from the endpoint ball condition *ii)* of Definition 4.17 and $r \leq r_{1,2} \wedge r_{2,3} \wedge r_{3,1}$ that $\mathcal{T}_{i,i+1}(t) \cap B_r(\mathcal{T}(t)) \subset \overline{\mathbb{H}_{\bar{\tau}_{i,i+1}}(t)} \cap B_r(\mathcal{T}(t))$. On the other side, based on the ball condition *i)* of Definition 4.17 at the triple junction $\mathcal{T}(t) = \{p(t)\}$, we may sharpen this inclusion to

$$\begin{aligned}&\mathcal{T}_{i,i+1}(t) \cap B_r(\mathcal{T}(t))\\&\subset \Big(\overline{\mathbb{H}_{\bar{\tau}_{i,i+1}}(t)} \cap B_r(\mathcal{T}(t))\Big) \setminus \Big(B_r\big(p(t)+r\bar{\mathrm{n}}_{i,i+1}(p(t),t)\big) \cup B_r\big(p(t)-r\bar{\mathrm{n}}_{i,i+1}(p(t),t)\big)\Big).\end{aligned}$$

Hence, the first inclusion of (4.75) follows after choosing $r \in (0, r_{1,2} \wedge r_{2,3} \wedge r_{3,1}]$ sufficiently small, with a proportionality constant depending only on the opening angles of the interface cones $C_{i,i+1}$.

We turn to the proof of the estimates (4.77)–(4.79). The estimate (4.78) is a consequence of the first inclusion of (4.75), the fact that the interface wedges $W_{i,i+1}$, $i \in \{1,2,3\}$, are separated from each other by the interpolation wedges W_i, $i \in \{1,2,3,\}$, and that within $B_r(\mathcal{T}(t))$ the distance to $\mathcal{T}_{i,i+1}$ equals the distance to $\bar{I}_{i,i+1}$ by Definition 4.17 and $r \in (0, r_{1,2} \wedge r_{2,3} \wedge r_{3,1}]$. The estimate (4.79) follows from similar considerations, exploiting again that the interface wedges are separated from each other by the interpolation wedges. Also the argument for the proof of (4.77) is analogous; at least once we improved the curve trapping condition (4.75) to a wedge which is strictly included in $W_{i,i+1}$. A possible choice for such a wedge is to simply bisect the angles formed by $\bar{\tau}_{i,i+1}, X^i_{i,i+1}$ and $\bar{\tau}_{i,i+1}, X^{i+1}_{i,i+1}$, respectively. The improvement of (4.75) then follows from possibly reducing $r \in (0, r_{1,2} \wedge r_{2,3} \wedge r_{3,1}]$ even further. This in turn can be done again at the cost of a proportionality constant depending only on the surface tensions at the triple junction. ☐

4.5.3 Local compatibility estimates

We conclude this section with a result verifying that the local constructions at a triple junction from Proposition 4.22 are (in a certain sense) suitable perturbations of the respective local constructions from Lemma 4.18 with respect to interfaces meeting at the triple junction. It is precisely at this stage where we rely on the freedom to choose a tangential component for the local velocity field from Lemma 4.18.

Proposition 4.29. *Let $d = 2$ and $P \in \mathbb{N}$, $P \geq 2$. Let $\bar{\Omega} = (\bar{\Omega}_1, \ldots, \bar{\Omega}_P)$ be a strong solution to multiphase mean curvature flow in the sense of Definition 4.14. Let $i, j \in \{1, \ldots, P\}$ such that $i \neq j$ and $\bar{I}_{i,j}$ is a non-trivial interface. Denote by $\mathcal{T}_c$ a space-time connected component of $\bar{I}_{i,j}$, and assume that $\mathcal{T}_c$ connects two evolving triple junctions $\mathcal{T}_{p_+}$ and $\mathcal{T}_{p_-}$, respectively. Let $\hat{r}_{p_+}, \hat{r}_{p_-} \in (0,1]$ denote the associated localization scales from Proposition 4.22, respectively. Finally, denote by $(\xi^c_{i,j}, B^c)$ the local vector fields from Lemma 4.18.*

Then there exists a choice of the tangential component α_c of B^c satisfying

$$\max_{k=0,1,2} (\hat{r}_{p_+} \wedge \hat{r}_{p_-} \wedge \ell)^{k+1} |\nabla^k \alpha_c| \leq C, \quad 3\ell := \min_{t\in[0,T]} \mathrm{dist}(\mathcal{T}_{p_+}(t), \mathcal{T}_{p_-}(t)), \tag{4.169}$$

throughout $\mathrm{im}(\Psi_{\mathcal{T}_c})$, so that at each of the two triple junctions $\mathcal{T}_p$, $p \in \{p_+, p_-\}$, the local vector fields $(\xi^p_{i,j}, B^p)$ from Proposition 4.22 (at scale $\hat{r}_p$) may be chosen so that they are locally compatible with $(\xi^c_{i,j}, B^c)$ in the sense that

$$\big|\xi^c_{i,j} - \xi^p_{i,j}\big| + \hat{r}_p \big|(\nabla\xi^c_{i,j} - \nabla\xi^p_{i,j})^\mathsf{T} \xi^c_{i,j}\big| \leq C\hat{r}_p^{-1}\,\mathrm{dist}(\cdot, \bar{I}_{i,j}), \tag{4.170}$$

$$\big|(\xi^c_{i,j} - \xi^p_{i,j}) \cdot \xi^c_{i,j}\big| \leq C\hat{r}_p^{-2}\,\mathrm{dist}^2(\cdot, \bar{I}_{i,j}), \tag{4.171}$$

$$\big|B^p - B^c\big| \leq C\hat{r}_p^{-3}\,\mathrm{dist}^2(\cdot, \bar{I}_{i,j}), \tag{4.172}$$

$$\big|\nabla B^p - \nabla B^c\big| \leq C\hat{r}_p^{-3}\,\mathrm{dist}(\cdot, \bar{I}_{i,j}) \tag{4.173}$$

in the region $B_{\frac{1}{2}(\hat{r}_p\wedge\ell)}(\mathcal{T}_p(t))\cap\big(W^p_{i,j}(t)\cup W^p_i(t)\cup W^p_j(t)\big)$ *for all* $t\in[0,T]$ *(where the wedges* $W^p_{i,j},W^p_i,W^p_j$ *are the ones from Definition 4.20 with respect to the triple junction* $\mathcal{T}_p$*). The constant* $C>0$ *in the above estimates* (4.169)–(4.173) *may depend on* $\bar{\Omega}$*, but is independent of* $\hat{r}_{p+}$*,* $\hat{r}_{p-}$ *and* ℓ*.*

Proof. The proof is split into three steps.

Step 1: Choice of vector fields. We take $(\xi^{p\pm}_{i,j},B^{p\pm})$ as constructed in the proof of Proposition 4.22. Moreover, we take $(\xi^c_{i,j},B^c)$ as defined in Lemma 4.18 with the following choice of the tangential component α_c. Let θ be a smooth cutoff function with $\theta(r)=1$ for $|r|\leq\frac{1}{2}$ and $\theta\equiv 0$ for $|r|\geq 1$. We then define

$$\alpha_c:=\theta\Big(\frac{\mathrm{dist}(\cdot,\mathcal{T}_{p_+})}{\ell\wedge\hat{r}_{p+}}\Big)B^{p+}\cdot\bar{\tau}_{i,j}+\theta\Big(\frac{\mathrm{dist}(\cdot,\mathcal{T}_{p_-})}{\ell\wedge\hat{r}_{p_-}}\Big)B^{p-}\cdot\bar{\tau}_{i,j},\quad\text{in }\mathrm{im}(\Psi_{\mathcal{T}_c}).\tag{4.174}$$

By the choice of the cutoff θ, this is indeed well-defined. The regularity estimate (4.169) is a direct consequence of the definition (4.174) and the estimates (4.98) and (4.85). Note that (4.169) in turn updates the estimate (4.59) to

$$\max_{k=0,1,2}(\hat{r}_{p_+}\wedge\hat{r}_{p_-}\wedge\ell)^{k+1}|\nabla^k B^c|\leq C\quad\text{in }\mathrm{im}(\Psi_{\mathcal{T}_c}),\tag{4.175}$$

with the constant $C>0$ being independent of $\hat{r}_{p_+}$, $\hat{r}_{p_-}$ and ℓ.

Step 2: Proof of (4.172) *and* (4.173). Let $p\in\{p_+,p_-\}$. First, we note that for all $t\in[0,T]$ it holds $B_{\frac{1}{2}(\hat{r}_p\wedge\ell)}(\mathcal{T}_p(t))\cap\big(W^p_{i,j}(t)\cup W^p_i(t)\cup W^p_j(t)\big)\subset\mathrm{im}(\Psi_{\mathcal{T}_c})$ due to (4.75)–(4.76). By means of the regularity estimates (4.85) and (4.175), the choice of the cutoff function θ, and the definition (4.174) of the tangential velocity of B^c, it thus suffices to prove $B^c=B^p$ within the interface wedge $W^p_{i,j}(t)\cap B_{\frac{1}{2}(\hat{r}_p\wedge\ell)}(\mathcal{T}_p(t))$ for all $t\in[0,T]$. However, by (4.174) the two vector fields agree in tangential direction. Their normal component in turn equals $H_{i,j}\bar{\mathrm{n}}_{i,j}$, which is evident for B^c from definition (4.56), and for B^p from the definitions (4.120) and (4.150).

Step 3: Proof of (4.170) *and* (4.171). Let again $p\in\{p_+,p_-\}$. Thanks to the regularity estimates (4.57) resp. (4.84) and the fact $(\nabla\xi^c_{i,j})^\mathsf{T}\xi^c_{i,j}=\frac{1}{2}\nabla|\xi^c_{i,j}|^2=0$, the asserted bounds (4.170) and (4.171) follow once we assured ourselves of the validity of $\xi^c_{i,j}-\xi^p_{i,j}=0$ and $(\nabla\xi^p_{i,j})^\mathsf{T}\xi^c_{i,j}=0$ along the local patch $\mathcal{T}_c(t)\cap B_{\frac{1}{2}(\hat{r}_p\wedge\ell)}(\mathcal{T}_p(t))$ for all $t\in[0,T]$. The former is immediate from both vector fields being extensions of the unit normal $\bar{\mathrm{n}}_{i,j}|_{\bar{I}_{i,j}}$, whereas the latter then follows from adding zero and $|\xi^p_{i,j}|^2\equiv 1$ in form of $(\nabla\xi^p_{i,j})^\mathsf{T}\xi^c_{i,j}=(\nabla\xi^p_{i,j})^\mathsf{T}\xi^p_{i,j}=\frac{1}{2}\nabla|\xi^p_{i,j}|^2=0$. □

4.6 Gradient flow calibrations for a regular network

The aim of this section is to prove Proposition 4.6: Given a strong solution to multiphase mean curvature flow (in the sense of an evolving network of smooth curves meeting at triple junctions), we construct a gradient flow calibration by gluing together the local constructions from the previous two sections.

More precisely, in Section 4.6.1 we define a partition of unity which allows us to localize around each topological feature $\mathcal{T}_n$, i.e., a two phase interface or a triple junction, for some suitable index $n\in\mathbb{N}$. We then define the global vector fields $\xi_{i,j}$ for $i,j\in\{1,\dots,P\}$ with $i\neq j$ and B in Section 4.6.2 by gluing together suitable locally defined vector fields $\xi^n_{i,j}$ and B^n. Most of these vector fields were already constructed in Sections 4.4 and 4.5, so that in Section 4.6.2 we only need to define those vector fields $\xi^n_{i,j}$ for which at least one of the two phases i or j is not present at the selected topological feature $\mathcal{T}_n$. For their construction we crucially use the coercivity condition of Definition 4.8 on the matrix of surface tensions. In Section 4.6.3, we prove the compatibility between the local constructions of the vector fields of adjacent topological features, which then allows us in Section 4.6.4 to prove Proposition 4.6.

We first describe the necessary notation. Let $\bar\Omega = (\bar\Omega_1, \ldots, \bar\Omega_P)$ be a strong solution for multiphase mean curvature flow in the sense of Definition 4.14 on some time interval $[0,T]$. In particular, the family $\bar\Omega$ is a smoothly evolving regular partition and the family $\mathcal{I} = \bigcup_{i\neq j} \bar I_{i,j}$ is a smoothly evolving regular network of interfaces in the sense of Definition 4.13.

We decompose the network of interfaces of the strong solution according to its topological features, i.e., into smooth two-phase interfaces on the one hand and triple junctions on the other hand. Suppose that the strong solution has N of such topological features $\mathcal{T}_n$, $n \in \{1,\ldots,N\}$. We then split $\{1,\ldots,N\} =: \mathcal{C} \cup \mathcal{P}$ with the convention that $\mathcal{C}$ enumerates the connected components in space-time of the smooth two-phase interfaces (being time-evolving curves) and $\mathcal{P}$ enumerates the triple junctions (being time-evolving points). If $p \in \mathcal{P}$, we define $\mathcal{T}_p := \bigcup_{t\in[0,T]} \mathcal{T}_p(t) \times \{t\}$ to be the trajectory in space-time described by the triple junction. If $c \in \mathcal{C}$, we define $\mathcal{T}_c := \bigcup_{t\in[0,T]} \mathcal{T}_c(t) \times \{t\} \subset \bar I_{i,j}$ for some $i,j \in \{1,\ldots,P\}$ with $i \neq j$ to be the corresponding space-time connected component of a two-phase interface $\bar I_{i,j}$. We say that the i-th phase of the strong solution is *present at the topological feature* $\mathcal{T}_n$ for $n \in \{1,\ldots,N\}$ if $\partial\bar\Omega_i \cap \mathcal{T}_n \neq \emptyset$. Otherwise, we say that the phase is *absent at* $\mathcal{T}_n$. Finally, we write $c \sim p$ for $c \in \mathcal{C}$ and $p \in \mathcal{P}$ if and only if $\mathcal{T}_c$ has an endpoint at $\mathcal{T}_p$. Otherwise, we write $c \not\sim p$.

For each $p \in \mathcal{P}$, let $\hat r_p \in (0,1]$ denote the localization scale provided by Proposition 4.22, and for each $i,j \in \{1,\ldots,P\}$ such that $i \neq j$ let $r_{i,j} \in (0,1]$ be an admissible localization scale for the interface $\bar I_{i,j}$ in the sense of Definition 4.17. We also define

$$3\ell_{\mathcal{P}} := 1 \wedge \min_{t\in[0,T]} \min_{p,p'\in\mathcal{P},\, p\neq p'} \operatorname{dist}(\mathcal{T}_p(t), \mathcal{T}_{p'}(t)).$$

In words, $\ell_{\mathcal{P}}$ keeps track of the separation of the triple junctions. Moreover, for each $c \in \mathcal{C}$ we let

$$3\ell_c := 1 \wedge \min_{t\in[0,T]} \min_{c'\in\mathcal{C}\setminus\{c\}\colon \mathcal{T}_c\cap\mathcal{T}_{c'}=\emptyset} \operatorname{dist}(\mathcal{T}_c(t), \mathcal{T}_{c'}(t)).$$

If $c \in \mathcal{C}$ refers to a closed loop, then ℓ_c measures the separation to all other topological features. Otherwise, $c \in \mathcal{C}$ refers to a two-phase interface with two triple junction endpoints, and in this case ℓ_c represents the minimal distance to all other topological features except for the two triple junctions at its endpoints and the set of two-phase interfaces also having an endpoint at these triple junctions. We then define

$$2r_{\mathcal{P}} := \min_{p\in\mathcal{P}} \hat r_p \wedge \ell_{\mathcal{P}} \wedge \min_{c\in\mathcal{C}} \ell_c \in (0,1]. \tag{4.176}$$

Note that $r_{\mathcal{P}}$ allows for the application of all the results from Section 4.5, and that distinct triple junctions are well separated. In addition, the $r_{\mathcal{P}}$-ball around a triple junction $\mathcal{T}_p$ intersects with the $r_{\mathcal{P}}$-neighborhood of a two-phase interface $\mathcal{T}_c$ if and only if $c \sim p$.

Next, in case $c \in \mathcal{C}$ does not refer to a closed loop, i.e., there exists exactly two $p_+, p_- \in \mathcal{P}$ such that $c \sim p_+$ and $c \sim p_-$, we consider

$$3\ell_c' := 1 \wedge \min_{t\in[0,T]} \min_{\substack{c'\in\mathcal{C}\setminus\{c\} \\ c'\sim p,\, p\in\{p_\pm\}}} \operatorname{dist}\Big(\mathcal{T}_c(t) \setminus \bigcup_{p\in\{p_\pm\}} B_{r_{\mathcal{P}}}(\mathcal{T}_p(t)), \mathcal{T}_{c'}(t)\Big).$$

The purpose of ℓ_c' is to separate interfaces which meet at the same triple junction; at least outside of a neighborhood of the latter. We then define

$$2r_{\mathcal{C}} := \min_{i,j\in\{1,\ldots,P\},\, i\neq j} r_{i,j} \wedge \min_{c\in\mathcal{C}} \ell_c \wedge \min_{c\in\mathcal{C}\colon \exists p\in\mathcal{P} \text{ s.t. } c\sim p} \ell_c' \in (0,1].$$

Observe that the scale $r_{\mathcal{C}}$ allows for the application of all the results from Section 4.4, and that distinct interfaces are well separated at this scale in the previously described sense.

Finally, it is convenient to define a minimal localization scale by means of

$$\bar{r}_{\min} := r_{\mathcal{C}} \wedge r_{\mathcal{P}} > 0. \tag{4.177}$$

4.6.1 Localization of topological features

We now introduce a partition of unity $(\eta_{\mathrm{bulk}}, \eta_1, \ldots, \eta_N)$, where each η_n for $n = 1, \ldots, N$ localizes in a neighborhood of the corresponding topological feature $\mathcal{T}_n$ as follows:

Lemma 4.30. *Let $d = 2$ and $P \in \mathbb{N}$, $P \geq 2$. Let $\bar{\Omega} = (\bar{\Omega}_1, \ldots, \bar{\Omega}_P)$ be a strong solution to multiphase mean curvature flow in the sense of Definition 4.14, whose network of interfaces decomposes into N topological features $\mathcal{T}_n$, $n \in \{1, \ldots, N\}$. Let $r_{\mathcal{P}}, \bar{r}_{\min} \in (0, 1]$ be the localization scales defined by (4.176) and (4.177), and let $\mathcal{T}_{\mathcal{P}} := \bigcup_{p \in \mathcal{P}} \mathcal{T}_p$.*

Then, for each $n \in \{1, \ldots, N\}$ there exists a continuous function

$$\eta_n \colon \mathbb{R}^2 \times [0, T] \to [0, 1]$$

satisfying $\eta_n \in (C_t^0 C_x^2 \cap C_t^1 C_x^0)(\mathbb{R}^2 \times [0, T] \setminus \mathcal{T}_{\mathcal{P}})$ with corresponding estimates

$$\max_{k=1,2} \bar{r}_{\min}^k |\nabla^k \eta_n| + \bar{r}_{\min}^2 |\partial_t \eta_n| \leq C \quad \text{in } \mathbb{R}^2 \times [0, T] \setminus \mathcal{T}_{\mathcal{P}}, \tag{4.178}$$

for some constant $C > 0$, depending only on $\bar{\Omega}$ but not on $\bar{r}_{\min}$, so that the family $(\eta_1, \ldots, \eta_N)$ is a partition of unity in the following sense:

i) Let $\eta_{\mathrm{bulk}} := 1 - \sum_{n=1}^N \eta_n$. Then $\eta_{\mathrm{bulk}} \in [0, 1]$ throughout $\mathbb{R}^2 \times [0, T]$. On the evolving network of interfaces $\mathcal{I} := \bigcup_{i \neq j} \bar{I}_{i,j}$ we have $\eta_{\mathrm{bulk}} \equiv 0$. Moreover, there exists a constant $C \geq 1$, depending only on $\bar{\Omega}$ but not on $\bar{r}_{\min}$, such that it holds

$$C^{-1} \big(\bar{r}_{\min}^{-2} \operatorname{dist}^2(\cdot, \mathcal{I}) \wedge 1 \big) \leq \eta_{\mathrm{bulk}} \quad \text{in } \mathbb{R}^2 \times [0, T] \setminus \mathcal{T}_{\mathcal{P}}, \tag{4.179}$$

$$\eta_{\mathrm{bulk}} \leq C \big(\bar{r}_{\min}^{-2} \operatorname{dist}^2(\cdot, \mathcal{I}) \wedge 1 \big) \quad \text{in } \mathbb{R}^2 \times [0, T] \setminus \mathcal{T}_{\mathcal{P}}, \tag{4.180}$$

$$|\nabla \eta_{\mathrm{bulk}}| \leq C \bar{r}_{\min}^{-1} \big(\bar{r}_{\min}^{-1} \operatorname{dist}(\cdot, \mathcal{I}) \wedge 1 \big) \quad \text{in } \mathbb{R}^2 \times [0, T] \setminus \mathcal{T}_{\mathcal{P}}, \tag{4.181}$$

$$|\partial_t \eta_{\mathrm{bulk}}| \leq C \bar{r}_{\min}^{-2} \big(\bar{r}_{\min}^{-1} \operatorname{dist}(\cdot, \mathcal{I}) \wedge 1 \big) \quad \text{in } \mathbb{R}^2 \times [0, T] \setminus \mathcal{T}_{\mathcal{P}}, \tag{4.182}$$

and if either phase i or phase j is absent at a given topological feature $n \in \{1, \ldots, N\}$ we have the estimates

$$\eta_n \leq C \big(\bar{r}_{\min}^{-2} \operatorname{dist}^2(\cdot, \bar{I}_{i,j}) \wedge 1 \big) \quad \text{in } \mathbb{R}^2 \times [0, T] \setminus \mathcal{T}_{\mathcal{P}}, \tag{4.183}$$

$$|\nabla \eta_n| \leq C \bar{r}_{\min}^{-1} \big(\bar{r}_{\min}^{-1} \operatorname{dist}(\cdot, \bar{I}_{i,j}) \wedge 1 \big) \quad \text{in } \mathbb{R}^2 \times [0, T] \setminus \mathcal{T}_{\mathcal{P}}, \tag{4.184}$$

$$|\partial_t \eta_n| \leq C \bar{r}_{\min}^{-2} \big(\bar{r}_{\min}^{-1} \operatorname{dist}(\cdot, \bar{I}_{i,j}) \wedge 1 \big) \quad \text{in } \mathbb{R}^2 \times [0, T] \setminus \mathcal{T}_{\mathcal{P}}. \tag{4.185}$$

ii) For all $c \in \mathcal{C}$ and $t \in [0, T]$ it holds

$$\operatorname{supp} \eta_c(\cdot, t) \subset \Psi_{\mathcal{T}_c}(\mathcal{T}_c(t) \times \{t\} \times [\bar{r}_{\min}, \bar{r}_{\min}]) =: \operatorname{im}_{\bar{r}_{\min}}(\Psi_{\mathcal{T}_c})(t), \tag{4.186}$$

with $\Psi_{\mathcal{T}_c}$ denoting the restriction to $\mathcal{T}_c$ of the diffeomorphism (4.45).

iii) For all $p \in \mathcal{P}$ and $t \in [0, T]$ it holds

$$\operatorname{supp} \eta_p(\cdot, t) \subset B_{r_{\mathcal{P}}}(\mathcal{T}_p(t)). \tag{4.187}$$

iv) Let $p, p' \in \mathcal{P}$ be two distinct triple junctions. Then for all $t \in [0,T]$ we have

$$\operatorname{supp}\eta_p(\cdot,t) \cap \operatorname{supp}\eta_{p'}(\cdot,t) \subset B_{r_{\mathcal{P}}}(\mathcal{T}_p(t)) \cap B_{r_{\mathcal{P}}}(\mathcal{T}_{p'}(t)) = \emptyset. \tag{4.188}$$

v) Let $p \in \mathcal{P}$ be a triple junction and let $c \in \mathcal{C}$ be a two-phase interface. Then $\operatorname{supp}\eta_p \cap \operatorname{supp}\eta_c \neq \emptyset$ if and only if $\mathcal{T}_c$ has an endpoint at $\mathcal{T}_p$. In this case and assuming $\mathcal{T}_c \subset \bar{I}_{i,j}$ for $i \neq j \in \{1,\dots,P\}$, it holds for all $t \in [0,T]$ that

$$\operatorname{supp}\eta_p(\cdot,t) \cap \operatorname{supp}\eta_c(\cdot,t) \subset B_{r_{\mathcal{P}}}(\mathcal{T}_p(t)) \cap (W_{i,j}(t) \cup W_i(t) \cup W_j(t)), \tag{4.189}$$

where $W_{i,j}$, W_i and W_j are as in Definition 4.20.

vi) Let $c, c' \in \mathcal{C}$ be two distinct two-phase interfaces. Then we have $\operatorname{supp}\eta_c \cap \operatorname{supp}\eta_{c'} \neq \emptyset$ if and only if both interfaces have an endpoint at the same triple junction $\mathcal{T}_p$, $p \in \mathcal{P}$. In this case, it holds for all $t \in [0,T]$ that

$$\operatorname{supp}\eta_c(\cdot,t) \cap \operatorname{supp}\eta_{c'}(\cdot,t) \subset B_{r_{\mathcal{P}}}(\mathcal{T}_p(t)) \cap W_i(t), \tag{4.190}$$

where we assume that $\mathcal{T}_c \subset \bar{I}_{i,j}$ and $\mathcal{T}_{c'} \subset \bar{I}_{k,i}$.

Proof. An illustration of the constructed functions close to a triple junction can be found in Figure 4.9. For the definition of a partition of unity $(\eta_{\mathrm{bulk}}, \eta_1, \dots, \eta_N)$ with the required localization and coercivity properties we proceed in several steps.

Step 1: Definition of auxiliary cutoffs. Let θ be a smooth and even cutoff function with $\theta(s) = 1$ for $|s| \leq \frac{1}{2}$ and $\theta \equiv 0$ for $|s| \geq 1$. Let $\zeta\colon \mathbb{R} \to [0,\infty)$ be another smooth cutoff function defined by

$$\zeta(s) = (1-s^2)\theta(s^2), \tag{4.191}$$

see Figure 4.8. Let $\delta \in (0,1]$ be a constant to be determined later (independent of $\bar{r}_{\min}$). Based on the profile ζ, we then introduce for each topological feature $\mathcal{T}_n$, $n \in \{1,\dots,N\}$, a corresponding cutoff function ζ_n as follows. First, for a given triple junction $p \in \mathcal{P}$ we define the associated triple junction cutoff

$$\zeta_p(x,t) := \zeta\Big(\frac{\operatorname{dist}(x,\mathcal{T}_p(t))}{r_{\mathcal{P}}}\Big), \quad (x,t) \in \mathbb{R}^2 \times [0,T]. \tag{4.192}$$

Second, for a given connected component $c \in \mathcal{C}$ of a two-phase interface, say $\mathcal{T}_c \subset \bar{I}_{i,j}$ for some $i,j \in \{1,\dots,P\}$ with $i \neq j$, we define the associated interface cutoff function

$$\zeta_c(x,t) := \begin{cases} \zeta\big(\frac{s_{i,j}(x,t)}{\delta\bar{r}_{\min}}\big), & (x,t) \in \overline{\operatorname{im}(\Psi_{\mathcal{T}_c})}, \\ 0 & \text{else}. \end{cases} \tag{4.193}$$

where $s_{i,j}$ is the signed distance function defined in (4.47) and $\operatorname{im}(\Psi_{\mathcal{T}_c})$ is the image of the diffeomorphism $\Psi_{\mathcal{T}_c}$, i.e., the restriction to $\mathcal{T}_c$ of the diffeomorphism (4.45).

It follows directly from the definitions (4.191)–(4.193), the regularity of the signed distance in form of (4.50), (4.97) and (4.101), as well as (4.102) that

$$\operatorname{supp}\zeta_p(\cdot,t) \subset B_{r_{\mathcal{P}}}(\mathcal{T}_p(t)), \qquad t \in [0,T], \tag{4.194}$$

$$\operatorname{supp}\zeta_c(\cdot,t) \subset \Psi_{\mathcal{T}_c}(\mathcal{T}_c(t) \times \{t\} \times [-\delta\bar{r}_{\min}, \delta\bar{r}_{\min}]), \qquad t \in [0,T], \tag{4.195}$$

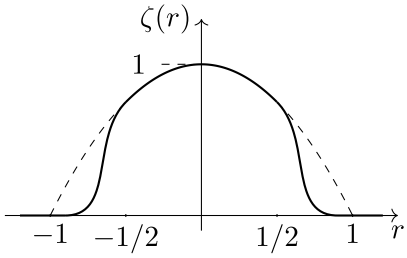

Figure 4.8: The profile ζ used to construct the cutoff functions for two-phase interfaces and triple junctions.

and $\zeta_p \in (C_t^0C_x^2 \cap C_t^1C_x^0)(\mathbb{R}^2\times[0,T] \setminus \mathcal{T}_p)$ as well as $\zeta_c \in (C_t^0C_x^2 \cap C_t^1C_x^0)(\overline{\operatorname{im}(\Psi_{\mathcal{T}_c})})$ with corresponding estimates (assuming $\mathcal{T}_c \subset \bar{I}_{i,j}$)

$$\begin{aligned}
|1-\zeta_p| &\leq C\big(\bar{r}_{\min}^{-2}\operatorname{dist}^2(\cdot,\mathcal{T}_p)\wedge 1\big) && \text{on } \mathbb{R}^2\times[0,T]\setminus\mathcal{T}_p, && (4.196)\\
|\nabla^k\zeta_p| &\leq C\bar{r}_{\min}^{-k}\big(\bar{r}_{\min}^{-(2-k)}\operatorname{dist}^{2-k}(\cdot,\mathcal{T}_p)\wedge 1\big) && \text{on } \mathbb{R}^2\times[0,T]\setminus\mathcal{T}_p,\ k\in\{1,2\}, && (4.197)\\
|\partial_t\zeta_p| &\leq C\bar{r}_{\min}^{-2}\big(\bar{r}_{\min}^{-1}\operatorname{dist}(\cdot,\mathcal{T}_p)\wedge 1\big) && \text{on } \mathbb{R}^2\times[0,T]\setminus\mathcal{T}_p, && (4.198)\\
|1-\zeta_c| &\leq C\big(\bar{r}_{\min}^{-2}\operatorname{dist}^2(\cdot,\bar{I}_{i,j})\wedge 1\big) && \text{on } \overline{\operatorname{im}(\Psi_{\mathcal{T}_c})}, && (4.199)\\
|\nabla^k\zeta_c| &\leq C\bar{r}_{\min}^{-k}\big(\bar{r}_{\min}^{-(2-k)}\operatorname{dist}^{2-k}(\cdot,\bar{I}_{i,j})\wedge 1\big) && \text{on } \overline{\operatorname{im}(\Psi_{\mathcal{T}_c})},\ k\in\{1,2\}, && (4.200)\\
|\partial_t\zeta_c| &\leq C\bar{r}_{\min}^{-2}\big(\bar{r}_{\min}^{-1}\operatorname{dist}(\cdot,\bar{I}_{i,j})\wedge 1\big) && \text{on } \overline{\operatorname{im}(\Psi_{\mathcal{T}_c})}. && (4.201)
\end{aligned}$$

Step 2: Define η_p for triple junctions $p \in \mathcal{P}$. Let us assume that the phases $i,j,k \in \{1,\dots,P\}$ are present at the triple junction $\mathcal{T}_p$, and the corresponding interfaces are denoted by $\mathcal{T}_{c_{i,j}} \subset \bar{I}_{i,j}$, $\mathcal{T}_{c_{j,k}} \subset \bar{I}_{j,k}$ and $\mathcal{T}_{c_{k,i}} \subset \bar{I}_{k,i}$.

We want to define η_p such that (4.187) holds true. Recall from Definition 4.20 that $B_{r_{\mathcal{P}}}(\mathcal{T}_p)$ decomposes into six wedges. Three of them, namely the interface wedges $W_{i,j}$, $W_{j,k}$ resp. $W_{k,i}$, contain the interfaces $\mathcal{T}_{c_{i,j}}$, $\mathcal{T}_{c_{j,k}}$ resp. $\mathcal{T}_{c_{k,i}}$. The other three are interpolation wedges denoted by W_i, W_j resp. W_k.

We now have everything in place to move on with the definition of η_p. We note that $B_{r_{\mathcal{P}}}(\mathcal{T}_p(t)) \cap W_{i,j}(t) \subset \operatorname{im}(\Psi_{\mathcal{T}_{c_{i,j}}})$ for all $t \in [0,T]$ due to (4.75) and (4.176). Therefore, we can begin by setting

$$\eta_p(x,t) := \zeta_p(x,t)\zeta_{c_{i,j}}(x,t), \quad t\in[0,T],\ x\in B_{r_{\mathcal{P}}}(\mathcal{T}_p(t))\cap W_{i,j}(t), \tag{4.202}$$

and analogously on the other interface wedges $W_{j,k}$ and $W_{k,i}$. To define η_p on the interpolation wedges, we use the interpolation parameter built in Lemma 4.28. To clarify the direction of interpolation, i.e., on which boundary of the interpolation wedge the corresponding interpolation function is equal to one or zero, we make use of the following notational convention. For the interpolation wedge W_i, say, we denote by $\lambda_i^{j,k}$ the interpolation function as built in Lemma 4.28 and which interpolates from j to k in the sense that it is equal to one on $(\partial W_{i,j} \cap \partial W_i) \setminus \mathcal{T}_p$ and which vanishes on $(\partial W_{k,i} \cap \partial W_i) \setminus \mathcal{T}_p$. We also define $\lambda_i^{k,j} := 1 - \lambda_i^{j,k}$ which interpolates on W_i in the opposite direction from k to j. Analogously, one introduces the interpolation functions on the other interpolation wedges. We may then define

$$\begin{aligned}
\eta_p(x,t) := \lambda_i^{j,k}(x,t)\zeta_p(x,t)\zeta_{c_{i,j}}(x,t) + (1-\lambda_i^{j,k})(x,t)\zeta_p(x,t)\zeta_{c_{k,i}}(x,t),\\
t\in[0,T],\ x\in B_{r_{\mathcal{P}}}(\mathcal{T}_p(t))\cap W_i(t),
\end{aligned}\tag{4.203}$$

due to $B_{r_{\mathcal{P}}}(\mathcal{T}_p(t)) \cap W_i(t) \subset \operatorname{im}(\Psi_{\mathcal{T}_{c_{i,j}}}) \cap \operatorname{im}(\Psi_{\mathcal{T}_{c_{k,i}}})$ for all $t \in [0,T]$, which follows from (4.76) and (4.176). We can analogously define η_p on the other two interpolation wedges W_j and W_k.

Finally, we define

$$\eta_p(x,t) := 0, \quad t\in[0,T],\, x\notin B_{r_{\mathcal{P}}}(\mathcal{T}_p(t)). \tag{4.204}$$

We refer to Figure 4.9 for an illustration of the construction.

The localization property (4.187) is immediate from the definitions (4.202)–(4.204) and the property (4.194), whereas (4.188) follows from the definition (4.176) of the localization scale $r_{\mathcal{P}}$. Moreover, as a consequence of the estimates (4.143)–(4.144) for the interpolation parameter, the estimates (4.196)–(4.201) for the auxiliary cutoffs, the definitions (4.202)–(4.204) and the trivial estimate $\operatorname{dist}(\cdot,\bar I_{i,j})\vee\operatorname{dist}(\cdot,\bar I_{j,k})\vee\operatorname{dist}(\cdot,\bar I_{k,i})\le\operatorname{dist}(\cdot,\mathcal{T}_p)$ throughout $B_{r_{\mathcal{P}}}(\mathcal{T}_p(t))$ for all $t\in[0,T]$ (assuming that the phases $i,j,k\in\{1,\ldots,P\}$ are present at $\mathcal{T}_p$) we obtain

$$|1-\eta_p|\le C\big(\bar r_{\min}^{-2}\operatorname{dist}^2(\cdot,\mathcal{T}_p)\wedge 1\big) \qquad \text{on } \mathbb{R}^2\times[0,T]\setminus\mathcal{T}_p, \tag{4.205}$$

$$|\nabla^k\eta_p|\le C\bar r_{\min}^{-k}\big(\bar r_{\min}^{-(2-k)}\operatorname{dist}^{2-k}(\cdot,\mathcal{T}_p)\wedge 1\big) \qquad \text{on } \mathbb{R}^2\times[0,T]\setminus\mathcal{T}_p,\ k\in\{1,2\}, \tag{4.206}$$

$$|\partial_t\eta_p|\le C\bar r_{\min}^{-2}\big(\bar r_{\min}^{-1}\operatorname{dist}(\cdot,\mathcal{T}_p)\wedge 1\big) \qquad \text{on } \mathbb{R}^2\times[0,T]\setminus\mathcal{T}_p. \tag{4.207}$$

These estimates of course imply the asserted bound (4.178) for $n=p\in\mathcal{P}$. Note also that the error estimates (4.183)–(4.185) are trivially fulfilled by definition (4.176) of the localization scale $r_{\mathcal{P}}$, the property (4.187) and the estimate (4.178).

Step 3: Define η_c for $c\in\mathcal{C}$. Let $i,j\in\{1,\ldots,P\}$ with $i\neq j$ be such that $\mathcal{T}_c\subset\bar I_{i,j}$. If the interface $\mathcal{T}_c$ has no endpoint at a triple junction, i.e., it is a closed loop, we simply set

$$\eta_c(x,t) := \begin{cases}\zeta_c(x,t) & \text{if } (x,t)\in\operatorname{im}(\Psi_{\mathcal{T}_c}),\\ 0, & \text{else},\end{cases} \tag{4.208}$$

where the cutoff ζ_c was already defined in (4.193).

Otherwise, the interface ends in two different triple junctions corresponding to $p,p'\in\mathcal{P}$ with $p\neq p'$. We will only describe the construction close to $\mathcal{T}_p$, as by (4.176) the triple junctions are separated on scale $r_{\mathcal{P}}$ and can thus also be treated separately. Away from the triple junctions $\mathcal{T}_p$ and $\mathcal{T}_{p'}$, we still define

$$\eta_c(x,t) := \begin{cases}\zeta_c(x,t) & (x,t)\in\operatorname{im}(\Psi_{\mathcal{T}_c})\setminus\bigcup_{t\in[0,T]}\big(B_{r_{\mathcal{P}}}(\mathcal{T}_p(t))\cup B_{r_{\mathcal{P}}}(\mathcal{T}_{p'}(t))\big)\times\{t\}\\ 0 & \text{in } \big(\mathbb{R}^2\times[0,T]\setminus\operatorname{im}(\Psi_{\mathcal{T}_c})\big)\setminus\bigcup_{t\in[0,T]}\big(B_{r_{\mathcal{P}}}(\mathcal{T}_p(t))\cup B_{r_{\mathcal{P}}}(\mathcal{T}_{p'}(t))\big)\times\{t\}.\end{cases} \tag{4.209}$$

Near the triple junction, i.e., on $B_{r_{\mathcal{P}}}(\mathcal{T}_p(t))$ for all $t\in[0,T]$, we aim to modify the definition such that η_c is supported within the set $W_i\cup W_j\cup W_{i,j}$. To this end, we define

$$\eta_c(x,t) := \big(1-\zeta_p(x,t)\big)\zeta_c(x,t), \quad t\in[0,T],\, x\in B_{r_{\mathcal{P}}}(\mathcal{T}_p(t))\cap W_{i,j}(t), \tag{4.210}$$

which is indeed possible in analogy to (4.202), and where the auxiliary cutoff ζ_p was introduced in (4.192). On the interpolation wedges W_i resp. W_j, we again make use of the arguments enabling (4.203) and set

$$\begin{aligned}
\eta_c(x,t) &:= \lambda_i^{j,k}(x,t)\big(1-\zeta_p(x,t)\big)\zeta_c(x,t), \quad t\in[0,T],\, x\in B_{r_{\mathcal{P}}}(\mathcal{T}_p(t))\cap W_i(t),\\
\eta_c(x,t) &:= \lambda_j^{i,k}(x,t)\big(1-\zeta_p(x,t)\big)\zeta_c(x,t), \quad t\in[0,T],\, x\in B_{r_{\mathcal{P}}}(\mathcal{T}_p(t))\cap W_j(t),\\
\eta_c(x,t) &:= 0, \qquad t\in[0,T],\, x\in B_{r_{\mathcal{P}}}(\mathcal{T}_p(t))\setminus\big(W_{i,j}(t)\cup W_i(t)\cup W_j(t)\big),
\end{aligned} \tag{4.211}$$

where $k\in\{1,\ldots,P\}$ corresponds to the third phase present at p. We refer again to Figure 4.9 for an illustration of the construction.

In terms of the required qualitative regularity for η_c, the only obstruction might be the compatibility of (4.209) with (4.211). This is precisely the point where we rely on a suitable choice of the scale $\delta \in (0,1]$. As we have seen in the proof of Lemma 4.21, the curve trapping condition of (4.75) in fact holds on scale $r_{\mathcal{P}}$ for a wedge strictly contained in the interface wedge $W_{i,j}$ (e.g., a wedge obtained by angle bisection). Hence, due to the ball condition of Definition 4.17, this improved curve trapping condition, and the definition (4.176) of the localization scale $r_{\mathcal{P}}$ we may choose the constant $\delta \in (0,1]$ small enough, depending only on the surface tensions associated with $\bar{\Omega}$, such that

$$\overline{\Psi_{\mathcal{T}_c}(\mathcal{T}_c(t)\times\{t\}\times[-\delta r_{\mathcal{P}}, \delta r_{\mathcal{P}}])} \cap \partial B_{r_{\mathcal{P}}}(\mathcal{T}_p(t)) \subset\subset W_{i,j}(t)$$

uniformly over all $t \in [0,T]$. This choice in turn ensures continuity of η_c, and then based on the definitions (4.208)–(4.211) that $\eta_c \in (C^0_t C^2_x \cap C^1_t C^0_x)(\mathbb{R}^2\times[0,T] \setminus \mathcal{T}_{\mathcal{P}})$ since all the constituents of η_c enjoy this regularity (cf. *Step 1* for the auxiliary cutoffs and Lemma 4.28 for the interpolation parameter, respectively).

Next, we may infer the localization property (4.186) from the definitions (4.208)–(4.211) and the property (4.195). Moreover, based on the choice (4.177) of the localization scale $\bar{r}_{\min}$, the localization property (4.187) and the definitions (4.210)–(4.211), one may deduce (4.189) and (4.190).

We move on with the proof of the estimates (4.178) and (4.183)–(4.185) in terms of $n = c \in \mathcal{C}$. First, a straightforward application of the definitions (4.208)–(4.211), the estimates (4.143)–(4.144) for the interpolation parameter, and the estimates (4.196)–(4.201) for the auxiliary cutoffs implies (4.178). Consider then $c \in \mathcal{C}$ and distinct $i,j \in \{1,\ldots,P\}$ such that $\mathcal{T}_c \not\subset \bar{I}_{i,j}$, i.e., either phase i or phase j is absent at $\mathcal{T}_c$. Without loss of generality, we may assume that there exists $c' \in \mathcal{C}\setminus\{c\}$ and $p \in \mathcal{P}$ such that $\mathcal{T}_{c'} \subset \bar{I}_{i,j}$, $c \sim p$ and $c' \sim p$; and in this regime, it even suffices to restrict to the domain $B_{r_{\mathcal{P}}}(\mathcal{T}_p(t))$ for all $t \in [0,T]$. Otherwise, the error estimates (4.183)–(4.185) are trivially fulfilled because of (4.186), the estimate (4.178) and definition (4.177) of the localization scale $\bar{r}_{\min}$.

To prove the error estimates in the remaining regime, we now fully exploit the fact that a factor of $1-\zeta_p$ always appears in the definitions (4.210) and (4.211). In particular, by means of the estimates (4.143)–(4.144) for the interpolation parameter, the estimates (4.196)–(4.201) for the auxiliary cutoffs, and the trivial estimate $\mathrm{dist}(\cdot,\mathcal{T}_c) \leq \mathrm{dist}(\cdot,\mathcal{T}_p)$ throughout $B_{r_{\mathcal{P}}}(\mathcal{T}_p(t))$ for all $t \in [0,T]$, it follows

$$\eta_c \leq C\big(\bar{r}_{\min}^{-2}\,\mathrm{dist}^2(\cdot,\mathcal{T}_p) \wedge 1\big) \qquad \text{in } B_{r_{\mathcal{P}}}(\mathcal{T}_p(t)),\ t \in [0,T], \tag{4.212}$$
$$|\nabla\eta_c| \leq C\bar{r}_{\min}^{-1}\big(\bar{r}_{\min}^{-1}\,\mathrm{dist}(\cdot,\mathcal{T}_p) \wedge 1\big) \qquad \text{in } B_{r_{\mathcal{P}}}(\mathcal{T}_p(t)),\ t \in [0,T], \tag{4.213}$$
$$|\partial_t\eta_c| \leq C\bar{r}_{\min}^{-2}\big(\bar{r}_{\min}^{-1}\,\mathrm{dist}(\cdot,\mathcal{T}_p) \wedge 1\big) \qquad \text{in } B_{r_{\mathcal{P}}}(\mathcal{T}_p(t)),\ t \in [0,T]. \tag{4.214}$$

These estimates upgrade to (4.183)–(4.185) thanks to the bounds (4.79) and (4.77).

Step 4: Partition of unity. Next, we validate the partition of unity property for the family of localization functions $(\eta_1,\ldots,\eta_N)$. First of all, it is clear from our definitions (4.202)–(4.211) that $\eta_n \in [0,1]$ for each topological feature $n \in \{1,\ldots,N\}$. Together with the already established localization properties (4.186)–(4.190) and the definitions (4.202)–(4.211), it also follows that $\sum_{n=1}^N \eta_n \leq 1$ on $\mathbb{R}^2\times[0,T]$ as well as $\sum_{n=1}^N \eta_n \equiv 1$ on the evolving network of interfaces $\mathcal{I} = \bigcup_{i\neq j}\bar{I}_{i,j}$. Hence, we may define the bulk term $\eta_{\mathrm{bulk}} := 1-\sum_{n=1}^N \eta_n \in [0,1]$ and obtain that the extended family $(\eta_{\mathrm{bulk}},\eta_1,\ldots,\eta_N)$ is indeed a partition of unity on $\mathbb{R}^2\times[0,T]$.

Step 5: Estimates for the bulk cutoff. By the localization properties (4.186)–(4.190) as well as the choices (4.176) and (4.177) of the localization scales $r_{\mathcal{P}}$ and $\bar{r}_{\min}$, it suffices to prove (4.180)–(4.181) in $\bigcup_{c\in\mathcal{C}} \mathrm{im}_{\bar{r}_{\min}}(\Psi_{\mathcal{T}_c}) \setminus \bigcup_{p\in\mathcal{P}}\bigcup_{t\in[0,T]} B_{r_{\mathcal{P}}}(\mathcal{T}_p(t))\times\{t\}$ and in $\bigcup_{p\in\mathcal{P}}\bigcup_{t\in[0,T]} B_{r_{\mathcal{P}}}(\mathcal{T}_p(t))\times\{t\}$, respectively. We in fact may argue separately for each $c \in \mathcal{C}$

and each $p \in \mathcal{P}$. Moreover, for all $c \in \mathcal{C}$ and all distinct $i, j \in \{1, \ldots, P\}$ such that $\mathcal{T}_c \subset \bar{I}_{i,j}$ it holds

$$\operatorname{dist}(\cdot, \bar{I}_{i,j}) = \operatorname{dist}(\cdot, \mathcal{I}) \quad \text{in } \operatorname{im}_{\bar{r}_{\min}}(\Psi_{\mathcal{T}_c}) \setminus \bigcup_{p\in\mathcal{P}} \bigcup_{t\in[0,T]} B_{r_{\mathcal{P}}}(\mathcal{T}_p(t)) \times \{t\}, \tag{4.215}$$

and similarly for all $p \in \mathcal{P}$ with present phases $i, j, k \in \{1, \ldots, P\}$, it holds

$$\operatorname{dist}(\cdot, \bar{I}_{i,j}) \wedge \operatorname{dist}(\cdot, \bar{I}_{j,k}) \wedge \operatorname{dist}(\cdot, \bar{I}_{k,i}) = \operatorname{dist}(\cdot, \mathcal{I}) \ \text{ in } \bigcup_{t\in[0,T]} B_{r_{\mathcal{P}}}(\mathcal{T}_p(t)) \times \{t\}. \tag{4.216}$$

First, let $c \in \mathcal{C}$. Due to the localization properties (4.186)–(4.190), the choices (4.176) and (4.177) of the localization scales $r_{\mathcal{P}}$ and $\bar{r}_{\min}$, as well as the definitions (4.208) and (4.209) it holds

$$\eta_{\text{bulk}} = 1 - \eta_c = 1 - \zeta_c \quad \text{in } \operatorname{im}_{\bar{r}_{\min}}(\Psi_{\mathcal{T}_c}) \setminus \bigcup_{p\in\mathcal{P}} \bigcup_{t\in[0,T]} B_{r_{\mathcal{P}}}(\mathcal{T}_p(t)) \times \{t\}. \tag{4.217}$$

The upper bounds (4.180)–(4.182) are therefore an immediate consequence of the bounds (4.199)–(4.201), respectively, together with (4.215) and (4.216). The coercivity estimate (4.179) in turn follows from the choice (4.191) of the quadratic cutoff profile.

Second, consider $p \in \mathcal{P}$ and assume that the pairwise distinct phases $i, j, k \in \{1, \ldots, P\}$ are present at $\mathcal{T}_p$. Modulo a permutation of the indices, it suffices to consider the two unique two-phase interfaces $\mathcal{T}_{c_{i,j}} \subset \bar{I}_{i,j}$ and $\mathcal{T}_{c_{k,i}} \subset \bar{I}_{k,i}$ so that $c_{i,j} \sim p$ and $c_{k,i} \sim p$, and then to prove the desired estimates on the interface wedge $W_{i,j}$ and the interpolation wedge W_i. In this regime, due to the localization properties (4.186)–(4.190), the choices (4.176) and (4.177) of the localization scales $r_{\mathcal{P}}$ and $\bar{r}_{\min}$, as well as the definitions (4.202)–(4.203) resp. (4.210)–(4.211), it holds

$$\eta_{\text{bulk}} = 1 - \eta_{c_{i,j}} - \eta_p = 1 - \zeta_{c_{i,j}} \qquad \text{in } B_{r_{\mathcal{P}}}(\mathcal{T}_p(t)) \cap W_{i,j}(t), \tag{4.218}$$

$$\begin{aligned} \eta_{\text{bulk}} &= 1 - \eta_{c_{i,j}} - \eta_{c_{k,i}} - \eta_p \\ &= \lambda_i^{j,k}(1 - \zeta_{c_{i,j}}) + (1 - \lambda_i^{j,k})(1 - \zeta_{c_{k,i}}) \qquad \text{in } B_{r_{\mathcal{P}}}(\mathcal{T}_p(t)) \cap W_i(t) \end{aligned} \tag{4.219}$$

for all $t \in [0, T]$. The upper bounds (4.180)–(4.182) therefore follow from the estimates (4.199)–(4.201), the bound (4.143) for the interpolation parameter, the estimates (4.77) and (4.78), as well as the estimates (4.215) and (4.216). The coercivity estimate (4.179) in turn is again implied by (4.191). □

4.6.2 Global construction of the calibration

In this section, we glue together the local constructions to define the global extensions $\xi_{i,j}$ and B of the normal vector fields and velocity field, respectively.

The idea for the construction of the vector fields $\xi_{i,j}$ for $i, j \in \{1, \ldots, P\}$ with $i \neq j$ is as follows. First, we provide the definition of local vector fields $\xi^n_{i,j}$ for $n \in \{1, \ldots, N\}$ in the support of the associated localization function η_n for each topological feature $\mathcal{T}_n$. If both phases i and j are *present* at $\mathcal{T}_n$, we define $\xi^n_{i,j}$ by means of the local constructions provided in Section 4.4 for the model problem of a smooth manifold and Section 4.5 for the model problem of a triple junction. This, however, leaves open the question of the definition of the vector fields $\xi^n_{i,j}$ for phases *absent* at $\mathcal{T}_n$. It turns out that this issue is related to the conditions of global stability between the phases. In particular, we would like to ensure that at a given topological feature $\mathcal{T}_n$, our relative entropy functional provides a length control for those interfaces which are not present at $\mathcal{T}_n$. For this purpose, we rely on the stability condition for an admissible matrix of surface tensions in the sense of Definition 4.8 *iii)*.

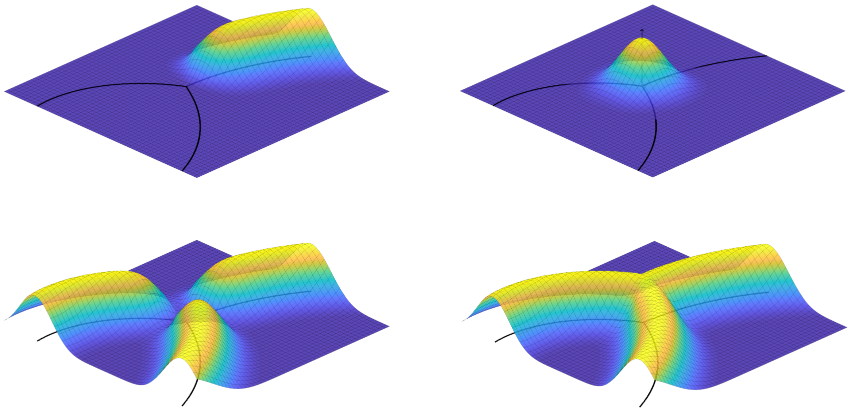

Figure 4.9: The different functions η_n for $n \in \mathcal{C} \cup \mathcal{P}$ in the partition of unity at a single triple junction $\mathcal{T}_p$ for $p \in \mathcal{P}$: The function η_c for a single two-phase interface $c \in \mathcal{C}$ ending at the triple junction (top left), the function η_p for the triple junction itself (top right), the sum of all two-phase localization functions at a triple junction (bottom left), and the sum of all localization functions $\sum_n \eta_n$ (bottom right). Observe that the sum of all localization functions equals 1 on the interfaces in the strong solution, but decays quadratically away from them.

Lemma 4.31. *Let $d = 2$ and $P \in \mathbb{N}$, $P \geq 2$. Let $\bar{\Omega} = (\bar{\Omega}_1, \dots, \bar{\Omega}_P)$ be a strong solution to multiphase mean curvature flow in the sense of Definition 4.14. Let $(\eta_{\text{bulk}}, \eta_1, \dots, \eta_N)$ be a partition of unity as constructed in Lemma 4.30. In particular, let $\bar{r}_{\min} \in (0, 1]$ be the localization scale defined by (4.177), and $\mathcal{T}_{\mathcal{P}} := \bigcup_{p \in \mathcal{P}} \mathcal{T}_p$. Let $i, j \in \{1, \dots, P\}$ be distinct phases and let $n \in \{1, \dots, N\}$ correspond to a topological feature. Given*

$$\mathcal{U}_n := \bigcup_{t \in [0,T]} \{x \in \mathbb{R}^2 : \eta_n(x, t) > 0\} \times \{t\} \tag{4.220}$$

there exist continuous vector fields

$$\xi^n_{i,j} : \mathcal{U}_n \to \mathbb{R}^2,$$
$$\xi^n_i : \mathcal{U}_n \to \mathbb{R}^2,$$

satisfying the following properties:

i) It holds $\xi^n_{i,j}, \xi^n_i \in \big(C^0_t C^2_x \cap C^1_t C^0_x\big)(\overline{\mathcal{U}_n} \setminus \mathcal{T}_{\mathcal{P}})$, and there exists $C > 0$, which may depend on $\bar{\Omega}$ but not on $\bar{r}_{\min}$, such that throughout $\mathcal{U}_n \setminus \mathcal{T}_{\mathcal{P}}$

$$\max_{k=0,1,2} \bar{r}^k_{\min} |\nabla^k \xi^n_{i,j}| + \bar{r}^2_{\min} |\partial_t \xi^n_{i,j}| \leq C. \tag{4.221}$$

ii) On $\mathcal{U}_n$ we have $\xi^n_{i,j} = -\xi^n_{j,i}$, $|\xi^n_{i,j}| \leq 1$ as well as

$$\sigma_{i,j} \xi^n_{i,j} = \xi^n_i - \xi^n_j. \tag{4.222}$$

iii) If the phases i and j are both present at the topological feature $\mathcal{T}_n$, then $\xi^n_{i,j}$ coincides on $\mathcal{U}_n$ with the explicit two-phase construction from Lemma 4.18 in case of $n \in \mathcal{C}$, respectively the triple junction construction from Proposition 4.22 in case of $n \in \mathcal{P}$.

iv) There exists a constant $b = b(\sigma) \in (0,1)$, depending only on the surface tension matrix associated with the strong solution $\bar{\Omega}$, with the property that if either phase i or j is absent at the topological feature $\mathcal{T}_n$, then throughout $\mathcal{U}_n$ we have

$$|\xi^n_{i,j}| \leq b < 1. \tag{4.223}$$

Proof. The proof consists of two parts distinguishing between the topological features present in the network of interfaces of the strong solution.

Step 1: Consider the case $n = c \in \mathcal{C}$. We first assume that both phases i and j are present at the two-phase interface $\mathcal{T}_c$, i.e., $\mathcal{T}_c \subset \bar{I}_{i,j}$. We then define the vector field $\xi^c_{i,j}$ on $\mathcal{U}_c$ as in Lemma 4.18. Note that by the localization property (4.186) and the definition (4.177), we are indeed in the setting of Section 4.4. In particular, $\xi^c_{i,j} = -\xi^c_{j,i}$ and $\xi^c_{i,j}$ coincides with $\bar{\mathrm{n}}_{i,j}$ on $\operatorname{supp}\eta_c \cap \bar{I}_{i,j}$. Furthermore, let us define the vector fields ξ^c_i and ξ^c_j as $\xi^c_i := \frac{\sigma_{i,j}}{2}\xi^c_{i,j}$ resp. as $\xi^c_j := \frac{\sigma_{i,j}}{2}\xi^c_{j,i}$. This ensures that the desired formula (4.222) is indeed satisfied. Moreover, the regularity estimate (4.221) follows from (4.57) and (4.58).

Now, let us assume that at least one of the phases i or j is absent at the two-phase interface $\mathcal{T}_c$. To be specific, we fix $m, l \in \{1, \ldots, P\}$ with $m \neq l$ such that $\mathcal{T}_c \subset \bar{I}_{m,l}$. The idea now is to first define vector fields ξ^c_i and ξ^c_j and then define $\xi^c_{i,j}$ by means of (4.222) such that (4.223) holds true. To this end, we rely on the strict triangle inequality (4.8) for the given matrix of surface tensions, a direct consequence of our stability assumption Definition 4.8 *iii)*. Let us define

$$\xi^c_i := \frac{1}{2}(\sigma_{l,i}\xi^c_{m,l} + \sigma_{m,i}\xi^c_{l,m}),$$

and analogously for ξ^c_j. Note that this is indeed well-defined since we have already provided a definition of the vector fields $\xi^c_{m,l} = -\xi^c_{l,m}$ on the right-hand side as they are assumed to be associated to phases present at $\mathcal{T}_c$. This definition is also consistent with the previous one because of the convention $\sigma_{l,l} = \sigma_{m,m} = 0$. We may then compute plugging in the definitions

$$\xi^c_{i,j} := \frac{\xi^c_i - \xi^c_j}{\sigma_{i,j}} = \frac{1}{2}\Big(\frac{\sigma_{l,i} - \sigma_{l,j}}{\sigma_{i,j}}\xi^c_{m,l} + \frac{\sigma_{m,i} - \sigma_{m,j}}{\sigma_{i,j}}\xi^c_{l,m}\Big).$$

Hence, (4.223) holds true because we have $|\frac{\sigma_{l,i}-\sigma_{l,j}}{\sigma_{i,j}}| < 1$ and $|\frac{\sigma_{m,i}-\sigma_{m,j}}{\sigma_{i,j}}| < 1$ due to the strict triangle inequality (4.8), whereas (4.221) follows because $\xi^c_{m,l} = -\xi^c_{l,m}$ is already subject to the same bound.

Step 2: Consider the case $n = p \in \mathcal{P}$. Again, we first assume that both phases i and j are present at the triple junction $\mathcal{T}_p$, i.e., a connected component of the interface $\bar{I}_{i,j}$ has an endpoint at $\mathcal{T}_p$. Note that by the localization property (4.187) and the definition (4.176), we may apply Proposition 4.22. Therein, we constructed a vector field in the support of η_p we now call $\xi^p_{i,j}$. In particular, $\xi^p_{i,j} = -\xi^p_{j,i}$ and $\xi^p_{i,j}$ coincides with $\bar{\mathrm{n}}_{i,j}$ on $\operatorname{supp}\eta_p \cap \bar{I}_{i,j}$.

Assume now that $k \in \{1, \ldots, P\}$ is the third phase being present at the triple junction $\mathcal{T}_p$. By construction, we have $\sigma_{i,j}\xi^p_{i,j} + \sigma_{j,k}\xi^p_{j,k} + \sigma_{k,i}\xi^p_{k,i} = 0$ on the support of η_p. Defining then the vector field ξ^p_i as $\xi^p_i := \frac{1}{3}(\sigma_{i,j}\xi^p_{i,j} + \sigma_{i,k}\xi^p_{i,k})$, and analogously for ξ^p_j and ξ^p_k, we indeed obtain (4.222). The remaining claimed properties follow from Proposition 4.22.

In order to define $\xi^p_{i,j}$ if at least one of the phases i or j is absent at the triple junction, we define the vector fields ξ^p_i and ξ^p_j as time-independent affine combinations of the previously defined vector fields using the stability condition Definition 4.8 *iii)*.

To be specific, we assume that the distinct phases $k, l, m \in \{1, \ldots, P\}$ are present at $\mathcal{T}_p$. We then employ the stability condition Definition 4.8 *iii)*, that is, there exists a non-degenerate $(P-1)$-simplex $(q_1, \ldots, q_P)$ in $\mathbb{R}^{P-1}$ such that $\sigma_{i',j'} = |q_{i'} - q_{j'}|$ for all $i', j' \in \{1, \ldots, P\}$. In particular, the triangle (q_k, q_l, q_m) is non-degenerate and spans a

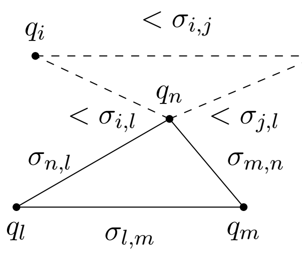

Figure 4.10: Sketch of the l^2-embedding of σ in the case that i and j correspond to absent phases, projected into the plane E containing q_k, q_l and q_m.

plane E in $\mathbb{R}^{P-1}$, which we may isometrically identify with $\mathbb{R}^2$ via an affine map $\phi\colon E \to \mathbb{R}^2$. We furthermore denote the orthogonal projection onto E by π. See Figure 4.10 for a sketch.

In order to prepare the proof of the coercivity condition (4.223) we claim

$$|\pi q_i - \pi q_j| < b\sigma_{i,j} \tag{4.224}$$

for some $b \in (0,1)$, which we prove by considering two cases:

If exactly one of the two indices, say, j corresponds to a phase being present at $\mathcal{T}_p$, then $\pi q_j = q_j$. Note that due to the simplex $(q_1, \ldots, q_P)$ being non-degenerate, also the 3-simplex (q_k, q_l, q_m, q_i) is non-degenerate, so that q_i cannot lie in the plane E. Therefore, we have $\pi q_i \neq q_i$, so that

$$|\pi q_i - \pi q_j|^2 < |q_i - \pi q_j|^2 + |\pi q_j - q_j|^2 = |q_i - q_j|^2 = \sigma_{i,j}^2,$$

the latter by Definition 4.8 *iii)*. This implies the strict inequality in this subcase.

If both i and j correspond to phases being absent at $\mathcal{T}_p$, we consider the orthogonal projection on the three dimensional affine space $\tilde{E}$ spanned by (q_j, q_k, q_l, q_m), as well as the orthogonal projection $\tilde{\pi}$ onto $\tilde{E}$. As the 4-simplex $(q_i, q_j, q_k, q_l, q_m)$ is non-degenerate, we have $\tilde{\pi} q_i \neq q_i$ and $\pi q_i = \pi \circ \tilde{\pi} q_i$. Therefore, we have

$$|\pi q_i - \pi q_j|^2 \le |q_i - \tilde{\pi} q_j|^2 < |q_i - \tilde{\pi} q_j|^2 + |\tilde{\pi} q_j - q_i|^2 = |q_i - q_j|^2 = \sigma_{i,j}^2,$$

allowing us to conclude as in the previous case.

We now proceed with the definition of $\xi_{i'}^p$ for all $i' \in \{1, \ldots, P\}$. As (q_k, q_l, q_m) is non-degenerate and ϕ is isometric, also the triangle $(\phi q_k, \phi q_l, \phi q_m)$ is non-degenerate. Therefore, there exist unique $\hat{\lambda}_k^{i'}, \hat{\lambda}_l^{i'}, \hat{\lambda}_m^{i'} \in \mathbb{R}$ such that $\hat{\lambda}_k^{i'} + \hat{\lambda}_l^{i'} + \hat{\lambda}_m^{i'} = 1$ and

$$\phi \circ \pi q_{i'} = \hat{\lambda}_k^{i'} \phi q_k + \hat{\lambda}_l^{i'} \phi q_l + \hat{\lambda}_m^{i'} \phi q_m.$$

We may then on $\mathcal{U}_p$ define

$$\xi_{i'}^p := \hat{\lambda}_k^{i'} \xi_k^p + \hat{\lambda}_l^{i'} \xi_l^p + \hat{\lambda}_m^{i'} \xi_m^p, \tag{4.225}$$

as well as $\xi_{i,j}^p$ and $\xi_{j,i}^p$ via (4.222). By uniqueness of the coefficient, these definitions are consistent with the previous ones.

The claimed properties *i)* and *iii)* immediately follow from Proposition 4.22. The identity $\xi_{i,j}^p = -\xi_{j,i}^p$, (4.222), and (4.221) are straightforward consequences of the definition and again Proposition 4.22. Therefore, we only have to prove (4.223) in order to get $|\xi_{i,j}^p| \le 1$. To this end, we argue as follows:

Again by non-degeneracy of $(\phi q_k, \phi q_l, \phi q_m)$ for all $(x,t) \in \mathcal{U}_p$ there exist a unique matrix $A(x,t) \in \mathbb{R}^{2\times 2}$ and $y(x,t) \in \mathbb{R}^2$ such that

$$\xi_{i'}^p(x,t) = A(x,t)\phi \circ \pi q_{i'} + y(x,t). \tag{4.226}$$

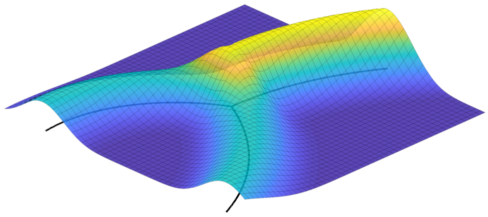

Figure 4.11: Plot of the length of the vector field $\xi_{i,j}$. Observe that the length is 1 on the interface $\bar{I}_{i,j}$ of the strong solution, but decays quadratically away from it to a value strictly smaller than 1, even on the other interfaces $\bar{I}_{i,p}$ and $\bar{I}_{j,p}$. As a consequence, the integral $\int_{I_{i,j}} 1 - \mathrm{n}_{i,j} \cdot \xi_{i,j} \,\mathrm{d}\mathcal{H}^1$ provides an upper bound for the interface error functional $c\int_{I_{i,j}} \min\{\mathrm{dist}^2(x, \bar{I}_{i,j}), 1\} \,\mathrm{d}\mathcal{H}^1$.

for all $i' = k, l, m$. As (4.225) constitutes an affine combination, this equality even holds for all $i' \in \{1, \ldots, P\}$. Furthermore, we have that the matrix A is orthogonal, i.e., $A(x,t) \in \mathcal{O}$ for all $(x,t) \in \mathcal{U}_p$, since by Proposition 4.22 *i)* we have

$$|A(\phi \circ \pi q_{i'} - \phi \circ \pi q_{j'})| = |\xi^p_{i'} - \xi^p_{j'}| = \sigma_{i',j'} |\xi^p_{i',j'}| = \sigma_{i',j'} = |\phi \circ \pi q_{i'} - \phi \circ \pi q_{j'}|$$

and the triangle $(\phi q_k, \phi q_l, \phi q_m)$ is non-degenerate. As A is orthogonal and ϕ is isometric, we have by (4.224) that

$$|\xi^p_i - \xi^p_i| = |A(\phi \circ \pi q_i - \phi \circ \pi q_j)| = |\pi q_i - \pi q_j| < b|q_i - q_j| = b\sigma_{ij}, \tag{4.227}$$

which together with (4.222) gives *iv)*. □

Now we may define the global extensions $\xi_{i,j} = -\xi_{j,i}$ of the unit normal vector fields between the phases i and j in the strong solution by gluing the local definitions by means of the partition of unity $(\eta_{\mathrm{bulk}}, \eta_1, \ldots, \eta_N)$ from Lemma 4.30.

Construction 4.32. Let $d = 2$ and $P \in \mathbb{N}$, $P \geq 2$. Let $\bar{\Omega} = (\bar{\Omega}_1, \ldots, \bar{\Omega}_P)$ be a strong solution to multiphase mean curvature flow in the sense of Definition 4.14. Let $(\eta_{\mathrm{bulk}}, \eta_1, \ldots, \eta_N)$ be a partition of unity as constructed in Lemma 4.30. Let $i, j \in \{1, \ldots, P\}$ with $i \neq j$, and let for all $n \in \{1, \ldots, N\}$ the local vector fields $\xi^n_{i,j} = -\xi^n_{j,i}$ be given as in Lemma 4.31. We then define

$$\xi_{i,j}(x,t) := \sum_{n=1}^{N} \eta_n(x,t) \xi^n_{i,j}(x,t) \tag{4.228}$$

for all $x \in \mathbb{R}^2$ and all $t \in [0,T]$.

We proceed with the derivation of the coercivity condition provided by the length of the vector fields $\xi_{i,j}$ as defined by Construction 4.32. For an illustration we refer to Figure 4.11.

Lemma 4.33. *Let $d = 2$ and $P \in \mathbb{N}$, $P \geq 2$. Let $\bar{\Omega} = (\bar{\Omega}_1, \ldots, \bar{\Omega}_P)$ be a strong solution to multiphase mean curvature flow in the sense of Definition 4.14. Let $(\eta_{\mathrm{bulk}}, \eta_1, \ldots, \eta_N)$ be a partition of unity as constructed in Lemma 4.30. In particular, let $\bar{r}_{\min} \in (0,1]$ be the localization scale defined by (4.177). Let $\xi_{i,j}$ for $i, j \in \{1, \ldots, P\}$ with $i \neq j$ be the family of vector fields provided by Construction 4.32. Then there exists a constant $C \geq 1$, depending only on $\bar{\Omega}$ but not on $\bar{r}_{\min}$, such that for all $i, j \in \{1, \ldots, P\}$ with $i \neq j$ it holds*

$$\frac{1}{C}\big(\bar{r}_{\min}^{-2} \,\mathrm{dist}^2(\cdot, \bar{I}_{i,j}) \wedge 1\big) \leq 1 - |\xi_{i,j}|. \tag{4.229}$$

Proof. Let $(x,t) \in \mathbb{R}^2 \times [0,T]$ and $i,j \in \{1,\ldots,P\}$ with $i \neq j$. The asserted estimate (4.229) is trivially fulfilled for $(x,t) \notin \operatorname{supp} \xi_{i,j}$. By the definition (4.228) we may therefore assume that there exists a topological feature $n \in \{1,\ldots,N\}$ such that $(x,t) \in \operatorname{supp} \eta_n$ and that $\eta_n(x,t) = \max\{\eta_{n'}(x,t) : 1 \leq n' \leq N\}$. Because of the localization properties (4.188)–(4.190), we may additionally assume $\eta_n(x,t) \geq \frac{1}{4}$. Otherwise, $|\xi_{i,j}| \leq \frac{3}{4}$ on account of the local vector fields having at most unit length.

If either phase i or phase j is absent at the topological feature $\mathcal{T}_n$, we argue as follows. Using $b \in (0,1)$ from (4.223) we compute

$$\begin{aligned} |\xi_{i,j}| &= \Bigg|\eta_n \xi^n_{i,j} + \sum_{n' \in \{1,\ldots,N\} \setminus \{n\}} \eta_{n'} \xi^{n'}_{i,j}\Bigg| \leq \eta_n b + \sum_{n' \in \{1,\ldots,N\} \setminus \{n\}} \eta_{n'} \\ &\leq 1 - \eta_n(1-b). \end{aligned}$$

Due to $\eta_n(x,t) \geq \frac{1}{4}$ we deduce $1 - |\xi_{i,j}(x,t)| \geq \frac{1}{4}(1-b) \in (0,1)$. Therefore the estimate (4.229) holds in this case.

Next, we assume that both phases i and j are present at $\mathcal{T}_n$. In the regime $n = c \in \mathcal{C}$, it follows from $(x,t) \in \operatorname{supp} \eta_c$, the localization properties (4.186) and (4.189), the definitions (4.177) and (4.176) of the localization scales $r_{\mathcal{P}}$ and $\bar{r}_{\min}$, as well as the estimates (4.77) and (4.78) that $\operatorname{dist}(x, \bar{I}_{i,j}(t)) \leq C \operatorname{dist}(x, \mathcal{I}(t))$. Hence, (4.229) is implied by the coercivity estimate (4.179) for the bulk cutoff and the definition (4.228).

If $n = p \in \mathcal{P}$, denote by $k \in \{1,\ldots,P\}$ the third phase present at $\mathcal{T}_p$ next to the phases i and j. If $x \in B_{r_{\mathcal{P}}}(\mathcal{T}_p(t)) \setminus \big(W_{j,k}(t) \cup W_{k,i}(t) \cup W_k(t)\big)$, then by (4.77) and (4.78) it again holds $\operatorname{dist}(x, \bar{I}_{i,j}(t)) \leq C \operatorname{dist}(x, \mathcal{I}(t))$ so that (4.229) follows as before. Thus, assume that $x \in B_{r_{\mathcal{P}}}(\mathcal{T}_p(t)) \cap \big(W_{j,k}(t) \cup W_{k,i}(t) \cup W_k(t)\big)$. Figure 4.11 serves as an illustration for the subsequent argument, for which we in fact assume that $x \in W_k(t)$ (the argument in case of interface wedges is similar). Based on the definition (4.228), the localization properties (4.188)–(4.190), the coercivity estimate (4.223), and the definitions (4.203), (4.211) as well as (4.192), we estimate at (x,t)

$$\begin{aligned} 1 - |\xi_{i,j}| &\geq 1 - \big(\eta_p + b\eta_{c_{k,i}} + b\eta_{c_{j,k}}\big) \\ &= \lambda^{i,j}_k \Big(1 - \big(b\zeta_{c_{k,i}} + (1-b)\zeta_p\zeta_{c_{k,i}}\big)\Big) + (1-\lambda^{i,j}_k)\Big(1 - \big(b\zeta_{c_{j,k}} + (1-b)\zeta_p\zeta_{c_{j,k}}\big)\Big) \\ &\geq (1-b)(1-\zeta_p) \geq (1-b)\big(\bar{r}^{-2}_{\min} \operatorname{dist}^2(x, \mathcal{T}_p) \wedge 1\big). \end{aligned}$$

The trivial estimate $\operatorname{dist}(x, \mathcal{T}_p(t)) \geq \operatorname{dist}(x, \bar{I}_{i,j}(t))$ therefore allows to conclude. □

For a global definition of the velocity field B, we proceed analogously, i.e., we first provide a definition for local velocity fields B^n for each topological feature $\mathcal{T}_n$ with $n \in \{1,\ldots,N\}$ and then glue them together by means of the partition of unity $(\eta_{\text{bulk}}, \eta_1, \ldots, \eta_N)$ from Lemma 4.30.

Construction 4.34. Let $d = 2$ and $P \in \mathbb{N}$, $P \geq 2$. Let $\bar{\Omega} = (\bar{\Omega}_1, \ldots, \bar{\Omega}_P)$ be a strong solution to multiphase mean curvature flow in the sense of Definition 4.14. Let $(\eta_{\text{bulk}}, \eta_1, \ldots, \eta_N)$ be a partition of unity as constructed in Lemma 4.30.

Let $n \in \{1,\ldots,N\}$, and recalling the notation (4.220), we define a continuous vector field

$$B^n : \mathcal{U}_n \mapsto \mathbb{R}^2$$

as follows: in case of $n \in \mathcal{C}$ we take B^n as the restriction to $\mathcal{U}_n$ of the two-phase velocity field from Lemma 4.18. More precisely, in case the curve $\mathcal{T}_c$ connects two triple junctions, the tangential component of B^n is chosen as in Proposition 4.29; otherwise, we simply let the tangential component vanish. In case of $n \in \mathcal{P}$ we take B^n as the restriction to $\mathcal{U}_n$ of the triple junction velocity field from Proposition 4.22.

We finally define a global velocity field by means of

$$B(x,t) := \sum_{n=1}^{N} \eta_n(x,t) B^n(x,t) \tag{4.230}$$

for all $x \in \mathbb{R}^2$ and all $t \in [0,T]$.

We briefly present the regularity properties of the family of local velocity fields from Construction 4.34.

Lemma 4.35. *In the setting of Construction 4.34, for all $n \in \{1,\dots,N\}$ the associated local velocity field satisfies $B^n \in C_t^0 C_x^2(\overline{\mathcal{U}_n} \setminus \mathcal{T}_{\mathcal{P}})$, $\mathcal{T}_{\mathcal{P}} := \bigcup_{p\in\mathcal{P}} \mathcal{T}_p$. Moreover, there exists $C > 0$, which may depend on $\bar\Omega$ but not on the localization scale $\bar r_{\min}$ from (4.177), such that throughout $\mathcal{U}_n \setminus \mathcal{T}_{\mathcal{P}}$ it holds*

$$\max_{k=0,1,2} \bar r_{\min}^k |\nabla^k B^n| \leq C \bar r_{\min}^{-1}. \tag{4.231}$$

Proof. For $n = c \in \mathcal{C}$ the estimate (4.231) follows from (4.59) and (4.169), which in turn are indeed applicable thanks to the localization property (4.186) and the definition (4.177). In case of $n = p \in \mathcal{P}$, we may apply Proposition 4.22 due to the localization property (4.187) and the definition (4.176), so that (4.85) implies (4.231). □

Equipped with the definition of the global velocity field B, we may now prove a suitable estimate on the advective derivative of the bulk cutoff η_{bulk} from Lemma 4.30.

Lemma 4.36. *Let $d = 2$ and $P \in \mathbb{N}$, $P \geq 2$. Let $\bar\Omega = (\bar\Omega_1, \dots, \bar\Omega_P)$ be a strong solution to multiphase mean curvature flow in the sense of Definition 4.14. Let η_{bulk} be the bulk cutoff from Lemma 4.30, $\bar r_{\min} \in (0,1]$ the localization scale defined by (4.177), and $\mathcal{T}_{\mathcal{P}} := \bigcup_{p\in\mathcal{P}} \mathcal{T}_p$. Let B be the global velocity field from Construction 4.34. Denote by $\mathcal{I} := \bigcup_{t\in[0,T]} \bigcup_{i\neq j} \bar I_{i,j}(t) \times \{t\}$ the evolving network of interfaces. Then there exists a constant $C > 0$, depending only on the strong solution $\bar\Omega$ but not on $\bar r_{\min}$, such that*

$$|\partial_t \eta_{\mathrm{bulk}} + (B\cdot\nabla)\eta_{\mathrm{bulk}}| \leq C \bar r_{\min}^{-2} \big(\bar r_{\min}^{-2} \operatorname{dist}^2(\cdot, \mathcal{I}) \wedge 1\big) \tag{4.232}$$

in $\mathbb{R}^2 \times [0,T] \setminus \mathcal{T}_{\mathcal{P}}$. Moreover, for all $n \in \{1,\dots,N\}$ and all distinct $i,j \in \{1,\dots,P\}$ such that either phase i or phase j is absent at $\mathcal{T}_n$ it holds

$$|\partial_t \eta_n + (B\cdot\nabla)\eta_n| \leq C \bar r_{\min}^{-2} \big(\bar r_{\min}^{-2} \operatorname{dist}^2(\cdot, \bar I_{i,j}) \wedge 1\big) \tag{4.233}$$

in $\mathbb{R}^2 \times [0,T] \setminus \mathcal{T}_{\mathcal{P}}$.

Proof. The estimate (4.233) is trivially fulfilled in case of $n = p \in \mathcal{P}$ by (4.178), (4.187) and the definition (4.176) of the localization scale $r_{\mathcal{P}}$. Hence, let us reserve notation for the proof of (4.233) by fixing $c'' \in \mathcal{C}$ and distinct phases $i', j' \in \{1,\dots,P\}$ such that at least one of them is absent at $\mathcal{T}_{c''}$.

We now split the proof into two parts, first establishing the asserted estimates along two-phase interfaces $\mathcal{T}_c$ and away from triple junctions, and second in the vicinity of triple junctions adjacent to $\mathcal{T}_c$. More precisely, by the localization properties (4.186)–(4.190) and the choices (4.176)–(4.177) of the localization scales $r_{\mathcal{P}}$ and $\bar r_{\min}$, it suffices to prove (4.232) in $\bigcup_{c\in\mathcal{C}} \operatorname{im}_{\bar r_{\min}}(\Psi_{\mathcal{T}_c}) \setminus \bigcup_{p\in\mathcal{P}} \bigcup_{t\in[0,T]} B_{r_{\mathcal{P}}}(\mathcal{T}_p(t)) \times \{t\}$ and in $\bigcup_{p\in\mathcal{P}} \bigcup_{t\in[0,T]} B_{r_{\mathcal{P}}}(\mathcal{T}_p(t)) \times \{t\}$, respectively. We in fact may argue separately for each $c \in \mathcal{C}$ and each $p \in \mathcal{P}$.

Step 1: Estimates close to $\mathcal{T}_c$ and away from triple junctions. In this step, we restrict ourselves to the region $\mathrm{im}_{\bar{r}_{\min}}(\Psi_{\mathcal{T}_c}) \setminus \bigcup_{p\in\mathcal{P}} \bigcup_{t\in[0,T]} B_{r_\mathcal{P}}(\mathcal{T}_p(t))\times\{t\}$. To fix notation, let $i,j \in \{1,\ldots,P\}$ be such that c refers to a two-phase interface $\mathcal{T}_c \subset \bar{I}_{i,j}$. Recalling (4.217), we register that

$$\eta_{\mathrm{bulk}} = 1 - \eta_c, \tag{4.234}$$

$$\eta_c = \zeta_c = \zeta\Big(\frac{s_{i,j}}{\delta\bar{r}_{\min}}\Big), \tag{4.235}$$

$$B = \eta_c B^c, \tag{4.236}$$

in $\mathrm{im}_{\bar{r}_{\min}}(\Psi_{\mathcal{T}_c}) \setminus \bigcup_{p\in\mathcal{P}} \bigcup_{t\in[0,T]} B_{r_\mathcal{P}}(\mathcal{T}_p(t))\times\{t\}$.

For (4.232), we first observe that the signed distance function is transported by B^c, cf. (4.60). By the chain rule, this also holds for ζ_c, i.e.,

$$\partial_t\zeta_c + (B^c\cdot\nabla)\zeta_c = 0 \quad \text{in } \mathrm{im}(\Psi_{\mathcal{T}_c}). \tag{4.237}$$

Hence, using (4.236), (4.234), the quadratic order of η_{bulk} from (4.180), and the regularity estimates (4.178) and (4.231) we obtain

$$|\partial_t\zeta_c + (B\cdot\nabla)\zeta_c| = \eta_{\mathrm{bulk}}|(B^c\cdot\nabla)\zeta_c| \leq C\bar{r}_{\min}^{-2}\big(\bar{r}_{\min}^{-2}\operatorname{dist}^2(\cdot,\mathcal{I})\wedge 1\big) \tag{4.238}$$

in the region $\mathrm{im}_{\bar{r}_{\min}}(\Psi_{\mathcal{T}_c}) \setminus \bigcup_{p\in\mathcal{P}} \bigcup_{t\in[0,T]} B_{r_\mathcal{P}}(\mathcal{T}_p(t))\times\{t\}$. By (4.234) and (4.235), this is equivalent to (4.232).

For a proof of (4.233) throughout $\mathrm{im}_{\bar{r}_{\min}}(\Psi_{\mathcal{T}_c}) \setminus \bigcup_{p\in\mathcal{P}} \bigcup_{t\in[0,T]} B_{r_\mathcal{P}}(\mathcal{T}_p(t))\times\{t\}$, we may assume without loss of generality that $c'' = c$; otherwise, the estimate (4.233) is trivially fulfilled by (4.186) and the definition (4.177) of the localization scale $\bar{r}_{\min}$. However, if $c'' = c$ then the above argument already yields the claim thanks to (4.234), (4.235) and (4.238).

Step 2: Estimates close to $\mathcal{T}_c$ and in the vicinity of triple junctions. Now, consider $p \in \mathcal{P}$ and assume that the pairwise distinct phases $i,j,k \in \{1,\ldots,P\}$ are present at $\mathcal{T}_p$. Modulo a permutation of the indices, it suffices to consider the two unique two-phase interfaces $\mathcal{T}_{c_{i,j}} \subset \bar{I}_{i,j}$ and $\mathcal{T}_{c_{k,i}} \subset \bar{I}_{k,i}$ so that $c := c_{i,j} \sim p$ and $c' := c_{k,i} \sim p$, and then to prove the desired estimate (4.232) on the interface wedge $W_{i,j}$ and the interpolation wedge W_i.

In this step, let us turn to the interface wedge $W_{i,j}$. The interpolation wedge W_i will be discussed in *Step 3*. With respect to (4.233), it then suffices to work in the regime $c'' \sim p$ and $c'' = c$; otherwise, the estimate (4.233) is again fulfilled for trivial reasons thanks to (4.189) and (4.190). Based on (4.210) and (4.218) we then have

$$\eta_c = (1-\zeta_p)\zeta_c, \tag{4.239}$$

$$\eta_{\mathrm{bulk}} = 1 - \eta_c - \eta_p = 1 - \zeta_c, \tag{4.240}$$

$$B = \eta_c B^c + \eta_p B^p, \tag{4.241}$$

throughout $B_{r_\mathcal{P}}(\mathcal{T}_p(t)) \cap W_{i,j}(t)$ for all $t \in [0,T]$.

For the estimate on the advective derivative of the bulk cutoff, using (4.240) and the transport equation for the interface cutoff (4.237) (which is applicable throughout $B_{r_\mathcal{P}}(\mathcal{T}_p(t)) \cap W_{i,j}(t)$ for all $t \in [0,T]$ due to (4.75)) we obtain

$$\partial_t\zeta_c = -(B^c\cdot\nabla)\zeta_c = -\big(B\cdot\nabla\big)\zeta_c - \eta_{\mathrm{bulk}}(B^c\cdot\nabla)\zeta_c - \eta_p\big((B^p - B^c)\cdot\nabla\big)\zeta_c$$

in $B_{r_\mathcal{P}}(\mathcal{T}_p(t)) \cap W_{i,j}(t)$ for all $t \in [0,T]$. In particular, because of (4.180), (4.231), (4.200), (4.75), (4.178), (4.172), (4.78), and finally (4.216) this entails

$$\big|\partial_t\zeta_c + (B\cdot\nabla)\zeta_c\big| \leq C\bar{r}_{\min}^{-2}\big(\bar{r}_{\min}^{-2}\operatorname{dist}^2(\cdot,\mathcal{I})\wedge 1\big) \tag{4.242}$$

in $B_{r_{\mathcal{P}}}(\mathcal{T}_p(t)) \cap W_{i,j}(t)$ for all $t \in [0,T]$. By the representation (4.240), this is equivalent to (4.232).

To obtain the asserted bound on the advective derivative of the interface cut-off η_c, we use that since ζ_p is only a smooth function of the distance to the triple point $\mathcal{T}_p(t) = \{p(t)\}$ (performing an excusable abuse of notation), it satisfies the transport equation $\partial_t\zeta_p + (\frac{\mathrm{d}}{\mathrm{d}t}p(t)\cdot\nabla)\zeta_p = 0$ throughout $\mathbb{R}^2\times[0,T]\setminus\mathcal{T}_{\mathcal{P}}$. By Proposition 4.22 *i)*, the partition of unity property of the family $(\eta_1,\ldots,\eta_N)$, and the regularity estimates (4.231) resp. (4.178), it follows that $|B - B(p(t),t)| \leq C\bar{r}_{\min}^{-2}\operatorname{dist}(\cdot,\mathcal{T}_p)$ in $B_{r_{\mathcal{P}}}(\mathcal{T}_p(t)) \cap (W_{i,j}(t)\cup W_i(t)\cup W_j(t))$ for all $t\in[0,T]$. This in turn implies by means of (4.192)

$$|\partial_t\zeta_p + \big(B\cdot\nabla\big)\zeta_p| \leq C\bar{r}_{\min}^{-2}r_{\mathcal{P}}^{-2}\operatorname{dist}^2(\cdot,\mathcal{T}_p(t)) \leq C\bar{r}_{\min}^{-2}(1-\zeta_p) \tag{4.243}$$

in $B_{r_{\mathcal{P}}}(\mathcal{T}_p(t)) \cap (W_{i,j}(t)\cup W_i(t)\cup W_j(t))$ for all $t\in[0,T]$. Hence, when restricting to the interface wedge we obtain from the combination of (4.239), the product rule, (4.242), (4.243) and finally (4.183) that the desired estimate (4.233) indeed holds true in $B_{r_{\mathcal{P}}}(\mathcal{T}_p(t))\cap W_{i,j}(t)$ for all $t\in[0,T]$.

Step 3: Estimates in interpolation wedges at triple junctions. We turn to the proof of (4.232) and (4.233) on the interpolation wedge W_i. Recall to this end the notation fixed at the beginning of *Step 2*. With respect to proving (4.233), it suffices to consider $c'' \sim p$ and $c'' \in \{c,c'\}$, and thus up to a relabeling $c''=c$; otherwise, the estimate (4.233) follows trivially because of (4.189) and (4.190).

Because of (4.211) and (4.219), it then holds (abbreviating $\lambda := \lambda_i^{j,k}$)

$$\eta_c = \lambda(1-\zeta_p)\zeta_c, \tag{4.244}$$

$$\eta_{\text{bulk}} = 1-\eta_c-\eta_{c'}-\eta_p = \lambda(1-\zeta_c) + (1-\lambda)(1-\zeta_{c'}), \tag{4.245}$$

$$B = \eta_c B^c + \eta_{c'}B^{c'} + \eta_p B^p, \tag{4.246}$$

throughout $B_{r_{\mathcal{P}}}(\mathcal{T}_p(t))\cap W_i(t)$ for all $t\in[0,T]$.

Based on the second identity of (4.245) and (4.246), we may split the task of estimating the advective derivative of the bulk cutoff as follows:

$$\partial_t\eta_{\text{bulk}} + (B\cdot\nabla)\eta_{\text{bulk}} =: I + II,$$

where we defined

$$I := (1-\zeta_c)(\partial_t + B\cdot\nabla)\lambda + (1-\zeta_{c'})(\partial_t + B\cdot\nabla)(1-\lambda),$$
$$II := \lambda\big(\partial_t + B\cdot\nabla\big)(1-\zeta_c) + (1-\lambda)\big(\partial_t + B\cdot\nabla\big)(1-\zeta_{c'})$$

We estimate term by term. For an estimate of II, we argue in a similar fashion to *Step 2*. More precisely, applying (4.245) and the transport equation for the interface cutoff (4.237) (which is applicable throughout $B_{r_{\mathcal{P}}}(\mathcal{T}_p(t))\cap W_i(t)$ for all $t\in[0,T]$ due to (4.76)) we have

$$\partial_t\zeta_c = -\big(B\cdot\nabla\big)\zeta_c - \eta_{\text{bulk}}(B^c\cdot\nabla)\zeta_c - \eta_{c'}\big((B^{c'}-B^c)\cdot\nabla\big)\zeta_c - \eta_p\big((B^p-B^c)\cdot\nabla\big)\zeta_c$$

in $B_{r_{\mathcal{P}}}(\mathcal{T}_p(t))\cap W_i(t)$ for all $t\in[0,T]$. Replacing the use of (4.75) by (4.76) and the use of (4.78) by (4.77), we may rely on the otherwise same argument entailing (4.242) to deduce that (adding also zero in form of $B^{c'}-B^c = (B^{c'}-B^p) + (B^p-B^c)$)

$$\big|\partial_t\zeta_c + \big(B\cdot\nabla\big)\zeta_c\big| \leq C\bar{r}_{\min}^{-2}\big(\bar{r}_{\min}^{-2}\operatorname{dist}^2(\cdot,\mathcal{I})\wedge 1\big) \tag{4.247}$$

in $B_{r_{\mathcal{P}}}(\mathcal{T}_p(t))\cap W_i(t)$ for all $t\in[0,T]$. Of course, the same estimate holds true in terms of $\zeta_{c'}$. Hence, $|II| \leq C\bar{r}_{\min}^{-2}\big(\bar{r}_{\min}^{-2}\operatorname{dist}^2(\cdot,\mathcal{I})\wedge 1\big)$ in $B_{r_{\mathcal{P}}}(\mathcal{T}_p(t))\cap W_i(t)$ for all $t\in[0,T]$ as desired.

We turn to the estimate of I. Adding zero and relying on (4.245) as well as (4.246), we observe that it holds

$$(\partial_t{+}B\cdot\nabla)\lambda = (\partial_t{+}B^p\cdot\nabla)\lambda + \big(\eta_c(B^c{-}B^p){+}\eta_{c'}(B^{c'}{-}B^p){-}\eta_{\mathrm{bulk}}B^p\big)\cdot\nabla\lambda.$$

By familiar arguments in combination with the controlled blowup (4.143) of the derivative of the interpolation parameter, one checks that the second right hand side term of the previous display is of the order $O(\bar r_{\min}^{-2})$. The first right hand side term is of the same order thanks to (4.77) and the bound (4.147) on the advective derivative of the interpolation parameter (for which we may freely pass from B^p to $B^p(p(t),t)$, abusing again notation in form of $\mathcal{T}_p(t) = \{p(t)\}$, cf. Proposition 4.22 *i)* and the estimate (4.85)). Hence,

$$|\partial_t\lambda + (B\cdot\nabla)\lambda| \leq C\bar r_{\min}^{-2}. \tag{4.248}$$

By (4.199) and (4.77), we thus obtain $|(1{-}\zeta_c)(\partial_t{+}B\cdot\nabla)\lambda| \leq C\bar r_{\min}^{-2}(r_{\min}^{-2}\operatorname{dist}^2(\cdot,\mathcal{I})\wedge 1)$. Arguing analogously one also bounds the term $(1{-}\zeta_{c'})(\partial_t{+}B\cdot\nabla)(1{-}\lambda)$ to the same order, so that in summary (4.232) follows in the region $B_{r_{\mathcal{P}}}(\mathcal{T}_p(t))\cap W_i(t)$ for all $t\in[0,T]$.

We finally provide the proof of (4.233) in the given interpolation wedge. When computing the advective derivative of η_c, it follows from (4.244), the product rule, (4.247), (4.243) and (4.183) that we only need to additionally control the term when the derivative falls onto the interpolation parameter. However, since we already have (4.248) at our disposal, it follows from (4.196) that

$$|(\partial_t{+}B\cdot\nabla)\lambda|(1-\zeta_p)\zeta_c \leq C\bar r_{\min}^{-2}\big(\bar r_{\min}^{-2}\operatorname{dist}^2(\cdot,\mathcal{T}_p)\wedge 1\big),$$

which by (4.77) (or a trivial argument if either i' or j' is absent at $\mathcal{T}_p$) entails a bound of required order. This in turn concludes the proof. □

4.6.3 Global compatibility estimates

We next lift the local compatibility estimates from Proposition 4.29 to compatibility estimates between the global and local constructions. These technical estimates will be needed in order to derive the estimates (4.1c)–(4.1e) for the global constructions from the corresponding ones for the local constructions in Lemma 4.18 and Proposition 4.22.

Lemma 4.37. *Let $d=2$ and $P\in\mathbb{N}$, $P\geq 2$. Let $\bar\Omega=(\bar\Omega_1,\ldots,\bar\Omega_P)$ be a strong solution to multiphase mean curvature flow in the sense of Definition 4.14. Let $(\eta_{\mathrm{bulk}},\eta_1,\ldots,\eta_N)$ be a partition of unity as constructed in Lemma 4.30. In particular, let $\bar r_{\min}\in(0,1]$ be the localization scale defined by (4.177) and $\mathcal{T}_{\mathcal{P}} := \bigcup_{p\in\mathcal{P}}\mathcal{T}_p$. Let $(\xi^n_{i,j})_{n\in\{1,\ldots,N\}}$ be the local vector fields from Lemma 4.31 as well as $(B^n)_{n\in\{1,\ldots,N\}}$ be the local velocity fields from Construction 4.34. Let $\xi_{i,j}$ be the global vector fields from Construction 4.32, and let B be the global velocity field from Construction 4.34.*

Then, the local and global constructions are compatible in the sense that for all topological features $n\in\{1,\ldots,N\}$, and all distinct phases $i,j\in\{1,\ldots,P\}$ such that both i and j are present at $\mathcal{T}_n$, the following estimates are satisfied

$$\mathbb{1}_{\operatorname{supp}\eta_n}\big|\xi_{i,j}-\xi^n_{i,j}\big| \leq C\big(\bar r_{\min}^{-1}\operatorname{dist}(\cdot,\bar I_{i,j})\wedge 1\big), \tag{4.249}$$

$$\mathbb{1}_{\operatorname{supp}\eta_n}\big|(\xi_{i,j}-\xi^n_{i,j})\cdot\xi^n_{i,j}\big| \leq C\big(\bar r_{\min}^{-2}\operatorname{dist}^2(\cdot,\bar I_{i,j})\wedge 1\big), \tag{4.250}$$

$$\mathbb{1}_{\operatorname{supp}\eta_n}\big|B-B^n\big| \leq C\bar r_{\min}^{-1}\big(\bar r_{\min}^{-2}\operatorname{dist}^2(\cdot,\bar I_{i,j})\wedge 1\big), \tag{4.251}$$

$$\mathbb{1}_{\operatorname{supp}\eta_n}\big|\nabla B-\nabla B^n\big| \leq C\bar r_{\min}^{-2}\big(\bar r_{\min}^{-1}\operatorname{dist}(\cdot,\bar I_{i,j})\wedge 1\big) \tag{4.252}$$

throughout $\mathbb{R}^2{\times}[0,T]\setminus\mathcal{T}_{\mathcal{P}}$. The constant $C>0$ may depend on the strong solution $\bar\Omega$, but is independent of $\bar r_{\min}$.

For the proof of Lemma 4.37, recall that we decomposed $\{1,\ldots,N\} =: \mathcal{C} \cup \mathcal{P}$ with the convention that $\mathcal{C}$ enumerates the connected components in space-time of the smooth two-phase interfaces and $\mathcal{P}$ enumerates the triple junctions. If $p \in \mathcal{P}$, we defined $\mathcal{T}_p$ to be the trajectory in space-time described by the triple junction. If $c \in \mathcal{C}$, we defined $\mathcal{T}_c \subset \bar{I}_{i,j}$ for some $i,j \in \{1,\ldots,P\}$ with $i \neq j$ to be the corresponding space-time connected component of a two-phase interface $\bar{I}_{i,j}$. We further write $c \sim p$ for $c \in \mathcal{C}$ and $p \in \mathcal{P}$ if and only if $\mathcal{T}_c$ has an endpoint at $\mathcal{T}_p$. Note finally that two distinct phases $i,j \in \{1,\ldots,P\}$ are simultaneously present at a topological feature $\mathcal{T}_n$, $n \in \{1,\ldots,N\}$, if and only if $\mathcal{T}_n \subset \bar{I}_{i,j}$.

Proof. We aim to reduce the situation to the local compatibility estimates from Proposition 4.29. Such a reduction argument turns out to be possible due to the localization properties (4.188)–(4.190), the estimates (4.180)–(4.184), and our assumption that both phases i and j are present at the selected topological feature. For all what follows, let $n \in \{1,\ldots,N\}$ and $i,j \in \{1,\ldots,P\}$ such that $i \neq j$ as well as $\mathcal{T}_n \subset \bar{I}_{i,j}$. For notational convenience, we abbreviate for the purpose of the proof $\bar{r} := \bar{r}_{\min}$ and $d_{i,j} := \operatorname{dist}(\cdot, \bar{I}_{i,j})$.

Step 1: Proof of (4.249). We insert the definition (4.228) which in combination with the estimates (4.180), (4.183) and (4.221) yields

$$\begin{aligned}
\mathbb{1}_{\operatorname{supp}\eta_n}(\xi_{i,j}-\xi^n_{i,j}) &= -\mathbb{1}_{\operatorname{supp}\eta_n}\eta_{\mathrm{bulk}}\xi^n_{i,j} + \sum_{n'=1,n'\neq n}^{N} \mathbb{1}_{\operatorname{supp}\eta_n}\eta_{n'}(\xi^{n'}_{i,j}-\xi^n_{i,j}) \\
&= \sum_{\substack{n'=1,n'\neq n\\ \mathcal{T}_{n'}\subset \bar{I}_{i,j}}}^{N} \mathbb{1}_{\operatorname{supp}\eta_n}\eta_{n'}(\xi^{n'}_{i,j}-\xi^n_{i,j}) + O(\bar{r}^{-2}d^2_{i,j}\wedge 1).
\end{aligned} \tag{4.253}$$

Next, the localization properties (4.188)–(4.190) allow to represent the remaining right hand side terms in form of

$$\begin{aligned}
\sum_{\substack{n'=1,n'\neq n\\ \mathcal{T}_{n'}\subset \bar{I}_{i,j}}}^{N} \mathbb{1}_{\operatorname{supp}\eta_n}\eta_{n'}(\xi^{n'}_{i,j}-\xi^n_{i,j}) &= \sum_{p\in\mathcal{P},\mathcal{T}_p\subset\bar{I}_{i,j}} \sum_{c\in\mathcal{C},c\sim p} \mathbb{1}_{n=c}\mathbb{1}_{\operatorname{supp}\eta_c}\eta_p(\xi^p_{i,j}-\xi^c_{i,j}) \\
&\quad + \sum_{c\in\mathcal{C},\mathcal{T}_c\subset\bar{I}_{i,j}} \sum_{p\in\mathcal{P},c\sim p} \mathbb{1}_{n=p}\mathbb{1}_{\operatorname{supp}\eta_p}\eta_c(\xi^c_{i,j}-\xi^p_{i,j}) \\
&\quad + \sum_{c\in\mathcal{C},\mathcal{T}_c\subset\bar{I}_{i,j}} \sum_{p\in\mathcal{P},c\sim p} \sum_{\substack{c'\in\mathcal{C},c'\neq c\\ c'\sim p}} \mathbb{1}_{n=c'}\mathbb{1}_{\operatorname{supp}\eta_{c'}}\eta_c(\xi^c_{i,j}-\xi^{c'}_{i,j}).
\end{aligned}$$

The *assumption* $\mathcal{T}_n \subset \bar{I}_{i,j}$ furthermore enables us to post-process the previous identity as follows

$$\begin{aligned}
\sum_{\substack{n'=1,n'\neq n\\ \mathcal{T}_{n'}\subset \bar{I}_{i,j}}}^{N} \mathbb{1}_{\operatorname{supp}\eta_n}\eta_{n'}(\xi^{n'}_{i,j}-\xi^n_{i,j}) &= \sum_{p\in\mathcal{P},\mathcal{T}_p\subset\bar{I}_{i,j}} \sum_{\substack{c\in\mathcal{C},\mathcal{T}_c\subset\bar{I}_{i,j}\\ c\sim p}} \mathbb{1}_{n=c}\mathbb{1}_{\operatorname{supp}\eta_c}\eta_p(\xi^p_{i,j}-\xi^c_{i,j}) \\
&\quad + \sum_{c\in\mathcal{C},\mathcal{T}_c\subset\bar{I}_{i,j}} \sum_{\substack{p\in\mathcal{P},\mathcal{T}_p\subset\bar{I}_{i,j}\\ c\sim p}} \mathbb{1}_{n=p}\mathbb{1}_{\operatorname{supp}\eta_p}\eta_c(\xi^c_{i,j}-\xi^p_{i,j}).
\end{aligned}$$

We are now in a position to apply Proposition 4.29. More precisely, thanks to the localization property (4.189) and the definition (4.177) we have the estimate (4.170) at our disposal, implying that

$$\sum_{\substack{n'=1,n'\neq n\\ \mathcal{T}_{n'}\subset \bar{I}_{i,j}}}^{N} \mathbb{1}_{\operatorname{supp}\eta_n}\eta_{n'}(\xi^{n'}_{i,j}-\xi^n_{i,j}) = O(\bar{r}^{-1}d_{i,j}\wedge 1),$$

at least under our assumption of $\mathcal{T}_n \subset \bar{I}_{i,j}$. This concludes the argument for (4.249).

Step 2: Proof of (4.250). Multiplying (4.253) by $\xi^n_{i,j}$ and afterwards running through the same argument as in *Step 1* entails

$$\begin{aligned}
&\sum_{\substack{n'=1,n'\neq n\\ \mathcal{T}_{n'}\subset\bar{I}_{i,j}}}^{N} \mathbb{1}_{\operatorname{supp}\eta_n}\eta_{n'}(\xi^{n'}_{i,j}-\xi^{n}_{i,j})\cdot\xi^n_{i,j}\\
&=\sum_{p\in\mathcal{P},\mathcal{T}_p\subset\bar{I}_{i,j}}\ \sum_{\substack{c\in\mathcal{C},\mathcal{T}_c\subset\bar{I}_{i,j}\\ c\sim p}} \mathbb{1}_{n=c}\mathbb{1}_{\operatorname{supp}\eta_c}\eta_p(\xi^p_{i,j}-\xi^c_{i,j})\cdot\xi^c_{i,j}\\
&\quad+\sum_{c\in\mathcal{C},\mathcal{T}_c\subset\bar{I}_{i,j}}\ \sum_{\substack{p\in\mathcal{P},\mathcal{T}_p\subset\bar{I}_{i,j}\\ c\sim p}} \mathbb{1}_{n=p}\mathbb{1}_{\operatorname{supp}\eta_p}\eta_c(\xi^c_{i,j}-\xi^p_{i,j})\cdot\xi^p_{i,j}+O(\bar{r}^{-2}d^2_{i,j}\wedge 1).
\end{aligned}$$

Adding zero in the second right hand side term of the previous display in form of $(\xi^c_{i,j}-\xi^p_{i,j})\cdot\xi^p_{i,j}=-|\xi^c_{i,j}-\xi^p_{i,j}|^2+(\xi^c_{i,j}-\xi^p_{i,j})\cdot\xi^c_{i,j}$, and then applying the local compatibility estimates (4.171) and (4.170), we deduce (4.250).

Step 3: Proof of (4.251). Using the definition (4.230), the regularity estimates (4.231) and the local compatibility estimate (4.172) instead of (4.228), (4.221) and (4.170), respectively, and substituting (B,B^n) for $(\xi_{i,j},\xi^n_{i,j})$ in the argument of *Step 1* directly implies (4.251).

Step 4: Proof of (4.252). We give some details here, as in comparison to *Step 1* or *Step 3* the argument in favor of (4.252) involves an additional (though simple) reduction step. Starting with the definition (4.230), the estimates (4.180), (4.183) and (4.231), and in addition the product rule we obtain

$$\begin{aligned}
&\mathbb{1}_{\operatorname{supp}\eta_n}(\nabla B-\nabla B^n)\\
&=-\mathbb{1}_{\operatorname{supp}\eta_n}\eta_{\mathrm{bulk}}\nabla B^n+\sum_{n'=1,n'\neq n}^{N}\mathbb{1}_{\operatorname{supp}\eta_n}\eta_{n'}(\nabla B^{n'}-\nabla B^n)+\sum_{n'=1}^{N}\mathbb{1}_{\operatorname{supp}\eta_n}B^{n'}\otimes\nabla\eta_{n'}\\
&=\sum_{\substack{n'=1,n'\neq n\\ \mathcal{T}_{n'}\subset\bar{I}_{i,j}}}^{N}\mathbb{1}_{\operatorname{supp}\eta_n}\eta_{n'}(\nabla B^{n'}-\nabla B^n)+\sum_{n'=1}^{N}\mathbb{1}_{\operatorname{supp}\eta_n}B^{n'}\otimes\nabla\eta_{n'}\\
&\quad+O\big(\bar{r}^{-2}(\bar{r}^{-2}d^2_{i,j}\wedge 1)\big).
\end{aligned}$$

The first right hand side term is estimated to desired order $O\big(\bar{r}^{-2}(\bar{r}^{-1}d_{i,j}\wedge 1)\big)$ based on the local compatibility estimate (4.173) and the above familiar reduction arguments. Adding zero in the second right hand side term moreover entails

$$\begin{aligned}
&\sum_{n'=1}^{N}\mathbb{1}_{\operatorname{supp}\eta_n}B^{n'}\otimes\nabla\eta_{n'}\\
&=\sum_{n'=1,n'\neq n}^{N}\mathbb{1}_{\operatorname{supp}\eta_n}(B^{n'}-B^n)\otimes\nabla\eta_{n'}-\mathbb{1}_{\operatorname{supp}\eta_n}B^n\otimes\nabla\eta_{\mathrm{bulk}}.
\end{aligned}$$

The previous reduction arguments in combination with the local compatibility estimate (4.172), the upper bound (4.181) for the gradient of the bulk cutoff, as well as the regularity estimates (4.178) and (4.231) thus show that $\sum_{n'=1}^{N}\mathbb{1}_{\operatorname{supp}\eta_n}B^{n'}\otimes\nabla\eta_{n'}$ is of order $O\big(\bar{r}^{-2}(\bar{r}^{-1}d_{i,j}\wedge 1)\big)$. This concludes the proof. □

4.6.4 Approximate transport and mean curvature flow equations

We derive the global (or network) version of our previous bounds from Lemma 4.18 and Proposition 4.22, which are valid for the model problem of a smooth manifold and a triple junction, respectively.

Lemma 4.38. *Let $d = 2$ and $P \in \mathbb{N}$, $P \geq 2$. Let $\bar\Omega = (\bar\Omega_1, \ldots, \bar\Omega_P)$ be a strong solution to multiphase mean curvature flow in the sense of Definition 4.14. Let next $\bar r_{\min} \in (0,1]$ be the localization scale defined by (4.177) and $\mathcal{T}_{\mathcal{P}} := \bigcup_{p\in\mathcal{P}} \mathcal{T}_p$. Let $(\xi^n_{i,j})_{n\in\{1,\ldots,N\}}$ be the local vector fields from Lemma 4.31 as well as $(B^n)_{n\in\{1,\ldots,N\}}$ be the local velocity fields from Construction 4.34. Let $\xi_{i,j}$ be the global vector fields from Construction 4.32, and let B be the global velocity field from Construction 4.34.*

Then there exists a constant $C > 0$, depending only on the strong solution $\bar\Omega$ but not on $\bar r_{\min}$, so that we have the estimates

$$|\partial_t \xi_{i,j} + (B\cdot\nabla)\xi_{i,j} + (\nabla B)^{\mathsf{T}}\xi_{i,j}| \leq C\bar r_{\min}^{-2}\big(\bar r_{\min}^{-1}\operatorname{dist}(\cdot, \bar I_{i,j}) \wedge 1\big), \tag{4.254}$$

$$|(\nabla\cdot\xi_{i,j}) + B\cdot\xi_{i,j}| \leq C\bar r_{\min}^{-1}\big(\bar r_{\min}^{-1}\operatorname{dist}(\cdot, \bar I_{i,j}) \wedge 1\big), \tag{4.255}$$

$$\big|\xi_{i,j}\cdot\partial_t\xi_{i,j} + \xi_{i,j}\cdot(B\cdot\nabla)\xi_{i,j}\big| \leq C\bar r_{\min}^{-2}\big(\bar r_{\min}^{-2}\operatorname{dist}^2(\cdot, \bar I_{i,j}) \wedge 1\big) \tag{4.256}$$

in $\mathbb{R}^2\times[0,T]\setminus\mathcal{T}_{\mathcal{P}}$, for all $i,j\in\{1,\ldots,P\}$ with $i\neq j$.

Proof. Let $i,j\in\{1,\ldots,P\}$ such that $i\neq j$. For notational convenience, we again abbreviate for the purpose of the proof $\bar r := \bar r_{\min}$ and $d_{i,j} := \operatorname{dist}(\cdot, \bar I_{i,j})$. Recall that the distinct phases i and j are both present at a given topological feature $\mathcal{T}_n$, $n\in\{1,\ldots,N\}$, if and only if $\mathcal{T}_n \subset \bar I_{i,j}$.

Step 1: Proof of (4.254). By the product rule, the definition (4.228), the regularity estimates (4.221) and (4.231), as well as the error estimates (4.183)–(4.185) we compute

$$\begin{aligned}\partial_t\xi_{i,j} + (B\cdot\nabla)\xi_{i,j} = \sum_{n=1,\mathcal{T}_n\subset\bar I_{i,j}}^{N} \eta_n(\partial_t + B\cdot\nabla)\xi^n_{i,j} + \sum_{n=1,\mathcal{T}_n\subset\bar I_{i,j}}^{N} \xi^n_{i,j}(\partial_t + B\cdot\nabla)\eta_n \\ + O\big(\bar r^{-2}(\bar r^{-1}d_{i,j}\wedge 1)\big).\end{aligned}$$

Next, it follows from adding zero, the compatibility estimate (4.249), the regularity bound (4.178), and again (4.231), (4.184) and (4.185) that

$$\begin{aligned}\sum_{n=1,\mathcal{T}_n\subset\bar I_{i,j}}^{N} \xi^n_{i,j}(\partial_t + B\cdot\nabla)\eta_n &= \sum_{n=1,\mathcal{T}_n\subset\bar I_{i,j}}^{N} \xi_{i,j}(\partial_t + B\cdot\nabla)\eta_n + O\big(\bar r^{-2}(\bar r^{-1}d_{i,j}\wedge 1)\big) \\ &= -\xi_{i,j}(\partial_t + B\cdot\nabla)\eta_{\mathrm{bulk}} + O\big(\bar r^{-2}(\bar r^{-1}d_{i,j}\wedge 1)\big).\end{aligned}$$

Thanks to the compatibility estimate (4.251) and the regularity estimate (4.221), we also have

$$\sum_{n=1,\mathcal{T}_n\subset\bar I_{i,j}}^{N} \eta_n(B\cdot\nabla)\xi^n_{i,j} = \sum_{n=1,\mathcal{T}_n\subset\bar I_{i,j}}^{N} \eta_n(B^n\cdot\nabla)\xi^n_{i,j} + O\big(\bar r^{-2}(\bar r^{-1}d_{i,j}\wedge 1)\big).$$

Together with the upper bounds (4.181) resp. (4.182) for the bulk cutoff and the regularity estimate (4.231), the previous three displays combine to

$$\partial_t\xi_{i,j} + (B\cdot\nabla)\xi_{i,j} = \sum_{n=1,\mathcal{T}_n\subset\bar I_{i,j}}^{N} \eta_n(\partial_t + B^n\cdot\nabla)\xi^n_{i,j} + O\big(\bar r^{-2}(\bar r^{-1}d_{i,j}\wedge 1)\big). \tag{4.257}$$

In a next step, we compute based on the product rule, the definitions (4.228) and (4.230), the error estimate (4.183), the regularity estimates (4.231) and (4.178), as well as the compatibility estimate (4.252)

$$
\begin{aligned}
(\nabla B)^{\mathsf{T}}\xi_{i,j} &= \sum_{n=1,\mathcal{T}_n\subset\bar{I}_{i,j}}^{N} \eta_n(\nabla B)^{\mathsf{T}}\xi_{i,j}^n + O\big(\bar{r}^{-2}(\bar{r}^{-1}d_{i,j}\wedge 1)\big) \\
&= \sum_{n=1,\mathcal{T}_n\subset\bar{I}_{i,j}}^{N} \eta_n(\nabla B^n)^{\mathsf{T}}\xi_{i,j}^n + O\big(\bar{r}^{-2}(\bar{r}^{-1}d_{i,j}\wedge 1)\big). \qquad (4.258)
\end{aligned}
$$

Hence, in view of (4.257) and (4.258) we reduced the task to the local evolution equations at topological features for which both phases i and j are present:

$$
\begin{aligned}
\partial_t\xi_{i,j}+(B\cdot\nabla)\xi_{i,j}+(\nabla B)^{\mathsf{T}}\xi_{i,j} = \sum_{n=1,\mathcal{T}_n\subset\bar{I}_{i,j}}^{N} &\eta_n\big(\partial_t\xi_{i,j}^n+(B^n\cdot\nabla)\xi_{i,j}^n+(\nabla B^n)^{\mathsf{T}}\xi_{i,j}^n\big) \\
&+ O\big(\bar{r}^{-2}(\bar{r}^{-1}d_{i,j}\wedge 1)\big).
\end{aligned}
$$

To conclude that (4.254) holds, it thus only remains to observe that the bounds on the local evolution equations (4.61) and (4.81), respectively, are applicable due to the localization properties (4.186)–(4.187) and the definitions (4.176)–(4.177).

Step 2: Proof of (4.255). We proceed in the same style as for the proof of (4.254). On one side, it is immediate from the definitions (4.228) and (4.230), the error estimate (4.183), the regularity estimates (4.221) and (4.231), as well as the compatibility estimate (4.251)

$$
\begin{aligned}
B\cdot\xi_{i,j} &= \sum_{n=1,\mathcal{T}_n\subset\bar{I}_{i,j}}^{N} \eta_n B\cdot\xi_{i,j}^n + O\big(\bar{r}^{-1}(\bar{r}^{-1}d_{i,j}\wedge 1)\big) \\
&= \sum_{n=1,\mathcal{T}_n\subset\bar{I}_{i,j}}^{N} \eta_n B^n\cdot\xi_{i,j}^n + O\big(\bar{r}^{-1}(\bar{r}^{-1}d_{i,j}\wedge 1)\big).
\end{aligned}
$$

On the other side, we have by the definition (4.228), the product rule, the error estimates (4.183)–(4.184), the regularity estimates (4.221) and (4.178), the compatibility estimate (4.249), and finally the upper bound (4.181) for the bulk cutoff

$$
\begin{aligned}
\nabla\cdot\xi_{i,j} &= \sum_{n=1,\mathcal{T}_n\subset\bar{I}_{i,j}}^{N} \eta_n(\nabla\cdot\xi_{i,j}^n) + \sum_{n=1,\mathcal{T}_n\subset\bar{I}_{i,j}}^{N} (\xi_{i,j}^n\cdot\nabla)\eta_n + O\big(\bar{r}^{-1}(\bar{r}^{-1}d_{i,j}\wedge 1)\big) \\
&= \sum_{n=1,\mathcal{T}_n\subset\bar{I}_{i,j}}^{N} \eta_n(\nabla\cdot\xi_{i,j}^n) + \sum_{n=1,\mathcal{T}_n\subset\bar{I}_{i,j}}^{N} (\xi_{i,j}\cdot\nabla)\eta_n + O\big(\bar{r}^{-1}(\bar{r}^{-1}d_{i,j}\wedge 1)\big) \\
&= \sum_{n=1,\mathcal{T}_n\subset\bar{I}_{i,j}}^{N} \eta_n(\nabla\cdot\xi_{i,j}^n) - (\xi_{i,j}\cdot\nabla)\eta_{\mathrm{bulk}} + O\big(\bar{r}^{-1}(\bar{r}^{-1}d_{i,j}\wedge 1)\big) \\
&= \sum_{n=1,\mathcal{T}_n\subset\bar{I}_{i,j}}^{N} \eta_n(\nabla\cdot\xi_{i,j}^n) + O\big(\bar{r}^{-1}(\bar{r}^{-1}d_{i,j}\wedge 1)\big).
\end{aligned}
$$

The previous two displays in total imply

$$
\nabla\cdot\xi_{i,j} + B\cdot\xi_{i,j} = \sum_{n=1,\mathcal{T}_n\subset\bar{I}_{i,j}}^{N} \eta_n\big(\nabla\cdot\xi_{i,j}^n+B^n\cdot\xi_{i,j}^n\big) + O\big(\bar{r}^{-1}(\bar{r}^{-1}d_{i,j}\wedge 1)\big),
$$

so that (4.255) follows due to its local counterparts (4.63) and (4.82), respectively.

Step 3: Proof of (4.256). We first claim that

$$\begin{aligned} &\xi_{i,j}\cdot(\partial_t{+}B\cdot\nabla)\xi_{i,j} \\ &= \sum_{\substack{n,n'=1\\ \mathcal{T}_n,\mathcal{T}_{n'}\subset\bar I_{i,j}}}^{N} \eta_n\eta_{n'}\xi^n_{i,j}\cdot(\partial_t{+}B^{n'}\cdot\nabla)\xi^{n'}_{i,j} + O\big(\bar r^{-2}(\bar r^{-2}d^2_{i,j}\wedge 1)\big). \end{aligned} \tag{4.259}$$

For a proof of (4.259) one may argue as follows. First, plugging in the definition (4.228), applying the product rule, and making use of the error estimate (4.183) as well as the regularity estimates (4.221) and (4.178) entails

$$\begin{aligned} \xi_{i,j}\cdot\partial_t\xi_{i,j} &= \sum_{n=1,\mathcal{T}_n\subset\bar I_{i,j}}^{N} \eta_n\xi^n_{i,j}\cdot\partial_t\xi_{i,j} + O\big(\bar r^{-2}(\bar r^{-2}d^2_{i,j}\wedge 1)\big) \\ &= \sum_{\substack{n,n'=1\\ \mathcal{T}_n,\mathcal{T}_{n'}\subset\bar I_{i,j}}}^{N} \eta_n\eta_{n'}\xi^n_{i,j}\cdot\partial_t\xi^{n'}_{i,j} + \sum_{n=1,\mathcal{T}_n\subset\bar I_{i,j}}^{N}\sum_{n'=1}^{N}\eta_n(\xi^n_{i,j}\cdot\xi^{n'}_{i,j})\partial_t\eta_{n'} \\ &\quad + O\big(\bar r^{-2}(\bar r^{-2}d^2_{i,j}\wedge 1)\big). \end{aligned}$$

Substituting the differential operator $(B\cdot\nabla)$ for ∂_t, and recalling in addition to the above ingredients the regularity estimate (4.231) as well as the compatibility estimate (4.251) (which allows to switch from B to $B^{n'}$) then also yields

$$\begin{aligned} \xi_{i,j}\cdot(B\cdot\nabla)\xi_{i,j} &= \sum_{\substack{n,n'=1\\ \mathcal{T}_n,\mathcal{T}_{n'}\subset\bar I_{i,j}}}^{N} \eta_n\eta_{n'}\xi^n_{i,j}\cdot(B^{n'}\cdot\nabla)\xi^{n'}_{i,j} \\ &\quad + \sum_{n=1,\mathcal{T}_n\subset\bar I_{i,j}}^{N}\sum_{n'=1}^{N}\eta_n(\xi^n_{i,j}\cdot\xi^{n'}_{i,j})(B\cdot\nabla)\eta_{n'} + O\big(\bar r^{-2}(\bar r^{-2}d^2_{i,j}\wedge 1)\big). \end{aligned}$$

Observe that the combination of the previous two displays already generates the first right hand side term of (4.259).

We proceed by first splitting the sum over topological features $n'\in\{1,\ldots,N\}$, adding zero several times in the resulting first term, then applying the compatibility estimates (4.249), (4.250) and (4.170), and finally recalling the regularity estimate (4.178) which results in the estimate (of course, only terms with $\operatorname{supp}\eta_n\cap\operatorname{supp}\eta_{n'}\neq\emptyset$ are relevant in the subsequent sums)

$$\begin{aligned} &\sum_{n=1,\mathcal{T}_n\subset\bar I_{i,j}}^{N}\sum_{n'=1}^{N}\eta_n(\xi^n_{i,j}\cdot\xi^{n'}_{i,j})\partial_t\eta_{n'} \\ &= \sum_{\substack{n,n'=1\\ \mathcal{T}_n,\mathcal{T}_{n'}\subset\bar I_{i,j}}}^{N} \eta_n(\xi^n_{i,j}\cdot\xi^{n'}_{i,j})\partial_t\eta_{n'} + \sum_{\substack{n,n'=1\\ \mathcal{T}_n\subset\bar I_{i,j},\mathcal{T}_{n'}\not\subset\bar I_{i,j}}}^{N} \eta_n(\xi^n_{i,j}\cdot\xi^{n'}_{i,j})\partial_t\eta_{n'} \\ &= \sum_{\substack{n,n'=1\\ \mathcal{T}_n,\mathcal{T}_{n'}\subset\bar I_{i,j}}}^{N} \eta_n\big(|\xi_{i,j}|^2{-}|\xi^n_{i,j}{-}\xi_{i,j}|^2 + (\xi^n_{i,j}{-}\xi_{i,j})\cdot\xi^n_{i,j} + \xi^{n'}_{i,j}\cdot(\xi^{n'}_{i,j}{-}\xi_{i,j})\big)\partial_t\eta_{n'} \end{aligned}$$

$$+\sum_{\substack{n,n'=1\\ \mathcal{T}_n,\mathcal{T}_{n'}\subset \bar{I}_{i,j}}}^{N} \eta_n(\xi^n_{i,j}-\xi^{n'}_{i,j})\cdot(\xi^{n'}_{i,j}-\xi_{i,j})\partial_t\eta_{n'} + \sum_{\substack{n,n'=1\\ \mathcal{T}_n\subset \bar{I}_{i,j},\mathcal{T}_{n'}\not\subset \bar{I}_{i,j}}}^{N} \eta_n(\xi^n_{i,j}\cdot\xi^{n'}_{i,j})\partial_t\eta_{n'}$$
$$=\sum_{\substack{n,n'=1\\ \mathcal{T}_n,\mathcal{T}_{n'}\subset \bar{I}_{i,j}}}^{N} \eta_n|\xi_{i,j}|^2\partial_t\eta_{n'} + \sum_{\substack{n,n'=1\\ \mathcal{T}_n\subset \bar{I}_{i,j},\mathcal{T}_{n'}\not\subset \bar{I}_{i,j}}}^{N} \eta_n(\xi^n_{i,j}\cdot\xi^{n'}_{i,j})\partial_t\eta_{n'} + O\big(\bar{r}^{-2}(\bar{r}^{-2}d^2_{i,j}\wedge 1)\big).$$

Based on the regularity estimates (4.178) and (4.231), we may again substitute the differential operator $(B\cdot\nabla)$ for ∂_t in the previous computation, which in turn by two applications of the crucial estimate (4.233) and finally an application of the bulk cutoff estimates (4.232) resp. (4.180) allows to deduce

$$\sum_{n=1,\mathcal{T}_n\subset \bar{I}_{i,j}}^{N}\sum_{n'=1}^{N} \eta_n(\xi^n_{i,j}\cdot\xi^{n'}_{i,j})(\partial_t{+}B\cdot\nabla)\eta_{n'}$$
$$=\sum_{n,\mathcal{T}_n\subset \bar{I}_{i,j}}^{N}\sum_{n'=1}^{N} \eta_n|\xi_{i,j}|^2(\partial_t{+}B\cdot\nabla)\eta_{n'}$$
$$+\sum_{\substack{n,n'=1\\ \mathcal{T}_n\subset \bar{I}_{i,j},\mathcal{T}_{n'}\not\subset \bar{I}_{i,j}}}^{N} \eta_n(\xi^n_{i,j}\cdot\xi^{n'}_{i,j})(\partial_t{+}B\cdot\nabla)\eta_{n'} + O\big(\bar{r}^{-2}(\bar{r}^{-2}d^2_{i,j}\wedge 1)\big)$$
$$=-(1{-}\eta_{\mathrm{bulk}})|\xi_{i,j}|^2(\partial_t{+}B\cdot\nabla)\eta_{\mathrm{bulk}} + O\big(\bar{r}^{-2}(\bar{r}^{-2}d^2_{i,j}\wedge 1)\big) = O\big(\bar{r}^{-2}(\bar{r}^{-2}d^2_{i,j}\wedge 1)\big).$$

In particular, we obtain the asserted estimate (4.259).

It remains to post-process the right hand side term of (4.259). In view of (4.62) and (4.83), it suffices to get rid of the "off-diagonal" terms $n\neq n'\in\{1,\dots,N\}$ with $\mathcal{T}_n\subset \bar{I}_{i,j}$, $\mathcal{T}_{n'}\subset \bar{I}_{i,j}$ and $\operatorname{supp}\eta_n\cap\operatorname{supp}\eta_{n'}\neq\emptyset$. For each such pair of topological features we may add zero several times to rewrite (recall again the local identities (4.62) and (4.83))

$$\begin{aligned}
&\xi^n_{i,j}\cdot(\partial_t{+}B^{n'}\cdot\nabla)\xi^{n'}\\
&=\xi^n_{i,j}\cdot\big(\partial_t\xi^{n'}_{i,j}{+}(B^{n'}\cdot\nabla)\xi^{n'}_{i,j}{+}(\nabla B^{n'})^\mathsf{T}\xi^{n'}_{i,j}\big)-\xi^n_{i,j}(\nabla B^{n'})^\mathsf{T}\xi^{n'}_{i,j}\\
&=(\xi^n_{i,j}{-}\xi^{n'}_{i,j})\cdot\big(\partial_t\xi^{n'}_{i,j}{+}(B^{n'}\cdot\nabla)\xi^{n'}_{i,j}{+}(\nabla B^{n'})^\mathsf{T}\xi^{n'}_{i,j}\big)+(\xi^{n'}_{i,j}{-}\xi^n_{i,j})(\nabla B)^\mathsf{T}\xi^{n'}_{i,j}\\
&\quad+(\xi^{n'}_{i,j}{-}\xi^n_{i,j})(\nabla B^{n'}{-}\nabla B)^\mathsf{T}\xi^{n'}_{i,j}.
\end{aligned}$$

Hence, summing the previous identity over the relevant topological features, then matching terms which correspond to the previous computation but with the roles of n and n' being reversed, and finally using the compatibility estimates (4.252) resp. (4.170) as well as the local evolution equations (4.61) and (4.81) we infer that

$$\sum_{\substack{n,n'=1\\ \mathcal{T}_n,\mathcal{T}_{n'}\subset \bar{I}_{i,j}}}^{N} \eta_n\eta_{n'}\xi^n_{i,j}\cdot(\partial_t{+}B^{n'}\cdot\nabla)\xi^{n'}_{i,j} = O\big(\bar{r}^{-2}(\bar{r}^{-2}d^2_{i,j}\wedge 1)\big).$$

This in turn constitutes the required upgrade of (4.259). □

4.6.5 Existence of gradient flow calibrations: Proof of Proposition 4.6

Let us summarize our results from the previous sections to conclude with a proof of the main result.

Proof of Proposition 4.6. Let $(\xi_{i,j})_{i\neq j}$ be the family of global vector fields from Construction 4.32. Let $i,j \in \{1,\ldots,P\}$ with $i \neq j$. The coercivity condition (4.1b) immediately follows from Lemma 4.33. The formula (4.1a) follows from the corresponding local version (4.222) and the definition (4.228). Moreover, that $\xi_{i,j}(x,t) = \bar{\mathrm{n}}_{i,j}(x,t)$ holds true for all $t \in [0,T]$ and $x \in \bar{I}_{i,j}(t)$ is a consequence of Lemma 4.31 *iii)* and that $(\eta_1,\ldots,\eta_N)$ is a partition of unity on the network of interfaces of the strong solution (see Lemma 4.30 *i)*).

Finally, let B be the global velocity field from Construction 4.34. The validity of the equations (4.1c), (4.1d) and (4.1e) is then the content of Lemma 4.38. □

4.7 Existence of transported weights

The aim of this section is to establish the existence of a family of transported weights in the case of $d = 2$ and an underlying strong solution of multiphase mean curvature flow.

Proof of Lemma 4.7. We again make use of the description of the network of interfaces of the strong solution in terms of its underlying topological features, namely two-phase interfaces and triple junctions. Assume that there is a total of $N \in \mathbb{N}$ such topological features present. Recall then that we decomposed $\{1,\ldots,N\} =: \mathcal{C} \,\dot\cup\, \mathcal{P}$ with the convention that $\mathcal{C}$ enumerates the connected components in space-time of the smooth two-phase interfaces and $\mathcal{P}$ enumerates the triple junctions. If $p \in \mathcal{P}$, we defined $\mathcal{T}_p$ to be the trajectory in space-time described by the triple junction. If $c \in \mathcal{C}$, we defined $\mathcal{T}_c \subset \bar{I}_{i,j}$ for some $i,j \in \{1,\ldots,P\}$ with $i \neq j$ to be the corresponding space-time connected component of a two-phase interface $\bar{I}_{i,j}$. We further write $c \sim p$ for $c \in \mathcal{C}$ and $p \in \mathcal{P}$ if and only if $\mathcal{T}_c$ has an endpoint at $\mathcal{T}_p$.

Let now $r_{\mathcal{P}}$ and $\bar{r}_{\min}$ be the localization scales from (4.176) and (4.177). We then choose a large-scale cutoff $R > 0$ such that for all $t \in [0,T]$ a suitable neighborhood of the network of interfaces at time t is compactly supported in the ball $B_R(0)$:

$$\bigcup_{p\in\mathcal{P}} B_{r_{\mathcal{P}}}(\mathcal{T}_p(t)) \cup \Big(\bigcup_{c\in\mathcal{C}} \mathrm{im}_{\bar{r}_{\min}}(\Psi_{\mathcal{T}_c})(t) \setminus \bigcup_{p\in\mathcal{P}} B_{r_{\mathcal{P}}}(\mathcal{T}_p(t)) \Big) \subset\subset B_R(0), \tag{4.260}$$

where we abbreviated $\mathrm{im}_{\bar{r}_{\min}}(\Psi_{\mathcal{T}_c})(t) := \Psi_{\mathcal{T}_c}(\mathcal{T}_c(t)\times\{t\}\times[-\bar{r}_{\min},\bar{r}_{\min}])$ for $t \in [0,T]$, and where $\Psi_{\mathcal{T}_c}$ refers to the restriction of the diffeomorphism (4.45) to $\mathcal{T}_c$ (assuming $\mathcal{T}_c \subset \bar{I}_{i,j}$).

The idea for the proof is to construct in the first part a family of weight functions $(\hat{\vartheta}_i)_{i\in\{1,\ldots,P\}}$ which satisfies all the requirements of Definition 4.4 but violates the integrability condition $\hat{\vartheta}_i \in L^1_{x,t}(\mathbb{R}^2 \times [0,T])$. To overcome the integrability issue at the end of the proof, we introduce a smooth and concave function $\kappa\colon [0,\infty) \to [0,1]$ such that $\kappa(r) = 1$ for $r \geq 1$, $\kappa'(r) \in (0,2)$ for $r \in (0,1)$ and $\kappa(0) = 0$. Note that κ represents an upper concave approximation of $r \mapsto r \wedge 1$ on the interval $[0,\infty)$. We next define an integrable weight $\eta_R \in W^{1,\infty}_x(\mathbb{R}^2) \cap W^{1,1}_x(\mathbb{R}^2)$ by means of

$$\eta_R(x) := \kappa(\exp(R - |x|)), \quad x \in \mathbb{R}^2, \tag{4.261}$$

whose spatial gradient is now subject to the following convenient estimate

$$|\nabla\eta_R| \leq C|\eta_R| \quad \text{in } \mathbb{R}^2. \tag{4.262}$$

We will then define $\vartheta_i := \eta_R\hat{\vartheta}_i$, and verify in a second part that all the requirements of Definition 4.4 are indeed satisfied for this choice of weight functions.

Step 1: Construction of $(\hat{\vartheta}_i)_{i\in\{1,\ldots,P\}}$. Let $\vartheta\colon \mathbb{R} \to \mathbb{R}$ be a truncation of the identity with $\vartheta(r) = r$ for $|r| \leq \frac{1}{2}$, $\vartheta(r) = -1$ for $r \leq -1$, $\vartheta(r) = 1$ for $r \geq 1$, $0 \leq \vartheta' \leq 2$ as well as $|\vartheta''| \leq C$. Fix $i \in \{1,\ldots,P\}$. For purely technical reasons (similar to the one described in *Step 3*, Proof of Lemma 4.30), we need to introduce another constant $\delta \in (0,1]$ which will be determined in the course of the proof (depending only on the surface tensions associated with the strong solution).

We start with the definition of $\hat{\vartheta}_i$ away from the (relevant part of the) network of interfaces. To this end, we define subsets $\mathcal{P}_i \subset \mathcal{P}$ and $\mathcal{C}_i \subset \mathcal{C}$ which collect those triple junctions and two-phase interfaces for which the phase i is present, respectively. We then define for all $t \in [0,T]$

$$\hat{\vartheta}_i(\cdot,t) := -1 \tag{4.263}$$
$$\text{in } \bar{\Omega}_i(t) \setminus \bigcup_{p\in\mathcal{P}_i} B_{r_{\mathcal{P}}}(\mathcal{T}_p(t)) \cup \Big(\bigcup_{c\in\mathcal{C}_i} \mathrm{im}_{\bar{r}_{\min}}(\Psi_{\mathcal{T}_c})(t) \setminus \bigcup_{p\in\mathcal{P}_i} B_{r_{\mathcal{P}}}(\mathcal{T}_p(t))\Big),$$
$$\hat{\vartheta}_i(\cdot,t) := \quad 1 \tag{4.264}$$
$$\text{in } \big(\mathbb{R}^2 \setminus \bar{\Omega}_i(t)\big) \setminus \bigcup_{p\in\mathcal{P}_i} B_{r_{\mathcal{P}}}(\mathcal{T}_p(t)) \cup \Big(\bigcup_{c\in\mathcal{C}_i} \mathrm{im}_{\bar{r}_{\min}}(\Psi_{\mathcal{T}_c})(t) \setminus \bigcup_{p\in\mathcal{P}_i} B_{r_{\mathcal{P}}}(\mathcal{T}_p(t))\Big).$$

By the definitions (4.176) and (4.177) of the scales $r_{\mathcal{P}}$ and $\bar{r}_{\min}$, we may provide the further construction of $\hat{\vartheta}_i$ separately within $\mathrm{im}_{\bar{r}_{\min}}(\Psi_{\mathcal{T}_c})(t) \setminus \bigcup_{p\in\mathcal{P}_i} B_{r_{\mathcal{P}}}(\mathcal{T}_p(t))$ for each $c \in \mathcal{C}_i$ and within $B_{r_{\mathcal{P}}}(\mathcal{T}_p(t))$ for each $p \in \mathcal{P}_i$, respectively.

For each $c \in \mathcal{C}_i$, and assuming for notational concreteness that $\mathcal{T}_c \subset \bar{I}_{i,j}$ for some $j \in \{1\ldots,P\} \setminus \{i\}$, we simply define for all $t \in [0,T]$

$$\hat{\vartheta}_i(\cdot,t) := \vartheta\Big(\frac{s_{i,j}(\cdot,t)}{\delta\bar{r}_{\min}}\Big), \quad \text{in } \mathrm{im}_{\bar{r}_{\min}}(\Psi_{\mathcal{T}_c})(t) \setminus \bigcup_{p\in\mathcal{P}_i} B_{r_{\mathcal{P}}}(\mathcal{T}_p(t)), \tag{4.265}$$

where the signed distance $s_{i,j}$ was introduced in (4.47).

Now, consider a triple junction $p \in \mathcal{P}_i$. We assume that the pairwise distinct phases present at $\mathcal{T}_p$ are given by $i,j,k \in \{1,\ldots,P\}$. Recall from Definition 4.20 that $B_{r_{\mathcal{P}}}(\mathcal{T}_p)$ decomposes into six wedges. Three of them, namely the interface wedges $W_{i,j}$, $W_{j,k}$ resp. $W_{k,i}$, contain the interfaces $\mathcal{T}_{c_{i,j}}$, $\mathcal{T}_{c_{j,k}}$ resp. $\mathcal{T}_{c_{k,i}}$. The other three are interpolation wedges denoted by W_i, W_j resp. W_k. For the definition of $\hat{\vartheta}_i$ on the latter wedges, we rely on the interpolation parameter built in Lemma 4.28. To clarify the direction of interpolation, i.e., on which boundary of the interpolation wedge the corresponding interpolation function is equal to one or zero, we make use of the following notational convention. For the interpolation wedge W_i, say, we denote by $\lambda_i^{j,k}$ the interpolation function as built in Lemma 4.28 and which interpolates from j to k in the sense that it is equal to one on $(\partial W_{i,j} \cap \partial W_i) \setminus \mathcal{T}_p$ and which vanishes on $(\partial W_{k,i} \cap \partial W_i) \setminus \mathcal{T}_p$. We also define $\lambda_i^{k,j} := 1 - \lambda_i^{j,k}$ which interpolates on W_i in the opposite direction from k to j. Analogously, one introduces the interpolation functions on the other interpolation wedges.

We now define the weight function $\hat{\vartheta}_i$ for all $t \in [0,T]$ on the ball $B_{r_{\mathcal{P}}}(\mathcal{T}_p(t))$ as follows:

$$\hat{\vartheta}_i(\cdot,t) := \vartheta\Big(\frac{s_{i,j}(\cdot,t)}{\delta\bar{r}_{\min}}\Big), \quad \text{in } W_{i,j}(t) \cap B_{r_{\mathcal{P}}}(\mathcal{T}_p(t)), \tag{4.266}$$

and analogously on the interface wedge $W_{i,k}$, whereas we interpolate on the interpolation wedge W_i by means of

$$\hat{\vartheta}_i(\cdot,t) := \lambda_i^{j,k}(\cdot,t)\vartheta\Big(\frac{s_{i,j}(\cdot,t)}{\delta\bar{r}_{\min}}\Big) + \lambda_i^{k,j}(\cdot,t)\vartheta\Big(\frac{s_{i,k}(\cdot,t)}{\delta\bar{r}_{\min}}\Big), \text{ in } W_i(t) \cap B_{r_{\mathcal{P}}}(\mathcal{T}_p(t)). \tag{4.267}$$

Furthermore, we define

$$\hat{\vartheta}_i(\cdot,t) := \vartheta\Big(\frac{\mathrm{dist}(\cdot,\mathcal{T}_p(t))}{\delta\bar{r}_{\min}}\Big), \quad \text{in } W_{j,k}(t) \cap B_{r_{\mathcal{P}}}(\mathcal{T}_p(t)), \tag{4.268}$$

whereas we again interpolate on the interpolation wedge W_j via

$$\hat{\vartheta}_i(\cdot,t) := \lambda_j^{k,i}(\cdot,t)\vartheta\Big(\frac{\mathrm{dist}(\cdot,\mathcal{T}_p(t))}{\delta\bar{r}_{\min}}\Big) + \lambda_j^{i,k}(\cdot,t)\vartheta\Big(\frac{s_{i,j}(\cdot,t)}{\delta\bar{r}_{\min}}\Big), \text{ in } W_j(t)\cap B_{r_{\mathcal{P}}}(\mathcal{T}_p(t)), \quad (4.269)$$

and analogously for the interpolation wedge W_k.

Step 2: Regularity of $(\hat{\vartheta}_i)_{i\in\{1,\dots,P\}}$. First of all, it is immediate from the above definitions (4.263)–(4.269) that the coercivity properties of Definition 4.4 hold true as required. Choosing $\delta\in(0,1]$ as in *Step 3*, Proof of Lemma 4.30, ensures that the definitions (4.266)–(4.269) close to triple junctions are compatible with the bulk definitions (4.263)–(4.264). In particular, the asserted regularity $\hat{\vartheta}_i\in W^{1,\infty}_{x,t}(\mathbb{R}^2\times[0,T])$ for the auxiliary weight functions is now a consequence of the regularity (4.50) of the signed distance functions as well as the controlled blowup (4.143) of the first-order derivatives of the interpolation parameter. In terms of estimates, it holds

$$\max_{k=0,1}\bar{r}_{\min}^k|\nabla^k\hat{\vartheta}_i| + \bar{r}_{\min}^2|\partial_t\hat{\vartheta}_i| \leq C \quad \text{in } \mathbb{R}^2\times[0,T]\setminus\bigcup_{p\in\mathcal{P}_i}\mathcal{T}_p, \quad (4.270)$$

for a constant $C>0$ which may depend on the strong solution $\bar{\Omega}$, but which is independent of $\bar{r}_{\min}$.

Step 3: Estimate for the advective derivatives of $(\hat{\vartheta}_i)_{i\in\{1,\dots,P\}}$. For a proof of the bound (4.4) on the advective derivative with respect to the auxiliary weight $\hat{\vartheta}_i$, it suffices to work in the regions $\bigcup_{c\in\mathcal{C}_i}\mathrm{im}_{\bar{r}_{\min}}(\Psi_{\mathcal{T}_c})\setminus\bigcup_{p\in\mathcal{P}_i}\bigcup_{t\in[0,T]}B_{r_{\mathcal{P}}}(\mathcal{T}_p(t))\times\{t\}$ and $\bigcup_{p\in\mathcal{P}_i}\bigcup_{t\in[0,T]}B_{r_{\mathcal{P}}}(\mathcal{T}_p(t))\times\{t\}$, respectively. We in fact may argue separately for each $c\in\mathcal{C}_i$ and each $p\in\mathcal{P}_i$. The argument turns out to be almost analogous to the one for the proof of (4.232); a connection which we will make precise in the subsequent steps to avoid unnecessary repetition.

Substep 1: Estimate near $\partial\bar{\Omega}_i$ but away from triple junctions. Let $c\in\mathcal{C}_i$, and assume for concreteness that $\mathcal{T}_c\subset\bar{I}_{i,j}$. It follows from the definition (4.265) that $\hat{\vartheta}_i$ is a smooth function of the signed distance $s_{i,j}$ throughout the space-time domain $\mathrm{im}_{\bar{r}_{\min}}(\Psi_{\mathcal{T}_c})\setminus\bigcup_{p\in\mathcal{P}_i}\bigcup_{t\in[0,T]}B_{r_{\mathcal{P}}}(\mathcal{T}_p(t))\times\{t\}$. Hence, due to (4.270) the otherwise exact same argument guaranteeing (4.238) entails

$$|\partial_t\hat{\vartheta}_i + (B\cdot\nabla)\hat{\vartheta}_i| \leq C\bar{r}_{\min}^{-2}(\bar{r}_{\min}^{-1}\,\mathrm{dist}(\cdot,\bar{I}_{i,j})\wedge 1) \leq C\bar{r}_{\min}^{-2}|\hat{\vartheta}_i| \quad (4.271)$$

in $\mathrm{im}_{\bar{r}_{\min}}(\Psi_{\mathcal{T}_c})\setminus\bigcup_{p\in\mathcal{P}_i}\bigcup_{t\in[0,T]}B_{r_{\mathcal{P}}}(\mathcal{T}_p(t))\times\{t\}$. The last inequality follows due to ϑ being a truncation of unity.

Substep 2: Estimate at triple junction in interface wedges containing $\partial\bar{\Omega}_i$. Consider $p\in\mathcal{P}_i$, and let $c\in\mathcal{C}$ such that $c\sim p$ and $\mathcal{T}_c\subset\bar{I}_{i,j}$. We provide the required estimate in the interface wedge $W_{i,j}(t)\cap B_{r_{\mathcal{P}}}(\mathcal{T}_p(t))$ for all $t\in[0,T]$. In this case, definition (4.266) applies so that $\hat{\vartheta}$ is again a smooth function of the signed distance $s_{i,j}$. Recalling (4.270), we may thus apply the argument in favor of (4.242) to deduce again

$$|\partial_t\hat{\vartheta}_i + (B\cdot\nabla)\hat{\vartheta}_i| \leq C\bar{r}_{\min}^{-2}(\bar{r}_{\min}^{-1}\,\mathrm{dist}(\cdot,\bar{I}_{i,j})\wedge 1) \leq C\bar{r}_{\min}^{-2}|\hat{\vartheta}_i|, \quad (4.272)$$

this time throughout $W_{i,j}(t)\cap B_{r_{\mathcal{P}}}(\mathcal{T}_p(t))$ for all $t\in[0,T]$.

Substep 3: Estimate at triple junction in interface wedge not containing $\partial\bar{\Omega}_i$. Let $p\in\mathcal{P}_i$, and let $j,k\in\{1,\dots,P\}$ denote the other two distinct phases which are present at $\mathcal{T}_p$ next to i. We aim to estimate the advective derivative of $\hat{\vartheta}_i$ in the interface wedge $W_{j,k}(t)\cap B_{r_{\mathcal{P}}}(\mathcal{T}_p(t))$ for all $t\in[0,T]$. Note that thanks to (4.268), the auxiliary weight $\hat{\vartheta}_i$ is a smooth function of the distance to the triple junction. Hence, we may simply follow the argument resulting in (4.243) and obtain together with (4.270) that

$$|\partial_t\hat{\vartheta}_i + (B\cdot\nabla)\hat{\vartheta}_i| \leq C\bar{r}_{\min}^{-2}(\bar{r}_{\min}^{-1}\,\mathrm{dist}(\cdot,\mathcal{T}_p)\wedge 1) \leq C\bar{r}_{\min}^{-2}|\hat{\vartheta}_i| \quad (4.273)$$

in the region $W_{j,k}(t) \cap B_{r_{\mathcal{P}}}(\mathcal{T}_p(t))$ for all $t \in [0,T]$.

Substep 4: Estimate at triple junction in interpolation wedges. Let the notation of *Substep 3* in place. On the interpolation wedge W_i, the auxiliary weight is defined by means of (4.267), i.e., one interpolates between two smooth functions of the signed distances $s_{i,j}$ and $s_{k,i}$, respectively. Hence, we may estimate based on the product rule, the estimate (4.248), the bound (4.270), the fact that $\lambda_i^{j,k} = 1-\lambda_i^{k,j}$, the argument establishing (4.247), and finally (4.77)

$$\begin{aligned}
|\partial_t \hat{\vartheta}_i + (B\cdot\nabla)\hat{\vartheta}_i| \leq C\bar{r}_{\min}^{-2}\Big|\vartheta\Big(\frac{s_{i,j}(\cdot,t)}{\delta\bar{r}_{\min}}\Big)-\vartheta\Big(\frac{s_{k,i}(\cdot,t)}{\delta\bar{r}_{\min}}\Big)\Big| \\
+ C\bar{r}_{\min}^{-2}\lambda_i^{j,k}(\bar{r}_{\min}^{-1}\operatorname{dist}(\cdot,\bar{I}_{i,j})\wedge 1) \\
+ C\bar{r}_{\min}^{-2}\lambda_i^{k,j}(\bar{r}_{\min}^{-1}\operatorname{dist}(\cdot,\bar{I}_{k,i})\wedge 1) \\
\leq C\bar{r}_{\min}^{-2}(\bar{r}_{\min}^{-1}\operatorname{dist}(\cdot,\mathcal{T}_p)\wedge 1) + C\bar{r}_{\min}^{-2}|\hat{\vartheta}_i| \leq C\bar{r}_{\min}^{-2}|\hat{\vartheta}_i| \qquad (4.274)
\end{aligned}$$

throughout $W_i(t) \cap B_{r_{\mathcal{P}}}(\mathcal{T}_p(t))$ for all $t \in [0,T]$. In view of the definition (4.269) and the argument for (4.243) (carefully noting that the latter is established also on interpolation wedges), the otherwise same ingredients and computations employed for the proof of (4.274) also imply

$$|\partial_t \hat{\vartheta}_i + (B\cdot\nabla)\hat{\vartheta}_i| \leq C\bar{r}_{\min}^{-2}|\hat{\vartheta}_i| \qquad (4.275)$$

in $W_j(t) \cap B_{r_{\mathcal{P}}}(\mathcal{T}_p(t))$ for all $t \in [0,T]$.

Substep 5: Conclusion. In summary, the estimates (4.271)–(4.275) imply the asserted bound (4.4) for the advective derivative in terms of the auxiliary weights $\hat{\vartheta}_i$. In particular, the family of auxiliary weights $(\hat{\vartheta}_i)_{i\in\{1,\dots,P\}}$ satisfies all the required properties of Definition 4.4 with the only exception being $\hat{\vartheta}_i \in L^1_{x,t}(\mathbb{R}^2\times[0,T])$.

Step 4: Construction and properties of ϑ_i. As already mentioned at the beginning of the proof, we may now define $\vartheta_i := \eta_R\hat{\vartheta}_i$ for all $i \in \{1,\dots,P\}$. The regularity and the required coercivity properties for ϑ_i are then immediate consequences of its definition and the previous step. The estimate (4.4) on the advective derivative also carries over since η_R is time-independent and by (4.262)

$$|\hat{\vartheta}_i||(B\cdot\nabla)\eta_R| \leq C|\vartheta_i| \quad \text{in } \mathbb{R}^2\times[0,T],$$

so that the product rule together with the previous step implies (4.4) on the level of the weight ϑ_i. This in turn concludes the proof of Lemma 4.7. □

Glossary of notation

$d \geq 2$	ambient dimension
D	open set
$\partial_t v$	distributional partial derivative w.r.t. time of $v : D\times[0,T) \to \mathbb{R}^d$
∇v	distributional partial derivative w.r.t. space, $(\nabla v)_{i,j} = \partial_j v_i$
$C^\infty_{\mathrm{cpt}}(D)$	space of compactly supported and infinitely differentiable functions on D
$C^l_t C^k_x(U)$	space of functions on $U \subset \mathbb{R}^d\times[0,T]$ with continuous and bounded partial derivatives $\partial_t^{l'}\partial_x^{k'}$, $0 \leq l' \leq l$, $0 \leq k' \leq k$.

$u \otimes v$	tensor product of $u, v \in \mathbb{R}^d$, $(u \otimes v)_{i,j} = u_i v_j$
$A : B$	$\sum_{i,j} A_{ij} B_{ij}$, scalar product of tensors
$\mathcal{L}^d$	d-dimensional Lebesgue measure
$\mathcal{H}^k$	k-dimensional Hausdorff measure on $\mathbb{R}^d$ for $k \in [0, d]$
$L^p(\Omega, \mu)$	Lebesgue space w.r.t. to a measure μ on $\Omega \subset \mathbb{R}^d$ for $p \in [1, \infty]$
$L^p(D)$	Lebesgue space w.r.t. Lebesgue measure
$L^p(D; \mathbb{R}^d)$	Lebesgue space for vector valued functions
$L^p([0,T]; X)$	Bochner–Lebesgue space for a Banach space X and $T \in (0, \infty)$
$W^{k,p}(D)$	Sobolev spaces with $p \in [1, \infty)$ and $k \in \mathbb{N}$
$BV(D)$	Functions of bounded variation [12] on Lipschitz domain $D \subset \mathbb{R}^d$
$\partial^* \Omega$	reduced boundary of a set of finite perimeter $\Omega \subset D$
$\mathrm{n} = -\frac{\nabla \chi_\Omega}{\lvert\nabla \chi_\Omega\rvert}$	outward pointing unit normal vector field along $\partial^* \Omega$
$s_{i,j}$	signed distance function to $\bar{I}_{i,j}$ with $\nabla s_{i,j} = \bar{\mathrm{n}}_{i,j}$
$\mathrm{dist}(\cdot, A)$	distance function $\mathbb{R}^d \times [0,T] \ni (x,t) \mapsto \mathrm{dist}(x, A(t))$ for a domain $A = \bigcup_{t \in [0,T]} A(t) \times \{t\}$, $A(t) \subset \mathbb{R}^d$, $t \in [0,T]$.
$P \geq 2$	number of phases
Ω_i	region occupied by phase $i = 1, \ldots, P$ in *weak solutions*
χ_i	characteristic function of Ω_i
$I_{i,j}$	interface between phases Ω_i and Ω_j
$\mathrm{n}_{i,j}$	unit normal vectors along $I_{i,j}$ pointing from phase i to phase j
V_i	normal velocity of $I_{i,j}$ with $V_i > 0$ for expanding Ω_i, see (4.12b)
$\bar{\Omega}_i$, $\bar{\chi}_i$, ...	corresponding quantities of the *strong solution*
$\mathrm{H}_{i,j}$	mean curvature vector of $\bar{I}_{i,j}$
$H_{i,j}$	scalar mean curvature of $\bar{I}_{i,j}$ given by $\mathrm{H}_{i,j} \cdot \bar{\mathrm{n}}_{i,j} = -\nabla^{\tan} \cdot \bar{\mathrm{n}}_{i,j} = -\Delta s_{i,j}$
$s_{i,j}$	signed distance function to $\bar{I}_{i,j}$ with $\nabla s_{i,j} = \bar{\mathrm{n}}_{i,j}$
$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$	counter-clockwise rotation by 90°
$\bar{\tau}_{i,j}$	tangent vector along $\bar{I}_{i,j}$ given by $J^{-1} \bar{\mathrm{n}}_{i,j}$
$O(\cdot)$	Landau symbol, implicit constant only depends on strong solution

CHAPTER 5

Weak-strong uniqueness for the mean curvature flow of double bubbles

Abstract. We derive a weak-strong uniqueness principle for BV solutions to multiphase mean curvature flow of triple line clusters in three dimensions. Our proof is based on the explicit construction of a gradient-flow calibration in the sense of our recent work [Fischer et al., arXiv:2003.05478] for any such cluster. This extends our two-dimensional construction to the three-dimensional case of surfaces meeting along triple junctions.

5.1 Main results & definitions

We developed in Chapter 4 a general approach to the question of weak-strong uniqueness of BV solutions to multiphase mean curvature flow in arbitrary ambient dimension $d \geq 2$. This approach splits into a two-step procedure.

In a first step, we introduced a novel concept of calibrated flows with respect to the gradient flow of the interface energy functional given by the (weighted) sum of the surface areas of the interfaces, cf. (5.8) below. This concept can be interpreted as the evolutionary analogue of the well-known notion of paired calibrations due to Lawlor and Morgan [102] from their study of the minimization problem of interfacial surface area of networks. Indeed, the main merit of a calibrated flow is that its existence (essentially) implies qualitative uniqueness and quantitative stability of BV solutions to multiphase mean curvature flow in arbitrary ambient dimension $d \geq 2$.

In a second step, we then put our theory to use by showing that any sufficiently regular network of interfaces in the plane $\mathbb{R}^2$, which in addition is subject to the correct angle condition at triple junctions, is in fact calibrated in the precise sense of Definition 4.2. The purpose of the present work is to extend this second step of our approach to the three-dimensional setting of mean curvature flow of sufficiently regular double bubbles (again with the correct angle condition along the triple line). The main contributions are summarized in the following result.

Theorem 5.1. *Let $T \in (0,\infty)$ be a time horizon, and let $(\bar\Omega_1, \bar\Omega_2, \bar\Omega_3)$ be a regular double bubble smoothly evolving by MCF on $[0,T]$ in the sense of Definition 5.10. The evolution of*

$(\bar{\Omega}_1, \bar{\Omega}_2, \bar{\Omega}_3)$ *on* $[0,T]$ *is then calibrated in the sense that there exists an associated gradient-flow calibration* $((\xi_i)_{i\in\{1,2,3\}}, B)$ *on* $[0,T]$*, cf. Definition 5.2. Moreover, the smoothly evolving regular double bubble* $(\bar{\Omega}_1, \bar{\Omega}_2, \bar{\Omega}_3)$ *admits a family of transported weights* $(\vartheta_i)_{i\in\{1,2,3\}}$ *on* $[0,T]$ *in the sense of Definition 5.5.*

As a corollary, we obtain a weak-strong uniqueness and stability of evolutions principle for BV *solutions* $(\Omega_1, \Omega_2, \Omega_3)$ *to multiphase MCF on* $[0,T]$ *(cf. Definition 4.11) with respect to the class of regular double bubbles smoothly evolving by MCF on* $[0,T]$ *in the sense of Definition 5.10. We refer to Theorem 5.6 for a more detailed statement of this corollary, and to the discussion right below it for an account on the general regime of* $P \geq 3$ *phases on the level of the BV solution.*

Proof. The existence of a gradient-flow calibration $((\xi_i)_{i\in\{1,2,3\}}, B)$ on $[0,T]$ is the content of Theorem 5.3. Its proof occupies almost the whole paper and is carried out from Section 5.2 to Section 5.4. We emphasize in this context that the local construction at a triple line performed in Section 5.3 represents the core contribution of the present work. The existence of transported weights $(\vartheta_i)_{i\in\{1,2,3\}}$ on $[0,T]$ is proven in Section 5.5 in form of Proposition 5.5.

These two existence results in turn realize the assumptions of our general conditional weak-strong uniqueness and stability of evolutions principle Proposition 4.5 for BV solutions to multiphase mean curvature flow (with respect to the setting of $P = 3$ phases and $d = 3$ dimensions), which therefore establishes the claim of the corollary. □

The results of Chapter 4 together with Theorem 5.1 admittedly only cover two thirds of the story concerning weak-strong uniqueness for general clusters in $\mathbb{R}^3$ evolving by multiphase mean curvature flow. Indeed, one also has to allow for quadruple junctions at which four distinct phases meet (cf. the structure result on minimizer of interfacial surface energy by Taylor [148]). We expect that a suitable generalization of our ideas for the construction at a triple point ($d = 2$) or a triple line ($d = 3$) should also lead to the correct construction in the case of a quadruple junction, and thus to a full-fledged weak-strong uniqueness result in $\mathbb{R}^3$.

5.1.1 Existence of gradient-flow calibrations

For the sake of completeness, let us first restate the precise definition of the concept of a gradient-flow calibration.

Definition 5.2 (Gradient-flow calibration). *Let* $T \in (0,\infty)$ *be a time horizon, and let* $\sigma \in \mathbb{R}^{P\times P}$ *be an admissible matrix of surface tensions, cf. Remark 5.7, for* $P \geq 2$ *phases. Moreover, let* $(\bar{\Omega}_1, \ldots, \bar{\Omega}_P)$ *be an evolving partition of finite interface energy on* $\mathbb{R}^d \times [0,T]$ *in the sense of Definition 5.8 in dimension* $d \geq 2$*, and denote by* $\bigcup_{i\neq j} \bar{I}_{i,j}$ *the associated network of evolving interfaces.*

A tuple of vector fields

$$(\xi_i)_{i\in\{1,\ldots,P\}} \colon \mathbb{R}^d \times [0,T] \to (\mathbb{R}^d)^P,$$
$$B \colon \mathbb{R}^d \times [0,T] \to \mathbb{R}^d$$

is called a calibration for the L^2-gradient flow of the interface energy (5.8) on $[0,T]$ *with respect to the evolving partition* $(\bar{\Omega}_1, \ldots, \bar{\Omega}_P)$*—or in short a* gradient-flow calibration*—if it is subject to the following requirements:*

i) It holds $\xi_i, B \in C^0([0,T]; C^0_{\mathrm{cpt}}(\mathbb{R}^d;\mathbb{R}^d))$ *for all* $i \in \{1,\ldots,P\}$*. Moreover, for each time* $t \in [0,T]$*, there exists an* $\mathcal{H}^{d-1}$ *null set* $\Gamma_t \subset \mathbb{R}^d$ *such that for* $\Gamma := \bigcup_{t\in[0,T]} \Gamma_t \times \{t\}$ *it holds* $\xi_i \in (C^0_t C^1_x \cap C^1_t C^0_x)(\mathbb{R}^d \times [0,T] \setminus \Gamma)$ *for all* $i \in \{1,\ldots,P\}$ *and* $B \in C^0_t C^1_x(\mathbb{R}^d \times [0,T] \setminus \Gamma)$*. Finally, there exists* $C > 0$ *such that*

$$\sup_{t\in[0,T]} \sup_{x\in\mathbb{R}^d\setminus\Gamma_t} |\nabla B(x,t)| + |\nabla \xi_i(x,t)| + |\partial_t \xi_i(x,t)| \leq C.$$

ii) For $i, j \in \{1, \dots, P\}$ with $i \neq j$, define the vector field

$$\xi_{i,j} := \frac{1}{\sigma_{i,j}}(\xi_i - \xi_j) \quad \text{in } \mathbb{R}^d \times [0, T]. \tag{5.1a}$$

Denoting by $\bar{\mathrm{n}}_{i,j}$ the unit normal vector field along the interface $\bar{I}_{i,j}$ (pointing from the ith into the jth phase), it is then required that

$$\xi_{i,j} = \bar{\mathrm{n}}_{i,j} \quad \text{along } \bar{I}_{i,j}. \tag{5.1b}$$

Moreover, there exists $c \in (0, 1)$ such that a coercivity estimate in terms of the length of the vector field $\xi_{i,j}$ holds true:

$$|\xi_{i,j}(x, t)| \leq 1 - c \min\{\mathrm{dist}^2(x, \bar{I}_{i,j}(t)), 1\}, \quad (x, t) \in \mathbb{R}^d \times [0, T]. \tag{5.1c}$$

iii) The vector field B represents a velocity field for the partition $(\bar{\Omega}_1, \dots, \bar{\Omega}_P)$ in the sense that the following two approximate evolution equations hold true for the vector fields $\xi_{i,j}$, $i, j \in \{1, \dots, P\}$ with $i \neq j$,

$$\big|\partial_t \xi_{i,j} + (B \cdot \nabla)\xi_{i,j} + (\nabla B)^{\mathsf{T}} \xi_{i,j}\big|(x, t) \leq C \min\{\mathrm{dist}(x, \bar{I}_{i,j}(t)), 1\}, \tag{5.1d}$$

$$\big|\partial_t |\xi_{i,j}|^2 + (B \cdot \nabla)|\xi_{i,j}|^2\big|(x, t) \leq C \min\{\mathrm{dist}^2(x, \bar{I}_{i,j}(t)), 1\}, \tag{5.1e}$$

for some $C > 0$ and all $(x, t) \in \mathbb{R}^d \times [0, T]$.

iv) The velocity B represents motion by multiphase mean curvature (i.e., the L^2-gradient flow with respect to the interface energy (5.8)) in the sense that there exists a constant $C > 0$ such that

$$\big|\xi_{i,j} \cdot B + \nabla \cdot \xi_{i,j}\big| \leq C \min\{\mathrm{dist}(x, \bar{I}_{i,j}(t)), 1\}, \quad (x, t) \in \mathbb{R}^d \times [0, T]. \tag{5.1f}$$

If a gradient-flow calibration exists, we say that the evolving partition $(\bar{\Omega}_1, \dots, \bar{\Omega}_P)$ is calibrated on $[0, T]$.

Note that the required regularity from the first item of the above definition is on one side slightly less than what is actually stated in Definition 4.2 from the previous chapter, but on the other side still sufficient to ensure the validity of Proposition 4.3.

The main result of the present work is now that any sufficiently regular and smoothly evolving double bubble admits an associated gradient-flow calibration.

Theorem 5.3 (Existence of gradient-flow calibrations)**.** *Let $T \in (0, \infty)$, let $\sigma \in \mathbb{R}^{3 \times 3}$ be an admissible matrix of surface tensions, and let $(\bar{\Omega}_1, \bar{\Omega}_2, \bar{\Omega}_3)$ be a regular double bubble smoothly evolving by MCF on $[0, T]$ in the sense of Definition 5.10. Then $(\bar{\Omega}_1, \bar{\Omega}_2, \bar{\Omega}_3)$ is calibrated on $[0, T]$ in the sense of Definition 5.2.*

It turns out that the existence of a gradient-flow calibration already implies a quantitative inclusion principle for the surface cluster of general BV solutions to multiphase mean curvature flow, see Proposition 4.3. More precisely, if at the initial time each interface of a BV solution is contained in the corresponding interface of a calibrated flow, then this inclusion property remains to be satisfied as long as the calibrated flow exists. Furthermore, this qualitative property is in fact a consequence of a quantitative stability estimate for the interface error between a general BV solution and a calibrated flow (formulated in terms of an error functional of the form (5.3) below).

The inclusion principle, however, is of course consistent with the vanishing of a phase in the BV solution, so that weak-strong uniqueness cannot be derived by means of a gradient-flow calibration alone. In order to get a control on the bulk deviations of the phases, one relies on an additional input which can be formalized as follows.

Definition 5.4 (Family of transported weights). *Let $T \in (0,\infty)$ be a time horizon, and let $\sigma \in \mathbb{R}^{P\times P}$ be an admissible matrix of surface tensions satisfying the strict triangle inequality for $P \geq 2$ phases. Let $d \geq 2$, and let $(\bar\Omega_1,\ldots,\bar\Omega_P)$ be an evolving partition of finite interface energy on $\mathbb{R}^d\times[0,T]$ in the sense of Definition 5.8, and denote by $(\bar\chi_1,\ldots,\bar\chi_P)$ the associated family of indicator functions. We then in addition assume that the measure $\partial_t\bar\chi_i$ is absolutely continuous with respect to the measure $|\nabla\bar\chi_i|$, and that $\partial\bar\Omega_i(\cdot,t)$ is Lipschitz regular for all $t \in [0,T]$. Consider finally a velocity vector field $B \in C^0([0,T];C^1_{\mathrm{cpt}}(\mathbb{R}^d;\mathbb{R}^d))$.*

A map $\vartheta = (\vartheta_i)_{i\in\{1,\ldots,P\}} : \mathbb{R}^d \times [0,T] \to [-1,1]^P$ is called a family of transported weights *for $((\bar\Omega_1,\ldots,\bar\Omega_P),B)$ if it satisfies the following list of properties:*

i) In terms of regularity, we require $\vartheta_i \in (W^{1,1}\cap W^{1,\infty})(\mathbb{R}^d\times[0,T];[-1,1])$ for all $i \in \{1,\ldots,P\}$.

ii) We require that $\vartheta_i(\cdot,t) = 0$ on $\partial\bar\Omega_i(t)$, and $\vartheta_i(\cdot,t) > 0$ in the essential exterior resp. $\vartheta_i(\cdot,t) < 0$ in the essential interior of $\bar\Omega_i(\cdot,t)$ for all $i \in \{1,\ldots,P\}$ and all $t \in [0,T]$.

iii) Each weight is approximately advected by the velocity B in form of

$$|\partial_t\vartheta_i + (B\cdot\nabla)\vartheta_i| \leq C|\vartheta_i| \quad \text{on } \mathbb{R}^d\times[0,T],\ i \in \{1,\ldots,P\}. \tag{5.2}$$

The existence of a family of transported weights is precisely what is needed to derive a quantitative stability estimate for the bulk error between a general BV solution and a calibrated flow (formulated in terms of an error functional of the form (5.4) below), which together with the already mentioned quantitative inclusion principle then implies a weak-strong uniqueness principle for BV solutions of multiphase mean curvature flow, see Proposition 4.5.

It is therefore of interest to extend our 2D existence result from Chapter 4 to the 3D setting of any sufficiently regular and smoothly evolving double bubble.

Proposition 5.5 (Existence of a family of transported weights). *Let $T \in (0,\infty)$ be a time horizon, and let $(\bar\Omega_1,\bar\Omega_2,\bar\Omega_3)$ be a regular double bubble smoothly evolving by MCF on $[0,T]$ in the sense of Definition 5.10. Let B denote the velocity field from the gradient-flow calibration on $[0,T]$ associated with $(\bar\Omega_1,\bar\Omega_2,\bar\Omega_3)$, whose existence in turn is guaranteed by Theorem 5.3. Then there exists an associated family of transported weights $(\vartheta_i)_{i\in\{1,2,3\}}$ on $[0,T]$ with respect to the data $((\bar\Omega_1,\bar\Omega_2,\bar\Omega_3),B)$ in the precise sense of Definition 5.4.*

5.1.2 Weak-strong uniqueness and stability of evolutions

Combining Theorem 5.3 and Proposition 5.5 with the conditional stability of any calibrated MCF in arbitrary dimensions Proposition 4.5, we obtain the following weak-strong uniqueness principle for distributional (i.e., BV) solutions to multiphase MCF in three dimensions.

Theorem 5.6 (Weak-strong uniqueness and quantitative stability). *Let $T \in (0,\infty)$ be a time horizon, $d = 3$, $P = 3$, and $\sigma \in \mathbb{R}^{3\times3}$ be a surface tension matrix satisfying the strict triangle inequality. Let $\bar\Omega = (\bar\Omega_1,\bar\Omega_2,\bar\Omega_3)$ be a regular double bubble smoothly evolving by MCF on $[0,T]$ in the sense of Definition 5.10 (with respect to σ), and let $\Omega = (\Omega_1,\Omega_2,\Omega_3)$ be a BV solution to multiphase MCF in the sense of Definition 4.11 (again with respect to σ).*

If the initial conditions of the regular double bubble and the BV solution coincide, then the solutions also coincide for later times on $[0,T]$. More precisely,

$$\begin{aligned}&\mathcal{L}^3\big((\Omega_i(0)\setminus\bar\Omega_i(0))\cup(\bar\Omega_i(0)\setminus\Omega_i(0))\big) = 0 \text{ for all } i \in \{1,2,3\}\\ &\Rightarrow\quad \mathcal{L}^3\big((\Omega_i(t)\setminus\bar\Omega_i(t))\cup(\bar\Omega_i(t)\setminus\Omega_i(t))\big) = 0 \text{ for a.e. } t \in [0,T] \text{ and all } i \in \{1,2,3\}.\end{aligned}$$

Moreover, we have quantitative stability estimates in the following sense. Denote by $(\xi := (\xi_i)_{i\in\{1,2,3\}}, B)$ the gradient-flow calibration on the time interval $[0,T]$ from Theorem 5.3 with *respect to* $(\bar{\Omega}_1, \bar{\Omega}_2, \bar{\Omega}_3)$*, and denote by* $(\vartheta_i)_{i\in\{1,2,3\}}$ *the corresponding family of transported weights on* $[0,T]$ *from Proposition 5.5. Let* $\mathrm{n}_{i,j}(\cdot,t)$ *be the measure theoretic unit normal along the interface* $\partial^*\Omega_i(t) \cap \partial^*\Omega_j(t)$ *pointing from* $\Omega_i(t)$ *into* $\Omega_j(t)$*,* $t \in [0,T]$*. Then, the error functionals defined for all* $t \in [0,T]$ *by*

$$E[\Omega|\xi](t) := \sum_{i,j\in\{1,2,3\},\, i\neq j} \sigma_{i,j} \int_{\partial^*\Omega_i(t)\cap\partial^*\Omega_j(t)} 1-\mathrm{n}_{i,j}(\cdot,t)\cdot\xi_{i,j}(\cdot,t)\,\mathrm{d}\mathcal{H}^2, \tag{5.3}$$

$$E[\Omega|\bar{\Omega}](t) := \sum_{i=1}^{3} \int_{(\Omega_i(t)\setminus\bar{\Omega}_i(t))\cup(\bar{\Omega}_i(t)\setminus\Omega_i(t))} |\vartheta_i(\cdot,t)|\,\mathrm{d}x \tag{5.4}$$

satisfy the stability estimates

$$E[\Omega|\xi](t) \leq E[\Omega|\xi](0)e^{Ct}, \tag{5.5}$$

$$E[\Omega|\bar{\Omega}](t) \leq \big(E[\Omega|\xi](0)+E[\Omega|\bar{\Omega}](0)\big)e^{Ct} \tag{5.6}$$

for almost every $t \in [0,T]$*. The constant* $C > 0$ *in these estimates depends only on the data of the smoothly evolving regular double bubble* $(\bar{\Omega}_1, \bar{\Omega}_2, \bar{\Omega}_3)$ *on* $[0,T]$ *through the explicit constructions* $((\xi_i)_{i\in\{1,2,3\}}, B)$ *and* $(\vartheta_i)_{i\in\{1,2,3\}}$*.*

Proof. As mentioned above, this is a straightforward application of Theorem 5.3, Proposition 5.5 and Proposition 4.5. ☐

Remark 5.7 (Admissible surface tensions)**.** *Let us briefly comment on the matrix of surface tensions* $\sigma \in \mathbb{R}^{P\times P}$*. We say* σ *is admissible if it satisfies precisely the assumption in Definition 4.8. More concretely, we require that there exists a non-degenerate* $(P-1)$*-simplex* $(q_1, \ldots, q_P)$ *in* $\mathbb{R}^{P-1}$ *which represents the surface tensions in form of* $\sigma_{i,j} = |q_i - q_j|$*.*

In the framework of the present paper, i.e., the case $P = 3$*, this is equivalent to the strict triangle inequality*

$$\sigma_{i,j} < \sigma_{i,k} + \sigma_{k,j} \quad \textit{for all choices } \{i,j,k\} = \{1,2,3\}. \tag{5.7}$$

In the general case $P \geq 3$*, the* ℓ^2*-embeddability is in fact stronger than (5.7), and it constitutes the key ingredient to construct the missing calibration vector fields* $(\xi_i)_{i\in\{4,\ldots,P\}}$*, for which one may in fact argue along the same lines as in the proof of Lemma 4.31 without requiring any additional ingredients from the constructions.*

We emphasize that only for simplicity, we considered in Theorem 5.6 the case of $P = 3$ phases on the level of the BV solution. Let us briefly outline the additional ingredients which are needed to establish the stability estimates (5.5) and (5.6) in terms of general BV solutions $(\Omega_1, \ldots, \Omega_P)$, $P > 3$, defined on $\mathbb{R}^3\times[0,T]$ with respect to a given ℓ^2-embeddable matrix of surface tensions $\sigma = (\sigma_{i,j})_{i,j\in\{1,\ldots,P\}} \in \mathbb{R}^{P\times P}$, and a fixed regular double bubble $(\bar{\Omega}_1, \bar{\Omega}_2, \bar{\Omega}_3)$ smoothly evolving by MCF with respect to the restriction $(\sigma_{i,j})_{i,j\in\{1,2,3\}}$ of the surface tension matrix $\sigma \in \mathbb{R}^{P\times P}$.

Recalling the definitions (5.3) and (5.4) of the error functionals (in which one only needs to replace 3 by P in the case $P > 3$), it is clear that we have to augment the gradient-flow calibration provided by Theorem 5.3 with additional calibrating vector fields $(\xi_i)_{i\in\{4,\ldots,P\}}$, and the family of transported weights by Proposition 5.5 with additional weights $(\vartheta_i)_{i\in\{4,\ldots,P\}}$, such that the resulting augmented families adhere to the requirements of Definition 5.2 and Definition 5.4, respectively, in order to allow for the desired application of Proposition 4.5. For consistency with our definitions, let us interpret to this end the smoothly evolving regular double bubble as a partition $(\bar{\Omega}_1, \ldots, \bar{\Omega}_P)$ with the convention that $\bar{\Omega}_i := \emptyset$ for all additional phases $i \in \{4, \ldots, P\}$ in the BV solution.

Extending the family of transported weights is trivial since we may define $\vartheta_i := 1$ for all $i \in \{4, \dots, P\}$, thus representing consistently the fact that the additional phases on the level of the smoothly evolving regular double bubble are empty.

Furthermore, the missing calibration vector fields can be constructed along the lines of the proof of Lemma 4.31. It is then straightforward that the associated additional vector fields

$$\sigma_{i,j}\xi_{i,j} := \xi_i - \xi_j, \quad i \in \{4, \dots, P\} \text{ or } j \in \{4, \dots, P\},$$

satisfy (5.1b)–(5.1f) (together with the desired regularity). Indeed, except for the coercivity condition (5.1c), all these properties are trivially satisfied in terms of the relevant additional pairs of indices since the associated interfaces on the level of the smoothly evolving regular double bubble are empty. With respect to (5.1c), the proof of Lemma 4.33 applies verbatim without requiring any additional ingredients from the constructions of this work.

We decided to restrict ourselves to the case $P = 3$ in the formulation of Theorem 5.6 because we view the main contribution of this paper to be the first part of Theorem 5.1 (i.e., the combination of Theorem 5.3 and Proposition 5.5), and thus aim to shift the focus on the required $3D$ generalization of those results of Chapter 4 which are concerned with the given strong solution only (i.e., in the present work a regular double bubble smoothly evolving by MCF).

5.1.3 Definition of a regular double bubble smoothly evolving by MCF

This part is concerned with the formulation of a “strong solution concept” for a (topologically standard) double bubble moving by mean curvature flow, for which we are then able to show that its flow is calibrated in the precise sense of Definition 5.2. We start with the associated energy functional.

Definition 5.8 (Partition with finite interface energy, see Definition 4.10)**.** *Consider $d \geq 2$, $P \geq 2$, and an admissible matrix of surface tensions $\sigma \in \mathbb{R}^{P\times P}$. Let $(\bar\Omega_1, \dots, \bar\Omega_P)$ be a family of measurable subsets of $\mathbb{R}^d$ such that $\mathcal{L}^d(\mathbb{R}^d \setminus \bigcup_{i=1}^P \bar\Omega_i) = 0$ and $\mathcal{L}^d(\bar\Omega_i \cap \bar\Omega_j) = 0$ for all $i, j \in \{1, \dots, P\}$ with $i \neq j$. We then call $(\bar\Omega_1, \dots, \bar\Omega_P)$ a* partition of $\mathbb{R}^d$ with finite interface energy *if*

$$E[(\bar\Omega_1, \dots, \bar\Omega_P)] := \sum_{i,j\in\{1,\dots,P\},\, i\neq j} \sigma_{i,j}\mathcal{H}^{d-1}(\partial^*\bar\Omega_i \cap \partial^*\bar\Omega_j) < \infty. \tag{5.8}$$

Let next $T \in (0, \infty)$ be a time horizon, and consider a family $(\bar\Omega_1, \dots, \bar\Omega_P)$ of open subsets of $\mathbb{R}^d\times[0, T]$ in the form of $\bar\Omega_i = \bigcup_{t\in[0,T]} \bar\Omega_i(t)\times\{t\}$ for all $i \in \{1, \dots, P\}$. In this evolutionary setting, we call $(\bar\Omega_1, \dots, \bar\Omega_P)$ an evolving partition on $\mathbb{R}^d\times[0, T]$ with finite interface energy, *if for all $t \in [0, T]$ the family $(\bar\Omega_1(t), \dots, \bar\Omega_P(t))$ is a partition of $\mathbb{R}^d$ with finite interface energy in the above sense and it holds*

$$\sup_{t\in[0,T]} E[(\bar\Omega_1(t), \dots, \bar\Omega_P(t))] < \infty. \tag{5.9}$$

For such an evolving partition, we denote the associated evolving interfaces by $\bar I_{i,j} := \bigcup_{t\in[0,T]} \bar I_{i,j}(t)\times(t)$, where $\bar I_{i,j}(t) := \partial^*\bar\Omega_i(t) \cap \partial^*\bar\Omega_j(t)$ for all $t \in [0, T]$ and all pairs $i, j \in \{1, \dots, P\}$, $i \neq j$.

In a next step, we formalize the topological setup as well as the main regularity assumptions. We also state the main compatibility condition in form of the Herring angle condition.

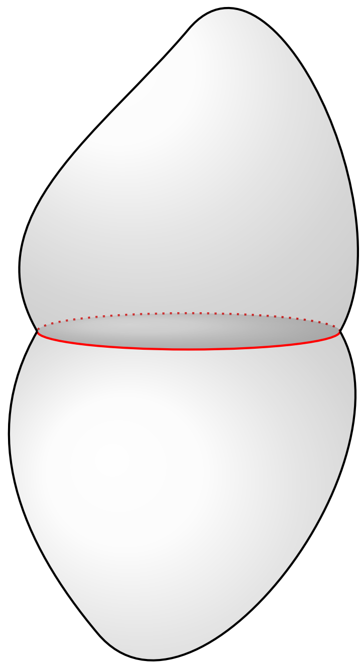

Figure 5.1: An illustration of a double bubble in three dimensions. The triple line $\bar{\Gamma}$ along which all three interfaces meet is marked in red.

Definition 5.9 (Regular double bubble)**.** *Let $\sigma \in \mathbb{R}^{3\times 3}$ be an admissible matrix of surface tensions, and consider a partition $(\bar{\Omega}_1, \bar{\Omega}_2, \bar{\Omega}_3)$ of $\mathbb{R}^3$ with finite interface energy in the sense of Definition 5.8. Assume in addition that $\bar{\Omega}_i$ is an open, non-empty and simply connected subset of $\mathbb{R}^3$ such that $\partial\bar{\Omega}_i$ is the closure of $\partial^*\bar{\Omega}_i$ for all $i \in \{1,2,3\}$. Define then for each $i,j \in \{1,2,3\}$ with $i \neq j$ the associated interface $\bar{I}_{i,j} := \partial\bar{\Omega}_i \cap \partial\bar{\Omega}_j$, which is assumed to be non-empty.*

We call $(\bar{\Omega}_1, \bar{\Omega}_2, \bar{\Omega}_3)$ a regular double bubble *if the following additional regularity conditions are satisfied:*

i) Each interface $\bar{I}_{i,j}$ is a two-dimensional, compact and simply connected manifold with boundary of class C^5. The interior of each interface is embedded.

ii) The three interfaces $\bar{I}_{1,2}$, $\bar{I}_{2,3}$, and $\bar{I}_{3,1}$ intersect precisely along their respective boundary, which in turn is a non-empty one-dimensional, compact and connected manifold $\bar{\Gamma}$ without boundary of class C^5.

iii) Along the triple line $\bar{\Gamma}$, the Herring angle condition has to be satisfied:

$$\sigma_{1,2}\bar{\mathrm{n}}_{1,2} + \sigma_{2,3}\bar{\mathrm{n}}_{2,3} + \sigma_{3,1}\bar{\mathrm{n}}_{3,1} = 0, \tag{5.10}$$

where we denote by $\bar{\mathrm{n}}_{i,j}$ the associated unit normal vector field along $\bar{I}_{i,j}$ pointing from $\bar{\Omega}_i$ into $\bar{\Omega}_j$.

With the notion of a regular double bubble in place, we finally clarify what we mean by a (sufficiently) smooth evolution of a regular double bubble with respect to mean curvature flow. It turns out that the construction of an associated gradient-flow calibration in the vicinity of the evolving triple line requires two additional higher-order compatibility conditions. For a sufficiently smooth evolution of a regular double bubble, these two compatibility conditions are consequences of differentiating in time the assumed zeroth order compatibility condition (i.e., the triple line being the common boundary of the three interfaces) or first order compatibility condition (i.e., the Herring angle condition), respectively. Since we will require regularity down to time $t = 0$, we have to include the resulting compatibility conditions for the initial double bubble.

Definition 5.10 (Regular double bubble smoothly evolving by MCF). *Let $\sigma \in \mathbb{R}^{3\times 3}$ be an admissible matrix of surface tensions. Consider an associated initial partition $(\bar\Omega_1^0, \bar\Omega_2^0, \bar\Omega_3^0)$ of $\mathbb{R}^3$ representing a regular double bubble in the sense of Definition 5.9. Assume in addition that $(\bar\Omega_1^0, \bar\Omega_2^0, \bar\Omega_3^0)$ satisfies the following two higher-order compatibility conditions:*

First, we require for the scalar mean curvatures in form of $H_{i,j}^0 := -\nabla^{\tan} \cdot \bar{\mathrm{n}}_{i,j}^0$ that along the initial triple line $\bar\Gamma^0$ it holds

$$\sigma_{1,2} H_{1,2}^0 + \sigma_{2,3} H_{2,3}^0 + \sigma_{3,1} H_{3,1}^0 = 0, \tag{5.11}$$

which by (5.10) is equivalent to the existence of a unique vector field $\mathrm{V}_{\bar\Gamma^0}$ along $\bar\Gamma^0$, which takes values in the normal bundle $\mathrm{Tan}^\perp \bar\Gamma^0$ such that

$$\bar{\mathrm{n}}_{i,j}^0 \cdot \mathrm{V}_{\bar\Gamma^0} = H_{i,j}^0 \quad \text{along } \bar\Gamma^0 \text{ for all } i,j \in \{1,2,3\} \text{ with } i \neq j.$$

Second, denoting by $\bar{\mathrm{t}}^0$ a unit length tangent vector field along the initial triple line $\bar\Gamma^0$ and defining $\bar\tau_{i,j}^0 := \bar{\mathrm{n}}_{i,j}^0 \times \bar{\mathrm{t}}^0$ along $\bar\Gamma^0$ for all $i,j \in \{1,2,3\}$ with $i \neq j$, we require that the quantity

$$-\big(\bar\tau_{i,j}^0 \otimes \bar\tau_{i,j}^0 : \nabla^{\tan} \bar{\mathrm{n}}_{i,j}^0\big)\big(\bar\tau_{i,j}^0 \cdot \mathrm{V}_{\bar\Gamma^0}\big) + \big(\bar\tau_{i,j}^0 \cdot \nabla^{\tan}\big) H_{i,j}^0 \tag{5.12}$$

is independent of the choice of distinct $i,j \in \{1,2,3\}$ at each point on $\bar\Gamma^0$.

Let now $T \in (0,\infty)$ be a time horizon, and consider an evolving partition $(\bar\Omega_1, \bar\Omega_2, \bar\Omega_3)$ on $\mathbb{R}^3 \times [0,T]$ with finite interface energy in the sense of Definition 5.8. We call $(\bar\Omega_1, \bar\Omega_2, \bar\Omega_3)$ a regular double bubble smoothly evolving by MCF on $[0,T]$ *with initial data $(\bar\Omega_1^0, \bar\Omega_2^0, \bar\Omega_3^0)$ if it satisfies:*

i) For each $t \in [0,T]$, the family $(\bar\Omega_1(t), \bar\Omega_2(t), \bar\Omega_3(t))$ is a regular double bubble in the sense of Definition 5.9. Furthermore, the initial condition is attained in the sense that $(\bar\Omega_1(0), \bar\Omega_2(0), \bar\Omega_3(0)) = (\bar\Omega_1^0, \bar\Omega_2^0, \bar\Omega_3^0)$.

ii) There exists a family of diffeomorphisms $\psi^t \colon \mathbb{R}^3 \to \mathbb{R}^3$, $t \in [0,T]$, such that it holds $\psi^0(x) = x$ for all $x \in \mathbb{R}^3$, and $\bar\Omega_i(t) = \psi^t(\bar\Omega_i^0)$ as well as $\bar I_{i,j}(t) = \psi^t(\bar I_{i,j}^0)$ for all $t \in [0,T]$ and all $i,j \in \{1,2,3\}$, $i \neq j$. In addition, the map

$$\psi_{i,j} \colon \bar I_{i,j}^0 \times [0,T] \to \bar I_{i,j}, \quad (x,t) \mapsto (\psi^t(x), t)$$

is a diffeomorphism of class $(C_t^0 C_x^5 \cap C_t^1 C_x^3)(\bar I_{i,j}^0 \times [0,T])$.

iii) For each $i,j \in \{1,2,3\}$ with $i \neq j$ and each $(x,t) \in \bar I_{i,j}$ denote by $\mathrm{V}_{\bar I_{i,j}}(x,t)$ the normal velocity vector of $\bar I_{i,j}(t)$ at $x \in \bar I_{i,j}(t)$. We then require motion by MCF for each interface, i.e.,

$$\big(\bar{\mathrm{n}}_{i,j} \cdot \mathrm{V}_{\bar I_{i,j}}\big)(x,t) = H_{i,j}(x,t), \quad (x,t) \in \bar I_{i,j},\ i,j \in \{1,2,3\} \text{ with } i \neq j. \tag{5.13}$$

5.1.4 Notation

We briefly review the standard notation employed throughout the present work. The notation of geometric quantities will be introduced in the course of the paper.

We write $\mathcal{L}^d$ for the d-dimensional Lebesgue measure, $\mathcal{H}^s$ for the s-dimensional Hausdorff measure, as well as $\partial^* D$ for the reduced boundary of a set of finite perimeter. The standard Lebesgue spaces with respect to the Lebesgue measure are denoted as always by $L^p(D)$ for any $p \in [0,\infty]$ and any measurable $D \subset \mathbb{R}^d$, whereas in addition for any $k \in \mathbb{N}$ we denote by $W^{k,p}(D)$ the standard Sobolev space. We further write $C^k(D)$, $k \geq 0$, for the space of functions with bounded and continuous derivatives up to order k on $D \subset \mathbb{R}^d$. The intersection with the space $C^0_{\mathrm{cpt}}(D)$ of continuous and compactly supported functions on D is denoted by $C^k_{\mathrm{cpt}}(D)$. Vector-valued versions of these function spaces are denoted by $L^p(D;\mathbb{R}^d)$ and so on. For a differentiable function $f\colon D \to \mathbb{R}^m$ we write $\nabla f \in \mathbb{R}^{m\times d}$ for its Jacobian matrix, i.e., it holds $(\nabla f)_{i,j} = \partial_j f_i$. If $f\colon M \to \mathbb{R}^m$ is a differentiable function along a given C^1 manifold M, we denote by $\nabla^{\tan}$ its tangential gradient.

For a space-time domain $D \subset \mathbb{R}^d\times[0,T]$ of the form $D = \bigcup_{t\in[0,T]} D(t)\times\{t\}$ we write $C^l_t C^k_x(D)$, $l,k \geq 0$, for the space of continuous functions f on D with continuous and bounded partial derivatives $\partial_t^{l'}\partial_x^{k'} f$ on D for any $0 \leq l' \leq l$ and any multi-index k' such that $0 \leq |k'| \leq k$. With a slight abuse of notation, the distance function $\mathrm{dist}(\cdot, D)$ with respect to such a space-time domain D is understood as the distance to the corresponding time slice, i.e., $(x,t) \mapsto \mathrm{dist}(x, D(t))$ for all $(x,t) \in \mathbb{R}^d\times[0,T]$.

In terms of vector and tensor notation, we denote by $v\times w$ the cross product between two vectors $v,w \in \mathbb{R}^3$, by $v \wedge w := v\otimes w - w \otimes v$ the exterior product of $v,w \in \mathbb{R}^3$, and by $A : B := \sum_{i,j} A_{i,j}B_{i,j}$ the complete contraction of two matrices $A,B \in \mathbb{R}^{m\times n}$. Abusing notation we will also write $a \wedge b := \min\{a,b\}$ for the minimum of two numbers $a,b \in \mathbb{R}$; however, it will always be perfectly clear from the context what the symbol $\wedge$ represents. We also occasionally use $a \vee b := \max\{a,b\}$ for the maximum of two numbers $a,b \in \mathbb{R}$.

5.2 Local gradient flow calibration at a smooth interface

The aim of this section is to provide the local building block of a gradient-flow calibration in the vicinity of an interface present in a smoothly evolving double bubble. To this end, we introduce the following geometric setup.

Definition 5.11 (Localization radius for interface)**.** *Let $(\bar\Omega_1,\bar\Omega_2,\bar\Omega_3)$ be a regular double bubble smoothly evolving by MCF in the sense of Definition 5.10 on a time interval $[0,T]$. Fix $i,j \in \{1,2,3\}$ with $i \neq j$. We call a scale $r_{i,j} \in (0,1]$ an* admissible localization radius for the interface *$\bar I_{i,j}$ if*

$$\Psi_{i,j}\colon \bar I_{i,j} \times (-r_{i,j}, r_{i,j}) \to \mathbb{R}^3 \times [0,T], \quad (x,t,s) \mapsto (x+s\bar{\mathrm{n}}_{i,j}(x,t),t) \tag{5.14}$$

is bijective onto its image $\mathrm{im}(\Psi_{i,j}) := \Psi_{i,j}(\bar I_{i,j}\times(-r_{i,j},r_{i,j}))$. Moreover, it is required that the inverse Ψ^{-1} is a diffeomorphism of class $(C^0_tC^4_x \cap C^1_tC^2_x)(\mathrm{im}(\Psi_{i,j}))$, and that it splits in form of

$$\Psi^{-1}_{i,j}\colon \mathrm{im}(\Psi_{i,j}) \to \bar I_{i,j} \times (-r_{i,j}, r_{i,j}), \quad (x,t) \mapsto (P_{i,j}(x,t), t, s_{i,j}(x,t)),$$

where the map $s_{i,j}$ represents a signed distance function (oriented by means of $\bar{\mathrm{n}}_{i,j}$, i.e., $\nabla s_{i,j} = \bar{\mathrm{n}}_{i,j}$ along the interface $\bar I_{i,j}$)

$$s_{i,j}(x,t) = \begin{cases} \mathrm{dist}(x, \bar I_{i,j}(t)) & \text{if } (x,t) \in \Psi_{i,j}(\bar I_{i,j}\times[0,r_{i,j})), \\ -\mathrm{dist}(x, \bar I_{i,j}(t)) & \text{if } (x,t) \in \Psi_{i,j}(\bar I_{i,j}\times(-r_{i,j},0)), \end{cases} \tag{5.15}$$

and the map $P_{i,j}$ *being (in each time slice) the nearest-point projection onto* $\bar{I}_{i,j}$

$$P_{i,j}(x,t) = \operatorname*{arg\,min}_{y\in \bar{I}_{i,j}(t)} |y-x|, \quad (x,t) \in \operatorname{im}(\Psi_{i,j}). \tag{5.16}$$

In view of Definition 5.10 of a regular double bubble smoothly evolving by MCF, it follows from the tubular neighborhood theorem that all interfaces admit an admissible localization radius in the sense of Definition 5.11.

We introduce some further notation and consequences with respect to Definition 5.11. First, the nearest-point projection onto the interface admits the representation

$$P_{i,j}(x,t) = x - s_{i,j}(x,t)\nabla s_{i,j}(x,t), \quad (x,t) \in \operatorname{im}(\Psi_{i,j}). \tag{5.17}$$

Second, it holds in terms of regularity

$$s_{i,j} \in (C^0_tC^5_x \cap C^1_tC^3_x)(\operatorname{im}(\Psi_{i,j})), \quad P_{i,j} \in (C^0_tC^4_x \cap C^1_tC^2_x)(\operatorname{im}(\Psi_{i,j})). \tag{5.18}$$

The scalar mean curvature of the interface $\bar{I}_{i,j}$ with respect to the orientation induced by $\bar{\mathrm{n}}_{i,j}$ is denoted by $H_{i,j}$. We extend these geometric quantities away from the interface, performing a slight abuse of notation, by means of

$$\bar{\mathrm{n}}_{i,j}\colon \operatorname{im}(\Psi_{i,j}) \to \mathbb{S}^2, \quad (x,t) \mapsto \nabla s_{i,j}(x,t), \tag{5.19}$$

$$H_{i,j}\colon \operatorname{im}(\Psi_{i,j}) \to \mathbb{R}, \quad (x,t) \mapsto -\Delta s_{i,j}(P_{i,j}(x,t),t). \tag{5.20}$$

Observe that these definitions immediately imply that

$$\bar{\mathrm{n}}_{i,j} \in (C^0_tC^4_x \cap C^1_tC^2_x)(\operatorname{im}(\Psi_{i,j})), \quad H_{i,j} \in (C^0_tC^3_x \cap C^1_tC^1_x)(\operatorname{im}(\Psi_{i,j})). \tag{5.21}$$

Construction 5.12 (Gradient-flow calibration along smooth interfaces)**.** Let the assumptions and notation of Definition 5.11 be in place, and let $\mathcal{Y}_{i,j}\colon \operatorname{im}(\Psi_{i,j}) \to \mathbb{R}^3$ be an arbitrary vector field of class $C^0_tC^1_x(\operatorname{im}(\Psi_{i,j}))$. We then define a pair of vector fields $(\xi_{i,j}, B)\colon \operatorname{im}(\Psi_{i,j}) \to \mathbb{S}^2 \times \mathbb{R}^3$ as follows:

$$\xi_{i,j} := \bar{\mathrm{n}}_{i,j}, \quad B := H_{i,j}\bar{\mathrm{n}}_{i,j} + (\mathrm{Id}-\bar{\mathrm{n}}_{i,j}\otimes\bar{\mathrm{n}}_{i,j})\mathcal{Y}_{i,j}. \tag{5.22}$$

We call $(\xi_{i,j}, B)$ a *local gradient-flow calibration for the interface* $\bar{I}_{i,j}$. ◇

We now register the properties of the pair of vector fields $(\xi_{i,j}, B)$, i.e., that it satisfies locally the requirements of Definition 5.2 with the exception of (5.1c). The latter will only be satisfied once we glued together the local constructions in Section 5.4 by means of a suitable family of cutoff functions.

Lemma 5.13. *Let the assumptions and notation of Construction 5.12 be in place. Then it holds*

$$\xi_{i,j} \in (C^0_tC^4_x \cap C^1_tC^2_x)(\operatorname{im}(\Psi_{i,j})), \quad B \in C^0_tC^1_x(\operatorname{im}(\Psi_{i,j})). \tag{5.23}$$

Moreover, there exists a constant $C > 0$ *which depends only on the data of the smoothly evolving regular double bubble* $(\bar{\Omega}_1, \bar{\Omega}_2, \bar{\Omega}_3)$ *on* $[0,T]$*, such that we have throughout the space-time domain* $\operatorname{im}(\Psi_{i,j})$

$$|\nabla\xi_{i,j}| + |\partial_t\xi_{i,j}| \leq C, \tag{5.24}$$

$$|B| + |\nabla B| \leq C, \tag{5.25}$$

$$\partial_t\xi_{i,j} + (B\cdot\nabla)\xi_{i,j} + (\nabla B)^{\mathsf{T}}\xi_{i,j} = 0, \tag{5.26}$$

$$|\nabla\cdot\xi_{i,j} + B\cdot\xi_{i,j}| \leq C\operatorname{dist}(\cdot\bar{I}_{i,j}), \tag{5.27}$$

$$\partial_t|\xi_{i,j}|^2 + (B\cdot\nabla)|\xi_{i,j}|^2 = 0. \tag{5.28}$$

Proof. The asserted regularity follows immediately from the definitions (5.22) and the regularity (5.21) of the constituents. The equation (5.26) for the time evolution of $\xi_{i,j}$ follows from differentiating in the spatial variable the PDE satisfied by the signed distance function $s_{i,j}$, i.e.,

$$\partial_t s_{i,j} = -H_{i,j} = -(B \cdot \nabla) s_{i,j}, \tag{5.29}$$

relying in the process on the product rule and $\bar{\mathrm{n}}_{i,j} = \nabla s_{i,j}$. The divergence constraint (5.27) is a direct consequence of the definitions (5.19), (5.20) and (5.22) in combination with the regularity (5.18) of the signed distance. Finally, equation (5.28) is satisfied for trivial reasons since $\xi_{i,j} \in \mathbb{S}^2$. □

5.3 Local gradient flow calibration at a triple line

This section constitutes the core of the present work. We establish the existence of a gradient-flow calibration in the vicinity of the triple line for a double bubble smoothly evolving by MCF in the sense of Definition 5.10. The main result of this section reads as follows.

Proposition 5.14 (Existence of gradient-flow calibration at triple line)**.** *Consider a regular double bubble $(\bar{\Omega}_1, \bar{\Omega}_2, \bar{\Omega}_3)$ smoothly evolving by MCF on a time interval $[0,T]$ in the sense of Definition 5.10. Let $r \in (0,1]$ be an associated admissible localization radius for the triple line in the sense of Definition 5.15 below. There then exists a potentially smaller radius $\hat{r} \in (0,r]$, only depending on the data of the smoothly evolving regular double bubble $(\bar{\Omega}_1, \bar{\Omega}_2, \bar{\Omega}_3)$ on $[0,T]$, which gives rise to the following assertions:*

Denote by $\mathcal{N}_{\hat{r}}(\bar{\Gamma}) := \bigcup_{t\in[0,T]} B_{\hat{r}}(\bar{\Gamma}(t)) \times \{t\}$ the neighborhood if the evolving triple line. For all $i,j \in \{1,2,3\}$ with $i \neq j$ there exists a continuous local extension

$$\xi_{i,j} \colon \mathcal{N}_{\hat{r}}(\bar{\Gamma}) \to \overline{B_1(0)}$$

of the unit normal vector field $\bar{\mathrm{n}}_{i,j}|_{\bar{I}_{i,j}}$ of $\bar{I}_{i,j}$, and a continuous local extension

$$B \colon \mathcal{N}_{\hat{r}}(\bar{\Gamma}) \to \mathbb{R}^3$$

of the velocity vector field of the network $\mathcal{I} = \bigcup_{i,j\in\{1,2,3\}, i\neq j} \bar{I}_{i,j}$, such that the pair of vector fields $((\xi_{i,j})_{i,j\in\{1,2,3\},i\neq j}, B)$ satisfies the following list of requirements:

i) For all $i,j \in \{1,2,3\}$ with $i \neq j$ it holds $\xi_{i,j} \in (C_t^0 C_x^1 \cap C_t^1 C_x^0)(\mathcal{N}_{\hat{r}}(\bar{\Gamma}) \setminus \bar{\Gamma})$ and $B \in C_t^0 C_x^1(\mathcal{N}_{\hat{r}}(\bar{\Gamma}) \setminus \bar{\Gamma})$, with corresponding estimates throughout $\mathcal{N}_{\hat{r}}(\bar{\Gamma}) \setminus \bar{\Gamma}$

$$|\nabla \xi_{i,j}| + |\partial_t \xi_{i,j}| \leq C, \tag{5.30}$$

$$|B| + |\nabla B| \leq C \tag{5.31}$$

for some constant $C > 0$ which depends only on the data of the smoothly evolving regular double bubble $(\bar{\Omega}_1, \bar{\Omega}_2, \bar{\Omega}_3)$ on $[0,T]$.

ii) We have for all $i,j \in \{1,2,3\}$ with $i \neq j$

$$\xi_{i,j}(\cdot,t) = \bar{\mathrm{n}}_{i,j}(\cdot,t) \qquad \text{along } \bar{I}_{i,j}(t) \cap B_{\hat{r}}(\bar{\Gamma}(t)), \tag{5.32}$$

$$B(\cdot,t) = \mathrm{V}_{\bar{\Gamma}}(\cdot,t) \qquad \text{along } \bar{\Gamma}(t) \tag{5.33}$$

for all $t \in [0,T]$, where $\mathrm{V}_{\bar{\Gamma}}$ denotes the normal velocity of the triple line $\bar{\Gamma}$. Moreover, the skew-symmetry relation $\xi_{i,j} = -\xi_{j,i}$ holds true.

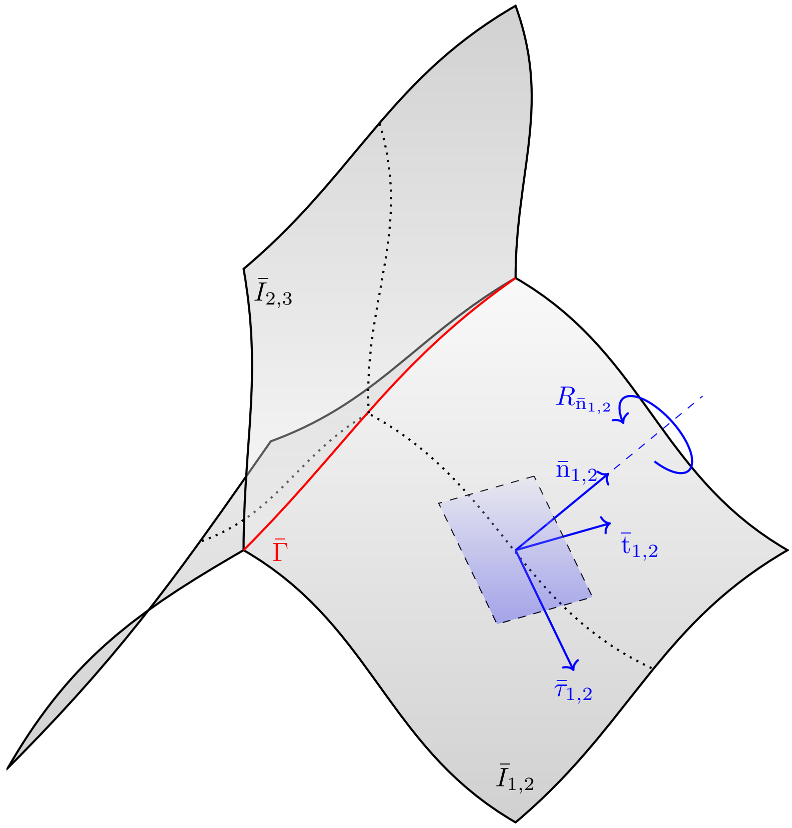

Figure 5.2: The smooth solution close to the triple line $\bar{\Gamma}$. Three sheets come together at fixed angles along $\bar{\Gamma}$ (here 120°). In this general situation, one needs to introduce an additional gauge rotation field $R_{\bar{n}_{1,2}}$. At each point, this matrix is a rotation in the tangent plane spanned by $\bar{\tau}_{1,2}$ and $\bar{t}_{1,2}$, illustrated here by a shaded (blue) rectangle.

iii) The Herring angle condition is satisfied in the whole space-time tubular neighborhood $\mathcal{N}_{\hat{r}}(\bar{\Gamma})$ of the triple line, i.e.,

$$\sigma_{1,2}\xi_{1,2} + \sigma_{2,3}\xi_{2,3} + \sigma_{3,1}\xi_{3,1} = 0 \quad \text{in } \mathcal{N}_{\hat{r}}(\bar{\Gamma}). \tag{5.34}$$

iv) There exists a constant $C > 0$, depending only on the data of the smoothly evolving regular double bubble $(\bar{\Omega}_1, \bar{\Omega}_2, \bar{\Omega}_3)$ on $[0,T]$, such that for all $i,j \in \{1,2,3\}$ with $i \neq j$ the estimates

$$|\partial_t \xi_{i,j} + (B\cdot\nabla)\xi_{i,j} + (\nabla B)^{\mathsf{T}}\xi_{i,j}| \leq C \operatorname{dist}(\cdot, \bar{I}_{i,j}), \tag{5.35}$$

$$|B\cdot\xi_{i,j} + \nabla\cdot\xi_{i,j}| \leq C \operatorname{dist}(\cdot, \bar{I}_{i,j}), \tag{5.36}$$

$$\partial_t|\xi_{i,j}|^2 + (B\cdot\nabla)|\xi_{i,j}|^2 \leq C \operatorname{dist}^2(\cdot, \bar{I}_{i,j}) \tag{5.37}$$

hold true within $\mathcal{N}_{\hat{r}}(\bar{\Gamma}) \setminus \bar{\Gamma}$.

A pair $((\xi_{i,j})_{i,j\in\{1,2,3\},i\neq j}, B)$ subject to these conditions is called a local gradient-flow calibration at the triple line $\bar{\Gamma}$.

The remainder of this section is organized as follows. In Subsection 5.3.1 we introduce the necessary notation employed in the construction of the desired vector fields. Subsection 5.3.2 implements the construction of the main building blocks for the vector fields $((\xi_{i,j})_{i\neq j}, B)$, which will then be glued together in Subsection 5.3.3. Subsection 5.3.4 contains the proof of

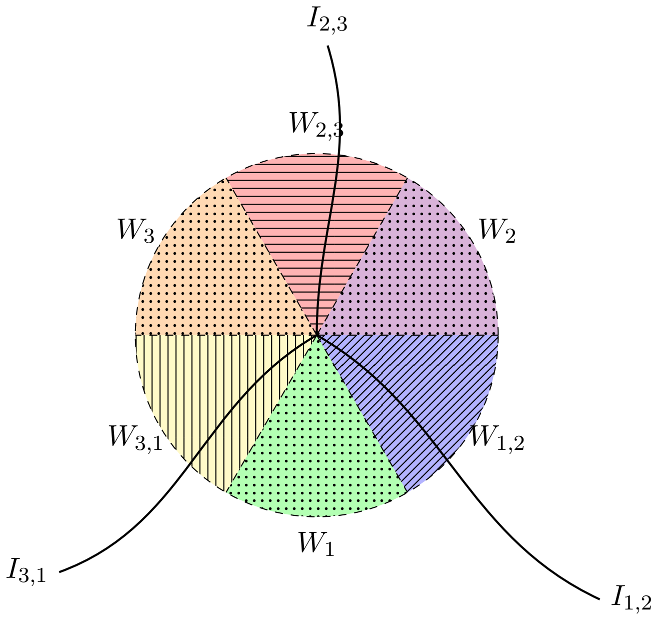

Figure 5.3: A cross-section orthorgonal to the triple line illustrating the wedge decomposition in Definition 5.15. The "interpolation wedges" are marked with a dotted pattern, the "interface wedges" with striped patterns.

Proposition 5.14. In the final Subsection 5.3.5, we formalize the fact that the local gradient-flow calibration at the triple line due to Proposition 5.14 represents an admissible perturbation of the local gradient-flow calibrations at the interfaces in a suitable sense.

5.3.1 Local geometry at a triple line

We first provide a suitable decomposition of the space-time tubular neighborhood of the triple line of a smoothly evolving regular double bubble in the sense of Definition 5.10. The main ingredient is given by the following notion of an admissible localization radius for the triple line, cf. Figure 5.3.

Definition 5.15 (Localization radius for triple line)**.** *Let $(\bar\Omega_1, \bar\Omega_2, \bar\Omega_3)$ be a regular double bubble smoothly evolving by MCF in the sense of Definition 5.10 on a time interval $[0,T]$. For each $i,j \in \{1,2,3\}$ with $i \neq j$, let $r_{i,j} \in (0,1]$ be an admissible localization radius for the interface $\bar I_{i,j}$ in the sense of Definition 5.11. We call $r = r_{\bar\Gamma} \in \big(0, \min\{r_{i,j} \colon i,j \in \{1,2,3\},\, i \neq j\}\big)$ an* admissible localization radius for the triple line *$\bar\Gamma$ if the following properties are satisfied:*

i) (Regularity of triple line) Define $\mathcal{N}_r(\bar\Gamma) := \bigcup_{t\in[0,T]} B_r(\bar\Gamma(t)) \times \{t\}$. The squared distance to $\bar\Gamma$ satisfies $\operatorname{dist}^2(\cdot,\bar\Gamma) \in C_t^0 C_x^4(\mathcal{N}_r(\bar\Gamma)) \cap C_t^1 C_x^2(\mathcal{N}_r(\bar\Gamma))$, and for the nearest-point projection onto $\bar\Gamma$ it holds $P_{\bar\Gamma} \in C_t^0 C_x^4(\mathcal{N}_r(\bar\Gamma)) \cap C_t^1 C_x^2(\mathcal{N}_r(\bar\Gamma))$.

ii) (Wedge decomposition) For each $i,j \in \{1,2,3\}$ with $i \neq j$, there exist sets $W_{\bar I_{i,j}} := \bigcup_{t\in[0,T]} W_{\bar I_{i,j}(t)} \times \{t\}$, $W_{\bar I_{j,i}} := W_{\bar I_{i,j}}$, and $W_{\bar\Omega_i} := \bigcup_{t\in[0,T]} W_{\bar\Omega_i}(t) \times \{t\}$ subject to the following conditions:

First, for each $t \in [0,T]$ the sets $(W_{\bar I_{i,j}}(t))_{i,j\in\{1,2,3\},i\neq j}$ and $(W_{\bar\Omega_i}(t))_{i\in\{1,2,3\}}$ are non-empty, pairwise disjoint open subsets of $B_r(\bar\Gamma(t))$. For each $x \in \bar\Gamma(t)$, the slice of each of these sets in the normal plane $x + \operatorname{Tan}_x^\perp \bar\Gamma(t)$ is the intersection of $B_r(\bar\Gamma(t))$ and a cone with apex x, cf. Figure 5.3. More precisely, there exist unit length vector fields $(X^\pm_{\bar I_{i,j}})_{i,j\in\{1,2,3\},i\neq j}$

and $(X^{\pm}_{\bar\Omega_i})_{i\in\{1,2,3\}}$ *along* $\bar\Gamma$*, taking values for each* $t\in[0,T]$ *in the normal bundle* $\mathrm{Tan}^{\perp}\bar\Gamma(t)$ *and being of class* $C^0_tC^4_x(\bar\Gamma)\cap C^1_tC^2_x(\bar\Gamma)$*, so that for all* $i,j\in\{1,2,3\}$ *with* $i\neq j$ *and all* $(x,t)\in\bar\Gamma$ *it holds*

$$\begin{aligned}&W_{\bar I_{i,j}}(t)\cap\big(x+\mathrm{Tan}^{\perp}_x\bar\Gamma(t)\big)\\&=\big(x+\{\alpha X^+_{\bar I_{i,j}}(x,t)+\beta X^-_{\bar I_{i,j}}(x,t)\colon\alpha,\beta\in(0,\infty)\}\big)\cap B_r(\bar\Gamma(t)),\end{aligned}\tag{5.38}$$

as well as

$$\begin{aligned}&W_{\bar\Omega_i}(t)\cap\big(x+\mathrm{Tan}^{\perp}_x\bar\Gamma(t)\big)\\&=\big(x+\{\alpha X^+_{\bar\Omega_i}(x,t)+\beta X^-_{\bar\Omega_i}(x,t)\colon\alpha,\beta\in(0,\infty)\}\big)\cap B_r(\bar\Gamma(t)).\end{aligned}\tag{5.39}$$

Moreover, $X^{\pm}_{\bar I_{i,j}}=X^{\pm}_{\bar I_{j,i}}$ *and* $(X^+_{\bar\Omega_i},X^-_{\bar\Omega_i})\in\big\{(X^+_{\bar I_{i,j}},X^-_{\bar I_{k,i}}),(X^+_{\bar I_{k,i}},X^-_{\bar I_{i,j}})\big\}$ *for all* $i,j,k\in\{1,2,3\}$ *such that* $\{i,j,k\}=\{1,2,3\}$*. The opening angles of these cones are constant along* $\bar\Gamma$ *and take values in* $(0,\pi)$*.*

Second, for each $t\in[0,T]$ *these sets provide a decomposition of the tubular neighborhood of the triple line in the sense that*

$$\overline{B_r(\bar\Gamma(t))}=\overline{W_{\bar I_{1,2}}(t)}\cup\overline{W_{\bar I_{2,3}}(t)}\cup\overline{W_{\bar I_{3,1}}(t)}\cup\bigcup_{i\in\{1,2,3\}}\overline{W_{\bar\Omega_i}(t)}.\tag{5.40}$$

Third, for all $t\in[0,T]$ *and all distinct* $i,j\in\{1,2,3\}$ *it holds*

$$\bar I_{i,j}(t)\cap B_r(\bar\Gamma(t))\subset W_{\bar I_{i,j}}(t)\cup\bar\Gamma(t)\subset\{x\in\mathbb{R}^3\colon(x,t)\in\mathrm{im}(\Psi_{i,j})\},\tag{5.41}$$

$$W_{\bar\Omega_i}(t)\subset\bigcap_{j\in\{1,2,3\}\setminus\{i\}}\{x\in\mathbb{R}^3\colon(x,t)\in\mathrm{im}(\Psi_{i,j})\},\tag{5.42}$$

where we refer to Definition 5.11 for the diffeomorphisms $\Psi_{i,j}$*.*

iii) (Comparability of distances) There exists $C>0$ *such that for all pairwise distinct* $i,j,k\in\{1,2,3\}$ *it holds (recall that* $\mathcal{I}=\bigcup_{i,j\in\{1,2,3\},i\neq j}\bar I_{i,j}$*)*

$$\mathrm{dist}(\cdot,\bar\Gamma)+\mathrm{dist}(\cdot,\bar I_{i,j})+\mathrm{dist}(\cdot,\bar I_{k,i})\leq C\,\mathrm{dist}(\cdot,\mathcal{I})\qquad\text{in } W_{\bar\Omega_i},\tag{5.43}$$

$$\mathrm{dist}(\cdot,\bar\Gamma)\leq C\,\mathrm{dist}(\cdot,\bar I_{i,j})\qquad\text{in } W_{\bar I_{j,k}}\cup W_{\bar I_{k,i}},\tag{5.44}$$

$$\mathrm{dist}(\cdot,\bar I_{i,j})\leq C\,\mathrm{dist}(\cdot,\mathcal{I})\qquad\text{in } W_{\bar I_{i,j}}.\tag{5.45}$$

We refer from here onwards to the sets $(W_{\bar I_{i,j}})_{i,j\in\{1,2,3\},i\neq j}$ *as* interface wedges*, and to the sets* $(W_{\bar\Omega_i})_{i\in\{1,2,3\}}$ *as* interpolation wedges.

Equations (5.38) and (5.39) simply mean that the domains $W_{\bar\Omega_i}(t)$ and $W_{\bar I_{i,j}}(t)$ are "wedges" in the sense that their respective slices across the normal space $x+\mathrm{Tan}^{\perp}\bar\Gamma(t)$ of the triple line have a cone structure close to $\bar\Gamma(t)$. The comparability (5.43)–(5.45) of distance functions in the various slices can be already guessed from Figure 5.3.

Let us first briefly discuss the existence of an admissible localization radius.

Lemma 5.16. *Let the assumptions and notation of Definition 5.15 be in place. Then there exists an admissible localization radius* $r=r_{\bar\Gamma}\in(0,1]$ *for the triple line. The radius* r *and the associated data only depends on the data of the smoothly evolving regular double bubble* $(\bar\Omega_1,\bar\Omega_2,\bar\Omega_3)$ *on* $[0,T]$*.*

Proof. We argue how to arrange the vector fields $(X^{\pm}_{\bar{I}_{i,j}})_{i,j\in\{1,2,3\},i\neq j}$ and $(X^{\pm}_{\bar{\Omega}_i})_{i\in\{1,2,3\}}$ in order to ensure the properties (5.38)–(5.40). The remaining conditions are a consequence of exploiting the uniform space-time regularity of the interfaces present in the smoothly evolving regular double bubble $(\bar{\Omega}_1, \bar{\Omega}_2, \bar{\Omega}_3)$ on $[0,T]$, cf. Definition 5.10, and choosing the scale $r \in (0, r_{1,2} \wedge r_{2,3} \wedge r_{3,1}]$ sufficiently small.

Fix $(x,t) \in \bar{\Gamma}$, and up to a translation and rotation we may assume without loss of generality that $x = 0$ and $\mathrm{Tan}^{\perp}_x \bar{\Gamma}(t) = \{0\}\times\mathbb{R}^2 = \langle e_2, e_3\rangle$, where $\{e_1, e_2, e_3\}$ denotes the standard basis of $\mathbb{R}^3$ and $\langle e_2, e_3\rangle$ the $\mathbb{R}$-linear span of $\{e_2, e_3\}$. Denote then by $\bar{\tau}_{1,2}, \bar{\tau}_{2,3}, \bar{\tau}_{3,1} \in \langle e_2, e_3\rangle$ the tangent vectors at $x = 0$ to the interfaces $\bar{I}_{1,2}$, $\bar{I}_{2,3}$ and $\bar{I}_{3,1}$, respectively, with the orientation chosen so that along $\bar{\Gamma}$ all of them point in the direction of the associated interface (which is also described in more detail in Construction 5.17 below). These tangent vectors define associated half-spaces

$$\mathbb{H}_{1,2} := \big\{ y \in \langle e_2, e_3\rangle \colon y \cdot \bar{\tau}_{1,2} > 0 \big\}, \tag{5.46}$$

where $\mathbb{H}_{2,3}$ and $\mathbb{H}_{3,1}$ are defined analogously.

We now construct a set of pairwise disjoint open cones $C_{\bar{\Omega}_1}, C_{\bar{\Omega}_2}, C_{\bar{\Omega}_3} \subset \langle e_2, e_3\rangle$, which will provide the cone structure of the interpolation wedges, by means of the following procedure: If the cone given by $\mathbb{H}_{1,2} \cap \mathbb{H}_{3,1}$ has an opening angle strictly greater than 90°, we define $C_{\bar{\Omega}_1} := \mathbb{H}_{1,2} \cap \mathbb{H}_{3,1}$. In the other case, we define $C_{\bar{\Omega}_1}$ to be the middle third of the cone with opening vectors $\bar{\tau}_{1,2}$ and $\bar{\tau}_{3,1}$. The remaining two cones $C_{\bar{\Omega}_2}$ and $C_{\bar{\Omega}_3}$ are defined in the same way.

Note that the opening angles of the cones $(C_{\bar{\Omega}_i})_{i\in\{1,2,3\}}$ are always strictly smaller than 180° since the surface tensions satisfy the strict triangle inequality. We proceed by selecting cones $C_{\bar{I}_{1,2}} =: C_{\bar{I}_{2,1}}, C_{\bar{I}_{2,3}} =: C_{\bar{I}_{3,2}}, C_{\bar{I}_{3,1}} =: C_{\bar{I}_{1,3}} \subset \langle e_2, e_3\rangle$, which are uniquely determined by the requirement that together with $(C_{\bar{\Omega}_i})_{i\in\{1,2,3\}}$ they form a family of pairwise disjoint open cones in $\langle e_2, e_3\rangle$ such that

$$\langle e_2, e_3\rangle = \overline{C_{\bar{I}_{1,2}}} \cup \overline{C_{\bar{I}_{2,3}}} \cup \overline{C_{\bar{I}_{3,1}}} \cup \bigcup_{i\in\{1,2,3\}} \overline{C_{\bar{\Omega}_i}}, \tag{5.47}$$

$$\bar{\tau}_{1,2} \in C_{\bar{I}_{1,2}}, \quad \bar{\tau}_{2,3} \in C_{\bar{I}_{2,3}}, \quad \bar{\tau}_{3,1} \in C_{\bar{I}_{3,1}}. \tag{5.48}$$

We finally define $(X^{\pm}_{\bar{I}_{i,j}})_{i,j\in\{1,2,3\},i\neq j}$ and $(X^{\pm}_{\bar{\Omega}_i})_{i\in\{1,2,3\}}$ by means of the unit length opening vectors of the cones $(C_{\bar{I}_{i,j}})_{i,j\in\{1,2,3\},i\neq j}$ and $(C_{\bar{\Omega}_i})_{i\in\{1,2,3\}}$, respectively. The right hand sides of properties (5.38) and (5.39) now serve as the defining objects for the interface and interpolation wedges, respectively. □

In a second preparatory step, we proceed with the definition of a preliminary orthonormal frame along each of the three respective interfaces in the vicinity of the triple line, cf. Figure 5.2.

Construction 5.17 (Preliminary choice of tangent frame)**.** Let the assumptions and notation of Definition 5.15 be in place. In particular, let $r \in (0, r_{1,2} \wedge r_{2,3} \wedge r_{3,1}]$ be an associated admissible localization radius for the triple line $\bar{\Gamma}$. We then provide for all $t \in [0,T]$ and all distinct phases $i, j \in \{1,2,3\}$ two tangent vector fields $\bar{\tau}_{i,j}(\cdot,t), \bar{\mathfrak{t}}_{i,j}(\cdot,t) \in \mathbb{S}^2$ along the local interface patch $\bar{I}_{i,j}(t) \cap B_r(\bar{\Gamma}(t))$ by means of the following procedure:

First, slicing the interface $\bar{I}_{i,j}(t)$ along the planes $y + \mathrm{Tan}^{\perp}_y \bar{\Gamma}(t)$ produces a family of curves $\bar{I}^y_{i,j}(t) := \bar{I}_{i,j}(t) \cap \big(y + \mathrm{Tan}^{\perp}_y \bar{\Gamma}(t)\big) \cap B_r(\bar{\Gamma}(t))$ for all $y \in \bar{\Gamma}(t)$. Second, for each $t \in [0,T]$ and each $y \in \bar{\Gamma}(t)$ denote by $\bar{\tau}^y_{i,j}(\cdot,t) \in \mathbb{S}^2$ the tangent vector field along the curve $\bar{I}^y_{i,j}(t)$ which is oriented by $y + \frac{r}{2}\bar{\tau}^y_{i,j}(y,t) \in W_{\bar{I}_{i,j}}(t) \cap \big(y + \mathrm{Tan}^{\perp}_y \bar{\Gamma}(t)\big)$. We then define two tangent vector fields on the local interface patch $\bar{I}_{i,j} \cap \mathcal{N}_r(\bar{\Gamma})$ by means of

$$\bar{\tau}_{i,j}(x,t) := \bar{\tau}^y_{i,j}(x,t)\big|_{y=P_{\bar{\Gamma}}(x,t)} \in \mathbb{S}^2, \qquad (x,t) \in \bar{I}_{i,j} \cap \mathcal{N}_r(\bar{\Gamma}),$$
$$\bar{\mathrm{t}}_{i,j}(x,t) := (\bar{\mathrm{n}}_{i,j} \times \bar{\tau}_{i,j})(x,t) \in \mathbb{S}^2, \qquad (x,t) \in \bar{I}_{i,j} \cap \mathcal{N}_r(\bar{\Gamma}).$$

This yields an orthonormal frame $(\bar{\mathrm{n}}_{i,j}, \bar{\tau}_{i,j}, \bar{\mathrm{t}}_{i,j})$ on $\bar{I}_{i,j} \cap \mathcal{N}_r(\bar{\Gamma})$. Observe also that it holds

$$\bar{\mathrm{t}}_{1,2}|_{\bar{\Gamma}} = \bar{\mathrm{t}}_{2,3}|_{\bar{\Gamma}} = \bar{\mathrm{t}}_{3,1}|_{\bar{\Gamma}}. \tag{5.49}$$

By a minor abuse of notation, we finally introduce extensions of these tangential vector fields away from the interfaces. Namely, we define

$$(\bar{\tau}_{i,j}, \bar{\mathrm{t}}_{i,j})(x,t) := (\bar{\tau}_{i,j}, \bar{\mathrm{t}}_{i,j})(y,t)\big|_{y=P_{i,j}(x,t)}, \quad (x,t) \in \mathrm{im}(\Psi_{i,j}) \cap \mathcal{N}_r(\bar{\Gamma}). \tag{5.50}$$

We refer to Definition 5.11 for the diffeomorphism $\Psi_{i,j}$ and the projection $P_{i,j}$ onto the interface $\bar{I}_{i,j}$. We register in terms of regularity that

$$\bar{\tau}_{i,j}, \bar{\mathrm{t}}_{i,j} \in (C^0_t C^4_x \cap C^1_t C^2_x)(\mathrm{im}(\Psi_{i,j}) \cap \mathcal{N}_r(\bar{\Gamma})). \tag{5.51}$$

This concludes our construction of *orthonormal frames* $(\bar{\mathrm{n}}_{i,j}, \bar{\tau}_{i,j}, \bar{\mathrm{t}}_{i,j})$. ♢

In the sequel we will repeatedly rely on an explicit representation of the gradients for the normal and tangential vector fields. These formulas are the content of the following result.

Lemma 5.18. *Let the assumptions and notation of Definition 5.15 and Construction 5.17 be in place. To ease notation, let $\bar{I} := \bar{I}_{1,2}$, $\bar{I}' := \bar{I}_{2,3}$ and $\bar{I}'' := \bar{I}_{3,1}$ for the three interfaces present in the smoothly evolving regular double bubble $(\bar{\Omega}_1, \bar{\Omega}_2, \bar{\Omega}_3)$. We proceed accordingly for the associated orthonormal frames $(\bar{\mathrm{n}}, \bar{\tau}, \bar{\mathrm{t}})$, $(\bar{\mathrm{n}}', \bar{\tau}', \bar{\mathrm{t}}')$ and $(\bar{\mathrm{n}}'', \bar{\tau}'', \bar{\mathrm{t}}'')$, respectively.*

Using also the abbreviations $\kappa_{\bar{\tau}\bar{\tau}} := -\bar{\tau} \otimes \bar{\tau} : \nabla\bar{\mathrm{n}}$, $\kappa_{\bar{\mathrm{t}}\bar{\mathrm{t}}} := -\bar{\mathrm{t}} \otimes \bar{\mathrm{t}} : \nabla\bar{\mathrm{n}}$ as well as $\kappa_{\bar{\tau}\bar{\mathrm{t}}} := -\bar{\tau} \otimes \bar{\mathrm{t}} : \nabla\bar{\mathrm{n}}$, it holds $\kappa_{\bar{\tau}\bar{\mathrm{t}}} = -\bar{\mathrm{t}} \otimes \bar{\tau} : \nabla\bar{\mathrm{n}}$ and

$$\nabla\bar{\mathrm{n}} = -\kappa_{\bar{\tau}\bar{\tau}}\, \bar{\tau} \otimes \bar{\tau} - \kappa_{\bar{\mathrm{t}}\bar{\mathrm{t}}}\, \bar{\mathrm{t}} \otimes \bar{\mathrm{t}} - \kappa_{\bar{\tau}\bar{\mathrm{t}}}\, (\bar{\mathrm{t}} \otimes \bar{\tau} + \bar{\tau} \otimes \bar{\mathrm{t}}), \tag{5.52}$$
$$\nabla\bar{\tau} = \kappa_{\bar{\tau}\bar{\tau}}\, \bar{\mathrm{n}} \otimes \bar{\tau} - (\nabla \cdot \bar{\mathrm{t}})\, \bar{\mathrm{t}} \otimes \bar{\tau} + \kappa_{\bar{\tau}\bar{\mathrm{t}}}\, \bar{\mathrm{n}} \otimes \bar{\mathrm{t}} + (\nabla \cdot \bar{\tau})\, \bar{\mathrm{t}} \otimes \bar{\mathrm{t}}, \tag{5.53}$$
$$\nabla\bar{\mathrm{t}} = \kappa_{\bar{\mathrm{t}}\bar{\mathrm{t}}}\, \bar{\mathrm{n}} \otimes \bar{\mathrm{t}} + \kappa_{\bar{\tau}\bar{\mathrm{t}}}\, \bar{\mathrm{n}} \otimes \bar{\tau} + (\nabla \cdot \bar{\mathrm{t}})\, \bar{\tau} \otimes \bar{\tau} - (\nabla \cdot \bar{\tau})\, \bar{\tau} \otimes \bar{\mathrm{t}} \tag{5.54}$$

along the local interface patch $\bar{I} \cap \mathcal{N}_r(\bar{\Gamma})$. Analogous formulas of course hold true for $(\bar{\mathrm{n}}', \bar{\tau}', \bar{\mathrm{t}}')$ along $\bar{I}' \cap \mathcal{N}_r(\bar{\Gamma})$ in terms of $(\kappa'_{\bar{\tau}'\bar{\tau}'}, \kappa'_{\bar{\mathrm{t}}'\bar{\mathrm{t}}'}, \kappa'_{\bar{\tau}'\bar{\mathrm{t}}'})$, and for $(\bar{\mathrm{n}}'', \bar{\tau}'', \bar{\mathrm{t}}'')$ along $\bar{I}'' \cap \mathcal{N}_r(\bar{\Gamma})$ in terms of $(\kappa''_{\bar{\tau}''\bar{\tau}''}, \kappa''_{\bar{\mathrm{t}}''\bar{\mathrm{t}}''}, \kappa''_{\bar{\tau}''\bar{\mathrm{t}}''})$.

Proof. The representation (5.52) is essentially just a rephrasing of the definition of the coefficients $\kappa_{\bar{\tau}\bar{\tau}}$, $\kappa_{\bar{\mathrm{t}}\bar{\mathrm{t}}}$ and $\kappa_{\bar{\tau}\bar{\mathrm{t}}}$. The only additional ingredients needed for the validity of (5.52) are $(\nabla\bar{\mathrm{n}})^\mathsf{T}\bar{\mathrm{n}} = \frac{1}{2}\nabla|\bar{\mathrm{n}}|^2 = 0$ and the symmetry of $\nabla\bar{\mathrm{n}} = \nabla^2 s_{1,2}$, cf. (5.19).

For a proof of (5.53), we write $\bar{\tau} = J\bar{\mathrm{n}}$ where $J = \bar{\tau} \wedge \bar{\mathrm{n}} + \bar{\mathrm{t}} \otimes \bar{\mathrm{t}}$ denotes the associated rotation matrix around the $\bar{\mathrm{t}}$-axis. Based on $(\bar{\mathrm{n}} \cdot \nabla)\bar{\tau} = 0$, $(\nabla\bar{\tau})^\mathsf{T}\bar{\tau} = \frac{1}{2}\nabla|\bar{\tau}|^2 = 0$ and (5.52) we then obtain

$$\begin{aligned}
\nabla\bar{\tau} &= \kappa_{\bar{\tau}\bar{\tau}}\, \bar{\mathrm{n}} \otimes \bar{\tau} - \kappa_{\bar{\mathrm{t}}\bar{\mathrm{t}}}\, \bar{\mathrm{t}} \otimes \bar{\mathrm{t}} - \kappa_{\bar{\tau}\bar{\mathrm{t}}}\, (\bar{\mathrm{t}} \otimes \bar{\tau} - \bar{\mathrm{n}} \otimes \bar{\mathrm{t}}) \\
&\quad + \big((\bar{\tau} \cdot \nabla)J\big)\bar{\mathrm{n}} \otimes \bar{\tau} + \big((\bar{\mathrm{t}} \cdot \nabla)J\big)\bar{\mathrm{n}} \otimes \bar{\mathrm{t}} \\
&= \kappa_{\bar{\tau}\bar{\tau}}\, \bar{\mathrm{n}} \otimes \bar{\tau} - \kappa_{\bar{\mathrm{t}}\bar{\mathrm{t}}}\, \bar{\mathrm{t}} \otimes \bar{\mathrm{t}} - \kappa_{\bar{\tau}\bar{\mathrm{t}}}\, (\bar{\mathrm{t}} \otimes \bar{\tau} - \bar{\mathrm{n}} \otimes \bar{\mathrm{t}}) \\
&\quad + \big(\bar{\mathrm{n}} \otimes \bar{\mathrm{n}} : (\bar{\tau} \cdot \nabla)J\big)\, \bar{\mathrm{n}} \otimes \bar{\tau} + \big(\bar{\mathrm{t}} \otimes \bar{\mathrm{n}} : (\bar{\tau} \cdot \nabla)J\big)\, \bar{\mathrm{t}} \otimes \bar{\tau} \\
&\quad + \big(\bar{\mathrm{n}} \otimes \bar{\mathrm{n}} : (\bar{\mathrm{t}} \cdot \nabla)J\big)\, \bar{\mathrm{n}} \otimes \bar{\mathrm{t}} + \big(\bar{\mathrm{t}} \otimes \bar{\mathrm{n}} : (\bar{\mathrm{t}} \cdot \nabla)J\big)\, \bar{\mathrm{t}} \otimes \bar{\mathrm{t}}.
\end{aligned}$$

For the two appearing $(\bar{\mathrm{n}}\otimes\bar{\mathrm{n}})$-components of ∇J, it suffices to take the symmetric part of J into account, which is $\bar{\mathrm{t}}\otimes\bar{\mathrm{t}}$. It then follows from $\bar{\mathrm{t}}\cdot\bar{\mathrm{n}}=0$ that

$$\bar{\mathrm{n}}\otimes\bar{\mathrm{n}} : (\bar{\tau}\cdot\nabla)J = \bar{\mathrm{n}}\otimes\bar{\mathrm{n}} : (\bar{\mathrm{t}}\cdot\nabla)J = 0.$$

Based on (5.52), $\bar{\mathrm{t}}\cdot(\bar{\mathrm{t}}\cdot\nabla)\bar{\mathrm{t}} = \frac{1}{2}(\bar{\mathrm{t}}\cdot\nabla)|\bar{\mathrm{t}}|^2 = 0$, and $\bar{\mathrm{t}} = \bar{\mathrm{n}}\times\bar{\tau}$ we may further compute

$$\begin{aligned}
\bar{\mathrm{t}}\otimes\bar{\mathrm{n}} : (\bar{\mathrm{t}}\cdot\nabla)J &= (\bar{\mathrm{t}}\otimes\bar{\mathrm{n}}) : (\bar{\mathrm{t}}\cdot\nabla)(\bar{\tau}\wedge\bar{\mathrm{n}}) + \bar{\mathrm{n}}\cdot(\bar{\mathrm{t}}\cdot\nabla)\bar{\mathrm{t}} \\
&= (\bar{\mathrm{t}}\otimes\bar{\mathrm{n}}) : (\bar{\mathrm{t}}\cdot\nabla)(\bar{\tau}\wedge\bar{\mathrm{n}}) - \bar{\mathrm{t}}\otimes\bar{\mathrm{t}} : \nabla\bar{\mathrm{n}} \\
&= (\bar{\mathrm{t}}\otimes\bar{\mathrm{n}}) : (\bar{\mathrm{t}}\cdot\nabla)(\bar{\tau}\wedge\bar{\mathrm{n}}) + \kappa_{\bar{\mathrm{t}}\bar{\mathrm{t}}}.
\end{aligned}$$

Based on (5.52), $\bar{\mathrm{t}}\cdot\bar{\mathrm{n}}=0$, $(\bar{\mathrm{t}}\otimes\bar{\mathrm{n}}) : (\bar{\tau}\cdot\nabla)(\bar{\tau}\wedge\bar{\mathrm{n}}) = (\bar{\mathrm{t}}\otimes\bar{\tau}) : \nabla\bar{\tau}$, and $\bar{\mathrm{t}} = \bar{\mathrm{n}}\times\bar{\tau}$ we in addition have

$$\begin{aligned}
\bar{\mathrm{t}}\otimes\bar{\mathrm{n}} : (\bar{\tau}\cdot\nabla)J &= (\bar{\mathrm{t}}\otimes\bar{\tau}) : \nabla\bar{\tau} + \bar{\mathrm{n}}\cdot(\bar{\tau}\cdot\nabla)\bar{\mathrm{t}} \\
&= (\bar{\mathrm{t}}\otimes\bar{\tau}) : \nabla\bar{\tau} - \bar{\mathrm{t}}\otimes\bar{\tau} : \nabla\bar{\mathrm{n}} \\
&= (\bar{\mathrm{t}}\otimes\bar{\tau}) : \nabla\bar{\tau} + \kappa_{\bar{\tau}\bar{\mathrm{t}}}.
\end{aligned}$$

The combination of the previous four displays yields

$$\nabla\bar{\tau} = \kappa_{\bar{\tau}\bar{\tau}}\,\bar{\mathrm{n}}\otimes\bar{\tau} + \big((\bar{\mathrm{t}}\otimes\bar{\tau}) : \nabla\bar{\tau}\big)\,\bar{\mathrm{t}}\otimes\bar{\tau} + \kappa_{\bar{\tau}\bar{\mathrm{t}}}\,\bar{\mathrm{n}}\otimes\bar{\mathrm{t}} + (\nabla\cdot\bar{\tau})\,\bar{\mathrm{t}}\otimes\bar{\mathrm{t}}, \tag{5.55}$$

$$\nabla\cdot\bar{\tau} = (\bar{\mathrm{t}}\otimes\bar{\mathrm{n}}) : (\bar{\mathrm{t}}\cdot\nabla)(\bar{\tau}\wedge\bar{\mathrm{n}}). \tag{5.56}$$

Moreover, exploiting that $\bar{\mathrm{t}} = \bar{\mathrm{n}}\times\bar{\tau}$ yields by the product rule, (5.52) and the previous display

$$\nabla\bar{\mathrm{t}} = \kappa_{\bar{\mathrm{t}}\bar{\mathrm{t}}}\,\bar{\mathrm{n}}\otimes\bar{\mathrm{t}} + (\nabla\cdot\bar{\mathrm{t}})\,\bar{\tau}\otimes\bar{\tau} + \kappa_{\bar{\tau}\bar{\mathrm{t}}}\,\bar{\mathrm{n}}\otimes\bar{\tau} - (\nabla\cdot\bar{\tau})\,\bar{\tau}\otimes\bar{\mathrm{t}}, \tag{5.57}$$

$$\nabla\cdot\bar{\mathrm{t}} = -(\bar{\mathrm{t}}\otimes\bar{\tau}) : \nabla\bar{\tau}. \tag{5.58}$$

The previous two displays in turn directly imply (5.53) and (5.54). □

The orthonormal frames provided by Construction 5.17 together with the signed distance functions (5.15) constitute all ingredients for the construction of a suitable building block $\widetilde{\xi}_{i,j}$ for the vector field $\xi_{i,j}$; at least in $\mathcal{N}_r(\bar{\Gamma})\cap\mathrm{im}(\Psi_{i,j})$, see Construction 5.21 below. However, we also have to provide a construction of the vector field $\xi_{i,j}$ outside of the domain $\mathcal{N}_r(\bar{\Gamma})\cap\mathrm{im}(\Psi_{i,j})$, i.e., where this vector field a priori does not have a "natural" definition. The guiding principle is to mimic the Herring angle condition valid on the triple line:

$$\sigma_{1,2}\bar{\mathrm{n}}_{1,2} + \sigma_{2,3}\bar{\mathrm{n}}_{2,3} + \sigma_{3,1}\bar{\mathrm{n}}_{3,1} = 0.$$

This condition motivates to appropriately rotate the already defined candidate vector fields $\widetilde{\xi}_{j,k}$ and $\widetilde{\xi}_{k,i}$ to provide the building blocks for the vector field $\xi_{i,j}$ throughout $\mathcal{N}_r(\bar{\Gamma})\cap\mathrm{im}(\Psi_{j,k})$ and $\mathcal{N}_r(\bar{\Gamma})\cap\mathrm{im}(\Psi_{k,i})$, respectively.

The rotations used in this procedure have to be chosen carefully so that our constructions will satisfy the requirements of a local gradient-flow calibration at the triple line, e.g., sufficiently high regularity (in particular, adequate compatibility along the triple line) and the validity of the required evolution equations (up to a desired error in the distance to the interface).

Construction 5.19 (Gauged Herring rotation fields)**.** Let the assumptions and notation of Definition 5.15, Construction 5.17 and Lemma 5.18 be in place. Consistent with the notational conventions of the latter, denote by Ψ, Ψ' and Ψ'' the diffeomorphisms from Definition 5.11 with respect to the interfaces $\bar{I}$, $\bar{I}'$ and $\bar{I}''$, respectively. We proceed accordingly for the surface tensions $(\sigma,\sigma',\sigma'')$ and the projections (P,P',P'').

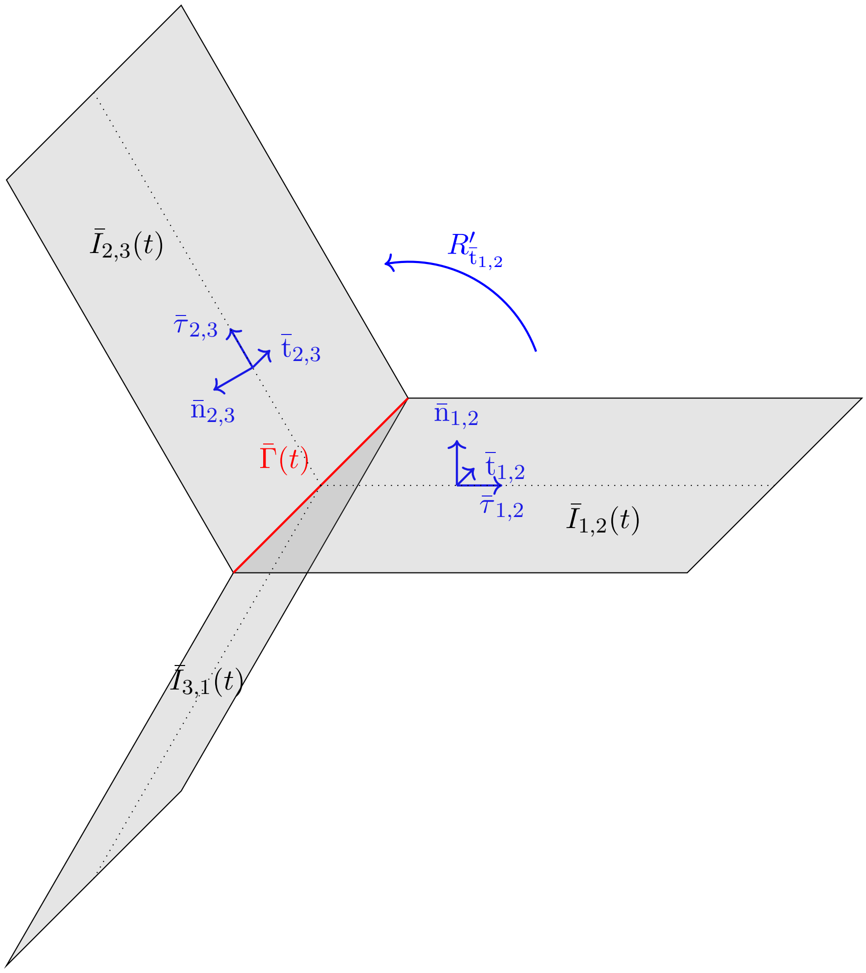

Figure 5.4: Local geometry at the triple line and preliminary construction of tangent frame. For simplicity, we illustrate here the case of three flat sheets coming together at equal angles of 120° along a straight triple line $\bar{\Gamma}(t)$. In this case, the "Herring" rotation $R'(y,t)$ is a rotation by 120° about the axis given by the tangent vector $\bar{t} = \bar{t}_{1,2}(y,t)$ of $\bar{\Gamma}(t)$. The dotted lines represent the three slices $\bar{I}^y_{i,j}(t)$ of the interfaces $\bar{I}_{i,j}$.

We now define a pair of *Herring rotation fields*

$$R'_{\bar{t}}, R''_{\bar{t}} \colon \mathcal{N}_r(\bar{\Gamma}) \cap \mathrm{im}(\Psi) \to SO(3) \subset \mathbb{R}^{3\times 3} \tag{5.59}$$

around the $\bar{t}$-axis by means of

$$R'_{\bar{t}}(x,t) := \cos\theta' \,\mathrm{Id} + \sin\theta' \,(\bar{\tau} \wedge \bar{n})(x,t) + (1-\cos\theta')\,(\bar{t}\otimes\bar{t})(x,t), \tag{5.60}$$

$$R''_{\bar{t}}(x,t) := \cos\theta'' \,\mathrm{Id} + \sin\theta'' \,(\bar{\tau} \wedge \bar{n})(x,t) + (1-\cos\theta'')\,(\bar{t}\otimes\bar{t})(x,t) \tag{5.61}$$

for all $(x,t) \in \mathcal{N}_r(\bar{\Gamma}) \cap \mathrm{im}(\Psi)$, cf. Figure 5.4. The associated angles $\theta', \theta'' \in (0,\pi)$ are independent of $(x,t) \in \mathcal{N}_r(\bar{\Gamma}) \cap \mathrm{im}(\Psi)$ and chosen based on the triple of surface tensions $(\sigma, \sigma', \sigma'')$ such that the relations

$$R'_{\bar{t}}\bar{n} = \bar{n}', \tag{5.62}$$

$$R''_{\bar{t}}\bar{n} = \bar{n}'' \tag{5.63}$$

hold true along the triple line $\bar{\Gamma}$. Hence, the Herring condition (5.10) implies that for all $(x,t) \in \mathcal{N}_r(\bar{\Gamma}) \cap \mathrm{im}(\Psi)$ and all $v \in \mathbb{R}^3$ such that $v \cdot \bar{t}(x,t) = 0$ it holds

$$\sigma v + \sigma' R'_{\bar{t}}(x,t)v + \sigma'' R''_{\bar{t}}(x,t)v = 0. \tag{5.64}$$

Analogously, one defines a pair of rotations $(R_{\bar{\mathrm{t}}'}, R''_{\bar{\mathrm{t}}'})$, resp. $(R_{\bar{\mathrm{t}}''}, R'_{\bar{\mathrm{t}}''})$, throughout the region $\mathcal{N}_r(\bar{\Gamma}) \cap \mathrm{im}(\Psi')$, resp. $\mathcal{N}_r(\bar{\Gamma}) \cap \mathrm{im}(\Psi'')$.

Apart from the Herring rotation fields, we also introduce the *gauge rotation field*

$$R_{\bar{\mathrm{n}}} := R^{(2)}_{\bar{\mathrm{n}}} R^{(1)}_{\bar{\mathrm{n}}} \colon \mathcal{N}_r(\bar{\Gamma}) \cap \mathrm{im}(\Psi) \to SO(3) \subset \mathbb{R}^{3\times 3} \tag{5.65}$$

around the $\bar{\mathrm{n}}$-axis, cf. Figure 5.2. The auxiliary rotation fields $R^{(1)}_{\bar{\mathrm{n}}}$ and $R^{(2)}_{\bar{\mathrm{n}}}$ around the $\bar{\mathrm{n}}$-axis are defined via

$$\begin{aligned} R^{(1)}_{\bar{\mathrm{n}}}(x,t) &:= \cos\delta(x,t)\,\mathrm{Id} + \sin\delta(x,t)\,(\bar{\mathrm{t}} \wedge \bar{\tau})(x,t) \\ &\qquad + (1 - \cos\delta(x,t))\,(\bar{\mathrm{n}} \otimes \bar{\mathrm{n}})(x,t), \end{aligned} \tag{5.66}$$

$$\begin{aligned} R^{(2)}_{\bar{\mathrm{n}}}(x,t) &:= \cos\omega(x,t)\,\mathrm{Id} + \sin\omega(x,t)\,(\bar{\mathrm{t}} \wedge \bar{\tau})(x,t) \\ &\qquad + (1 - \cos\omega(x,t))\,(\bar{\mathrm{n}} \otimes \bar{\mathrm{n}})(x,t). \end{aligned} \tag{5.67}$$

Here the rotation angle $\delta(x,t)$ is given explicitly by

$$\delta(x,t) := s(x,t)\kappa_{\bar{\tau}\bar{\mathrm{t}}}(x,t), \quad (x,t) \in \mathcal{N}_r(\bar{\Gamma}) \cap \mathrm{im}(\Psi), \tag{5.68}$$

and the angle $\omega(x,t)$ is given by the extension

$$\omega(x,t) := \widehat{\omega}(P(x,t),t), \quad (x,t) \in \mathcal{N}_r(\bar{\Gamma}) \cap \mathrm{im}(\Psi) \tag{5.69}$$

of $\widehat{\omega}(x,t)$, which in turn is defined by the one-parameter family of ODEs

$$\begin{cases} \widehat{\omega}(x,t) = 0, & (x,t) \in \bar{\Gamma}, \\ (\bar{\tau}(x,t)\cdot\nabla)\,\widehat{\omega}(x,t) = (\nabla\cdot\bar{\mathrm{t}})(x,t), & (x,t) \in \bar{I} \cap \mathcal{N}_r(\bar{\Gamma}). \end{cases} \tag{5.70}$$

Analogously, one defines a gauge rotation $R_{\bar{\mathrm{n}}'} := R^{(2)}_{\bar{\mathrm{n}}'} R^{(1)}_{\bar{\mathrm{n}}'}$, resp. $R_{\bar{\mathrm{n}}''} := R^{(2)}_{\bar{\mathrm{n}}''} R^{(1)}_{\bar{\mathrm{n}}''}$, throughout the region $\mathcal{N}_r(\bar{\Gamma}) \cap \mathrm{im}(\Psi')$, resp. $\mathcal{N}_r(\bar{\Gamma}) \cap \mathrm{im}(\Psi'')$.

We finally define via conjugation a pair of *gauged Herring rotation fields*

$$\widetilde{R}'_{\bar{I}} := R_{\bar{\mathrm{n}}} R'_{\bar{\mathrm{t}}} R^{\mathsf{T}}_{\bar{\mathrm{n}}} \colon \mathcal{N}_r(\bar{\Gamma}) \cap \mathrm{im}(\Psi) \to SO(3) \subset \mathbb{R}^{3\times 3}, \tag{5.71}$$

$$\widetilde{R}''_{\bar{I}} := R_{\bar{\mathrm{n}}} R''_{\bar{\mathrm{t}}} R^{\mathsf{T}}_{\bar{\mathrm{n}}} \colon \mathcal{N}_r(\bar{\Gamma}) \cap \mathrm{im}(\Psi) \to SO(3) \subset \mathbb{R}^{3\times 3}, \tag{5.72}$$

and analogously a pair $(\widetilde{R}_{\bar{I}'}, \widetilde{R}''_{\bar{I}'})$, resp. $(\widetilde{R}_{\bar{I}''}, \widetilde{R}'_{\bar{I}''})$, of gauged Herring rotation fields throughout the region $\mathcal{N}_r(\bar{\Gamma}) \cap \mathrm{im}(\Psi')$, resp. $\mathcal{N}_r(\bar{\Gamma}) \cap \mathrm{im}(\Psi'')$. ◇

In a symmetric setting with either rotational or translational symmetry, cf. Figure 5.4, the gauge rotations $R_{\bar{\mathrm{n}}}$, $R_{\bar{\mathrm{n}}'}$, and $R_{\bar{\mathrm{n}}''}$ are not needed and, in fact, reduce to the identity matrix. In the general case, cf. Figure 5.2, they account for the fact that, for instance, the normal vector field $\bar{\mathrm{n}}(\cdot,t)$ evaluated along a slice $\bar{I}(t) \cap (x + \mathrm{Tan}^{\perp}_x \bar{\Gamma}(t))$ for some $x \in \bar{\Gamma}(t)$ will in general rotate out of the plane $x + \mathrm{Tan}^{\perp}_x \bar{\Gamma}(t)$ as one moves away from the triple line point x.

We conclude this section with the derivation of compatibility conditions along the triple line. These represent the last missing ingredients to ensure compatibility of the main building blocks $\widetilde{\xi}_{i,j}$ (cf. Construction 5.21 below) for the vector field $\xi_{i,j}$ and its rotated counterparts along the triple line (see Lemma 5.22 below).

Lemma 5.20. *Let the assumptions and notation of Definition 5.15, Construction 5.17, Lemma 5.18, and Construction 5.19 be in place. Consistently with the notational conventions of the latter two, denote by H, H' and H'' the extended scalar mean curvatures defined by (5.20) with respect to the interfaces $\bar{I}$, $\bar{I}'$ and $\bar{I}''$, respectively. Denote by $\mathrm{V}_{\bar{\Gamma}}$ the normal velocity vector field of the triple line.*

Then, the following compatibility conditions are satisfied along the triple line $\bar\Gamma$:

$$\bar\tau' = R'_{\bar{\mathrm t}}\bar\tau, \quad \bar\tau'' = R''_{\bar{\mathrm t}}\bar\tau, \tag{5.73}$$

$$\kappa'_{\bar\tau'\bar{\mathrm t}'} = \kappa_{\bar\tau\bar{\mathrm t}}, \quad \kappa''_{\bar\tau''\bar{\mathrm t}''} = \kappa_{\bar\tau\bar{\mathrm t}}, \tag{5.74}$$

$$\kappa'_{\bar{\mathrm t}'\bar{\mathrm t}'} = (R'_{\bar{\mathrm t}}\bar{\mathrm n}\cdot\bar{\mathrm n})\kappa_{\bar{\mathrm t}\bar{\mathrm t}} - (R'_{\bar{\mathrm t}}\bar{\mathrm n}\cdot\bar\tau)\nabla\cdot\bar\tau, \tag{5.75}$$

$$\kappa''_{\bar{\mathrm t}''\bar{\mathrm t}''} = (R''_{\bar{\mathrm t}}\bar{\mathrm n}\cdot\bar{\mathrm n})\kappa_{\bar{\mathrm t}\bar{\mathrm t}} - (R''_{\bar{\mathrm t}}\bar{\mathrm n}\cdot\bar\tau)\nabla\cdot\bar\tau, \tag{5.76}$$

$$\nabla\cdot\bar\tau' = (R'_{\bar{\mathrm t}}\bar{\mathrm n}\cdot\bar\tau)\kappa_{\bar{\mathrm t}\bar{\mathrm t}} + (R'_{\bar{\mathrm t}}\bar{\mathrm n}\cdot\bar{\mathrm n})\nabla\cdot\bar\tau, \tag{5.77}$$

$$\nabla\cdot\bar\tau'' = (R''_{\bar{\mathrm t}}\bar{\mathrm n}\cdot\bar\tau)\kappa_{\bar{\mathrm t}\bar{\mathrm t}} + (R''_{\bar{\mathrm t}}\bar{\mathrm n}\cdot\bar{\mathrm n})\nabla\cdot\bar\tau, \tag{5.78}$$

$$\sigma H + \sigma' H' + \sigma'' H'' = 0, \tag{5.79}$$

$$\begin{aligned}\kappa''_{\bar\tau''\bar\tau''}(\bar\tau''\cdot \mathrm{V}_{\bar\Gamma}) + (\bar\tau''\cdot\nabla)H'' &= \kappa'_{\bar\tau'\bar\tau'}(\bar\tau'\cdot \mathrm{V}_{\bar\Gamma}) + (\bar\tau'\cdot\nabla)H' \\ &= \kappa_{\bar\tau\bar\tau}(\bar\tau\cdot \mathrm{V}_{\bar\Gamma}) + (\bar\tau\cdot\nabla)H.\end{aligned} \tag{5.80}$$

Of course, the analogues of (5.73) as well as (5.75)–(5.78) hold true for the appropriate relabellings of the associated data.

Introduce next a gauged orthonormal frame on $\mathcal{N}_r(\bar\Gamma)\cap\mathrm{im}(\Psi)$ by means of

$$(\bar{\mathrm n}, \bar\tau_*, \bar{\mathrm t}_*) := (\bar{\mathrm n}, R_{\bar{\mathrm n}}\bar\tau, R_{\bar{\mathrm n}}\bar{\mathrm t}). \tag{5.81}$$

Then, the following compatibility condition holds true:

$$(\bar{\mathrm n}, \bar\tau_*, \bar{\mathrm t}_*) = (\bar{\mathrm n}, \bar\tau, \bar{\mathrm t}) \quad \textit{along the triple line } \bar\Gamma. \tag{5.82}$$

The analogue of (5.82) with respect to the gauged frame $(\bar{\mathrm n}', \bar\tau'_*, \bar{\mathrm t}'_*) := (\bar{\mathrm n}', R_{\bar{\mathrm n}'}\bar\tau', R_{\bar{\mathrm n}'}\bar{\mathrm t}')$ on $\mathcal{N}_r(\bar\Gamma)\cap\mathrm{im}(\Psi')$, resp. $(\bar{\mathrm n}'', \bar\tau''_*, \bar{\mathrm t}''_*) := (\bar{\mathrm n}'', R_{\bar{\mathrm n}''}\bar\tau'', R_{\bar{\mathrm n}''}\bar{\mathrm t}'')$ on $\mathcal{N}_r(\bar\Gamma)\cap\mathrm{im}(\Psi'')$, is also satisfied.

Proof. Except for the conditions (5.73) and (5.82), the asserted compatibility conditions are consequences of differentiating the existing zeroth and first order compatibility conditions along the triple line.

Step 1: Proof of (5.73). By (5.49) and the choice of the orientation for the tangent fields $(\bar\tau, \bar\tau', \bar\tau'')$ along the triple line, cf. Construction 5.17, it holds

$$\bar\tau = J\bar{\mathrm n}, \quad \bar\tau' = J\bar{\mathrm n}', \quad \bar\tau'' = J\bar{\mathrm n}'' \quad \text{on } \bar\Gamma \tag{5.83}$$

in terms of a single 90° rotation field around the $\bar{\mathrm t}$-axis

$$J = (\bar\tau\wedge\bar{\mathrm n}) + \bar{\mathrm t}\otimes\bar{\mathrm t} = (\bar\tau'\wedge\bar{\mathrm n}') + \bar{\mathrm t}'\otimes\bar{\mathrm t}' = (\bar\tau''\wedge\bar{\mathrm n}'') + \bar{\mathrm t}''\otimes\bar{\mathrm t}'' \quad \text{on } \bar\Gamma. \tag{5.84}$$

Hence, it follows from (5.62) and the fact that the Herring rotation $R'_{\bar{\mathrm t}}$ is a rotation around the same axis

$$R'_{\bar{\mathrm t}}\bar\tau = R'_{\bar{\mathrm t}}J\bar{\mathrm n} = JR'_{\bar{\mathrm t}}\bar{\mathrm n} = J\bar{\mathrm n}' = \bar\tau' \quad \text{on } \bar\Gamma.$$

This proves the first asserted identity of (5.73); the second of course follows analogously based on (5.63).

Step 2: Proof of (5.74)–(5.76). Since the Herring rotation $R'_{\bar{\mathrm t}}$ defined by (5.60) is a rotation around the $\bar{\mathrm t}$-axis with constant angle, the coefficients in the representation $R'_{\bar{\mathrm t}}\bar{\mathrm n} = (R'_{\bar{\mathrm t}}\bar{\mathrm n}\cdot\bar{\mathrm n})\bar{\mathrm n} + (R'_{\bar{\mathrm t}}\bar{\mathrm n}\cdot\bar\tau)\bar\tau$ are constant. Hence we may compute along $\bar\Gamma$ together with the formulas (5.52) and (5.53)

$$\begin{aligned}(\bar{\mathrm t}\cdot\nabla)R'_{\bar{\mathrm t}}\bar{\mathrm n} &= (R'_{\bar{\mathrm t}}\bar{\mathrm n}\cdot\bar{\mathrm n})(\bar{\mathrm t}\cdot\nabla)\bar{\mathrm n} + (R'_{\bar{\mathrm t}}\bar{\mathrm n}\cdot\bar\tau)(\bar{\mathrm t}\cdot\nabla)\bar\tau \\ &= \big((R'_{\bar{\mathrm t}}\bar{\mathrm n}\cdot\bar\tau)(\nabla\cdot\bar\tau) - (R'_{\bar{\mathrm t}}\bar{\mathrm n}\cdot\bar{\mathrm n})\kappa_{\bar{\mathrm t}\bar{\mathrm t}}\big)\bar{\mathrm t} - (R'_{\bar{\mathrm t}}\bar{\mathrm n}\cdot\bar{\mathrm n})\kappa_{\bar\tau\bar{\mathrm t}}\bar\tau + (R'_{\bar{\mathrm t}}\bar{\mathrm n}\cdot\bar\tau)\kappa_{\bar\tau\bar{\mathrm t}}\bar{\mathrm n}.\end{aligned}$$

Furthermore, by the analogue of (5.53) for the tangent field $\bar{\tau}'$ as well as the identities (5.49) and (5.73), and again the fact that $R'_{\bar{\mathrm{t}}}$ and J commute, we obtain along the triple line $\bar{\Gamma}$

$$\begin{aligned}(\bar{\mathrm{t}}'\cdot\nabla)\bar{\mathrm{n}}' &= -\kappa'_{\bar{\mathrm{t}}'\bar{\mathrm{t}}'}\bar{\mathrm{t}}' - \kappa'_{\bar{\tau}'\bar{\mathrm{t}}'}\bar{\tau}' \\ &= -\kappa'_{\bar{\mathrm{t}}'\bar{\mathrm{t}}'}\bar{\mathrm{t}} - (R'_{\bar{\mathrm{t}}}\bar{\tau}\cdot\bar{\tau})\kappa'_{\bar{\tau}'\bar{\mathrm{t}}'}\bar{\tau} - (R'_{\bar{\mathrm{t}}}\bar{\tau}\cdot\bar{\mathrm{n}})\kappa'_{\bar{\tau}'\bar{\mathrm{t}}'}\bar{\mathrm{n}} \\ &= -\kappa'_{\bar{\mathrm{t}}'\bar{\mathrm{t}}'}\bar{\mathrm{t}} - (R'_{\bar{\mathrm{t}}}\bar{\mathrm{n}}\cdot\bar{\mathrm{n}})\kappa'_{\bar{\tau}'\bar{\mathrm{t}}'}\bar{\tau} + (R'_{\bar{\mathrm{t}}}\bar{\mathrm{n}}\cdot\bar{\tau})\kappa'_{\bar{\tau}'\bar{\mathrm{t}}'}\bar{\mathrm{n}}.\end{aligned}$$

Hence, the defining condition (5.62) of the Herring rotation $R'_{\bar{\mathrm{t}}}$ and matching coefficients in the previous two displays implies the first identity of (5.74) as well as (5.75) (note that of course, either $(R'_{\bar{\mathrm{t}}}\bar{\mathrm{n}}\cdot\bar{\mathrm{n}})$ or $(R'_{\bar{\mathrm{t}}}\bar{\mathrm{n}}\cdot\bar{\tau})$ is non-zero). The second identity of (5.74) as well as (5.76) in turn follow from an analogous computation based on (5.63).

Step 3: Proof of (5.77)–(5.78). These two compatibility conditions are derived as in the previous step, this time computing the tangential derivative along the triple line for both sides of the identities from (5.73), respectively.

Step 4: Proof of (5.79)–(5.80). By (5.13), the normal velocity $\mathrm{V}_{\bar{\Gamma}}$ of the triple line satisfies along $\bar{\Gamma}$

$$\mathrm{V}_{\bar{\Gamma}}\cdot\sigma\bar{\mathrm{n}} = \sigma H, \quad \mathrm{V}_{\bar{\Gamma}}\cdot\sigma'\bar{\mathrm{n}}' = \sigma' H', \quad \mathrm{V}_{\bar{\Gamma}}\cdot\sigma''\bar{\mathrm{n}}'' = \sigma'' H''. \tag{5.85}$$

Summing these identities results in (5.79) thanks to the Herring angle condition (5.10) being satisfied at each time.

To derive the compatibility condition (5.80), we differentiate the Herring angle condition and obtain

$$(\partial_t + \mathrm{V}_{\bar{\Gamma}}\cdot\nabla)\big(\sigma\bar{\mathrm{n}} + \sigma'\bar{\mathrm{n}}' + \sigma''\bar{\mathrm{n}}''\big) = 0.$$

Now we compute using (5.19) and and (5.29) for the first term and (5.52) for the second one

$$\partial_t\bar{\mathrm{n}} + (\mathrm{V}_{\bar{\Gamma}}\cdot\nabla)\bar{\mathrm{n}} = -(\bar{\mathrm{t}}\cdot\nabla H)\bar{\mathrm{t}} - (\bar{\tau}\cdot\nabla H)\bar{\tau} - (\mathrm{V}_{\bar{\Gamma}}\cdot\bar{\tau})(\kappa_{\bar{\tau}\bar{\tau}}\bar{\tau}+\kappa_{\bar{\tau}\bar{\mathrm{t}}}\bar{\mathrm{t}}) \tag{5.86}$$

on $\bar{\Gamma}$. The analogous equations hold for $\bar{\mathrm{n}}'$ and $\bar{\mathrm{n}}''$. Plugging those into (5.86), using (5.49) and (5.74), we obtain

$$\begin{aligned}0 = \big(\bar{\mathrm{t}}\cdot\nabla(\sigma H + \sigma' H' + \sigma'' H'')\big)\bar{\mathrm{t}} + (\bar{\tau}\cdot\nabla H)\sigma\bar{\tau} + (\bar{\tau}'\cdot\nabla H')\sigma'\bar{\tau}' + (\bar{\tau}''\cdot\nabla H'')\sigma''\bar{\tau}'' \\ + \kappa_{\bar{\tau}\bar{\tau}}(\mathrm{V}_{\bar{\Gamma}}\cdot\bar{\tau})\sigma\bar{\tau} + \kappa'_{\bar{\tau}'\bar{\tau}'}(\mathrm{V}_{\bar{\Gamma}}\cdot\bar{\tau}')\sigma'\bar{\tau}' + \kappa''_{\bar{\tau}''\bar{\tau}''}(\mathrm{V}_{\bar{\Gamma}}\cdot\bar{\tau}'')\sigma''\bar{\tau}'' \\ + \mathrm{V}_{\bar{\Gamma}}\cdot(\sigma\bar{\tau}+\sigma'\bar{\tau}'+\sigma''\bar{\tau}'')(\kappa_{\bar{\tau}\bar{\mathrm{t}}}\bar{\mathrm{t}})\end{aligned}$$

on $\bar{\Gamma}$. Differentiating (5.79) along $\bar{\Gamma}$, we see that the first term vanishes. The last term vanishes by applying the fixed rotation J to the Herring condition (5.10). Thus, since the three vectors $\bar{\tau}$, $\bar{\tau}'$, and $\bar{\tau}''$ lie in one plane, we deduce (5.80) from the previous display.

Step 5: Proof of (5.82). The requirement (5.82) is immediate from the definitions (5.65)–(5.70) in form of

$$R_{\bar{\mathrm{n}}} = \mathrm{Id} \tag{5.87}$$

along the triple line $\bar{\Gamma}$. □

With all of these ingredients in place, we may eventually move on with the construction of a local gradient-flow calibration at a triple line.

5.3.2 Extension of vector fields close to each interface

The aim of this section is to provide auxiliary extensions of the unit normal vector fields and an auxiliary extension of the normal velocity vector field which are defined in the neighborhood $\mathcal{N}_r(\bar{\Gamma}) \cap \operatorname{im}(\Psi_{i,j})$ for each interface $\bar{I}_{i,j}$, respectively. These extensions constitute the main building blocks for the desired extensions from Proposition 5.14.

Throughout this whole subsection, let the assumptions of Proposition 5.14 and the notation of Section 5.2 and Subsection 5.3.1 be in place. In particular, let us again make use of the following notational conventions which basically aim to drop the indices $i,j \in \{1,2,3\}$. We denote by $\bar{I} := \bar{I}_{1,2}, \bar{I}' := \bar{I}_{2,3}, \bar{I}'' := \bar{I}_{3,1}$ the three interfaces present in the given smoothly evolving regular double bubble $(\bar{\Omega}_1, \bar{\Omega}_2, \bar{\Omega}_3)$ on $[0,T]$. We proceed accordingly for the associated orthonormal frames $(\bar{\mathrm{n}}, \bar{\tau}, \bar{\mathrm{t}})$, $(\bar{\mathrm{n}}', \bar{\tau}', \bar{\mathrm{t}}')$ $(\bar{\mathrm{n}}'', \bar{\tau}'', \bar{\mathrm{t}}'')$ due to Construction 5.17, the surface tensions $(\sigma, \sigma', \sigma'')$, the signed distances (s, s', s''), the projections (P, P', P''), the scalar mean curvatures (H, H', H'') and the diffeomorphisms (Ψ, Ψ', Ψ'') from Definition 5.11.

Construction 5.21 (Extension of normal vector fields close to their associated interfaces)**.** Define a coefficient function $\alpha\colon \mathcal{N}_r(\bar{\Gamma}) \cap \operatorname{im}(\Psi) \to \mathbb{R}$ by

$$\alpha(x,t) := \alpha_{\mathrm{vel}}(x,t) + (\nabla \cdot \bar{\tau})(x,t), \quad (x,t) \in \mathcal{N}_r(\bar{\Gamma}) \cap \operatorname{im}(\Psi), \tag{5.88}$$

where $\alpha_{\mathrm{vel}}\colon \mathcal{N}_r(\bar{\Gamma}) \cap \operatorname{im}(\Psi) \to \mathbb{R}$ denotes, for the time being, an arbitrary coefficient function of class $C^0_t C^2_x(\mathcal{N}_r(\bar{\Gamma}) \cap \operatorname{im}(\Psi))$ such that along the triple line it holds

$$\alpha_{\mathrm{vel}}(x,t) = \bar{\tau}(x,t) \cdot \mathrm{V}_{\bar{\Gamma}}(x,t), \quad (x,t) \in \bar{\Gamma}. \tag{5.89}$$

Here, $\mathrm{V}_{\bar{\Gamma}}$ denotes again the normal velocity vector field of the triple line $\bar{\Gamma}$. Recall finally the definition (5.81) of the gauged orthonormal frame $(\bar{\mathrm{n}}, \bar{\tau}_*, \bar{\mathrm{t}}_*)$.

We then define an initial extension $\widetilde{\xi}\colon \mathcal{N}_r(\bar{\Gamma}) \cap \operatorname{im}(\Psi) \to \mathbb{R}^3$ for the normal vector field $\bar{\mathrm{n}}|_{\bar{I}}$ of the interface $\bar{I}$ by means of the *gauged expansion ansatz*

$$\begin{aligned} \widetilde{\xi}(x,t) := \bar{\mathrm{n}}(x,t) & \\ &+ \alpha(P_{\bar{\Gamma}}(x,t),t) s(x,t) \bar{\tau}_*(x,t) \\ &- \frac{1}{2}\alpha^2(P_{\bar{\Gamma}}(x,t),t) s^2(x,t) \bar{\mathrm{n}}(x,t) \end{aligned} \tag{5.90}$$

for all $(x,t) \in \mathcal{N}_r(\bar{\Gamma}) \cap \operatorname{im}(\Psi)$.

Analogously, one defines initial extensions $\widetilde{\xi}'\colon \mathcal{N}_r(\bar{\Gamma}) \cap \operatorname{im}(\Psi') \to \mathbb{R}^3$ as well as $\widetilde{\xi}''\colon \mathcal{N}_r(\bar{\Gamma}) \cap \operatorname{im}(\Psi'') \to \mathbb{R}^3$ of the normal vector fields $\bar{\mathrm{n}}'|_{\bar{I}'}$ and $\bar{\mathrm{n}}''|_{\bar{I}''}$. ◊

The following result shows that, after applying the correct gauged Herring rotation as provided by Construction 5.19, the initial extensions of our normal vector fields are regular and compatible to first order along the triple line $\bar{\Gamma}$.

Lemma 5.22. *Let $(\widetilde{\xi}, \widetilde{\xi}', \widetilde{\xi}'')$ be the initial extensions from Construction 5.21 of the normal vector fields $(\bar{\mathrm{n}}|_{\bar{I}}, \bar{\mathrm{n}}'|_{\bar{I}'}, \bar{\mathrm{n}}''|_{\bar{I}''})$. Moreover, let $(\widetilde{R}'_{\bar{I}}, \widetilde{R}''_{\bar{I}})$, $(\widetilde{R}_{\bar{I}'}, \widetilde{R}''_{\bar{I}'})$ and $(\widetilde{R}_{\bar{I}''}, \widetilde{R}'_{\bar{I}''})$ be the gauged Herring rotations as provided by Construction 5.19.*

Then it holds $(\widetilde{\xi}, \widetilde{R}'_{\bar{I}}\widetilde{\xi}, \widetilde{R}''_{\bar{I}}\widetilde{\xi}) \in (C^0_t C^2_x \cap C^1_t C^0_x)(\mathcal{N}_r(\bar{\Gamma}) \cap \operatorname{im}(\Psi))$ with corresponding estimates

$$|(\nabla, \nabla^2, \partial_t)(\widetilde{\xi}, \widetilde{R}'_{\bar{I}}\widetilde{\xi}, \widetilde{R}''_{\bar{I}}\widetilde{\xi})| \leq C \quad \text{in } \mathcal{N}_r(\bar{\Gamma}) \cap \operatorname{im}(\Psi), \tag{5.91}$$

where the constant $C > 0$ only depends on the data of the smoothly evolving regular double bubble $(\bar{\Omega}_1, \bar{\Omega}_2, \bar{\Omega}_3)$ on $[0,T]$. Moreover, the constructions are compatible to first order in the sense that along the triple line $\bar{\Gamma}$

$$\widetilde{R}'_{\bar{I}}\widetilde{\xi} = \widetilde{\xi}', \quad \widetilde{R}''_{\bar{I}}\widetilde{\xi} = \widetilde{\xi}'', \tag{5.92}$$

$$\nabla\big(\widetilde{R}'_{\bar{I}}\widetilde{\xi}\big) = \nabla\widetilde{\xi}', \quad \nabla\big(\widetilde{R}''_{\bar{I}}\widetilde{\xi}\big) = \nabla\widetilde{\xi}''. \tag{5.93}$$

Analogous claims are satisfied in terms of the vector fields $(\widetilde{R}_{\bar{I}'}\widetilde{\xi}', \widetilde{\xi}', \widetilde{R}''_{\bar{I}'}\widetilde{\xi}')$*, resp. the vector fields* $(\widetilde{R}_{\bar{I}''}\widetilde{\xi}'', \widetilde{R}'_{\bar{I}''}\widetilde{\xi}'', \widetilde{\xi}'')$*, throughout the region* $\mathcal{N}_r(\bar{\Gamma}) \cap \mathrm{im}(\Psi')$*, resp. the region* $\mathcal{N}_r(\bar{\Gamma}) \cap \mathrm{im}(\Psi'')$*.*

Proof. We split the proof into two steps.

Step 1: Regularity estimates. We first claim that for each $\mathcal{R} \in \{R'_{\bar{\mathrm{t}}}, R''_{\bar{\mathrm{t}}}, R_{\bar{\mathrm{n}}}\}$

$$|(\nabla, \nabla^2, \partial_t)\mathcal{R}| \leq C \quad \text{in } \mathcal{N}_r(\bar{\Gamma}) \cap \mathrm{im}(\Psi) \tag{5.94}$$

for some constant $C > 0$ which depends only on the data of the smoothly evolving regular double bubble $(\bar{\Omega}_1, \bar{\Omega}_2, \bar{\Omega}_3)$ on $[0, T]$, and that analogous estimates hold true for $\mathcal{R} \in \{R_{\bar{\mathrm{t}}'}, R''_{\bar{\mathrm{t}}'}, R_{\bar{\mathrm{n}}'}\}$ in $\mathcal{N}_r(\bar{\Gamma}) \cap \mathrm{im}(\Psi')$, or for $\mathcal{R} \in \{R_{\bar{\mathrm{t}}''}, R'_{\bar{\mathrm{t}}''}, R_{\bar{\mathrm{n}}''}\}$ in $\mathcal{N}_r(\bar{\Gamma}) \cap \mathrm{im}(\Psi'')$.

For a Herring rotation $\mathcal{R} \in \{R'_{\bar{\mathrm{t}}}, R''_{\bar{\mathrm{t}}}\}$, the claim (5.94) follows directly from the regularity of the frame $(\bar{\mathrm{n}}, \bar{\tau}, \bar{\mathrm{t}})$, see (5.21) and (5.51), since the associated angles θ', θ'' are independent of $(x, t) \in \mathcal{N}_r(\bar{\Gamma}) \cap \mathrm{im}(\Psi)$, see Construction 5.19. In terms of the gauge rotation $\mathcal{R} = R_{\bar{\mathrm{n}}}$, it suffices to show that

$$|(\nabla, \nabla^2, \partial_t)(\delta, \omega)| \leq C \quad \text{in } \mathcal{N}_r(\bar{\Gamma}) \cap \mathrm{im}(\Psi) \tag{5.95}$$

for the associated angles (δ, ω) defined in (5.68) and (5.69), respectively. For the angle δ, the regularity estimate from the previous display can be deduced from the regularity (5.21) of the normal $\bar{\mathrm{n}}$. The regularity estimate for the angle ω in turn follows from the regularity (5.18) of the projection onto the interface $\bar{I}$, the regularity (5.51) of the tangent vector fields $(\bar{\tau}, \bar{\mathrm{t}})$, and from explicitly integrating (in each time slice) the ODE (5.70) along the integral lines of the tangent vector field $\bar{\tau}$.

We next claim that there exist constants $c_1, c_2 \in (-1, 1)$ only depending on the surface tensions such that

$$\alpha_{\mathrm{vel}}(x, t) = (1 - c_1^2)^{-1} c_2 \big(H'(x, t) - c_1 H(x, t)\big) \tag{5.96}$$

for all $(x, t) \in \bar{\Gamma}$. For a proof of (5.96), we define $c_1 := \bar{\tau}(x, t) \cdot \bar{\tau}'(x, t)$ and $c_2 := \bar{\mathrm{n}}'(x, t) \cdot \bar{\tau}(x, t) = -\bar{\mathrm{n}}(x, t) \cdot \bar{\tau}'(x, t)$, and then simply observe from (5.85) and (5.89) that

$$\begin{aligned} \alpha_{\mathrm{vel}}(x, t) &= c_2 H'(x, t) + c_1 \alpha'_{\mathrm{vel}}(x, t), \\ \alpha'_{\mathrm{vel}}(P_{\bar{\Gamma}}(x, t), t) &= -c_2 H(x, t) + c_1 \alpha_{\mathrm{vel}}(x, t) \quad \text{on } \bar{\Gamma}. \end{aligned}$$

Inserting the second identity of the previous display into the first one then directly yields the claim (5.96).

The upshot of (5.94) and (5.96) is now the following. First, it follows from (5.88), the regularity of the projection onto the triple line $\bar{\Gamma}$ (cf. Definition 5.15 *i)*), the regularity (5.51) of the tangent $\bar{\tau}$, the representation (5.96) and finally the regularity (5.21) of the extended scalar mean curvatures that $\alpha_{\bar{\Gamma}}(x, t) := \alpha(P_{\bar{\Gamma}}(x, t), t)$ satisfies

$$|\alpha_{\bar{\Gamma}}| + |(\nabla, \nabla^2, \partial_t)\alpha_{\bar{\Gamma}}| \leq C \quad \text{in } \mathcal{N}_r(\bar{\Gamma}) \cap \mathrm{im}(\Psi).$$

The previous display in combination with (5.94) and the expansion ansatz (5.90) finally implies the asserted regularity estimate (5.91).

Step 2: First order compatibility along triple line. The zeroth order conditions (5.92) are immediate from the definitions (5.90) as well as the identities (5.62) and (5.63), respectively. For a proof of the first order condition, we focus on deriving the first identity of (5.93). The second follows along the same lines.

Recalling the definition (5.71) of the gauged Herring rotation and the gauged expansion ansatz (5.90) we compute on the interface $\bar{I}$ (abbreviating $\alpha_{\bar{\Gamma}}(\cdot,t) := \alpha(P_{\bar{\Gamma}}(\cdot,t),t)$ for $t \in [0,T]$)

$$\nabla\big(\widetilde{R}'_{\bar{I}}\,\widetilde{\xi}\,\big) = (\nabla R_{\bar{\mathrm{n}}})R'_{\bar{\mathrm{t}}}\bar{\mathrm{n}} + R_{\bar{\mathrm{n}}}\nabla\big(R'_{\bar{\mathrm{t}}}\bar{\mathrm{n}}\big) + \alpha_{\bar{\Gamma}}\big(R_{\bar{\mathrm{n}}}R'_{\bar{\mathrm{t}}}\bar{\tau}\big)\otimes\bar{\mathrm{n}}. \tag{5.97}$$

Let us now first compute $\nabla\big(R'_{\bar{\mathrm{t}}}\bar{\mathrm{n}}\big)$ and neglect the gauge rotations for a while. Recalling the fact that $R'_{\bar{\mathrm{t}}}$ is a rotation around the $\bar{\mathrm{t}}$-axis with constant angle, see (5.60), we obtain on the interface $\bar{I}$

$$\nabla\big(R'_{\bar{\mathrm{t}}}\bar{\mathrm{n}}\big) = \nabla\big((R'_{\bar{\mathrm{t}}}\bar{\mathrm{n}}\cdot\bar{\mathrm{n}})\bar{\mathrm{n}} + (R'_{\bar{\mathrm{t}}}\bar{\mathrm{n}}\cdot\bar{\tau})\bar{\tau}\big) = (R'_{\bar{\mathrm{t}}}\bar{\mathrm{n}}\cdot\bar{\mathrm{n}})\,\nabla\bar{\mathrm{n}} + (R'_{\bar{\mathrm{t}}}\bar{\mathrm{n}}\cdot\bar{\tau})\,\nabla\bar{\tau}.$$

Plugging in the identities (5.52) and (5.53), and using in a second step that $R'_{\bar{\mathrm{t}}}\bar{\mathrm{n}}\cdot\bar{\mathrm{n}} = R'_{\bar{\mathrm{t}}}\bar{\tau}\cdot\bar{\tau}$ as well as $R'_{\bar{\mathrm{t}}}\bar{\mathrm{n}}\cdot\bar{\tau} = -R'_{\bar{\mathrm{t}}}\bar{\tau}\cdot\bar{\mathrm{n}}$, we further compute

$$\begin{aligned}
\nabla\big(R'_{\bar{\mathrm{t}}}\bar{\mathrm{n}}\big) &= -\kappa_{\bar{\tau}\bar{\tau}}\big((R'_{\bar{\mathrm{t}}}\bar{\mathrm{n}}\cdot\bar{\mathrm{n}})\,\bar{\tau}\otimes\bar{\tau} - (R'_{\bar{\mathrm{t}}}\bar{\mathrm{n}}\cdot\bar{\tau})\,\bar{\mathrm{n}}\otimes\bar{\tau}\big)\\
&\quad - \big((R'_{\bar{\mathrm{t}}}\bar{\mathrm{n}}\cdot\bar{\mathrm{n}})\kappa_{\bar{\tau}\bar{\mathrm{t}}} + (R'_{\bar{\mathrm{t}}}\bar{\mathrm{n}}\cdot\bar{\tau})(\nabla\cdot\bar{\mathrm{t}})\big)\,\bar{\mathrm{t}}\otimes\bar{\tau}\\
&\quad - \kappa_{\bar{\tau}\bar{\mathrm{t}}}\big((R'_{\bar{\mathrm{t}}}\bar{\mathrm{n}}\cdot\bar{\mathrm{n}})\,\bar{\tau}\otimes\bar{\mathrm{t}} - (R'_{\bar{\mathrm{t}}}\bar{\mathrm{n}}\cdot\bar{\tau})\,\bar{\mathrm{n}}\otimes\bar{\mathrm{t}}\big)\\
&\quad - \big((R'_{\bar{\mathrm{t}}}\bar{\mathrm{n}}\cdot\bar{\mathrm{n}})\kappa_{\bar{\mathrm{t}}\bar{\mathrm{t}}} - (R'_{\bar{\mathrm{t}}}\bar{\mathrm{n}}\cdot\bar{\tau})(\nabla\cdot\bar{\tau})\big)\,\bar{\mathrm{t}}\otimes\bar{\mathrm{t}}\\
&= -\kappa_{\bar{\tau}\bar{\tau}}R'_{\bar{\mathrm{t}}}\bar{\tau}\otimes\bar{\tau}\\
&\quad - \big((R'_{\bar{\mathrm{t}}}\bar{\mathrm{n}}\cdot\bar{\mathrm{n}})\kappa_{\bar{\tau}\bar{\mathrm{t}}} + (R'_{\bar{\mathrm{t}}}\bar{\mathrm{n}}\cdot\bar{\tau})(\nabla\cdot\bar{\mathrm{t}})\big)\,\bar{\mathrm{t}}\otimes\bar{\tau}\\
&\quad - \kappa_{\bar{\tau}\bar{\mathrm{t}}}R'_{\bar{\mathrm{t}}}\bar{\tau}\otimes\bar{\mathrm{t}}\\
&\quad - \big((R'_{\bar{\mathrm{t}}}\bar{\mathrm{n}}\cdot\bar{\mathrm{n}})\kappa_{\bar{\mathrm{t}}\bar{\mathrm{t}}} - (R'_{\bar{\mathrm{t}}}\bar{\mathrm{n}}\cdot\bar{\tau})(\nabla\cdot\bar{\tau})\big)\,\bar{\mathrm{t}}\otimes\bar{\mathrm{t}},
\end{aligned} \tag{5.98}$$

which holds true on the interface $\bar{I}$.

Recalling the choice (5.88) for α, we may infer from the formula (5.98) for $\nabla\big(R'_{\bar{\mathrm{t}}}\bar{\mathrm{n}}\big)$, substituting $\kappa_{\bar{\tau}\bar{\tau}} = H - \kappa_{\bar{\mathrm{t}}\bar{\mathrm{t}}}$ along $\bar{I}$, the identity (5.87), and the formula (5.97) the following representation for the gradient of $\widetilde{R}'_{\bar{I}}\,\widetilde{\xi}$ along the triple line $\bar{\Gamma}$

$$\begin{aligned}
\nabla\big(\widetilde{R}'_{\bar{I}}\,\widetilde{\xi}\,\big) &= R'_{\bar{\mathrm{t}}}\bar{\tau}\otimes(-H\bar{\tau} + \alpha_{\mathrm{vel}}\bar{\mathrm{n}}) + R'_{\bar{\mathrm{t}}}\bar{\tau}\otimes(\kappa_{\bar{\mathrm{t}}\bar{\mathrm{t}}}\bar{\tau} + (\nabla\cdot\bar{\tau})\bar{\mathrm{n}})\\
&\quad + \big((\bar{\mathrm{t}}\cdot\nabla)\widetilde{R}'_{\bar{I}}\,\widetilde{\xi}\,\big)\otimes\bar{\mathrm{t}}\\
&\quad - \big((R'_{\bar{\mathrm{t}}}\bar{\mathrm{n}}\cdot\bar{\mathrm{n}})\kappa_{\bar{\tau}\bar{\mathrm{t}}} + (R'_{\bar{\mathrm{t}}}\bar{\mathrm{n}}\cdot\bar{\tau})(\nabla\cdot\bar{\mathrm{t}})\big)\,\bar{\mathrm{t}}\otimes\bar{\tau}\\
&\quad + (\nabla R_{\bar{\mathrm{n}}})R'_{\bar{\mathrm{t}}}\bar{\mathrm{n}}.
\end{aligned} \tag{5.99}$$

A direct computation based on the ansatz (5.90), the identities (5.52), (5.53), and (5.87), and substituting $\kappa'_{\bar{\tau}'\bar{\tau}'} = H' - \kappa'_{\bar{\mathrm{t}}'\bar{\mathrm{t}}'}$ also yields along the triple line $\bar{\Gamma}$

$$\begin{aligned}
\nabla\widetilde{\xi}' &= \bar{\tau}'\otimes(-H'\bar{\tau}' + \alpha'_{\mathrm{vel}}\bar{\mathrm{n}}') + \bar{\tau}'\otimes(\kappa'_{\bar{\mathrm{t}}'\bar{\mathrm{t}}'}\bar{\tau}' + (\nabla\cdot\bar{\tau}')\bar{\mathrm{n}}')\\
&\quad + (\bar{\mathrm{t}}'\cdot\nabla)\widetilde{\xi}'\otimes\bar{\mathrm{t}}'\\
&\quad - \kappa'_{\bar{\tau}'\bar{\mathrm{t}}'}\,\bar{\mathrm{t}}'\otimes\bar{\tau}'\\
&\quad + (\nabla R_{\bar{\mathrm{n}}'})\bar{\mathrm{n}}'.
\end{aligned} \tag{5.100}$$

We proceed by comparing the respective formulas (5.99) and (5.100). Recalling that we denoted by $\mathrm{V}_{\bar{\Gamma}}$ the normal velocity vector field of the triple line, we obtain from (5.85), the choice of α_{vel} (5.89), the identities (5.83) and (5.84), as well as the zeroth order compatibility (5.73) along the triple line that the first terms in (5.99) and (5.100) are identical:

$$R'_{\bar{\mathrm{t}}}\bar{\tau}\otimes(-H\bar{\tau} + \alpha_{\mathrm{vel}}\bar{\mathrm{n}}) = -\bar{\tau}'\otimes J\mathrm{V}_{\bar{\Gamma}} = \bar{\tau}'\otimes(-H'\bar{\tau}' + \alpha'_{\mathrm{vel}}\bar{\mathrm{n}}') \quad \text{along } \bar{\Gamma}.$$

Moreover, by the compatibility conditions (5.73), (5.75) and (5.77) along the triple line, as well as $R'_{\bar{\mathrm{t}}}\bar{\tau}\cdot\bar{\tau} = R'_{\bar{\mathrm{t}}}\bar{\mathrm{n}}\cdot\bar{\mathrm{n}}$ and $R'_{\bar{\mathrm{t}}}\bar{\tau}\cdot\bar{\mathrm{n}} = -R'_{\bar{\mathrm{t}}}\bar{\mathrm{n}}\cdot\bar{\tau}$, we may infer that the second terms agree, too:

$$R'_{\bar{\mathrm{t}}}\bar{\tau}\otimes(\kappa_{\bar{\mathrm{t}}\bar{\mathrm{t}}}\bar{\tau}+(\nabla\cdot\bar{\tau})\bar{\mathrm{n}}) = \bar{\tau}'\otimes(\kappa'_{\bar{\mathrm{t}}'\bar{\mathrm{t}}'}\bar{\tau}'+(\nabla\cdot\bar{\tau}')\bar{\mathrm{n}}') \quad \text{along } \bar{\Gamma}.$$

From the last two identities together with (5.99), (5.100), (5.92), and (5.49) we therefore obtain along the triple line $\bar{\Gamma}$

$$\begin{aligned}\nabla\big(\widetilde{R}_{\bar{I}}\,\widetilde{\xi}\big) - \nabla\widetilde{\xi}' = -\big((R'_{\bar{\mathrm{t}}}\bar{\mathrm{n}}\cdot\bar{\mathrm{n}})\kappa_{\bar{\tau}\bar{\mathrm{t}}} + (R'_{\bar{\mathrm{t}}}\bar{\mathrm{n}}\cdot\bar{\tau})(\nabla\cdot\bar{\mathrm{t}})\big)\,\bar{\mathrm{t}}\otimes\bar{\tau} + \kappa'_{\bar{\tau}'\bar{\mathrm{t}}'}\,\bar{\mathrm{t}}\otimes\bar{\tau}' \\ + (\nabla R_{\bar{\mathrm{n}}})R'_{\bar{\mathrm{t}}}\bar{\mathrm{n}} - (\nabla R_{\bar{\mathrm{n}}'})\bar{\mathrm{n}}'.\end{aligned} \tag{5.101}$$

In the rotationally symmetric case, the right hand side terms in the first line of (5.101) actually vanish. However, there is no reason in general why these terms should vanish without assuming additional symmetry. This is the motivation for the introduction of the additional gauge rotation matrices around the normal axis. Their definition is arranged in such a way so that their contribution in (5.101) exactly cancels the right hand side terms of the first line.

First, we obtain from the definitions (5.65)–(5.70) along the triple line

$$(\nabla R_{\bar{\mathrm{n}}})R'_{\bar{\mathrm{t}}}\bar{\mathrm{n}} = \big((\bar{\tau}\cdot\nabla)R^{(2)}_{\bar{\mathrm{n}}}\big)R'_{\bar{\mathrm{t}}}\bar{\mathrm{n}}\otimes\bar{\tau} + \big((\bar{\mathrm{n}}\cdot\nabla)R^{(1)}_{\bar{\mathrm{n}}}\big)R'_{\bar{\mathrm{t}}}\bar{\mathrm{n}}\otimes\bar{\mathrm{n}}. \tag{5.102}$$

Let us next compute the two relevant directional derivatives of the gauge rotation matrices. We first observe that due to (5.66) and (5.68)

$$(\bar{\mathrm{n}}\cdot\nabla)R^{(1)}_{\bar{\mathrm{n}}} = \kappa_{\bar{\tau}\bar{\mathrm{t}}}\,\bar{\mathrm{t}}\wedge\bar{\tau} \tag{5.103}$$

along the interface $\bar{I}$. This in turn entails by $R'_{\bar{\mathrm{t}}}\bar{\tau}\cdot\bar{\mathrm{n}} = -R'_{\bar{\mathrm{t}}}\bar{\mathrm{n}}\cdot\bar{\tau}$

$$\big((\bar{\mathrm{n}}\cdot\nabla)R^{(1)}_{\bar{\mathrm{n}}}\big)R'_{\bar{\mathrm{t}}}\bar{\mathrm{n}}\otimes\bar{\mathrm{n}} = -\kappa_{\bar{\tau}\bar{\mathrm{t}}}(R'_{\bar{\mathrm{t}}}\bar{\tau}\cdot\bar{\mathrm{n}})\,\bar{\mathrm{t}}\otimes\bar{\mathrm{n}} \quad \text{along } \bar{\Gamma}. \tag{5.104}$$

Moreover, we may compute based on (5.67), (5.69), and (5.70) on the triple line $\bar{\Gamma}$

$$(\bar{\tau}\cdot\nabla)R^{(2)}_{\bar{\mathrm{n}}} = (\nabla\cdot\bar{\mathrm{t}})\,\bar{\mathrm{t}}\wedge\bar{\tau}, \tag{5.105}$$

from which we deduce

$$\big((\bar{\tau}\cdot\nabla)R^{(2)}_{\bar{\mathrm{n}}}\big)R'_{\bar{\mathrm{t}}}\bar{\mathrm{n}}\otimes\bar{\tau} = \big((R'_{\bar{\mathrm{t}}}\bar{\mathrm{n}}\cdot\bar{\tau})(\nabla\cdot\bar{\mathrm{t}})\big)\,\bar{\mathrm{t}}\otimes\bar{\tau} \quad \text{along } \bar{\Gamma}. \tag{5.106}$$

A straightforward computation shows that along the triple line $\bar{\Gamma}$ it holds

$$(\nabla R_{\bar{\mathrm{n}}'})\bar{\mathrm{n}}' = \nabla\big(R_{\bar{\mathrm{n}}'}\bar{\mathrm{n}}'\big) - R_{\bar{\mathrm{n}}'}\nabla\bar{\mathrm{n}}' = \nabla\bar{\mathrm{n}}' - \nabla\bar{\mathrm{n}}' = 0. \tag{5.107}$$

Combining (5.102), (5.104), (5.106), and (5.107) with the compatibility conditions (5.73) and (5.74) finally yields the desired cancellation

$$-\big((R'_{\bar{\mathrm{t}}}\bar{\mathrm{n}}\cdot\bar{\mathrm{n}})\kappa_{\bar{\tau}\bar{\mathrm{t}}} + (R'_{\bar{\mathrm{t}}}\bar{\mathrm{n}}\cdot\bar{\tau})(\nabla\cdot\bar{\mathrm{t}})\big)\,\bar{\mathrm{t}}\otimes\bar{\tau} + \kappa'_{\bar{\tau}'\bar{\mathrm{t}}'}\,\bar{\mathrm{t}}\otimes\bar{\tau}' + (\nabla R_{\bar{\mathrm{n}}})R'_{\bar{\mathrm{t}}}\bar{\mathrm{n}} - (\nabla R_{\bar{\mathrm{n}}'})\bar{\mathrm{n}}' = 0$$

along the triple line $\bar{\Gamma}$. By (5.101), this in turn concludes the proof of Lemma 5.22. □

We proceed with the construction of suitable candidate velocity fields.

Construction 5.23 (Extension of velocity fields close to their associated interfaces)**.** Recall that $\mathrm{V}_{\bar{\Gamma}}$ denotes the normal velocity of the triple line $\bar{\Gamma}$, and recall the definition (5.81) of the gauged orthonormal frame $(\bar{\mathrm{n}}, \bar{\tau}_*, \bar{\mathrm{t}}_*)$. We then define a coefficient function

$$\alpha_{\mathrm{vel}}\colon \mathcal{N}_r(\bar{\Gamma})\cap\mathrm{im}(\Psi)\to\mathbb{R}, \quad (x,t)\mapsto\widehat{\alpha}_{\mathrm{vel}}(P(x,t),t), \tag{5.108}$$

where the coefficient α_{vel} is defined by projection onto the interface $\bar{I}$ in terms of the solution of the following family of ODEs, solved along the integral lines of the tangent vector field $\bar{\tau}_*$ with initial condition posed on the triple line $\bar{\Gamma}$

$$\begin{cases} \widehat{\alpha}_{\mathrm{vel}}(x,t) & = (\bar{\tau}_* \cdot \mathrm{V}_{\bar{\Gamma}})(x,t), \quad (x,t) \in \bar{\Gamma}, \\ (\bar{\tau}_* \cdot \nabla)\, \widehat{\alpha}_{\mathrm{vel}}(x,t) & = (H\kappa_{\bar{\tau}_*\bar{\tau}_*})(x,t), \quad (x,t) \in \bar{I} \cap \mathcal{N}_r(\bar{\Gamma}). \end{cases} \tag{5.109}$$

Note that the choice of the initial value in (5.109) is consistent with (5.89). Next, we define another coefficient function

$$\beta \colon \mathcal{N}_r(\bar{\Gamma}) \cap \mathrm{im}(\Psi) \to \mathbb{R}, \quad (x,t) \mapsto -\big((\bar{\tau}_* \cdot \nabla)H\big)(x,t) - (\alpha_{\mathrm{vel}}\kappa_{\bar{\tau}_*\bar{\tau}_*})(x,t). \tag{5.110}$$

We now define a preliminary extension $\widetilde{B} \colon \mathcal{N}_r(\bar{\Gamma}) \cap \mathrm{im}(\Psi) \to \mathbb{R}^3$ of the normal velocity vector field $(H\bar{\mathrm{n}})|_{\bar{I}}$ for the interface $\bar{I}$ in terms of the *gauged expansion ansatz*

$$\begin{aligned} \widetilde{B}(x,t) := {} & H(x,t)\,\bar{\mathrm{n}}(x,t) \\ & + \alpha_{\mathrm{vel}}(x,t)\,\bar{\tau}_*(x,t) \\ & + \beta(x,t)s(x,t)\,\bar{\tau}_*(x,t) \end{aligned} \tag{5.111}$$

for all $(x,t) \in \mathcal{N}_r(\bar{\Gamma}) \cap \mathrm{im}(\Psi)$.

Analogously, one defines preliminary extensions $\widetilde{B}' \colon \mathcal{N}_r(\bar{\Gamma}) \cap \mathrm{im}(\Psi') \to \mathbb{R}^3$ as well as $\widetilde{B}'' \colon \mathcal{N}_r(\bar{\Gamma}) \cap \mathrm{im}(\Psi'') \to \mathbb{R}^3$ of the normal velocity vector fields $(H'\bar{\mathrm{n}}')|_{\bar{I}'}$ and $(H''\bar{\mathrm{n}}'')|_{\bar{I}''}$, respectively. $\diamondsuit$

Note carefully that even away from the triple line we do not introduce a tangential velocity in $\bar{\mathrm{t}}_*$-direction. As the proof of the following result shows, this will entail that the gradients of the auxiliary velocities $\widetilde{B}$, $\widetilde{B}'$ and $\widetilde{B}''$ do not fully match along the triple line. However, the only mismatch appears in, at least for our purposes, inessential components. More precisely, in terms of, say, $\nabla\widetilde{B}$ the only non-matching terms result from its $\bar{\mathrm{t}}_* \otimes \bar{\tau}_*$ resp. $\bar{\mathrm{t}}_* \otimes \bar{\mathrm{n}}$ component. In view of the desired evolution equation (5.1d) and the fact that $\widetilde{\xi} \perp \bar{\mathrm{t}}_*$ due to (5.90), this specific component of $\nabla\widetilde{B}$ is intrinsically irrelevant for a gradient-flow calibration (this argument turns out to be robust even with respect to the interpolation construction from Subsection 5.3.3).

Lemma 5.24. *Let $(\widetilde{B}, \widetilde{B}', \widetilde{B}'')$ be the preliminary extensions from Construction 5.23 of the normal velocity vector fields $((H\bar{\mathrm{n}})|_{\bar{I}}, (H'\bar{\mathrm{n}}')|_{\bar{I}'}, (H''\bar{\mathrm{n}}'')|_{\bar{I}''})$.*

Then it holds $\widetilde{B} \in C^0_tC^2_x(\mathcal{N}_r(\bar{\Gamma}) \cap \mathrm{im}(\Psi))$ with corresponding estimate

$$|\widetilde{B}| + |\nabla\widetilde{B}| + |\nabla^2\widetilde{B}| \leq C \quad \text{in } \mathcal{N}_r(\bar{\Gamma}) \cap \mathrm{im}(\Psi), \tag{5.112}$$

where the constant $C > 0$ only depends on the data of the smoothly evolving regular double bubble $(\bar{\Omega}_1, \bar{\Omega}_2, \bar{\Omega}_3)$ on $[0,T]$. Analogous claims hold true for $\widetilde{B}'$ resp. $\widetilde{B}''$ throughout $\mathcal{N}_r(\bar{\Gamma}) \cap \mathrm{im}(\Psi')$ resp. $\mathcal{N}_r(\bar{\Gamma}) \cap \mathrm{im}(\Psi'')$.

Moreover, the constructions are essentially compatible to first order in the sense that along the triple line $\bar{\Gamma}$ it holds

$$\widetilde{B} = \widetilde{B}' = \widetilde{B}'' = \mathrm{V}_{\bar{\Gamma}}, \tag{5.113}$$

$$(\mathrm{Id} - \bar{\mathrm{t}} \otimes \bar{\mathrm{t}})(\nabla\widetilde{B}) = (\mathrm{Id} - \bar{\mathrm{t}}' \otimes \bar{\mathrm{t}}')(\nabla\widetilde{B}') = (\mathrm{Id} - \bar{\mathrm{t}}'' \otimes \bar{\mathrm{t}}'')(\nabla\widetilde{B}''), \tag{5.114}$$

for which one should also recall that $\bar{\mathrm{t}} = \bar{\mathrm{t}}' = \bar{\mathrm{t}}''$ along $\bar{\Gamma}$, cf. (5.49).

Note that here the projection $\mathrm{Id} - \bar{\mathrm{t}} \otimes \bar{\mathrm{t}}$ acts on the components of $\widetilde{B}$, not ∇.

Proof. Step 1: Regularity estimates. Due to the definition (5.110), the regularity estimates (5.94) for the gauge rotations, the regularity of the frame $(\bar{\mathrm{n}}, \bar{\tau}, \bar{\mathrm{t}})$, see (5.21) and (5.51), the regularity (5.21) of the extended scalar mean curvatures, and finally the expansion ansatz (5.111) it suffices to prove that

$$|\alpha_{\mathrm{vel}}| + |(\nabla, \nabla^2)\alpha_{\mathrm{vel}}| \leq C \quad \text{in } \mathcal{N}_r(\bar{\Gamma}) \cap \mathrm{im}(\Psi), \tag{5.115}$$

where $C > 0$ is a constant which depends only on the data of the smoothly evolving regular double bubble $(\bar{\Omega}_1, \bar{\Omega}_2, \bar{\Omega}_3)$ on $[0, T]$.

The estimate (5.115) in turn follows directly from explicitly integrating (in each time slice) the ODEs (5.109) along the integral lines of the tangent field $\bar{\tau}_*$, and exploiting as before the regularity of the associated geometric quantities.

Step 2: Zeroth order compatibility at triple line. The condition (5.113) is immediate from the definition (5.111), the identities (5.85), as well as the specific choices (5.108)–(5.109).

Step 3: First order compatibility at triple line. We proceed with the proof of (5.114). Observe that we have on the interface $\bar{I}$ by direct analogy to the proofs of (5.52) and (5.53) that

$$\nabla\bar{\mathrm{n}} = -\kappa_{\bar{\tau}_*\bar{\tau}_*}\,\bar{\tau}_* \otimes \bar{\tau}_* - \kappa_{\bar{\mathrm{t}}_*\bar{\mathrm{t}}_*}\,\bar{\mathrm{t}}_* \otimes \bar{\mathrm{t}}_* - \kappa_{\bar{\tau}_*\bar{\mathrm{t}}_*}\,(\bar{\mathrm{t}}_* \otimes \bar{\tau}_* + \bar{\tau}_* \otimes \bar{\mathrm{t}}_*), \tag{5.116}$$

$$\begin{aligned}\nabla\bar{\tau}_* = {} & \kappa_{\bar{\tau}_*\bar{\tau}_*}\,\bar{\mathrm{n}} \otimes \bar{\tau}_* - (\nabla \cdot \bar{\mathrm{t}}_*)\,\bar{\mathrm{t}}_* \otimes \bar{\tau}_* + \kappa_{\bar{\tau}_*\bar{\mathrm{t}}_*}\,\bar{\mathrm{n}} \otimes \bar{\mathrm{t}}_* + (\nabla \cdot \bar{\tau}_*)\,\bar{\mathrm{t}}_* \otimes \bar{\mathrm{t}}_* \\ & + (\bar{\mathrm{n}} \cdot \nabla)\bar{\tau}_* \otimes \bar{\mathrm{n}}.\end{aligned} \tag{5.117}$$

It follows directly from the definitions (5.50) resp. (5.81) of our orthonormal frames, the definitions (5.65)–(5.70) of the gauge rotations, as well as the formula (5.103) being valid along the interface $\bar{I}$ that

$$(\bar{\mathrm{n}} \cdot \nabla)\bar{\tau}_* = R^{(2)}_{\bar{\mathrm{n}}}\big((\bar{\mathrm{n}} \cdot \nabla)R^{(1)}_{\bar{\mathrm{n}}}\big)\bar{\tau} = \kappa_{\bar{\tau}\bar{\mathrm{t}}}R_{\bar{\mathrm{n}}}\bar{\mathrm{t}} = \kappa_{\bar{\tau}\bar{\mathrm{t}}}\bar{\mathrm{t}}_* \quad \text{along } \bar{I}.$$

Starting now from the definition (5.111), the previous display, the choices (5.108)–(5.110) of the coefficient functions, as well as the formulas (5.116) and (5.117) directly entail along the interface $\bar{I}$

$$\begin{aligned}\nabla\widetilde{B} &= \beta\,\bar{\tau}_* \otimes \bar{\mathrm{n}} + \big((\bar{\tau}_* \cdot \nabla)H + \widehat{\alpha}_{\mathrm{vel}}\kappa_{\bar{\tau}_*\bar{\tau}_*}\big)\,\bar{\mathrm{n}} \otimes \bar{\tau}_* \\ &\quad + \big((\bar{\tau}_* \cdot \nabla)\widehat{\alpha}_{\mathrm{vel}} - H\kappa_{\bar{\tau}_*\bar{\tau}_*}\big)\,\bar{\tau}_* \otimes \bar{\tau}_* \\ &\quad + (\bar{\mathrm{t}}_* \cdot \nabla)\widetilde{B} \otimes \bar{\mathrm{t}}_* \\ &\quad - \big(H\kappa_{\bar{\tau}_*\bar{\mathrm{t}}_*} + \widehat{\alpha}_{\mathrm{vel}}(\nabla \cdot \bar{\mathrm{t}}_*)\big)\,\bar{\mathrm{t}}_* \otimes \bar{\tau}_* + \kappa_{\bar{\tau}\bar{\mathrm{t}}}\bar{\mathrm{t}}_* \otimes \bar{\mathrm{n}} \\ &= \beta\,\bar{\tau}_* \wedge \bar{\mathrm{n}} + (\bar{\mathrm{t}}_* \cdot \nabla)\widetilde{B} \otimes \bar{\mathrm{t}}_* \\ &\quad - \big(H\kappa_{\bar{\tau}_*\bar{\mathrm{t}}_*} + \widehat{\alpha}_{\mathrm{vel}}(\nabla \cdot \bar{\mathrm{t}}_*)\big)\,\bar{\mathrm{t}}_* \otimes \bar{\tau}_* + \kappa_{\bar{\tau}\bar{\mathrm{t}}}\bar{\mathrm{t}}_* \otimes \bar{\mathrm{n}}.\end{aligned} \tag{5.118}$$

Hence, the already established zeroth order condition (5.113) together with the compatibility conditions (5.80) and (5.82) in form of $\beta = \beta' = \beta''$ along $\bar{\Gamma}$ imply (5.114). ☐

The following result provides the approximate evolution equations for our auxiliary constructions $(\widetilde{\xi}, \widetilde{R}'_{\bar{I}}\widetilde{\xi}, \widetilde{R}''_{\bar{I}}\widetilde{\xi})$ in terms of the associated auxiliary velocity $\widetilde{B}$, which will eventually lead us to (5.1d)–(5.1f).

Lemma 5.25. *Let $(\widetilde{\xi}, \widetilde{\xi}', \widetilde{\xi}'')$ be the initial extensions from Construction 5.21 of the normal vector fields $(\bar{\mathrm{n}}|_{\bar{I}}, \bar{\mathrm{n}}'|_{\bar{I}'}, \bar{\mathrm{n}}''|_{\bar{I}''})$. Moreover, let $(\widetilde{R}'_{\bar{I}}, \widetilde{R}''_{\bar{I}})$, $(\widetilde{R}_{\bar{I}'}, \widetilde{R}''_{\bar{I}'})$ and $(\widetilde{R}_{\bar{I}''}, \widetilde{R}'_{\bar{I}''})$ be the gauged Herring rotations as provided by Construction 5.19, respectively. Finally, let $(\widetilde{B}, \widetilde{B}', \widetilde{B}'')$ be the initial extensions from Construction 5.23 of the corresponding normal velocity vector fields $((H\bar{\mathrm{n}})|_{\bar{I}}, (H'\bar{\mathrm{n}}')|_{\bar{I}'}, (H''\bar{\mathrm{n}}'')|_{\bar{I}''})$.*

Then there exists a constant $C > 0$, which depends only on the data of the smoothly evolving regular double bubble $(\bar\Omega_1, \bar\Omega_2, \bar\Omega_3)$ on $[0,T]$, such that for each rotation $\mathcal{R} \in \{\mathrm{Id}, \widetilde{R}'_{\bar I}, \widetilde{R}''_{\bar I}\}$ it holds

$$\big|1 - |\mathcal{R}\widetilde{\xi}|^2\big| \leq C \operatorname{dist}^4(\cdot, \bar I), \tag{5.119}$$

$$\big|\nabla |\mathcal{R}\widetilde{\xi}|^2\big| \leq C \operatorname{dist}^3(\cdot, \bar I), \tag{5.120}$$

$$\big|\partial_t |\mathcal{R}\widetilde{\xi}|^2\big| \leq C \operatorname{dist}^3(\cdot, \bar I), \tag{5.121}$$

$$\big|\partial_t \mathcal{R}\widetilde{\xi} + (\widetilde{B}\cdot\nabla)\mathcal{R}\widetilde{\xi} + (\nabla\widetilde{B})^{\mathsf{T}}\mathcal{R}\widetilde{\xi}\,\big| \leq C \begin{cases} \operatorname{dist}(\cdot,\bar I) & \textit{if } \mathcal{R} = \mathrm{Id}, \\ \operatorname{dist}(\cdot,\bar\Gamma) & \textit{else}, \end{cases} \tag{5.122}$$

$$\big|\nabla\cdot\mathcal{R}\widetilde{\xi} + \widetilde{B}\cdot\mathcal{R}\widetilde{\xi}\,\big| \leq C \begin{cases} \operatorname{dist}(\cdot,\bar I) & \textit{if } \mathcal{R} = \mathrm{Id}, \\ \operatorname{dist}(\cdot,\bar\Gamma) & \textit{else}, \end{cases} \tag{5.123}$$

throughout the domain $\mathcal{N}_r(\bar\Gamma) \cap \operatorname{im}(\Psi)$.

Analogous estimates hold true throughout the domain $\mathcal{N}_r(\bar\Gamma)\cap\operatorname{im}(\Psi')$ in terms of the vector fields $(\mathcal{R}\widetilde{\xi}', \widetilde{B}')$ for each rotation $\mathcal{R} \in \{\widetilde{R}_{\bar I'}, \mathrm{Id}, \widetilde{R}''_{\bar I'}\}$, as well as throughout $\mathcal{N}_r(\bar\Gamma)\cap\operatorname{im}(\Psi'')$ in terms of $(\mathcal{R}\widetilde{\xi}'', \widetilde{B}'')$ for each $\mathcal{R} \in \{\widetilde{R}_{\bar I''}, \widetilde{R}'_{\bar I''}, \mathrm{Id}\}$.

Proof. Fix a rotation $\mathcal{R} \in \{\mathrm{Id}, \widetilde{R}'_{\bar I}, \widetilde{R}''_{\bar I}\}$, and for the purposes of the proof abbreviate $\alpha_{\bar\Gamma}(\cdot,t) := \alpha(P_{\bar\Gamma}(\cdot,t),t)$, $t \in [0,T]$.

Step 1: Proof of (5.119)–(5.121). It follows immediately from the ansatz (5.90) and the orthogonality $\bar\tau_* \cdot \bar{\mathrm{n}} = 0$ that

$$|\mathcal{R}\widetilde{\xi}|^2 = |\widetilde{\xi}|^2 = \Big(1 - \frac{1}{2}\alpha_{\bar\Gamma}^2 s^2\Big)^2 + \alpha_{\bar\Gamma}^2 s^2 = 1 + \frac{1}{4}\alpha_{\bar\Gamma}^4 s^4.$$

The previous display of course immediately implies the estimates (5.119)–(5.121).

Step 2: Proof of (5.123). By the regularity estimates (5.91) and (5.112), it suffices to show that (5.123) is exact *on* the interface $\bar I$ if $\mathcal{R} = \mathrm{Id}$, or otherwise that (5.123) is exact *on* the triple line $\bar\Gamma$. To this end, let us first assume that $\mathcal{R}_{\bar{\mathrm{t}}} = \mathrm{Id}$. Then we also have $\mathrm{R} = \mathrm{Id}$ and hence we may directly infer from the definitions (5.90) and (5.111) of $\widetilde{\xi}$ and $\widetilde{B}$, respectively, that $\nabla\cdot\widetilde{\xi} = H = \widetilde{\xi}\cdot\widetilde{B}$ on the interface $\bar I$. In the remaining cases, we express $\mathcal{R} = R_{\bar{\mathrm{n}}}\mathcal{R}_{\bar{\mathrm{t}}}R_{\bar{\mathrm{n}}}^{\mathsf{T}}$ in terms of the associated Herring rotation $\mathcal{R}_{\bar{\mathrm{t}}} \in \{R'_{\bar{\mathrm{t}}}, R''_{\bar{\mathrm{t}}}\}$, and then simply read off from (5.97), (5.98), (5.102), (5.104) and (5.106) that

$$\nabla\cdot\mathcal{R}\widetilde{\xi} = -H(\mathcal{R}_{\bar{\mathrm{t}}}\bar{\mathrm{n}}\cdot\bar{\mathrm{n}}) + (\nabla\cdot\bar\tau)(\mathcal{R}_{\bar{\mathrm{t}}}\bar{\mathrm{n}}\cdot\bar\tau) - \alpha_{\bar\Gamma}(\mathcal{R}_{\bar{\mathrm{t}}}\bar{\mathrm{n}}\cdot\bar\tau)$$

along the triple line $\bar\Gamma$. Moreover, the definitions (5.90) and (5.111) directly imply that

$$\widetilde{B}\cdot\mathcal{R}\widetilde{\xi} = H(\mathcal{R}_{\bar{\mathrm{t}}}\bar{\mathrm{n}}\cdot\bar{\mathrm{n}}) + \alpha_{\mathrm{vel}}(\mathcal{R}_{\bar{\mathrm{t}}}\bar{\mathrm{n}}\cdot\bar\tau)$$

holds true on the interface $\bar I$. Hence, the estimate (5.123) follows from the previous two displays in combination with the choice (5.88).

Step 3: Proof of (5.122). It suffices again to check that (5.122) is exact on the interface $\bar I$ if $\mathcal{R} = \mathrm{Id}$, or otherwise that (5.122) is exact on the triple line $\bar\Gamma$. Let us also again express $\mathcal{R} = R_{\bar{\mathrm{n}}}\mathcal{R}_{\bar{\mathrm{t}}}R_{\bar{\mathrm{n}}}^{\mathsf{T}}$ in terms of the associated Herring rotation $\mathcal{R}_{\bar{\mathrm{t}}} \in \{\mathrm{Id}, R'_{\bar{\mathrm{t}}}, R''_{\bar{\mathrm{t}}}\}$.

Using that the vector field $\mathcal{R}\bar{\mathrm{n}} = R_{\bar{\mathrm{n}}}\mathcal{R}_{\bar{\mathrm{t}}}\bar{\mathrm{n}}$ lies in the $(\bar{\mathrm{n}}, R_{\bar{\mathrm{n}}}\bar\tau)$-plane and has constant coefficients in this frame, we compute along the interface $\bar I$ relying also on (5.90)

$$\begin{aligned} &\partial_t\mathcal{R}\widetilde{\xi} + (\widetilde{B}\cdot\nabla)\mathcal{R}\widetilde{\xi} + (\nabla\widetilde{B})^{\mathsf{T}}\mathcal{R}\widetilde{\xi} \\ &= (R_{\bar{\mathrm{n}}}\mathcal{R}_{\bar{\mathrm{t}}}\bar{\mathrm{n}}\cdot\bar{\mathrm{n}})\big(\partial_t\bar{\mathrm{n}} + (\widetilde{B}\cdot\nabla)\bar{\mathrm{n}} + (\nabla\widetilde{B})^{\mathsf{T}}\bar{\mathrm{n}}\big) \\ &\quad + (R_{\bar{\mathrm{n}}}\mathcal{R}_{\bar{\mathrm{t}}}\bar{\mathrm{n}}\cdot R_{\bar{\mathrm{n}}}\bar\tau)\big(\partial_t\bar\tau_* + (\widetilde{B}\cdot\nabla)\bar\tau_* + (\nabla\widetilde{B})^{\mathsf{T}}\bar\tau_*\big) \\ &\quad + \alpha_{\bar\Gamma}\big(\partial_t s + (\widetilde{B}\cdot\nabla)s\big)\,\bar\tau_*. \end{aligned} \tag{5.124}$$

The last right hand side term of (5.124) vanishes due to $(\widetilde{B}\cdot\nabla)s = H$ and (5.29). Differentiating this equation in space yields because of $\nabla s = \bar{\mathrm{n}}$

$$0 = \nabla\big(\partial_t s + (\widetilde{B}\cdot\nabla)s\big) = \partial_t\bar{\mathrm{n}} + (\widetilde{B}\cdot\nabla)\bar{\mathrm{n}} + (\nabla\widetilde{B})^{\mathsf{T}}\bar{\mathrm{n}}.$$

Hence, also the first right hand side term of (5.124) vanishes. Since $\mathcal{R}_{\bar{\mathrm{t}}} = \mathrm{Id}$ if and only if $\mathcal{R} = \mathrm{Id}$, the estimate (5.122) already follows from these arguments in the case $\mathcal{R} = \mathrm{Id}$. Hence, let us restrict to the case $\mathcal{R} \neq \mathrm{Id}$ in the following. Recall from the claim (5.122) that it then suffices to estimate in terms of the distance to the triple line.

It follows from $|\bar{\tau}_*| = 1$ that $\bar{\tau}_*\cdot\big(\partial_t\bar{\tau}_* + (\widetilde{B}\cdot\nabla)\bar{\tau}_*\big) = 0$. Furthermore, the ansatz for the velocity field $\widetilde{B}$ is arranged such that $\bar{\tau}_*\otimes\bar{\tau}_* : \nabla\widetilde{B} = 0$; cf. the identity (5.118). Hence, in the evolution equation for the tangent vector field $\bar{\tau}_*$ we may neglect the $\bar{\tau}_*$-component. The $\bar{\mathrm{n}}$-component also vanishes as a consequence of the orthogonality $\bar{\tau}_*\cdot\bar{\mathrm{n}} = 0$, the skew-symmetry $\bar{\tau}_*\otimes\bar{\mathrm{n}} : \nabla\widetilde{B} = -\bar{\mathrm{n}}\otimes\bar{\tau}_* : \nabla\widetilde{B}$, cf. again (5.118), and the already established evolution equation for the unit normal vector field $\bar{\mathrm{n}}$

$$\bar{\mathrm{n}}\cdot\big(\partial_t\bar{\tau}_* + (\widetilde{B}\cdot\nabla)\bar{\tau}_* + (\nabla\widetilde{B})^{\mathsf{T}}\bar{\tau}_*\big) = -\bar{\tau}_*\cdot\big(\partial_t\bar{\mathrm{n}} + (\widetilde{B}\cdot\nabla)\bar{\mathrm{n}} + (\nabla\widetilde{B})^{\mathsf{T}}\bar{\mathrm{n}}\big) = 0.$$

It therefore suffices to check that the velocity field $\widetilde{B}$ correctly captures the translation and rotation of the tangent vector field $\bar{\tau}_*$ in $\bar{\mathrm{t}}_*$-direction *on* the triple line $\bar{\Gamma}$, i.e., $\bar{\mathrm{t}}_*\cdot\big(\partial_t\bar{\tau}_* + (\widetilde{B}\cdot\nabla)\bar{\tau}_* + (\nabla\widetilde{B})^{\mathsf{T}}\bar{\tau}_*\big) = 0$, or equivalently by exploiting the orthogonality $\bar{\tau}_*\cdot\bar{\mathrm{t}}_* = 0$ that

$$\bar{\tau}_*\cdot\big(\partial_t\bar{\mathrm{t}}_* + (\widetilde{B}\cdot\nabla)\bar{\mathrm{t}}_*\big) = \bar{\mathrm{t}}_*\cdot(\nabla\widetilde{B})^{\mathsf{T}}\bar{\tau}_* \tag{5.125}$$

along the triple line $\bar{\Gamma}$.

In order to prove (5.125), we start by noticing that as a consequence of the definition (5.111), as well as the formulas (5.116) and (5.117) we have

$$\bar{\mathrm{t}}_*\cdot(\nabla\widetilde{B})^{\mathsf{T}}\bar{\tau}_* = \bar{\tau}_*\cdot(\bar{\mathrm{t}}_*\cdot\nabla)\widetilde{B} = -H\kappa_{\bar{\tau}_*\bar{\mathrm{t}}_*} + (\bar{\mathrm{t}}_*\cdot\nabla)\alpha_{\mathrm{vel}} \quad \text{on } \bar{I}. \tag{5.126}$$

That this expression equals $\bar{\tau}_*\cdot\big(\partial_t\bar{\mathrm{t}}_* + (\widetilde{B}\cdot\nabla)\bar{\mathrm{t}}_*\big)$ on the triple line $\bar{\Gamma}$ is a consequence of the following considerations. Let $\psi_{\bar{\Gamma}}(\cdot,t)\colon \bar{\Gamma}^0\times[0,T] \to \bar{\Gamma}(t)$, $t\in[0,T]$, be a normal parametrization of the triple line, i.e., it holds $\partial_t\psi_{\bar{\Gamma}}(x_0,t) = \mathrm{V}_{\bar{\Gamma}}(\psi_{\bar{\Gamma}}(x_0,t),t)$ for all $(x_0,t)\in\bar{\Gamma}^0\times[0,T]$. Choose moreover a C^5 diffeomorphic parametrization $\varphi_0\colon[0,1]\to\bar{\Gamma}^0$ of the initial triple line, and define for all $t\in[0,T]$ the dynamic parametrizations

$$\varphi\colon[0,1]\times[0,T]\to\bar{\Gamma}(t), \quad (s,t)\mapsto\psi_{\bar{\Gamma}}(\varphi_0(s),t).$$

Observe then that due to the zeroth order compatibility condition (5.113) and the definition of $\widetilde{B}$ (5.111) it holds for all $(s,t)\in[0,1]\times[0,T]$

$$\partial_t\varphi(s,t) = \widetilde{B}(\varphi(s,t),t) = (H\bar{\mathrm{n}})(\varphi(s,t),t) + (\alpha_{\mathrm{vel}}\bar{\tau}_*)(\varphi(s,t),t). \tag{5.127}$$

Define finally the differential operator $\partial_v := \frac{\partial_s}{|\partial_s\varphi|}$. Note that $\partial_v\varphi(\cdot,t)$ is a unit tangent vector field along the triple line $\bar{\Gamma}(t)$ for all $t\in[0,T]$, and we may choose the orientation such that $\partial_v\varphi(\cdot,t) = \bar{\mathrm{t}}_*(\varphi(\cdot,t),t)$ for all $t\in[0,T]$. A straightforward computation now yields

$$\partial_t\partial_v\varphi = \partial_v\partial_t\varphi - (\partial_v\partial_t\varphi\cdot\partial_v\varphi)\partial_v\varphi.$$

In particular, the commutator $[\partial_t\partial_v,\partial_v\partial_t]\varphi$ vanishes in $\bar{\tau}_*$-direction along the triple line. Using the chain rule and the first identity in (5.127), we thus obtain for all $(s,t)\in[0,1]\times[0,T]$, by the orthogonality of the frame $(\bar{\mathrm{n}}, \bar{\tau}_*, \bar{\mathrm{t}}_*)$, the second identity in (5.127), as well as (5.116) and (5.117)

$$\begin{aligned}
&\big(\bar{\tau}_* \cdot \big(\partial_t \bar{\mathrm{t}}_* + (\widetilde{B} \cdot \nabla)\bar{\mathrm{t}}_*\big)\big)(\varphi(s,t),t) \\
&= \bar{\tau}_*(\varphi(s,t),t) \cdot \partial_t \partial_v \varphi(s,t) \\
&= \bar{\tau}_*(\varphi(s,t),t) \cdot \partial_v \partial_t \varphi(s,t) \\
&= \big(\bar{\tau}_* \cdot (\bar{\mathrm{t}}_* \cdot \nabla)(H\bar{\mathrm{n}} + \alpha_{\mathrm{vel}}\bar{\tau}_*)\big)(\varphi(s,t),t) \\
&= -(H\kappa_{\bar{\tau}_*\bar{\mathrm{t}}_*})(\varphi(s,t),t) + \big((\bar{\mathrm{t}}_* \cdot \nabla)\alpha_{\mathrm{vel}}\big)(\varphi(s,t),t).
\end{aligned}$$

Hence, we may obtain (5.125) by (5.126), which concludes the proof. □

5.3.3 Global construction by interpolation

Throughout this whole subsection, let the assumptions of Proposition 5.14 and the notation of Section 5.2, Subsection 5.3.1 and Subsection 5.3.2 be in place. The next results provide the last missing ingredient for the construction of a local gradient-flow calibration at the triple line. We refer to Definition 5.15 and Figure 5.3 to recall the geometric setup.

Lemma 5.26. *Let $i, j, k \in \{1,2,3\}$ such that $\{i,j,k\} = \{1,2,3\}$. For each interpolation wedge $W_{\bar{\Omega}_i}$ there exists a pair of associated interpolation functions*

$$\lambda_{\bar{\Omega}_i}^{\bar{I}_{i,j}}, \lambda_{\bar{\Omega}_i}^{\bar{I}_{k,i}} \colon \bigcup_{t\in[0,T]} \big(\overline{W_{\bar{\Omega}_i}(t)} \setminus \bar{\Gamma}(t)\big) \times \{t\} \to [0,1]$$

of class $(C_t^0 C_x^1 \cap C_t^1 C_x^0)\big(\bigcup_{t\in[0,T]} \big(\overline{W_{\bar{\Omega}_i}(t)} \setminus \bar{\Gamma}(t)\big)\times\{t\}\big)$ such that $\lambda_{\bar{\Omega}_i}^{\bar{I}_{k,i}} = 1 - \lambda_{\bar{\Omega}_i}^{\bar{I}_{i,j}}$, and where $\lambda_{\bar{\Omega}_i}^{\bar{I}_{i,j}}$ is subject to the following additional requirements:

i) On the boundary of the interpolation wedge $W_{\bar{\Omega}_i}$, the values of $\lambda_{\bar{\Omega}_i}^{\bar{I}_{i,j}}$ and its derivatives are given by

$$\begin{aligned}
\lambda_{\bar{\Omega}_i}^{\bar{I}_{i,j}}(\cdot,t) &= 0, && \text{on } \big(\partial W_{\bar{\Omega}_i}(t) \cap \partial W_{\bar{I}_{k,i}}(t)\big) \setminus \bar{\Gamma}(t), \\
\lambda_{\bar{\Omega}_i}^{\bar{I}_{i,j}}(\cdot,t) &= 1, && \text{on } \big(\partial W_{\bar{\Omega}_i}(t) \cap \partial W_{\bar{I}_{i,j}}(t)\big) \setminus \bar{\Gamma}(t), \\
\nabla\lambda_{\bar{\Omega}_i}^{\bar{I}_{i,j}}(\cdot,t) &= 0, \quad \partial_t \lambda_{\bar{\Omega}_i}^{\bar{I}_{i,j}}(\cdot,t) = 0, && \text{on } \big(B_r(\bar{\Gamma}(t)) \cap \partial W_{\bar{\Omega}_i}(t)\big) \setminus \bar{\Gamma}(t),
\end{aligned}$$

for all $t \in [0,T]$.

ii) There exists a constant $C > 0$, which depends only on the data of the smoothly evolving regular double bubble $(\bar{\Omega}_1, \bar{\Omega}_2, \bar{\Omega}_3)$ on $[0,T]$, such that the estimate

$$|\partial_t \lambda_{\bar{\Omega}_i}^{\bar{I}_{i,j}}| + |\nabla \lambda_{\bar{\Omega}_i}^{\bar{I}_{i,j}}| \le C \operatorname{dist}^{-1}(\cdot, \bar{\Gamma}) \tag{5.128}$$

holds true on $\bigcup_{t\in[0,T]} \big(\overline{W_{\bar{\Omega}_i}(t)} \setminus \bar{\Gamma}(t)\big)\times\{t\}$.

iii) Denoting again by $\mathrm{V}_{\bar{\Gamma}}$ the normal velocity vector field of the triple line $\bar{\Gamma}$, we have an improved estimated on the advective derivative

$$\big|\partial_t \lambda_{\bar{\Omega}_i}^{\bar{I}_{i,j}}(\cdot,t) + \big(\mathrm{V}_{\bar{\Gamma}}(P_{\bar{\Gamma}}(\cdot,t),t) \cdot \nabla\big)\lambda_{\bar{\Omega}_i}^{\bar{I}_{i,j}}(\cdot,t)\big| \le C \tag{5.129}$$

on $\overline{W_{\bar{\Omega}_i}(t)} \setminus \bar{\Gamma}(t)$ for all $t \in [0,T]$. The constant $C > 0$ depends only on the data of the smoothly evolving regular double bubble $(\bar{\Omega}_1, \bar{\Omega}_2, \bar{\Omega}_3)$ on $[0,T]$.

Proof. Let $i,j,k \in \{1,2,3\}$ such that $\{i,j,k\} = \{1,2,3\}$. For the construction of the interpolation function $\lambda^{\bar{I}_{i,j}}_{\bar{\Omega}_i} =: 1-\lambda^{\bar{I}_{k,i}}_{\bar{\Omega}_i}$, we first choose a smooth function $\widetilde{\lambda}\colon \mathbb{R} \to [0,1]$ such that $\widetilde{\lambda} \equiv 0$ on $[\frac{2}{3},\infty)$ and $\widetilde{\lambda} \equiv 1$ on $(-\infty,\frac{1}{3}]$. Denote next by $\theta_i \in (0,\pi)$ the constant opening angle of the interpolation wedge W_i, cf. the representation (5.39). We then define $\lambda_i\colon [-1,1] \to [0,1]$ by $\lambda_i(u) := \widetilde{\lambda}(\frac{1-u}{1-\cos(\theta_i)})$, and based on this auxiliary map an interpolation function

$$\lambda_i^+(x,t) := \lambda_i\bigg(X^+_{\bar{\Omega}_i}(P_{\bar{\Gamma}}(x,t),t)\cdot\frac{x-P_{\bar{\Gamma}}(x,t)}{|x-P_{\bar{\Gamma}}(x,t)|}\bigg),\, t\in[0,T],\, x\in\overline{W_{\bar{\Omega}_i}(t)}\setminus\bar{\Gamma}(t).$$

The interpolation function $\lambda^{\bar{I}_{i,j}}_{\bar{\Omega}_i}$ is then either defined by λ_i^+ or by $1-\lambda_i^+$, depending on the right choice of "orientation" to satisfy the first item of (5.26), which in turn is then an immediate consequence of the definitions. For the proof of (5.128) and (5.129), it anyhow suffices to work on the level of the interpolation function λ_i^+.

The qualitative regularity of λ_i^+ and the corresponding regularity estimate (5.128) follow directly from the chain rule, the definition of λ_i^+, and the regularity requirements of Definition 5.15. For the improved estimate (5.129) on the advective derivative, we need an appropriate representation of $\partial_t P_{\bar{\Gamma}}$ in $\mathcal{N}_r(\bar{\Gamma})$. Abbreviating $g(x,t) := \frac{1}{2}\mathrm{dist}^2(x,\bar{\Gamma}(t))$ as well as $g_{\bar{\Gamma}}(x,t) := g(P_{\bar{\Gamma}}(x,t),t)$ for all $(x,t)\in\mathcal{N}_r(\bar{\Gamma})$, we obtain by the chain rule

$$\begin{aligned}0 &= \frac{\mathrm{d}}{\mathrm{d}t}\Big(\nabla g_{\bar{\Gamma}}(x,t)\Big)\\ &= (\nabla\partial_t g)(y,t)\big|_{y=P_{\bar{\Gamma}}(x,t)} + (\nabla^2 g)(y,t)\big|_{y=P_{\bar{\Gamma}}(x,t)}\partial_t P_{\bar{\Gamma}}(x,t), \quad (x,t)\in\mathcal{N}_r(\bar{\Gamma}).\end{aligned}$$

However, it is a well-known fact that $-\nabla\partial_t g$ evaluated along $\bar{\Gamma}$ precisely represents the normal velocity of $\bar{\Gamma}$ (cf. [11, Theorem 7 *ii)*, p. 18]). Hence, the previous display updates to

$$\mathrm{V}_{\bar{\Gamma}}\big(P_{\bar{\Gamma}}(x,t),t\big) = \nabla^2 g(y,t)\big|_{y=P_{\bar{\Gamma}}(x,t)}\partial_t P_{\bar{\Gamma}}(x,t)$$

for all $(x,t)\in\mathcal{N}_r(\bar{\Gamma})$. Moreover, $\nabla^2 g(\cdot,t)$ evaluated along the triple line $\bar{\Gamma}(t)$ represents for all $t\in[0,T]$ the projection onto the normal bundle $\mathrm{Tan}^\perp\bar{\Gamma}(t)$ for all $t\in[0,T]$ (cf. [11, Theorem 2 *ii)*, p. 12]). In other words,

$$\mathrm{V}_{\bar{\Gamma}}\big(P_{\bar{\Gamma}}(x,t),t\big) = (\mathrm{Id}-\bar{\mathrm{t}}\otimes\bar{\mathrm{t}})(y,t)\big|_{y=P_{\bar{\Gamma}}(x,t)}\partial_t P_{\bar{\Gamma}}(x,t) \tag{5.130}$$

for all $(x,t)\in\mathcal{N}_r(\bar{\Gamma})$.

Abbreviating $u_i^+ := u_i^+(x,t) := X^+_{\bar{\Omega}_i}\big(P_{\bar{\Gamma}}(x,t),t\big)\cdot\frac{x-P_{\bar{\Gamma}}(x,t)}{|x-P_{\bar{\Gamma}}(x,t)|}$ we may now compute by an application of the chain rule

$$\begin{aligned}&\partial_t\lambda_i^+(x,t)\\ &= \lambda_i'(u_i^+)X^+_{\bar{\Omega}_i}\big(P_{\bar{\Gamma}}(x,t),t\big)\cdot\partial_t\frac{x-P_{\bar{\Gamma}}(x,t)}{|x-P_{\bar{\Gamma}}(x,t)|}\\ &\quad+\lambda_i'(u_i^+)\frac{x-P_{\bar{\Gamma}}(x,t)}{|x-P_{\bar{\Gamma}}(x,t)|}\cdot\big((\bar{\mathrm{t}}\cdot\nabla)X^+_{\bar{\Omega}_i}\big)(y,t)\big|_{y=P_{\bar{\Gamma}}(x,t)}\Big(\bar{\mathrm{t}}(y,t)\big|_{y=P_{\bar{\Gamma}}(x,t)}\cdot\partial_t P_{\bar{\Gamma}}(x,t)\Big)\\ &\quad+\lambda_i'(u_i^+)\frac{x-P_{\bar{\Gamma}}(x,t)}{|x-P_{\bar{\Gamma}}(x,t)|}\cdot\big(\partial_t X^+_{\bar{\Omega}_i}\big)(y,t)\big|_{y=P_{\bar{\Gamma}}(x,t)}\end{aligned}$$

for all $(x,t)\in\bigcup_{t\in[0,T]}\big(\overline{W_{\bar{\Omega}_i}(t)}\setminus\bar{\Gamma}(t)\big)\times\{t\}$. Observe that the last two right hand side terms in the previous display are bounded by the regularity of the projection $P_{\bar{\Gamma}}$ and the regularity of the vector field $X^+_{\bar{\Omega}_i}$, cf. Definition 5.15. Next, for all $(x,t)\in\mathcal{N}_r(\bar{\Gamma})\setminus\bar{\Gamma}$

$$\partial_t\frac{x-P_{\bar{\Gamma}}(x,t)}{|x-P_{\bar{\Gamma}}(x,t)|} = -\frac{1}{|x-y|}\Big(\mathrm{Id}-\frac{x-y}{|x-y|}\otimes\frac{x-y}{|x-y|}\Big)\Big|_{y=P_{\bar{\Gamma}}(x,t)}\partial_t P_{\bar{\Gamma}}(x,t),$$

so that together with (5.130), $X^+_{\bar\Omega_i}(y,t), \mathrm{V}_{\bar\Gamma}(y,t) \in \mathrm{Tan}^\perp_y\bar\Gamma(t)$ for all $(y,t)\in\bar\Gamma$, as well as $\nabla P_{\bar\Gamma}(x,t) = \big(\bar{\mathrm{t}}(y,t)|_{y=P_{\bar\Gamma}(x,t)}\cdot\nabla\big)P_{\bar\Gamma}(x,t)\otimes\bar{\mathrm{t}}(y,t)|_{y=P_{\bar\Gamma}(x,t)}$ for all $(x,t)\in\mathcal{N}_r(\bar\Gamma)$

$$
\begin{aligned}
&\partial_t\lambda_i^+(x,t)\\
&= -\lambda_i'(u_i^+)\frac{1}{|x-y|}\Big(\mathrm{Id}-\frac{x-y}{|x-y|}\otimes\frac{x-y}{|x-y|}\Big)X^+_{\bar\Omega_i}(y,t)\Big|_{y=P_{\bar\Gamma}(x,t)}\cdot\partial_tP_{\bar\Gamma}(x,t)+O(1)\\
&= -\lambda_i'(u_i^+)\frac{1}{|x-y|}\Big(\mathrm{Id}-\frac{x-y}{|x-y|}\otimes\frac{x-y}{|x-y|}\Big)X^+_{\bar\Omega_i}(y,t)\cdot\mathrm{V}_{\bar\Gamma}(y,t)\Big|_{y=P_{\bar\Gamma}(x,t)}+O(1)\\
&= -\big(\mathrm{V}_{\bar\Gamma}\big(P_{\bar\Gamma}(x,t),t\big)\cdot\nabla\big)\lambda_i^+(x,t)+O(1)
\end{aligned}
$$

for all $(x,t)\in\bigcup_{t\in[0,T]}\big(\overline{W_{\bar\Omega_i}(t)}\setminus\bar\Gamma(t)\big)\times\{t\}$ as asserted. □

We may now provide the desired extensions $(\xi_{i,j})_{i,j\in\{1,2,3\},i\neq j}$ for the unit normal vector fields as well as the desired extension B of the velocity vector field within a space-time tubular neighborhood $\mathcal{N}_{\hat r}(\bar\Gamma)$ of the evolving triple line $\bar\Gamma$, where the radius $\hat r>0$ has to be chosen suitably and is potentially smaller than the admissible localization radius r.

Construction 5.27 (Gradient-flow calibration at triple line)**.** Let $(\widetilde\xi,\widetilde\xi',\widetilde\xi'')$ be the preliminary extensions from Construction 5.21 of the normal vector fields $(\bar{\mathrm{n}}|_{\bar I},\bar{\mathrm{n}}'|_{\bar I'},\bar{\mathrm{n}}''|_{\bar I''})$. Let $(\widetilde R'_{\bar I},\widetilde R''_{\bar I})$, $(\widetilde R_{\bar I'},\widetilde R''_{\bar I'})$ and $(\widetilde R_{\bar I''},\widetilde R'_{\bar I''})$ be the gauged Herring rotations as provided by Construction 5.19, and let $(\widetilde B,\widetilde B',\widetilde B'')$ be the preliminary extensions of the normal velocity vector fields from Construction 5.23. We also introduce the abbreviations $\bar\Omega:=\bar\Omega_1$, $\bar\Omega':=\bar\Omega_2$ and $\bar\Omega'':=\bar\Omega_3$.

With these ingredients in place, we first define a scale $\hat r:=r\wedge(2C)^{-\frac14}$, where $C>0$ denotes the (maximum of the) constant(s) from the estimate(s) (5.119). This choice of $\hat r\in(0,r]$ then entails due to (5.119) that

$$|\widetilde\xi|^2\in\big[\tfrac12,\tfrac32\big]\qquad\text{in } \mathcal{N}_{\hat r}(\bar\Gamma)\cap\mathrm{im}(\Psi), \tag{5.131}$$
$$|\widetilde\xi'|^2\in\big[\tfrac12,\tfrac32\big]\qquad\text{in } \mathcal{N}_{\hat r}(\bar\Gamma)\cap\mathrm{im}(\Psi'), \tag{5.132}$$
$$|\widetilde\xi''|^2\in\big[\tfrac12,\tfrac32\big]\qquad\text{in } \mathcal{N}_{\hat r}(\bar\Gamma)\cap\mathrm{im}(\Psi''). \tag{5.133}$$

Based on these non-degeneracy conditions and the properties (5.40)–(5.42) from the wedge decomposition of $\mathcal{N}_r(\bar\Gamma)$, we construct a well-defined set of vector fields

$$\xi,\xi',\xi''\colon\mathcal{N}_{\hat r}(\bar\Gamma)\to\overline{B_1(0)}, \tag{5.134}$$
$$B\colon\mathcal{N}_{\hat r}(\bar\Gamma)\to\mathbb{R}^3 \tag{5.135}$$

by the following procedure: On the closure of the interface wedges we define

$$(\xi,\xi',\xi''):=|\widetilde\xi|^{-1}\big(\widetilde\xi,\,\widetilde R'_{\bar I}\widetilde\xi,\,\widetilde R''_{\bar I}\widetilde\xi\big)\qquad\text{on }\overline{W_{\bar I}}, \tag{5.136}$$
$$(\xi,\xi',\xi''):=|\widetilde\xi'|^{-1}\big(\widetilde R_{\bar I'}\widetilde\xi',\,\widetilde\xi',\,\widetilde R''_{\bar I'}\widetilde\xi'\big)\qquad\text{on }\overline{W_{\bar I'}}, \tag{5.137}$$
$$(\xi,\xi',\xi''):=|\widetilde\xi''|^{-1}\big(\widetilde R_{\bar I''}\widetilde\xi'',\,\widetilde R'_{\bar I''}\widetilde\xi'',\,\widetilde\xi''\big)\qquad\text{on }\overline{W_{\bar I''}}, \tag{5.138}$$

as well as

$$B:=\widetilde B\text{ on }\overline{W_{\bar I}},\quad B:=\widetilde B'\text{ on }\overline{W_{\bar I'}},\quad B:=\widetilde B''\text{ on }\overline{W_{\bar I''}}. \tag{5.139}$$

On the interpolation wedges, say $W_{\bar\Omega}$, we define

$$\xi:=\lambda^{\bar I}_{\bar\Omega}|\widetilde\xi|^{-1}\widetilde\xi+\lambda^{\bar I''}_{\bar\Omega}|\widetilde\xi''|^{-1}\widetilde R_{\bar I''}\widetilde\xi'', \tag{5.140}$$
$$\xi':=\lambda^{\bar I}_{\bar\Omega}|\widetilde\xi|^{-1}\widetilde R'_{\bar I}\widetilde\xi+\lambda^{\bar I''}_{\bar\Omega}|\widetilde\xi''|^{-1}\widetilde R'_{\bar I''}\widetilde\xi'', \tag{5.141}$$
$$\xi'':=\lambda^{\bar I}_{\bar\Omega}|\widetilde\xi|^{-1}\widetilde R''_{\bar I}\widetilde\xi+\lambda^{\bar I''}_{\bar\Omega}|\widetilde\xi''|^{-1}\widetilde\xi'', \tag{5.142}$$
$$B:=\lambda^{\bar I}_{\bar\Omega}\widetilde B+\lambda^{\bar I''}_{\bar\Omega}\widetilde B''. \tag{5.143}$$

On the remaining two interpolation wedges $W_{\bar\Omega'}$ and $W_{\bar\Omega''}$, one proceeds analogously for the definition of these vector fields. ♢

5.3.4 Proof of Proposition 5.14

Let (ξ,ξ',ξ'',B) be the vector fields from Construction 5.27. We aim to show that this tuple of vector fields gives rise to a local gradient flow calibration at the triple line $\bar\Gamma$ in the sense of Proposition 5.14 after defining

$$\xi_{1,2} := \xi, \quad \xi_{2,3} := \xi', \quad \xi_{3,1} := \xi'' \quad \text{in } \mathcal{N}_{\hat r}(\bar\Gamma), \tag{5.144}$$

as well as $\xi_{j,i} := -\xi_{i,j}$ for the remaining set of distinct phases $i,j \in \{1,2,3\}$. The proof is now split into several steps.

In *Step 1* of the proof, we will derive the following useful compatibility estimates valid throughout interpolation wedges and which are needed in all subsequent steps:

$$\bigg|\frac{\widetilde\xi}{|\widetilde\xi|} - \frac{\widetilde R_{\bar I''}\widetilde\xi''}{|\widetilde\xi''|}\bigg| + \bigg|\frac{\widetilde R'_{\bar I}\widetilde\xi}{|\widetilde\xi|} - \frac{\widetilde R'_{\bar I''}\widetilde\xi''}{|\widetilde\xi''|}\bigg| + \bigg|\frac{\widetilde R''_{\bar I}\widetilde\xi}{|\widetilde\xi|} - \frac{\widetilde\xi''}{|\widetilde\xi''|}\bigg| \le C\operatorname{dist}^2(\cdot,\bar\Gamma) \tag{5.145}$$

in $W_{\bar\Omega}\cap\mathcal{N}_{\hat r}(\bar\Gamma)$, with analogous estimates being satisfied in the other two interpolation wedges. Moreover, the constant $C>0$ only depends on the data of the smoothly evolving regular double bubble $(\bar\Omega_1,\bar\Omega_2,\bar\Omega_3)$ on $[0,T]$.

In *Step 2*, we will verify that (ξ,ξ',ξ'',B) are continuous vector fields throughout $\mathcal{N}_{\hat r}(\bar\Gamma)$, that the extensions of the unit normals (ξ,ξ',ξ'') are of class $(C^0_tC^1_x\cap C^1_tC^0_x)(\mathcal{N}_{\hat r}(\bar\Gamma)\setminus\bar\Gamma)$ whereas the extended velocity B is of class $C^0_tC^1_x(\mathcal{N}_{\hat r}(\bar\Gamma)\setminus\bar\Gamma)$, and that there exists a constant $C>0$ depending only on the data of the smoothly evolving regular double bubble $(\bar\Omega_1,\bar\Omega_2,\bar\Omega_3)$ on $[0,T]$ such that the estimate

$$|(\partial_t,\nabla)(\xi,\xi',\xi'')| + |B| + |\nabla B| \le C \tag{5.146}$$

holds true throughout $\mathcal{N}_{\hat r}(\bar\Gamma)\setminus\bar\Gamma$. Moreover, we will show that

$$\xi = \bar{\mathrm n}|_{\bar I} \qquad \text{along } \bar I\cap\mathcal{N}_{\hat r}(\bar\Gamma), \tag{5.147}$$
$$B = \mathrm V_{\bar\Gamma} \qquad \text{along } \bar\Gamma, \tag{5.148}$$
$$\sigma\xi + \sigma'\xi' + \sigma''\xi'' = 0 \qquad \text{in } \mathcal{N}_{\hat r}(\bar\Gamma), \tag{5.149}$$

where property (5.147) is also satisfied in terms of $(\xi',\bar{\mathrm n}'|_{\bar I'})$ along $\bar I'\cap\mathcal{N}_{\hat r}(\bar\Gamma)$, or in terms of $(\xi'',\bar{\mathrm n}''|_{\bar I''})$ along $\bar I''\cap\mathcal{N}_{\hat r}(\bar\Gamma)$.

Step 3 of the proof is then devoted to the verification of the approximate evolution equation

$$|\partial_t\xi + (B\cdot\nabla)\xi + (\nabla B)^{\mathsf T}\xi| \le C\operatorname{dist}(\cdot,\bar I) \quad \text{in } \mathcal{N}_{\hat r}(\bar\Gamma)\setminus\bar\Gamma, \tag{5.150}$$

whereas in *Step 4* we will prove the estimate

$$|\nabla\cdot\xi + B\cdot\xi| \le C\operatorname{dist}(\cdot,\bar I) \quad \text{in } \mathcal{N}_{\hat r}(\bar\Gamma)\setminus\bar\Gamma. \tag{5.151}$$

We finally conclude in *Step 5* by deducing the estimate

$$(\partial_t + B\cdot\nabla)|\xi|^2 \le C\operatorname{dist}^2(\cdot,\bar I) \quad \text{in } \mathcal{N}_{\hat r}(\bar\Gamma). \tag{5.152}$$

We record for completeness that analogous estimates with respect to (5.150)–(5.152) are satisfied for (ξ',B), resp. (ξ'',B), in terms of $\operatorname{dist}(\cdot,\bar I')$, resp. $\operatorname{dist}(\cdot,\bar I'')$, and that the constant $C>0$ again only depends on the data of the smoothly evolving regular double bubble $(\bar\Omega_1,\bar\Omega_2,\bar\Omega_3)$ on $[0,T]$.

Step 1: Proof of (5.145). Adding zero, making use of the reverse triangle inequality and recalling the non-degeneracy condition (5.131)–(5.133), we may estimate

$$\begin{aligned}\Big|\frac{\widetilde{\xi}}{|\widetilde{\xi}|}-\frac{\widetilde{R}_{\bar{I}''}\widetilde{\xi}''}{|\widetilde{\xi}''|}\Big| &\leq \frac{1}{|\widetilde{\xi}|}\big|\widetilde{\xi}-\widetilde{R}_{\bar{I}''}\widetilde{\xi}''\big| + \Big|\frac{1}{|\widetilde{\xi}|}-\frac{1}{|\widetilde{R}_{\bar{I}''}\widetilde{\xi}''|}\Big|\big|\widetilde{R}_{\bar{I}''}\widetilde{\xi}''\big| \\ &\leq \frac{1}{|\widetilde{\xi}|}\big|\widetilde{\xi}-\widetilde{R}_{\bar{I}''}\widetilde{\xi}''\big| + \frac{1}{|\widetilde{\xi}|}\Big||\widetilde{\xi}|-|\widetilde{R}_{\bar{I}''}\widetilde{\xi}''|\Big| \leq 2\sqrt{2}\big|\widetilde{\xi}-\widetilde{R}_{\bar{I}''}\widetilde{\xi}''\big|.\end{aligned}$$

Due to the compatibility conditions (5.92) and (5.93) as well as the regularity estimates (5.91), the previous estimate then easily upgrades to

$$\Big|\frac{\widetilde{\xi}}{|\widetilde{\xi}|}-\frac{\widetilde{R}_{\bar{I}''}\widetilde{\xi}''}{|\widetilde{\xi}''|}\Big| \leq C\,\mathrm{dist}^2(\cdot,\bar{\Gamma}) \quad \text{in } W_{\bar{\Omega}}\cap\mathcal{N}_{\hat{r}}(\bar{\Gamma})$$

by inserting a second-order Taylor expansion with base point located at the unique nearest point on the triple line $\bar{\Gamma}$. The other two terms on the left hand side of (5.145) are treated analogously.

Step 2: Proof of (5.146)–(5.149). In terms of the asserted qualitative regularity, we observe that the first item of Lemma 5.26 together with the definitions from Construction 5.27 ensure that the vector fields (ξ,ξ',ξ'',B) and their required derivatives are continuous across the boundaries of the interpolation wedges (away from the triple line). Continuity of B throughout the whole space-time neighborhood $\mathcal{N}_r(\bar{\Gamma})$ with the asserted representation (5.148) along the triple line $\bar{\Gamma}$ follows from the compatibility condition (5.113). The unit normal extensions (ξ,ξ',ξ'') are continuous throughout $\mathcal{N}_r(\bar{\Gamma})$ due to the compatibility estimates (5.145). The representation (5.147) along the associated interface in turn is a consequence of the expansion ansatz (5.90) and the inclusion (5.41).

Next, on interface wedges the regularity estimate (5.146) follows directly from the estimates (5.91) and (5.112). For the derivation of (5.146) throughout an interpolation wedge, say $W_{\bar{\Omega}}\cap\mathcal{N}_{\hat{r}}(\bar{\Gamma})$, we simply compute by plugging in the definitions from Construction 5.27 and recalling from Lemma 5.26 that $\lambda_{\bar{\Omega}}^{\bar{I}''}=1-\lambda_{\bar{\Omega}}^{\bar{I}}$

$$\begin{aligned}(\partial_t,\nabla)\xi &= \lambda_{\bar{\Omega}}^{\bar{I}}(\partial_t,\nabla)\frac{\widetilde{\xi}}{|\widetilde{\xi}|}+\lambda_{\bar{\Omega}}^{\bar{I}''}(\partial_t,\nabla)\frac{\widetilde{R}_{\bar{I}''}\widetilde{\xi}''}{|\widetilde{\xi}''|}+\Big(\frac{\widetilde{\xi}}{|\widetilde{\xi}|}-\frac{\widetilde{R}_{\bar{I}''}\widetilde{\xi}''}{|\widetilde{\xi}''|}\Big)\otimes(\partial_t,\nabla)\lambda_{\bar{\Omega}}^{\bar{I}}, \\ \nabla B &= \lambda_{\bar{\Omega}}^{\bar{I}}\nabla\widetilde{B}+\lambda_{\bar{\Omega}}^{\bar{I}''}\nabla\widetilde{B}''+(\widetilde{B}-\widetilde{B}'')\otimes\nabla\lambda_{\bar{\Omega}}^{\bar{I}}.\end{aligned}$$

We thus infer (5.146) from the chain rule in form of $\nabla\frac{1}{|f|}=-\frac{(\nabla f)^{\mathsf{T}}f}{|f|^3}$, the regularity estimates (5.91), (5.112) and (5.128), and the compatibility conditions (5.145) and (5.113).

We turn to the proof of (5.149). Recalling the expansion ansatz (5.90) and the definitions (5.71) resp. (5.72) of the gauged Herring rotations, we deduce from (5.64)

$$\sigma\widetilde{\xi}+\sigma'\widetilde{R}'_{\bar{I}}\widetilde{\xi}+\sigma''\widetilde{R}''_{\bar{I}}\widetilde{\xi}=0 \quad \text{throughout } \mathcal{N}_r(\bar{\Gamma})\cap\mathrm{im}(\Psi), \tag{5.153}$$

and analogously throughout $\mathcal{N}_r(\bar{\Gamma})\cap\mathrm{im}(\Psi')$ in terms of $(\widetilde{R}_{\bar{I}'}\widetilde{\xi}',\widetilde{\xi}',\widetilde{R}''_{\bar{I}'}\widetilde{\xi}')$, or throughout $\mathcal{N}_r(\bar{\Gamma})\cap\mathrm{im}(\Psi'')$ in terms of $(\widetilde{R}_{\bar{I}''}\widetilde{\xi}'',\widetilde{R}'_{\bar{I}''}\widetilde{\xi}'',\widetilde{\xi}'')$. Due to the inclusion (5.41) and the definitions from Construction 5.27, we thus obtain from (5.153)

$$\sigma\xi+\sigma'\xi'+\sigma''\xi''=|\widetilde{\xi}|^{-1}\big(\sigma\widetilde{\xi}+\sigma'\widetilde{R}'_{\bar{I}}\widetilde{\xi}+\sigma''\widetilde{R}''_{\bar{I}}\widetilde{\xi}\big)=0 \quad \text{in } W_{\bar{I}}\cap\mathcal{N}_{\hat{r}}(\bar{\Gamma}).$$

An analogous argument works in the case of the other two interface wedges.

On interpolation wedges, say $W_{\bar{\Omega}}$, the extended Herring angle condition (5.149) follows from a linear combination of the previous ingredients. More precisely, the definitions from Construction 5.27 and the cancellations (5.153) directly imply

$$\begin{aligned}&\sigma\xi+\sigma'\xi'+\sigma''\xi''\\&=\lambda_{\bar{\Omega}}^{\bar{I}}|\widetilde{\xi}|^{-1}\big(\sigma\widetilde{\xi}+\sigma'\widetilde{R}'_{\bar{I}}\widetilde{\xi}+\sigma''\widetilde{R}''_{\bar{I}}\widetilde{\xi}\big)+\lambda_{\bar{\Omega}}^{\bar{I}''}|\widetilde{\xi}''|^{-1}\big(\sigma\widetilde{R}_{\bar{I}''}\widetilde{\xi}''+\sigma'\widetilde{R}'_{\bar{I}''}\widetilde{\xi}''+\sigma''\widetilde{\xi}''\big)=0\end{aligned}$$

throughout $W_{\bar{\Omega}}\cap\mathcal{N}_{\hat{r}}(\bar{\Gamma})$ as desired. This concludes the proof of (5.149), and thus *Step 2* of the proof, as on the other interpolation wedges (5.149) follows analogously.

Step 3: Proof of (5.150). We first claim that for each rotation $\mathcal{R}\in\{\mathrm{Id},\widetilde{R}'_{\bar{I}},\widetilde{R}''_{\bar{I}}\}$ it holds throughout $\mathcal{N}_{\hat{r}}(\bar{\Gamma})\cap\mathrm{im}(\Psi)$

$$\Big|\big(\partial_t+(\widetilde{B}\cdot\nabla)+(\nabla\widetilde{B})^{\mathsf{T}}\big)\frac{\mathcal{R}\widetilde{\xi}}{|\mathcal{R}\widetilde{\xi}|}\Big|\leq C\begin{cases}\mathrm{dist}(\cdot,\bar{I}) & \text{if } \mathcal{R}=\mathrm{Id},\\ \mathrm{dist}(\cdot,\bar{\Gamma}) & \text{else,}\end{cases}\tag{5.154}$$

$$\Big|\widetilde{B}\cdot\frac{\mathcal{R}\widetilde{\xi}}{|\mathcal{R}\widetilde{\xi}|}+\nabla\cdot\frac{\mathcal{R}\widetilde{\xi}}{|\mathcal{R}\widetilde{\xi}|}\Big|\leq C\begin{cases}\mathrm{dist}(\cdot,\bar{I}) & \text{if } \mathcal{R}=\mathrm{Id},\\ \mathrm{dist}(\cdot,\bar{\Gamma}) & \text{else,}\end{cases}\tag{5.155}$$

for some constant $C>0$ which depends only on the smoothly evolving regular double bubble $(\bar{\Omega}_1,\bar{\Omega}_2,\bar{\Omega}_3)$ on $[0,T]$. Moreover, analogous estimates hold true throughout the domain $\mathcal{N}_{\hat{r}}(\bar{\Gamma})\cap\mathrm{im}(\Psi')$ in terms of the vector fields $(\mathcal{R}\widetilde{\xi}',\widetilde{B}')$ for each rotation $\mathcal{R}\in\{\widetilde{R}_{\bar{I}'},\mathrm{Id},\widetilde{R}''_{\bar{I}'}\}$, as well as throughout $\mathcal{N}_{\hat{r}}(\bar{\Gamma})\cap\mathrm{im}(\Psi'')$ in terms of $(\mathcal{R}\widetilde{\xi}'',\widetilde{B}'')$ for each $\mathcal{R}\in\{\widetilde{R}_{\bar{I}''},\widetilde{R}'_{\bar{I}''},\mathrm{Id}\}$.

The estimate (5.154) follows from the straightforward computation

$$\begin{aligned}&\big(\partial_t+(\widetilde{B}\cdot\nabla)+(\nabla\widetilde{B})^{\mathsf{T}}\big)\frac{\mathcal{R}\widetilde{\xi}}{|\mathcal{R}\widetilde{\xi}|}\\&=\frac{1}{|\mathcal{R}\widetilde{\xi}|}\big(\partial_t+(\widetilde{B}\cdot\nabla)+(\nabla\widetilde{B})^{\mathsf{T}}\big)\mathcal{R}\widetilde{\xi}-\frac{\partial_t|\mathcal{R}\widetilde{\xi}|^2+(\widetilde{B}\cdot\nabla)|\mathcal{R}\widetilde{\xi}|^2}{2|\mathcal{R}\widetilde{\xi}|^3}\mathcal{R}\widetilde{\xi}\end{aligned}$$

together with the condition (5.131) and the estimates (5.120), (5.121) and (5.122). The estimate (5.155) in turn can be deduced from the same ingredients as well as

$$\widetilde{B}\cdot\frac{\mathcal{R}\widetilde{\xi}}{|\mathcal{R}\widetilde{\xi}|}+\nabla\cdot\frac{\mathcal{R}\widetilde{\xi}}{|\mathcal{R}\widetilde{\xi}|}=\frac{1}{|\mathcal{R}\widetilde{\xi}|}\big(B\cdot\mathcal{R}\widetilde{\xi}+\nabla\cdot\mathcal{R}\widetilde{\xi}\big)-\frac{(\mathcal{R}\widetilde{\xi}\cdot\nabla)|\mathcal{R}\widetilde{\xi}|^2}{2|\mathcal{R}\widetilde{\xi}|^3}.$$

On interface wedges, facilitated by the inclusion (5.41) the claim (5.150) now follows from an application of the estimate (5.154) and, if needed, a simple post-processing by means of (5.44). Hence, let us directly move on with the verification of (5.150) throughout interpolation wedges, say $W_{\bar{\Omega}}\cap\mathcal{N}_{\hat{r}}(\bar{\Gamma})$. Plugging in the definitions (5.140)–(5.143) from Construction 5.27 we may compute based on the product rule, adding zero, and recalling from Lemma 5.26 that $\lambda_{\bar{\Omega}}^{\bar{I}''}=1-\lambda_{\bar{\Omega}}^{\bar{I}}$

$$\begin{aligned}\big(\partial_t+(B\cdot\nabla)+(\nabla B)^{\mathsf{T}}\big)\xi&=\lambda_{\bar{\Omega}}^{\bar{I}}\big(\partial_t+(\widetilde{B}\cdot\nabla)+(\nabla\widetilde{B})^{\mathsf{T}}\big)\frac{\widetilde{\xi}}{|\widetilde{\xi}|}\\&\quad+\big(1-\lambda_{\bar{\Omega}}^{\bar{I}}\big)\big(\partial_t+(\widetilde{B}''\cdot\nabla)+(\nabla\widetilde{B}'')^{\mathsf{T}}\big)\frac{\widetilde{R}_{\bar{I}''}\widetilde{\xi}''}{|\widetilde{\xi}''|}\\&\quad+\Big(\frac{\widetilde{\xi}}{|\widetilde{\xi}|}-\frac{\widetilde{R}_{\bar{I}''}\widetilde{\xi}''}{|\widetilde{\xi}''|}\Big)\big(\partial_t+(B\cdot\nabla)\big)\lambda_{\bar{\Omega}}^{\bar{I}}\\&\quad+\lambda_{\bar{\Omega}}^{\bar{I}}\big((B-\widetilde{B})\cdot\nabla\big)\frac{\widetilde{\xi}}{|\widetilde{\xi}|}+\big(1-\lambda_{\bar{\Omega}}^{\bar{I}}\big)\big((B-\widetilde{B}'')\cdot\nabla\big)\frac{\widetilde{R}_{\bar{I}''}\widetilde{\xi}''}{|\widetilde{\xi}''|}\\&\quad+\lambda_{\bar{\Omega}}^{\bar{I}}\big(\nabla B-\nabla\widetilde{B}\big)^{\mathsf{T}}\frac{\widetilde{\xi}}{|\widetilde{\xi}|}+\big(1-\lambda_{\bar{\Omega}}^{\bar{I}}\big)\big(\nabla B-\nabla\widetilde{B}''\big)^{\mathsf{T}}\frac{\widetilde{R}_{\bar{I}''}\widetilde{\xi}''}{|\widetilde{\xi}''|}.\end{aligned}\tag{5.156}$$

The first two right hand side terms of the previous display are at least of order $O(\mathrm{dist}(\cdot,\bar{\Gamma}))$ due to the estimates (5.154), which in turn are available this time due to the inclusion (5.42). The third, fourth and fifth right hand side terms are of the same order thanks to the compatibility conditions (5.145) and (5.113), the regularity estimates (5.91), (5.112) and (5.146), the estimate (5.129) on the advective derivative of an interpolation function, as well as the non-degeneracy conditions (5.131)–(5.133).

Regarding the two right hand side terms from the last line of the previous display, we may argue as follows. Plugging in the definition of B from Construction 5.27, we compute by the product rule, the identity $\lambda_{\bar{\Omega}}^{\bar{I}}+\lambda_{\bar{\Omega}}^{\bar{I}''}=1$ and by carefully noting that $\widetilde{\xi}\perp\bar{\mathrm{t}}_*$ throughout $\mathcal{N}_r(\bar{\Gamma})\cap\mathrm{im}(\Psi)$ due to the expansion ansatz (5.90)

$$
\begin{aligned}
(\nabla B-\nabla\widetilde{B})^\mathsf{T}\widetilde{\xi}=(1-\lambda_{\bar{\Omega}}^{\bar{I}})(\nabla\widetilde{B}''-\nabla\widetilde{B})^\mathsf{T}(\mathrm{Id}-\bar{\mathrm{t}}_*\otimes\bar{\mathrm{t}}_*)\widetilde{\xi}\\
+\big((\widetilde{B}-\widetilde{B}'')\cdot(\mathrm{Id}-\bar{\mathrm{t}}_*\otimes\bar{\mathrm{t}}_*)\widetilde{\xi}\big)\nabla\lambda_{\bar{\Omega}}^{\bar{I}}.
\end{aligned}
$$

Abbreviating $\bar{\mathrm{t}}_{\bar{\Gamma}}(x,t):=\bar{\mathrm{t}}(P_{\bar{\Gamma}}(x,t),t)$ for all $(x,t)\in\mathcal{N}_r(\bar{\Gamma})\cap\mathrm{im}(\Psi)$ and recalling the compatibility conditions (5.81) resp. (5.113) as well as the regularity estimate (5.128) for the interpolation function, we may switch from $\bar{\mathrm{t}}_*$ to $\bar{\mathrm{t}}_{\bar{\Gamma}}$ in the previous display at the cost of an admissible error:

$$
\begin{aligned}
(\nabla B-\nabla\widetilde{B})^\mathsf{T}\widetilde{\xi}=(1-\lambda_{\bar{\Omega}}^{\bar{I}})(\nabla\widetilde{B}''-\nabla\widetilde{B})^\mathsf{T}(\mathrm{Id}-\bar{\mathrm{t}}_{\bar{\Gamma}}\otimes\bar{\mathrm{t}}_{\bar{\Gamma}})\widetilde{\xi}\\
+\big((\widetilde{B}-\widetilde{B}'')\cdot(\mathrm{Id}-\bar{\mathrm{t}}_{\bar{\Gamma}}\otimes\bar{\mathrm{t}}_{\bar{\Gamma}})\widetilde{\xi}\big)\nabla\lambda_{\bar{\Omega}}^{\bar{I}}+O(\mathrm{dist}(\cdot,\bar{\Gamma})).
\end{aligned}
$$

It then follows from the compatibility conditions (5.49), (5.113) and (5.114), and again the regularity estimate (5.128) for the interpolation function that

$$
(\nabla B-\nabla\widetilde{B})^\mathsf{T}\widetilde{\xi}=O(\mathrm{dist}(\cdot,\bar{\Gamma})).
$$

One may argue similarly for the second term after replacing $|\widetilde{\xi}''|^{-1}\widetilde{R}_{\bar{I}''}\widetilde{\xi}''$ by $|\widetilde{\xi}|^{-1}\widetilde{\xi}$ using the compatibility estimate (5.145).

In summary, the asserted estimate (5.150) in terms of ξ now follows from the previously derived estimates for the right hand side terms of (5.156) and a subsequent post-processing of them by means of (5.43). We finally remark that the argument proceeds analogously for the other two vector fields ξ' and ξ'', respectively.

Step 4: Proof of (5.151). Thanks to the inclusion (5.41), the estimate (5.155), and, if needed, the estimate (5.44), it again suffices to provide additional details only for the argument for (5.151) on interpolation wedges, say $W_{\bar{\Omega}}\cap\mathcal{N}_{\hat{r}}(\bar{\Gamma})$. Plugging in the definitions (5.140)–(5.143) from Construction 5.27, applying the product rule, recalling from Lemma 5.26 that $\lambda_{\bar{\Omega}}^{\bar{I}''}=1-\lambda_{\bar{\Omega}}^{\bar{I}}$, and adding zero yields

$$
\begin{aligned}
B\cdot\xi+\nabla\cdot\xi=\lambda_{\bar{\Omega}}^{\bar{I}}\Big(\widetilde{B}\cdot\frac{\widetilde{\xi}}{|\widetilde{\xi}|}+\nabla\cdot\frac{\widetilde{\xi}}{|\widetilde{\xi}|}\Big)+\big(1-\lambda_{\bar{\Omega}}^{\bar{I}}\big)\Big(\widetilde{B}''\cdot\frac{\widetilde{R}_{\bar{I}''}\widetilde{\xi}''}{|\widetilde{\xi}''|}+\nabla\cdot\frac{\widetilde{R}_{\bar{I}''}\widetilde{\xi}''}{|\widetilde{\xi}''|}\Big)\\
+\lambda_{\bar{\Omega}}^{\bar{I}}(B-\widetilde{B})\cdot\frac{\widetilde{\xi}}{|\widetilde{\xi}|}+\big(1-\lambda_{\bar{\Omega}}^{\bar{I}}\big)(B-\widetilde{B}'')\cdot\frac{\widetilde{R}_{\bar{I}''}\widetilde{\xi}''}{|\widetilde{\xi}''|}\\
+\Big(\frac{\widetilde{\xi}}{|\widetilde{\xi}|}-\frac{\widetilde{R}_{\bar{I}''}\widetilde{\xi}''}{|\widetilde{\xi}''|}\Big)\cdot\nabla\lambda_{\bar{\Omega}}^{\bar{I}}.
\end{aligned}
$$

The right hand side terms of the previous display are all at least of order $O(\mathrm{dist}(\cdot,\bar{\Gamma}))$—and thus of required order due to (5.43)—by an application of the inclusion (5.42), the estimates (5.155), the compatibility conditions (5.113) and (5.145), as well as the regularity estimate (5.128) for the interpolation function.

This proves (5.151) in terms of ξ. The argument proceeds again analogously for the other two vector fields ξ' and ξ''.

Step 5: Proof of (5.152). There is nothing to prove throughout interface wedges since the unit normal extensions (ξ,ξ',ξ'') are unit length vectors, cf. the definitions from Construction 5.27. On interpolation wedges, say $W_{\bar\Omega}\cap\mathcal{N}_{\hat r}(\bar\Gamma)$, we may compute by the definition (5.140) from Construction 5.27

$$|\xi|^2 = 1 - \lambda_{\bar\Omega}^{\bar I}\lambda_{\bar\Omega}^{\bar I''}\left|\frac{\widetilde\xi}{|\widetilde\xi|} - \frac{\widetilde R_{\bar I''}\widetilde\xi''}{|\widetilde\xi''|}\right|^2. \tag{5.157}$$

The estimate (5.152) ist thus a consequence of (5.145), (5.128), (5.146) and (5.43). One may argue analogously for the other two vector fields ξ' and ξ'', respectively. □

5.3.5 Compatibility of local gradient-flow calibrations

A regular double bubble is built out of two distinct topological features: the three two-phase interfaces and the triple line. For each of these topological features, we so far constructed a tuple of vector fields living in a space-time neighborhood of the feature and locally mimicking the requirements of a gradient-flow calibration. The remaining step in the construction consists of pasting together these local vector fields into globally defined ones. This task will be carried out in Section 5.4. The key issue is to transfer properties from the local to the global level, which turns out to be possible because, among other things, the local constructions for the two distinct topological features can be arranged to be sufficiently compatible. We formalize this as follows.

Proposition 5.28. *Let $(\bar\Omega_1,\bar\Omega_2,\bar\Omega_3)$ be a regular double bubble smoothly evolving by MCF in the sense of Definition 5.10 on a time interval $[0,T]$. Let $\hat r\in(0,1]$ be the localization scale of Proposition 5.14, and for each pair of distinct phases $i,j\in\{1,2,3\}$, denote by $(\xi_{i,j}^{\bar I_{i,j}},B^{\bar I_{i,j}})$ the local gradient-flow calibration for the interface $\bar I_{i,j}$ from Construction 5.12.*

For all $i,j\in\{1,2,3\}$ with $i\neq j$, there exists a choice of the tangential component $\mathcal{Y}_{i,j}$ of $B^{\bar I_{i,j}}$ and a local gradient-flow calibration $((\xi_{i,j}^{\bar\Gamma})_{i,j\in\{1,2,3\},i\neq j},B^{\bar\Gamma})$ at the triple line in the sense of Proposition 5.14 such that in addition the following compatibility estimates hold true

$$\big|\xi_{i,j}^{\bar I_{i,j}}-\xi_{i,j}^{\bar\Gamma}\big| + \big|(\nabla\xi_{i,j}^{\bar I_{i,j}})^\mathsf{T}\xi_{i,j}^{\bar\Gamma}\big| + \big|(\nabla\xi_{i,j}^{\bar\Gamma})^\mathsf{T}\xi_{i,j}^{\bar I_{i,j}}\big| \le C\operatorname{dist}(\cdot,\bar I_{i,j}), \tag{5.158}$$

$$\big|(\xi_{i,j}^{\bar I_{i,j}}-\xi_{i,j}^{\bar\Gamma})\cdot\xi_{i,j}^{\bar I_{i,j}}\big| \le C\operatorname{dist}^2(\cdot,\bar I_{i,j}), \tag{5.159}$$

$$\big|B^{\bar I_{i,j}}-B^{\bar\Gamma}\big| \le C\operatorname{dist}(\cdot,\bar I_{i,j}), \tag{5.160}$$

$$\big|(\nabla B^{\bar I_{i,j}}-\nabla B^{\bar\Gamma})^\mathsf{T}\xi_{i,j}^{\bar I_{i,j}}\big| \le C\operatorname{dist}(\cdot,\bar I_{i,j}) \tag{5.161}$$

throughout $\mathcal{N}_{\frac{\hat r}{2}}(\bar\Gamma)\cap(W_{\bar I_{i,j}}\cup W_{\bar\Omega_i}\cup W_{\bar\Omega_j})$, where $C>0$ is a constant which depends only on the smoothly evolving regular double bubble $(\bar\Omega_1,\bar\Omega_2,\bar\Omega_3)$ on $[0,T]$.

Proof. Let $((\xi_{i,j}^{\bar\Gamma})_{i,j\in\{1,2,3\},i\neq j},B^{\bar\Gamma})$ be the local gradient-flow calibration at the triple line $\bar\Gamma$ as constructed in the proof of Proposition 5.14, and let $i,j\in\{1,2,3\}$ be distinct phases.

Step 1: Proof of (5.158). The estimate $|\xi_{i,j}^{\bar I_{i,j}}-\xi_{i,j}^{\bar\Gamma}|\le C\operatorname{dist}(\cdot,\bar I_{i,j})$ is an immediate consequence of the regularity estimates (5.24) and (5.30), the inclusions (5.41)–(5.42), as well as the extension property $\xi_{i,j}^{\bar I_{i,j}}=\bar{\mathrm{n}}_{i,j}=\xi_{i,j}^{\bar\Gamma}$ along $\bar I_{i,j}\cap\mathcal{N}_{\hat r}(\bar\Gamma)$.

The estimate $|(\nabla\xi_{i,j}^{\bar I_{i,j}})^\mathsf{T}\xi_{i,j}^{\bar\Gamma}|\le C\operatorname{dist}(\cdot,\bar I_{i,j})$ follows from adding zero, the already established estimate for the first left hand side term of (5.158), and $\xi_{i,j}^{\bar I_{i,j}}$ being a unit length vector field due to (5.22).

For the remaining part of (5.158), it suffices to estimate $\frac{1}{2}\nabla|\xi^{\bar\Gamma}_{i,j}|^2$ due to (5.30) and the already established estimate for the first left hand side term of (5.158). Throughout the interpolation wedge $W_{\bar I_{i,j}} \cap \mathcal{N}_{\hat r}(\bar\Gamma)$, we have $|\xi^{\bar\Gamma}_{i,j}| \equiv 1$ in view of the definitions (5.136)–(5.138), so that the desired estimate is satisfied for trivial reasons. Within the relevant interpolation wedges, one may employ the representation (5.157) and then deduce $\big|\frac{1}{2}\nabla|\xi^{\bar\Gamma}_{i,j}|^2\big| \leq C\,\mathrm{dist}(\cdot,\bar I_{i,j})$ from (5.145), (5.128) and (5.43).

Step 2: Proof of (5.159). Denote by $\widetilde\xi^{\bar I_{i,j}}$ the auxiliary extension of the unit normal $\bar{\mathrm{n}}_{i,j}|_{\bar I_{i,j}}$ from Construction 5.21. Due to (5.136)–(5.138), (5.140)–(5.142), and the compatibility estimates (5.145) it holds

$$\xi^{\bar\Gamma}_{i,j} = |\widetilde\xi^{\bar I_{i,j}}|^{-1}\widetilde\xi^{\bar I_{i,j}} + O(\mathrm{dist}^2(\cdot,\bar I_{i,j})) \quad \text{in } \mathcal{N}_{\hat r}(\bar\Gamma) \cap (W_{\bar I_{i,j}} \cup W_{\bar\Omega_i} \cup W_{\bar\Omega_j}).$$

Making use of the non-degeneracy conditions (5.131)–(5.133) and the estimate (5.119) we also obtain

$$|\widetilde\xi^{\bar I_{i,j}}|^{-1} - 1 = \frac{1-|\widetilde\xi^{\bar I_{i,j}}|^2}{|\widetilde\xi^{\bar I_{i,j}}|\big(1+|\widetilde\xi^{\bar I_{i,j}}|\big)} = O(\mathrm{dist}^2(\cdot,\bar I_{i,j})) \quad \text{in } \mathcal{N}_{\hat r}(\bar\Gamma) \cap \mathrm{im}(\Psi_{i,j}).$$

Recalling the precise representations (5.22) and (5.90), we thus infer from the previous two displays that

$$\big|(\xi^{\bar I_{i,j}}_{i,j} - \xi^{\bar\Gamma}_{i,j}) \cdot \xi^{\bar I_{i,j}}_{i,j}\big| \leq \big|1 - |\widetilde\xi^{\bar I_{i,j}}|^{-1}\big| + O(\mathrm{dist}^2(\cdot,\bar I_{i,j})) = O(\mathrm{dist}^2(\cdot,\bar I_{i,j}))$$

throughout $\mathcal{N}_{\hat r}(\bar\Gamma) \cap (W_{\bar I_{i,j}} \cup W_{\bar\Omega_i} \cup W_{\bar\Omega_j})$ as asserted.

Step 3: Construction of the tangential component $\mathcal{Y}_{i,j}$ *of* $B^{\bar I_{i,j}}$. Let θ be a smooth and even cutoff function with $\theta(r) = 1$ for $|r| \leq \frac{1}{2}$ and $\theta(r) = 0$ for $|r| \geq 1$. Denote by $\widetilde B^{\bar I_{i,j}}$ the auxiliary local velocity field from Construction 5.23 with respect to the interface $\bar I_{i,j}$. The tangential component $\mathcal{Y}_{i,j}$ of $B^{\bar I_{i,j}}$ is then simply defined by means of

$$\mathcal{Y}_{i,j} := \theta\Big(\frac{\mathrm{dist}(\cdot,\bar\Gamma)}{\hat r}\Big)(\mathrm{Id} - \bar{\mathrm{n}}_{i,j} \otimes \bar{\mathrm{n}}_{i,j})\widetilde B^{\bar I_{i,j}} \quad \text{in } \mathrm{im}(\Psi_{i,j}). \tag{5.162}$$

Note that $\mathcal{Y}_{i,j} \in C^0_t C^1_x(\mathrm{im}(\Psi_{i,j}))$ as required by Construction 5.12 due to the regularity (5.21) of the normal $\bar{\mathrm{n}}_{i,j}$ and the regularity estimate (5.112) for $\widetilde B^{\bar I_{i,j}}$.

Step 4: Proof of (5.160)–(5.161). It follows from the expansion ansatz (5.111), the definitions (5.139) and (5.22), the choice of the tangential component (5.162), as well as the choice of the cutoff θ from the previous step that

$$B^{\bar I_{i,j}} = B^{\bar\Gamma} \quad \text{throughout } W_{\bar I_{i,j}} \cap \mathcal{N}_{\frac{\hat r}{2}}(\bar\Gamma). \tag{5.163}$$

More precisely, denoting by $\widetilde B^{\bar I_{i,j}}$ the auxiliary local velocity field from Construction 5.23 with respect to the interface $\bar I_{i,j}$, we in fact have

$$B^{\bar I_{i,j}} = \widetilde B^{\bar I_{i,j}} \quad \text{throughout } (W_{\bar I_{i,j}} \cup W_{\bar\Omega_i} \cup W_{\bar\Omega_j}) \cap \mathcal{N}_{\frac{\hat r}{2}}(\bar\Gamma). \tag{5.164}$$

Now (5.160) follows directly from a Taylor-expansion argument exploiting the regularity estimates (5.25) resp. (5.31) as well as the inclusions (5.41)–(5.42).

In view of (5.163), the estimate (5.161) is satisfied for trivial reasons throughout the interface wedge $\mathcal{N}_{\frac{\hat r}{2}}(\bar\Gamma) \cap W_{\bar I_{i,j}}$. Within the relevant interpolation wedges, say for concreteness $\mathcal{N}_{\frac{\hat r}{2}}(\bar\Gamma) \cap W_{\bar\Omega_i}$, we make use of (5.164). Let $k \in \{1,2,3\} \setminus \{i,j\}$ denote the third phase. It then follows from (5.164) and expressing the definition (5.143) in form of $B^{\bar\Gamma} = \lambda^{\bar I_{i,j}}_{\bar\Omega_i}\widetilde B^{\bar I_{i,j}} + (1-\lambda^{\bar I_{i,j}}_{\bar\Omega_i})\widetilde B^{\bar I_{k,i}}$

$$\nabla B^{\bar I_{i,j}} - \nabla B^{\bar\Gamma} = \big(1-\lambda^{\bar I_{i,j}}_{\bar\Omega_i}\big)\big(\nabla\widetilde B^{\bar I_{i,j}} - \nabla\widetilde B^{\bar I_{k,i}}\big) - \big(\widetilde B^{\bar I_{i,j}} - \widetilde B^{\bar I_{k,i}}\big)\otimes\nabla\lambda^{\bar I_{i,j}}_{\bar\Omega_i}.$$

Since $\xi^{\bar I_{i,j}}_{i,j} = \bar{\mathrm{n}}_{i,j}$ due to (5.22), the estimate (5.161) now follows throughout the interpolation wedge $\mathcal{N}_{\frac{\hat r}{2}}(\bar\Gamma)\cap W_{\bar\Omega_i}$ by the same argument which deals with estimating the last two right hand side terms of (5.156). We recall for convenience that the essential input for the latter is given by the compatibility conditions (5.113) and (5.114) for the auxiliary velocity fields $\widetilde B^{\bar I_{i,j}}$ and $\widetilde B^{\bar I_{k,i}}$. □

5.4 Gradient flow calibrations for double bubbles

5.4.1 Localization of topological features

We start by introducing a family of suitable cutoff functions localizing around the interfaces and the triple line in a smoothly evolving regular double bubble. This family will be used to provide the construction of a gradient-flow calibration by means of gluing together the local constructions from the previous two sections.

Lemma 5.29. *Let $(\bar\Omega_1,\bar\Omega_2,\bar\Omega_3)$ be a regular double bubble smoothly evolving by MCF in the sense of Definition 5.10 on a time interval $[0,T]$. Let the notation of Definition 5.11 resp. Definition 5.15 be in place, and let $\hat r\in(0,1]$ be the radius of Proposition 5.14. In particular, let $(r_{i,j})_{i,j\in\{1,2,3\},i\neq j}$ be admissible localization radii for the interfaces in the sense of Definition 5.11 such that $\hat r\leq r_{1,2}\wedge r_{2,3}\wedge r_{3,1}$. We next define for each pair $i,j\in\{1,2,3\}$ with $i\neq j$ a scale*

$$3\ell_{i,j} := \min_{t\in[0,T]}\ \min_{\substack{k,l\in\{1,2,3\},\,k\neq l,\\ (k,l)\notin\{(i,j),(j,i)\}}} \mathrm{dist}\,\big(\bar I_{i,j}(t)\setminus B_{\hat r}(\bar\Gamma(t)), \bar I_{k,l}(t)\big) > 0,$$

and based on these a localization scale $\bar r\in(0,r_{1,2}\wedge r_{2,3}\wedge r_{3,1}]$ by means of

$$2\bar r := \hat r\wedge\min_{i,j\in\{1,2,3\},\,i\neq j}\ell_{i,j}. \tag{5.165}$$

There then exists a collection of continuous cutoff functions

$$\eta_{\bar\Gamma},\,\eta_{\bar I_{1,2}},\,\eta_{\bar I_{2,3}},\,\eta_{\bar I_{3,1}}:\mathbb{R}^3\times[0,T]\to[0,1]$$

satisfying the following properties:

i) The cutoff functions are of class $(C^0_tC^1_x\cap C^1_tC^0_x)(\mathbb{R}^3\times[0,T]\setminus\bar\Gamma)$ with corresponding regularity estimates

$$|(\partial_t,\nabla)(\eta_{\bar\Gamma},\eta_{\bar I_{1,2}},\eta_{\bar I_{2,3}},\eta_{\bar I_{3,1}})|\leq C\quad\textit{in } \mathbb{R}^3\times[0,T]\setminus\bar\Gamma \tag{5.166}$$

for some constant $C>0$ depending only on the data of the smoothly evolving regular double bubble $(\bar\Omega_1,\bar\Omega_2,\bar\Omega_3)$ on $[0,T]$.

ii) The family $(\eta_{\bar\Gamma},(\eta_{\bar I_{i,j}})_{i,j\in\{1,2,3\},i\neq j})$ is a partition of unity for the evolving surface cluster in the sense that $\eta_{\bar\Gamma}+\eta_{\bar I_{1,2}}+\eta_{\bar I_{2,3}}+\eta_{\bar I_{3,1}}\equiv 1$ holds true on the surface cluster $\mathcal{I} := \bigcup_{i,j\in\{1,2,3\},i\neq j}\bar I_{i,j}$.

Moreover, for all pairwise distinct $i,j,k \in \{1,2,3\}$ it holds

$$\eta_{\bar{I}_{k,i}} \leq C(\operatorname{dist}^2(\cdot,\bar{I}_{i,j}) \wedge 1) \qquad \text{in } \mathbb{R}^3\times[0,T], \tag{5.167}$$

$$|\nabla\eta_{\bar{I}_{k,i}}| \leq C(\operatorname{dist}(\cdot,\bar{I}_{i,j}) \wedge 1) \qquad \text{in } \mathbb{R}^3\times[0,T]\setminus\bar{\Gamma}, \tag{5.168}$$

$$|\partial_t\eta_{\bar{I}_{k,i}}| \leq C(\operatorname{dist}(\cdot,\bar{I}_{i,j}) \wedge 1) \qquad \text{in } \mathbb{R}^3\times[0,T]\setminus\bar{\Gamma}. \tag{5.169}$$

Defining $\eta_{\mathrm{bulk}} := 1 - \eta_{\bar{\Gamma}} - \eta_{\bar{I}_{1,2}} - \eta_{\bar{I}_{2,3}} - \eta_{\bar{I}_{3,1}}$ we have $\eta_{\mathrm{bulk}} \in [0,1]$ on $\mathbb{R}^3\times[0,T]$, and the bulk cutoff is subject to the estimates

$$\frac{1}{C}(\operatorname{dist}^2(\cdot,\mathcal{I}) \wedge 1) \leq \eta_{\mathrm{bulk}} \leq C(\operatorname{dist}^2(\cdot,\mathcal{I}) \wedge 1) \qquad \text{in } \mathbb{R}^3\times[0,T], \tag{5.170}$$

$$|\nabla\eta_{\mathrm{bulk}}| \leq C(\operatorname{dist}(\cdot,\mathcal{I}) \wedge 1) \qquad \text{in } \mathbb{R}^3\times[0,T]\setminus\bar{\Gamma}, \tag{5.171}$$

$$|\partial_t\eta_{\mathrm{bulk}}| \leq C(\operatorname{dist}(\cdot,\mathcal{I}) \wedge 1) \qquad \text{in } \mathbb{R}^3\times[0,T]\setminus\bar{\Gamma}. \tag{5.172}$$

The constant $C \geq 1$ in the estimates (5.167)–(5.172) depends only on the data of the smoothly evolving regular double bubble $(\bar{\Omega}_1,\bar{\Omega}_2,\bar{\Omega}_3)$ on $[0,T]$.

iii) For all pairwise distinct $i,j,k \in \{1,2,3\}$ and all $t \in [0,T]$ it holds

$$\operatorname{supp}\eta_{\bar{I}_{i,j}}(\cdot,t) \subset \Psi_{i,j}(\bar{I}_{i,j}(t)\times\{t\}\times[-\bar{r},\bar{r}]), \tag{5.173}$$

$$B_{\hat{r}}(\bar{\Gamma}(t)) \cap \operatorname{supp}\eta_{\bar{I}_{i,j}}(\cdot,t) \subset B_{\hat{r}}(\bar{\Gamma}(t)) \cap \big(W_{\bar{I}_{i,j}}(t) \cup W_{\bar{\Omega}_i}(t) \cup W_{\bar{\Omega}_j}(t)\big), \tag{5.174}$$

$$\operatorname{supp}\eta_{\bar{I}_{i,j}}(\cdot,t) \cap \operatorname{supp}\eta_{\bar{I}_{j,k}}(\cdot,t) \subset B_{\hat{r}}(\bar{\Gamma}(t)) \cap W_{\bar{\Omega}_j}(t), \tag{5.175}$$

$$\operatorname{supp}\eta_{\bar{\Gamma}}(\cdot,t) \subset B_{\hat{r}/2}(\bar{\Gamma}(t)). \tag{5.176}$$

Proof. The proof is split into several steps.

Step 1: Definition of building blocks. Let θ be a smooth and even cutoff function with $\theta(r) = 1$ for $|r| \leq \frac{1}{2}$ and $\theta(r) = 0$ for $|r| \geq 1$. Then define a smooth quadratic profile $\zeta\colon \mathbb{R} \to [0,1]$ by means of

$$\zeta(r) = (1-r^2)\theta(r^2), \quad r \in \mathbb{R}. \tag{5.177}$$

Let $\delta \in (0,1]$ be a constant whose value will be determined in subsequent steps of the proof. For all distinct $i,j \in \{1,2,3\}$ we define auxiliary cutoff functions

$$\zeta_{\bar{I}_{i,j}} := \zeta\Big(\frac{s_{i,j}}{\delta\bar{r}}\Big) \qquad \text{in } \operatorname{im}(\Psi_{i,j}), \tag{5.178}$$

$$\zeta_{\bar{\Gamma}} := \zeta\Big(\frac{\operatorname{dist}(\cdot,\bar{\Gamma})}{\hat{r}/2}\Big) \qquad \text{in } \mathbb{R}^3\times[0,T]. \tag{5.179}$$

Note that as a consequence of the regularity (5.18) of the signed distance, expressing $\operatorname{dist}(x,\bar{\Gamma}(t)) = |x - P_{\bar{\Gamma}}(x,t)|$ for all $x \in B_{\hat{r}}(\bar{\Gamma}(t))$ and all $t \in [0,T]$, the regularity of the projection $P_{\bar{\Gamma}}$ onto the triple line $\bar{\Gamma}$ from Definition 5.15, and (5.177) it holds

$$|(\partial_t,\nabla)\zeta_{\bar{I}_{i,j}}| \leq C\operatorname{dist}(\cdot,\bar{I}_{i,j}) \qquad \text{in } \operatorname{im}(\Psi_{i,j}), \tag{5.180}$$

$$|(\partial_t,\nabla)\zeta_{\bar{\Gamma}}| \leq C\operatorname{dist}(\cdot,\bar{\Gamma}) \qquad \text{in } \mathbb{R}^3\times[0,T]. \tag{5.181}$$

Step 2: Definition of interface cutoffs. Fix distinct $i,j \in \{1,2,3\}$. We define the cutoff $\eta_{\bar{I}_{i,j}}\colon \mathbb{R}^3\times[0,T] \to [0,1]$ for the two-phase interface $\bar{I}_{i,j}$ by means of

$$\eta_{\bar{I}_{i,j}}(\cdot,t) := \zeta_{\bar{I}_{i,j}}(\cdot,t) \qquad \text{in } \operatorname{im}(\Psi_{i,j}(t)) \setminus B_{\hat{r}}(\bar{\Gamma}(t)), \tag{5.182}$$

$$\eta_{\bar{I}_{i,j}}(\cdot,t) := (1-\zeta_{\bar{\Gamma}}(\cdot,t))\zeta_{\bar{I}_{i,j}}(\cdot,t) \qquad \text{in } B_{\hat{r}}(\bar{\Gamma}(t)) \cap W_{\bar{I}_{i,j}}(t), \tag{5.183}$$

$$\eta_{\bar{I}_{i,j}}(\cdot,t) := \lambda^{\bar{I}_{i,j}}_{\bar{\Omega}_i}(\cdot,t)(1-\zeta_{\bar{\Gamma}}(\cdot,t))\zeta_{\bar{I}_{i,j}}(\cdot,t) \qquad \text{in } B_{\hat{r}}(\bar{\Gamma}(t)) \cap W_{\bar{\Omega}_i}(t), \tag{5.184}$$

$$\eta_{\bar{I}_{i,j}}(\cdot,t) := \lambda^{\bar{I}_{i,j}}_{\bar{\Omega}_j}(\cdot,t)(1-\zeta_{\bar{\Gamma}}(\cdot,t))\zeta_{\bar{I}_{i,j}}(\cdot,t) \qquad \text{in } B_{\hat{r}}(\bar{\Gamma}(t)) \cap W_{\bar{\Omega}_j}(t), \tag{5.185}$$

$$\eta_{\bar{I}_{i,j}}(\cdot,t) := 0 \qquad \text{else} \tag{5.186}$$

for all $t \in [0,T]$. Here, the maps $\lambda^{\bar{I}_{i,j}}_{\bar{\Omega}_i}$ resp. $\lambda^{\bar{I}_{i,j}}_{\bar{\Omega}_j}$ are the interpolation functions of Lemma 5.26 on the interpolation wedges $W_{\bar{\Omega}_i}$ resp. $W_{\bar{\Omega}_j}$. Observe that (5.183) is well-defined because of (5.41), and that (5.184) resp. (5.185) are well-defined as a consequence of (5.42). In particular, the properties (5.173)–(5.175) are immediate consequences of the definitions (5.182)–(5.186) and the choice (5.165) of the localization scale $\bar{r}$. Finally, in order to ensure continuity of $\eta_{\bar{I}_{i,j}}$ throughout $\mathbb{R}^3\times[0,T]$ (i.e., compatibility of the definition (5.182) resp. the definition (5.186) with the definitions (5.183)–(5.185)) we choose the constant $\delta \in (0,\frac{1}{2}]$ small enough such that for all $t \in [0,T]$ and all distinct $i,j \in \{1,2,3\}$ it holds

$$\partial B_{\hat{r}}(\bar{\Gamma}(t)) \cap \overline{\Psi_{i,j}(\bar{I}_{i,j}(t)\times\{t\}\times[-\delta\hat{r},\delta\hat{r}])} \subset\subset W_{\bar{I}_{i,j}}(t). \tag{5.187}$$

Step 3: Definition of triple line cutoff. We construct a cutoff for the triple line $\eta_{\bar{\Gamma}}\colon \mathbb{R}^3\times[0,T] \to [0,1]$ as follows: for all distinct $i,j,k \in \{1,2,3\}$ and all $t \in [0,T]$ we define

$$\eta_{\bar{\Gamma}}(\cdot,t) := \zeta_{\bar{\Gamma}}(\cdot,t)\zeta_{\bar{I}_{i,j}}(\cdot,t) \qquad \text{in } B_{\hat{r}}(\bar{\Gamma}(t)) \cap W_{\bar{I}_{i,j}}(t), \tag{5.188}$$

$$\begin{aligned}\eta_{\bar{\Gamma}}(\cdot,t) :=\ & \lambda^{\bar{I}_{i,j}}_{\bar{\Omega}_i}(\cdot,t)\zeta_{\bar{\Gamma}}(\cdot,t)\zeta_{\bar{I}_{i,j}}(\cdot,t) \\ &+ \lambda^{\bar{I}_{k,i}}_{\bar{\Omega}_i}(\cdot,t)\zeta_{\bar{\Gamma}}(\cdot,t)\zeta_{\bar{I}_{k,i}}(\cdot,t) \qquad \text{in } B_{\hat{r}}(\bar{\Gamma}(t)) \cap W_{\bar{\Omega}_i}(t),\end{aligned} \tag{5.189}$$

$$\eta_{\bar{\Gamma}}(\cdot,t) := 0 \qquad \text{in } \mathbb{R}^3 \setminus B_{\hat{r}}(\bar{\Gamma}(t)). \tag{5.190}$$

Because of (5.40), the definitions (5.188)–(5.190) provide a definition of $\eta_{\bar{\Gamma}}$ on the whole space-time domain $\mathbb{R}^3\times[0,T]$. Property (5.176) is obviously satisfied in view of (5.190). Since $\lambda^{\bar{I}_{i,k}}_{\bar{\Omega}_i} = 1-\lambda^{\bar{I}_{i,j}}_{\bar{\Omega}_i}$ on interpolation wedges $W_{\bar{\Omega}_i}$, we indeed have $\eta_{\bar{\Gamma}}(x,t) \in [0,1]$ for all $(x,t) \in \mathbb{R}^3\times[0,T]$.

Step 4: Partition of unity property along the surface cluster. Define the bulk cutoff $\eta_{\mathrm{bulk}} := 1 - \eta_{\bar{\Gamma}} - \eta_{\bar{I}_{1,2}} - \eta_{\bar{I}_{2,3}} - \eta_{\bar{I}_{3,1}}$. We claim that

$$\eta_{\mathrm{bulk}} = 0 \quad \text{along } \mathcal{I} = \bigcup_{i,j\in\{1,2,3\},\, i\neq j} \bar{I}_{i,j}. \tag{5.191}$$

Fix $t \in [0,T]$ and a point $x \in \mathcal{I}(t) \setminus B_{\hat{r}}(\bar{\Gamma}(t))$. There exists a unique pair of distinct phases $i,j \in \{1,2,3\}$ such that $x \in \bar{I}_{i,j}(t)$ and, because of the localization properties (5.175) and (5.176), $\eta_{\mathrm{bulk}}(x,t) = 1-\eta_{\bar{I}_{i,j}}(x,t)$. It then follows from the definitions (5.182) and (5.178) that $\eta_{\mathrm{bulk}}(x,t) = 0$.

Now fix $t \in [0,T]$ and consider a point $x \in \mathcal{I}(t) \cap B_{\hat{r}}(\bar{\Gamma}(t))$. Let $i,j \in \{1,2,3\}$ be the unique pair of distinct phases such that $x \in \bar{I}_{i,j}(t)$. As a consequence of (5.41), the localization properties (5.174)–(5.176), and the definitions (5.183) resp. (5.188), we obtain that $\eta_{\mathrm{bulk}}(x,t) = 1 - \eta_{\bar{\Gamma}}(x,t) - \eta_{\bar{I}_{i,j}}(x,t) = 1 - \zeta_{\bar{I}_{i,j}}(x,t)$. Hence, $\eta_{\mathrm{bulk}}(x,t) = 0$ due to the definition (5.178). This concludes the proof of (5.191).

Step 5: Regularity of cutoff functions. Fix $i,j \in \{1,2,3\}$ such that $i \neq j$. The required derivatives of $\eta_{\bar{I}_{i,j}}$ exist in $\mathbb{R}^3 \setminus \overline{B_{\hat{r}}(\bar{\Gamma}(t))}$ resp. in $B_{\hat{r}}(\bar{\Gamma}(t)) \setminus \bar{\Gamma}(t)$ in a pointwise sense for all $t \in [0,T]$ due to the definition of $\eta_{\bar{I}_{i,j}}$ from *Step 2* of this proof, the definitions (5.178) and (5.179), the properties of the interpolation functions from Lemma 5.26, and the regularity (5.180) and (5.181) of the auxiliary cutoff functions. By the choice (5.187) of the scale $\delta \in (0,1]$, these derivatives do not jump across the boundary of $B_{\hat{r}}(\bar{\Gamma}(t))$. Hence, $\partial_t\eta_{\bar{I}_{i,j}}$ and $\nabla\eta_{\bar{I}_{i,j}}$ exist in a pointwise sense in $\mathbb{R}^3\times[0,T] \setminus \bar{\Gamma}$.

In terms of the required bounds (5.166) for these derivatives, the only possibly critical cases are those for which at least one derivative hits an interpolation function present in the definitions (5.184) resp. (5.185). The blow-up of these derivatives (see Lemma 5.26), however, is always cured by the presence of the term $1-\zeta_{\bar{\Gamma}}$. In summary, $\eta_{\bar{I}_{i,j}} \in (C_t^0C_x^1 \cap C_t^1C_x^0)(\mathbb{R}^3{\times}[0,T] \setminus \bar{\Gamma})$ and (5.166) holds true.

Along similar lines, one checks that $\partial_t\eta_{\bar{\Gamma}}$ and $\nabla\eta_{\bar{\Gamma}}$ exist in a pointwise sense in $\mathbb{R}^3{\times}[0,T] \setminus \bar{\Gamma}$. The required cancellations to counteract the blow-up of derivatives of the interpolation parameter in interpolation wedges this time comes from recalling $\lambda_{\bar{\Omega}_i}^{\bar{I}_{k,i}} = 1-\lambda_{\bar{\Omega}_i}^{\bar{I}_{i,j}}$, which in turn ensures that potentially critical terms always involve the term $\zeta_{\bar{I}_{i,j}} - \zeta_{\bar{I}_{k,i}}$. As the latter vanishes to first order at the triple line and has a bounded second-order spatial derivative within interpolation wedges, it follows that $\eta_{\bar{\Gamma}} \in (C_t^0C_x^1 \cap C_t^1C_x^0)(\mathbb{R}^3{\times}[0,T] \setminus \bar{\Gamma})$, and that (5.166) holds true.

Step 6: Estimates for the bulk cutoff. By construction it holds $\eta_{\mathrm{bulk}}(\cdot,t) \equiv 1$ outside of the space-time domain $B_{\hat{r}}(\bar{\Gamma}(t)) \cup \bigcup_{i,j\in\{1,2,3\},i\neq j} \Psi_{i,j}(\bar{I}_{i,j}(t){\times}\{t\}{\times}[-\bar{r},\bar{r}])$ for all $t \in [0,T]$. Hence, for a proof of $\eta_{\mathrm{bulk}} \in [0,1]$ and the estimates (5.170)–(5.172), we may restrict our attention to $\bigcup_{i,j\in\{1,2,3\},i\neq j} \Psi_{i,j}(\bar{I}_{i,j}(t){\times}\{t\}{\times}[-\bar{r},\bar{r}]) \setminus B_{\hat{r}}(\bar{\Gamma}(t))$ and $B_{\hat{r}}(\bar{\Gamma}(t))$ for all $t \in [0,T]$.

In view of the choice (5.165) of the localization scale $\bar{r}$, one may argue separately on $\Psi_{i,j}(\bar{I}_{i,j}(t){\times}\{t\}{\times}[-\bar{r},\bar{r}]) \setminus B_{\hat{r}}(\bar{\Gamma}(t))$ for each pair of distinct phases $i,j \in \{1,2,3\}$ and all $t \in [0,T]$. Because of the localization properties (5.175) and (5.176) it holds

$$\begin{aligned} \eta_{\mathrm{bulk}}(\cdot,t) &= 1 - \eta_{\bar{I}_{i,j}}(\cdot,t) \\ &= 1 - \zeta_{\bar{I}_{i,j}}(\cdot,t) \quad \text{in } \Psi_{i,j}(\bar{I}_{i,j}(t){\times}\{t\}{\times}[-\bar{r},\bar{r}]) \setminus B_{\hat{r}}(\bar{\Gamma}(t)) \end{aligned} \tag{5.192}$$

for all $t \in [0,T]$. Hence, $\eta_{\mathrm{bulk}} \in [0,1]$ and the estimates (5.170)–(5.172) follow from the definitions (5.182) and (5.178) in combination with the quadratic behaviour around the origin of the profile (5.177). Note in this context that (5.165) precisely ensures that the error can be expressed in terms of $\mathrm{dist}(\cdot,\mathcal{I})$ as required.

We move on to the argument in the ball $B_{\hat{r}}(\bar{\Gamma}(t))$ for all $t \in [0,T]$. On interface wedges, we infer from the localization properties (5.174) and (5.175) as well as the definitions (5.183) and (5.188) that

$$\begin{aligned} \eta_{\mathrm{bulk}}(\cdot,t) &= 1 - \eta_{\bar{\Gamma}}(\cdot,t) - \eta_{\bar{I}_{i,j}}(\cdot,t) \\ &= 1 - \zeta_{\bar{I}_{i,j}}(\cdot,t) \quad \text{in } B_{\hat{r}}(\bar{\Gamma}(t)) \cap W_{\bar{I}_{i,j}}(t) \end{aligned} \tag{5.193}$$

for all $t \in [0,T]$, so that the asserted bounds follow as in the previous case together with the bound (5.45) to express the error in terms of $\mathrm{dist}(\cdot,\mathcal{I})$.

On interpolation wedges, we may compute based on (5.174) and (5.175) as well as (5.184) and (5.189) that (recall the relation $\lambda_{\bar{\Omega}_i}^{\bar{I}_{k,i}} = 1-\lambda_{\bar{\Omega}_i}^{\bar{I}_{i,j}}$)

$$\begin{aligned} \eta_{\mathrm{bulk}}(\cdot,t) &= 1 - \eta_{\bar{\Gamma}}(\cdot,t) - \eta_{\bar{I}_{i,j}}(\cdot,t) - \eta_{\bar{I}_{k,i}}(\cdot,t) \\ &= \lambda_{\bar{\Omega}_i}^{\bar{I}_{i,j}}(1-\zeta_{\bar{I}_{i,j}})(\cdot,t) + (1-\lambda_{\bar{\Omega}_i}^{\bar{I}_{i,j}})(1-\zeta_{\bar{I}_{k,i}})(\cdot,t) \quad \text{in } B_{\hat{r}}(\bar{\Gamma}(t)) \cap W_{\bar{\Omega}_i}(t) \end{aligned} \tag{5.194}$$

for all $t \in [0,T]$. It follows immediately that $\eta_{\mathrm{bulk}}(\cdot,t) \in [0,1]$. Moreover, the definition (5.178), the quadratic behavior around the origin of the profile (5.177), and the estimate (5.43) directly imply (5.170). Finally, since

$$\begin{aligned} \nabla\eta_{\mathrm{bulk}}(\cdot,t) &= -\lambda_{\bar{\Omega}_i}^{\bar{I}_{i,j}}(\cdot,t)\nabla\zeta_{\bar{I}_{i,j}}(\cdot,t) - (1-\lambda_{\bar{\Omega}_i}^{\bar{I}_{i,j}})(\cdot,t)\nabla\zeta_{\bar{I}_{k,i}}(\cdot,t) \\ &\quad - (\zeta_{\bar{I}_{i,j}}-\zeta_{\bar{I}_{k,i}})(\cdot,t)\nabla\lambda_i^{\bar{I}_{i,j}}(\cdot,t) \quad \text{in } B_{\hat{r}}(\bar{\Gamma}(t)) \cap W_{\bar{\Omega}_i}(t) \end{aligned} \tag{5.195}$$

for all $t \in [0,T]$, we obtain (5.171) and (5.172) because the blow-up of $\nabla\lambda_{\bar{\Omega}_i}^{\bar{I}_{i,j}}$, see Lemma 5.26, is cancelled to required order by the term $\zeta_{\bar{I}_{i,j}}-\zeta_{\bar{I}_{k,i}}$. Indeed, the latter vanishes to first order at the triple line and has a bounded second-order spatial derivative within interpolation wedges.

Step 7: Error estimates for interface cutoffs. The bounds (5.167)–(5.169) are trivially fulfilled outside of $B_{\hat{r}}(\bar{\Gamma}(t))$ for all $t \in [0,T]$ by construction and the choice (5.165) of the localization scale $\bar{r}$. In view of the definitions (5.183)–(5.185) and the definition (5.179), we also have $\eta_{\bar{I}_{k,i}}(\cdot,t) \leq 1-\zeta_{\bar{\Gamma}}(\cdot,t) \leq C\,\mathrm{dist}^2(\cdot,\bar{\Gamma}(t))$ in $B_{\hat{r}}(\bar{\Gamma}(t)) \cap \big(W_{\bar{I}_{k,i}}(t) \cup W_{\bar{\Omega}_i}(t) \cup W_{\bar{\Omega}_k}(t)\big)$ for all $t \in [0,T]$. Recalling the bounds (5.43) and (5.44), this in turn implies (5.167) throughout $B_{\hat{r}}(\bar{\Gamma}(t))$ for all $t \in [0,T]$.

For a proof of (5.168) and (5.169), note that

$$|(\partial_t,\nabla)\eta_{\bar{I}_{k,i}}(\cdot,t)| \leq C(1-\zeta_{\bar{\Gamma}}(\cdot,t)) + C|(\partial_t,\nabla)\,\mathrm{dist}(\cdot,\bar{\Gamma}(t))|\,\mathrm{dist}(\cdot,\bar{\Gamma}(t))$$

in $B_{\hat{r}}(\bar{\Gamma}(t)) \cap \big(W_{\bar{I}_{k,i}}(t) \cup W_{\bar{\Omega}_i}(t) \cup W_{\bar{\Omega}_k}(t)\big)$ for all $t \in [0,T]$. The first right hand side term is estimated as before, while the second one is of required order due to the bounds (5.43) resp. (5.44) and the regularity of the projection onto the triple line $\bar{\Gamma}$, see Definition 5.15, which in turn one may employ throughout $B_{\hat{r}}(\bar{\Gamma}(t))$ based on the representation $|x-P_{\bar{\Gamma}}(x,t)| = \mathrm{dist}(x,\bar{\Gamma}(t))$. □

5.4.2 Construction of a gradient-flow calibration

We have everything in place to provide the construction of a gradient-flow calibration for a regular double bubble smoothly evolving by MCF. We first introduce a global definition for the vector fields $\xi_{i,j}$ extending the unit normal vector fields $\bar{\mathrm{n}}_{i,j}|_{\bar{I}_{i,j}}$ of the interfaces $\bar{I}_{i,j}$.

Construction 5.30 (Global extensions of the unit normal vector fields $\bar{\mathrm{n}}_{i,j}|_{\bar{I}_{i,j}}$)**.** Consider a regular double bubble $(\bar{\Omega}_1,\bar{\Omega}_2,\bar{\Omega}_3)$ smoothly evolving by MCF in the sense of Definition 5.10 on a time interval $[0,T]$. Let $(\eta_{\bar{\Gamma}},(\eta_{\bar{I}_{i,j}})_{i,j\in\{1,2,3\},i\neq j})$ be the partition of unity from the proof of Lemma 5.29. Fix $i,j \in \{1,2,3\}$ with $i \neq j$. We then define a family of vector fields

$$\xi_{i,j}^{\bar{I}_{k,l}}\colon \bigcup_{t\in[0,T]} \operatorname{supp}\eta_{\bar{I}_{k,l}}(\cdot,t)\times\{t\} \to \overline{B_1(0)}, \quad k,l\in\{1,2,3\},\, k\neq l, \tag{5.196}$$

$$\xi_{i,j}^{\bar{\Gamma}}\colon \bigcup_{t\in[0,T]} \operatorname{supp}\eta_{\bar{\Gamma}}(\cdot,t)\times\{t\} \to \overline{B_1(0)} \tag{5.197}$$

by means of the following procedure:

For $k,l \in \{1,2,3\}$ with $(k,l) \in \{(i,j),(j,i)\}$ we let $\xi_{i,j}^{\bar{I}_{k,l}}$ be the corresponding vector field from Construction 5.12 for the interface $\bar{I}_{k,l}$. For $k,l \in \{1,2,3\}$ with $(k,l) \notin \{(i,j),(j,i)\}$ and $k \neq l$ we define $\xi_{i,j}^{\bar{I}_{k,l}} := \frac{1}{2}\big(\frac{\sigma_{l,i}-\sigma_{l,j}}{\sigma_{i,j}}\xi_{k,l}^{\bar{I}_{k,l}} + \frac{\sigma_{k,i}-\sigma_{k,j}}{\sigma_{i,j}}\xi_{l,k}^{\bar{I}_{k,l}}\big)$, which is well-defined reversing the roles of i,j and k,l in the previous step. Finally, we denote by $\xi_{i,j}^{\bar{\Gamma}}$ the corresponding vector field from the proof of Proposition 5.28.

With this family of local vector fields in place, we now define a global vector field $\xi_{i,j}\colon \mathbb{R}^3\times[0,T] \to \mathbb{R}^3$ by means of

$$\xi_{i,j} := \eta_{\bar{\Gamma}}\xi_{i,j}^{\bar{\Gamma}} + \eta_{\bar{I}_{1,2}}\xi_{i,j}^{\bar{I}_{1,2}} + \eta_{\bar{I}_{2,3}}\xi_{i,j}^{\bar{I}_{2,3}} + \eta_{\bar{I}_{3,1}}\xi_{i,j}^{\bar{I}_{3,1}} \tag{5.198}$$

for all distinct pairs of phases $i,j \in \{1,2,3\}$. ♢

We proceed by showing that the vector fields from the previous construction satisfy the structural assumption (5.1a) and the coercivity estimate (5.1c) of a gradient-flow calibration.

Lemma 5.31. *Let the assumptions and notation of Construction 5.30 be in place. Fix $i,j \in \{1,2,3\}$ such that $i \neq j$. The vector field $\xi_{i,j}$ is then subject to the following list of properties:*

i) *It holds* $\xi_{i,j} \in (C^0_tC^1_x \cap C^1_tC^0_x)(\mathbb{R}^3\times[0,T]\setminus\bar{\Gamma})$, *and there exists a constant* $C > 0$ *which depends only on the data of the smoothly evolving regular double bubble* $(\bar{\Omega}_1, \bar{\Omega}_2, \bar{\Omega}_3)$ *on* $[0,T]$ *such that*

$$|(\partial_t, \nabla)\xi_{i,j}| \leq C \quad \text{in } \mathbb{R}^3\times[0,T]\setminus\bar{\Gamma}. \tag{5.199}$$

Moreover, it holds $\xi_{i,j} = \bar{\mathrm{n}}_{i,j}$ *along* $\bar{I}_{i,j}$.

ii) *For each phase* $i \in \{1,2,3\}$, *there exists a vector field* $\xi_i\colon \mathbb{R}^3\times[0,T] \to \mathbb{R}^3$ *of class* $(C^0_tC^1_x \cap C^1_tC^0_x)(\mathbb{R}^3\times[0,T]\setminus\bar{\Gamma})$ *such that* $\sigma_{i,j}\xi_{i,j} = \xi_i - \xi_j$ *holds true on* $\mathbb{R}^3\times[0,T]$.

iii) *There exists a constant* $c \in (0,1)$, *which depends only on the data of the smoothly evolving regular double bubble* $(\bar{\Omega}_1, \bar{\Omega}_2, \bar{\Omega}_3)$ *on* $[0,T]$, *such that*

$$c(\mathrm{dist}^2(\cdot, \bar{I}_{i,j}) \wedge 1) \leq 1 - |\xi_{i,j}| \quad \text{in } \mathbb{R}^3\times[0,T]. \tag{5.200}$$

Proof. The proof is performed in three steps.

Step 1: Regularity and structural properties. The asserted qualitative regularity of the vector fields $\xi_{i,j}$ together with the estimate (5.199) follows from the definition (5.198), the regularity (5.166) of the cutoff functions, as well as the regularity of the local building blocks (5.196) and (5.197) in form of

$$\big|(\partial_t, \nabla)(\xi^{\bar{I}_{k,l}}_{i,j}, \xi^{\bar{\Gamma}}_{i,j})\big| \leq C \quad \text{in } \mathbb{R}^3\times[0,T], \tag{5.201}$$

which in turn is a consequence of the definitions from Construction 5.30 and the regularity estimates (5.24) and (5.30). The property $\xi_{i,j}|_{\bar{I}_{i,j}} \equiv \bar{\mathrm{n}}_{i,j}$ is immediate from the definition (5.198), the fact that $(\eta_{\bar{\Gamma}}, (\eta_{\bar{I}_{i,j}})_{i,j\in\{1,2,3\},i\neq j})$ constitutes a partition of unity along the network $\mathcal{I}$, and the corresponding property in terms of the local constructions from Lemma 5.13 and Proposition 5.14.

The existence of vector fields $(\xi_i)_{i\in\{1,2,3\}}$ of class $(C^0_tC^1_x \cap C^1_tC^0_x)(\mathbb{R}^3\times[0,T]\setminus\bar{\Gamma})$ such that $\sigma_{i,j}\xi_{i,j} = \xi_i - \xi_j$ holds true on $\mathbb{R}^3\times[0,T]$ follows from the following considerations. Let $i,j,k \in \{1,2,3\}$ be pairwise distinct. We define $\xi^{\bar{\Gamma}}_i := \frac{1}{3}(\sigma_{i,j}\xi^{\bar{\Gamma}}_{i,j} + \sigma_{i,k}\xi^{\bar{\Gamma}}_{i,k})$. Since $\sigma_{1,2}\xi^{\bar{\Gamma}}_{1,2} + \sigma_{2,3}\xi^{\bar{\Gamma}}_{2,3} + \sigma_{3,1}\xi^{\bar{\Gamma}}_{3,1} = 0$ holds true in the support of $\eta_{\bar{\Gamma}}$, see Proposition 5.14, we indeed obtain $\sigma_{i,j}\xi^{\bar{\Gamma}}_{i,j} = \xi^{\bar{\Gamma}}_i - \xi^{\bar{\Gamma}}_j$. Next, fix $k,l \in \{1,2,3\}$ with $k \neq l$, and let $i \in \{1,2,3\}$. We may then define $\xi^{\bar{I}_{k,l}}_i := \frac{1}{2}(\sigma_{l,i}\xi^{\bar{I}_{k,l}}_{k,l} + \sigma_{k,i}\xi^{\bar{I}_{k,l}}_{l,k})$. Again, plugging in the definitions immediately shows $\sigma_{i,j}\xi^{\bar{I}_{k,l}}_{i,j} = \xi^{\bar{I}_{k,l}}_i - \xi^{\bar{I}_{k,l}}_j$ for all $i,j \in \{1,2,3\}$ such that $i \neq j$. Defining $\xi_i := \eta_{\bar{\Gamma}}\xi^{\bar{\Gamma}}_i + \eta_{\bar{I}_{1,2}}\xi^{\bar{I}_{1,2}}_i + \eta_{\bar{I}_{2,3}}\xi^{\bar{I}_{2,3}}_i + \eta_{\bar{I}_{3,1}}\xi^{\bar{I}_{3,1}}_i$ therefore entails the desired conclusion.

Step 2: A coercivity condition. As a preparation for the proof of (5.200), we claim that there exists a constant $\varepsilon = \varepsilon(\sigma) \in (0,1)$ such that for all $i,j \in \{1,2,3\}$ with $i \neq j$, as well as all $k,l \in \{1,2,3\}$ with $(k,l) \notin \{(i,j),(j,i)\}$ and $k \neq l$ it holds

$$\big|\xi^{\bar{I}_{k,l}}_{i,j}\big| \leq \varepsilon < 1. \tag{5.202}$$

Indeed, the estimate (5.202) is an immediate consequence of the definition of the vector field $\xi^{\bar{I}_{k,l}}_{i,j} = \frac{1}{2}(\frac{\sigma_{l,i}-\sigma_{l,j}}{\sigma_{i,j}}\xi^{\bar{I}_{k,l}}_{k,l} + \frac{\sigma_{k,i}-\sigma_{k,j}}{\sigma_{i,j}}\xi^{\bar{I}_{k,l}}_{l,k})$, see Construction 5.30, and the fact that $|\frac{\sigma_{l,i}-\sigma_{l,j}}{\sigma_{i,j}}| < 1$ resp. $|\frac{\sigma_{k,i}-\sigma_{k,j}}{\sigma_{i,j}}| < 1$, which in turn is true since the matrix of surface tensions satisfies the strict triangle inequality by assumption.

Step 3: Proof of the estimate (5.200). Fix $i,j \in \{1,2,3\}$ such that $i \neq j$. By the localization properties (5.173)–(5.176) and the choice (5.165) of the localization scale $\bar{r}$, it suffices to establish the desired estimate throughout $\mathrm{supp}\,\eta_{\bar{I}_{k,l}}(\cdot,t) \setminus B_{\hat{r}}(\bar{\Gamma}(t))$, $B_{\hat{r}}(\bar{\Gamma}(t)) \cap W_{\bar{I}_{k,l}}(t)$

or $B_{\hat r}(\bar\Gamma(t)) \cap W_{\bar\Omega_l}(t)$ for all distinct phases $k,l \in \{1,2,3\}$ and all $t \in [0,T]$. Hence, fix such $k,l \in \{1,2,3\}$ with $k \neq l$ and $t \in [0,T]$, and then observe that due to the definition (5.198) and the localization properties (5.173)–(5.176) it holds

$$\xi_{i,j} = \begin{cases} \eta_{\bar I_{k,l}} \xi_{i,j}^{\bar I_{k,l}} & \text{on } \operatorname{supp} \eta_{\bar I_{k,l}}(\cdot,t) \setminus B_{\hat r}(\bar\Gamma(t)), \\ \eta_{\bar\Gamma}\xi_{i,j}^{\bar\Gamma} + \eta_{\bar I_{k,l}}\xi_{i,j}^{\bar I_{k,l}} & \text{on } B_{\hat r}(\bar\Gamma(t)) \cap W_{\bar I_{k,l}}(t), \\ \eta_{\bar\Gamma}\xi_{i,j}^{\bar\Gamma} + \eta_{\bar I_{k,l}}\xi_{i,j}^{\bar I_{k,l}} + \eta_{\bar I_{l,m}}\xi_{i,j}^{\bar I_{l,m}} & \text{on } B_{\hat r}(\bar\Gamma(t)) \cap W_{\bar\Omega_l}(t),\, m \in \{1,2,3\} \setminus \{k,l\}. \end{cases} \tag{5.203}$$

Based on (5.203), we now distinguish between two cases.

Substep 3.1: Assume that $(k,l) \in \{(i,j),(j,i)\}$. In other words, both the phases k and l are present at the interface $\bar I_{i,j}$. In this case, observe first that throughout the three domains represented in (5.203) it holds due to (5.43), (5.45) and (5.165) that the distance to $\mathcal{I}$ is comparable to the distance to $\bar I_{i,j}$: $\frac{1}{C}\operatorname{dist}(\cdot,\bar I_{i,j}) \leq \operatorname{dist}(\cdot,\mathcal{I}) \leq C\operatorname{dist}(\cdot,\bar I_{i,j})$ for some constant $C \geq 1$. Furthermore, it follows from (5.203) and the triangle inequality that $|\xi_{i,j}| \leq 1-\eta_{\mathrm{bulk}}$ throughout the three domains represented in (5.203). Hence, the bound (5.200) follows from the lower bound in (5.170).

Substep 3.2: Assume that $(k,l) \notin \{(i,j),(j,i)\}$. In the first case of (5.203), the estimate (5.200) follows immediately from the coercivity condition (5.202). In the third case of (5.203), we may additionally assume that $(l,m) \notin \{(i,j),(j,i)\}$; otherwise, we are again in the setting of the argument from *Substep 3.1* above. Plugging in the definitions (5.184), (5.185) and (5.189), as well as exploiting the coercivity condition (5.202) for both the vector fields $\xi_{i,j}^{\bar I_{k,l}}$ and $\xi_{i,j}^{\bar I_{l,m}}$ (which is admissible due to our assumptions), we may estimate from below

$$\begin{aligned} 1 - |\xi_{i,j}| &\geq 1 - \big(\eta_{\bar\Gamma} + \varepsilon\eta_{\bar I_{k,l}} + \varepsilon\eta_{\bar I_{l,m}}\big) \\ &\geq (1-\varepsilon)(1-\zeta_{\bar\Gamma}) \geq (1-\varepsilon)(\operatorname{dist}^2(\cdot,\bar\Gamma) \wedge 1) \end{aligned}$$

on $B_{\hat r}(\bar\Gamma(t)) \cap W_{\bar\Omega_l}(t)$ for all $t \in [0,T]$, so that (5.200) follows again. Since the argument proceeds similarly in the second case of (5.203), we may conclude the proof. □

The next step consists of providing the global definition of a suitable velocity field along which a smoothly evolving regular double bubble and our associated constructions are transported.

Construction 5.32 (Global extension of velocity vector field)**.** Let $(\bar\Omega_1,\bar\Omega_2,\bar\Omega_3)$ be a regular double bubble smoothly evolving by MCF in the sense of Definition 5.10 on a time interval $[0,T]$. Let $(\eta_{\bar\Gamma},(\eta_{\bar I_{i,j}})_{i,j\in\{1,2,3\},i\neq j})$ be the partition of unity from the proof of Lemma 5.29. We then introduce a family of vector fields

$$B^{\bar I_{i,j}} \colon \bigcup_{t\in[0,T]} \operatorname{supp}\eta_{\bar I_{i,j}}(\cdot,t) \times \{t\} \to \mathbb{R}^3 \quad \text{for all } i,j \in \{1,2,3\},\, i \neq j, \tag{5.204}$$

$$B^{\bar\Gamma} \colon \bigcup_{t\in[0,T]} \operatorname{supp}\eta_{\bar\Gamma}(\cdot,t) \times \{t\} \to \mathbb{R}^3 \tag{5.205}$$

as follows: the velocity field $B^{\bar\Gamma}$ denotes the corresponding vector field from the proof of Proposition 5.28, whereas $B^{\bar I_{i,j}}$ is the velocity field from Construction 5.12 with tangential component chosen as in the proof of Proposition 5.28.

With this family of local vector fields in place, we now define a global velocity field by means of

$$B := \eta_{\bar\Gamma}B^{\bar\Gamma} + \eta_{\bar I_{1,2}}B^{\bar I_{1,2}} + \eta_{\bar I_{2,3}}B^{\bar I_{2,3}} + \eta_{\bar I_{3,1}}B^{\bar I_{3,1}} \tag{5.206}$$

throughout $\mathbb{R}^3 \times [0,T]$. ◇

A crucial ingredient for the proof of the estimates (5.1d) and (5.1e) are the following bounds on the advective derivatives of the partition of unity from Lemma 5.29.

Lemma 5.33. *Let the assumptions and notation of Construction 5.32 be in place. In particular,* $(\eta_{\bar{\Gamma}}, (\eta_{\bar{I}_{i,j}})_{i,j\in\{1,2,3\},i\neq j})$ *denotes the partition of unity from the proof of Lemma 5.29. Then* $B \in C^0_tC^1_x(\mathbb{R}^3{\times}[0,T] \setminus \bar{\Gamma})$ *with corresponding estimate*

$$|B| + |\nabla B| \leq C \quad \text{in } \mathbb{R}^3{\times}[0,T] \setminus \bar{\Gamma}. \tag{5.207}$$

Moreover, the velocity field B *gives rise to an improved estimate on the advective derivative of the bulk cutoff in form of*

$$|\partial_t\eta_{\mathrm{bulk}} + (B\cdot\nabla)\eta_{\mathrm{bulk}}| \leq C(\mathrm{dist}^2(\cdot,\mathcal{I}) \wedge 1) \quad \text{in } \mathbb{R}^3 \times [0,T], \tag{5.208}$$

and similarly for all pairwise distinct phases $i,j,k \in \{1,2,3\}$

$$|\partial_t\eta_{\bar{I}_{k,i}} + (B\cdot\nabla)\eta_{\bar{I}_{k,i}}| \leq C(\mathrm{dist}^2(\cdot,\bar{I}_{i,j}) \wedge 1) \quad \text{in } \mathbb{R}^3 \times [0,T]. \tag{5.209}$$

The constant $C > 0$ *in the estimates* (5.207)–(5.209) *depends only on the data of the smoothly evolving regular double bubble* $(\bar{\Omega}_1, \bar{\Omega}_2, \bar{\Omega}_3)$ *on* $[0,T]$.

Proof. The proof is decomposed into three steps.

Step 1: Regularity estimates. The asserted qualitative regularity of the velocity field B together with the associated estimate (5.207) follow from its definition (5.206), the regularity (5.166) of the cutoff functions, as well as the regularity of the local building blocks (5.204) and (5.205) in form of

$$\big|(B^{\bar{\Gamma}}, B^{\bar{I}_{i,j}})\big| + \big|\nabla(B^{\bar{\Gamma}}, B^{\bar{I}_{i,j}})\big| \leq C \quad \text{in } \mathbb{R}^3{\times}[0,T], \tag{5.210}$$

which is a consequence of (5.25) and (5.31).

Step 2: Proof of (5.208). It holds $\eta_{\mathrm{bulk}}(\cdot,t) \equiv 1$ outside of the space-time domain $B_{\hat{r}}(\bar{\Gamma}(t)) \cup \bigcup_{i,j\in\{1,2,3\},i\neq j} \Psi_{i,j}(\bar{I}_{i,j}(t){\times}\{t\}{\times}[-\bar{r},\bar{r}])$ for all $t \in [0,T]$ by construction. Hence, for a proof of the estimate (5.208), we may restrict our attention to $\bigcup_{i,j\in\{1,2,3\},i\neq j} \Psi_{i,j}(\bar{I}_{i,j}(t){\times}\{t\}{\times}[-\bar{r},\bar{r}]) \setminus B_{\hat{r}}(\bar{\Gamma}(t))$ and $B_{\hat{r}}(\bar{\Gamma}(t))$ for all $t \in [0,T]$. By the choice (5.165) of the localization scale $\bar{r}$, one may even argue separately on $\Psi_{i,j}(\bar{I}_{i,j}(t){\times}\{t\}{\times}[-\bar{r},\bar{r}]) \setminus B_{\hat{r}}(\bar{\Gamma}(t))$ for each pair of distinct phases $i,j \in \{1,2,3\}$ and all $t \in [0,T]$.

Substep 2.1: Proof of (5.208) *on* $\Psi_{i,j}(\bar{I}_{i,j}(t){\times}\{t\}{\times}[-\bar{r},\bar{r}]) \setminus B_{\hat{r}}(\bar{\Gamma}(t))$. It follows from the representation (5.192) and the definition (5.206) that $B = \eta_{\bar{I}_{i,j}}B^{\bar{I}_{i,j}}$ and

$$|\partial_t\eta_{\mathrm{bulk}} + (B\cdot\nabla)\eta_{\mathrm{bulk}}| \leq \big|\partial_t\zeta_{\bar{I}_{i,j}} + (B^{\bar{I}_{i,j}}\cdot\nabla)\zeta_{\bar{I}_{i,j}}\big| + \eta_{\mathrm{bulk}}\big|(B^{\bar{I}_{i,j}}\cdot\nabla)\zeta_{\bar{I}_{i,j}}\big| \tag{5.211}$$

throughout $\Psi_{i,j}(\bar{I}_{i,j}(t){\times}\{t\}{\times}[-\bar{r},\bar{r}]) \setminus B_{\hat{r}}(\bar{\Gamma}(t))$ for all $t \in [0,T]$.

Recall that the signed distance $s_{i,j}$ satisfies

$$\partial_t s_{i,j} + (B^{\bar{I}_{i,j}}\cdot\nabla)s_{i,j} = 0 \quad \text{in } \mathrm{im}(\Psi_{i,j}) \tag{5.212}$$

as a consequence of the choice of the local velocity $B^{\bar{I}_{i,j}}$, cf. Construction 5.32, Construction 5.12 and (5.29). Hence, we infer from the definition (5.178) and an application of the chain rule that

$$\partial_t\zeta_{\bar{I}_{i,j}} + (B^{\bar{I}_{i,j}}\cdot\nabla)\zeta_{\bar{I}_{i,j}} = 0 \quad \text{in } \mathrm{im}(\Psi_{i,j}). \tag{5.213}$$

For an estimate of the second right hand side term of (5.211), we simply make use of the upper bound for the bulk cutoff (5.170) as well as the regularity estimates (5.210) and (5.180) of $B^{\bar{I}_{i,j}}$ and $\zeta_{\bar{I}_{i,j}}$, respectively.

Substep 2.2: Proof of (5.208) *on* $B_{\hat r}(\bar\Gamma(t)) \cap W_{\bar I_{i,j}}(t)$. In the interface wedge $W_{\bar I_{i,j}}(t) \cap B_{\hat r}(\bar\Gamma(t))$, it holds $B = \eta_{\bar\Gamma} B^{\bar\Gamma} + \eta_{\bar I_{i,j}} B^{\bar I_{i,j}}$ thanks to the representation (5.193) and the definition (5.206). We may then estimate, making use again of (5.193),

$$
\begin{aligned}
|\partial_t \eta_{\mathrm{bulk}} + (B\cdot\nabla)\eta_{\mathrm{bulk}}| &\leq \big|\partial_t \zeta_{\bar I_{i,j}} + (B^{\bar I_{i,j}}\cdot\nabla)\zeta_{\bar I_{i,j}}\big| \\
&\quad + \eta_{\mathrm{bulk}}\big|(B^{\bar I_{i,j}}\cdot\nabla)\zeta_{\bar I_{i,j}}\big| + \eta_{\bar\Gamma}\big|B^{\bar\Gamma} - B^{\bar I_{i,j}}\big||\nabla\zeta_{\bar I_{i,j}}|
\end{aligned}
\tag{5.214}
$$

on $W_{\bar I_{i,j}}(t) \cap B_{\hat r}(\bar\Gamma(t))$ for all $t \in [0,T]$. Thanks to (5.41), the identity (5.213) is still applicable on an interface wedge. In particular, the first two right hand side terms of (5.214) can be estimated along the same lines as in *Substep 2.1.* The third right hand side term is of required order due to the compatibility estimate (5.160), the bound (5.45), and the regularity estimate (5.180).

Substep 2.3: Proof of (5.208) *on* $B_{\hat r}(\bar\Gamma(t)) \cap W_{\bar\Omega_i}(t)$. Throughout $W_{\bar\Omega_i}(t) \cap B_{\hat r}(\bar\Gamma(t))$, we may represent, as a consequence of the identity (5.194), the global velocity defined by (5.206) in form of $B = \eta_{\bar\Gamma} B^{\bar\Gamma} + \eta_{\bar I_{i,j}} B^{\bar I_{i,j}} + \eta_{\bar I_{k,i}} B^{\bar I_{k,i}}$. Plugging in (5.194) and adding zero twice then entails

$$
\begin{aligned}
&|\partial_t \eta_{\mathrm{bulk}} + (B\cdot\nabla)\eta_{\mathrm{bulk}}| \\
&\leq \big|\partial_t \lambda^{\bar I_{i,j}}_{\bar\Omega_i} + (B\cdot\nabla)\lambda^{\bar I_{i,j}}_{\bar\Omega_i}\big||\zeta_{\bar I_{i,j}} - \zeta_{\bar I_{k,i}}| \\
&\quad + \lambda^{\bar I_{i,j}}_{\bar\Omega_i}\big|\partial_t\zeta_{\bar I_{i,j}} + (B^{\bar I_{i,j}}\cdot\nabla)\zeta_{\bar I_{i,j}}\big| + (1-\lambda^{\bar I_{i,j}}_{\bar\Omega_i})\big|\partial_t\zeta_{\bar I_{i,k}} + (B^{\bar I_{k,i}}\cdot\nabla)\zeta_{\bar I_{k,i}}\big| \\
&\quad + \lambda^{\bar I_{i,j}}_{\bar\Omega_i}\eta_{\bar\Gamma}\big|B^{\bar I_{i,j}} - B^{\bar\Gamma}\big||\nabla\zeta_{\bar I_{i,j}}| + (1-\lambda^{\bar I_{i,j}}_{\bar\Omega_i})\eta_{\bar\Gamma}\big|B^{\bar I_{k,i}} - B^{\bar\Gamma}\big||\nabla\zeta_{\bar I_{k,i}}| \\
&\quad + \lambda^{\bar I_{i,j}}_{\bar\Omega_i}\eta_{\bar I_{k,i}}\big|B^{\bar I_{i,j}} - B^{\bar I_{k,i}}\big||\nabla\zeta_{\bar I_{i,j}}| + (1-\lambda^{\bar I_{i,j}}_{\bar\Omega_i})\eta_{\bar I_{i,j}}\big|B^{\bar I_{k,i}} - B^{\bar I_{i,j}}\big||\nabla\zeta_{\bar I_{k,i}}| \\
&\quad + \lambda^{\bar I_{i,j}}_{\bar\Omega_i}\eta_{\mathrm{bulk}}\big|(B^{\bar I_{i,j}}\cdot\nabla)\zeta_{\bar I_{i,j}}\big| + (1-\lambda^{\bar I_{i,j}}_{\bar\Omega_i})\eta_{\mathrm{bulk}}\big|(B^{\bar I_{k,i}}\cdot\nabla)\zeta_{\bar I_{k,i}}\big|.
\end{aligned}
\tag{5.215}
$$

The last eight right hand side terms of (5.215) can be estimated by means of the same ingredients as in the previous two substeps, relying in the process also on (5.42) and (5.43). Hence, we focus only on the first right hand side term of (5.215). Since the difference $\zeta_{\bar I_{i,j}} - \zeta_{\bar I_{k,i}}$ vanishes to first order at the triple line and has a bounded second-order spatial derivative within interpolation wedges, we have the bound

$$
|\zeta_{\bar I_{i,j}} - \zeta_{\bar I_{k,i}}| \leq C\,\mathrm{dist}^2(\cdot,\bar\Gamma) \tag{5.216}
$$

on $W_{\bar\Omega_i}(t) \cap B_{\hat r}(\bar\Gamma(t))$ for all $t \in [0,T]$. Since the advective derivative of the interpolation parameter is bounded within interpolation wedges in form of (5.129), we may add zero and exploit the property (5.33) as well as the regularity estimates (5.207) and (5.128) to obtain

$$
\big|\partial_t \lambda^{\bar I_{i,j}}_{\bar\Omega_i} + (B\cdot\nabla)\lambda^{\bar I_{i,j}}_{\bar\Omega_i}\big| \leq C \tag{5.217}
$$

throughout $W_{\bar\Omega_i}(t) \cap B_{\hat r}(\bar\Gamma(t))$ for all $t \in [0,T]$. Post-processing (5.216) by means of (5.43) thus entails (5.208) on $W_{\bar\Omega_i}(t) \cap B_{\hat r}(\bar\Gamma(t))$ for all $t \in [0,T]$.

Step 3: Proof of (5.209). Fix $i,j,k \in \{1,2,3\}$ such that $\{i,j,k\} = \{1,2,3\}$. Due to the localization properties (5.173)–(5.175), the choice (5.165) of the localization scale $\bar r$, and the regularity estimates (5.166) and (5.207), the estimate (5.209) is satisfied for trivial reasons outside of $B_{\hat r}(\bar\Gamma(t)) \cap (W_{\bar\Omega_k}(t) \cup W_{\bar\Omega_i}(t) \cup W_{\bar I_{k,i}}(t))$ for all $t \in [0,T]$.

Substep 3.1: Proof of (5.209) *on* $B_{\hat r}(\bar\Gamma(t)) \cap W_{\bar I_{k,i}}(t)$. Based on the representation (5.193) as well as the definition (5.183), it holds $\eta_{\bar I_{k,i}} = (1-\zeta_{\bar\Gamma})(1-\eta_{\mathrm{bulk}})$ on $B_{\hat r}(\bar\Gamma(t)) \cap W_{\bar I_{k,i}}(t)$ for all $t \in [0,T]$. By an application of the product rule and the already established estimate (5.208) for the advective derivative of the bulk cutoff we thus infer

$$
\big|\partial_t \eta_{\bar I_{k,i}} + (B\cdot\nabla)\eta_{\bar I_{k,i}}\big| \leq |\partial_t \zeta_{\bar\Gamma} + (B\cdot\nabla)\zeta_{\bar\Gamma}| + C(\mathrm{dist}^2(\cdot,\bar I_{i,j}) \wedge 1)
$$

on $B_{\hat{r}}(\bar{\Gamma}(t)) \cap W_{\bar{I}_{k,i}}(t)$ for all $t \in [0,T]$. Expressing $\operatorname{dist}(x,\bar{\Gamma}(t)) = |x - P_{\bar{\Gamma}}(x,t)|$ for all $x \in B_{\hat{r}}(\bar{\Gamma}(t))$ and all $t \in [0,T]$, as well as recalling the relations (5.130) and (5.33), we may compute

$$\begin{aligned} \partial_t \operatorname{dist}(x,\bar{\Gamma}(t)) &= -\frac{x - P_{\bar{\Gamma}}(x,t)}{|x - P_{\bar{\Gamma}}(x,t)|} \cdot B(P_{\bar{\Gamma}}(x,t),t) \\ &= -\big(B(P_{\bar{\Gamma}}(x,t),t) \cdot \nabla\big) \operatorname{dist}(x,\bar{\Gamma}(t)) \end{aligned} \tag{5.218}$$

for all $x \in B_{\hat{r}}(\bar{\Gamma}(t)) \setminus \bar{\Gamma}(t)$ and all $t \in [0,T]$. It is now a consequence of the chain rule and the regularity estimates (5.207) resp. (5.181) that

$$|\partial_t \zeta_{\bar{\Gamma}} + (B \cdot \nabla)\zeta_{\bar{\Gamma}}| \leq C(\operatorname{dist}^2(\cdot,\bar{\Gamma}) \wedge 1) \tag{5.219}$$

throughout $B_{\hat{r}}(\bar{\Gamma}(t)) \setminus \bar{\Gamma}(t)$ for all $t \in [0,T]$. Post-processing the previous display by means of (5.44) then yields (5.209) on $B_{\hat{r}}(\bar{\Gamma}(t)) \cap W_{\bar{I}_{k,i}}(t)$ for all $t \in [0,T]$.

Substep 3.2: Proof of (5.209) *on* $B_{\hat{r}}(\bar{\Gamma}(t)) \cap W_{\bar{\Omega}_i}(t)$. Recall (5.184)–(5.185), i.e., $\eta_{\bar{I}_{k,i}} = \lambda_{\bar{\Omega}_i}^{\bar{I}_{k,i}}(1-\zeta_{\bar{\Gamma}})\zeta_{\bar{I}_{k,i}}$ on $B_{\hat{r}}(\bar{\Gamma}(t)) \cap W_{\bar{\Omega}_i}(t)$ for all $t \in [0,T]$. It then directly follows from the product rule, the trivial estimate $1-\zeta_{\bar{\Gamma}} \leq C(\operatorname{dist}^2(\cdot,\bar{\Gamma}) \wedge 1)$, the estimate (5.217) on the advective derivative of the interpolation function $\lambda_{\bar{\Omega}_i}^{\bar{I}_{k,i}} = 1 - \lambda_{\bar{\Omega}_i}^{\bar{I}_{i,j}}$, the regularity estimates (5.180) and (5.207), the estimate (5.219), and finally the bound (5.43) that (5.209) holds true on $B_{\hat{r}}(\bar{\Gamma}(t)) \cap W_{\bar{\Omega}_i}(t)$ for all $t \in [0,T]$.

This concludes the proof of Lemma 5.33 since the argument on the other relevant interpolation wedge proceeds analogously. □

5.4.3 Approximate transport equations and motion by mean curvature

We establish the validity of the estimates (5.1d)–(5.1f) in terms of the global extensions $(\xi_{i,j})_{i,j \in \{1,2,3\}, i \neq j}$ of the unit normal vector fields from Construction 5.30 and the global extension B of the velocity field from Construction 5.32.

Lemma 5.34. *Let the assumptions and notation from Construction 5.30 and Construction 5.32 be in place. There exists a constant $C > 0$, which depends only on the data of the smoothly evolving regular double bubble $(\bar{\Omega}_1, \bar{\Omega}_2, \bar{\Omega}_3)$ on $[0,T]$, such that for all $i,j \in \{1,2,3\}$ with $i \neq j$ it holds throughout $\mathbb{R}^3 \times [0,T]$*

$$|\partial_t \xi_{i,j} + (B \cdot \nabla)\xi_{i,j} + (\nabla B)^{\mathsf{T}} \xi_{i,j}| \leq C(\operatorname{dist}(\cdot,\bar{I}_{i,j}) \wedge 1), \tag{5.220}$$

$$|B \cdot \xi_{i,j} + \nabla \cdot \xi_{i,j}| \leq C(\operatorname{dist}(\cdot,\bar{I}_{i,j}) \wedge 1), \tag{5.221}$$

$$|\xi_{i,j} \cdot (\partial_t \xi_{i,j} + (B \cdot \nabla)\xi_{i,j})| \leq C(\operatorname{dist}^2(\cdot,\bar{I}_{i,j}) \wedge 1). \tag{5.222}$$

Proof. The main point of the proof is the reduction to the corresponding assertions on the level of the local constructions $(\xi_{i,j}^{\bar{I}_{i,j}}, B^{\bar{I}_{i,j}})$ at two-phase interfaces (see Lemma 5.13) and the local construction $(\xi^{\bar{\Gamma}}, B^{\bar{\Gamma}})$ at a triple line (see Proposition 5.14). The reduction argument is facilitated by an interplay of the estimates (5.167)–(5.172) resp. (5.208) and (5.209) with sufficient compatibility of the local and global constructions. We list and prove the required compatibility estimates in a first step before starting with the proof of the bounds (5.220)–(5.222).

Step 1: Compatibility estimates. We claim that for all $i,j \in \{1,2,3\}$ with $i \neq j$ it holds on $\mathbb{R}^3 \times [0,T]$

$$\mathbb{1}_{\operatorname{supp}\eta_{\bar{I}_{i,j}}}\big|\xi_{i,j}-\xi_{i,j}^{\bar{I}_{i,j}}\big| + \mathbb{1}_{\operatorname{supp}\eta_{\bar{\Gamma}}}\big|\xi_{i,j}-\xi_{i,j}^{\bar{\Gamma}}\big| \leq C(\operatorname{dist}(\cdot,\bar{I}_{i,j}) \wedge 1), \tag{5.223}$$

$$\mathbb{1}_{\operatorname{supp}\eta_{\bar{I}_{i,j}}}\big|B-B^{\bar{I}_{i,j}}\big| + \mathbb{1}_{\operatorname{supp}\eta_{\bar{\Gamma}}}\big|B-B^{\bar{\Gamma}}\big| \leq C(\operatorname{dist}(\cdot,\bar{I}_{i,j}) \wedge 1), \tag{5.224}$$

$$\begin{aligned}\mathbb{1}_{\operatorname{supp}\eta_{\bar{I}_{i,j}}}\big|(\nabla B-\nabla B^{\bar{I}_{i,j}})^{\mathsf{T}}\xi_{i,j}^{\bar{I}_{i,j}}\big| + \mathbb{1}_{\operatorname{supp}\eta_{\bar{\Gamma}}}\big|(\nabla B-\nabla B^{\bar{\Gamma}})^{\mathsf{T}}\xi_{i,j}^{\bar{\Gamma}}\big| \\ \leq C(\operatorname{dist}(\cdot,\bar{I}_{i,j}) \wedge 1),\end{aligned} \tag{5.225}$$

$$\begin{aligned}\mathbb{1}_{\operatorname{supp}\eta_{\bar{I}_{i,j}}}\big|\big(\xi_{i,j}-\xi_{i,j}^{\bar{I}_{i,j}}\big)\cdot\xi_{i,j}^{\bar{I}_{i,j}}\big| + \mathbb{1}_{\operatorname{supp}\eta_{\bar{\Gamma}}}\big|\big(\xi_{i,j}-\xi_{i,j}^{\bar{\Gamma}}\big)\cdot\xi_{i,j}^{\bar{\Gamma}}\big| \\ \leq C(\operatorname{dist}^2(\cdot,\bar{I}_{i,j}) \wedge 1),\end{aligned} \tag{5.226}$$

$$\begin{aligned}\mathbb{1}_{\operatorname{supp}\eta_{\bar{I}_{i,j}}}\big|\xi_{i,j}^{\bar{I}_{i,j}}\cdot\big((B-B^{\bar{I}_{i,j}})\cdot\nabla\big)\xi_{i,j}^{\bar{I}_{i,j}}\big| + \mathbb{1}_{\operatorname{supp}\eta_{\bar{\Gamma}}}\big|\xi_{i,j}^{\bar{\Gamma}}\cdot\big((B-B^{\bar{\Gamma}})\cdot\nabla\big)\xi_{i,j}^{\bar{\Gamma}}\big| \\ \leq C(\operatorname{dist}^2(\cdot,\bar{I}_{i,j}) \wedge 1).\end{aligned} \tag{5.227}$$

For a proof of these compatibility estimates, we only focus on the respective first left hand side terms. The proof for the second left hand side terms follows along the same lines switching the roles of $\bar{I}_{i,j}$ and $\bar{\Gamma}$ in the process.

Inserting the definition (5.198) and exploiting the estimate (5.167) yields $\xi_{i,j}-\xi_{i,j}^{\bar{I}_{i,j}} = \eta_{\bar{\Gamma}}(\xi_{i,j}^{\bar{\Gamma}}-\xi_{i,j}^{\bar{I}_{i,j}}) - \eta_{\mathrm{bulk}}\xi_{i,j}^{\bar{I}_{i,j}} + O(\operatorname{dist}^2(\cdot,\bar{I}_{i,j}) \wedge 1)$ on $\operatorname{supp}\eta_{\bar{I}_{i,j}}$. Hence, we obtain the asserted bound (5.223) thanks to the estimates (5.158) and (5.170).

Next, the definition (5.206) together with the estimates (5.160), (5.167), (5.170) and (5.210) implies $B-B^{\bar{I}_{i,j}} = \eta_{\bar{\Gamma}}(B^{\bar{\Gamma}}-B^{\bar{I}_{i,j}})-\eta_{\mathrm{bulk}}B^{\bar{I}_{i,j}}+O(\operatorname{dist}(\cdot,\bar{I}_{i,j})\wedge 1) = O(\operatorname{dist}(\cdot,\bar{I}_{i,j})\wedge 1)$ on $\operatorname{supp}\eta_{\bar{I}_{i,j}}$ as required.

Moreover, it holds on $\operatorname{supp}\eta_{\bar{I}_{i,j}}$ as a consequence of the definition (5.206), the product rule, the already established compatibility estimate (5.224), as well as the estimates (5.167), (5.168) and (5.210) that

$$\begin{aligned}(\nabla B-\nabla B^{\bar{I}_{i,j}})^{\mathsf{T}}\xi_{i,j}^{\bar{I}_{i,j}} &= \eta_{\bar{\Gamma}}(\nabla B^{\bar{\Gamma}}-\nabla B^{\bar{I}_{i,j}})^{\mathsf{T}}\xi_{i,j}^{\bar{I}_{i,j}} - \eta_{\mathrm{bulk}}(\nabla B^{\bar{I}_{i,j}})^{\mathsf{T}}\xi_{i,j}^{\bar{I}_{i,j}} \\ &\quad + (B^{\bar{\Gamma}}\cdot\xi_{i,j}^{\bar{I}_{i,j}})\nabla\eta_{\bar{\Gamma}} + (B^{\bar{I}_{i,j}}\cdot\xi_{i,j}^{\bar{I}_{i,j}})\nabla\eta_{\bar{I}_{i,j}} + O(\operatorname{dist}(\cdot,\bar{I}_{i,j}) \wedge 1) \\ &= \eta_{\bar{\Gamma}}(\nabla B^{\bar{\Gamma}}-\nabla B^{\bar{I}_{i,j}})^{\mathsf{T}}\xi_{i,j}^{\bar{I}_{i,j}} \\ &\quad - (B\cdot\xi_{i,j}^{\bar{I}_{i,j}})\nabla\eta_{\mathrm{bulk}} - \eta_{\mathrm{bulk}}(\nabla B^{\bar{I}_{i,j}})^{\mathsf{T}}\xi_{i,j}^{\bar{I}_{i,j}} + O(\operatorname{dist}(\cdot,\bar{I}_{i,j}) \wedge 1).\end{aligned}$$

The previous display in turn implies (5.225) in view of the bounds (5.161), (5.170), (5.171), (5.210) and (5.207).

By the argument for (5.223) we also have $(\xi_{i,j}-\xi_{i,j}^{\bar{I}_{i,j}})\cdot\xi_{i,j}^{\bar{I}_{i,j}} = \eta_{\bar{\Gamma}}(\xi_{i,j}^{\bar{\Gamma}}-\xi_{i,j}^{\bar{I}_{i,j}})\cdot\xi_{i,j}^{\bar{I}_{i,j}} - \eta_{\mathrm{bulk}}|\xi_{i,j}^{\bar{I}_{i,j}}|^2 + O(\operatorname{dist}^2(\cdot,\bar{I}_{i,j}) \wedge 1)$ on $\operatorname{supp}\eta_{\bar{I}_{i,j}}$. Hence, we deduce from (5.159) and (5.170) that (5.226) holds true.

Finally, based on the definition (5.206) and the estimates (5.167), (5.201) and (5.210), we may bound on $\operatorname{supp}\eta_{\bar{I}_{i,j}}$

$$\begin{aligned}&\xi_{i,j}^{\bar{I}_{i,j}}\cdot\big((B-B^{\bar{I}_{i,j}})\cdot\nabla\big)\xi_{i,j}^{\bar{I}_{i,j}} \\ &= \eta_{\bar{\Gamma}}\big(\xi_{i,j}^{\bar{I}_{i,j}}-\xi_{i,j}^{\bar{\Gamma}}\big)\cdot\big((B^{\bar{\Gamma}}-B^{\bar{I}_{i,j}})\cdot\nabla\big)\xi_{i,j}^{\bar{I}_{i,j}} + \eta_{\bar{\Gamma}}\xi_{i,j}^{\bar{\Gamma}}\cdot\big((B^{\bar{\Gamma}}-B^{\bar{I}_{i,j}})\cdot\nabla\big)\xi_{i,j}^{\bar{I}_{i,j}} \\ &\quad - \eta_{\mathrm{bulk}}\xi_{i,j}^{\bar{I}_{i,j}}\cdot(B^{\bar{I}_{i,j}}\cdot\nabla)\xi_{i,j}^{\bar{I}_{i,j}} + O(\operatorname{dist}^2(\cdot,\bar{I}_{i,j}) \wedge 1),\end{aligned}$$

so that (5.158), (5.160), (5.170), (5.201) and (5.210) entail the desired estimate (5.227).

Step 2: Proof of (5.220). For the sake of brevity, from now on we refrain from explicitly spelling out the application of the regularity estimates (5.199), (5.201), (5.207) or (5.210), and thus solely concentrate on the error contributions in terms of the distance to the interface $\bar{I}_{i,j}$.

We start estimating based on the definition (5.198), the product rule, as well as the bounds (5.167) and (5.169)

$$\partial_t \xi_{i,j} = \eta_{\bar{\Gamma}} \partial_t \xi_{i,j}^{\bar{\Gamma}} + \eta_{\bar{I}_{i,j}} \partial_t \xi_{i,j}^{\bar{I}_{i,j}} + \xi_{i,j}^{\bar{\Gamma}} \partial_t \eta_{\bar{\Gamma}} + \xi_{i,j}^{\bar{I}_{i,j}} \partial_t \eta_{\bar{I}_{i,j}} + O(\mathrm{dist}(\cdot, \bar{I}_{i,j}) \wedge 1).$$

As a consequence of the compatibility estimate (5.223) and the bounds (5.169), we may add zero twice and obtain

$$\begin{aligned}
\xi_{i,j}^{\bar{\Gamma}} \partial_t \eta_{\bar{\Gamma}} + \xi_{i,j}^{\bar{I}_{i,j}} \partial_t \eta_{\bar{I}_{i,j}} &= \xi_{i,j} (\partial_t \eta_{\bar{\Gamma}} + \partial_t \eta_{\bar{I}_{i,j}}) + O(\mathrm{dist}(\cdot, \bar{I}_{i,j}) \wedge 1) \\
&= -\xi_{i,j} \partial_t \eta_{\mathrm{bulk}} + O(\mathrm{dist}(\cdot, \bar{I}_{i,j}) \wedge 1).
\end{aligned}$$

The previous two displays combine to

$$\partial_t \xi_{i,j} = \eta_{\bar{\Gamma}} \partial_t \xi_{i,j}^{\bar{\Gamma}} + \eta_{\bar{I}_{i,j}} \partial_t \xi_{i,j}^{\bar{I}_{i,j}} - \xi_{i,j} \partial_t \eta_{\mathrm{bulk}} + O(\mathrm{dist}(\cdot, \bar{I}_{i,j}) \wedge 1). \tag{5.228}$$

Replacing the differential operator ∂_t by $(B \cdot \nabla)$ in the previous argument entails

$$\begin{aligned}
(B \cdot \nabla) \xi_{i,j} = \eta_{\bar{\Gamma}} (B \cdot \nabla) \xi_{i,j}^{\bar{\Gamma}} &+ \eta_{\bar{I}_{i,j}} (B \cdot \nabla) \xi_{i,j}^{\bar{I}_{i,j}} \\
&- \xi_{i,j} (B \cdot \nabla) \eta_{\mathrm{bulk}} + O(\mathrm{dist}(\cdot, \bar{I}_{i,j}) \wedge 1).
\end{aligned}$$

Making use of the compatibility estimate (5.224) updates the previous display to

$$\begin{aligned}
(B \cdot \nabla) \xi_{i,j} = \eta_{\bar{\Gamma}} (B^{\bar{\Gamma}} \cdot \nabla) \xi_{i,j}^{\bar{\Gamma}} &+ \eta_{\bar{I}_{i,j}} (B^{\bar{I}_{i,j}} \cdot \nabla) \xi_{i,j}^{\bar{I}_{i,j}} \\
&- \xi_{i,j} (B \cdot \nabla) \eta_{\mathrm{bulk}} + O(\mathrm{dist}(\cdot, \bar{I}_{i,j}) \wedge 1).
\end{aligned} \tag{5.229}$$

Inserting the definition (5.198), recalling the estimate (5.167), and adding zero based on the compatibility estimate (5.225) moreover allows to estimate

$$\begin{aligned}
(\nabla B)^{\mathsf{T}} \xi_{i,j} &= \eta_{\bar{\Gamma}} (\nabla B)^{\mathsf{T}} \xi_{i,j}^{\bar{\Gamma}} + \eta_{\bar{I}_{i,j}} (\nabla B)^{\mathsf{T}} \xi_{i,j}^{\bar{I}_{i,j}} + O(\mathrm{dist}(\cdot, \bar{I}_{i,j}) \wedge 1) \\
&= \eta_{\bar{\Gamma}} (\nabla B^{\bar{\Gamma}})^{\mathsf{T}} \xi_{i,j}^{\bar{\Gamma}} + \eta_{\bar{I}_{i,j}} (\nabla B^{\bar{I}_{i,j}})^{\mathsf{T}} \xi_{i,j}^{\bar{I}_{i,j}} + O(\mathrm{dist}(\cdot, \bar{I}_{i,j}) \wedge 1).
\end{aligned} \tag{5.230}$$

The desired estimate (5.220) thus follows from (5.228)–(5.230), the estimate (5.208) of the advective derivative of the bulk cutoff, as well as the local versions (5.26) and (5.35) of (5.220), respectively.

Step 3: Proof of (5.221). We compute as a consequence of the definition (5.198), the estimate (5.167), and the compatibility estimate (5.224)

$$\begin{aligned}
B \cdot \xi_{i,j} &= \eta_{\bar{\Gamma}} B \cdot \xi_{i,j}^{\bar{\Gamma}} + \eta_{\bar{I}_{i,j}} B \cdot \xi_{i,j}^{\bar{I}_{i,j}} + O(\mathrm{dist}(\cdot, \bar{I}_{i,j}) \wedge 1) \\
&= \eta_{\bar{\Gamma}} B^{\bar{\Gamma}} \cdot \xi_{i,j}^{\bar{\Gamma}} + \eta_{\bar{I}_{i,j}} B^{\bar{I}_{i,j}} \cdot \xi_{i,j}^{\bar{I}_{i,j}} + O(\mathrm{dist}(\cdot, \bar{I}_{i,j}) \wedge 1).
\end{aligned} \tag{5.231}$$

We also directly estimate by means of the definition (5.198), the estimate (5.168), as well as the compatibility estimate (5.223)

$$\begin{aligned}
\nabla \cdot \xi_{i,j} &= \eta_{\bar{\Gamma}} \nabla \cdot \xi_{i,j}^{\bar{\Gamma}} + \eta_{\bar{I}_{i,j}} \nabla \cdot \xi_{i,j}^{\bar{I}_{i,j}} + (\xi_{i,j}^{\bar{\Gamma}} \cdot \nabla) \eta_{\bar{\Gamma}} + (\xi_{i,j}^{\bar{I}_{i,j}} \cdot \nabla) \eta_{\bar{I}_{i,j}} + O(\mathrm{dist}(\cdot, \bar{I}_{i,j}) \wedge 1) \\
&= \eta_{\bar{\Gamma}} \nabla \cdot \xi_{i,j}^{\bar{\Gamma}} + \eta_{\bar{I}_{i,j}} \nabla \cdot \xi_{i,j}^{\bar{I}_{i,j}} - (\xi_{i,j} \cdot \nabla) \eta_{\mathrm{bulk}} + O(\mathrm{dist}(\cdot, \bar{I}_{i,j}) \wedge 1).
\end{aligned} \tag{5.232}$$

Hence, the estimate (5.221) follows by combining (5.231)–(5.232), the estimate (5.171) for the bulk cutoff, and the local versions of (5.221) given by (5.27) and (5.36), respectively.

Step 4: Proof of (5.222). Plugging in the definition (5.198), recalling the estimate (5.167), and denoting by $k \in \{1,2,3\} \setminus \{i,j\}$ the remaining phase yields

$$\begin{aligned}\xi_{i,j} \cdot \partial_t \xi_{i,j} &= \eta_{\bar\Gamma} \xi^{\bar\Gamma}_{i,j} \cdot \partial_t \xi_{i,j} + \eta_{\bar I_{i,j}} \xi^{\bar I_{i,j}}_{i,j} \cdot \partial_t \xi_{i,j} + O(\mathrm{dist}^2(\cdot, \bar I_{i,j}) \wedge 1) \\ &= \eta^2_{\bar\Gamma} \xi^{\bar\Gamma}_{i,j} \cdot \partial_t \xi^{\bar\Gamma}_{i,j} + \eta^2_{\bar I_{i,j}} \xi^{\bar I_{i,j}}_{i,j} \cdot \partial_t \xi^{\bar I_{i,j}}_{i,j} \\ &\quad + \eta_{\bar\Gamma}\eta_{\bar I_{i,j}} \xi^{\bar\Gamma}_{i,j} \cdot \partial_t \xi^{\bar I_{i,j}}_{i,j} + \eta_{\bar\Gamma}\eta_{\bar I_{i,j}} \xi^{\bar I_{i,j}}_{i,j} \cdot \partial_t \xi^{\bar\Gamma}_{i,j} \\ &\quad + \eta_{\bar\Gamma} \xi^{\bar\Gamma}_{i,j} \cdot \big(\xi^{\bar\Gamma}_{i,j} \partial_t \eta_{\bar\Gamma} + \xi^{\bar I_{i,j}}_{i,j} \partial_t \eta_{\bar I_{i,j}} + \xi^{\bar I_{j,k}}_{j,k} \partial_t \eta_{\bar I_{j,k}} + \xi^{\bar I_{k,i}}_{k,i} \partial_t \eta_{\bar I_{k,i}}\big) \\ &\quad + \eta_{\bar I_{i,j}} \xi^{\bar I_{i,j}}_{i,j} \cdot \big(\xi^{\bar\Gamma}_{i,j} \partial_t \eta_{\bar\Gamma} + \xi^{\bar I_{i,j}}_{i,j} \partial_t \eta_{\bar I_{i,j}} + \xi^{\bar I_{j,k}}_{j,k} \partial_t \eta_{\bar I_{j,k}} + \xi^{\bar I_{k,i}}_{k,i} \partial_t \eta_{\bar I_{k,i}}\big) \\ &\quad + O(\mathrm{dist}^2(\cdot, \bar I_{i,j}) \wedge 1).\end{aligned}$$

The compatibility estimates (5.223) and (5.226) in combination with the bounds (5.167), and (5.170) provide an upgrade of the previous display in form of

$$\begin{aligned}\xi_{i,j} \cdot \partial_t \xi_{i,j} &= \eta^2_{\bar\Gamma} \xi^{\bar\Gamma}_{i,j} \cdot \partial_t \xi^{\bar\Gamma}_{i,j} + \eta^2_{\bar I_{i,j}} \xi^{\bar I_{i,j}}_{i,j} \cdot \partial_t \xi^{\bar I_{i,j}}_{i,j} \\ &\quad + \eta_{\bar\Gamma}\eta_{\bar I_{i,j}} \xi^{\bar\Gamma}_{i,j} \cdot \partial_t \xi^{\bar I_{i,j}}_{i,j} + \eta_{\bar\Gamma}\eta_{\bar I_{i,j}} \xi^{\bar I_{i,j}}_{i,j} \cdot \partial_t \xi^{\bar\Gamma}_{i,j} \\ &\quad + \eta_{\bar\Gamma} (\xi^{\bar\Gamma}_{i,j} \cdot \xi_{i,j}) \partial_t (\eta_{\bar\Gamma} + \eta_{\bar I_{i,j}}) \\ &\quad + \eta_{\bar I_{i,j}} (\xi^{\bar I_{i,j}}_{i,j} \cdot \xi_{i,j}) \partial_t (\eta_{\bar\Gamma} + \eta_{\bar I_{i,j}}) \\ &\quad + \eta_{\bar\Gamma} \xi^{\bar\Gamma}_{i,j} \cdot \big(\xi^{\bar I_{j,k}}_{j,k} \partial_t \eta_{\bar I_{j,k}} + \xi^{\bar I_{k,i}}_{k,i} \partial_t \eta_{\bar I_{k,i}}\big) \\ &\quad + \eta_{\bar I_{i,j}} \xi^{\bar I_{i,j}}_{i,j} \cdot \big(\xi^{\bar I_{j,k}}_{j,k} \partial_t \eta_{\bar I_{j,k}} + \xi^{\bar I_{k,i}}_{k,i} \partial_t \eta_{\bar I_{k,i}}\big) \\ &\quad + O(\mathrm{dist}^2(\cdot, \bar I_{i,j}) \wedge 1).\end{aligned} \tag{5.233}$$

Substituting the differential operator $(B\cdot\nabla)$ for ∂_t in the previous argument, making use of the compatibility estimates (5.227), (5.223) and (5.224), and exploiting twice the estimate (5.209) then shows that

$$\begin{aligned}&\xi_{i,j} \cdot (\partial_t + B \cdot \nabla) \xi_{i,j} \\ &= \eta^2_{\bar\Gamma} \xi^{\bar\Gamma}_{i,j} \cdot (\partial_t + B^{\bar\Gamma} \cdot \nabla) \xi^{\bar\Gamma}_{i,j} + \eta^2_{\bar I_{i,j}} \xi^{\bar I_{i,j}}_{i,j} \cdot (\partial_t + B^{\bar I_{i,j}} \cdot \nabla) \xi^{\bar I_{i,j}}_{i,j} \\ &\quad + \eta_{\bar\Gamma}\eta_{\bar I_{i,j}} \xi^{\bar\Gamma}_{i,j} \cdot (\partial_t + B^{\bar I_{i,j}} \cdot \nabla) \xi^{\bar I_{i,j}}_{i,j} + \eta_{\bar\Gamma}\eta_{\bar I_{i,j}} \xi^{\bar I_{i,j}}_{i,j} \cdot (\partial_t + B^{\bar\Gamma} \cdot \nabla) \xi^{\bar\Gamma}_{i,j} \\ &\quad - \eta_{\bar\Gamma} (\xi^{\bar\Gamma}_{i,j} \cdot \xi_{i,j}) (\partial_t + B \cdot \nabla) \eta_{\mathrm{bulk}} - \eta_{\bar I_{i,j}} (\xi^{\bar I_{i,j}}_{i,j} \cdot \xi_{i,j}) (\partial_t + B \cdot \nabla) \eta_{\mathrm{bulk}} \\ &\quad + O(\mathrm{dist}^2(\cdot, \bar I_{i,j}) \wedge 1).\end{aligned}$$

Hence, employing the local versions (5.28) and (5.37) of (5.222) and making use of the estimate (5.208) for the bulk cutoff shows that

$$\begin{aligned}&\xi_{i,j} \cdot (\partial_t + B \cdot \nabla) \xi_{i,j} \\ &= \eta_{\bar\Gamma}\eta_{\bar I_{i,j}} \xi^{\bar\Gamma}_{i,j} \cdot (\partial_t + B^{\bar I_{i,j}} \cdot \nabla) \xi^{\bar I_{i,j}}_{i,j} + \eta_{\bar\Gamma}\eta_{\bar I_{i,j}} \xi^{\bar I_{i,j}}_{i,j} \cdot (\partial_t + B^{\bar\Gamma} \cdot \nabla) \xi^{\bar\Gamma}_{i,j} \\ &\quad + O(\mathrm{dist}^2(\cdot, \bar I_{i,j}) \wedge 1).\end{aligned} \tag{5.234}$$

Adding zero, making use of the local evolution equations (5.26) resp. (5.28), and exploiting the compatibility estimates (5.223) and (5.225) further implies that

$$\begin{aligned}&\eta_{\bar\Gamma}\eta_{\bar I_{i,j}} \xi^{\bar\Gamma}_{i,j} \cdot \big(\partial_t \xi^{\bar I_{i,j}}_{i,j} + (B^{\bar I_{i,j}} \cdot \nabla) \xi^{\bar I_{i,j}}_{i,j}\big) \\ &= \eta_{\bar\Gamma}\eta_{\bar I_{i,j}} \xi^{\bar\Gamma}_{i,j} \cdot \big(\partial_t \xi^{\bar I_{i,j}}_{i,j} + (B^{\bar I_{i,j}} \cdot \nabla) \xi^{\bar I_{i,j}}_{i,j} + (\nabla B^{\bar I_{i,j}})^{\mathsf T} \xi^{\bar I_{i,j}}_{i,j}\big) - \eta_{\bar\Gamma}\eta_{\bar I_{i,j}} \xi^{\bar\Gamma}_{i,j} \cdot (\nabla B^{\bar I_{i,j}})^{\mathsf T} \xi^{\bar I_{i,j}}_{i,j} \\ &= -\eta_{\bar\Gamma}\eta_{\bar I_{i,j}} (\xi^{\bar I_{i,j}}_{i,j} - \xi^{\bar\Gamma}_{i,j}) (\nabla B)^{\mathsf T} \xi^{\bar I_{i,j}}_{i,j} + O(\mathrm{dist}^2(\cdot, \bar I_{i,j}) \wedge 1).\end{aligned}$$

Switching the roles of $\bar{\Gamma}$ and $\bar{I}_{i,j}$ in the argument leading to the previous display, relying in the process on the local evolution equations (5.35) resp. (5.37), we then in summary obtain together with (5.223)

$$\begin{aligned}
&\eta_{\bar{\Gamma}}\eta_{\bar{I}_{i,j}}\xi_{i,j}^{\bar{\Gamma}}\cdot\big(\partial_t\xi_{i,j}^{\bar{I}_{i,j}}+(B^{\bar{I}_{i,j}}\cdot\nabla)\xi_{i,j}^{\bar{I}_{i,j}}\big)+\eta_{\bar{\Gamma}}\eta_{\bar{I}_{i,j}}\xi_{i,j}^{\bar{I}_{i,j}}\cdot\big(\partial_t\xi_{i,j}^{\bar{\Gamma}}+(B^{\bar{\Gamma}}\cdot\nabla)\xi_{i,j}^{\bar{\Gamma}}\big)\\
&=-\eta_{\bar{\Gamma}}\eta_{\bar{I}_{i,j}}\big(\xi_{i,j}^{\bar{I}_{i,j}}-\xi_{i,j}^{\bar{\Gamma}}\big)(\nabla B)^{\mathsf{T}}\big(\xi_{i,j}^{\bar{I}_{i,j}}-\xi_{i,j}^{\bar{\Gamma}}\big)+O(\operatorname{dist}^2(\cdot,\bar{I}_{i,j})\wedge 1)\\
&=O(\operatorname{dist}^2(\cdot,\bar{I}_{i,j})\wedge 1).
\end{aligned}\tag{5.235}$$

The combination of the estimates (5.234) and (5.235) thus entails the bound (5.222). □

5.4.4 Existence of a gradient-flow calibration: Proof of Theorem 5.3

This is only a matter of collecting already established facts. More precisely, the required regularity for $((\xi_{i,j})_{i,j\in\{1,2,3\},i\neq j},B)$ is part of Lemma 5.31 and Lemma 5.33, respectively. The calibration resp. extension property (5.1a) as well as the coercivity estimate (5.1c) for the extensions of the unit normal vector fields follow from Lemma 5.31. The estimates (5.1d)–(5.1f) are finally the content of Lemma 5.34. □

5.5 Existence of transported weights

Proof of Proposition 5.5. The proof proceeds in several steps.

Step 1: Construction of an auxiliary family of transported weights. We first fix a smooth truncation of the identity. More precisely, let $\vartheta\colon\mathbb{R}\to\mathbb{R}$ be a smooth and non-decreasing map such that $\vartheta(r)=r$ for $|r|\leq\frac{1}{2}$, $\vartheta(r)=1$ for $r\geq 1$ and $\vartheta(r)=-1$ for $r\leq -1$. Let $\hat{r}\in(0,1]$ be the localization scale of Proposition 5.14, let $\bar{r}\in(0,1]$ be the localization scale defined by (5.165), and let finally $\delta\in(0,1]$ be the constant from *Step 2* of the proof of Lemma 5.29 (cf. the defining property (5.187) for all $i,j\in\{1,2,3\}$, $i\neq j$). We then define building blocks

$$\vartheta_{i,j}:=\vartheta\Big(\frac{s_{i,j}}{\delta\bar{r}}\Big)\qquad\text{in }\operatorname{im}(\Psi_{i,j}),\tag{5.236}$$

$$\vartheta_{\mathrm{ext}}:=\vartheta\Big(\frac{\operatorname{dist}(\cdot,\bar{\Gamma})}{\hat{r}}\Big)\qquad\text{in }\mathbb{R}^3\times[0,T].\tag{5.237}$$

Note that by definition (5.165) of the localization scale $\bar{r}$, we have for all phases $i\in\{1,2,3\}$ a covering of $\partial\bar{\Omega}_i$ in form of

$$\partial\bar{\Omega}_i\subset B_{\hat{r}}(\bar{\Gamma}(t))\cup\bigcup_{j\in\{1,2,3\},j\neq i}\operatorname{im}_{\bar{r}}(\Psi_{i,j})(t)\setminus B_{\hat{r}}(\bar{\Gamma}(t))=:\mathcal{N}_{\hat{r},\bar{r}}^{\partial\bar{\Omega}_i}(t),\tag{5.238}$$

for all $t\in[0,T]$, and where we abbreviated

$$\operatorname{im}_{\bar{r}}(\Psi_{i,j})(t):=\Psi_{i,j}(\bar{I}_{i,j}(t)\times\{t\}\times[-\bar{r},\bar{r}]),\quad t\in[0,T].$$

Note that this also implies a disjoint covering of $\mathbb{R}^3$ by means of

$$\mathbb{R}^3=\mathcal{N}_{\hat{r},\bar{r}}^{\partial\bar{\Omega}_i}(t)\cup\big(\bar{\Omega}_i(t)\setminus\mathcal{N}_{\hat{r},\bar{r}}^{\partial\bar{\Omega}_i}(t)\big)\cup\big((\mathbb{R}^3\setminus\bar{\Omega}_i(t))\setminus\mathcal{N}_{\hat{r},\bar{r}}^{\partial\bar{\Omega}_i}(t)\big)\tag{5.239}$$

for all $t\in[0,T]$.

For each phase $i\in\{1,2,3\}$, denote by $j,k\in\{1,2,3\}\setminus\{i\}$ the remaining two phases. We then define, based on the building blocks (5.236) and (5.237), a weight $\hat{\vartheta}_i\colon\mathbb{R}^3\times[0,T]\to[-1,1]$

by means of

$$\hat{\vartheta}_i(\cdot,t) := \vartheta_{i,\ell}(\cdot,t) \qquad \text{in } \mathrm{im}_{\bar{r}}(\Psi_{i,j})(t) \setminus B_{\hat{r}}(\bar{\Gamma}(t)),\ \ell \neq i, \tag{5.240}$$

$$\hat{\vartheta}_i(\cdot,t) := \vartheta_{i,\ell}(\cdot,t) \qquad \text{in } \overline{W_{\bar{I}_{i,j}}(t)} \cap B_{\hat{r}}(\bar{\Gamma}(t)),\ \ell \neq i, \tag{5.241}$$

$$\hat{\vartheta}_i(\cdot,t) := \lambda^{\bar{I}_{i,j}}_{\bar{\Omega}_i}(\cdot,t)\vartheta_{i,j}(\cdot,t) + \lambda^{\bar{I}_{k,i}}_{\bar{\Omega}_i}(\cdot,t)\vartheta_{i,k}(\cdot,t) \qquad \text{in } W_{\bar{\Omega}_i}(t) \cap B_{\hat{r}}(\bar{\Gamma}(t)), \tag{5.242}$$

$$\hat{\vartheta}_i(\cdot,t) := \vartheta_{\mathrm{ext}}(\cdot,t) \qquad \text{in } \overline{W_{\bar{I}_{j,k}}(t)} \cap B_{\hat{r}}(\bar{\Gamma}(t)), \tag{5.243}$$

$$\hat{\vartheta}_i(\cdot,t) := \lambda^{\bar{I}_{i,j}}_{\bar{\Omega}_j}(\cdot,t)\vartheta_{i,j}(\cdot,t) + \lambda^{\bar{I}_{j,k}}_{\bar{\Omega}_j}(\cdot,t)\vartheta_{\mathrm{ext}}(\cdot,t) \qquad \text{in } W_{\bar{\Omega}_j}(t) \cap B_{\hat{r}}(\bar{\Gamma}(t)), \tag{5.244}$$

$$\hat{\vartheta}_i(\cdot,t) := \lambda^{\bar{I}_{k,i}}_{\bar{\Omega}_k}(\cdot,t)\vartheta_{i,k}(\cdot,t) + \lambda^{\bar{I}_{j,k}}_{\bar{\Omega}_k}(\cdot,t)\vartheta_{\mathrm{ext}}(\cdot,t) \qquad \text{in } W_{\bar{\Omega}_k}(t) \cap B_{\hat{r}}(\bar{\Gamma}(t)), \tag{5.245}$$

$$\hat{\vartheta}_i(\cdot,t) := -1 \qquad \text{in } \bar{\Omega}_i(t) \setminus \mathcal{N}^{\partial\bar{\Omega}_i}_{\hat{r},\bar{r}}(t), \tag{5.246}$$

$$\hat{\vartheta}_i(\cdot,t) := 1 \qquad \text{else} \tag{5.247}$$

for all $t \in [0,T]$. For the construction and properties of the interpolation functions, we refer to Lemma 5.29. Note that $\hat{\vartheta}_i$ is well-defined in view of (5.238), (5.239) and (5.40). Moreover, due to the defining property (5.187) of the constant $\delta \in (0,1]$, we infer that $\hat{\vartheta}_i$ is continuous throughout $\mathbb{R}^3 \times [0,T]$.

Step 2: Properties of the auxiliary family of transported weights. In this step, we verify that the auxiliary family $\hat{\vartheta} = (\hat{\vartheta}_i)_{i\in\{1,2,3\}}$ satisfies all the requirements of Definition 5.4 with the (obvious) exception that $\hat{\vartheta}_i \in L^1(\mathbb{R}^3 \times [0,T])$. The $W^{1,\infty}$-regularity on $\mathbb{R}^3 \times [0,T]$ as well as the required conditions from item *ii)* of Definition 5.4 are immediate from the definitions (5.240)–(5.247). Hence, we focus in the following on the deduction of the advection estimate (5.2).

Substep 2.1: Preliminary estimates. We first claim that for all $i,j \in \{1,2,3\}$ with $i \neq j$ and all $t \in [0,T]$ it holds

$$|\partial_t\vartheta_{i,j} + (B\cdot\nabla)\vartheta_{i,j}|(\cdot,t) \leq C\,\mathrm{dist}(\cdot,\partial\bar{\Omega}_i(t)) \quad \text{in } \mathrm{im}_{\bar{r}}(\Psi_{i,j})(t) \setminus B_{\hat{r}}(\bar{\Gamma}(t)), \tag{5.248}$$

$$|\partial_t\vartheta_{i,j} + (B\cdot\nabla)\vartheta_{i,j}|(\cdot,t) \leq C\,\mathrm{dist}(\cdot,\partial\bar{\Omega}_i(t)) \quad \text{in } B_{\hat{r}}(\bar{\Gamma}(t)) \cap \big(W_{\bar{I}_{i,j}}(t) \cup W_{\bar{\Omega}_i}(t) \cup W_{\bar{\Omega}_j}(t)\big), \tag{5.249}$$

$$|\partial_t\vartheta_{\mathrm{ext}} + (B\cdot\nabla)\vartheta_{\mathrm{ext}}|(\cdot,t) \leq C\,\mathrm{dist}(\cdot,\bar{\Gamma}(t)) \quad \text{in } B_{\hat{r}}(\bar{\Gamma}(t)) \setminus \bar{\Gamma}(t). \tag{5.250}$$

We start with a proof of (5.248). It follows from the representation (5.192) and the definition (5.206) that $B = \eta_{\bar{I}_{i,j}} B^{\bar{I}_{i,j}}$ in $\mathrm{im}_{\bar{r}}(\Psi_{i,j})(t) \setminus B_{\hat{r}}(\bar{\Gamma}(t))$ for all $t \in [0,T]$. We may then estimate by the chain rule, the definition (5.236), the identity (5.212), the representation (5.192), as well as the estimate (5.170)

$$|\partial_t\vartheta_{i,j} + (B\cdot\nabla)\vartheta_{i,j}| \leq \eta_{\mathrm{bulk}}|(B^{\bar{I}_{i,j}}\cdot\nabla)\vartheta_{i,j}| \leq C\,\mathrm{dist}(\cdot,\bar{\Omega}_i)$$

throughout $\mathrm{im}_{\bar{r}}(\Psi_{i,j})(t) \setminus B_{\hat{r}}(\bar{\Gamma}(t))$ for all $t \in [0,T]$.

We next prove (5.249). Throughout the interface wedge $W_{\bar{I}_{i,j}}(t) \cap B_{\hat{r}}(\bar{\Gamma}(t))$, it holds $B = \eta_{\bar{\Gamma}} B^{\bar{\Gamma}} + \eta_{\bar{I}_{i,j}} B^{\bar{I}_{i,j}}$ thanks to the representation (5.193) and the definition (5.206). Employing (5.193) once more, we then estimate making also use of the chain rule, the definition (5.236) and the identity (5.212)

$$|\partial_t\vartheta_{i,j} + (B\cdot\nabla)\vartheta_{i,j}| \leq \eta_{\mathrm{bulk}}|(B^{\bar{I}_{i,j}}\cdot\nabla)\vartheta_{i,j}| + \eta_{\bar{\Gamma}}\big|\big((B^{\bar{\Gamma}} - B^{\bar{I}_{i,j}})\cdot\nabla\big)\vartheta_{i,j}\big|$$

on $W_{\bar{I}_{i,j}}(t) \cap B_{\hat{r}}(\bar{\Gamma}(t))$ for all $t \in [0,T]$. Post-processing the previous display by means of (5.170), (5.160) and (5.45) thus yields (5.249) on $W_{\bar{I}_{i,j}}(t) \cap B_{\hat{r}}(\bar{\Gamma}(t))$, $t \in [0,T]$.

Throughout $W_{\bar{\Omega}_i}(t) \cap B_{\hat{r}}(\bar{\Gamma}(t))$, we may write, as a consequence of the representation (5.194), the global velocity defined by (5.206) in form of $B = \eta_{\bar{\Gamma}} B^{\bar{\Gamma}} + \eta_{\bar{I}_{i,j}} B^{\bar{I}_{i,j}} + \eta_{\bar{I}_{k,i}} B^{\bar{I}_{k,i}}$. Hence, based on the same ingredients as in the case of interface wedges we may estimate

$$
\begin{aligned}
&|\partial_t \vartheta_{i,j} + (B\cdot\nabla)\vartheta_{i,j}| \\
&\leq \eta_{\mathrm{bulk}} |(B^{\bar{I}_{i,j}}\cdot\nabla)\vartheta_{i,j}| + \eta_{\bar{\Gamma}} \big|\big((B^{\bar{\Gamma}} - B^{\bar{I}_{i,j}})\cdot\nabla\big)\vartheta_{i,j}\big| + \eta_{\bar{I}_{k,i}} \big|\big((B^{\bar{I}_{k,i}} - B^{\bar{I}_{i,j}})\cdot\nabla\big)\vartheta_{i,j}\big|
\end{aligned}
$$

on $W_{\bar{\Omega}_i}(t) \cap B_{\hat{r}}(\bar{\Gamma}(t))$ for all $t \in [0,T]$. The previous display in turn upgrades to the desired estimate (5.249) thanks to (5.170), (5.160) and (5.43).

Finally, the estimate (5.250) is a direct consequence of the chain rule, the definition (5.237), the identity (5.218) and the regularity estimate (5.207).

Substep 2.2: Proof of (5.2) *in terms of* $(\hat{\vartheta}_i)_{i\in\{1,2,3\}}$. We first observe that as a consequence of the definitions (5.240)–(5.247), there exists $C \geq 1$ such that

$$
\frac{1}{C}|\hat{\vartheta}_i| \leq \mathrm{dist}(\cdot, \partial\bar{\Omega}_i) \leq C|\hat{\vartheta}_i| \quad \text{in } \mathbb{R}^3 \times [0,T]. \tag{5.251}
$$

Modulo this post-processing, the claim (5.2) in terms of $\hat{\vartheta}_i$ is then directly implied for the regions (5.240), (5.241) and (5.243) by the estimates (5.248)–(5.250) and (5.44). Furthermore, the only additional ingredients needed in the interpolation regions (5.242), (5.244) and (5.245) are given by the estimate (5.217) for the interpolation functions as well as the bound (5.43). Since there is nothing to prove for the regions (5.246) and (5.247), this in turn concludes the proof of (5.2) in terms of $(\hat{\vartheta}_i)_{i\in\{1,2,3\}}$.

Step 3: Enforcing integrability of the weights. We slightly modify the construction from the previous step to take care of the integrability issue. To this end, we first choose a smooth and concave function $\kappa\colon [0,\infty) \to [0,1]$ such that $\kappa(0) = 0$ as well as $\kappa(r) = 1$ for $r \geq 1$. Which we think of as an upper concave approximation of the map $r \mapsto r \wedge 1$ on the interval $[0,\infty)$. Choose a sufficiently large radius $R > 0$ such that

$$
\bigcup_{t\in[0,T]} \bigcup_{i,j\in\{1,2,3\}, i\neq j} B_{\hat{r}}(\bar{I}_{i,j}(t)) \times \{t\} \subset\subset B_R(0). \tag{5.252}
$$

We then define a weight $\eta_R \in W^{1,\infty}(\mathbb{R}^3) \cap W^{1,1}(\mathbb{R}^3)$ by means of

$$
\eta_R(x) := \kappa(\exp(R - |x|)), \quad x \in \mathbb{R}^3, \tag{5.253}
$$

with its spatial gradient being bounded in form of

$$
|\nabla \eta_R| \leq C|\eta_R| \quad \text{in } \mathbb{R}^3. \tag{5.254}
$$

With all of these ingredients in place, we may finally define $\vartheta_i := \eta_R \hat{\vartheta}_i$ for all phases $i \in \{1,2,3\}$. Note that $\vartheta_i \in W^{1,1}(\mathbb{R}^3 \times [0,T]; [-1,1])$ as desired. Moreover, the weights ϑ_i directly inherit all the other required properties of Definition 5.4 from the auxiliary weights $\hat{\vartheta}_i$ of the previous step, as can be seen from the definitions. □

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